An explicit bound on ϵ for nonemptiness of ϵ-cores of games.[*]

Alexander Kovalenkov[†]
Department of Economics, Gardner Hall, University of North Carolina,
Chapel Hill, NC 27599-3305, U.S.A.

Myrna Holtz Wooders[‡]
Department of Economics, University of Toronto,
150 St.George St., Toronto, M5S 3G7, Canada
and
Department of Economics, University of Warwick,
Coventry, CV4 7Al, United Kingdom

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Abstract

We consider parameterized collections of games without side payments and determine a bound on ϵ so that all sufficiently large games in the collection

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have non-empty "-cores. Our result makes explicit the relationship between the required size of " for non-emptiness of the "-core, the parameters describing the collection of games, and the size of the total player set. Given the parameters describing the collection, the larger the game, the smaller the " that can be chosen.

1 Introduction.

We consider parameterized collections of games without side payments and obtain an explicit bound on " as a function of the parameters so that all su±ciently large games in the collection have non-empty "-cores. A parameterized collection of games is described by (a) a number of approximate player types and the accuracy of this approximation; (b) an upper bound on the size of near-e®ective groups and the closeness of these groups to being e®ective for the realization of all gains to collective activities; (c) a bound on the supremum of per-capita payo®s achievable in coalitions; and (d) a measure of the extent to which boundaries of payo® sets are bounded away from being \°at." Given these parameters and an arbitrary positive real number "; we obtain a lower bound on the number of players so that all games in the collection containing more players than the bound have non-empty "-cores. Since the bound on the number of players in the game is expressed in terms of the parameters describing the games, this bound induces the desired bound on ".

Two results, using di®erent notions of distance to describe nearly e®ective small groups, are established. Our Theorem uses the same notion of distance as in our prior work (Kovalenkov and Wooders (1997a,b)). Corollary 2 uses a second, much less demanding notion of distance, but nevertheless we obtain result analogous to those of our Theorem. Due to the di®erent notion of distance, the bound on " may be signi¯cantly improved by using Corollary 2 rather than the Theorem. An example is provided illustrating such improvement. The key for both results is Corollary 1, treating the central case of games with side payments.

The next section of this paper develops our model. Related literature and possible applications are discussed in the concluding section of the paper. We note here, however, that we use techniques related to those of Scarf (1965) (an earlier unpublished version of his well-known paper, Scarf (1967)), showing that balanced games have non-empty cores. (See also Billera (1970)). Also, our work is related in spirit to the \least "-core," introduced by Maschler, Peleg, and Shapley (1979), since we obtain a lower bound on " ensuring that the "-core is non-empty.

2 Definitions.

2.1 Cooperative games: description and notation.

Let $N = \{1, \ldots, n\}$ denote a set of players. A non-empty subset of N is called a coalition. For any coalition S let R^S denote the $|S|$-dimensional Euclidean space with coordinates indexed by elements of S. For $x \in R^N$; x_S will denote its restriction to R^S. To order vectors in R^S we use the symbols $\gg$, $>$ and $\geq$ with their usual interpretations. The non-negative orthant of R^S is denoted by R^S_+ and the strictly positive orthant by R^S_{++}. We denote by 1_S the vector of ones in R^S, that is, $1_S = (1, \ldots, 1) \in R^S$. Each coalition S has a feasible set of payoffs or utilities denoted by $V_S \subset R^S$. By agreement, $V_\emptyset = \{0\}$ and $V_{\{i\}}$ is non-empty, closed and bounded from above for any i. In addition, we will assume that

$$\max \left\{ x : x \in V_{\{i\}} \right\} = 0 \text{ for any } i \in N;$$

this is by no means restrictive since it can always be achieved by a normalization.

It is convenient to describe the feasible utilities of a coalition as a subset of R^N. For each coalition S let $V(S)$, called the payoff set for S, be defined by

$$V(S) := \left\{ x \in R^N : x_S \in V_S \text{ and } x_a = 0 \text{ for } a \notin S \right\}.$$

A game without side payments (called also an NTU game or simply a game) is a pair $(N; V)$ where the correspondence $V : 2^N \to R^N$ is such that $V(S) \subset \left\{ x \in R^N : x_a = 0 \text{ for } a \notin S \right\}$ for any $S \subset N$ and satisfies the following properties :

(2.1) $V(S)$ is non-empty and closed for all $S \subset N$.

(2.2) $V(S) \cap R^N_+$ is bounded for all $S \subset N$, in the sense that there is a real number $K > 0$ such that if $x \in V(S) \cap R^N_+$; then $x_i \leq K$ for all $i \in S$.

(2.3) $V(S_1 \bigcup S_2) \supset V(S_1) + V(S_2)$ for any disjoint $S_1, S_2 \subset N$ (superadditivity).

We next introduce the uniform version of strong comprehensiveness assumed for our results. Roughly, this notion dictates that payoff sets are both comprehensive and uniformly bounded away from having level segments in their boundaries. Consider a set $W \subset R^S$. We say that W is comprehensive if $x \in W$ and $y \leq x$ implies $y \in W$. The set W is strongly comprehensive if it is comprehensive, and whenever $x \in W$; $y \in W$; and $x < y$ there exists $z \in W$ such that $x \ll z$.[1] Given (i) $x \in R^S$,

[1] Informally, if one person can be made better off (while all the others remain at least as well off), then all persons can be made better off. This property has also been called "nonleveledness."

(ii) $i, j \in S$, (iii) $0 \le q \le 1$ and (iv) $\varepsilon \ge 0$, define a vector $x^q_{i,j}(\varepsilon) \in \mathbb{R}^S$, where

$$\begin{aligned} (x^q_{i,j}(\varepsilon))_i &= x_i - \varepsilon; \\ (x^q_{i,j}(\varepsilon))_j &= x_j + q\varepsilon; \text{ and} \\ (x^q_{i,j}(\varepsilon))_k &= x_k \text{ for } k \in S \setminus \{i, j\}. \end{aligned}$$

The set W is q-comprehensive if W is comprehensive and if, for any $x \in W$, it holds that $(x^q_{i,j}(\varepsilon)) \in W$ for any $i, j \in S$ and any $\varepsilon \ge 0$.[2] This condition for $q > 0$ uniformly bounds the slopes of the Pareto frontier of payoff sets away from zero. Note that for $q = 0$, 0-comprehensiveness is simply comprehensiveness. Also note that if a game is q-comprehensive for some $q > 0$ then the game is q'-comprehensive for all q' with $0 \le q' \le q$.

Let $V_S \subset \mathbb{R}^S$ be a payoff set for $S \subset N$. Given q, $0 \le q \le 1$, let $W^q_S \subset \mathbb{R}^S$ be the smallest q-comprehensive set that includes the set V_S. For $V(S) \subset \mathbb{R}^N$ let us define the set $c_q(V(S))$ in the following way:

$$c_q(V(S)) := \left\{ x \in \mathbb{R}^N : x_S \in W^q_S \text{ and } x_a = 0 \text{ for } a \notin S \right\}.$$

Notice that for the relevant components – those assigned to the members of S – the set $c_q(V(S))$ is q-comprehensive, but not for other components. With some abuse of the terminology, we will call this set the q-comprehensive cover of $V(S)$. When $q > 0$ we can think of a game as having some degree of "side-paymentness" or as allowing transfers between players, but not necessarily at a one-to-one rate. This is an eminently reasonable assumption for games derived from economic models.

2.2 Parameterized collections of games.

To introduce the notion of parameterized collections of games we will need the concept of Hausdorff distance. For every two non-empty subsets E and F of a metric space (M, d), define the Hausdorff distance between E and F (with respect to the metric d on M), denoted by $\mathrm{dist}(E, F)$, as

$$\mathrm{dist}(E, F) := \inf\{\varepsilon \in (0, \infty) : E \subset B_\varepsilon(F) \text{ and } F \subset B_\varepsilon(E)\},$$

where $B_\varepsilon(E) := \{x \in M : d(x, E) \le \varepsilon\}$ denotes an ε-neighborhood of E.

Since payoff sets are unbounded below, we will use a modification of the concept of the Hausdorff distance so that the distance between two payoff sets is the distance between the intersection of the sets and a subset of Euclidean space. Let m^* be a fixed positive real number. Let M^* be a subset of Euclidean space $\mathbb{R}^N$ defined by

[2] The notion of q-comprehensiveness can be found in Kaneko and Wooders (1996). For the purposes of the current paper, q-comprehensiveness can be relaxed outside the individually rational payoff sets.

