An explicit bound on ε for non-emptiness of the ε-core of an arbitrary game without side payments.*

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Abstract

We consider parameterized collections of games without side payments and determine a bound on ε so that all sufficiently large games in the collection have non-empty ε-cores. The required size of ε depends on: (a) the number of approximate player types and the accuracy of this approximation; (b) the size of approximately effective small groups; (c) the size of the bound on maximal per-capita payoffs achievable in coalitions; and (d) a measure on the extent to which the payoff sets of the games are bounded away from being "flat." Our

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result makes explicit the relationship between the required size of ε for non-emptiness of the ε-core, the parameters describing the collection of games, and the size of the total player set. Given the parameters describing the collection, the larger the game, the smaller the ε that can be chosen. Results are provided both for the case where the size of effective coalitions is bounded in absolute size of membership and for the case where relative sizes of effective coalitions are small.

1 Introduction.

In this paper we consider parameterized collections of games without side payments and obtain an explicit bound on ε so that all sufficiently large games in the collection have non-empty ε-cores. The required size of ε depends on the parameters describing the games: (a) the number of approximate player types and the accuracy of this approximation; (b) an upper bound on the size of near-effective groups and the closeness of these groups to being effective for the realization of all gains to collective activities; (c) the supremum of per-capita payoffs achievable in coalitions; and (d) a measure of the extent to which boundaries of payoff sets are bounded away from being "flat." Given these parameters, the larger the number of players in a game, the smaller the ε that can be chosen so that the game has a non-empty ε-cores and for very large numbers of players, ε can be very small. The bound on the number of players in the game is expressed in terms of the parameters describing the games; thus it is apparent how the bound on ε varies as the parameters are varied.

The framework of parameterized collections of games was introduced in Kovalenkov and Wooders (1997a). In the first paper, we provided a result showing that, given a parameterized collection of games, all sufficiently large games have non-empty ε-cores. An advantage of the result is that no convexity assumptions were required. No explicit bounds, however, on ε or on the number of players required to ensure that the ε-core be non-empty, were provided. Such bounds are of interest since they show how the required size of ε depends on the parameters describing the games. In this paper using significantly different techniques and the assumption of convexity of payoff sets, we are able to derive an explicit lower bound on ε.

We provide three results, two for games with nearly-effective groups bounded in absolute size of membership and a third for sequences of games where relative sizes of nearly-effective groups become small as the terms in the sequence grow large. The first two results are established using different notions of distance to describe nearly effective small groups. Our first Proposition uses the same notion of distance as in our prior work (Kovalenkov and Wooders (1997a)) and in Wooders (1983). Our main Theorem uses a second, much less demanding notion of distance, but nevertheless we obtain an analogous result and in fact, the lower bound on ε for non-emptiness of the ε-core may be significantly improved by using our Theorem rather than the

Proposition. An example is provided illustrating such improvement. Our third result, for sequences of games, is motivated primarily by the fact that continuum games may be viewed, in a number of circumstances, as limits of large finite games with small effective groups.

The next section of this paper develops our model. Related literature is discussed in the concluding section of the paper. We note here, however, that our results are obtained by extending similar results for games with side payments in Kovalenkov and Wooders (1997c) to games without side payments. We also use techniques related to those of Scarf (1965) (an earlier unpublished version of his well-known paper Scarf (1967)), showing that balanced games have non-empty cores. (See also Billera (1970)).

2 Definitions.

2.1 Cooperative games: description and notation.

Let $N = \{1, ..., n\}$ denote a *set of players.* A non-empty subset of N is called a *coalition.* For any coalition S let $\mathbf{R}^S$ denote the $|S|$-dimensional Euclidean space with coordinates indexed by elements of S. For $x \in \mathbf{R}^N, x_S$ will denote its restriction to $\mathbf{R}^S$. To order vectors in $\mathbf{R}^S$ we use the symbols $>>$, $>$ and $\geq$ with their usual interpretations. The non-negative orthant of $\mathbf{R}^S$ is denoted by $\mathbf{R}^S_+$ and the strictly positive orthant by $\mathbf{R}^S_{++}$. For $A \subseteq \mathbf{R}^S$, $intA$ denotes the interior of A and $co(A)$ denotes the convex hull. We denote by $\vec{1}_S$ the vector of ones in $\mathbf{R}^S$, that is, $\vec{1}_S = (1, ..., 1) \in \mathbf{R}^S$. Each coalition S has a feasible set of payoffs or utilities denoted by $V_S \subset \mathbf{R}^S$. By agreement, $V_\emptyset = \{0\}$ and $V_{\{i\}}$ is non-empty, closed and bounded from above for any i. In addition, we will assume that

$$\max\left\{x : x \in V_{\{i\}}\right\} = 0 \text{ for any } i \in N;$$

this is by no means restrictive since it can always be achieved by a normalization.

It is convenient to describe the feasible utilities of a coalition as a subset of $\mathbf{R}^N$. For each coalition S let $V(S)$, called the *payoff set for* S, be defined by

$$V(S) := \left\{x \in \mathbf{R}^N : x_S \in V_S \text{ and } x_a = 0 \text{ for } a \notin S\right\}.$$

A *game without side payments* (called also an *NTU game* or simply a *game*) is a pair (N, V) where the correspondence $V : 2^N \longrightarrow \mathbf{R}^N$ is such that $V(S) \subset \left\{x \in \mathbf{R}^N : x_a = 0 \text{ for } a \notin S\right\}$ for any $S \subset N$ and satisfies the following properties :

(2.1) $V(S)$ is non-empty and closed for all $S \subset N$.

(2.2) $V(S) \cap \mathbf{R}_+^N$ is bounded for all $S \subset N$, in the sense that there is a real number $K > 0$ such that if $x \in V(S) \cap \mathbf{R}_+^N$, then $x_i \leq K$ for all $i \in S$.

(2.3) $V(S_1 \bigcup S_2) \supset V(S_1) + V(S_2)$ for any disjoint $S_1, S_2 \subset N$ (superadditivity).

We next introduce the uniform version of strong comprehensiveness assumed for our results. Roughly, this notion dictates that payoff sets are both comprehensive and uniformly bounded away from having level segments in their boundaries. Consider a set $W \subset \mathbf{R}^S$. We say that W is *comprehensive* if $x \in W$ and $y \leq x$ implies $y \in W$. The set W is *strongly comprehensive* if it is comprehensive, and whenever $x \in W$, $y \in W$, and $x < y$ there exists $z \in W$ such that $x << z$.[1] Given (i) $x \in \mathbf{R}^S$, (ii) $i, j \in S$, (iii) $0 \leq q \leq 1$ and (iv) $\varepsilon > 0$, define a vector $x_{i,j}^q(\varepsilon) \in \mathbf{R}^S$, where

$$\begin{aligned}(x_{i,j}^q(\varepsilon))_i &= x_i - \varepsilon, \\ (x_{i,j}^q(\varepsilon))_j &= x_j + q\varepsilon, \text{ and} \\ (x_{i,j}^q(\varepsilon))_k &= x_k \text{ for } k \in S \backslash \{i,j\}.\end{aligned}$$

The set W is *q-comprehensive* if W is comprehensive and if, for any $x \in W$, it holds that $(x_{i,j}^q(\varepsilon)) \in W$ for any $i, j \in S$ and any $\varepsilon \geq 0$.[2] This condition for $q > 0$ uniformly bounds the slopes of the Pareto frontier of payoff sets away from zero. Note that for $q = 0$, 0-comprehensiveness is simply comprehensiveness. Also note that if a game is q-comprehensive for some $q > 0$ then the game is q'-comprehensive for all q' with $0 \leq q' \leq q$.

