Epsilon cores of games and economies with limited side payments.[*]

Abstract

We introduce the concept of a parameterized collection of games with limited side payments, ruling out large transfers of utility. Under the assumption that the payoff set of the grand coalition is convex, we show that a game with limited side payments has a nonempty ϵ-core. Our main result is that, when some degree of side-paymentness within nearly-effective small groups is assumed, then all payoffs in the ϵ-core treat similar players similarly. A bound on the distance between ϵ-core payoffs of any two similar players is given in

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terms of the parameters describing the game. These results add to the literature showing that games with many players and small e®ective groups have the properties of competitive markets.

1 Introduction.

In many game theoretic and economic situations side payments may be freely made only within small groups of players. For example, marriage games may permit some side-paymentness between matched pairs of players but may rule out side payments between marriages. Even if side payments were possible it may require cooperation and agreement among a large numbers of players to make large transfers of payo® and such agreements may be di±cult to obtain.[1] In this paper we introduce the concept of games with limited side payments and demonstrate conditions under which "-cores are nonempty and all payo®s in the "-core of a game treat similar players nearly equally. The "-core consists of those feasible payo®s that cannot be improved upon by any coalition by at least " for each member of the coalition. Side payments are limited in the sense that coalitions bounded in size are e®ective for achievement of almost all gains to increasing coalition size and large transfers of payo® are ruled out. We show that games with limited side payments and convex total payo® sets have nonempty "-cores, where for games with many players, " can be chosen to be small. Our main result is that any "-core payo® has approximately the equal-treatment property; that is, all players who are substitutes or approximate substitutes are treated nearly equally.

Our interest in approximate cores stems from our interest in competition in diverse economic settings, such as those with local public goods, clubs, production, location, indivisibilities, non-monotonicities, and other deviations from the classic Arrow-Debreu-McKenzie model. Both of our results contribute to a line of research investigating the \market-like" properties of large games under conditions limiting returns to group size, cf. Wooders (1983,1994b). An important precursor is Shapley and Shubik (1966), which shows that large exchange economies have non-empty approximate cores; another is Shapley and Shubik (1969), which demonstrates an equivalence between markets and totally balanced games. From an economic perspective, the remarkable paper of Tiebout (1956), in which he conjectured that large economies with local public goods were \market-like," contains insights similar to those that lead to the current research.[2]

In large games, the core is a \stand-in" for the competitive equilibrium.[3] Thus,

[1] This point is stressed in Kaneko and Wooders (1997).

[2] See Wooders (1997) and references therein.

[3] See, for example, see Shapley and Shubik (1969) and Wooders (1994a,b) for market-games with side payments. Also relevant are Billera (1974) and Mas-Colell (1975). Hildenbrand and Kirman (1988, Chapter 4), among others, treat the core more explicitly as the stand in for the competitive

a crucial result is that in large games cores or approximate cores are nonempty. Since the competitive equilibrium has the equal-treatment property, for games to approximate competitive markets it is also crucial that approximate cores converge to equal-treatment payo®s. Our result showing "-cores treat similar players similarly demonstrate that the equal treatment property of competitive equilibrium extends to approximate cores of large games with small e®ective groups and limited side payments.

Our results apply to parameterized collections of games. The parameters describing the games are: (a) a bound on the number of approximate player types and the accuracy of the approximation and (b) a bound on the size of nearly e®ective groups of players and their distance from exact e®ectiveness. Two players are approximately the same types if they make similar contributions to coalitions; that is, they are approximate substitutes. That nearly-e®ective groups are bounded in size means that all or almost all gains to collective activities can be realized by coalitions no larger than the bound. This can be viewed as a sort of \small group e®ectiveness" assumption.

To place our research in the context of the literature, we remark that prior to the introduction of parameterized collections of games, other papers showing nonemptiness of approximate cores of large games relied on the construct of a pregame.[4] Recall that a pregame consists of a set of player types and a single worth function assigning a payo® to any group of players. The payo® to a group of players depends on the numbers of players of each type in the group. Given a total player set and a speci¯cation of the numbers of players of each type in the set, the worth function of the pregame induces a worth function on the player set and thus determines a game. The pregame structure, which requires that all games considered are subgames of some larger game, has hidden consequences. To illustrate, in the pregame framework, when there are su±ciently many players of each type that appears in the games, the assumptions of small group e®ectiveness and per capita boundedness are equivalent (Wooders 1994b, Section 5); this does not hold for parameterized collections of games. In addition, the pregame framework dictates that the payo® to a group is independent of the player set in which it's embedded; thus, widespread externalities are ruled out.

Results in the literature on convergence of approximate cores to equal-treatment payo®s have also relied on the pregame structure. In the context of a pregame with side payments, when there are many close substitutes for each player, under the mild condition of per capita boundedness of average payo®, "-cores of large games with side payments are nonempty and approximately equal-treatment in the sense that most players of the same type receive nearly the same payo®s (Wooders 1980,1994a)[5]. In

equilibrium in non-classical exchange economies.

[4] See, for example, Wooders (1983,1994a,b) and Wooders and Zame (1984).

[5] See also Shubik and Wooders (1982), who apply the game-theoretic results of Wooders (1980) to economies with clubs and multiple memberships and provide a discussion of related literature.

large games with strictly effective small groups – that is, all gains to cooperation can be realized by groups bounded in size – and with strongly comprehensive payoff sets, all payoffs in the core have the equal treatment property (Wooders 1983, Theorem 3).

Since the games we consider in the current paper have limited side payments we are able to obtain a result close to Wooders' (1983) result for games with strictly effective small groups. Thus, our equal-treatment result is stronger than prior results in three important ways: First, although it is not required that small groups are strictly effective, we demonstrate that all players of the same approximate type are treated approximately equally. Second, when there are sufficiently many players of an approximate type, our result provides an explicit bound on the distance between the ϵ-core payoffs for any two players of that approximate type. Third, our results are not restricted to games with transferable utility nor to games derived from pregames.

In the next section we introduce required definitions. Section 3 states our nonemptiness result and provides a number of examples. Section 4 introduces games with "side-paymentness" uniformly bounded away from zero only for nearly-effective small groups, states our result that the ϵ-core is approximately equal-treatment, and provides additional examples. Section 5 provides some remarks on our model and results and relates our work to prior literature. Section 6 discusses related economies with limited side payments. Proofs are collected in Section 7 and two additional examples appear in an appendix.