$M^* := \left\{ x \in \mathbb{R}^N : x_a \geq -m^* \text{ for any } a \in N \right\}$. For every two non-empty subsets E and F of Euclidean space $\mathbb{R}^N$ let $H_\infty[E; F]$ denote the Hausdorff distance between $E \cap M^*$ and $F \cap M^*$ with respect to the metric $\|x - y\|_\infty := \max_i |x_i - y_i|$ on Euclidean space $\mathbb{R}^N$.

The concepts defined below lead to the definition of parameterized collections of games. To motivate the concepts, each is related to analogous concepts in the pregame framework. Recall that a pregame[3] is a specification of a set of player types (a finite set or, more generally, a compact metric space of player types) and a worth function ascribing a payoff to any group of players, where the group is described by the number of players of each type in the group.

<u>$\delta-$substitute partitions:</u> In our approach we approximate games with many players, all of whom may be distinct, by games with finite sets of player types. Observe that for a compact metric space of player types, given any real number $\delta > 0$ there is a partition (not necessarily unique) of the space of player types into a finite number of subsets, each containing players who are "δ-similar" to each other. Parameterized collections of games do not restrict to a compact metric space of player types, but do employ the idea of a finite number of approximate types.

Let $(N; V)$ be a game and let $\delta \geq 0$ be a non-negative real number. A δ-substitute partition is a partition of the player set N into subsets with the property that any two players in the same subset are "within δ" of being substitutes for each other. Formally, given a set $W \subset \mathbb{R}^N$ and a permutation π of N, let $\sigma_\pi(W)$ denote the set formed from W by permuting the values of the coordinates according to the associated permutation π. Given a partition $\{N[t] : t = 1, \ldots, T\}$ of N, a permutation π of N is type $-$ preserving if, for any $i \in N$, $\pi(i)$ belongs to the same element of the partition $\{N[t]\}$ as i. A δ-substitute partition of N is a partition $\{N[t] : t = 1, \ldots, T\}$ of N with the property that, for any type-preserving permutation π and any coalition S,

$$H_\infty\left[V(S); \sigma_\pi^{-1}(V(\pi(S)))\right] \leq \delta.$$

Note that in general a δ-substitute partition of N is not uniquely determined. Moreover, two games may have the same partitions but have no other relationship to each other (in contrast to games derived from a pregame).

<u>(δ, T)- type games.</u> The notion of a (δ, T)-type game is an extension of the notion of a game with a finite number of types to a game with approximate types. For our purposes, this is significantly less restrictive than the extension of a finite set of types to a compact metric space.

[3] See, for example, Wooders (1983) or Wooders and Zame (1984).

Let δ be a non-negative real number and let T be a positive integer. A game $(N;V)$ is a $(\delta;T)$-type game if there is a T-member δ-substitute partition $\{N[t] : t = 1, \ldots, T\}$ of N. The set $N[t]$ is interpreted as an approximate type. Players in the same element of a δ-substitute partition are δ-substitutes. When $\delta = 0$, they are exact substitutes.

<u>per capita boundedness.</u> Let C be a positive real number. A game $(N;V)$ has a per capita payoff bound of C if, for all coalitions $S \subset N$,

$$\sum_{a \in S} x_a \leq C|S| \text{ for any } x \in V(S).$$

<u>β-effective B-bounded groups:</u> Informally, groups of players containing no more than B members are β-effective if, by restricting coalitions to having fewer than B members, the loss to each player is no more than β. This is a form of "small group effectiveness" for arbitrary games. Let $(N;V)$ be a game. Let $\beta \geq 0$ be a given non-negative real number and let B be a given positive integer. For each group $S \subset N$, define a corresponding set $V(S;B) \subset \mathbb{R}^N$ in the following way:

$$V(S;B) := \bigcup \left[\sum_k V(S^k) : \left\{S^k\right\} \text{ is a partition of } S, \left|S^k\right| \leq B \right].$$

The set $V(S;B)$ is the payoff set of the coalition S when groups are restricted to have no more than B members. Note that, by superadditivity, $V(S;B) \subset V(S)$ for any $S \subset N$ and, by construction, $V(S;B) = V(S)$ for $|S| \leq B$. We might think of $c_q(V(S;B))$ as the payoff set to the coalition S when groups are restricted to have no more than B members and transfers are allowed between groups in the partition. If the game $(N;V)$ has q-comprehensive payoff sets then $c_q(V(S;B)) \subset V(S)$ for any $S \subset N$. The game $(N;V)$ with q-comprehensive payoff sets has β-effective B-bounded groups if for every group $S \subset N$

$$H_1\left[V(S); c_q(V(S;B))\right] \leq \beta.$$

When $\beta = 0$, 0-effective B-bounded groups are called strictly effective B-bounded groups.

<u>parameterized collections of games $G^q((\delta;T);C;(\beta;B))$.</u> With the above definitions in hand, we can now define parameterized collections of games. Let T and B be positive integers and let C and q be real numbers, $0 \leq q \leq 1$. Let $G^q((\delta;T);C;(\beta;B))$ be the collection of all $(\delta;T)$-type games that are superadditive, have q-comprehensive payoff sets, have per capita bound of C, and have β-effective B-bounded groups.

Less formally, given non-negative real numbers C, q, β and δ, and positive integers T and B, a game $(N;V)$ belongs to the class $G^q((\delta;T);C;(\beta;B))$ if:

(a) the payo® sets satisfy q-comprehensiveness;

(b) there is a partition of the total player set into T sets where each element of the partition contains players who are ±-substitutes for each other;

(c) maximum per capita gains are bounded by C; and

(c) almost all gains to collective activities (with a maximum possible loss of ¯ for each player) can be realized by partitions of the total player sets into groups containing fewer than B members.

3 The results.

First, we recall some de¯nitions.

The core and epsilon cores. Let (N; V) be a game. A payo® x is "-undominated if for all S ½ N and y 2 V(S) it is not the case that $y_S >> x_S + 1_S$". The payo® x is feasible if x 2 V(N). The "-core of a game (N; V) consists of all feasible and "-undominated allocations. When " = 0, the "-core is the core.

The equal treatment epsilon core. Given non-negative real numbers " and ±, we will de¯ne the equal treatment "-core of a game (N; V) relative to a partition fN[t]g of the player set into ±-substitutes as the set of payo®s x in the "-core with the property that for each t and all i and j in N[t], it holds that $x_i = x_j$.

To motivate our Theorem, notice that a feasible payo® is in the "-core if no coalition of players can improve upon the payo® by at least " for each member of the coalition. This suggests that the distance of coalitions containing fewer than B-members from being e®ective for the realization of all gains to coalition formation should be de¯ned using the Hausdor® distance with respect to the sup norm, and indeed this was the approach that we took in previous papers. Thus we ¯rst establish a form of our result for the case with ¯-e®ective groups de¯ned as above. We then consider another de¯nition of ¯-e®ective B-bounded groups and obtain a stronger form of our result.

3.1 The Theorem.

Let (N; V) 2 G^q((±; T); C; (¯; B)). The following Theorem provides a lower bound on " so that for any "^{0} ¸ ", the game (N; V) has a non-empty "^{0}-core. In fact, the Theorem shows non-emptiness of the equal-treatment "-core as well, de¯ned as the subset of payoffs in the ε-core that assign equal payoffs to all agents of the same approximate type. It is convenient to define first a constant:

$$K(T;B) := \sum_{l=1}^{B} \frac{(T+l-1)!}{(T-1)!(l-1)!}.$$

Then the lower bound on ε is given by

$$\alpha_N^q((\delta;T);C;(\beta;B)) := \frac{1}{q}\left(\frac{K(T;B)C}{|N|} + \beta\right) + \delta.$$

Of course the interesting cases are those where this bound is small. To avoid trivialities associated with large ε we restrict attention to the case $\alpha_N^q((\delta;T);C;(\beta;B)) \leq m^*$, where m^* is the positive real number fixed in Section 2.2.

Theorem. Let $(N;V) \in G^q((\delta;T);C;(\beta;B))$; where $q > 0$. Assume $V(N)$ is convex. Let ε be a positive real number. If $\varepsilon \geq \alpha_N^q((\delta;T);C;(\beta;B))$ then the equal treatment ε-core of $(N;V)$ is non-empty.

The relationships between the lower bound on ε, the parameters describing the game, and the number of players in the total player set are immediate. Note in particular, the smaller B, the size of β-effective groups, the smaller the lower bound. It is easy to see that increasing β increases the bound $\alpha_N^q((\delta;T);C;(\beta;B))$ proportionally while increasing B increases the bound much more rapidly.

Now let us consider the central case of games with side payments.