Let $V_S \subset \mathbf{R}^S$ be a payoff set for $S \subset N$. Given q, $0 \leq q \leq 1$, let $W_S^q \subset \mathbf{R}^S$ be the smallest q-comprehensive set that includes the set V_S. Then the *q-comprehensive cover* of $V(S) \subset \mathbf{R}^N$, denoted by $c_q(V(S))$, is defined in the following way:

$$c_q(V(S)) := \left\{x \in \mathbf{R}^N : x_S \in W_S^q \text{ and } x_a = 0 \text{ for } a \notin S\right\}.$$

When $q > 0$ we can think of a game as having some degree of "side-paymentness" or as allowing transfers between players, but not necessarily at a one-to-one rate. This is an eminently reasonable assumption for games derived from economic models.

A *game with side payments* (also called a *TU game*) is a game (N, V) with 1-comprehensive payoff sets, that is $V(S) = c_1(V(S))$ for any $S \subset N$. This implies that for any $S \subset N$ there exists $v(S) \geq 0$ such that $V_S = \left\{x \in \mathbf{R}^S : \sum_{i \in S} x_i \leq v(S)\right\}$. The numbers $v(S)$ for $S \subset N$ determine a function v mapping the subsets of N to $\mathbf{R}_+$. Then the TU game is represented as the pair (N, v).

[1] Informally, if one person can be made better off (while all the others remain at least as well off), then all persons can be made better off. Strong comprehensiveness has also been called "nonleveledness."

[2] The notion of q-comprehensiveness used here was introduced in Kaneko and Wooders (1996). Results similar to those presented in this paper can also be obtained when the payoff sets are unrestricted outside the individually rational subsets of the payoff sets; this is a relatively minor technical matter.

2.2 Parameterized collections of games.

To introduce the notion of parameterized collections of games we will need the concept of Hausdorff distance. For every two non-empty subsets E and F of a metric space (M, d), define the *Hausdorff distance between E and F* (with respect to the metric d on M), denoted by $dist(E, F)$, as

$$dist(E,F) := \inf \{\varepsilon \in (0,\infty) : \; E \subset B_\varepsilon(F) \text{ and } F \subset B_\varepsilon(E)\},$$

where $B_\varepsilon(E) := \{x \in M : d(x, E) \leq \varepsilon\}$ denotes an ε-neighborhood of E.

Since payoff sets are unbounded below, we will use a modification of the concept of the Hausdorff distance so that the distance between two payoff sets is the distance between the intersection of the sets and a compact subset of Euclidean space. Let m^* be a fixed positive real number. Let M^* be a subset of Euclidean space $\mathbf{R}^N$ defined by $M^* := \left\{x \in \mathbf{R}^N : x_a \geq -m^* \text{ for any } a \in N\right\}$. For every two non-empty subsets E and F of Euclidean space $\mathbf{R}^N$ let $H_\infty[E, F]$ denote the Hausdorff distance between $E \cap M^*$ and $F \cap M^*$ with respect to the metric $\|x - y\|_\infty := \max_i |x_i - y_i|$ on Euclidean space $\mathbf{R}^N$.

$\delta-$substitute partitions. Let (N, V) be a game and let $\delta \geq 0$ be a non-negative real number. A δ-substitute partition is a partition of the player set N into subsets with the property that any two players in the same subset are "within δ" of being substitutes for each other. Formally, given a set $W \subset \mathbf{R}^N$ and a permutation τ of N, let $\sigma_\tau(W)$ denote the set formed from W by permuting the values of the coordinates according to the associated permutation τ. Given a partition $\{N[t] : t = 1, .., T\}$ of N, a permutation τ of N is *$type-preserving$* if, for any $i \in N$, $\tau(i)$ belongs to the same element of the partition $\{N[t]\}$ as i. A *δ-substitute partition* of N is a partition $\{N[t] : t = 1, .., T\}$ of N with the property that, for any type-preserving permutation τ and any coalition S,

$$H_\infty\left[V(S), \sigma_\tau^{-1}(V(\tau(S)))\right] \leq \delta.$$

Note that in general a δ-substitute partition of N is not uniquely determined.

profiles. Let $\delta \geq 0$ be a non-negative real number and let (N, V) be a game. For a δ-substitute partition $\{N[t] : t = 1, .., T\}$ of N, the set $N[t]$ is interpreted as an *approximate type*. Players in the same element of a δ-substitute partition are *δ-substitutes.* When $\delta = 0$, they are *exact substitutes*. A *profile* relative to $\{N[t]\}$ is a vector of non-negative integers $f \in Z_+^T$ and a *subprofile* s of a profile f is a profile satisfying the condition that $s \leq f$. Given $S \subset N$ the *profile of S* is a profile, say $s \in Z_+^T$, where $s_t = |S \cap N[t]|$. A profile describes a group of players in terms of the numbers of players of each approximate type in the group. The *norm* of a profile f, denoted by $\|f\|$, is simply the number of players in a group described by f, that is, $\|f\| = \sum f_t$.

(δ,T)- type games. Let δ be a non-negative real number and let T be a positive integer. A game (N,V) is a (δ,T)*-type game* if there is a T-member δ-substitute partition $\{N[t]:t=1,..,T\}$ of N (in other words, if there are T approximate types).

$\varepsilon-$effective $B-$bounded groups. Informally, groups of players containing no more than B members are ε-effective if, by restricting coalitions to fewer than B members, the loss to each player is no more than ε. This is a form of "small group effectiveness" for arbitrary games. As we will discuss further, this is a mild condition. It does not rule out ever-increasing gains to coalitions size, for example.

Let (N,V) be a game. Let $\varepsilon\geq 0$ be a given non-negative number and let B be a given positive integer. For each group $S\subset N$, define a corresponding set $V(S;B)\subset \mathbf{R}^N$ in the following way:

$$V(S;B):=\bigcup\left[\sum_k V(S^k):\left\{S^k\right\}\text{ is a partition of }S,\ \left|S^k\right|\leq B\right].$$

The set $V(S;B)$ is the payoff set of the coalition S when groups are restricted to have no more than B members. Note that, by superadditivity, $V(S;B)\subset V(S)$ for any $S\subset N$ and, by construction, $V(S;B)=V(S)$ for $|S|\leq B$. We might think of $c_q(V(S;B))$ as the payoff set to the coalition S when groups are restricted to have no more than B members and transfers are allowed between groups in the partition. If the game (N,V) has q-comprehensive payoff sets then $c_q(V(S;B))\subset V(S)$ for any $S\subset N$. The game (N,V) with q-comprehensive payoff sets has *ε-effective B-bounded groups* if for every group $S\subset N$

$$H_\infty\left[V(S),c_q(V(S;B))\right]\leq\varepsilon.$$

When $\varepsilon=0$, 0-effective B-bounded groups are called *strictly effective B-bounded groups.*

per capita boundedness. Let C be a positive real number. A game (N,V) has a *per capita payoff bound* of C if, for all coalitions $S\subseteq N$,

$$\sum_{a\in S}x_a\leq C\left|S\right|\text{ for any }x\in V(S).$$

parameterized collections of games $G^q((\delta,T),C,(\varepsilon,B))$. Let T and B be positive integers and let C and q be a positive real numbers, $q\leq 1$. Let $G^q((\delta,T),C,(\varepsilon,B))$ be the collection of all (δ,T)-type games that are superadditive, have q-comprehensive payoff sets, have per capita bound of C, and have ε-effective B-bounded groups. Thus, given non-negative real numbers ε and δ, positive real numbers C and q, and positive integers T and B, a game (N,V) belongs to the class $G^q((\delta,T),C,(\varepsilon,B))$ if:

(a) the payoff sets satisfy q-comprehensiveness;

(b) there is a partition of the total player set into T sets where each element of the partition contains players who are δ-substitutes for each other;

(c) no coalition can get a per capita payoff for it's members of more than C; and

(d) almost all gains to collective activities (with a maximum possible loss of ε for each player) can be realized by partitions of the total player sets into groups containing fewer than B members.