2 Definitions.

2.1 Cooperative games: description and notations.

Let $N = \{1, \ldots, n\}$ denote a set of players. A nonempty subset of N is called a coalition. For any coalition S let $\mathbb{R}^S$ denote the $|S|$-dimensional Euclidean space with coordinates indexed by elements of S. For $x \in \mathbb{R}^N$, x_S will denote its restriction to $\mathbb{R}^S$. To order vectors in $\mathbb{R}^S$ we use the symbols $\gg$, $>$ and $\geq$ with their usual interpretations. The nonnegative orthant of $\mathbb{R}^S$ is denoted by $\mathbb{R}^S_+$ and the strictly positive orthant by $\mathbb{R}^S_{++}$. For $A \subset \mathbb{R}^S$, $co(A)$ denotes the convex hull of A. We denote by $1\!\!1_S$ the vector of ones in $\mathbb{R}^S$, that is, $1\!\!1_S = (1, \ldots, 1) \in \mathbb{R}^S$. Each coalition S has a feasible set of payoffs or utilities denoted by $V_S \subset \mathbb{R}^S$. By agreement $V_\emptyset = \{0\}$ and $V_{\{i\}}$ is nonempty, closed and bounded from above for any i. Moreover we will

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The results of Wooders (1980) should not be confused with other results published the same year in Econometrica, dealing with local public good (or club) economies with anonymous crowding. In that context, coalitions can be restricted to consist of identical individuals and the core will be unchanged; game-theoretically this is a very special case. In contrast, Wooders (1980), referenced herein, treats the general case of pregames with a finite number of types and requires only boundedness of per capita payoffs.

assume that

$$(*)\quad \max\left\{x : x \in V_{\{i\}}\right\} = 0 \text{ for any } i \in N.$$

It must be noted that $(*)$ is by no means restrictive since it can always be achieved by a normalization. It is convenient to describe the feasible utilities of a coalition as a subset of $\mathbb{R}^N$. For each coalition S let

$$V(S) := \left\{x \in \mathbb{R}^N : x_S \in V_S \text{ and } x_a = 0 \text{ for } a \notin S\right\}.$$

A game without side payments (called also an NTU game or simply a game) is a pair (N, V) where the correspondence $V : 2^N \to \mathbb{R}^N$ is such that $V(S) \subset \left\{x \in \mathbb{R}^N : x_a = 0 \text{ for } a \notin S\right\}$ for any $S \subset N$ and satisfies the following properties :

(2.1) $V(S)$ is non-empty and closed for all $S \subset N$.

(2.2) $V(S) \cap \mathbb{R}^N_+$ is bounded for all $S \subset N$, in the sense that there is real number $K > 0$ such that if $x \in V(S) \cap \mathbb{R}^N_+$ then $x_i \le K$ for all $i \in S$.

(2.3) $V(S_1 \bigcup S_2) \supset V(S_1) + V(S_2)$ for any disjoint $S_1, S_2 \subset N$ (superadditivity).

(2.4) $V(S) = V(S) - \left\{x \in \mathbb{R}^N_+ : x_a = 0 \text{ for } a \notin S\right\}$ for all $S \subset N$ (comprehensiveness).

2.2 Games with limited side payments.

To define parameterized collections of games we will need the concept of Hausdorff distance. For every two nonempty subsets E and F of a metric space (M, d) define the Hausdorff distance between E and F $\operatorname{dist}(E, F)$ (with respect to the metric d on M) by

$$\operatorname{dist}(E, F) := \inf\{\epsilon \in (0, \infty) : E \subset B_\epsilon(F) \text{ and } F \subset B_\epsilon(E)\},$$

where $B_\epsilon(E) := \{x \in M : d(x, E) \le \epsilon\}$ denotes the ϵ-neighborhood of E.

Since payoff sets are unbounded below, we will use a modification of the concept of the Hausdorff distance so that the distance between two payoff sets is the distance between the intersection of the sets and a subset of Euclidean space. Let m^* be a fixed positive real number. Let M^* be a subset of Euclidean space $\mathbb{R}^N$ defined by $M^* := \left\{x \in \mathbb{R}^N : x_a \ge -m^* \text{ for any } a \in N\right\}$. For every two non-empty subsets E and F of Euclidean space $\mathbb{R}^N$ let $H_\infty[E, F]$ denote the Hausdorff distance between $E \cap M^*$ and $F \cap M^*$ with respect to the metric $\|x - y\|_\infty := \max_i |x_i - y_i|$ on Euclidean space $\mathbb{R}^N$.

The concepts defined below lead to the definition of parameterized collections of games. To motivate the concepts, each is related to analogous concepts in the pregame framework.

$\delta-$substitute partitions: In our approach we approximate games with many players, all of whom may be distinct, by games with finite sets of player types. Observe that for a compact metric space of player types, given any real number $\delta > 0$ there is a partition (not necessarily unique) of the space of player types into a finite number of subsets, each containing players who are "δ-similar" to each other. Parameterized collections of games do not restrict to a compact metric space of player types, but do employ the idea of a finite number of approximate types.

Let $(N; V)$ be a game and let $\delta \geq 0$ be a non-negative real number. A δ-substitute partition is a partition of the player set N into subsets with the property that any two players in the same subset are "within δ" of being substitutes for each other. Formally, given a set $W \subset \mathbb{R}^N$ and a permutation τ of N, let $\sigma_\tau(W)$ denote the set formed from W by permuting the values of the coordinates according to the associated permutation τ. Given a partition $\{N[t] : t = 1, \ldots, T\}$ of N, a permutation τ of N is type $-$ preserving if, for any $i \in N$, $\tau(i)$ belongs to the same element of the partition $\{N[t]\}$ as i. A δ-substitute partition of N is a partition $\{N[t] : t = 1, \ldots, T\}$ of N with the property that, for any type-preserving permutation τ and any coalition S,

$$H_1\left[V(S), \sigma_\tau^{-1}(V(\tau(S)))\right] \leq \delta.$$

Note that in general a δ-substitute partition of N is not uniquely determined. Moreover, two games may have the same partitions but have no other relationship to each other (in contrast to games derived from a pregame).

(δ,T)- type games. The notion of a (δ,T)-type game is an extension of the notion of a game with a finite number of types to a game with approximate types.

Let δ be a non-negative real number and let T be a positive integer. A game $(N; V)$ is a $(\delta; T)$-type game if there is a T-member δ-substitute partition $\{N[t] : t = 1, \ldots, T\}$ of N. The set $N[t]$ is interpreted as an approximate type. Players in the same element of a δ-substitute partition are δ-substitutes. When $\delta = 0$, they are exact substitutes.

$\beta -$ effective $B -$ bounded groups: In all studies of approximate cores of large games, some conditions are required to limit gains to collective activities, such as boundedness of marginal contributions to coalitions, as in Wooders and Zame (1984) or the less restrictive conditions of per capita boundedness and/or small group effectiveness, as in Wooders (1980,1983,1994a,b), for example. Small groups are effective if all or almost all gains to collective activities can be realized by groups bounded in size of membership. The following notion formulates the idea of small effective groups in the context of parameterized collections of games.

