The Econometrics of Ascending Auctions *

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Abstract

We characterize Bayesian Nash equilibria for asymmetric ascending auctions in which both common and private value components are accommodated, and bidders' valuations and signals are drawn from asymmetric distributions. It is shown that the equilibrium inverse bid functions in each round of the auction can be solved from a system of equations formed from bidders' expectations of the value of the object conditional on all information available during that round. The computational tractability of this characterization allows us to analytically solve for the equilibrium bidding strategies in a log-normal model. This forms the basis of an econometric model which we estimate using data from the PCS spectrum auctions run by the U.S. Federal Communications Commission.

Two concerns — one theoretical, and one empirical — motivate this paper. First of all, the theoretical literature on ascending auctions has, to date, focused primarily on symmetric models (where the bidders' signals about the value of the object are assumed to be *i.i.d.* across bidders).[1] The empirical importance of differential information among bidders in real-world auctions (cf. Hendricks and Porter [23], Porter [22]) however, casts

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[1]cf. the English auction model presented in Milgrom and Weber [19]

some doubt on this theoretical assumption. In the first part of this paper, we characterize Bayesian Nash equilibria for ascending auctions in which both common and private value components are accommodated, and bidders' valuations and signals are drawn from asymmetric distributions. Focusing on the *inverse bid functions* allows us to verify the existence and uniqueness of an increasing-strategy equilibrium directly from primitive model assumptions about the joint distribution of the bidders' underlying valuations and private signals. From a computational point of view, the equilibrium inverse bid functions in each round of the auction can be solved from a system of equations formed from bidders' expectations of the value of the object conditional on all information available during that round. We provide sufficient conditions for the existence of a unique increasing-strategy equilibrium for models with separable private and common value components. For the lognormal model, we show that these sufficient conditions are satisfied and analytically solve for the equilibrium bidding strategies.

These form the basis for an econometric model which we derive in the second part of this paper. The multiple round aspect of our model contrasts with much of the previous empirical work on auctions employing structural models (two examples are Paarsch [21] and Laffont, Ossard and Vuong ([13], hereafter LOV)), which have focused on *single-round* auctions. We estimate this model using data from the PCS spectrum auctions run by the U.S. Federal Communications Commission(FCC). The empirical work illustrates and suggests solutions to problems which arise in estimating this model in practice. In particular, we show that in the asymmetric ascending auction model, the order of dropping out imposes truncation conditions on the joint distribution of the observed bids. These conditions involve model parameters and pose nonregularity problems for maximum likelihood estimation. Therefore, we adapt the simulated nonlinear least square methodology proposed by LOV to the asymmetric ascending auction model.

1 Bidding behavior in multiple-round auctions

Consider an auction in which N bidders compete for the possession of a single object. Each bidder has observed a signal concerning the value of the object up for sale; denote these by $(X_1, \dots, X_N)$. Assume these signals are private, so that bidder i observes only X_i before the auction begins. The actual value of the object to bidder i will be denoted by V_i; we assume that $V = (V_1, \dots, V_N)$ has a joint distribution with $X = (X_1, \dots, X_N)$, which renders information regarding the signals (the X's) useful to bidders in predicting their unknown valuations (the V's). Before the auction begins, bidder i only observes his estimate X_i. He does not observe any of the $V_j, j = 1, \dots, n$ or any of the other bidders' estimates $X_k, k \neq i$.

The auction model which we focus on in this paper is an asymmetric version of the "irrevocable dropout" auction described in MW ([19], pg. 1104):

> Initially, all bidders are active at a price of zero. As the auctioneer raises the price, bidders drop out one by one. No bidder who has dropped out can become active again. After any bidder quits, all remaining active bidders know the price at which he quit.

By observing the dropout prices, remaining bidders can subsequently infer the private information possessed by the bidders who have dropped out. This information affects the behavior of the bidders only insofar as the object up for sale has a common value (CV) component, which is the same for all bidders. This common component leads to correlation among the signals which the bidders privately observe which, in turn, makes bidder j's signal X_j useful to bidder i in estimating V_i, his valuation of the object.[2] The winner's curse suggests that naive bidders who simply bid their expected valuation of the object conditional only upon their private signal will tend to gain a negative expected profit. In equilibrium, a bidder needs to take into account the extra *ex post* information he would gain about losing bidders' private signals were he to win the auction.[3]

The main difference between multi-round and single-round auctions lies in the *amount* of information that a bidder would learn about his competitors' private signals were he to win the object, and which he conditions upon in calculating his expected profit from winning — in what follows, we call this *ex post* information. In this section we mainly develop the intuition of the decision problem faced by the bidders as specified by the equilibrium bid functions for the asymmetric second-price and ascending auctions. A formal equilibrium proof is presented in the next section.