We will use the special notation $\Gamma((\delta,T),C,(\varepsilon,B))$ for collections of parameterized games with side payments, defined and considered in Kovalenkov and Wooders (1997c). Note that $\Gamma((\delta,T),C,(\varepsilon,B)) = G^1((\delta,T),C,(\varepsilon,B))$, parameterized collections of games where $q=1$.

3 Non-emptiness of symmetric epsilon cores.

The core and epsilon cores. Let (N,V) be a game. A payoff x is *ε-undominated* if for all $S \subseteq N$ and $y \in V(S)$ it is not the case that $y_S >> x_S + \vec{1}_S\varepsilon$. The payoff x is *feasible* if $x \in V(N)$. The ε-core of a game (N,V) consists of all feasible and ε-undominated allocations. When $\varepsilon = 0$, the ε-core is the *core*.

The equal treatment epsilon core. Given non-negative real numbers ε and δ, we will define the *equal treatment ε-core* of a game (N,V) relative to a partition $\{N[t]\}$ of the player set into δ-substitutes as the set of payoffs x in the ε-core with the property that for each t and all i and j in $N[t]$, it holds that $x_i = x_j$.

To motivate our Proposition, recall that a feasible payoff is in the ε-core if no coalition of players can improve upon the payoff by at least ε for each member of the coalition. This suggests that the distance of coalitions containing fewer than B-members from being effective for the realization of all gains to coalition formation should be defined using the Hausdorff distance with respect to the sup norm, and indeed this was the approach that we took in prior papers (Kovalenkov and Wooders (1997a,b). Thus we first establish a form of our result for the case with ε-effective groups defined as above. We then consider another definition of ε-effective B-bounded groups and obtain a stronger form of our result.

3.1 The Proposition.

Let $(N,V) \in G^q((\delta,T),C,(\varepsilon,B))$. The following Proposition provides a lower bound on ε^* so that for any $\varepsilon' \geq \varepsilon^*$, the game (N,V) has a non-empty ε'-core. In fact, the Proposition shows non-emptiness of the symmetric ε-core as well, where the symmetric ε-core is defined as the subset of payoffs in the ε-core that assign equal payoffs to all agents of the same approximate type. It is convenient to define first a constant:

$$K(T,B) := \sum_{l=1}^{B} \frac{(T+l-1)!}{(T-1)!(l-1)!}.$$

Then the lower bound on ε^* is given by

$$\alpha_N^q((\delta,T),C,(\varepsilon,B)) := \frac{1}{q}(\frac{K(T,B)C}{|N|} + \varepsilon) + \delta.$$

Of course the only interesting cases are those where this bound is small. To avoid trivialities associated with large ε^* we restrict attention to the case $\alpha_N^q((\delta,T),C,(\varepsilon,B)) \leq m^*$, where m^* is the positive real number fixed in the beginning of the subsection 2.2.

Proposition. Let $(N,V) \in G^q((\delta,T),C,(\varepsilon,B))$, where $q > 0$. Assume $V(N)$ is convex. Let ε^* be a positive real number. If $\varepsilon^* \geq \alpha_N^q((\delta,T),C,(\varepsilon,B))$ then the equal treatment ε^*-core of (N,V) is nonempty.

The relationships between the lower bound on ε^*, the parameters describing the game, and the number of players in the total player set are immediate. Note in particular, the smaller B, the size of ε-effective groups, the smaller the lower bound. It is easy to see that increasing ε increases the bound $\alpha_N^q((\delta,T),C,(\varepsilon,B))$ proportionally while increasing B increases the bound much more rapidly.

Note that for $q = 1$ (TU games) our bound coincides with the bound presented in Kovalenkov and Wooders (1997c) for the class $\Gamma((\delta,T),C,(\varepsilon,B))$. Therefore the Proposition strictly generalizes the result achieved in Kovalenkov and Wooders (1997c).

3.2 Examples.

Our first example illustrates the statement of the Proposition.

Example 1: Let (N,V_0) be a superadditive game where for any two-person coalition $S=\{i,j\}$, $j \neq i$,

$$V_0(S) := \{x \in \mathbf{R}^N : \ x_i \leq 1, \ x_j \leq 1, \text{and } x_k = 0 \text{ for } k \neq i,j\}$$

and for each $i \in N$,

$$V_0(\{i\}) := \{x \in \mathbf{R}^N : x_i \leq 0 \text{ and } x_j = 0 \text{ for all } j \neq i\}.$$

For an arbitrary coalition S the payoff set $V_0(S)$ is given as the *superadditive cover*, that is,

$$V_0(S) := \bigcup_{\mathcal{P}(S)} \sum_{S' \in \mathcal{P}(S)} V_0(S'),$$

where the union is taken over all partitions $\mathcal{P}(S)$ of S.

Now let us define a game $(N, V_{\frac{1}{3}})$ in the following way. For any $S \subset N$ let $V_{\frac{1}{3}}(S)$ be the $\frac{1}{3}$-comprehensive cover of the convex cover of the payoff set $V_0(S)$; that is,

$$V_{\frac{1}{3}}(S) := c_{\frac{1}{3}}(co(V_0(S))).$$

Obviously the game $(N, V_{\frac{1}{3}})$ has $\frac{1}{3}$-comprehensive convex payoff sets, one player type, and per capita bound of 1. We leave it to the reader to verify that for any positive integer m the game $(N, V_{\frac{1}{3}})$ has $\frac{1}{m}$-effective $(m-1)$-bounded groups. Thus the game $(N, V_{\frac{1}{3}})$ is a member of the class $G^{\frac{1}{3}}((0,1), 1, (\frac{1}{m}, m-1))$.

Since $V_{\frac{1}{3}}(N)$ is convex and $K(1, m-1) = \frac{m(m-1)}{2}$, the Proposition states that for any $\varepsilon^* \geq 3(\frac{m(m-1)}{2|N|} + \frac{1}{m})$ the equal treatment ε^*-core of $(N, V_{\frac{1}{3}})$ is non-empty. This implies that for any $\varepsilon^0 > 0$ there is a positive integer $N(\varepsilon^0)$ such that for any $|N| \geq N(\varepsilon^0)$ the game $(N, V_{\frac{1}{3}})$ has a non-empty equal treatment ε^0-core. (For an explicit bound take an integer $m^0 \geq \frac{6}{\varepsilon^0}$ and define $N(\varepsilon^0) \geq \frac{3m^0(m^0-1)}{\varepsilon^0}$.)

The following example illustrates why either convexity or some degree of comprehensiveness is required for our result, even for games with just one exact player type.

Example 2: Recall the game (N, V_0) defined in Example 1. Let m be a positive integer. Let (N^m, V_0^m) be a game where the number of players in the set N^m is $2m+1$ and for any coalition $S \subset N^m$ $V_0^m(S) := V_0(S)$. Thus, each game (N^m, V_0^m) has an odd number of players.

It is easy to see that the core of the game is non-empty: any payoff giving 1 to each of $2m$ players is in the core. Since the total number of players is odd, at least one person must be "left out." In a game with side payments this player could upset the non-emptiness of the core. But the games of this example do not satisfy strong comprehensiveness. Thus, a payoff giving 1 to each of $2m$ players cannot be improved upon since the "left-out" player, in a coalition by himself, cannot make both himself and a player in a two-person coalition better off – the player in the two-person coalition cannot be given more than 1. The games, however, can be approximated arbitrarily closely by games with strongly comprehensive payoff sets.[3]

Let (N^m, V_{sc}^m) be a game with strongly comprehensive payoff sets that approximates the game (N^m, V_0^m). For a sufficiently close approximation, the game

[3]The set $W \subseteq \mathbf{R}^S$ is compactly generated if there exists a compact set $C \subseteq \mathbf{R}^S$ such that $W = C - \mathbf{R}^S_+$. The approximation can be carried out for any game with comprehensive and compactly generated payoff sets - see Wooders (1983, Appendix).