Informally, groups of players containing no more than B members are β-effective if, by restricting coalitions to having fewer than B members, the loss to each player is no more than β: This is a form of "small group effectiveness" for arbitrary games. Let $(N;V)$ be a game. Let $\beta \geq 0$ be a given non-negative real number and let B be a given positive integer. For each group $S \subset N$; define a corresponding set $V(S;B) \subset \mathbb{R}^N$ in the following way:

$$V(S;B) := \bigcup \left[\sum_k V(S^k) : \left\{ S^k \right\} \text{ is a partition of } S, \left| S^k \right| \leq B \right].$$

The set $V(S;B)$ is the payoff set of the coalition S when groups are restricted to have no more than B members. Note that, by superadditivity, $V(S;B) \subset V(S)$ for any $S \subset N$ and, by construction, $V(S;B) = V(S)$ for $|S| \leq B$. The game $(N;V)$ has β-effective B-bounded groups if for every group $S \subset N$

$$H_\infty[V(S);V(S;B)] \leq \beta.$$

When $\beta = 0$, 0-effective B-bounded groups are called strictly effective B-bounded groups.

games with limited side payments $G((\delta;T);(\beta;B))$. Let T and B be positive integers. Let $G((\delta;T);(\beta;B))$ be the collection of all $(\delta;T)$-type games that have β-effective B-bounded groups. Then $G((\delta;T);(\beta;B))$ is called a parameterized collection of games with limited side payments.

Our results hold for all parameters δ and β that are sufficiently small, specifically, for $2(\delta+\beta) < m^*$; where m^* is a positive real number used in the definition of the Hausdorff distance. In Section 4 we introduce one additional parameter to describe subcollections of games with limited side payments.

It may not be immediately apparent how side payments are limited by the above definition. To illustrate this feature, consider a game $(N^m;V^m)$ in which any two players can earn two dollars and side payments are unrestricted. Suppose there are $2m$ players in the game. Thus the total payoff to the grand coalition is $2m$ dollars. Although at first glance it may seem that this game has β-effective 2-bounded groups, it does not. Moreover, given any $\beta \geq 0$ and any positive integer B; for sufficiently large m the game $(N^m;V^m)$ has no β-effective B-bounded groups. To make this clear, consider a payoff x that gives $2m$ dollars to one selected player and nothing to the others. The payoff x is feasible in the game, that is $x \in V(N)$: But for $2m > B$, $x \notin V(N;B)$, defined above. Moreover, for large m, the Hausdorff distance H_∞ from x to $V(N;B)$ is large.: Thus, the form of β-effective B-bounded groups used in the current paper rules out games with side payments.[6] In fact, similar examples will show that q-comprehensiveness of payoff sets for all coalitions with $q > 0$ are ruled out by the definition of β-effective B-bounded groups introduced in this paper. This motivates our use of the term "limited side payments."

[6] The notion of small group effectiveness used in our prior papers permitted games with side payments.

3 Non-emptiness of equal-treatment ε-cores of games with limited side payments.

<u>the core and epsilon cores.</u> Let $(N;V)$ be a game. A payoff x is ε-undominated if for all $S \subset N$ and $y \in V(S)$ it is not the case that $y_S \gg x_S + \vec{1}_S\varepsilon$. The payoff x is feasible if $x \in V(N)$. The ε-core of a game $(N;V)$ consists of all feasible and ε-undominated allocations. When $\varepsilon = 0$, the ε-core is the core.

<u>the equal treatment epsilon core.</u> Given non-negative real numbers ε and δ, we will define the equal treatment ε-core of a game $(N;V)$ relative to a partition $\{N[t]\}$ of the player set into δ-substitutes as the set of payoffs x in the ε-core with the property that for each t and all i and j in $N[t]$, it holds that $x_i = x_j$. Thus, a payoff for a game is in the equal-treatment ε-core if all players of the same approximate type are assigned equal payoffs.

3.1 The Theorem.

Some additional restrictions are required to obtain non-emptiness of epsilon cores of large games with limited side payments. We will discuss the indispensability of these conditions in the next subsection.

<u>per capita boundedness.</u> Let C be a positive real number. A game $(N;V)$ has a per capita payoff bound of C if, for all coalitions $S \subset N$,

$$\sum_{a \in S} x_a \le C|S| \text{ for any } x \in V(S).$$

<u>thickness.</u> Let ρ be a positive real number, $0 < \rho \le 1$. A $(\delta;T)$-type game $(N;V)$ is ρ-thick if for each t it holds that $|N[t]| \ge \rho|N|$.

Our first Theorem demonstrates that, with convexity of payoff sets of grand coalitions, all sufficiently large games in a parameterized collection have non-empty equal-treatment ε-cores. The size of a game required to ensure non-emptiness of the equal-treatment ε-core depends on the parameters describing the games, a bound on feasible per capita payoffs, and a bound on the thickness (the percentage of players of each approximate type) of the player set.

Theorem 1. Let T and B be positive integers. Let C and ρ be positive real numbers, $0 < \rho \le 1$. Then for any $\varepsilon > 0$ exists an integer $\eta(\varepsilon;T;B;C;\rho)$ such that if
(a) $(N;V) \in G((\delta;T);(\beta;B))$,
(b) $V(N)$ is convex,
(c) $(N;V)$ is ρ-thick,

(d) $(N;V)$ has a per capita payoff bound of C, and

(e) $|N| \geq \eta(\varepsilon;T;B;C;\rho)$

then the equal treatment $(\varepsilon+\delta+\beta)$-core of $(N;V)$ is nonempty.

The proof of the Theorem uses the following definition of the ε-remainder core and another nonemptiness result. Informally the ε-remainder core allows some small proportion of agents to be ignored. A payoff is in the ε-remainder core if it is feasible and if it is in the core of a subgame whose player set contains all the remaining players. The ε-remainder core is defined as follows.

<u>the ε-remainder core.</u> Let $(N;V)$ be a game. A payoff x belongs to the ε-remainder core if, for some group $S \subset N$, $\frac{|N|-|S|}{|N|} \leq \varepsilon$ and x_S belongs to the core of the subgame $(S;V)$:

<u>non-emptiness of the ε-remainder core.</u> (Kovalenkov and Wooders 1999, Theorem 1 (subcase for $q=0$)). Let T and B be positive integers. For any $\varepsilon > 0$; there exists an integer $\eta_1(\varepsilon;T;B)$ such that if

(a) $(N;V) \in G((0;T);(0;B))$ and

(b) $|N| \geq \eta_1(\varepsilon;T;B)$

then the ε-remainder core of $(N;V)$ is non-empty.

A sketch of the proof: In the proof of Theorem 1 we first approximate a game $(N;V)$ with T approximate types and β-effective B-bounded groups by another game $(N;V^0)$ with T exact player types and strictly effective B-bounded groups, that is, by a game $(N;V^0) \in G((0;T);(0;B))$: Then we use the above result on nonemptiness of the ε-remainder core to obtain non-emptiness of the ε-remainder core of $(N;V^0)$. This result implies that there is a core payoff in a large subgame $(S;V)$ and that the proportion of "leftover" players is bounded by ε. Then using per capita boundedness and thickness of the player set, we can "smooth" payoffs in the ε-remainder core across players of the same approximate types and thus compensate leftovers. The convexity assumption allows us to obtain equal treatment by averaging the payoffs across players of the same type while maintaining feasibility. We present the formal proof of Theorem 1 in the special section. ■

In the next subsection we present examples clarifying Theorem 1.

