Epsilon cores of games and economies with limited side payments.*

Alexander Kovalenkov†
Department of Economics, Universitat Autònoma de Barcelona, 08193, Bellaterra (Barcelona), Spain

Myrna Holtz Wooders‡
Department of Economics, University of Toronto, 150 St.George St., Toronto, M5S 3G7, Canada

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Abstract

We introduce the concept of a parameterized collection of games with limited side payments, ruling out large transfers of utility. Under the assumption that the payoff set of the grand coalition is convex, we show that a large game with limited side payments has a nonempty ε-core. Our second result is that, when some degree of side-paymentness within nearly-effective small groups is assumed, then all payoffs in the ε-core treat all players of the same approximate type approximately equally. A bound on the distance between ε-core

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payoffs of any two players of the same approximate type is given in terms of the parameters describing the game.

1 Introduction.

In many game theoretic and economic situations side payments may be freely made only within small groups of players. For example, marriage games may permit some sidepaymentness between matched pairs of players but there may be no medium to make side payments between marriages. Even if side payments were possible it may require cooperation and agreement between a large number of players to make large transfers of payoff between members of arbitrarily large groups.[1] In this paper we introduce the concept of games with limited side payments and study the ε-core. Side payments are limited in the sense that coalitions bounded in size are nearly effective for achievement of almost all gains to increasing coalition size and large transfers cannot be made. The ε-core consists of those feasible payoffs that cannot be improved upon by any coalition by at least ε for each member of the coalition. We show that arbitrary games with limited side payments and convex total payoff sets have nonempty ε-cores, where for games with "many" players, ε can be chosen to be "small." In addition, we demonstrate conditions under which any ε-core payoff has approximately the equal-treatment property; that is, all players who are substitutes or approximate substitutes are treated approximately equally.

The games in a parameterized collection are described by (a) a bound on the number of approximate player types and the accuracy of the approximation and (b) a bound on the size of "nearly" effective groups of players. Two players have approximately the same types if they make similar contributions to coalitions and, when in the same coalition, their payoffs are interchangeable; that is, they are approximate substitutes. That nearly-effective groups are bounded in size means that all or almost all gains to collective activities can be realized by coalitions no larger than the bound. In keeping with the extant literature we call this an assumption of "small group effectiveness" (cf., Wooders 1994b).

Our first Theorem demonstrates that, with convexity of payoff sets of grand coalitions, all sufficiently large games in a parameterized collection have non-empty symmetric ε-cores. A payoff for a game is symmetric if all players of the same approximate type are treated equally. The symmetric ε-core consists of those payoffs in the ε-core that are symmetric. The size of a game required to ensure non-emptiness of the symmetric ε-core depends on the parameters describing the games, a bound on feasible per capita payoffs, and a bound on the thickness (the percentage of players of each approximate type) of the game.

Our second Theorem requires some degree of side-paymentness for small coalitions

[1]This point is stressed in Kaneko and Wooders (1997).

– those coalitions containing no more members than the given bound on nearly effective coalition sizes. Payoff sets for nearly-effective coalitions are required to satisfy "q-comprehensiveness" for some $q > 0$; that is, side payments can be made between players in nearly effective coalitions at a rate bounded below by q. The Theorem demonstrates that if a game with limited side payments has sufficiently many players of any particular approximate type, then all payoffs in the ε-core treat all members of that type nearly symmetrically – that is, any two players of such an approximate type receive approximately the same ε-core payoffs. The distance between ε-core payoffs of any of the two players is bounded above by a function of the parameters describing the game. From inspection of the bound, we can conclude that as approximate types become close to exact types, nearly effective groups become strictly effective and ε goes to zero, the ε-core "shrinks" to the symmetric ε-core, which treats all players of the same type identically.

Our interest in approximate cores stems from our interest in competition in diverse economic settings, such as those with local public goods, clubs, production, location, indivisibilities, non-monotonicities, and other deviations from the classic Arrow-Debreu-McKenzie model. Both of our results contribute to a line of research investigating the "market-like" properties of large games under conditions limiting returns to group size, cf. Wooders (1983,1994b). An important precursor is Shapley and Shubik (1966), which shows that large exchange economies have non-empty approximate cores; another is Shapley and Shubik (1969), which demonstrates an equivalence between markets and totally balanced games. From an economic perspective, the remarkable paper of Tiebout (1956), in which he conjectured that large economies with local public goods were "market-like," contains insights similar to those that lead to the current research.[2] In large games, the core is a "stand-in" for the competitive equilibrium.[3] Thus, a crucial result is that in large games cores or approximate cores are nonempty.

It has now been shown in a number of papers on economies with private goods only that cores and approximate cores converge to competitive outcomes. Such results have also been obtained in models of economies with local public goods or clubs; see Conley and Wooders (1997) for a recent survey. This implies that cores of large economies are symmetric. If two agents are identical in terms of all aspects that enter preferences of others and/or production possibilities, then the free competition embodied in the notion of the core rules out discrimination between these agents. In large games with strictly effective small groups and with strongly comprehensive payoff sets, all payoffs in the core are symmetric (Wooders 1983, Theorem 3).[4] It is

[2] See Wooders (1997) and references therein.

[3] See, for example, see Shapley and Shubik (1969) and Wooders (1994a,b) for market-games with side payments. Also relevant are Billera (1974) and Mas-Colell (1975). Hildenbrand and Kirman (1988, Chapter 4), among others, treat the core more explicitly as the stand in for the competitive equilibrium in non-classical exchange economies.

[4] A special case of this result, treating games with side payments and using an equivalent assump-

known that when there are many close substitutes for each player, under the mild condition of per capita boundedness of average payoff, ε-cores of large games with side payments are nonempty and approximately symmetric in the sense that *most* players of the same type receive nearly the same payoffs (Wooders 1980,1994a).[5] Our result showing approximate symmetry of ε-cores demonstrates that the symmetry property of competitive equilibrium extends to approximate cores of large games with small effective groups and limited side payments. Since the games we consider in the current paper have *limited* side payments we are able to obtain a result close to that for games with strictly effective small groups. Thus, our symmetry result is stronger than prior results in three important ways: First, although it is not required that small groups are *strictly* effective, we demonstrate that *all* players of the same approximate type are treated approximately equally. Second, our result is for arbitrary games, rather than games derived from pregames – more on this below. Third, when there are sufficiently many players of an approximate type, our result provides an explicit bound on the distance between the ε-core payoffs for any two players of that approximate type.

Finally, we remark that prior to our introduction of parameterized collections of games other papers showing competitive properties of large games relied on the pregame notion. Some of these results are discussed in Section 5. The pregame structure, which requires that all games considered are subgames of some larger game, has hidden consequences. To illustrate, when there are sufficiently many players of each type that appears in the games, the assumptions of small group effectiveness and per capita boundedness are equivalent (Wooders 1994b, Section 5). Our framework does not have such consequences. In the literature of economics, our approach is more closely related to that of Anderson (1978) and Manelli (1991a,b), for example, who treat arbitrary economies described by certain parameters, rather than that of Kannai (1970) and Hildenbrand (1974), who consider large economies approaching a continuum limit. The framework of parameterized collections of games allows us to avoid the hidden consequences of the pregame framework – more on this below – and demonstrate that the results depends on certain parameters describing the game.

In the next section we introduce required definitions. Section 3 states our nonemptiness result and provides a number of examples. Section 4 introduces games with "side-paymentness" uniformly bounded away from zero only for nearly-effective small groups, states our result that the ε-core is approximately symmetric, and provides additional examples. Section 5 provides some remarks on our model and results and relates our work to prior literature. Section 6 discusses related economies with limited side payments. Proofs are collected in Section 7 and two additional examples appear in an appendix.

tion bounding effective groups, appears in Scotchmer and Wooders (1988).

[5] This result also appears in Wooders (1979) with a stronger condition bounding effective group sizes and in the 1982 Cowles Foundation Discussion Paper No. 657 version of Shubik and Wooders (1986).