(N^m, V_{sc}^m) will have effective small groups and an empty core. This follows from the observations that any payoff must give at least one player less than one and the two worst-off players a total of less than two. The two worst-off players form an improving coalition and hence the core is empty.[4]

Our results rely on convexity and q-comprehensiveness. Since there is only one type of player, in this example either convexity or q-comprehensiveness will suffice. The role of convexity is to average payoffs over similar players. Consider the game (N, V_{conv}^m) where V_{conv}^m is defined as the convex hull of V_{sc}^m. Then the payoff $x = (\frac{2m}{2m+1}, ..., \frac{2m}{2m+1})$ is feasible and in the ε-core of (N^m, V_{conv}^m) for any $\varepsilon \geq \frac{1}{2m+1}$. Now instead of convexity of the total payoff set, suppose that payoff sets are q-comprehensive. In this case for any payoff giving one to each of $2m$ players, it is possible to take some small amount, say ε, away from each of $2m$ players and "transfer" $2m\varepsilon q$ to the leftover player. Thus, for any ε and q satisfying $2m\varepsilon q \geq 1 - \varepsilon$ the ε-core is nonempty.

A crucial feature of Example 2 is the restriction to one player type. Because of this feature and the fact that two-player coalitions are effective, through convexity or q-comprehensiveness we can construct symmetric payoffs in approximate cores. This example suggests that either convexity or q-comprehensiveness is sufficient to get non-emptiness of the epsilon core for large games. In fact, Kovalenkov and Wooders (1997a) supports this intuition in the case of q-comprehensiveness. Kovalenkov and Wooders (1997b) shows that convexity is sufficient but requires "thickness" of the player set (that is, the condition that the proportion of any approximate player type is bounded above zero). Neither of these papers, however, provide explicit bounds.

The Proposition shows that with *both* convexity and q-comprehensiveness, an explicit bound can be obtained on ε for non-emptiness of the ε-core. This bound appears to be very simple and easily computable from the parameters.

3.3 The Theorem.

We now generalize the Proposition using another notion of distance; we define ε-effective B-bounded groups using the Hausdorff distance with respect to the sum norm. This, of course, provides a much less demanding notion of near-effectiveness. Nevertheless, we are able to establish that all sufficiently large games have non-empty approximate cores. An example is provided showing that when ε-effective B-bounded groups are defined using the Hausdorff distance with respect to the sum norm, the lower bound on ε may be substantially smaller.

[4]It can be shown with a precise construction of (N^m, V_{sc}^m) that for a small but positive ε the ε-core of (N^m, V_{sc}^m) can be empty even for a great number of players. The reader may also find it interesting and informative to consider an example where any two-player coalition can distribute a total of two units of payoff in any agreed-upon way, while there is no transferability of utility between coalitions.

Let us first define this another notion of the Hausdorff distance, while maintaining the previous definition of the set M^*. For every two non-empty subsets E and F of Euclidean space $\mathbf{R}^N$ let $H_1(E,F)$ denote the Hausdorff distance between $E \cap M^*$ and $F \cap M^*$ with respect to the metric $\|x-y\|_1 := \sum_{i=1}^{N} |x_i - y_i|$. Now we can define a weaker notion of the ε-effective B-bounded groups.

$\underline{\textit{weakly } \varepsilon-\textit{effective } B-\textit{bounded groups.}}$ The game (N,V) with q-comprehensive payoff sets has *weakly ε-effective B-bounded groups* if for every group $S \subset N$

$$H_1\left[V(S), c_q(V(S;B))\right] \leq \varepsilon\, |S|\,.$$

Notice that ε-effective B-bounded groups are always weakly ε-effective B-bounded groups, but for TU games these two notions coincide. These notions also coincide in the case when $\varepsilon = 0$.

We now introduce a new definition of parametrized collections of games.

$\underline{\textit{parameterized collections of games } \mathcal{G}^q((\delta,T),C,(\varepsilon,B)).}$ Let T and B be positive integers and let C and q be a positive real numbers, $q \leq 1$. Let $\mathcal{G}^q((\delta,T),C,(\varepsilon,B))$ be the collection of all (δ,T)-type games that are superadditive, have q-comprehensive payoff sets, have per capita bound of C, and have weakly ε-effective B-bounded groups.

- Of course $G^q((\delta,T),C,(\varepsilon,B)) \subset \mathcal{G}^q((\delta,T),C,(\varepsilon,B))$, but these two classes coincide for $q = 1$ (games with side payments), that is $\mathcal{G}^1((\delta,T),C,(\varepsilon,B)) = \Gamma((\delta,T),C,(\varepsilon,B))$.

The following theorem is a generalization of the Proposition.

Theorem. Let $(N,V) \in \mathcal{G}^q((\delta,T),C,(\varepsilon,B))$, where $q > 0$. Assume $V(N)$ is convex. Let ε^* be a positive real number. If $\varepsilon^* \geq \alpha_N^q((\delta,T),C,(\varepsilon,B))$ then the equal treatment ε^*-core of (N,V) is nonempty.

Notice that the Theorem is a strict generalization of the Proposition. Two remarks should be done about it. The first is that the Theorem can be applied to the larger class of games than the Proposition . The second (and much less obvious) is that the use of the Theorem rather than the use of the Proposition can improve the bound significantly. The following example continues Example 1 and illustrates this feature.

Example 3: Recall the game $(N,V_{\frac{1}{3}}) \in G^{\frac{1}{3}}((0,1),1,(\frac{1}{m},m))$ defined in Example 1. We leave it to the reader to verify that the game $(N,V_{\frac{1}{3}})$ has weakly $\frac{1}{|N|}$-effective 2-bounded groups. Therefore the game $(N,V_{\frac{1}{3}})$ is a member of the class $\mathcal{G}^{\frac{1}{3}}((0,1),1,(\frac{1}{|N|},2))$. Recall that $V_{\frac{1}{3}}(N)$ is convex. Then, since $K(1,2) =$

$1! + 2! = 3$, the Theorem states that for any $\varepsilon^* \geq 3(\frac{3}{|N|} + \frac{1}{|N|}) = \frac{12}{|N|}$ the equal treatment ε^*-core of $(N, V_{\frac{1}{3}})$ is non-empty while the Proposition gave the bound $\varepsilon^* \geq 3(\frac{m(m-1)}{2|N|} + \frac{1}{m})$. (The bound of the Theorem implies that for any $\varepsilon^0 > 0$ and for any $|N| \geq \frac{12}{\varepsilon^0}$ the game $(N, V_{\frac{1}{3}})$ has a non-empty equal treatment ε^0-core.)

3.4 Games with relatively small effective coalitions.

The results of this paper, as those in Kovalenkov and Wooders (1997c) require that coalitions bounded in size can realize all or almost all gains to cooperation. Since relative sizes of effective coalitions are important in some models, such as voting games, and may be important in others, we provide a corollary treating sequences of arbitrary games where relative sizes of effective coalitions become small.

Corollary. Consider a sequence of games (N_m, V_m) satisfying the properties that:

(a) $|N_m| \to \infty$ as $m \to \infty$ (games in the sequence became large),

(b) $V_m(N_m)$ is a convex and q_m-comprehensive set for some $q_m > 0$.

(c) (N_m, V_m) is a (δ_m, T_m)-game, where $\frac{T_m}{|N_m|} \to 0$ and $\delta_m \to 0$ as $m \to \infty$ (players in the large games can be separated in relatively small number of the approximate types, and approximation is close),

(d) (N_m, V_m) has weakly ε_m-effective B_m-bounded groups, where $\frac{B_m}{|N_m|} \to 0$ and $\frac{\varepsilon_m}{q_m} \to 0$ as $m \to \infty$ (almost all gain to the collective activity can be captured by the groups small in the relative size),

(e) (N_m, V_m) has per capita bound of C_m , where $\frac{C_m}{|N_m|} \to 0$ as $m \to \infty$ (the gain in per capita terms to the collective activity can grow but slower than the number of players in the game),

(f) $q_m, T_m, B_m,$and C_m satisfy the condition that $\frac{K(T_m,B_m)C_m}{q_m|N_m|} \to 0$ as $m \to \infty$ (T_m, B_m,and C_m can grow altogether and q_m can go to 0, but these parameters are required to do it slowly relative to the size of the total player set).