3.2 Games with side payments.

A game with side payments (also called a TU game) is a game $(N;V)$ with 1-comprehensive payoff sets, that is $V(S) = c_1(V(S))$ for any $S \subset N$. This implies that for any $S \subset N$ there exists a real number $v(S) \geq 0$ such that $V_S = \left\{x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq v(S)\right\}$. The numbers $v(S)$ for $S \subset N$ determine a function v mapping the subsets of N to $\mathbb{R}_+$. Then the TU game is represented as the pair $(N;v)$. Therefore all the definitions that we have introduced can be stated for TU games through the characteristic functions v. Moreover some of these definitions are essentially simpler and more straightforward than in the general case. For the purposes of the illustration we state below the definitions for TU games that became essentially simpler:

1). A game $(N;v)$ is superadditive if $v(S) \geq \sum_k v(S^k)$ for all groups $S \subset N$ and for all partitions $\left\{S^k\right\}$ of S.

2). Let $(N;v)$ be a game and let $\delta \geq 0$ be a non-negative real number. A δ-substitute partition of N is a partition $\{N[t] : t = 1, ..., T\}$ of N with the property that, for any type-consistent permutation τ and any coalition S,

$$|v(S) - v(\tau(S))| \leq \delta |S|.$$

3). Let ε be a given non-negative real number, and let B be a given integer. A game $(N;v)$ has ε-effective B-bounded groups if for every group $S \subset N$ there is a partition $\{S^k\}$ of S into subgroups with $|S^k| \leq B$ for each k and

$$v(S) - \sum_k v(S^k) \leq \varepsilon |S|.$$

4). Let C be a positive real number. A game $(N;v)$ has a per capita bound of C if $\frac{v(S)}{|S|} \leq C$ for all coalitions $S \subset N$.

The case of TU games is central, since first we prove our result for these games and then we extend the result to games without side payments. To make notations simpler in the following sections, we denote parameterized collections of games with side payments, $G^{\infty}((\delta;T);C;(\varepsilon;B))$, by $\Gamma((\delta;T);C;(\varepsilon;B))$.[4] For the convenience of the reader a corollary of the Theorem corresponding to the case of TU games follows:

Corollary 1. Let $(N;v) \in \Gamma((\delta;T);C;(\beta;B))$ and let ε be a positive real number. If

$$\varepsilon \geq \frac{K(T;B)C}{|N|} + \delta + \beta$$

then the equal treatment ε-core of $(N;v)$ is non-empty.

We present the proof of Corollary 1 in the next section. The proof of the Theorem is provided in later sections. Now let us state some examples.

3.3 Examples.

Let us first concentrate on games with side payments.

Example 1. Exact types and strictly effective small groups. Let us consider a game $(N;v)$ with two types of players. Assume that any player alone can get only 0 units or less, that is $v(\{i\}) = 0$ for all $i \in N$. Let γ_{11}; $\gamma_{12} = \gamma_{21}$; and γ_{22} be some numbers from the interval $[0;1]$: Suppose that any coalition of the two players of types i and j can get up to γ_{ij} units of payoff to divide. An arbitrary coalition can gain only what it can obtain in partitions where no member of the

[4] Parameterized collections of games with side payments were introduced in Wooders (1994b) and the following Corollary obtained.

partition contains more than two players.

We leave it to the reader to check that $(N;v) \in \Gamma((0;2);\frac{1}{2};(0;2))$. Since $K(2;2) = 2! + 3! = 8$, we have from Corollary 1 that for $\varepsilon \geq \frac{4}{|N|}$ the equal treatment ε-core of $(N;v)$ is non-empty. Notice that this result holds uniformly for all possible numbers γ_{11}, $\gamma_{12} = \gamma_{21}$, and γ_{22}.

The following example illustrates how our result can apply to games derived from pregames with a compact metric space of player types. For brevity, our example is somewhat informal.

Example 2. Approximate player types. Consider a pregame with two sorts of players, firms and workers. The set of possible types of workers is given by the points in the interval $[0;1)$ and the set of possible types of firms is given by the points in the interval $[1;2]$. Formally, let N be any finite player set and let ξ be an attribute function, that is, a function from N into $[0;2]$. If $\xi(i) \in [0;1)$ then i is a worker and if $\xi(i) \in [1;2]$ then i is a firm.

Firms can profitably hire up to three workers and the payoff to a firm i and a set of workers $W(i) \subset N$, containing no more than 3 members, is given by $v(\{i\} \bigcup W(i)) = \xi(i) + \sum_{j \in W(i)} \xi(j)$. Workers and firms can earn positive payoff only by cooperating so $v(\{i\}) = 0$ for all $i \in N$. For any coalition $S \subset N$ define $v(S)$ as the maximum payoff the group S could realize by splitting into coalitions containing either workers only, or 1 firm and no more than 3 workers. This completes the specification of the game.

We leave it to the reader to verify that for any positive integer m every game derived from the pregame is a $(\frac{1}{m};2m)$-type game and even a member of the class $\Gamma((\frac{1}{m};2m);2;(0;4))$. Then Corollary 1 implies that for any $\varepsilon \geq \frac{2K(2m;4)}{|N|} + \frac{1}{m}$ the equal treatment ε-core of $(N;v)$ is non-empty.

This implies that for any $\varepsilon' > 0$ there is a positive integer $N(\varepsilon')$ such that for any $|N| \geq N(\varepsilon')$ the game $(N;v)$ has a non-empty equal treatment ε'-core. (For an explicit bound take an integer $m' \geq \frac{2}{\varepsilon'}$ and define $N(\varepsilon') \geq \frac{4}{\varepsilon'}K(2m';4)$.)

For completeness, we present a simple example with β-effective B-bounded groups where $\beta \neq 0$.

Example 3. Nearly effective groups. Call a game $(N;v)$ a k-quota game if any coalition $S \subset N$ of size less than k can realize only 0 units (that is, $v(S) = 0$ if $|S| < k$), any coalition of size k can realize 1 unit (that is, $v(S) = 1$ if $|S| = k$), and an arbitrary coalition can gain only what it can obtain in partitions where no member of the partition contains more than k players. Let Q be a collection, across all k, of all k-quota games with player set N.

We leave it to the reader to verify that for any positive integer m every game in the collection Q has $\frac{1}{m}$-effective $(m-1)$-bounded groups. Moreover the class Q is contained in the class $-((0,1);1;(\frac{1}{m};m-1))$. Then Corollary 1 implies that for any $\varepsilon \geq \frac{K(1;m-1)}{|N|}+\frac{1}{m}$ and for any $(N;v)\in Q$ the equal treatment ε-core of $(N;v)$ is non-empty. This implies that for any $\varepsilon' > 0$ there is a positive integer $N(\varepsilon')$ such that for any $|N| \geq N(\varepsilon')$ any game $(N;v)\in Q$ has a non-empty equal treatment ε'-core. (For an explicit bound take an integer $m' \geq \frac{2}{\varepsilon'}$ and define $N(\varepsilon') \geq \frac{2}{\varepsilon'}K(1;m'-1)$.)

Our next example demonstrates how our Theorem can be applied to games without side payments.

Example 4. Let $(N;V_0)$ be a superadditive game where for any two-person coalition $S=\{i,j\},\ j\neq i$,

$$V_0(S) := \{x \in \mathbb{R}^N : x_i \leq 1,\ x_j \leq 1, \text{ and } x_k = 0 \text{ for } k \neq i,j\}$$

and for each $i \in N$,

$$V_0(\{i\}) := \{x \in \mathbb{R}^N : x_i \leq 0 \text{ and } x_j = 0 \text{ for all } j \neq i\}.$$

For an arbitrary coalition S the payoff set $V_0(S)$ is given as the superadditive cover, that is,

$$V_0(S) := \bigcup_{P(S)} \sum_{S' \in P(S)} V_0(S');$$

where the union is taken over all partitions $P(S)$ of S in the sets with one or two elements.

Now let us define a game $(N;V_{\frac{1}{3}})$ in the following way. For any $S \subset N$ let $V_{\frac{1}{3}}(S)$ be the $\frac{1}{3}$-comprehensive cover of the convex cover of the payoff set $V_0(S)$; that is,

$$V_{\frac{1}{3}}(S) := c_{\frac{1}{3}}(co(V_0(S))).$$

Obviously the game $(N;V_{\frac{1}{3}})$ has $\frac{1}{3}$-comprehensive convex payoff sets, one player type, and per capita bound of 1. We leave it to the reader to verify that for any positive integer $m \geq 3$ the game $(N;V_{\frac{1}{3}})$ has $\frac{1}{m}$-effective m-bounded groups. Thus the game $(N;V_{\frac{1}{3}})$ is a member of the class $G^{\frac{1}{3}}((0,1);1;(\frac{1}{m};m))$.