2 Definitions.

2.1 Cooperative games: description and notations.

Let $N = \{1, .., n\}$ denote a *set of players.* A nonempty subset of N is called a *coalition.* For any coalition S let $\mathbf{R}^S$ denote the $|S|$-dimensional Euclidean space with coordinates indexed by elements of S. For $x \in \mathbf{R}^N, x_S$ will denote its restriction to $\mathbf{R}^S$. To order vectors in $\mathbf{R}^S$ we use the symbols $>>$, $>$ and $\geq$ with their usual interpretations. The nonnegative orthant of $\mathbf{R}^S$ is denoted by $\mathbf{R}^S_+$ and the strictly positive orthant by $\mathbf{R}^S_{++}$. For $A \subset \mathbf{R}^S$, $co(A)$ denotes the convex hull of A. We denote by $\vec{1}_S$ the vector of ones in $\mathbf{R}^S$, that is, $\vec{1}_S = (1, ..., 1) \in \mathbf{R}^S$. Each coalition S has a feasible set of payoffs or utilities denoted by $V_S \subset \mathbf{R}^S$. By agreement $V_\emptyset = \{0\}$ and $V_{\{i\}}$ is nonempty, closed and bounded from above for any i. Moreover we will assume that

$$(*)\ \max\left\{x : x \in V_{\{i\}}\right\} = 0 \text{ for any } i \in N.$$

It must be noted that $(*)$ is by no means restrictive since it can always be achieved by a normalization. It is convenient to describe the feasible utilities of a coalition as a subset of $\mathbf{R}^N$. For each coalition S let

$$V(S) := \left\{x \in \mathbf{R}^N : x_S \in V_S \text{ and } x_a = 0 \text{ for } a \notin S\right\}.$$

A *game without side payments* (called also an *NTU game* or simply a *game*) is a pair (N, V) where the correspondence $V : 2^N \longrightarrow \mathbf{R}^N$ is such that $V(S) \subset \left\{x \in \mathbf{R}^N : x_a = 0 \text{ for } a \notin S\right\}$ for any $S \subset N$ and satisfies the following properties :

(2.1) $V(S)$ is non-empty and closed for all $S \subset N$.

(2.2) $V(S) \cap \mathbf{R}^N_+$ is bounded for all $S \subset N$, in the sense that there is real number $K > 0$ such that if $x \in V(S) \cap \mathbf{R}^N_+$ then $x_i \leq K$ for all $i \in S$.

(2.3) $V(S_1 \bigcup S_2) \supset V(S_1) + V(S_2)$ for any disjoint $S_1, S_2 \subset N$ (superadditivity).

(2.4) $V(S) = V(S) - \left\{x \in \mathbf{R}^N_+ : x_a = 0 \text{ for } a \notin S\right\}$ for all $S \subset N$ (comprehensiveness).

2.2 Games with limited side payments.

To define parameterized collections of games we will need the concept of Hausdorff distance. For every two nonempty subsets E and F of a metric space (M, d) define the *Hausdorff distance between E and F* $dist(E, F)$ (with respect to the metric d on M) by

$$dist(E, F) := \inf\left\{\varepsilon \in (0, \infty) :\ E \subset B_\varepsilon(F) \text{ and } F \subset B_\varepsilon(E)\right\},$$

where $B_\varepsilon(E) := \{x \in M : d(x, E) \leq \varepsilon\}$ denotes the ε-neighborhood of E.

Since payoff sets are unbounded below, we will use a modification of the concept of the Hausdorff distance so that the distance between two payoff sets is the distance between the intersection of the sets and a compact subset of Euclidean space. Let m^* be a fixed positive real number. Let M^* be a subset of Euclidean space $\mathbf{R}^N$ defined by $M^* := \left\{x \in \mathbf{R}^N : x_a \geq -m^* \text{ for any } a \in N\right\}$. For every two nonempty subsets E and F of Euclidean space $\mathbf{R}^N$ let $H_\infty [E, F]$ denote the Hausdorff distance between $E \cap M^*$ and $F \cap M^*$ with respect to the metric $\|x - y\|_\infty := \max_i |x_i - y_i|$ on Euclidean space $\mathbf{R}^N$.

$\delta-$substitute partitions. Let (N, V) be a game and let $\delta \geq 0$ be a non-negative real number. A δ-substitute partition is a partition of the player set N into subsets with the property that any two players in the same subset are "within δ" of being substitutes for each other. For games with side payments, to state that players i and j are substitutes it is only necessary to state that their marginal contribution to any coalition is the same. For games without side payments, following Wooders (1983), players i and j are substitutes if (a) they make the same marginal contribution to any coalition and (b) for coalitions containing both i and j, interchanging the names and payoffs of players i and j does not change payoff sets.

Formally, given a set $W \subset \mathbf{R}^N$ and a permutation τ of N, let $\sigma_\tau(W)$ denote the set formed from W by permuting the values of the coordinates according to the associated permutation τ. Given a partition $\{N[t] : t = 1, .., T\}$ of N, a permutation τ of N is *type − preserving* if, for any $i \in N$, $\tau(i)$ belongs to the same element of the partition $\{N[t]\}$ as i. A *δ-substitute partition* of N is a partition $\{N[t] : t = 1, .., T\}$ of N with the property that, for any type-preserving permutation τ and any coalition S,

$$H_\infty \left[V(S), \sigma_\tau^{-1}(V(\tau(S)))\right] \leq \delta.$$

Note that in general a δ-substitute partition of N is not uniquely determined.

profiles. Let $\delta \geq 0$ be a non-negative real number and let (N, V) be a game. For a δ-substitute partition $\{N[t] : t = 1, .., T\}$ of N, the set $N[t]$ is interpreted as an *approximate type*. Players in the same element of a δ-substitute partition are *δ-substitutes*. When $\delta = 0$, they are *exact substitutes*. A *profile* relative to $\{N[t]\}$ is a vector of non-negative integers $f \in Z_+^T$ and a *subprofile* s of a profile f is a profile satisfying the condition that $s \leq f$. Given $S \subset N$ the *profile of* S is a profile, say $s \in Z_+^T$, where $s_t = |S \cap N[t]|$. A profile describes a group of players in terms of the numbers of players of each approximate type in the group. The *norm* of a profile f, denoted by $\|f\|$, is simply the number of players in a group described by f, that is, $\|f\| = \sum f_t$.

(δ,T)- type games. Let δ be a non-negative real number and let T be a positive integer. A game (N, V) is a *(δ, T)-type game* if there is a T-member δ-substitute partition $\{N[t] : t = 1, .., T\}$ of N (in other words, if there are T approximate types).

$\varepsilon-$*effective* $B-$*bounded groups.* Informally, groups of players containing no more than B members are ε-effective if, by restricting coalitions to fewer than B members, the loss to each player is no more than ε. This is a relatively strong form of "small group effectiveness" for arbitrary games. Let (N,V) be a game. Let $\varepsilon \geq 0$ be a given non-negative number and let B be a given positive integer. For each group $S \subset N$, define a corresponding set $V(S;B) \subset \mathbf{R}^N$ in the following way:

$$V(S;B) := \bigcup \left[\sum_k V(S^k) : \left\{ S^k \right\} \text{ is a partition of } S, \ \left| S^k \right| \leq B \right].$$

The set $V(S;B)$ is the payoff set to the coalition S when groups are restricted to have no more than B members. Note that, by superadditivity, $V(S;B) \subset V(S)$ for any $S \subset N$ and, by construction, $V(S;B) = V(S)$ for $|S| \leq B$. The game (N,V) has ε-*effective* B-*bounded groups* if for every group $S \subset N$

$$H_\infty \left[V(S), V(S;B) \right] \leq \varepsilon.$$

When $\varepsilon = 0$, 0-effective B-bounded groups are called *strictly effective* B-*bounded groups.*

games with limited side payments $G((\delta,T),(\varepsilon,B))$. Let T and B be positive integers. Let $G((\delta,T),(\varepsilon,B))$ be the collection of all (δ,T)-type games that have ε-effective B-bounded groups. Then $G((\delta,T),(\varepsilon,B))$ is called a *parameterized collection of games with limited side payments.* In Section 4 we introduce one additional parameter to describe subcollections of games with limited side payments.