THEN: for any $\varepsilon^* > 0$, there is $m(\varepsilon^*)$, such that for any $m \geq m(\varepsilon^*)$ the game (N_m, V_m) has a non-empty equal treatment ε^*-core.

We leave the proof of the Corollary to the reader. The reader may found it interesting to look on Example 4 in Kovalenkov and Wooders (1997c) which illustrates the statement of the Corollary in the case of the TU games ($q = 1$). We do not present here any example specific for NTU games, since it would be straightforward but lengthy.

4 The proofs.

4.1 The main part of the proofs.

We first provide a sketch of the proofs and the main argument. In the next subsection we relate NTU games to TU games and provide proofs of several results used in this subsection.

Proof of the Proposition:
We begin the proof of the Proposition by first treating games where all players are exact substitutes of each other. (Later in the proof we will construct such a game from an arbitrary game.) We will use the following terminology: A set $W \subset \mathbf{R}^N$ is *symmetric across substitute players* if for any player type the set W remains unchanged under any perturbations of the values associated with players of that type.

The symmetric case. Assume first that $(N, V) \in G^q((0,T),(\varepsilon, B))$. Note that in this case all payoff sets of the game (N, V) are symmetric across substitute players. Let us prove that for any $\varepsilon^* \geq \alpha_N^q((0,T), C, (\varepsilon, B))$ the equal treatment ε^*-core of (N, V) is nonempty.

The idea of the proof for the symmetric case. In the proof for the symmetric case we will use the following definitions. Let $A \subseteq R^m$. A *recession cone* corresponding to A, denoted by $c(A)$, is defined as follows:

$$c(A) := \{y \in R^m : \ x + \lambda y \in A \text{ for all } \lambda \geq 0 \text{ and } x \in A\}.$$

The scalar product of $x, y \in R^m$ is denoted by $x \cdot y$. The *negative dual cone* of $P \in R^m$ is denoted by $d(P)$ and defined as follows:

$$d(P) := \{z \in R^m : \ z \cdot y \leq 0 \text{ for any } y \in P\}.$$

A bound on the required size of the parameter ε^* for our result in the symmetric case is obtained by constructing a family of "λ-weighted transferable utility" games (N, V_λ) corresponding, in a certain way, to the initial game (N, V). Next we consider only those values of λ in a set L^*, defined as the intersection of the equal treatment payoffs in the simplex with the negative dual cone of the recession cone of the modified game. For each λ there is corresponding TU game (N, v_λ). We give the formal construction of (N, v_λ) in Step 1 of the next subsection.

In Step 2 of the following subsection we prove Lemma 1, that, for some parameters C' and ε', any game (N, v_λ) is a member of the parameterized collection of TU games $\Gamma((0,T), C', (B, \varepsilon'))$. This allows us to use the result of Kovalenkov and Wooders (1997c). In Lemma 2 we relate approximate cores of the game (N, v_λ) to approximate cores of the NTU game (N, V_λ). Using the fact that we consider only values of λ in L^*, we obtain an explicit bound on ε^* for the initially given parameters C and ε for non-emptiness of the equal treatment ε^*-core for all games (N, V_λ). This result will give us exactly the bound that we need to deduce for the conclusion of Proposition for the symmetric case.

Now we need only prove that if, given some ε^*, the equal treatment ε^*-core of (N, V_λ) is non-empty for all $\lambda \in L^*$, then the equal treatment ε^*-core will be non-empty for both the modified and initial games as well. With the help of Lemma 3, Lemma 4 and a Theorem about excess demand considered in Step 3, all in the following subsection, we complete the proof in the symmetric case.

Remark. The initial approach in the following proof is similar to that introduced in Scarf (1965) and usually used in proofs of the non-emptiness of the exact core for strongly balanced NTU games (for a definition of strong balancedness and for an example of this technique see Hildenbrand and Kirman (1988, Appendix to Chapter 4)). But the proof below departs from the typical approach in that we construct games (N, V_λ) and (N, v_λ) not for all λ in the simplex as usual, but only for λ belonging to a specific subset L^* of the simplex. The set L^* is the intersection of the equal treatment payoffs in the simplex with the dual negative cone to the recession cone of the payoff set for the grand coalition in the modified game. Later we use the structure of the set L^* and q-comprehensiveness to complete the proof.[5]

The general case. Now let us consider the general case with no additional restrictions. We first modify the game (N, V). For any $S \subset N$ define $V^0(S) := \bigcap \sigma_\tau^{-1}(V(\tau(S)))$, where the intersection is taken over all type-preserving permutations τ of the player set N. Then from the definition of $V^0(S)$ it follows that $V^0(S) \subset V(S)$. (Informally, taking the intersection over all type-preserving permutations makes all players of each approximate type no more productive than the least productive members of that type.) From the definition of δ-substitutes, it follows that $H_\infty[V^0(S), V(S)] \leq \delta$ for any $S \subset N$. Moreover,

$$(N, V^0) \in G^q((0, T), (\varepsilon, B)) \text{ and } V^0(N) \text{ is convex.}$$

Therefore, we can apply the result proved in the symmetric case and conclude that the game (N, V^0) has some payoff x in the equal treatment $\frac{1}{q}(\frac{K(T,B)BC}{|N|} + \varepsilon)$-core. Now define a payoff vector y by

$$y(\{i\}) := x(\{i\}) - \delta \text{ for each } i \in N.$$

[5]Note that our technique is a generalization of the usual one, which does not involve recession cone arguments. Nevertheless the usual approach is a special case of our technique. (The usual technique is applied for $T = N$ and to games having $-\mathbf{R}_+^N$ as the recession cone of the payoff set $V(N)$. The negative dual cone to $-\mathbf{R}_+^N$ is $\mathbf{R}_+^N$. Then $\mathbf{R}_+^N \supset \triangle_+$, so the relevant intersection is the simplex itself.)

The payoff y will be feasible and $\left(\frac{1}{q}(\frac{K(T,B)BC}{|N|}+\varepsilon)+\delta\right)$-undominated in the initial game (N,V). Obviously, y has the equal treatment property. Therefore for $\varepsilon^* \geq \alpha_N^q((\delta,T),C,(\varepsilon,B))$ the equal treatment ε^*-core of (N,V) is nonempty. ■

Proof of the Theorem: The proof follows the proof of the Proposition. The only place in the proof of Proposition where we were using the fact that the given game has ε-effective B-bounded groups was application of Lemma 1. Note that the corresponding generalization of Lemma 1 for weakly ε-effective B-bounded groups is true. The same exactly proof as it was for Lemma 1 applies. ■

4.2 Relating games with and without side payments.

Step 1: Construction of the TU games. Let us first modify the game (N,V). Consider the set

$$K := \left\{x \in \mathbf{R}^N : x_a \geq -\alpha_N^q((0,T),C,(\varepsilon,B)) \text{ for any } a \in N\right\}.$$

Define

$$K^* := K \bigcap V(N)$$

and observe that set K^* is a compact set. Let $c(V(N),K)$ be the smallest closed cone such that

$$V(N) \subset K^* + c(V(N),K).$$

Now let us define a modified game (N,V^1) so that

$$\begin{aligned} \text{(a) } V^1(S) &:= V(S) \text{ for } S \neq N \text{ and} \\ \text{(b) } V^1(N) &:= K^* + c(V(N),K). \end{aligned}$$

Notice that $c(V(N),K)$ is a recession cone of $V^1(N)$; that is,

$$c(V^1(N)) = c(V(N),K).$$

We are going to prove that the modified game (N,V^1) has an equal treatment ε^*-core payoff, which we will denote by x^*, and that this payoff x^* belongs to K^*. Since $V(S) \subset V^1(S)$ for any S, x^* will be ε^*-undominated in the game (N,V). Since $K^* \subset V(N)$ the payoff x^* will be feasible in the game (N,V). It follows that x^* is an equal treatment ε^*-core payoff for (N,V).