Since $V_{\frac{1}{3}}(N)$ is convex and $K(1;m)=\frac{m(m+1)}{2}$; the Theorem states that for any $\varepsilon \geq 3(\frac{m(m+1)}{2|N|}+\frac{1}{m})$ the equal treatment ε-core of $(N;V_{\frac{1}{3}})$ is non-empty. This implies that for any $\varepsilon' > 0$ there is a positive integer $N(\varepsilon')$ such that for any $|N| \geq N(\varepsilon')$ the game $(N;V_{\frac{1}{3}})$ has a non-empty equal treatment ε'-core. (For an explicit bound take an integer $m' \geq \frac{6}{\varepsilon'}$ and define $N(\varepsilon') \geq \frac{3m'(m'+1)}{\varepsilon'}$.)

The following example illustrates why either convexity or some degree of comprehensiveness is required for our result, even for games with just one exact player type.

Example 5. Recall the game $(N; V_0)$ defined in Example 4. Let m be a positive integer. Let $(N^m; V_0^m)$ be a game where the number of players in the set N^m is $2m + 1$ and for any coalition $S \subset N^m$ $V_0^m(S) := V_0(S)$. Thus, each game $(N^m; V_0^m)$ has an odd number of players.

It is easy to see that the core of the game is non-empty: any payoff giving 1 to each of $2m$ players is in the core. Since the total number of players is odd, at least one person must be "left out." In a game with side payments this player could upset the non-emptiness of the core. But the games of this example do not satisfy strong comprehensiveness. Thus, a payoff giving 1 to each of $2m$ players cannot be improved upon since the "left-out" player, in a coalition by himself, cannot make both himself and a player in a two-person coalition better off – the player in the two-person coalition cannot be given more than 1. The games, however, can be approximated arbitrarily closely by games with strongly comprehensive payoff sets.[5]

Let $(N^m; V_{sc}^m)$ be a game with strongly comprehensive payoff sets that approximates the game $(N^m; V_0^m)$. For a sufficiently close approximation, the game $(N^m; V_{sc}^m)$ will have effective small groups and an empty core. This follows from the observations that any payoff must give at least one player less than one and the two worst-off players a total of less than two. The two worst-off players form an improving coalition and hence the core is empty.[6]

Our results rely on convexity and q-comprehensiveness. Since there is only one type of player, in this example either q-comprehensiveness or convexity will suffice. The role of convexity is to average payoffs over similar players. Consider the game $(N; V_{conv}^m)$ where V_{conv}^m is defined as the convex hull of V_{sc}^m. Then the payoff $x = (\frac{2m}{2m+1}, \ldots, \frac{2m}{2m+1})$ is feasible and in the ε-core of $(N^m; V_{conv}^m)$ for any $\varepsilon \geq \frac{1}{2m+1}$. Now instead of convexity of the total payoff set, suppose that payoff sets are q-comprehensive. In this case for any payoff giving one to each of $2m$ players, it is possible to take some small amount, say ε, away from each of $2m$ players and "transfer" $2m\varepsilon q$ to the leftover player. Thus, for any ε and q satisfying $2m\varepsilon q \geq 1 - \varepsilon$ the ε-core is non-empty.

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[5] The set $W \subset R^S$ is compactly generated if there exists a compact set $C \subset R^S$ such that $W = C - R^S_+$. The approximation can be carried out for any game with comprehensive and compactly generated payoff sets - see Wooders (1983, Appendix).

[6] It can be shown with a precise construction of $(N^m; V_{sc}^m)$ that for a small but positive ε the ε-core of $(N^m; V_{sc}^m)$ can be empty even for a great number of players. The reader may also find it interesting and informative to consider an example where any two-player coalition can distribute a total of two units of payoff in any agreed-upon way, while there is no transferability of utility between coalitions.

A crucial feature of Example 5 is the restriction to one player type. Because of this feature and the fact that two-player coalitions are effective, through convexity or q-comprehensiveness we can construct equal-treatment payoffs in approximate cores. This example suggests that either convexity or q-comprehensiveness is sufficient to get non-emptiness of the epsilon core for large games. In fact, Theorem 3 in Kovalenkov and Wooders (1997a) supports this intuition in the case of q-comprehensiveness. Theorem 1 in Kovalenkov and Wooders (1997b) shows that convexity is sufficient for nonemptiness but requires "thickness" of the player set (that is, the condition that the proportion of any approximate player type is bounded above zero). Neither of these papers, however, provide explicit bounds.

The Theorem shows that with both convexity and q-comprehensiveness, an explicit bound can be obtained on ε for non-emptiness of the ε-core. This bound appears to be very simple and easily computable from the parameters.

3.4 A more general result.

We now generalize the Theorem using another notion of distance; we define β-effective B-bounded groups using the Hausdorff distance with respect to the sum norm. This, of course, provides a much less demanding notion of near-effectiveness. Nevertheless, we are able to establish that all sufficiently large games have non-empty approximate cores. An example is provided showing that when β-effective B-bounded groups are defined using the Hausdorff distance with respect to the sum norm, the lower bound on ε may be substantially smaller.

Let us first define this another notion of the Hausdorff distance, while maintaining the previous definition of the set M^*: For every two non-empty subsets E and F of Euclidean space $\mathbb{R}^N$ let $H_1(E;F)$ denote the Hausdorff distance between $E \cap M^*$ and $F \cap M^*$ with respect to the metric $\|x - y\|_1 := \sum_{i=1}^N |x_i - y_i|$. Now we can define a weaker notion of the β-effective B-bounded groups.

<u>weakly $\beta-$effective $B-$bounded groups:</u> The game $(N;V)$ with q-comprehensive payoff sets has weakly β-effective B-bounded groups if for every group $S \subset N$

$$H_1[V(S); c_q(V(S;B))] \leq \beta|S|.$$

Notice that β-effective B-bounded groups are always weakly β-effective B-bounded groups, but for TU games these two notions coincide. These notions also coincide in the case when $\beta = 0$.

We now introduce a new definition of parametrized collections of games.

<u>parameterized collections of games $\mathcal{G}^q((\delta;T);C;(\beta;B))$.</u> Let T and B be positive integers and let C and q be positive real numbers, $q \leq 1$. Let $\mathcal{G}^q((\delta;T);C;(\beta;B))$ be

the collection of all $(\delta;T)$-type games that are superadditive, have q-comprehensive payoff sets, have per capita bound of C, and have weakly β-effective B-bounded groups.

- Of course $G^q((\delta;T);C;(\beta;B)) \subset G^q((\delta;T);C;(\beta;B))$, but these two classes coincide for $q = 1$ (games with side payments), that is $G^1((\delta;T);C;(\beta;B)) = \Gamma((\delta;T);C;(\beta;B))$.

The following statement is a generalization of the Theorem. Although it is a generalization, we prefer to call it a corollary since the proof of this statement is a straightforward implication of the proof of the Theorem.

Corollary 2. Let $(N;V) \in G^q((\delta;T);C;(\beta;B))$; where $q > 0$. Assume $V(N)$ is convex. Let ε be a positive real number. If $\varepsilon \geq \alpha^q_N((\delta;T);C;(\beta;B))$ then the equal treatment ε-core of $(N;V)$ is non-empty.

Notice that Corollary 2 is a strict generalization of the Theorem. Two remarks should be done about it. The first is that Corollary 2 can be applied to the larger class of games than the Theorem. The second (and much less obvious) is that the use of Corollary 2 rather than the use of the Theorem can improve the bound significantly. The following example continues Example 5 and illustrates this feature.

Example 6. Recall the game $(N;V_{\frac{1}{3}}) \in G^{\frac{1}{3}}((0;1);1;(\frac{1}{m};m))$ defined in Example 4. We leave it to the reader to verify that the game $(N;V_{\frac{1}{3}})$ has weakly $\frac{1}{|N|}$-effective 2-bounded groups. Therefore the game $(N;V_{\frac{1}{3}})$ is a member of the class $G^{\frac{1}{3}}((0;1);1;(\frac{1}{|N|};2))$. Recall that $V_{\frac{1}{3}}(N)$ is convex. Then, since $K(1;2) = 1! + 2! = 3$; Corollary 2 states that for any $\varepsilon \geq 3(\frac{3}{|N|} + \frac{1}{|N|}) = \frac{12}{|N|}$ the equal treatment ε-core of $(N;V_{\frac{1}{3}})$ is non-empty while the Theorem gave the bound $\varepsilon \geq 3(\frac{m(m+1)}{2|N|} + \frac{1}{m})$. (The bound of Corollary 2 implies that for any $\varepsilon' > 0$ and for any $|N| \geq \frac{12}{\varepsilon'}$ the game $(N;V_{\frac{1}{3}})$ has a non-empty equal treatment ε'-core.)

4 Proofs for games with side payments.

Let us first prove the following Lemma, from Wooders (1994b).