It may not be immediately apparent how side payments are limited by the above definition. To illustrate this feature, consider a game (N^m, V^m) in which any two players can earn two dollars and side payments are unrestricted. Suppose there are $2m$ players in the game. Thus the total payoff to the grand coalition is $2m$ dollars. Although it may initially appear that the game has ϵ-effective 2-bounded groups, according to the definition of this paper it does not. Moreover, given any $\varepsilon \geq 0$ and any positive integer B, for sufficiently large m the game (N^m, V^m) has no ε-effective B-bounded groups. This is because in the above definition of $V(S;B)$ no side payments are allowed between small groups . To make this clear, consider a payoff x that gives $2m$ dollars to one selected player and nothing to the others. The payoff x is feasible in the game, that is $x \in V(N)$. But for $2m > B$, $x \notin V(S;B)$. Moreover, for large m, according to the Hausdorff distance H_∞, x is very far from $V(N;B)$. Thus, the form of ε-effective B-bounded groups used in the current paper rules out games with side payments.[6] In fact, similar examples will show that q-comprehensiveness of payoff sets for all coalitions with $q > 0$ are ruled out by the

[6]The notion of small group effectiveness used in our prior papers permitted games with side payments.

definition of ε-effective B-bounded groups introduced in this paper. This motivates our use of the term "limited side payments."

Our results hold for all parameters δ and ε that are sufficiently small, that is, $2(\delta+\varepsilon) < m^*$, where m^* is a positive real number used in the definition of the Hausdorff distance.

3 Non-emptiness of symmetric ε-cores of games with limited side payments.

the core and epsilon cores. Let (N,V) be a game. A payoff x is *ε-undominated* if for all $S \subset N$ and $y \in V(S)$ it is not the case that $y_S >> x_S + \vec{1}_S\varepsilon$. The payoff x is *feasible* if $x \in V(N)$. The ε-core of a game (N,V) consists of all feasible and ε-undominated allocations. When $\varepsilon = 0$, the ε-core is the *core.*

the equal treatment epsilon core. Given non-negative real numbers ε and δ, we will define the *equal treatment ε-core* of a game (N,V) relative to a partition $\{N[t]\}$ of the player set into δ-substitutes as the set of payoffs x in the ε-core with the property that for each t and all i and j in $N[t]$, it holds that $x_i = x_j$.

3.1 The Theorem.

Some additional restrictions are required to obtain non-emptiness of epsilon cores of large games with limited side payments. We will discuss the indispensability of these conditions in the next subsection.

per capita boundedness. Let C be a positive real number. A game (N,V) has a *per capita payoff bound* of C if, for all coalitions $S \subset N$,

$$\sum_{a\in S} x_a \le C\,|S| \text{ for any } x \in V(S).$$

thickness. Let ρ be a positive real number, $0 < \rho \le 1$. A (δ, T)-type game (N,V) is *ρ-thick* if for each t it holds that $|N[t]| \ge \rho\,|N|$.

The following Theorem establishes non-emptiness of the ε-core for a sufficiently large game with limited side payments. In fact, Theorem 1 shows non-emptiness of the symmetric (equal treatment of approximate substitutes) ε-core as well.

Theorem 1. Let T and B be positive integers. Let C and ρ be positive real numbers, $0 < \rho \le 1$. Then for any $\varepsilon^* > 0$ exists an integer $\eta(\varepsilon^*, T, B, C, \rho)$ such that if

(a) $(N,V) \in G((\delta,T),(\varepsilon,B))$,

(b) $V(N)$ is convex,
(c) (N,V) is ρ-thick,
(d) (N,V) has a per capita payoff bound of C, and
(e) $|N| \geq \eta(\varepsilon^*, T, B, C, \rho)$
then the equal treatment $(\varepsilon^* + \delta + \varepsilon)$-core of (N,V) is nonempty.

The proof of the Theorem uses the following definition of the ε-remainder core and another nonemptiness result. Informally the ε-remainder core allows some small proportion of agents to be ignored. A payoff is in the ε-remainder core if it is feasible and if it is in the core of a subgame whose player set contains all the remaining players. The ε-remainder core is defined as follows.

the ε-remainder core. Let (N,V) be a game. A payoff x belongs to the *ε-remainder core* if, for some group $S \subset N$, $\frac{|N|-|S|}{|N|} \leq \varepsilon$ and x_S belongs to the core of the subgame (S,V).

non-emptiness of the ε-remainder core. (Kovalenkov and Wooders 1997a, Theorem 1 (subcase for $q=0$)). Let T and B be positive integers. For any $\varepsilon^* > 0$, there exists an integer $\eta_1(\varepsilon^*, T, B)$ such that if
(a) $(N,V) \in G((0,T),(0,B))$ and
(b) $|N| \geq \eta_1(\varepsilon^*, T, B)$
then the ε^*-remainder core of (N,V) is non-empty.

A sketch of the proof: In the proof of Theorem 1 we first approximate a game (N,V) with T approximate types and ε-effective B-bounded groups by another game (N,V^0) with T exact player types and strictly effective B-bounded groups, that is, by a game $(N,V^0) \in G((0,T),(0,B))$. Then we use the above result on nonemptiness of the ε-remainder core to obtain non-emptiness of the ε^*-remainder core of (N,V^0). This result implies that there is a core payoff in a large subgame (S,V) and that the proportion of "leftover" players is bounded by ε^*. Then using per capita boundedness and thickness of the player set, we can "smooth" payoffs in the ε^*-remainder core across players of the same approximate types and thus compensate leftovers. The convexity assumption allows us to obtain equal treatment by averaging the payoffs across players of the same type while maintaining feasibility. We present the formal proof of Theorem 1 in the special section. ■

In the next subsection we present examples clarifying Theorem 1.

3.2 Examples.

Our first example illustrates the statement of Theorem 1.

Example 1: Let (N, V_0) be a superadditive game where, for any two-person coalition $S = \{i, j\}$, $j \neq i$,

$$V_0(S) := \{x \in \mathbf{R}^N : \ x_i \leq 1, \ x_j \leq 1, \text{and } x_k = 0 \text{ for } k \neq i, j\}$$

and for each $i \in N$,

$$V_0(\{i\}) := \{x \in \mathbf{R}^N : x_i \leq 0 \text{ and } x_j = 0 \text{ for all } j \neq i\}.$$

For an arbitrary coalition S the payoff set $V_0(S)$ is given as the *superadditive cover*, that is,

$$V_0(S) := \bigcup_{\mathcal{P}(S)} \sum_{S' \in \mathcal{P}(S)} V_0(S'),$$

where the union is taken over all partitions $\mathcal{P}(S)$ of S.

We now define a game (N, V_{co}) in the following way. For each $S \subset N$, let $V_{co}(S)$ denote the convex cover of the payoff set $V_0(S)$, that is

$$V_{co}(S) := co(V_0(S)).$$

Obviously the game (N, V_{co}) has convex payoff sets, one player type, and per capita bound of 1. We leave it to the reader to verify that for any positive integer $m \geq 3$ the game (N, V_{co}) has $\frac{1}{m}$-effective m-bounded groups. Thus the game (N, V_{co}) is a member of the class $G((0,1), (\frac{1}{m}, m))$ and is 1-thick. Then, by Theorem 1, for $|N| \geq \eta(\varepsilon^*, 1, m, 1, 1)$ the equal treatment $(\varepsilon^* + \frac{1}{m})$-core of (N, V_{co}) is nonempty.

Therefore for any $\varepsilon^0 > 0$ there is a positive integer $N(\varepsilon^0)$ such that for any $|N| \geq N(\varepsilon^0)$ the game (N, V_{co}) has a non-empty equal treatment ε^0-core. (For example take an integer $m^0 \geq \frac{2}{\varepsilon^0}$ and define $N(\varepsilon^0) \geq \eta(\frac{\varepsilon^0}{2}, 1, m^0, 1, 1)$.)

The next two examples illustrates that neither convexity of $V(N)$ nor thickness of the game can be omitted in the statement of Theorem 1.