Define

$$\mathcal{C} := co\left\{x \in \mathbf{R}^N : \exists i,j \in N,\ x_i = -qx_J \geq 0, x_k = 0, k \neq i,j\right\}$$

and observe that $\mathcal{C}$ is a cone. Since $V^1(N)$ is q-comprehensive and convex, the cone $c(V^1(N))$ will include $\mathcal{C}$ but will not be more than a half-space. Hence the negative dual cone to the recession cone $d(c(V^1(N)))$ will be closed, non-empty and included in the cone dual to $\mathcal{C}$:

$$d(\mathcal{C}) = \left\{ x \in \mathbf{R}^N_{++} : q \leq \frac{x_i}{x_j} \leq \frac{1}{q} \ \forall i, j \right\} \bigcup \{0\} .$$

Now let us consider the simplex in $\mathbf{R}^N_+$:

$$\triangle_+ := \left\{ \lambda \in \mathbf{R}^N_+ : \sum_{i=1}^{N} \lambda_i = 1 \right\} .$$

Define

$$L := d(c(V^1(N))) \bigcap \triangle_+ .$$

Given a partition $\{N[t]\}$ of the player set into T types of δ-substitutes, the set of equal treatment allocations is denoted by E^T and defined as follows:

$$E^T := \left\{ x \in \mathbf{R}^N : x_i = x_j \text{ for any } t \text{ and any } i, j \in N[t] \right\} .$$

Now define

$$L^* := L \cap E^T .$$

Observe that L^* is a compact and convex set.

For any $\lambda \in L^*$ there exists a tangent hyperplane to the set $V(N)$ with normal λ such that the whole set $V(N)$ is contained in a closed half-space, and at least one point of the set $V(N)$ lies on the hyperplane. Moreover, since the game is superadditive, for any $\lambda \in L^*$ and any $S \subset N$ there exists a hyperplane in $\mathbf{R}^S$ that has normal parallel to λ_S and that is tangent to V_S. Thus, for a fixed $\lambda^* \in L$ there is a finite real number

$$v_\lambda(S) := \max \left\{ \sum_{a \in S} \lambda_a x_a : x \in V(S) \right\} .$$

The pair (N, v_λ) is a TU game. We construct a "λ-weighted transferable utility" game (N, V_λ) by defining, for each coalition $S \subset N$:

$$V_\lambda(S) := \left\{ x \in \mathbf{R}^N : x_a = 0 \text{ for } a \notin S \text{ and } \sum_{a \in S} \lambda_a x_a \leq v_\lambda(S) \right\} .$$

□

Step 2 : Nonemptiness of the epsilon core for (N, V_λ) games.

Consider a fixed $\lambda \in L^*$. Define $\lambda_{\max} := \max\{\lambda_i\}$ and $\lambda_{\min} := \min\{\lambda_i\}$.

Lemma 1. Let $(N, V) \in G^q((0,T), C, (\varepsilon, B))$. Then

$$(N, v_\lambda) \in \Gamma((0,T), C\lambda_{\max}, (\varepsilon\lambda_{\max}, B)).$$

Proof of Lemma 1:

1). We will prove that the $(0,T)$-partition $\{N[t]\}$ of the game (N,V) is a $(0,T)$-partition of the game (N,v_λ). We must check that for any type-consistent permutation τ of N and any coalition S it holds that $v_\lambda(S)=v_\lambda(\tau(S))$. But we have:

$$\begin{aligned} v_\lambda(\tau(S)) &\equiv \max\left\{\sum_{a\in\tau(S)}\lambda_a x_a : x\in V(\tau(S))\right\} \\ &= \max\left\{\sum_{a\in S}\lambda_{\tau(a)}x_a : x\in V(S)\right\} \\ &= \max\left\{\sum_{a\in S}\lambda_a x_a : x\in V(S)\right\}\equiv v_\lambda(S). \end{aligned}$$

The second equality follows from the fact that $V(\tau(S))=V(S)$, since $\{N[t]\}$ is a $(0,T)$-partition of the game (N,V). The third equality holds since, by construction of L^* and τ, for any a we have $\lambda_a=\lambda_{\tau(a)}$.

2). To show that the number $\lambda_{\max}C$ is a per capita bound for the TU game (N,v_λ), it is necessary to show that $\frac{v_\lambda}{|S|}\le\lambda_{\max}C$ for each coalition group S. Observe that by the definition of $v_\lambda(S)$, for some $x_a\in V_S$ it holds that

$$\frac{v_\lambda}{|S|}=\frac{\sum_{a\in S}\lambda_a x_a}{|S|}\le\lambda_{\max}\frac{\sum_{a\in S}x_a}{|S|}\le\lambda_{\max}C.$$

The last inequality follows from per capita boundedness of the game (N,V).

3). To prove effectiveness of B-bounded $\lambda_{\max}\varepsilon$-effective groups for the TU game (N,v_λ) we need to show that for any $S\subseteq N$ there exists a partition $\{S_k\}$ of S satisfying $|S_k|\le B$ for each k and

$$\left|v_\lambda(S)-\sum_k v_\lambda(S_k)\right|\le\lambda_{\max}\varepsilon\,|S|\,.$$

By superadditivity

$$v_\lambda(S)\ge\sum_k v_\lambda(S_k).$$

By the definition of v_λ there exists a vector x such that $x\in V(S)$ and $v_\lambda(S)=\sum_{a\in S}\lambda_a x_a$. Since (N,V) has ε-effective B-bounded groups there exists a vector $y\in c_q(V(S;B))$ such that

$$\sum_{a\in S}|y_a-x_a|\le\varepsilon\,|S|\,.$$

Then there exists a vector $z \in V(S;B)$ such that $y \in c_q(z)$. Note that since $\lambda \in L^* \subset d(\mathcal{C})$ and $y \in c_q(z)$ we have

$$\sum_{a \in S_k} \lambda_a y_a \leq \sum_{a \in S_k} \lambda_a z_a.$$

Then since $z \in V(S;B)$ we have that $z_{S_k} \in V_{S_k}$ for some partition $\{S_k\}$ of S (with $|S_k| \leq B$) and we get

$$\sum_{a \in S_k} \lambda_a z_a \leq v_\lambda(S_k).$$

Hence

$$\begin{aligned} \left| v_\lambda(S) - \sum_k v(S_k) \right| &\leq \left| \sum_{a \in S} \lambda_a x_a - \sum_k \sum_{a \in S_k} \lambda_a z_a \right| \leq \left| \sum_{a \in S} \lambda_a x_a - \sum_k \sum_{a \in S_k} \lambda_a y_a \right| \\ &\leq \sum_{a \in S} \lambda_a \left| x_a - y_a \right| \leq \lambda_{\max} \varepsilon \left| S \right|. \end{aligned}$$

From by 1), 2), 3) it holds that $(N, v_\lambda) \in \Gamma((0,T), C\lambda_{\max}, (B, \varepsilon\lambda_{\max}))$. ■

Lemma 2. If the equal treatment ε-core of (N, v_λ) is nonempty, then the equal treatment $\frac{\varepsilon}{\lambda_{\min}}$-core of (N, V_λ) game is nonempty.