3.2 Examples.

Our first example illustrates the statement of Theorem 1.

Example 1: Let $(N;V_0)$ be a superadditive game where, for any two-person coalition $S=\{i;j\}$; $j \neq i$;

$$V_0(S) := \{x \in \mathbb{R}^N : x_i \leq 1;\ x_j \leq 1; \text{and } x_k = 0 \text{ for } k \neq i;j\}$$

and for each $i \in N$,

$$V_0(\{i\}) := \{x \in \mathbb{R}^N : x_i \le 0 \text{ and } x_j = 0 \text{ for all } j \neq i\}.$$

For an arbitrary coalition S the payoff set $V_0(S)$ is given as the superadditive cover, that is,

$$V_0(S) := \bigcup_{P(S)} \sum_{S' \in P(S)} V_0(S');$$

where the union is taken over all partitions $P(S)$ of S.

We now define a game $(N; V_{co})$ in the following way. For each $S \subset N$, let $V_{co}(S)$ denote the convex cover of the payoff set $V_0(S)$, that is

$$V_{co}(S) := co(V_0(S)).$$

Obviously the game $(N; V_{co})$ has convex payoff sets, one player type, and per capita bound of 1. We leave it to the reader to verify that for any positive integer $m \ge 3$ the game $(N; V_{co})$ has $\frac{1}{m}$-effective m-bounded groups. Thus the game $(N; V_{co})$ is a member of the class $G((0; 1); (\frac{1}{m}; m))$ and is 1-thick. Then, by Theorem 1, for $|N| \ge \eta(\epsilon; 1; m; 1; 1)$ the equal treatment $(\epsilon + \frac{1}{m})$-core of $(N; V_{co})$ is nonempty.

Therefore for any $\epsilon' > 0$ there is a positive integer $N(\epsilon')$ such that for any $|N| \ge N(\epsilon')$ the game $(N; V_{co})$ has a non-empty equal treatment ϵ'-core. (For example take an integer $m' \ge \frac{2}{\epsilon'}$ and define $N(\epsilon') \ge \eta(\frac{\epsilon'}{2}; 1; m'; 1; 1)$.)

The next two examples illustrates that neither convexity of $V(N)$ nor thickness of the game can be omitted in the statement of Theorem 1.

Example 2. Convexity of $V(N)$: Consider a sequence of games without side payments $(N^m; V^m)_{m=1}^{\infty}$ where the m^{th} game has $2m + 1$ players. Suppose that any player alone can earn at most 0 units. Suppose that any two-player coalition can distribute a total payoff of 2 units in any agreed-upon way, while there is no transferability of payoff between coalitions. Suppose only one- and two-player coalitions are effective. Then the game is described by the following parameters: $T = 1$; $\rho = 1$; $B = 2$; $\delta = \beta = 0$ and has per capita bound $C = 1$. Thus the game satisfies all the assumptions of Theorem 1 except convexity of $V(N)$. The $\frac{1}{3}$-core of the game, however, is empty for arbitrarily large values of m. (At any feasible payoff, at least one player gets 0 units and some other player no more than 1 unit. These two players can form a coalition and receive 2 units in total. This coalition can improve upon the given payoff for each of its members by $\frac{1}{2}$.)

Example 3. Thickness. Consider a sequence of games without side payments $(N^m; V^m)_{m=1}^{\infty}$ where the m^{th} game has $m + 3$ players. Suppose that there are two types of players. Let the m^{th} game have 3 players of type A and m players of type B. Assume that only players of the type A are essential in the following sense: Any coalition with less than two players of type A can get only 0 units or less for each of its players. Suppose that any coalition with two or three players of type A can get only 2 units to be distributed between players of type A in the coalition and 0 units or less for each of its players of type B. Note that there is no transferability of payoff from the players of type B to the players of type A. Then the game is described by the following parameters: $T = 2$, $B = 3$, $\delta = \beta = 0$, $C = 1$. Moreover, $V^m(N)$ is convex. The $\frac{1}{7}$-core of the game, however, is empty for arbitrarily large values of m. (For any feasible payoff there is a player of type A who receives no more than $\frac{2}{3}$ units. There is another player of type A that gets no more than 1 unit. These 2 players can form a coalition and receive 2 units in total. This coalition can improve upon the given payoff for each of its members by $\frac{1}{6}$, since $(2 - \frac{5}{3})\frac{1}{2} = \frac{1}{6}$.)

Remark. Per capita boundedness and small group effectiveness. Recall that for large games with side payments derived from pregames, Wooders (1994b) establishes an equivalence between small group effectiveness and per capita boundedness.[7] Within the framework of parametrized collections of games, this equivalence no longer holds. This raises the question of whether either one or the other of the conditions can be dropped. Example A1, in an appendix, shows that the condition of small group effectiveness cannot be disregarded. The necessity of the assumption of per capita boundedness is not clear. Example A2 in the appendix illustrates that per capita boundedness is not a consequence of the other conditions in Theorem 1 but we have no example showing per capita boundedness is indispensible. Note, however, that per capita boundedness is a very mild assumption satisfied by most economic examples.

4 The equal treatment property of ε-cores of games with limited side payments.

In the previous Section we presented results on non-emptiness of the ε-cores for large games with limited side payments. In fact, Theorem 1 shows non-emptiness of the equal-treatment ε-core as well. In the current Section we focus on games where some degree of side payments is required for small groups. Specifically, we require q-comprehensiveness for nearly-effective groups. We then demonstrate that under this condition, the ε-core of a game with limited side payments is equal-treatment in a very strong sense: any payoff in this ε-core treats all players that are approximate substitutes nearly equally.

[7] The games must be large in the sense that any player that appears in the game must have many close substitutes.

4.1 The Theorem.

Consider a set $W \subset \mathbb{R}^S$. Recall that W is comprehensive if $x \in W$ and $y \le x$ implies $y \in W$. The set W is strongly comprehensive if it is comprehensive, and whenever $x \in W$, $y \in W$, and $x < y$ there exists $z \in W$ such that $x << z$.[8]

Our second Theorem requires a uniform version of strong comprehensiveness. Given $x \in \mathbb{R}^S$, $i, j \in S$, $0 \le q \le 1$ and $\varepsilon \ge 0$, define a vector $x^q_{i,j}(\varepsilon) \in \mathbb{R}^S$, by

$$
\begin{aligned}
(x^q_{i,j}(\varepsilon))_i &= x_i - \varepsilon; \\
(x^q_{i,j}(\varepsilon))_j &= x_j + q\varepsilon; \text{ and} \\
(x^q_{i,j}(\varepsilon))_k &= x_k \text{ for } k \in S \setminus \{i, j\}.
\end{aligned}
$$

The set W is q-comprehensive if W is comprehensive and if, for any $x \in W$, it holds that $(x^q_{i,j}(\varepsilon)) \in W$ for any $i, j \in S$ and any $\varepsilon \ge 0$.