Example 2. *Convexity of* $V(N)$. Consider a sequence of games without side payments $(N^m, V^m)_{m=1}^{\infty}$ where the m^{th} game has $2m+1$ players. Suppose that any player alone can earn at most 0 units. Suppose that any two-player coalition can distribute a total payoff of 2 units in any agreed-upon way, while there is no transferability of payoff between coalitions. Suppose only one- and two-player coalitions are effective. Then the game is described by the following parameters: $T = 1, \ \rho = 1, \ B = 2, \ \delta = \varepsilon = 0$ and has per capita bound $C = 1$. Thus the game satisfies all the assumptions of Theorem 1 except convexity of $V(N)$. The $\frac{1}{3}$-core of the game, however, is empty for arbitrarily large values of m. (At any feasible payoff, at least one player gets 0 units and some other player no more than 1 unit. These two players can form a coalition and receive 2 units in total. This coalition can improve upon the given payoff for each of its members by $\frac{1}{2}$.)

Example 3. *Thickness.* Consider a sequence of games without side payments $(N^m, V^m)_{m=1}^{\infty}$ where the m^{th} game has $m+3$ players. Suppose that there are two types of players. Let the m^{th} game have 3 players of type A and m players of type B. Assume that only players of the type A are essential in the following sense: Any coalition with less than two players of type A can get only 0 units or less for each of its players. Suppose that any coalition with two or three players of type A can get only 2 units to be distributed between players of type A in the coalition and 0 units or less for each of its players of type B. Note that there is no transferability of payoff from the players of type B to the players of type A. Then the game is described by the following parameters: $T = 2,\ B = 3,\ \delta = \varepsilon = 0,\ C = 1$. Moreover, $V^m(N)$ is convex. The $\frac{1}{7}$-core of the game, however, is empty for arbitrarily large values of m. (For any feasible payoff there is a player of type A who receives no more than $\frac{2}{3}$ units. There is another player of type A that gets no more than 1 unit. These 2 players can form a coalition and receive 2 units in total. This coalition can improve upon the given payoff for each of its members by $\frac{1}{6}$, since $(2 - \frac{5}{3})\frac{1}{2} = \frac{1}{6}$.)

Remark. *Per capita boundedness and small group effectiveness.* Recall that for games with side payments derived from pregames, Wooders (1994b) establishes an equivalence between small group effectiveness and per capita boundedness. Kovalenkov and Wooders (1997a, Examples 4 and 5) demonstrate that, within the framework of parametrized collections of games, this equivalence no longer holds; both small group effectiveness and per capita boundedness are required for the results of that paper. In Theorem 1 of the current paper a number of assumptions different from those of Kovalenkov and Wooders (1997a) are used (specifically, convexity of $V(N)$, limited side payments and thickness). Nevertheless, since games with side payments have convex payoff sets, Example 4 from Kovalenkov and Wooders (1997a) shows that the condition of small group effectiveness in Theorem 1 can not be disregarded. For the convenience of the reader we present this example in an appendix, Example A1. The necessity of the assumption of per capita boundedness is not clear. Example A2 in the appendix illustrates that per capita boundedness is not a consequence of the other conditions in Theorem 1 but we have no example showing indispensability of per capita boundedness. Note, however, that per capita boundedness is a very mild assumption satisfied by most economic examples.

4 Symmetry of ε-cores of games with limited side payments.

In the previous Section we presented results on non-emptiness of the ε-cores for large games with limited side payments. In fact, Theorem 1 shows non-emptiness of the symmetric ε-core as well. In the current Section we focus on games where some degree of side payments is required for small groups. Specifically, we require q-comprehensiveness for nearly-effective groups. We then demonstrate that under this condition, the ε-core of a game with limited side payments is symmetric in the very strong sense: any payoff in this ε-core treats *all* players that are approximate substitutes approximately equally.

4.1 The Theorem.

Consider a set $W \subset \mathbf{R}^S$. Recall that W is *comprehensive* if $x \in W$ and $y \leq x$ implies $y \in W$. The set W is *strongly comprehensive* if it is comprehensive, and whenever $x \in W$, $y \in W$, and $x < y$ there exists $z \in W$ such that $x << z$.[7]

Our second Theorem requires a uniform version of strong comprehensiveness. Given $x \in \mathbf{R}^S$, $i, j \in S$, $0 \leq q \leq 1$ and $\varepsilon > 0$, define a vector $x^q_{i,j}(\varepsilon) \in \mathbf{R}^S$, by

$$\begin{aligned} (x^q_{i,j}(\varepsilon))_i &= x_i - \varepsilon, \\ (x^q_{i,j}(\varepsilon))_j &= x_j + q\varepsilon, \text{ and} \\ (x^q_{i,j}(\varepsilon))_k &= x_k \text{ for } k \in S \setminus \{i, j\}. \end{aligned}$$

The set W is *q-comprehensive* if W is comprehensive and if, for any $x \in W$, it holds that $(x^q_{i,j}(\varepsilon)) \in W$ for any $i, j \in S$ and any $\varepsilon \geq 0$.

This condition for $q > 0$ bounds the slope of the Pareto frontier of the set away from zero. For q equal to one, q-comprehensiveness is equivalent to the assumption of transferable utility. For q equal to zero, q-comprehensiveness means simply comprehensiveness. Note that if a set is q-comprehensive for some $q > 0$ then the set is q'-comprehensive for all q' with $q \geq q' \geq 0$.

<u>*games with side payments limited to nearly-effective groups* $G((\delta, T), (\varepsilon, B, q))$.</u> Let $G((\delta, T), (\varepsilon, B))$ be a parameterized collection games with limited side payments that was defined in Section 2. Let $0 \leq q \leq 1$ be given. Let $G((\delta, T), (\varepsilon, B, q))$ denote the subcollection of games in $G((\delta, T), (\varepsilon, B, q))$ with the property that if (N, V) is a member of the subcollection then for all $S \subset N$ with $|S| \leq B$, V_S is q-comprehensive. Note that

$$G((\delta, T), (\varepsilon, B, q)) \subset G((\delta, T), (\varepsilon, B, q')) \subset G((\delta, T), (\varepsilon, B, 0) = G((\delta, T), (\varepsilon, B))$$

for all q and q' such that $q \geq q' \geq 0$. When $q > 0$ will call this collection of games, *games with side payments limited to nearly-effective groups.*

[7]Informally, if one person can be made better off (while all the others remain at least as well off), then all persons can be made better off. Strong comprehensiveness has also been called "nonleveledness."

For games with limited side payments, providing that the payoff sets for effective small groups are q-comprehensive with $q > 0$, any two members of an approximate type that is not dramatically scarce (that is, there are more than B players of that type) are treated approximately equally by the ε^*-core.

Theorem 2. Let $(N, V) \in G((\delta, T), (\varepsilon, B, q))$ where $q > 0$ and let $\{N[t]\}$ be a partition of N into δ-substitutes. If some payoff x belongs to the ε^*-core of (N, V) and for some t it holds that $|N[t]| > B$, then for any $t_1, t_2 \in N[t]$,

$$|x_{t_1} - x_{t_2}| \leq \frac{2B}{q}(\varepsilon^* + \delta + \varepsilon).$$

The above theorem implies that, as ε^*, δ and ε "became small," providing there are more than B players of each type, the ε^*-core "shrinks" to the equal treatment ε^*-core. We present the proof of Theorem 2 in the special section. In the next subsection we present examples clarifying Theorem 2.

4.2 Examples.

The following simple example illustrates the statement of Theorem 2 and also shows that, in the central case of $\delta = \varepsilon = 0$, the bound provided can not be improved upon.

Example 4: Let (N, V^*) be a superadditive game where for any two-person coalition $S = \{i, j\}$, $j \neq i$,

$$V^*(S) := \{x \in \mathbf{R}^N : \ x_i + x_j \leq 2, \text{and } x_k = 0 \text{ for } k \neq i, j\}$$

and for each $i \in N$, let

$$V^*(\{i\}) := \{x \in \mathbf{R}^N : x_i \leq 0 \text{ and } x_j = 0 \text{ for all } j \neq i\}.$$

For an arbitrary coalition S the payoff set $V^*(S)$ is given as the *superadditive cover*, that is,

$$V^*(S) := \bigcup_{\mathcal{P}(S)} \sum_{S' \in \mathcal{P}(S)} V^*(S'),$$

where the union is taken over all partitions $\mathcal{P}(S)$ of S.