Throughout this paper, we restrict our attention to *increasing strategy equilibria* where each bidder i's equilibrium bidding strategy $b_i(X_i)$ is a strictly increasing functions of X_i.[4]

[2] In the *private values* (PV) paradigm, in contrast, where each bidder has a private value for the object (which he knows), the (undominated) equilibrium bidding strategy is independent of others' valuations: a bidder will bid (up to, in the ascending case) his private valuation of the object.

[3] Loosely speaking, a bidder needs to take into account the consideration that were he to win the auction, he would have observed a signal which is larger in magnitude than the signals of the losing bidders (i.e., he is initially "more optimistic" than the losing bidders).

[4] This rules out ad-hoc bidding rules such as "stay in no matter what", where $\beta_i(X_i) = +\infty$ regardless of the value of X_i. Section (2) will specify primitive sufficient conditions for the existence and uniqueness of equilibrium with strictly increasing bidding strategies, where it is shown that the monotonicity of the bid

Furthermore, we maintain two important assumptions concerning the joint distribution of the valuations and private signals $(V_1, \ldots, V_N, X_1, \ldots, X_N)$ of the bidders in the auction:

A1 All the bidders' valuations $(V_i,\ i = 1, \ldots, N)$ and private signals $(X_i,\ i = 1, \ldots, N)$ are drawn from distributions which share some region of common support.

This rules out extremely asymmetric cases where, ex ante, conditional on all possible realizations of X, V_j is almost surely lower than V_i. Bidder j will never win the auction in this case and, indeed, has little incentive to participate in the auction at all.[5]

A2 The conditional expectation $E[V_i \mid X_1, \ldots, X_N]$ is *strictly increasing* in X_i.

This is a weaker assumption than affiliation (cf. MW[19]). Note that if $(V_1, \ldots, V_N, X_1, \ldots, X_N)$ are assumed to be affiliated, the conditional expectation $E[V_i \mid X_1, \ldots, X_N]$ will be nondecreasing in *all* X_j, $j = 1, \ldots, N$.

In the rest of this section and most of the next section, we motivate and characterize equilibrium bidding strategies in an arbitrarily general asymmetric auction where the joint distribution of $(V_1, \ldots, V_N, X_1, \ldots, X_N)$ satisfies assumptions A1 and A2. Later, we place additional structure on the joint distribution of $(V_1, \ldots, V_N, X_1, \ldots, X_N)$ and demonstrate existence and uniqueness of equilibrium for a special but important case of the asymmetric ascending auction — that in which bidder i's valuation of an object is separable into private and common value components.

1.1 Equilibrium bidding in the second-price auction

Since the intuition embodied in the equilibrium bidding strategies functions in both the second-price (single-round) and ascending(multi-round) auctions are similar, it is instructive to briefly describe the intuition in the equilibrium construction of the second-price auction before proceeding to the ascending auction. However, in the subsequent sections, we formally characterize equilibria only for the ascending auction.

Let us consider the decision problem for bidder i in the second-price auction, given his signal X_i, and the equilibrium bid functions for the $N-1$ other bidders $\beta_j(X_j),\ j \neq i$, which are common knowledge. Would bidder i regret winning the object at a price of P? If he is risk-neutral, then he doesn't mind winning so long as his *ex post* expected valuation of the object exceeds P. What information does he have *ex post* to form his expectation of the object's value? Naturally, he knows his own private signal X_i, but winning at price P

function and therefore the existence of equilibrium, can be verified by the monotonicity of the inverse bid functions solved using the system of equation (3.15) presented below.

[5] Such extremely asymmetric conditions are discussed in detail in Klemperer [12]

in a second-price auction means that the bidder with the second highest bid has a signal which dictated a bid equal to P, i.e., $\max_{j \neq i} \beta_j(X_j) = P$. In other words, bidder i is willing to win the object at all prices P for which *ex post* expected revenue is positive:

$$E[V_i \mid X_i, \max_{j \neq i} \beta_j(X_j) = P] - P \geq 0. \tag{1.1}$$

If this conditional expectation is strictly increasing in its first argument, then in equilibrium there will be a critical point $P^*(X_i)$, a function of X_i, which uniquely satisfies equation (1.1) as an equality. This critical $P^*(X_i)$ is a strictly increasing function of X_i, and is in fact bidder i's equilibrium bid $\beta_i(X_i)$, given X_i:

$$E[V_i \mid X_i, \max_{j \neq i} \beta_j(X_j) = \beta_i(X_i)] - \beta_i(X_i) \ = \ 0.$$

Generalizing, the set of equilibrium bid functions for the N bidders are implicitly defined via the following system of equations:

$$\begin{aligned} \beta_1(X_1) \quad &= \quad E[V_i \mid X_i, \max\nolimits_{j \neq 1} \beta_j(X_j) = \beta_1(X_1)] \\ \ldots \quad & \quad\ \ \ldots \\ \beta_N(X_N) \quad &= \quad E[V_N \mid X_N, \max\nolimits_{j \neq N} \beta_j(X_N) = \beta_i(X_N)]. \end{aligned} \tag{1.2}$$