This condition for $q > 0$ bounds the slope of the Pareto frontier of the set away from zero. For q equal to one, q-comprehensiveness is equivalent to the assumption of transferable utility. For q equal to zero, q-comprehensiveness means simply comprehensiveness. Note that if a set is q-comprehensive for some $q > 0$ then the set is q'-comprehensive for all q' with $q \ge q' \ge 0$.

<u>games with side payments limited to nearly-effective groups $G((\delta,T),(\beta,B,q))$.</u> Let $G((\delta,T),(\beta,B))$ be a parameterized collection games with limited side payments that was defined in Section 2. Let $0 \le q \le 1$ be given. Let $G((\delta,T),(\beta,B,q))$ denote the subcollection of games in $G((\delta,T),(\beta,B))$ with the property that if (N,V) is a member of the subcollection then for all $S \subset N$ with $|S| \le B$, V_S is q-comprehensive. Note that

$$
G((\delta,T),(\beta,B,q)) \subset G((\delta,T),(\beta,B,q')) \subset G((\delta,T),(\beta,B,0)) = G((\delta,T),(\beta,B))
$$

for all q and q' such that $q \ge q' \ge 0$. When $q > 0$ will call this collection of games, games with side payments limited to nearly-effective groups.

Our main Theorem requires some degree of side-paymentness for small coalitions – those coalitions containing no more members than the given bound on nearly effective coalition sizes. Payoff sets for nearly-effective coalitions are required to satisfy q-comprehensiveness for some $q > 0$; that is, side payments can be made between players in nearly effective coalitions at a rate bounded below by q. For large coalitions, however, q can be arbitrarily small or zero. The Theorem demonstrates that if a game with limited side payments has sufficiently many players of any particular

[8] Informally, if one person can be made better off (while all the others remain at least as well off), then all persons can be made better off. This assumption, called "quasi-transferable utility" in Wooders (1983) has also been called "nonleveledness."

approximate type, then all payoffs in the ε-core treat all members of that type nearly equally – that is, any two players of such an approximate type receive approximately the same ε-core payoffs. The distance between ε-core payoffs of any of the two players is bounded above by a function of the parameters describing the game. From inspection of the bound, we can conclude that as approximate types become close to exact types, nearly effective groups become strictly effective and ε goes to zero, the ε-core "shrinks" to the equal-treatment ε-core.

Theorem 2. Let $(N; V) \in G((\delta; T); (\beta; B; q))$ where $q > 0$ and let $\{N[t]\}$ be a partition of N into δ-substitutes. If some payoff x belongs to the ε-core of $(N; V)$ and for some t it holds that $|N[t]| > B$, then for any $t_1; t_2 \in N[t]$,

$$|x_{t_1} - x_{t_2}| \leq \frac{2B}{q}(\varepsilon + \delta + \beta):$$

The above theorem implies that, as ε; δ and β "became small," providing there are more than B players of each type, the ε-core "shrinks" to the equal treatment ε-core. It is important to bear in mind in interpreting the theorem that for large groups, q-comprehensiveness is not assumed. As noted previously, we have in mind situations such as economies with many small clubs or marriages, for example, where transfers may be made within the group, but large transfers, which would require the cooperation of large groups of players, are ruled out.

We present the proof of Theorem 2 in the special section. In the next subsection we present examples clarifying Theorem 2.

4.2 Examples.

The following simple example illustrates the statement of Theorem 2 and also shows that, in the central case of $\delta = \beta = 0$; the bound provided can not be improved upon.

Example 4: Let $(N; V^*)$ be a superadditive game where for any two-person coalition $S = \{i; j\}$; $j \neq i$;

$$V^*(S) := \{x \in \mathbb{R}^N : x_i + x_j \leq 2; \text{and } x_k = 0 \text{ for } k \notin \{i; j\}\}$$

and for each $i \in N$, let

$$V^*(\{i\}) := \{x \in \mathbb{R}^N : x_i \leq 0 \text{ and } x_j = 0 \text{ for all } j \neq i\}:$$

For an arbitrary coalition S the payoff set $V^*(S)$ is given as the superadditive cover, that is,

$$V^*(S) := \bigcup_{P(S)} \sum_{S' \in P(S)} V^*(S');$$

where the union is taken over all partitions $P(S)$ of S:

Observe that the game $(N;V^*)$ has one exact player type and strictly effective 2-bounded groups. Therefore $(N;V^*) \in G((0;1);(0;2))$: Moreover V^*_S is 1-comprehensive for any $S \subset N$, $|S| \leq 2$. Let the game $(N;V^*)$ have an even number of players, that is $|N| = 2m$ for some positive integer m. Observe that the payoff which gives one unit for each player belongs to the core of $(N;V^*)$. Thus, for any $\varepsilon \geq 0$; the ε-core of $(N;V^*)$ is non-empty. Theorem 2 implies that for any $a;b \in N$ and any payoff x in the ε-core of $(N;V^*)$, providing that $|N| > 2$; we must have

$$|x_a - x_b| \leq 4\varepsilon:$$

It is easy to check that this statement can not be improved. Observe first that the restriction $|N| > 2$ is essential. If $|N| = 2$ then any division of two units between two players is a core payoff; thus, for $\varepsilon < \frac{1}{2}$ the bound of the Theorem is not applicable. Now consider a payoff y that gives 1 unit to each of the players except two selected players (say players a and b), $1 - 2\varepsilon$ to player a, and $1 + 2\varepsilon$ to player b. The reader can observe that even in the case $|N| > 2$ this payoff is in the ε-core of the game $(N;V^*)$. (The payoff is obviously feasible, but it is also ε-undominated, since any pair of players get at y at least $2 - 2\varepsilon$.) But at this payoff it holds that $|y_a - y_b| = 4\varepsilon$. Thus the bound given in Theorem 2 cannot be improved.

It is well known that a Theorem such as the above will not hold for games with side payments. The following example provides an illustration.

Example 5a: Fix any $\varepsilon > 0$: Consider a game with side payments $(N;v)$ that has one exact player type and a nonempty equal-treatment core. (There are many such games.) Take any payoff $x \in \mathbb{R}^N$ in the equal-treatment core of $(N;v)$. This determines a number $\overline{x}$ as the payoff for each player in N: Now select one player (say player a). Consider the payoff y that assigns $\overline{x} - \varepsilon$ for all players except player a and $\overline{x} + (|N| - 1)\varepsilon$ to player a. Obviously the payoff y is feasible and ε-undominated. Thus y belongs to the ε-core of $(N;v)$: However for a very large $|N|$; y treats player a much better than others.

The sharpest result that can be obtained for games with side payments induced by pregames satisfying per capita boundedness (or even strict small group effectiveness) is that for small values of ε, most players of the same type are treated approximately equally (Wooders 1980,1994a). For games without side payments that have q-comprehensive payoff sets for $q > 0$ (q-sidepayments) it also does not hold that the ε-core treats all players of the same type approximately equally. The following example, quite similar to the previous one, demonstrates this point.