Proof of Lemma 2: Consider a payoff y in the equal treatment ε-core of the game (N, v_λ) and define $x_a := \frac{1}{\lambda_a} y_a$. Note that x also has equal treatment property. Then

$$\sum_{a \in N} \lambda_a x_a = \sum_{a \in N} y_a \leq v_\lambda(N);$$

thus x is feasible for the game (N, V_λ). Moreover, for all $S \subseteq N$,

$$\sum_{a \in S} \lambda_a (x_a + \frac{\varepsilon}{\lambda_{\min}}) = \sum_{a \in S} y_a + \sum_{a \in S} \lambda_a \frac{\varepsilon}{\lambda_{\min}} \geq \sum_{a \in S} y_a + \varepsilon \left| S \right| \geq v_\lambda(S)$$

thus x is $\frac{\varepsilon}{\lambda_{\min}}$-undominated in the game (N, V_λ). Therefore, x is in the equal treatment $\frac{\varepsilon}{\lambda_{\min}}$-core of (N, V_λ). ■

We can now finish Step 2 : Since $(N, V) \in G^q((0,T), C, (\varepsilon, B))$, by Lemma 1 we have that

$$(N, v_\lambda) \in \Gamma((0,T), C\lambda_{\max}, (\varepsilon\lambda_{\max}, B)).$$

But from Theorem in Kovalenkov and Wooders (1997b), for any TU game in $\Gamma((\delta', T), C', (\varepsilon', B))$ and any $\varepsilon^0 \geq \frac{K(T,B)C'}{|N|} + \delta' + \varepsilon'$, the equal treatment ε^0-core is nonempty . Hence, if $\varepsilon^0 \geq \lambda_{\max}(\frac{K(T,B)C}{|N|} + \varepsilon)$, the equal treatment ε^0-core of (N, v_λ) is nonempty. From Lemma 2 this implies that the equal treatment $\frac{\varepsilon^0}{\lambda_{\min}}$-core of (N, V_λ) is nonempty. Thus, since

$$\alpha_N^q((0,T), C, (\varepsilon, B)) = \frac{1}{q}(\frac{K(T,B)C}{|N|} + \varepsilon) \geq \frac{\lambda_{\max}}{\lambda_{\min}}(\frac{K(T,B)C}{|N|} + \varepsilon)$$

($\frac{\lambda_{\max}}{\lambda_{\min}} \leq \frac{1}{q}$ because $\lambda \in L^* \subset d(\mathcal{C})$), we can conclude that if $\varepsilon^* \geq \alpha_N^q((0,T), C, (\varepsilon, B))$ the equal treatment ε^*-core of (N, V_λ) is nonempty. This is exactly the bound that we need in the symmetric case.$\square$

Step 3: Nonemptiness of the epsilon core for the initial game. We need only to prove that if the equal treatment ε^*-core of (N, V_λ) is non-empty for all $\lambda \in L^*$ then the equal treatment ε^*-core of (N, V) is nonempty. Define

$$C_{\varepsilon^*}(\lambda) := \left\{ x : \sum_{a \in N} \lambda_a x_a \leq v_\lambda(N), \sum_{a \in S} \lambda_a (x_a + \varepsilon^*) \geq v_\lambda(S) \right\} \bigcap E^T,$$

the equal treatment ε^*-core of the (N, V_λ) game. Note that the equal treatment ε^*-core of (N, V_λ) is non-empty for any $\lambda \in L^*$. For any $\lambda \in L^*$ and any $x \in C_{\varepsilon^*}(\lambda)$, x cannot be ε'-improved upon in the initial game (N, V) for any $\varepsilon' > \varepsilon^*$. (If a coalition S could improve, we would have $x_S + \varepsilon'_S \in V_S$ and $\sum_{a \in S} \lambda_a (x_a + \varepsilon'_a) > v_\lambda(S)$, contradicting the definition of $v_\lambda(S)$.) Hence, it remains to show that there exists $\lambda^* \in L^*$ such that some $x^* \in C_{\varepsilon^*}(\lambda)$ is feasible in the initial game.

Lemma 3. The correspondence $\lambda \longmapsto C_{\varepsilon^*}(\lambda)$ from L^* to $\mathbf{R}^N$ is bounded, convex-valued and has a closed graph. Moreover, for any $x \in C_{\varepsilon^*}(\lambda)$ and for any player a it holds that $x_a \geq -\alpha_N^q((0,T), C, (\varepsilon, B))$.

Proof of Lemma 3:

1). If $f, g \in C_{\varepsilon^*}(\lambda)$ then $\mu f + (1-\mu) g$ has equal treatment property and $\mu f + (1-\mu) g \in C_{\varepsilon^*}(\lambda)$ since:

$$\begin{aligned}
\text{(a)} \ \sum_{a \in N} \lambda_a (\mu f_a + (1-\mu) g_a) &= \mu \sum_{a \in N} \lambda_a f_a + (1-\mu) \sum_{a \in N} \lambda_a g_a \\
&\leq \mu v_\lambda(N) + (1-\mu) v_\lambda(N) = v_\lambda(N) \text{ and}
\end{aligned}$$

$$\begin{aligned}
\text{(b)} \ \sum_{a \in S} \lambda_a (\mu f_a + (1-\mu) g_a + \varepsilon^*) &= \mu \sum_{a \in S} \lambda_a (f_a + \varepsilon^*) + (1-\mu) \sum_{a \in S} \lambda_a (g_a + \varepsilon^*) \\
&\geq \mu v_\lambda(S) + (1-\mu) v_\lambda(S) = v_\lambda(S).
\end{aligned}$$

2). It is straightforward to see that graph is closed since $v_\lambda(S)$ depends continuously on λ.

3). Consider $x \in C_{\varepsilon^*}(\lambda)$. Since x is in the ε^*-core of (N, V_λ) game, x is ε^*-individually rational, that is,

$$x_a \geq -\varepsilon^* \geq -\alpha_N^q((0,T), C, (\varepsilon, B)).$$

4). Consider $x \in C_{\varepsilon^*}(\lambda)$. By construction,

$$\sum_{a \in N} \lambda_a x_a \leq v_\lambda(N) \leq \frac{1}{q} C\,|N|\,.$$

Since $\lambda \in L^* \subseteq L \subseteq \triangle_+$, there exists i such that $\lambda_i \geq \frac{1}{|N|}$. Then $\lambda \in L$ implies $\lambda_a \geq q\lambda_i \geq \frac{q}{|N|}$. Therefore, using 3) above we have that

$$\frac{q}{|N|} x_a \leq \lambda_a x_a \leq \frac{1}{q} C\,|N| - (1 - \frac{q}{|N|})(-\alpha_N^q((0,T), C, (\varepsilon, B))).$$

This proves that

$$x_a \leq \frac{1}{q^2} C\,|N|^2 + (\frac{|N|}{q} - 1)\alpha_N^q((0,T), C, (\varepsilon, B)). \blacksquare$$

Now let us define

$$\Psi(\lambda) := C_{\varepsilon^*}(\lambda) - \left\{ x \in K^* : \sum_{a \in N} \lambda_a x_a = \max_{z \in V(N)} \sum_{a \in N} \lambda_a z_a \right\} \bigcap E^T.$$

For $\lambda \in L^*$ both the first term and the second term of this sum are nonempty, bounded, convex-valued correspondences with closed graphs; this follows from Lemma 3 and the observations that (a) $V(N)$ is convex and symmetric across substitute players and (b) K^* is compact. Hence the sum $\Psi(\lambda)$ is also bounded, closed and convex-valued for $\lambda \in L^*$. By construction $\sum_{a \in N} z_a \lambda_a \leq 0$ for any $z \in \Psi(\lambda)$.

Now we can use the following theorem of excess demand, which is in fact a version of Kakutani's theorem. (For a proof see Hildenbrand and Kirman (1988), Lemma AIV.1)

Theorem (Debreu, Gale, Nikaido): Let $\triangle^*$ be a closed and convex subset of $\triangle_+$. If the correspondence Ψ from $\triangle^*$ is bounded, convex-valued, has closed graph and it holds that for all $p \in \triangle^*$, $p \cdot z \leq 0$ for all $z \in \Psi(p)$, then there exists $p^* \in \triangle^*$ and $z^* \in \Psi(p^*)$ such that $p \cdot z^* \leq 0$ for all $p \in \triangle^*$.