Observe that the game (N, V^*) has one exact player type and strictly effective 2-bounded groups. Therefore $(N, V^*) \in G((0, 1), (0, 2))$. Moreover V^*_S is 1-comprehensive for any $S \subset N$, $|S| \leq 2$. Let the game (N, V^*) have an even number of players, that is $|N| = 2m$ for some positive integer m. Observe that the payoff which gives one unit for each player belongs to the core of (N, V^*). Thus, for any $\varepsilon^* \geq 0$, the ε^*-core of (N, V^*) is non-empty. Theorem 2 implies that for any $a, b \in N$ and any payoff x in the ε^*-core of (N, V^*), providing that $|N| > 2$, we must have

$$|x_a - x_b| \leq 4\varepsilon^*.$$

It is easy to check that this statement can not be improved. Observe first that the restriction $|N| > 2$ is essential. If $|N| = 2$ then any division of two units between two players is a core payoff; thus, for $\varepsilon^* < \frac{1}{2}$ the bound of the Theorem is not applicable. Now consider a payoff y that gives 1 unit to each of the players except two selected players (say players a and b), $1-2\varepsilon^*$ to player a, and $1+2\varepsilon^*$ to player b. The reader can observe that even in the case $|N| > 2$ this payoff is in the ε^*-core of the game (N, V^*). (The payoff is obviously feasible, but it is also ε^*-undominated, since any pair of players get at y at least $2 - 2\varepsilon^*$.) But at this payoff it holds that $|y_a - y_b| = 4\varepsilon^*$. Thus the bound given in Theorem 2 cannot be improved.

It is well known that a Theorem such as the above will not hold for games with side payments. The following example provides an illustration.

Example 5a: Fix any $\varepsilon > 0$. Consider a game with side payments (N, v) that has one exact player type and a nonempty symmetric core. (There are plenty of such games.) Take any payoff $x \in \mathbf{R}^N$ in the symmetric core of (N, v). This determines a number $\overline{x}$ as the payoff for each player in N. Now select one player (say player a). Consider the payoff y that assigns $\overline{x} - \varepsilon$ for all players except player a and $\overline{x} + (|N| - 1)\varepsilon$ to player a. Obviously the payoff y is feasible and ε-undominated. Thus y belongs to the ε-core of (N, v). However for a very large $|N|$, y treats player a much better than others.

The sharpest result that can be obtained for games with side payments induced by pregames satisfying per capita boundedness (or even strict small group effectiveness) is that for small values of ε, *most* players of the same type are treated approximately equally (Wooders 1980,1994a). For games without side payments that have q-comprehensive payoff sets for $q > 0$ (q-sidepayments) it also does not hold that the ε-core treats all players of the same type approximately equally. The following example, quite similar to the previous one, demonstrates this point.

Example 5b: Fix any $\varepsilon > 0$, $q > 0$. Consider a game (N, V) with q-comprehensive payoff sets, one exact player type and a nonempty symmetric core. (As above, there are many such games.) Take any payoff in the symmetric core of (N, V) and define $\overline{x}$ as the payoff assigned to each player in N. Now select one player, say a. Consider the payoff y^q that assigns $\overline{x}-\varepsilon$ for all players except player a and $\overline{x}+q(|N|-1)\varepsilon$ for a player a. Then the payoff y^q is feasible and ε-undominated. Thus y^q belongs to the ε-core of (N, V). For very large $|N|$, however, y^q treats one player much better than the others.

The above examples show that Theorem 2 depends crucially on the fact that side payments are limited. The other condition of Theorem 2, q-comprehensiveness ($q > 0$) for payoff sets of small coalitions, is also crucial for symmetry of the ε-core. Example 6 below, which continues Example 1, shows that without this restriction the ε-core may be non-symmetric even for games with limited side payments.

Example 6: Recall the game (N, V_0) defined in Example 1. Observe that the game (N, V_0) has one exact player type, strictly effective 2-bounded groups, and a per capita bound of 1. Thus, $(N, V_0) \in G((0,1),(0,2))$. But, for any $q > 0$ and any coalition S of two players, the payoff sets $V(S)$ are not q-comprehensive. Let m be a positive integer. Let (N^m, V_0^m) be a game where the number of players in the set N^m is $2m+1$ and for any coalition $S \subset N^m$ $V_0^m(S) := V_0(S)$. Thus, each game (N^m, V_0^m) has an odd number of players.

It is easy to see that the payoff y giving 1 to each of $2m$ players and 0 to the last player (say player a) is in the core. (The "left-out" player, in a coalition by himself, cannot make both himself and a player in a two-person coalition better off – the player in the two-person coalition cannot be given more than 1.) However for any player other than a, say for player b, we have $y_b = 1$. Thus $|y_a - y_b| = 1$; even though all players are exact substitutes and $\delta = \varepsilon = 0$ but the core does not became symmetric, even for an arbitrary large m.

5 Remarks.

1. *Relationships to other papers using parameterized collections of games.* Our prior results for the ε-core depended on a notion of q-comprehensiveness. In fact, all our previous results require that payoff sets be q-comprehensive with $q > 0$; the case $q = 0$ has not been treated. The main goal of the current paper is to treat this case.

   First, let us consider the relationship between Theorem 1 of this paper and Kovalenkov and Wooders (1997a, Theorem 3), another nonemptiness result. In the prior paper, ε-effective B-bounded groups dictates that payoffs sets $V(S)$ can be approximated by the q-comprehensive covers of payoff sets of B-bounded coalitions, $c_q(V(S;B))$, thus permitting games with side payments. Kovalenkov and Wooders (1997a, Theorem 3) then shows that, if $q > 0$, all sufficiently large games have nonempty epsilon cores. In the current paper, when we limit side payments by our definition of ε-effective B-bounded groups, we require the assumptions of thickness of the player set and convexity of payoff sets of the total player set.

   In Kovalenkov and Wooders (1997c), we define ε-effective B-bounded groups as in Kovalenkov and Wooders (1997a) but instead of defining the Hausdorff distance with respect to the sup norm, we use the sum norm. This significantly broadens the class of games considered. With the assumption that payoff sets are convex we are able to derive an explicit lower bound on the required size of ε^* as a function of the parameters describing a game, so that the game has a nonempty ε^*-core. That result, like the nonemptiness of ε-cores result of Kovalenkov and Wooders (1997a) requires q-comprehensiveness with $q > 0$.

2. *Parameterized collections of games compared to pregames.* As noted earlier, a pregame consists of a compact metric space of player types and a worth function, assigning a payoff set to each finite list of player types (repetitions allowed). See, for example, Wooders (1980,1983) for pregames with a finite set of player types and Kaneko and Wooders (1984,1996) and Wooders and Zame (1984,1989) for compact metric spaces of types more generally. Theorem 1 of the current paper extends results of Wooders (1983) and Wooders and Zame (1989) on nonemptiness of approximate cores to arbitrary games. For NTU games Wooders and Zame (1989) provides two forms of nonemptiness of approximate cores, one for nonemptiness of the weak ε-remainder core[8] and another for the ε-core. Kovalenkov and Wooders (1997a, Theorem 2) provides an extension of the Wooders-Zame result on nonemptiness of the weak ε-remainder core from games with a compact metric space of player attributes to arbitrary games. Theorem 1 of the current paper and Kovalenkov and Wooders (1997c, Theorem) extend the Wooders-Zame result on nonemptiness of ε-cores to arbitrary games. Theorem 2 of the current paper extends the equal treatment results on cores and ε-cores of Wooders (1979,1983,1994a) to arbitrary games with limited side payments.

   Games induced by pregames have the property that the payoff set of a group of players is independent of the total player set of the game; this follows from the fact that all induced games are derived from the one worth function for the pregame. A consequence is that, within this framework, when there are sufficiently many players of each type that appears in the games, small group effectiveness is equivalent to the condition that per capita payoffs are bounded (Wooders 1994b). In contrast, in the framework of parameterized collections of games, as shown in Kovalenkov and Wooders (1997a), these two assumptions are no longer equivalent. This difference illustrates that the pregame framework itself has hidden consequences. An advantage of the framework of parameterized collections of games is that it is apparent what features of the games themselves ensure the results.

3. *Small group effectiveness for improvement.* There are several possibilities for defining ε-effective B-bounded groups. Our definition requires that partitions of the total player set into B-bounded groups can realize almost all payoffs in

[8]See Kovalenkov and Wooders (1997a) for a definition.