Lemma 1. Let $(N;v) \in \Gamma((0;T);C;(0;B))$. If $\varepsilon \geq \frac{K(T;B)C}{|N|}$ then the equal treatment ε-core of $(N;v)$ is non-empty.

Proof of Lemma 1: In this proof we will use the notion of a totally balanced cover for a game. Let us first define balanced collections and balancing weights. Let $(N;v)$

be a game, let $S \subset N$, and let Ω denote a collection of subsets of S. The collection Ω is a balanced collection of subsets of S if there is a collection of non-negative real numbers $(\omega_{S'})_{S' \in \Omega}$, called balancing weights, such that for each $i \in N$,

$$\sum_{S' : i \in S',\, S' \in \Omega} \omega_{S'} = 1.$$

Let $(N; v)$ be a game and let v^b be the characteristic function defined for each subset S of N by

$$v^b(S) := \max_{\Omega} \sum_{S' \in \Omega} \omega_{S'} v(S'),$$

where the maximum is taken over all balanced collections Ω of S with corresponding balancing weights $(\omega_{S'})_{S' \in \Omega}$. Then $(N; v^b)$ is a game, called the totally balanced cover of $(N; v)$.

Bondareva (1962) and Shapley (1967) have shown that a TU game has a non-empty core if and only if $v^b(N) = v(N)$. It follows easily from their results that a game has a non-empty ε-core if and only if $v^b(N) \le v(N) + \varepsilon |N|$.

To begin the proof of Lemma we place a bound on the difference $v^b(N) - v(N)$. Let $\left\{ m^k \right\}_{k=1}^{K(T;B)}$ denote the collection of all profiles m^k relative to the partition $\{N[t]\}$ of N; where $\left\| m^k \right\| \le B$ for each m^k in the collection. Let f denote the profile of N: Define a characteristic function $\bar{v}$ mapping profiles into $\mathbb{R}^T_+$ by

$$\bar{v}(m) := v(M) \text{ for any group } M \text{ with profile } m:$$

Since $(N; v)$ satisfies boundedness of effective group sizes with bound B there is a balanced collection Ω of profiles of N where each $m \in \Omega$ is in $\left\{ m^k \right\}$ and for some collection of balanced weights (ω_k);

$$v^b(N) = \sum_k \omega_k \bar{v}(m^k):$$

From balancedness, it holds that $\sum_k \omega_k m^k = f$.

Since there is a finite number of distinct profiles in the set $\left\{ m^k \right\}$, we can write each ω_k as an integer plus a fraction, say $\omega_k = l_k + q_k$, where $q_k \in [0; 1)$. Since the game $(N; v)$ satisfies boundedness of effective group sizes and superadditivity it holds that

$$\sum_k l_k \bar{v}(m^k) \le v(N):$$

Now

$$\begin{aligned} v^b(N) - v(N) &\le \sum_k \omega_k \bar{v}(m^k) - \sum_k l_k \bar{v}(m^k) = \sum_k (\omega_k - l_k) \bar{v}(m^k) \\ &= \sum_k q_k \bar{v}(m^k) = \sum_k \bar{v}(m^k) = \sum_k \left\| m^k \right\| C: \end{aligned}$$

Let us denote by $k(T;l)$ the number of distinct profiles with norm l. It is easy to check that $k(T;l) = \frac{(T+l-1)!}{(T-1)!(l)!}$. Thus $v^b(N) - v(N) \le \sum_k \|m^k\| C \le \sum_{l=1}^{B} k(T;l) l C = \sum_{l=1}^{B} \frac{(T+l-1)!}{(T-1)!(l-1)!} C = K(T;B)C$. Hence, for any $\varepsilon \ge \frac{K(T;B)C}{|N|}$, the ε-core of $(N;v)$ is non-empty.

Note that if some payoff x belongs to the ε-core then for any type-consistent permutation τ of N, a payoff y, defined by its components as $y_a := x_{\tau(a)}$, belongs to the ε-core (since all agents of one type are exact substitutes and the payoff sets are unaffected by any permutation of substitute players). Let us consider an ε-core payoff x. Then there is an equal treatment ε-core payoff z defined by its components $z_t := \frac{1}{|N[t]|} \sum_{a \in N[t]} x_a$ (since the ε-core of a TU game is convex). Therefore, the equal treatment ε-core of $(N;v)$ is non-empty. ■

Now we will prove Corollary 1.

Proof of Corollary 1: Let $(N;v) \in \Gamma((\delta;T);C;(\beta;B))$ and let ε be a positive real number. We first construct another game with strictly effective groups bounded in size by B. From the definition of effective groups, for any $S \subset N$ there exists a partition $\{S^k\}$ of S, $|S^k| \le B$ for each k, such that $v(S) - \sum_k v(S^k) \le \beta|S|$. Let us define $w(S) := \max_{\{S^k\}} \sum v(S^k)$ where the maximum is taken over all partitions $\{S^k\}$ of S with $|S^k| \le B$ for each k. Then $(N;w) \in \Gamma((\delta;T);C;(0;B))$ and $\beta|S| \ge v(S) - w(S) \ge 0$ for any $S \subset N$.

Next we construct a related game by identifying all players of the same approximate type. First, for the game $(N;w)$ let $\{N[t]\}$ be a δ-substitute partition of N. Given a group $S \subset N$ let s denote the profile of S. Define

$$w^*(S) := \max \{w(S') : S' \text{ has profile } s\}.$$

Define w^c as the superadditive cover of w^*, i.e. for any $S \subset N$,

$$w^c(S) := \max_{\{S^k\}} \sum_k w^*(S^k);$$

where the maximum is taken over all partitions of S. Then $(N;w^c) \in \Gamma((0;T);C;(0;B))$ and

$$\delta|S| \ge w^c(S) - w(S) \ge 0 \text{ for each } S \subset N.$$

By Lemma 1 the game $(N;w^c)$ has a non-empty equal treatment $\frac{K(T;B)C}{|N|}$-core. Let x belong to the equal treatment $\frac{K(T;B)C}{|N|}$-core of $(N;w^c)$. Hence

$$\sum_{a \in N} x_a \le w^c(N) \text{ and } \sum_{a \in S} x_a + \frac{K(T;B)C}{|N|}|S| \ge w^c(S).$$

Now define a payoff vector y by

$$y(\{i\}) := x(\{i\}) - \delta$$

for each $i \in N$. Then

$$\sum_{a \in N} y_a = \sum_{a \in N} x_a - \delta|N|$$
$$\leq w^c(N) - \delta|N| \leq w(N) \leq v(N)$$

and for any group S it holds that

$$\sum_{a \in S} y_a + \left(\frac{K(T;B)C}{|N|} + \delta + \beta \right) |S| = \sum_{a \in S} x_a + \frac{K(T;B)C}{|N|} |S| + \beta|S|$$
$$\geq w^c(S) + \varepsilon|S| \geq w(S) + \beta|S| \geq v(S).$$

It follows that y is in the ε-core for any $\varepsilon \geq \frac{K(T;B)C}{|N|} + \delta + \beta$. Since y has equal treatment property by construction, the equal treatment ε-core of $(N;v)$ is non-empty. ■

5 The main part of the proofs.

We first provide a sketch of the proofs and the main argument. In appendix we relate NTU games to TU games and provide proofs of several results used in this subsection.

Proof of the Theorem:
We begin the proof of the Theorem by first treating games where all players are exact substitutes of each other. (Later in the proof we will construct such a game from an arbitrary game.) We will use the following terminology: A set $W \subset \mathbb{R}^N$ is symmetric across substitute players if for any player type the set W remains unchanged under any perturbations of the values associated with players of that type.

The symmetric case. Assume first that $(N;V) \in G^q((0;T);(\beta;B))$: Note that in this case all payoff sets of the game $(N;V)$ are symmetric across substitute players. Let us prove that for any $\varepsilon \geq \alpha^q_N((0;T);C;(\beta;B))$ the equal treatment ε-core of $(N;V)$ is non-empty.

The idea of the proof for the symmetric case. In the proof for the symmetric case we will use the following definitions. Let $A \subset \mathbb{R}^m$: A recession cone corresponding to A, denoted by $c(A)$, is defined as follows:

$$c(A) := \{y \in \mathbb{R}^m : x + \lambda y \in A \text{ for all } \lambda \geq 0 \text{ and } x \in A\}:$$

The scalar product of $x, y \in \mathbb{R}^m$ is denoted by $x \cdot y$. The negative dual cone of $P \subset \mathbb{R}^m$ is denoted by $d(P)$ and defined as follows:

$$d(P) := \{z \in \mathbb{R}^m : z \cdot y \leq 0 \text{ for any } y \in P\}.$$

A bound on the required size of the parameter ε for our result in the symmetric case is obtained by constructing a family of "λ-weighted transferable utility" games $(N; V_\lambda)$ corresponding, in a certain way, to the initial game $(N; V)$. Next we consider only those values of λ in a set L^*, defined as the intersection of the equal treatment payoffs in the simplex with the negative dual cone of the recession cone of the modified game. For each λ there is corresponding TU game $(N; v_\lambda)$. We give the formal construction of $(N; v_\lambda)$ in Step 1 of appendix.