Example 5b: Fix any $\varepsilon > 0$, $q > 0$: Consider a game $(N;V)$ with q-comprehensive payoff sets, one exact player type and a nonempty equal-treatment core. (As above, there are many such games.) Take any payoff in the equal-treatment core of $(N;V)$ and define $\bar{x}$ as the payoff assigned to each player in N. Now select one player, say a. Consider the payoff y^q that assigns $\bar{x} - \varepsilon$ for all players except player a and $\bar{x} + q(|N| - 1)\varepsilon$ for a player a. Then the payoff y^q is feasible and ε-undominated. Thus y^q belongs to the ε-core of $(N;V)$. For very large $|N|$, however, y^q treats one player much better than the others.

The above examples show that Theorem 2 depends crucially on the fact that side payments are limited. The other condition of Theorem 2, q-comprehensiveness ($q > 0$) for payoff sets of small coalitions, is also crucial for equal-treatment property of the ε-core. Example 6 below, which continues Example 1, shows that without this restriction the ε-core may contain non-equal-treatment payoffs, even for games with limited side payments.

Example 6: Recall the game $(N;V_0)$ defined in Example 1. Observe that the game $(N;V_0)$ has one exact player type, strictly effective 2-bounded groups, and a per capita bound of 1. Thus, $(N;V_0) \in G((0;1);(0;2))$. But, for any $q > 0$ and any coalition S of two players; the payoff sets $V(S)$ are not q-comprehensive. Let m be a positive integer. Let $(N^m;V_0^m)$ be a game where the number of players in the set N^m is $2m + 1$ and for any coalition $S \subset N^m$ $V_0^m(S) := V_0(S)$. Thus, each game $(N^m;V_0^m)$ has an odd number of players.

It is easy to see that the payoff y giving 1 to each of $2m$ players and 0 to the last player (say player a) is in the core. (The "left-out" player, in a coalition by himself, cannot make both himself and a player in a two-person coalition better off – the player in the two-person coalition cannot be given more than 1.) However for any player other than a; say for player b; we have $y_b = 1$. Thus $|y_a - y_b| = 1$; even though all players are exact substitutes and $\delta = \beta = 0$ but the core does not converge to the equal-treatment core, even for an arbitrarily large m.

5 Remarks.

Remark 1. Relationships to other papers using parameterized collections of games. Our prior results for the ε-core depended on a notion of q-comprehensiveness. In fact, all our previous results require that payoff sets be q-comprehensive with $q > 0$; the case $q = 0$ has not been treated. The main goal of the current paper is to treat the case where $q > 0$ only for small coalitions, thus allowing transfers within small effective coalitions but ruling out arbitrarily large transfers.

Remark 2. Small group effectiveness for improvement. There are several possibilities for defining β-effective B-bounded groups. Our definition requires that partitions of the total player set into B-bounded groups can realize almost all payoffs in $V(N)$: This approach has the advantage of ease of exposition. Another possibility is to require that all improvement can be carried out by groups bounded in size.

<u>β-improvement-effective B-bounded groups.</u> Let us say that a payoff x can be β-improved upon by some coalition S in a game $(N;V)$ if there is a payoff $y \in V(S)$ such that $y_S >> x_S - \beta 1_S$. Suppose that a game $(N;V)$ satisfies the property that for each payoff $x \in V(N)$, if x can be $\frac{\beta}{2}$-improved by some coalition then there is a coalition S satisfying $|S| \leq B$ that can β-improve upon x. Then $(N;V)$ has β-improvement-effective B-bounded groups.

The condition that a game has bounded group sizes that are effective for improvement is obviously less restrictive than the condition defined earlier of β-effective B-bounded groups. The conditions are, however, interchangeable.[9] The only difference in our results, were we to use β-effectiveness for improvement, would be that the bounds change. Our motivation in choosing the condition of β-effective B-bounded groups is that the condition is more easily verifiable since it only depends on what groups of individuals can do rather than on the entire game structure and it seems more consistent with the sort of examples most frequently used in the literature.[10]

Remark 3. Effectiveness of relatively small groups. It may be possible to obtain analogues of the results of this paper for situations where bounds are placed on relative sizes of effective coalitions rather than absolute sizes. We've chosen to bound the absolute sizes of near-effective coalitions since familiar examples of games and economies have natural bounds on absolute sizes of effective coalitions, for example, marriage models.

6 Economies with limited side payments.

In application, the conditions of our results may not be unduly restrictive. Indeed, in some game theoretic and economic situations, limited side payments would appear to be natural, for example, in job matching models (Kelso and Crawford (1982), Roth and Sotomayor (1990) and others). Even if side payments were possible, in large games it may require cooperation and agreement between a large number of

[9] See Wooders (1994a, Proposition 3.8) for the relationship between these two notions of effectiveness in the context of games derived from pregames. In other contexts, an example of the interchangeability of these conditions is given by Mas-Colell (1979) and Kaneko and Wooders (1989). Mas-Colell uses the fact that almost all improvement in exchange economies can be carried out by groups bounded in size while Kaneko and Wooders use the fact that almost all feasible outcomes can be achieved by partitions of the total player set into groups bounded in size.

[10] For example: marriages involve two persons; softball teams have nine members; and when average costs of production are downward sloping and bounded away from zero, there is some size of plant that is "almost efficient."

players to make large transfers. In this case the framework of side payments, or even q-comprehensiveness for $q > 0$, may be quite problematic. On the other hand, the presumption of economics that there exists opportunities for gains to trade and exchange is at the heart of economics. Thus, the requirement of some degree of side paymentness within small coalitions may not be restrictive.

There are also specific examples to which our results apply. Consider a two-sided matching model where all gains to collective activities can be realized by groups consisting of one person from each side of the market and where utility is transferable within buyer-seller pairs. If there is some mechanism convexifying the total payoff set (for example, lotteries), then both our nonemptiness result and our equal-treatment result apply. These remarks also apply to partitioning games.[11]

Another class of economic models whose derived games may well fit into our assumptions include economies with local public goods or clubs. (A survey is provided in Conley and Wooders 1998). Kovalenkov and Wooders (1999) introduce a model of an economy with clubs (and multiple memberships) and, under remarkably nonrestrictive conditions, demonstrate nonemptiness of approximate cores. With appropriate conditions on the economic models, the results of the current paper could be applied to models of economies with clubs.

7 Proofs.

Proof of Theorem 1: Recall that we required $2(\beta + \delta) < m^*$. Assume for now that $\varepsilon \leq \min\left\{\frac{m^*}{2}; BC\right\}$, implying that $(\varepsilon + \beta + \delta) < m^*$. We first consider the case where the game $(N;V)$ has strongly comprehensive payoff sets and then use the result for this case to obtain the result for the general case.