$V(N)$. This approach has the advantage of ease of exposition. Another possibility is to require that all improvement can be carried out by groups bounded in size.[9]

ε-improvement-effective B-bounded groups. Let us say that a payoff x can be *ε-improved upon* by some coalition S in a game (N, V) if there is a payoff $y \in V(S)$ such that $y_S >> x_S - \varepsilon\vec{1}_S$. Suppose that a game (N, V) satisfies the property that for each payoff $x \in V(N)$, if x can be $\frac{\varepsilon}{2}$*-improved* by some coalition then there is a coalition S satisfying $|S| \leq B$ that can *ε-improve* upon x. Then (N, V) has *ε-improvement-effective B-bounded groups.*

The condition that a game has bounded group sizes that are effective for improvement is obviously less restrictive than the condition defined earlier of ε-effective B-bounded groups. The conditions are, however, interchangeable.[10] The only difference in our results, were we to use ε-effectiveness for improvement, would be that the bounds change. Our motivation in choosing the condition of ε-effective B-bounded groups is that the condition is more easily verifiable since it only depends on what groups of individuals can do rather than on the entire game structure and it seems more consistent with the sort of examples most frequently used in the literature. For example: marriages involve two persons; softball teams have nine members; and when average costs of production are downward sloping and bounded away from zero, there is some size of plant that is "almost efficient." All these examples only involve sizes of the groups themselves whereas exhaustion of opportunities for improvement by groups bounded in size depends in an intricate way on the entire payoff structure of the game. We refer the reader to Wooders (1994a, Proposition 3.8) for the relationship between these two notions of effectiveness in the context of games derived from pregames.

4. *Effectiveness of relatively small groups.* We've chosen to bound the absolute sizes of near-effective coalitions since familiar examples of games and economies have natural bounds on absolute sizes of effective coalitions, for example, marriage models. Relative sizes of effective coalitions are important, however, in some sorts of models, most notably voting games. Moreover, bounding returns to coalition formation in terms of relatively small group sizes is useful if questions of the limiting properties of games/economies are addressed, cf.,

[9] A similar definition appears in Wooders and Zame (1987).

[10] In other contexts, examples of the interchangeability of the conditions of small group effectiveness for improvement and small group effectiveness for feasibility is given by Mas-Colell (1979) and Kaneko and Wooders (1989). Mas-Colell uses the fact that almost all improvement in exchange economies can be carried out by groups bounded in size while Kaneko and Wooders use the fact that almost all feasible outcomes can be achieved by partitions of the total player set into groups bounded in size. (As "almost" is required to be closer to "exact," then the bounds must increase in size.)

Mas-Colell (1979) and Kaneko and Wooders (1989). Since we provide a corollary treating sequences of games where effective coalitions become relatively small in Kovalenkov and Wooders (1997c), we omit such an extension of the non-emptiness result of this paper.

6 Economies with limited side payments.

In application, the conditions of our results may not be unduly restrictive. Indeed, in some game theoretic and economic situations, limited side payments would appear to be natural, for example, in job matching models (Kelso and Crawford (1982), Roth and Sotomayor (1990) and others). Even if side payments were possible, in large games it may require cooperation and agreement between a large number of players to make large transfers. In this case the framework of side payments, or even q-comprehensiveness for $q > 0$, may be quite problematic. On the other hand, the presumption of economics that there exists opportunities for gains to trade and exchange is at the heart of economics. Thus, the requirement of some degree of side paymentness within small coalitions may not be restrictive.

There are also specific examples to which our results apply. Consider a two-sided matching model where all gains to collective activities can be realized by groups consisting of one person from each side of the market and where utility is transferable within buyer-seller pairs. If there is some mechanism convexifying the total payoff set (for example, lotteries), then both our nonemptiness result and our symmetry result apply. These remarks also apply to partitioning games.[11]

Another class of economic models whose derived games may well fit into our assumptions include economies with local public goods or clubs. Kovalenkov and Wooders (1997b) introduce a model of an economy with clubs (and multiple memberships) and, under remarkably nonrestrictive conditions, demonstrate nonemptiness of approximate cores. With appropriate conditions on the economic models, the results of the current paper could be applied to models of economies with clubs.

7 Proofs.

Proof of Theorem 1: Recall that we required $2(\varepsilon+\delta) < m^*$. Assume for now that $\varepsilon^* \leq \min\left\{\frac{m^*}{2}, BC\right\}$, implying that $(\varepsilon^*+\varepsilon+\delta) < m^*$. We first consider the case where the game (N, V) has strongly comprehensive payoff sets and then use the result for this case to obtain the result for the general case.

[11]See Garratt and Qin (1996) for examples of games with lotteries, Kaneko (1982) for NTU assignment markets and Kaneko and Wooders (1982) and le Breton, Owen and Weber (1992) for partitioning games.

Case 1. Suppose the game (N, V) has strongly comprehensive payoff sets. Define for each $S \subset N$

$$V'(S) := \bigcap \sigma_\tau^{-1}(V(\tau(S))),$$

where intersection is taken over all type-consistent permutations τ. Then $(N, V') \in G((0, T), (\varepsilon, B))$ and $V'(N)$ is convex. Moreover, $V'(S) \subset V(S)$ and

$$H_\infty(V'(S), V(S)) \leq \delta$$

for any $S \subset N$. Now let us define for any $S \subset N$

$$V^0(S) := V'(S; B).$$

Then $(N, V^0) \in G((0, T), (0, B))$. Moreover,

$$V^0(S) \subset V'(S) \subset V(S)$$

and

$$H_\infty(V^0(S), V(S)) \leq H_\infty(V^0(S), V'(S)) + H_\infty(V'(S), V(S)) \leq \varepsilon + \delta.$$

In addition, the game (N, V^0) has a per capita payoff bound of C and strongly comprehensive payoff sets.

Applying Kovalenkov and Wooders (1997a, Theorem 1) for the ε^0-remainder core, with

$$\varepsilon^0 := \frac{\rho}{BC}\varepsilon^*,$$

to the game (N, V^0) it follows that for $|N| \geq \eta_1(\varepsilon^0, T, B)$ there is some coalition S such that the subgame (S, V^0) has a non-empty core and $\frac{|S|}{|N|} \geq 1 - \varepsilon^0$. Let x be a payoff in the core of the subgame (S, V^0). Note that $\rho > \varepsilon^0$.

We now prove that for $|N| \geq \frac{(B+1)}{\rho - \varepsilon^0}$ the payoff x will have the equal treatment property, that is, for every t and any $i, j \in N[t]$, $x_i = x_j$. Suppose there is a type t and two players $i_1, i_2 \in N[t] \cap S$ satisfying $x_{i_1} > x_{i_2}$. By feasibility of x, there is some partition $\left\{S^k\right\}$ of S, $\left|S^k\right| \leq B$ for each k, such that $x \in \sum_k V^0(S^k)$. Then, since $|N| \geq \frac{(B+1)}{\rho - \varepsilon^0}$ we have that $|N[t] \cap S| \geq \rho |N| - \varepsilon^0 |N| > B$ and there must exist $j_1, j_2 \in N[t] \cap S$ such that $x_{j_1} \in S^{k_1}, x_{j_2} \in S^{k_2}$ for $S^{k_1}, S^{k_2} \subset \left\{S^k\right\}$, $S^{k_1} \neq S^{k_2}$ and $x_{j_1} > x_{j_2}$. (If x_{i_1}, x_{i_2} are in the same element of $\left\{S^k\right\}$, there exists $j \in N[t] \bigcap S$ in the different element of $\left\{S^k\right\}$ than i_1, i_2 and it can't be $x_j = x_{i_1}$ and $x_j = x_{i_2}$ since $x_{i_1} \neq x_{i_2}$. Thus we can suppose without loss of generality that j_1 and j_2 are in different elements in $\left\{S^k\right\}$.) But then the agents of $S^{k_1} \backslash \{j_1\}$ can form the coalition $S^{k_1} \backslash \{j_2\}$with player j_2 rather then j_1. By strong comprehensiveness of payoff sets and since players j_1, j_2 are exact substitutes in the game (S, V^0), all players in the new coalition can be strictly better off than in x. This contradicts the supposition that x is a core payoff for the game (S, V^0). Thus x has the equal treatment property.