In Step 2 of appendix we prove Lemma 2, that, for some parameters C' and ε', any game $(N; v_\lambda)$ is a member of the parameterized collection of TU games $\Gamma((0; T); C', (\beta', B))$. This allows us to use Corollary 1 proved in the previous Section. In Lemma 3 we relate approximate cores of the game $(N; v_\lambda)$ to approximate cores of the NTU game $(N; V_\lambda)$. Using the fact that we consider only values of λ in L^*, we obtain an explicit bound on ε for the initially given parameters C and β for non-emptiness of the equal treatment ε-core for all games $(N; V_\lambda)$. This result will give us exactly the bound that we need to deduce for the conclusion of Theorem for the symmetric case.

Now we need only prove that if, given some ε, the equal treatment ε-core of $(N; V_\lambda)$ is non-empty for all $\lambda \in L^*$, then the equal treatment ε-core will be non-empty for both the modified and initial games as well. With the help of Lemma 4, Lemma 5, and a theorem about excess demand considered in Step 3, all in appendix, we complete the proof in the symmetric case.

Remark. The initial approach in the following proof is similar to that introduced in Scarf (1965) and usually used in proofs of the non-emptiness of the exact core for strongly balanced NTU games (for a definition of strong balancedness and for an example of this technique see Hildenbrand and Kirman (1988, Appendix to Chapter 4)). But the proof below departs from the typical approach in that we construct games $(N; V_\lambda)$ and $(N; v_\lambda)$ not for all λ in the simplex as usual, but only for λ belonging to a specific subset L^* of the simplex. The set L^* is the intersection of the equal treatment payoffs in the simplex with the dual negative cone to the recession cone of the payoff set for the grand coalition in the modified game. Later we use the structure of the set L^* and q-comprehensiveness to complete the proof.[7]

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[7]Note that our technique is a generalization of the usual one, which does not involve recession cone arguments. Nevertheless the usual approach is a special case of our technique. (The usual technique is applied for $T = N$ and to games having $-\mathbb{R}^N_+$ as the recession cone of the payoff set $V(N)$. The negative dual cone to $-\mathbb{R}^N_+$ is $\mathbb{R}^N_+$. Then $\mathbb{R}^N_+ \supset \Delta_+$, so the relevant intersection is the simplex itself.)

The general case. Now let us consider the general case with no additional restrictions. We first modify the game $(N;V)$. For any $S \subset N$ define $V'(S) := \bigcap \sigma_\tau^{-1}(V(\tau(S)))$; where the intersection is taken over all type-preserving permutations τ of the player set N. Then from the definition of $V'(S)$ it follows that $V'(S) \subset V(S)$. (Informally, taking the intersection over all type-preserving permutations makes all players of each approximate type no more productive than the least productive members of that type.) From the definition of δ-substitutes, it follows that $H_1[V'(S); V(S)] \leq \delta$ for any $S \subset N$. Moreover,

$$(N;V') \in G^q((0;T);(\beta;B)) \text{ and } V'(N) \text{ is convex:}$$

Therefore, we can apply the result proved in the symmetric case and conclude that the game $(N;V')$ has some payoff x in the equal treatment $\frac{1}{q}(\frac{K(T;B)C}{|N|} + \beta)$-core. Now define a payoff vector y by

$$y(\{i\}) := x(\{i\}) - \delta \text{ for each } i \in N:$$

The payoff y will be feasible and $\left(\frac{1}{q}(\frac{K(T;B)C}{|N|} + \beta) + \delta\right)$-undominated in the initial game $(N;V)$. Obviously, y has the equal treatment property. Therefore for $\varepsilon \geq \alpha_N^q((\delta;T);C;(\beta;B))$ the equal treatment ε-core of $(N;V)$ is non-empty. ■

Proof of Corollary 2: The proof follows the proof of the Theorem. The only place in the proof of Theorem where we were using the fact that the given game has β-effective B-bounded groups was application of Lemma 2. Note that the corresponding generalization of Lemma 2 for weakly β-effective B-bounded groups is true. The same exactly proof as it was for Lemma 2 applies. ■

6 Relationships to the literature.

Recall that Shapley and Shubik (1966) showed that exchange economies with many players and with quasi-linear utility functions (transferable utility) have nonempty approximate cores.[8] There are now a number of results in the literature showing that large games without side payments have non-empty approximate cores.[9] These results, however, are all obtained in the context of pregames. Recall that a pregame specifies a topological space of player types and a payoff set (or number) for every possible coalition in any game induced by the pregame. More precisely, given a compact metric space of player "types" or "attributes" (possibly finite), the payoff function of a pregame assigns a payoff set to every finite list of player types, repetitions

[8] These results were obtained by convexifying preferences rather than by using assumptions of small group effectiveness.

[9] See, for example, Wooders (1983), Kaneko and Wooders (1982,1996), and Wooders and Zame (1984).

allowed. Given any finite player set and an attribute function, assigning a type to each player in the player set, the payoff function of the game is determined by the payoff function of the pregame. Thus, the payoff set to any collection of players having a certain set of attributes is independent of the total player set in which it is embedded. The pregame structure itself has hidden consequences. For example, within the context of pregame with side payments, there is an equivalence between per capita boundedness, finiteness of the supremum of average payoff, and small group effectiveness, the condition that all or almost all gains to collective activities can be realized by groups bounded in size (Wooders 1994a, Section 5). No such consequences can be hidden within parameterized collections of games since there is no necessary relationship between any of the games in the collection (other than that they are all described by the same parameters). The pregame framework also rules out widespread externalities, that is, the worth of any coalition of players is independent of the total player set is embedded. This is a significant limitation in economic applications.

To study large games generally, without the structure and implicit assumptions imposed by a pregame, Kovalenkov and Wooders (1997a,b) introduce the concept of parameterized collections of games without side payments and show non-emptiness of approximate cores of large games. No explicit bound, however, on the required size of the games is provided and the dependence of the required size on the parameters is not explicitly demonstrated. In this paper using significantly different techniques and the assumption of convexity of payoff sets, we are able to derive an explicit lower bound on ε.

We remark that the results of this work may have application in economies with local public goods and/or coalition production (see, for example, Conley and Wooders (2000)) and other sorts of situations with coalitions. A possible very exciting application is to economies with differential information, as in Allen (1994,1995), Forges and Minelli (1999), or Forges, Heifetz, and Minelli (1999), among others. It may be possible, for example, to derive a bound on the extent of the deviation of cores involving differential information from the full information core.

7 Appendix: Relating games with and without side payments.

<u>Step 1: Construction of the TU games.</u> Let us first modify the game $(N; V)$. Consider the set

$$K := \left\{ x \in \mathbb{R}^N : x_a \geq -\varepsilon \text{ for any } a \in N \right\}:$$

Define

$$K^* := K \bigcap V(N)$$

and observe that set K^* is a compact set. Let $c(V(N);K)$ be the smallest closed cone such that

$$V(N) \subset K^* + c(V(N);K).$$

Now let us define a modified game $(N;V^1)$ so that

$$\begin{aligned} \text{(a) } V^1(S) &:= V(S) \text{ for } S \neq N \text{ and} \\ \text{(b) } V^1(N) &:= K^* + c(V(N);K). \end{aligned}$$

Notice that $c(V(N);K)$ is a recession cone of $V^1(N)$; that is,

$$c(V^1(N)) = c(V(N);K).$$

We are going to prove that the modified game $(N;V^1)$ has an equal treatment ε-core payoff, which we will denote by x^*. Since $V(S) \subset V^1(S)$ for any S, x^* will be ε-undominated in the game $(N;V)$. Thus $x^* \in K^* \cap V^1(N) = K^* \subset V(N)$. So the payoff x^* will be feasible in the game $(N;V)$. It follows that x^* is an equal treatment ε-core payoff for $(N;V)$.