¿From the Debreu-Gale-Nikaido Theorem it follows that there exists $\lambda^* \in L^*$ and $z^* \in \Psi(\lambda^*)$ such that $\lambda \cdot z^* \leq 0$ for all $\lambda \in L^*$. Since $z^* \in \Psi(\lambda^*)$, z^* can be represented as $z^* = x^* - y^*$ with $x^* \in C_{\varepsilon^*}(\lambda)$, $y^* \in K^* \bigcap E^T$. Therefore $z^* \in E^T$. As we argued at the beginning of this Step, x^* is ε^*-undominated in the initial game (N, V). In addition, x^* has the equal treatment property.

We now deduce that x^* is feasible for the game (N, V). Observe that $x^* = y^* + z^*$, where $y^* \in K^* \bigcap E^T$ and $\lambda \cdot z^* \leq 0$ for all $\lambda \in L^*$. Hence $z^* \in d(L^*) \bigcap E^T$.

Lemma 4. Let X be a convex and symmetric across substitute players subset of $\mathbf{R}^N$. Let $X^* := X \bigcap E^T$. Then $d(X^*) \bigcap E^T \subset d(X)$.

Proof of Lemma 4: For any $x \in X$, let us construct $\bar{x} \in \mathbf{R}^N$ as follows: for each $1 \leq t \leq T$, for any $a \in N[t]$ define

$$\bar{x}_a := \frac{1}{|N[t]|} \sum_{i \in N[t]} x_i.$$

Since X is convex and symmetric across substitute players, $\bar{x} \in X$. Obviously, $\bar{x} \in E^T$. Therefore $\bar{x} \in X \bigcap E^T = X^*$.

Now consider any $y \in d(X^*) \bigcap E^T$. For any $x \in X$ we have

$$\begin{aligned} y \cdot x &= \sum_{i \in N} y_i x_i = \sum_{1 \leq t \leq T} y_t (\sum_{i \in N[t]} x_i) \\ &= \sum_{1 \leq t \leq T} |N[t]| \, y_t \bar{x}_t = \sum_{i \in N} y_i \bar{x}_i \leq 0, \end{aligned}$$

where the last inequality follows from the fact $y \in d(X^*)$ and $\bar{x} \in X^*$. Hence, by the definition of the dual negative cone, $d(X^*) \bigcap E^T \subset d(X)$. ■

Since $V(N)$ is convex and symmetric across substitute players, it follows from construction of $c(V^1(N))$ that $L = d(c(V^1(N))) \bigcap \triangle_+$ is convex and symmetric across substitute players. Therefore, by Lemma 4,

$$z^* \in d(L^*) \bigcap E^T \subset d(L) = c(V^1(N)).$$

Moreover

$$x^* \in K^* + c(V^1(N)) \subset V^1(N),$$

that is, x^* is feasible in the modified game. We also have $x^* \in C_{\varepsilon^*}(\lambda)$. It follows from Lemma 3 that

$$x_a^* \geq -\alpha_N^q((\delta, T), C, (\varepsilon, B)).$$

It now follows from the definition of K and K^* that $x^* \in K^* \subset V(N)$, that is, x^* is feasible in the initial game (N, V). We have now proven that x^* is in the equal treatment ε^*-core of the initial game; therefore the equal treatment ε^*-core is nonempty.□

5 Relationships to the literature.

There are a number of results in the literature showing that large games without side payments have non-empty approximate cores.[6] These results, however, are all obtained in the context of pregames. Recall that a pregame specifies a topological space of player types and a payoff set (or number) for every possible coalition in any game induced by the pregame. More precisely, given a compact metric space of player "types" or "attributes" (possibly finite), the payoff function of a pregame assigns a payoff set to every finite list of player types, repetitions allowed. Given any finite player set and an attribute function, assigning a type to each player in the player set, the payoff function of the game is determined by the payoff function of the pregame. Thus, the payoff set to any collection of players having a certain set of attributes is *independent* of the total player set in which it is embedded. The pregame structure itself has hidden consequences.[7] No such consequences can be hidden within parameterized collections of games since there is no necessary relationship between any of the games in the collection (other than that they are all described by the same parameters).

To study large games generally, without the structure and implicit assumptions imposed by a pregame, Kovalenkov and Wooders (1997a,b) introduce the concept of parameterized collections of games without side payments and show non-emptiness of approximate cores of arbitrary large games. The difference between the approach of arbitrary games and games derived from pregames may be viewed as analogous to the differences in approach of Kannai (1970) on one hand, who requires a topology on the space of preferences[8], and Anderson (1978) and Manelli (1991a,b) on the other hand, who consider arbitrary economies described by certain parameters.

In Kovalenkov and Wooders (1997a) we first extend the Kaneko and Wooders (1982) result for the ε-remainder core (which requires that the core of a subgame containing all but a fraction ε of the players be non-empty) to arbitrary games $G^q((0,T),C,(0,B))$ with strictly effective small groups and a finite number of exact types. Like the result of Kaneko and Wooders (1982) no other conditions are required on the payoff sets – C may be taken to be infinity. We then extend the Shubik and Wooders (1983) results for the weak ε-remainder core (which, roughly,

[6] See Wooders (1983), Shubik and Wooders (1983), Kaneko and Wooders (1982), Kankeo and Wooders (1984,1996) and Wooders and Zame (1989) (listed in order of completion). We refer the reader interested in related results for games with side payments to Kovalenkov and Wooders (1997a) and the references therein.

[7] For example, within the context of pregame with side payments, per capita boundedness is equivalent to small group effectiveness, the condition that all or almost all gains to collective activities can be realized by groups bounded in size (Wooders 1994b, Section 5).

[8] Of course Kannai (1970) was considering the convergence of approximate cores to the core of a continuum limit economy, the sort of situation where a topology on the space of player types seems natural.

requires that the ε-core be non-empty for a subgame containing all but a fraction ε of the players be non-empty) to $G^q((\delta, T), C, (\varepsilon, B))$. Finally, under the assumption that the games are q-comprehensive with $q > 0$, we show nonemptiness of the ε-core. Since we require $q > 0$ but not convexity, our results in Kovalenkov and Wooders (1997a) do not exactly provide an extension of the results of Wooders (1983) to arbitrary games, although a Corollary is a step in that direction.

In Kovalenkov and Wooders (1997b) we assume convexity of payoff sets and "thickness" of the total player set, dictating that the percentage of players of each approximate type is bounded away from zero. Thickness substitutes for the feature in Wooders (1983) that the games have a constant proportion of agents of each type and thus the nonemptiness result in Kovalenkov and Wooders (1997b) extends Wooders (1983) to arbitrary games.

An interesting aspect of large games is whether cores and approximate cores are symmetric. Symmetric outcomes are particularly interesting in the study of the "competitive market-like" properties of large games. In competitive markets, agents with the same characteristics receive equivalent outcomes (given their preferences). Thus, it is interesting to determine when there exists symmetric payoffs in cores and approximate cores of large games. In Kovalenkov and Wooders (1997b) we show that if side payments are "limited" then all payoffs in the ε-core are nearly symmetric and a maximal bound on the distance between ε-core payoffs to two players of the same approximate type is given. For the generality of the model of this paper, we cannot show that all outcomes in the ε-core are symmetric. At best, it appears that it may be possible to show that ε-cores of large games treat "most" players of the same approximate type approximately equally. (Such a result is shown for games with side payments derived from pregames in Wooders (1980,1994)).

While Kovalenkov and Wooders (1997a,b) show non-emptiness of approximate cores of arbitrary large games, no explicit bound on the required size of the games is provided and the dependence of the required size on the parameters is not explicitly demonstrated. Such a result was obtained in Kovalenkov and Wooders (1997c) for games with side payments. The current paper extends that result to games without side payments.

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