Now let $\eta'(\varepsilon^*, T, B, C, \rho) := \max\left\{\eta_1(\varepsilon^0, T, B), \frac{(B+1)}{\rho-\varepsilon^0}\right\}$, where, as above, $\varepsilon^0 = \frac{\rho}{BC}\varepsilon^*$. As we shown, for $|N| \geq \eta'(\varepsilon^*, T, B, C, \rho)$ there is a coalition $S \subset N$ satisfying $\frac{|S|}{|N|} \geq 1 - \varepsilon^0$ and, for each t, $|S \bigcap N[t]| > B$ and there is a payoff $x \in \mathbf{R}^S$ in the equal treatment core of (S, V^0).

Next, define a payoff $z \in \mathbf{R}^N$ that, for each t, assigns to each $a \in N[t]$ the same payoff assigned to players of type t by x. This payoff z is undominated in the game (N, V^0). (If payoff z were dominated by some coalition S_1 in (N, V^0) then z could be dominated by some coalition $S_2 \subset S_1$, $|S_2| \leq B$. But there exists a coalition $S_3 \subset S$ with the same profile as S_2. This coalition S_3 dominates the payoff x in (S, V^0) contradicting to the fact that x is in the core of (S, V^0).) Moreover, since $x \in \sum_k V^0(S^k)$ for some partition $\left\{S^k\right\}$ of S with $\left|S^k\right| \leq B$, by per capita boundedness $x_a \leq BC$ for any $a \in S$. Thus $z_a \leq BC$ for any $a \in S$.

Now construct a payoff $x' \in \mathbf{R}^N$ from the payoff for $x \in \mathbf{R}^S$ by

$$x'_S := x_S \text{ and } x'_a := 0 \text{ for all } a \notin S.$$

Let $y \in \mathbf{R}^N$ be the average of all payoffs $\sigma_\tau(x')$ across all type-consistent permutations τ of N. Then y has an equal treatment property. Moreover, for every t and any $a \in N[t]$

$$\begin{aligned} z_a - y_a &= z_a - \frac{|N[t] \bigcap S|}{|N[t]|} z_a \leq \frac{|N[t]| - |N[t] \bigcap S|}{|N[t]|} z_a \\ &\leq \frac{|N| - |S|}{|N[t]|} z_a \leq \frac{\varepsilon^0}{\rho} z_a \leq \frac{\varepsilon^0}{\rho} BC = \varepsilon^*. \end{aligned}$$

Since z is undominated in (N, V^0), y is ε^*-undominated in (N, V^0). Therefore for $|N| \geq \eta'(\varepsilon^*, T, B, C, \rho)$ the payoff y has equal treatment property and is $(\varepsilon^* + \varepsilon + \delta)$-undominated in the original game (N, V). In addition, observe that $x' \in V^0(N) \subset V'(N)$. Therefore $\sigma_\tau(x') \in V'(N)$ for any type-consistent permutation τ of N. By convexity of $V'(N)$ and by construction of y, it holds that $y \in V'(N) \subset V(N)$. Thus, for $|N| \geq \eta'(\varepsilon^*, T, B, C, \rho)$ the payoff y belongs to the equal treatment $(\varepsilon^* + \varepsilon + \delta)$-core of (N, V).

Case 2. Now we consider the general case, where the game (N, V) does not necessarily have strongly comprehensive payoff sets. We first approximate the game (N, V) by another game (N, V') with strongly comprehensive payoff sets. This approximation can be done sufficiently closely, so that $V(S) \subset V'(S)$ and $H_\infty(V(S), V'(S)) \leq \frac{\varepsilon^*}{2}$ for any $S \subset N$. Note that the approximation is easy since, by our definition of the Hausdorff distance, we must ensure close approximation only on a compact set. (For details of such an approximation see Wooders (1983), Appendix.) Let $\eta(\varepsilon^*, T, B, C, \rho) := \eta'(\frac{\varepsilon^*}{2}, T, B, C, \rho)$. By Case 1 above, for $|N| \geq \eta(\varepsilon^*, T, B, C, \rho)$ the equal treatment $(\frac{\varepsilon^*}{2}+\varepsilon+\delta)$-core of the game (N, V') is nonempty. But then for $|N| \geq \eta(\varepsilon^*, T, B, C, \rho)$ the game (N, V) has a non-empty equal treatment $(\varepsilon^*+\varepsilon+\delta)$-core. This is the conclusion we need for $\varepsilon^* \leq \min\left\{\frac{m^*}{2}, BC\right\}$.

Finally for $\varepsilon^* > \min\left\{\frac{m^*}{2}, BC\right\}$ consider $\varepsilon' := \min\left\{\frac{m^*}{2}, BC\right\}$ and let $\eta(\varepsilon^*, T, B, C, \rho) := \eta(\varepsilon', T, B, C, \rho)$. Since $\varepsilon^* \geq \varepsilon'$, for $|N| \geq \eta(\varepsilon^*, T, B, C, \rho)$ again the equal treatment $(\varepsilon^*+\varepsilon+\delta)$-core of the game (N, V) is nonempty. ■

Proof of Theorem 2: Consider some payoff vector x in the ε^*-core of (N, V). Define a payoff vector y by $y(\{i\}) := x(\{i\}) - \varepsilon$ for each $i \in N$. Since (N, V) has ε-effective B-bounded groups, there exists a partition $\left\{S^k\right\}$ of N, with $\left|S^k\right| \leq B$ for each k, such that $y \in \sum_k V(S^k)$. Moreover, for the game (N, V), by construction the payoff y cannot be improved upon by any coalition $S \subset N$ for each of it's member by more than $(\varepsilon^*+\varepsilon)$.

Case 1. Consider the case, where according to the partition $\left\{S^k\right\}$, two players t_1 and t_2 of the same approximate type (that is, $t_1, t_2 \in N[t]$) are in different coalitions. Suppose $t_1 \in S^1$ and $t_2 \in S^2$. Suppose also that $y_{t_1} > y_{t_2}$. Then agents in $S^1 \backslash \{t_1\}$ would prefer to form a new coalition with t_2 rather then t_1. Let $S = S^1 \backslash \{t_1\}$. Let τ denote a permutation of N that permutes only t_1 and t_2. Since players t_1 and t_2 are of the same approximate type, τ is a type-consistent permutation and

$$H_\infty\left[V(S^1), \sigma_\tau(V(S \cup \{t_2\})\right] \leq \delta.$$

Hence there exists a payoff z that is feasible for $S \bigcup \{t_2\}$ and close to the payoff y with the loss for any agent in $S \bigcup \{t_2\}$ of not more then δ. Since transfers at rate q are possible, this new coalition can improve upon the payoff vector y by more than $(\varepsilon^*+\varepsilon)$ for each player if

$$q(|y_{t_1} - y_{t_2}| - (\varepsilon^*+\varepsilon) - \delta) > |S|\,\delta + |S|\,(\varepsilon^*+\varepsilon).$$

Since y is an $(\varepsilon^*+\varepsilon)$-core payoff we must have: $|y_{t_1} - y_{t_2}| - (\varepsilon^*+\delta+\varepsilon) \leq \frac{1}{q}\,|S|\,(\varepsilon^*+\delta+\varepsilon) \leq \frac{1}{q}(B-1)(\varepsilon^*+\delta+\varepsilon)$. Hence, $|x_{t_1} - x_{t_2}| = |y_{t_1} - y_{t_2}| \leq \frac{B}{q}(\varepsilon^*+\delta+\varepsilon)$.

Case 2. Now suppose that t_1 and t_2 are in the same element of the partition $\{S^k\}$. Then, since $|N[t]| > B$, there exists $t_3 \in N[t]$ that belongs to a different member of the partition $\{S^k\}$. Hence by Case 1 above: $|x_{t_1} - x_{t_2}| \leq |x_{t_1} - x_{t_3}| + |x_{t_3} - x_{t_2}| \leq \frac{2B}{q}(\varepsilon^*+\delta+\varepsilon)$. ■

8 Appendix: Additional examples.

In this Section we present two examples that were discussed in Section 3. Both illustrates the importance of the conditions in Theorem 1. Example A1 appeared before in Kovalenkov and Wooders (1994a, Example 4).