Case 1. Suppose the game $(N;V)$ has strongly comprehensive payoff sets. Define for each $S \subset N$

$$V'(S) := \bigcap \sigma_\tau^{-1}(V(\tau(S)));$$

where intersection is taken over all type-consistent permutations τ. Then $(N;V') \in G((0;T);(\beta;B))$ and $V'(N)$ is convex. Moreover, $V'(S) \subset V(S)$ and

$$H_1(V'(S);V(S)) \leq \delta$$

for any $S \subset N$. Now let us define for any $S \subset N$

$$V^0(S) := V'(S;B):$$

Then $(N;V^0) \in G((0;T);(0;B))$. Moreover,

$$V^0(S) \subset V'(S) \subset V(S)$$

[11] See Garratt and Qin (1997) for examples of games with lotteries, Kaneko (1982) for NTU assignment markets and Kaneko and Wooders (1982) and le Breton, Owen and Weber (1992) for partitioning games.

and

$$H_1(V^0(S);V(S)) \le H_1(V^0(S);V^0(S)) + H_1(V^0(S);V(S)) \le \beta+\delta.$$

In addition, the game $(N;V^0)$ has a per capita payoff bound of C and strongly comprehensive payoff sets.

Applying Kovalenkov and Wooders (1999, Theorem 1) for the ε^0-remainder core, with

$$\varepsilon^0 := \frac{\rho}{BC}\varepsilon;$$

to the game $(N;V^0)$ it follows that for $|N| \ge \eta_1(\varepsilon^0;T;B)$ there is some coalition S such that the subgame $(S;V^0)$ has a non-empty core and $\frac{|S|}{|N|} \ge 1-\varepsilon^0$. Let x be a payoff in the core of the subgame $(S;V^0)$. Note that $\rho > \varepsilon^0$.

We now prove that for $|N| \ge \frac{(B+1)}{\rho-\varepsilon^0}$ the payoff x will have the equal treatment property, that is, for every t and any $i,j \in N[t]$, $x_i = x_j$. Suppose there is a type t and two players $i_1, i_2 \in N[t] \cap S$ satisfying $x_{i_1} > x_{i_2}$. By feasibility of x, there is some partition $\left\{S^k\right\}$ of S, $|S^k| \le B$ for each k, such that $x \in \sum_k V^0(S^k)$. Then, since $|N| \ge \frac{(B+1)}{\rho-\varepsilon^0}$ we have that $|N[t] \cap S| \ge \rho|N| - \varepsilon^0|N| > B$ and there must exist $j_1, j_2 \in N[t] \cap S$ such that $x_{j_1} \in S^{k_1}, x_{j_2} \in S^{k_2}$ for $S^{k_1}, S^{k_2} \subset \left\{S^k\right\}$, $S^{k_1} \ne S^{k_2}$ and $x_{j_1} > x_{j_2}$. (If x_{i_1}, x_{i_2} are in the same element of $\left\{S^k\right\}$, there exists $j \in N[t] \bigcap S$ in the different element of $\left\{S^k\right\}$ than i_1, i_2 and it can't be $x_j = x_{i_1}$ and $x_j = x_{i_2}$ since $x_{i_1} \ne x_{i_2}$. Thus we can suppose without loss of generality that j_1 and j_2 are in different elements in $\left\{S^k\right\}$.) But then the agents of $S^{k_1}\backslash\{j_1\}$ can form the coalition $S^{k_1}\backslash\{j_2\}$ with player j_2 rather then j_1. By strong comprehensiveness of payoff sets and since players j_1, j_2 are exact substitutes in the game $(S;V^0)$, all players in the new coalition can be strictly better off than in x. This contradicts the supposition that x is a core payoff for the game $(S;V^0)$. Thus x has the equal treatment property.

Now let $\eta^0(\varepsilon;T;B;C;\rho) := \max\left\{\eta_1(\varepsilon^0;T;B), \frac{(B+1)}{\rho-\varepsilon^0}\right\}$, where, as above, $\varepsilon^0 = \frac{\rho}{BC}\varepsilon$. As we shown, for $|N| \ge \eta^0(\varepsilon;T;B;C;\rho)$ there is a coalition $S \subset N$ satisfying $\frac{|S|}{|N|} \ge 1-\varepsilon^0$ and, for each t, $|S \bigcap N[t]| > B$ and there is a payoff $x \in \mathbb{R}^S$ in the equal treatment core of $(S;V^0)$.

Next, define a payoff $z \in \mathbb{R}^N$ that, for each t, assigns to each $a \in N[t]$ the same payoff assigned to players of type t by x. This payoff z is undominated in the game $(N;V^0)$. (If payoff z were dominated by some coalition S_1 in $(N;V^0)$ then z could be dominated by some coalition $S_2 \subset S_1$, $|S_2| \le B$. But there exists a coalition $S_3 \subset S$ with the same profile as S_2. This coalition S_3 dominates the payoff x in $(S;V^0)$ contradicting to the fact that x is in the core of $(S;V^0)$.) Moreover, since $x \in \sum_k V^0(S^k)$ for some partition $\left\{S^k\right\}$ of S with $|S^k| \le B$, by per capita boundedness $x_a \le BC$ for any $a \in S$. Thus $z_a \le BC$ for any $a \in S$.

Now construct a payoff $x' \in R^N$ from the payoff for $x \in R^S$ by

$$x'_S := x_S \text{ and } x'_a := 0 \text{ for all } a \notin S.$$

Let $y \in R^N$ be the average of all payoffs $\sigma_\tau(x')$ across all type-consistent permutations τ of N. Then y has an equal treatment property. Moreover, for every t and any $a \in N[t]$

$$\begin{aligned} z_a - y_a &= z_a - \frac{|N[t] \bigcap S|}{|N[t]|} z_a \le \frac{|N[t]| - |N[t] \bigcap S|}{|N[t]|} z_a \\ &\le \frac{|N| - |S|}{|N[t]|} z_a \le \frac{\varepsilon'}{\rho} z_a \le \frac{\varepsilon'}{\rho} BC = \varepsilon. \end{aligned}$$

Since z is undominated in $(N; V')$, y is ε-undominated in $(N; V')$. Therefore for $|N| \ge \eta'(\varepsilon; T; B; C; \rho)$ the payoff y has equal treatment property and is $(\varepsilon + \beta + \delta)$-undominated in the original game $(N; V)$. In addition, observe that $x' \in V'(N) \subset V'(N)$. Therefore $\sigma_\tau(x') \in \mathring{V}(N)$ for any type-consistent permutation τ of N. By convexity of $\mathring{V}(N)$ and by construction of y, it holds that $y \in \mathring{V}(N) \subset V(N)$. Thus, for $|N| \ge \eta'(\varepsilon; T; B; C; \rho)$ the payoff y belongs to the equal treatment $(\varepsilon + \beta + \delta)$-core of $(N; V)$.