Define

$$C := co\left\{ x \in \mathbb{R}^N : \exists i,j \in N;\ x_i = -q x_j \geq 0;\ x_k = 0;\ k \neq i,j \right\}$$

and observe that C is a cone. Since $V^1(N)$ is q-comprehensive and convex, the cone $c(V^1(N))$ will include C but will not be more than a half-space. Hence the negative dual cone to the recession cone $d(c(V^1(N)))$ will be closed, non-empty and included in the cone dual to C:

$$d(C) = \left\{ x \in \mathbb{R}^N_{++} : q \leq \frac{x_i}{x_j} \leq \frac{1}{q}\ \forall i,j \right\} \cup \{0\}.$$

Now let us consider the simplex in $\mathbb{R}^N_+$:

$$\Delta_+ := \left\{ \lambda \in \mathbb{R}^N_+ : \sum_{i=1}^{N} \lambda_i = 1 \right\}.$$

Define

$$L := d(c(V^1(N))) \cap \Delta_+.$$

Given a partition $\{N[t]\}$ of the player set into T types of δ-substitutes, the set of equal treatment allocations is denoted by E^T and defined as follows:

$$E^T := \left\{ x \in \mathbb{R}^N : x_i = x_j \text{ for any } t \text{ and any } i,j \in N[t] \right\}.$$

Now define

$$L^* := L \cap E^T.$$

Observe that L^* is a compact and convex set.

For any $\lambda \in L^*$ there exists a tangent hyperplane to the set $V(N)$ with normal λ such that the whole set $V(N)$ is contained in a closed half-space, and at least one point of the set $V(N)$ lies on the hyperplane. Moreover, since the game is superadditive, for any $\lambda \in L^*$ and any $S \subset N$ there exists a hyperplane in R^S that has normal parallel to λ_S and that is tangent to V_S. Thus, for a fixed $\lambda \in L^*$ there is a finite real number

$$v_\lambda(S) := \max \left\{ \sum_{a \in S} \lambda_a x_a : x \in V(S) \right\}.$$

The pair (N, v_λ) is a TU game. We construct a "λ-weighted transferable utility" game (N, V_λ) by defining, for each coalition $S \subset N$:

$$V_\lambda(S) := \left\{ x \in R^N : x_a = 0 \text{ for } a \notin S \text{ and } \sum_{a \in S} \lambda_a x_a \leq v_\lambda(S) \right\}.$$

■

Step 2 : Nonemptiness of the epsilon core for (N, V_λ) games.

Consider a fixed $\lambda \in L^*$. Define $\lambda_{max} := \max\{\lambda_i\}$ and $\lambda_{min} := \min\{\lambda_i\}$.

Lemma 2. Let $(N, V) \in G^q((0, T), C, (\beta, B))$. Then

$$(N, v_\lambda) \in \Gamma((0, T), C\lambda_{max}, (\beta\lambda_{max}, B)).$$

Proof of Lemma 2:

1). We will prove that the $(0, T)$-partition $\{N[t]\}$ of the game (N, V) is a $(0, T)$-partition of the game (N, v_λ). We must check that for any type-consistent permutation τ of N and any coalition S it holds that $v_\lambda(S) = v_\lambda(\tau(S))$. But we have:

$$\begin{aligned} v_\lambda(\tau(S)) &\equiv \max \left\{ \sum_{a \in \tau(S)} \lambda_a x_a : x \in V(\tau(S)) \right\} \\ &= \max \left\{ \sum_{a \in S} \lambda_{\tau(a)} x_a : x \in V(S) \right\} \\ &= \max \left\{ \sum_{a \in S} \lambda_a x_a : x \in V(S) \right\} \equiv v_\lambda(S). \end{aligned}$$

The second equality follows from the fact that $V(\tau(S)) = V(S)$, since $\{N[t]\}$ is a $(0, T)$-partition of the game (N, V). The third equality holds since, by construction of L^* and τ, for any a we have $\lambda_a = \lambda_{\tau(a)}$.

2). To show that the number $\lambda_{max}C$ is a per capita bound for the TU game $(N; v_\lambda)$, it is necessary to show that $\frac{v_\lambda(S)}{|S|} \leq \lambda_{max}C$ for each coalition group S. Observe that by the definition of $v_\lambda(S)$; for some $x_a \in V_S$ it holds that

$$\frac{v_\lambda(S)}{|S|} = \frac{\sum_{a \in S} \lambda_a x_a}{|S|} \leq \lambda_{max} \frac{\sum_{a \in S} x_a}{|S|} \leq \lambda_{max} C.$$

The last inequality follows from per capita boundedness of the game $(N; V)$.

3). To prove effectiveness of B-bounded $\lambda_{max}\beta$-effective groups for the TU game $(N; v_\lambda)$ we need to show that for any $S \subset N$ there exists a partition $\{S_k\}$ of S satisfying $|S_k| \leq B$ for each k and

$$\left| v_\lambda(S) - \sum_k v_\lambda(S_k) \right| \leq \lambda_{max} \beta |S|.$$

By superadditivity

$$v_\lambda(S) \geq \sum_k v_\lambda(S_k).$$

By the definition of v_λ there exists a vector x such that $x \in V(S)$ and $v_\lambda(S) = \sum_{a \in S} \lambda_a x_a$. Since $(N; V)$ has β-effective B-bounded groups there exists a vector $y \in c_q(V(S; B))$ such that

$$\sum_{a \in S} |y_a - x_a| \leq \beta |S|.$$

Then there exists a vector $z \in V(S; B)$ such that $y \in c_q(z)$. Note that since $\lambda \in L^* \subset d(C)$ and $y \in c_q(z)$ we have

$$\sum_{a \in S_k} \lambda_a y_a \leq \sum_{a \in S_k} \lambda_a z_a.$$

Then since $z \in V(S; B)$ we have that $z_{S_k} \in V_{S_k}$ for some partition $\{S_k\}$ of S (with $|S_k| \leq B$) and we get

$$\sum_{a \in S_k} \lambda_a z_a \leq v_\lambda(S_k).$$

Hence

$$\left| v_\lambda(S) - \sum_k v(S_k) \right| \leq \left| \sum_{a \in S} \lambda_a x_a - \sum_k \sum_{a \in S_k} \lambda_a z_a \right| \leq \left| \sum_{a \in S} \lambda_a x_a - \sum_k \sum_{a \in S_k} \lambda_a y_a \right|$$

$$\leq \sum_{a \in S} \lambda_a |x_a - y_a| \leq \lambda_{max} \beta |S|.$$

By 1), 2), 3) it holds that $(N; v_\lambda) \in \Gamma((0; T); C\lambda_{max}; (\beta\lambda_{max}; B))$. ■

Lemma 3. If the equal treatment ε-core of $(N;v_\lambda)$ is non-empty, then the equal treatment $\frac{\varepsilon}{\lambda_{min}}$-core of $(N;V_\lambda)$ game is non-empty.

Proof of Lemma 3: Consider a payoff y in the equal treatment ε-core of the game $(N;v_\lambda)$ and define $x_a := \frac{1}{\lambda_a} y_a$. Note that x also has equal treatment property. Then

$$\sum_{a\in N} \lambda_a x_a = \sum_{a\in N} y_a \le v_\lambda(N);$$

thus x is feasible for the game $(N;V_\lambda)$. Moreover, for all $S \subset N$;

$$\sum_{a\in S} \lambda_a\left(x_a + \frac{\varepsilon}{\lambda_{min}}\right) = \sum_{a\in S} y_a + \sum_{a\in S} \lambda_a \frac{\varepsilon}{\lambda_{min}} \ge \sum_{a\in S} y_a + \varepsilon|S| \ge v_\lambda(S)$$

thus x is $\frac{\varepsilon}{\lambda_{min}}$-undominated in the game $(N;V_\lambda)$. Therefore, x is in the equal treatment $\frac{\varepsilon}{\lambda_{min}}$-core of $(N;V_\lambda)$. ■

We can now finish Step 2 : Since $(N;V) \in G^q((0;T);C;(\beta;B))$, by Lemma 2 we have that

$$(N;v_\lambda) \in \Gamma((0;T);C\lambda_{max};(\beta\lambda_{max};B)):$$

But from Corollary 1 for any game with side payments in $\Gamma((\delta';T);C';(\beta';B))$ and any $\varepsilon' \ge \frac{K(T;B)C'}{|N|} + \delta' + \beta'$; the equal treatment ε'-core is non-empty. Hence, if $\varepsilon' \ge \lambda_{max}(\frac{K(T;B)C}{|N|} + \beta)$; the equal treatment ε'-core of $(N;v_\lambda)$ is non-empty. From Lemma 3 this implies that the equal treatment $\frac{\varepsilon'}{\lambda_{min}}$-core of $(N;V_\lambda)$ is non-empty. Thus, since

$$\alpha^q_N((0;T);C;(\beta;B)) = \frac{1}{q}\left(\frac{K(T;B)C}{|N|} + \beta\right) \ge \frac{\lambda_{max}}{\lambda_{min}}\left(\frac{K(T;B)C}{|N|} + \beta\right)$$