Example A1: *The indispensability of the small group effectiveness assumption.* Consider a sequence of games $(N^m, v^m)_{m=1}^{\infty}$ with side payments and where the m^{th} game has $3m$ players. Suppose that any coalition S consisting of at least $2m$ players can get up to $2m$ units of payoff to divide among its members, that is, $v^m(S) = 2m$. Assume that if $|S| < 2m$, then $v^m(S) = 0$. Observe that each game has one exact player type and a per capita bound of 1. That is, $\rho = 1, T = 1, C = 1$, and $\delta = 0$. However, the $\frac{1}{7}$-core of the game is empty for arbitrarily large values of m.

For any feasible payoff there are m players that are assigned in total no more than $\frac{2m}{3m}m = \frac{2}{3}m$. There are another m players that are assigned in total no more than $\frac{2m}{2m}m = m$. These $2m$ players can form a coalition and receive $2m$ in total. This coalition can improve upon the given payoff for each of its members by $\frac{1}{6}$, since $(2m - \frac{5}{3}m)\frac{1}{2m} = \frac{1}{6}$.

Example A2: *The independence of the per capita boundedness assumption.* Let m be an arbitrary positive integer. Let (N, V^m) be a superadditive game where, for any coalition S, with $|S| > 1$,

$$V^m(S) := \{x \in \mathbf{R}^N : \ x_i \leq m \text{ for } i \in S, \text{ and } x_k = 0 \text{ for } k \notin S\}$$

and for each $i \in N$,

$$V^m(\{i\}) := \{x \in \mathbf{R}^N : x_i \leq 0 \text{ and } x_j = 0 \text{ for all } j \neq i\}.$$

Now let us define a collection of games

$$G^* := \{(N, V^m) : m \in \mathbf{Z}_+\}.$$

Obviously any game $(N, V) \in G^*$ has convex payoff sets and non-empty core. Moreover, any game $(N, V) \in G^*$ has one (exact) player type and thus is 1-thick. We leave it to the reader to verify that any game $(N, V) \in G^*$ has strictly effective 3-bounded groups. Thus, $G^* \subset G((0,1),(0,3))$ and all games in G^* are 1-thick, have convex payoff sets and non-empty cores.

However the class G^* is not restricted by any common per capita boundedness condition. For any positive integer C and any $m > C$ the game $(N, V^m) \in G^*$ does not satisfy per capita bound of C. Hence the condition of the common per capita bound on the collection of games is independent of all other conditions (and the implication) in Theorem 1.

References

[1] Anderson, R.M. (1978) "An elementary core equivalence theorem," *Econometrica* 46, 1438-1487.

[2] Billera, L.J. (1974) "On games without side payments arising from general class of games," *Journal of Mathematical Economics* 1, 129-139.

[3] le Breton, M., G. Owen and S. Weber (1992) "Strongly balanced cooperative games," *International Journal of Game Theory* 20, 419-428.

[4] Garratt, R. and C-Z Qin (1996) "On a market for coalitions with indivisible agents and lotteries," UCSB Working paper #14-95 revised.

[5] Conley, J. and M.H. Wooders (1997) "Anonymous pricing in Tiebout economies and economies with clubs," *Topics in Public Finance* (to appear).

[6] Hildenbrand, W. and A.P. Kirman (1988) *Equilibrium Analysis*, Advanced Textbooks in Economics, Amsterdam- New York- Oxford- Tokyo: North-Holland.

[7] Kaneko, M. (1982) "The central assignment game and the assignment markets," *Journal of Mathematical Economics* 10, 205-232.

[8] Kaneko, M. and M.H. Wooders (1982) "Cores of partitioning games," *Mathematical Social Sciences* 3, 313-327.

[9] Kaneko, M. and M.H. Wooders (1984) "The core of a game with a continuum of players and finite coalitions; Nonemptiness with bounded sizes of coalitions," IMA Preprint Series No. 126, Institute of Mathematics and Its Applications, University of Minnesota.

[10] Kaneko, M. and M.H. Wooders (1989) "The core of a continuum economy with widespread externalities and finite coalitions: From finite to continuum economics," *Journal of Economic Theory* 49, 135-168.

[11] Kaneko, M. and M.H. Wooders (1996) "The non-emptiness of the f-core of a game without side payments," *International Journal of Game Theory* 25, 245-258.

[12] Kaneko, M. and M.H. Wooders (1997) "Utility theory in cooperative games," *Handbook of Utility Theory*, P. Hammond and C. Seidel eds., Klauwer Academic Press (to appear).

[13] Kannai, Y. (1970) "Continuity properties of the core of a market,"*Econometrica* 38, 791-815.

[14] Kelso, A.S. and V.P. Crawford (1982) "Job matching, coalition formation, and gross substitutes," *Econometrica* 50, 1483-1504.

[15] Kovalenkov, A. and M.H. Wooders (1997a) "Three theorems on non-emptiness of approximate cores: Part 1. Game theoretic results," typescript.

[16] Kovalenkov, A. and M.H. Wooders (1997b) "Three theorems on non-emptiness of approximate cores: Part 2. Economies with clubs," typescript.

[17] Kovalenkov, A. and M.H. Wooders (1997c) "An explicit bound on ε for non-emptiness of the ε-core of an arbitrary game without side payments," typescript.

[18] Manelli, A.M. (1991a) "Core convergence without monotone preferences and free disposal", *Journal of Economic Theory* 55, 400-415.

[19] Manelli, A.M. (1991b) "Monotonic preferences and core equivalence," *Econometrica* 59, 123-138.

[20] Mas-Colell, A. (1975) "A further result on the representation of games by markets," *Journal of Economic Theory* 10, 117-122.

[21] Mas-Colell, A. (1979) "A refinement of the Core Equivalence Theorem," *Economic Letters*, 307-310.

[22] Shapley, L.S. and Shubik, M. (1966) "Quasi-cores in a monetary economy with nonconvex preferences," *Econometrica* 34, 805-827.

[23] Shapley, L.S. and Shubik, M. (1969) "On market games," *Journal of Economic Theory* 1, 9-25.

[24] Shubik, M. and M.H. Wooders (1983) "Approximate cores of replica games and economies: Part I. Replica games, externalities, and approximate cores," *Mathematical Social Sciences* 6, 27-48.

[25] Shubik, M. and Wooders, M.H. (1986) 'Near-markets and market-games', *Economic Studies Quarterly* 37, 289-299.

[26] Tiebout, C. (1956) "A pure theory of local expenditures," *Journal of Political Economy* 64, 416-424.

[27] Roth, A. and Sotomayor, M. (1990) *Two-sided Matching; A Study in Game-theoretic Modeling and Analysis*, Cambridge University Press, Cambridge.

[28] Scotchmer, S. and Wooders, M.H. (1988) "Monotonicity in games that exhaust gains to scale," IMSSS Technical Report No. 525, Stanford University.

[29] Wooders, M.H. (1979) "A characterization of approximate equilibria and cores in a class of coalition economies," Stony Brook Department of Economics Working Paper No. 184, revised. (On line at the home page of Myrna Wooders.)

[30] Wooders, M.H. (1980) "Asymptotic cores and asymptotic balancedness of large replica games," Stony Brook Department of Economics Working Paper No. 215, revised.

[31] Wooders, M.H. (1983) "The epsilon core of a large replica game," *Journal of Mathematical Economics* 11, 277-300.

[32] Wooders, M. H. (1994a) "Large games and economies with effective small groups," in *Game Theoretic Aproaches to General Equilibrium Theory*, J.-F. Mertens and S. Sorin eds., Dordrecht- Boston- London: Kluwer Academic Publishers.

[33] Wooders, M. H. (1994b) "Equivalence of games and markets," *Econometrica* 62, 1141-1160.

[34] Wooders, M.H. (1997) "Equivalence of Lindahl equilibrium with participation prices and the core," *Economic Thery* 9, 115-127.

[35] Wooders, M. H. and W.R. Zame (1984) "Approximate cores of large games," *Econometrica* 52, 1327-1350.

[36] Wooders, M.H. and W.R. Zame (1987) "Large games; fair and stable outcomes," *Journal of Economic Theory* 42, 59-93.

[37] Wooders, M. H. and W.R. Zame (1989) "Approximate cores of large games without side payments," typescript.