Case 2. Now we consider the general case, where the game $(N; V)$ does not necessarily have strongly comprehensive payoff sets. We first approximate the game $(N; V)$ by another game $(N; V')$ with strongly comprehensive payoff sets. This approximation can be done sufficiently closely, so that $V(S) \subset V'(S)$ and $H_1(V(S); V'(S)) \le \frac{\varepsilon}{2}$ for any $S \subset N$. Note that the approximation is easy since, by our definition of the Hausdorff distance, we must ensure close approximation only on a compact set. (For details of such an approximation see Wooders (1983), Appendix.) Let $\eta(\varepsilon; T; B; C; \rho) := \eta'(\frac{\varepsilon}{2}; T; B; C; \rho)$. By Case 1 above, for $|N| \ge \eta(\varepsilon; T; B; C; \rho)$ the equal treatment $(\frac{\varepsilon}{2} + \beta + \delta)$-core of the game $(N; V')$ is nonempty. But then for $|N| \ge \eta(\varepsilon; T; B; C; \rho)$ the game $(N; V)$ has a non-empty equal treatment $(\varepsilon + \beta + \delta)$-core. This is the conclusion we need for $\varepsilon \le \min\left\{\frac{m^*}{2}; BC\right\}$.

Finally for $\varepsilon > \min\left\{\frac{m^*}{2}; BC\right\}$ consider $\varepsilon' := \min\left\{\frac{m^*}{2}; BC\right\}$ and let $\eta(\varepsilon; T; B; C; \rho) := \eta(\varepsilon'; T; B; C; \rho)$. Since $\varepsilon \ge \varepsilon'$, for $|N| \ge \eta(\varepsilon; T; B; C; \rho)$ again the equal treatment $(\varepsilon + \beta + \delta)$-core of the game $(N; V)$ is nonempty. ■

Proof of Theorem 2: Consider some payoff vector x in the ε-core of $(N; V)$. Define a payoff vector y by $y(\{i\}) := x(\{i\}) - \beta$ for each $i \in N$. Since $(N; V)$ has β-effective B-bounded groups, there exists a partition $\left\{S^k\right\}$ of N, with $|S^k| \le B$ for each k, such that $y \in \sum_k V(S^k)$. Moreover, for the game $(N; V)$, by construction the payoff y cannot be improved upon by any coalition $S \subset N$ for each of it's member by more than $(\varepsilon + \beta)$.

Case 1. Consider the case, where according to the partition $\left\{S^k\right\}$, two players t_1 and t_2 of the same approximate type (that is, $t_1, t_2 \in N[t]$) are in different coalitions. Suppose $t_1 \in S^1$ and $t_2 \in S^2$. Suppose also that $y_{t_1} > y_{t_2}$. Then agents in $S^1 \setminus \{t_1\}$ would prefer to form a new coalition with t_2 rather then t_1. Let $S = S^1 \setminus \{t_1\}$. Let τ denote a permutation of N that permutes only t_1 and t_2. Since players t_1 and t_2 are of the same approximate type, τ is a type-consistent permutation and

$$H_1\left[V(S^1), \sigma_\tau(V(S \cup \{t_2\})\right] \le \delta.$$

Hence there exists a payoff z that is feasible for $S \bigcup \{t_2\}$ and close to the payoff y with the loss for any agent in $S \bigcup \{t_2\}$ of not more then δ. Since transfers at rate q are possible, this new coalition can improve upon the payoff vector y by more than $(\varepsilon + \beta)$ for each player if

$$q(|y_{t_1} - y_{t_2}| - (\varepsilon + \beta) - \delta) > |S|\delta + |S|(\varepsilon + \beta).$$

Since y is an $(\varepsilon + \beta)$-core payoff we must have: $|y_{t_1} - y_{t_2}| - (\varepsilon + \delta + \beta) \le \frac{1}{q}|S|(\varepsilon + \delta + \beta) \le \frac{1}{q}(B - 1)(\varepsilon + \delta + \beta)$. Hence, $|x_{t_1} - x_{t_2}| = |y_{t_1} - y_{t_2}| \le \frac{B}{q}(\varepsilon + \delta + \beta)$.

Case 2. Now suppose that t_1 and t_2 are in the same element of the partition $\{S^k\}$. Then, since $|N[t]| > B$, there exists $t_3 \in N[t]$ that belongs to a different member of the partition $\{S^k\}$. Hence by Case 1 above: $|x_{t_1} - x_{t_2}| \le |x_{t_1} - x_{t_3}| + |x_{t_3} - x_{t_2}| \le \frac{2B}{q}(\varepsilon + \delta + \beta)$. ■

8 Appendix: Additional examples.

In this Section we present two examples that were discussed in Section 3. Both illustrates the importance of the conditions in Theorem 1. Example A1 appeared previously in Kovalenkov and Wooders (1999, Example 4).

Example A1: The indispensability of the small group effectiveness assumption. Consider a sequence of games $(N^m, v^m)_{m=1}^{\infty}$ with side payments and where the m^{th} game has $3m$ players. Suppose that any coalition S consisting of at least $2m$ players can get up to $2m$ units of payoff to divide among its members, that is, $v^m(S) = 2m$. Assume that if $|S| < 2m$, then $v^m(S) = 0$. Observe that each game has one exact player type and a per capita bound of 1. That is, $\rho = 1, T = 1, C = 1$, and $\delta = 0$. However, the $\frac{1}{7}$-core of the game is empty for arbitrarily large values of m.

For any feasible payoff there are m players that are assigned in total no more than $\frac{2m}{3m}m = \frac{2}{3}m$. There are another m players that are assigned in total no more than $\frac{2m}{2m}m = m$. These $2m$ players can form a coalition and receive $2m$ in total. This coalition can improve upon the given payoff for each of its members by $\frac{1}{6}$, since $(2m - \frac{5}{3}m)\frac{1}{2m} = \frac{1}{6}$.

Example A2: The independence of the per capita boundedness assumption. Let m be an arbitrary positive integer. Let $(N; V^m)$ be a superadditive game where, for any coalition S, with $|S| > 1$,

$$V^m(S) := \{x \in \mathbb{R}^N : x_i \leq m \text{ for } i \in S; \text{ and } x_k = 0 \text{ for } k \notin S\}$$

and for each $i \in N$,

$$V^m(\{i\}) := \{x \in \mathbb{R}^N : x_i \leq 0 \text{ and } x_j = 0 \text{ for all } j \neq i\}.$$

Now let us define a collection of games

$$G^* := \{(N; V^m) : m \in \mathbb{Z}_+\}.$$

Obviously any game $(N; V) \in G^*$ has convex payoff sets and non-empty core. Moreover, any game $(N; V) \in G^*$ has one (exact) player type and thus is 1-thick. We leave it to the reader to verify that any game $(N; V) \in G^*$ has strictly effective 3-bounded groups. Thus, $G^* \subset G((0, 1), (0, 3))$ and all games in G^* are 1-thick, have convex payoff sets and non-empty cores.

However the class G^* is not restricted by any common per capita boundedness condition. For any positive integer C and any $m > C$ the game $(N; V^m) \in G^*$ does not satisfy per capita bound of C. Hence the condition of the common per capita bound on the collection of games is independent of all other conditions (and the implication) in Theorem 1.

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