($\frac{\lambda_{max}}{\lambda_{min}} \le \frac{1}{q}$ because $\lambda \in L^* \subset d(C)$), we can conclude that if

$$\varepsilon \ge \alpha^q_N((0;T);C;(\beta;B))$$

the equal treatment ε-core of $(N;V_\lambda)$ is non-empty. This is exactly the bound that we need in the symmetric case. ■

<u>Step 3: Nonemptiness of the epsilon core for the initial game.</u> We need only to prove that if the equal treatment ε-core of $(N;V_\lambda)$ is non-empty for all $\lambda \in L^*$ then the equal treatment ε-core of $(N;V)$ is non-empty. Define

$$C_\varepsilon(\lambda) := \left\{ x : \sum_{a\in N} \lambda_a x_a \le v_\lambda(N); \sum_{a\in S} \lambda_a(x_a + \varepsilon) \ge v_\lambda(S) \right\} \cap E^T;$$

the equal treatment ε-core of the $(N; V_\lambda)$ game. Note that the equal treatment ε-core of $(N; V_\lambda)$ is non-empty for any $\lambda \in L^*$. For any $\lambda \in L^*$ and any $x \in C_\varepsilon(\lambda)$, x cannot be ε'-improved upon in the initial game $(N; V)$ for any $\varepsilon' > \varepsilon$. (If a coalition S could improve, we would have $x_S + \varepsilon'_S \in V_S$ and $\sum_{a \in S} \lambda_a (x_a + \varepsilon'_a) > v_\lambda(S)$; contradicting the definition of $v_\lambda(S)$.) Hence, it remains to show that there exists $\lambda^* \in L^*$ such that some $x^* \in C_\varepsilon(\lambda)$ is feasible in the initial game.

Lemma 4. The correspondence $\lambda \mapsto C_\varepsilon(\lambda)$ from L^* to $\mathbb{R}^N$ is bounded, convex-valued and has a closed graph. Moreover, for any $x \in C_\varepsilon(\lambda)$ and for any player a it holds that $x_a \geq -\varepsilon$.

Proof of Lemma 4:

1). If $f, g \in C_\varepsilon(\lambda)$ then $\mu f + (1-\mu) g$ has equal treatment property and $\mu f + (1-\mu) g \in C_\varepsilon(\lambda)$ since:

$$\text{(a)} \quad \sum_{a \in N} \lambda_a (\mu f_a + (1-\mu) g_a) = \mu \sum_{a \in N} \lambda_a f_a + (1-\mu) \sum_{a \in N} \lambda_a g_a \leq \mu v_\lambda(N) + (1-\mu) v_\lambda(N) = v_\lambda(N) \text{ and}$$

$$\text{(b)} \quad \sum_{a \in S} \lambda_a (\mu f_a + (1-\mu) g_a + \varepsilon) = \mu \sum_{a \in S} \lambda_a (f_a + \varepsilon) + (1-\mu) \sum_{a \in S} \lambda_a (g_a + \varepsilon) \geq \mu v_\lambda(S) + (1-\mu) v_\lambda(S) = v_\lambda(S).$$

2). It is straightforward to see that graph is closed since $v_\lambda(S)$ depends continuously on λ.

3). Consider $x \in C_\varepsilon(\lambda)$. Since x is in the ε-core of $(N; V_\lambda)$ game, x is ε-individually rational, that is, $x_a \geq -\varepsilon$.

4). Consider $x \in C_\varepsilon(\lambda)$. By construction,

$$\sum_{a \in N} \lambda_a x_a \leq v_\lambda(N) \leq \frac{1}{q} C |N|.$$

Since $\lambda \in L^* \subset L \subset \Delta_+$, there exists i such that $\lambda_i \geq \frac{1}{|N|}$. Then $\lambda \in L$ implies $\lambda_a \geq q \lambda_i \geq \frac{q}{|N|}$. Therefore, using 3) above we have that

$$\frac{q}{|N|} x_a \leq \lambda_a x_a \leq \frac{1}{q} C |N| - \left(1 - \frac{q}{|N|}\right)(-\varepsilon).$$

This proves that

$$x_a \leq \frac{1}{q^2} C |N|^2 + \left(\frac{|N|}{q} - 1\right)\varepsilon. \blacksquare$$

Now let us define

$$\zeta(\lambda) := C_\varepsilon(\lambda) - \left\{ x \in K^* : \sum_{a \in N} \lambda_a x_a = \max_{z \in V(N)} \sum_{a \in N} \lambda_a z_a \right\} \bigcap E^T:$$

For $\lambda \in L^*$ both the first term and the second term of this sum are non-empty, bounded, convex-valued correspondences with closed graphs; this follows from Lemma 3 and the observations that (a) $V(N)$ is convex and symmetric across substitute players and (b) K^* is compact. Hence the sum $\zeta(\lambda)$ is also bounded, closed and convex-valued for $\lambda \in L^*$. By construction $\sum_{a \in N} z_a \lambda_a \leq 0$ for any $z \in \zeta(\lambda)$.

Now we can use the following theorem of excess demand, which is in fact a version of Kakutani's theorem. (For a proof see Hildenbrand and Kirman (1988), Lemma AIV.1)

Theorem (Debreu, Gale, Nikaido): Let Δ^* be a closed and convex subset of Δ_+. If the correspondence ζ from Δ^* is bounded, convex-valued, has closed graph and it holds that for all $p \in \Delta^*$, $p \cdot z \leq 0$ for all $z \in \zeta(p)$; then there exists $p^* \in \Delta^*$ and $z^* \in \zeta(p^*)$ such that $p \cdot z^* \leq 0$ for all $p \in \Delta^*$.

It follows, from the Debreu-Gale-Nikaido Theorem, that there exists $\lambda^* \in L^*$ and $z^* \in \zeta(\lambda^*)$ such that $\lambda \cdot z^* \leq 0$ for all $\lambda \in L^*$. Since $z^* \in \zeta(\lambda^*)$, z^* can be represented as $z^* = x^* - y^*$ with $x^* \in C_\varepsilon(\lambda)$; $y^* \in K^* \bigcap E^T$. Therefore $z^* \in E^T$. As we argued at the beginning of this Step, x^* is ε-undominated in the initial game $(N;V)$. In addition, x^* has the equal treatment property.

We now deduce that x^* is feasible for the game $(N;V)$. Observe that $x^* = y^* + z^*$, where $y^* \in K^* \bigcap E^T$ and $\lambda \cdot z^* \leq 0$ for all $\lambda \in L^*$: Hence $z^* \in d(L^*) \bigcap E^T$.

Lemma 5. Let X be a convex and symmetric across substitute players subset of $\mathbb{R}^N$. Let $X^* := X \bigcap E^T$. Then $d(X^*) \bigcap E^T \subset d(X)$.

Proof of Lemma 5: For any $x \in X$, let us construct $\bar{x} \in \mathbb{R}^N$ as follows: for each $1 \leq t \leq T$, for any $a \in N[t]$ define

$$\bar{x}_a := \frac{1}{|N[t]|} \sum_{i \in N[t]} x_i:$$

Since X is convex and symmetric across substitute players, $\bar{x} \in X$. Obviously, $\bar{x} \in E^T$. Therefore $\bar{x} \in X \bigcap E^T = X^*$.

Now consider any $y \in d(X^*) \bigcap E^T$. For any $x \in X$ we have

$$\begin{aligned} y \cdot x &= \sum_{i \in N} y_i x_i = \sum_{1 \leq t \leq T} y_t \Big(\sum_{i \in N[t]} x_i \Big) \\ &= \sum_{1 \leq t \leq T} |N[t]| \, y_t \bar{x}_t = \sum_{i \in N} y_i \bar{x}_i \leq 0; \end{aligned}$$

where the last inequality follows from the fact $y \in d(X^*)$ and $\bar{x} \in X^*$. Hence, by the definition of the dual negative cone, $d(X^*) \cap E^T \subset d(X)$. ■

Since $V(N)$ is convex and symmetric across substitute players, it follows from construction of $c(V^1(N))$ that $L = d(c(V^1(N))) \cap \mathbb{R}_+$ is convex and symmetric across substitute players. Therefore, by Lemma 5,

$$z^* \in d(L^*) \cap E^T \subset d(L) = c(V^1(N)).$$

Moreover

$$x^* \in K^* + c(V^1(N)) \subset V^1(N);$$

that is, x^* is feasible in the modified game. We also have $x^* \in C_\varepsilon(\lambda)$. It follows from Lemma 3 that $x^*_a \geq \lambda - \varepsilon$. It now follows from the definition of K and K^* that $x^* \in K^* \subset V(N)$; that is, x^* is feasible in the initial game $(N; V)$. We have now proven that x^* is in the equal treatment ε-core of the initial game; therefore the equal treatment ε-core is non-empty. ■

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