By rewriting these equations for each P in the common range of bids[6], the inverse bid functions $\phi_i(P) = \beta_i^{-1}(P), i = 1, \ldots, N$, which specifies the value of bidder i's private signal which corresponds to an equilibrium bid P, are defined via the following system of equations:

$$E\left(V_i \middle| X_i = \phi_i(P), \left[\bigcap_{j \neq i}(X_j) \leq \phi_j(P)\right] \cap \left[\bigcup_{j \neq i}(X_j) = \phi_j(P)\right]\right) - P = 0 \tag{1.3}$$

Therefore the conditions for the existence and uniqueness of a strictly increasing strategy equilibrium are reducible to requiring that the solutions $\phi_i(P)$ of the above system of equations (where P is taken to be the parameter) is unique and stricting increasing.

1.2 Equilibrium bidding in the ascending auction

Similar intuition motivates the structure of the equilibrium bid functions in the multi-round ascending (or English) auction. The only difference is that in the ascending format, upon winning the auction bidder i knows *all* the private signals of his losing competitors; in other words, his *ex post* information consists of all the signals of his competitors, not just the signal observed by the next-highest bidder, as in the second price auction described above.

[6] This common range of bids will be specified in section 2.

Before proceeding, let us introduce the indexing convention we will follow in this paper. The ascending auction proceeds in rounds. It enters a new round whenever another bidder drops out. N bidders are present in the auction; there will be $N-1$ "rounds" in the auction, indexed $k = 0, \ldots, N-2$. In round 0, all N bidders are active; in round k, only $N-k$ bidders are active. Each round ends when a bidder drops out; bidders are indexed by $i = 1, \ldots, N$. Without loss of generality, the ordering $1, \ldots, N$ indicates the order of dropout. In other words, bidder N drops out in round 0, and bidder 1 wins the auction; generally, bidder $N-k$ drops out at the end of round k. The dropout prices are indexed by rounds, i.e., $P_0, \ldots, P_{N-2}$. To sum up, bidder i drops out at the end of round $N-i$, at the price P_{N-i}.

Equilibrium bidding strategies in the ascending auction game specify, for each bidder i, bid functions $\beta_i^k(X_i)$ for each round k, $k = 0, \ldots, N-2$, i.e., $\beta_i^0(X_i), \ldots, \beta_i^{N-2}(X_i)$. Given a realization of the private signal X_i, the bid function $\beta_i^k(X_i)$ tells bidder i which price he should drop out at during round k. The collections of bid functions $\beta_i^0(X_i), \ldots, \beta_i^{N-2}(X_i)$ for bidders $i = 1, \ldots, N$ are common knowledge.

Again, consider bidder i, who is active during round k. As of round k, bidders $N-k+1, \ldots, N$ have already dropped out, at prices $P_k, \ldots, P_0$, respectively. Since the equilibrium bid functions are common knowledge, bidder i can use this information on the identity of the dropout bidders and their dropout prices to infer the private signals $X_{N-k+1}, \ldots, X_N$ observed by these bidders by inverting these bid functions: $X_j = (\beta_j^{N-j})^{-1}(P_{N-j})$, for $j = N-k+1, \ldots, N$.

Bidder i observes the price level rising; at each price level P he asks himself: would I regret winning the object at price P? As in the second-price auction, he won't regret winning so long as his *ex post* expected valuation of the object exceeds P. In the case of the ascending auction, however, bidder i wins at price P only when *all* the other active bidders in round k (bidders $1, \ldots, N-k$) instantaneously drop out at this price. But if this happens, then bidder i gains the *ex post* information that all the other bidders observed private signals which would have made them drop out at price P. In other words, since the equilibrium strategies are common knowledge, bidder i can infer that the other bidders' signals must be $X_j = (\beta_j^k)^{-1}(P)$, for $j = 1, \ldots, N-k$ and $j \neq i$. [7]

[7] In Wilson [26], the conditioning event $\beta_j^k(X_j) = \beta_i^k(X_i)$ is bidder i equilibrium bid function for round k is described as bidder i's equilibrium conjecture that all other active bidders j have the same targeted dropout price as he (bidder i) does. The discussion so far has shown that this equality arises because if bidder i wins the object at price $\beta_i^k(X_i)$, it must be that all other remaining bidders j have dropped out at this price, allowing bidder i to infer *ex post* that all these bidders had private signals which caused them to drop out at this price, i.e., $X_j = (\beta_j^k)^{-1}(\beta_i^k(X_i)) \leftrightarrow \beta_j^k(X_j) = \beta_i^k(X_i)$. Given the bidder asymmetries, this

The expected profit to bidder i when he wins the auction at price P during round k is therefore

$$E[V_i \mid X_i, X_j = (\beta_j^k)^{-1}(P), j = 1, \dots, N-k, j \neq i; \\ X_j = (\beta_j^{N-j})^{-1}(P_{N-j}), j = N-k+1, \dots, N] - P.$$

If this is positive, then bidder i doesn't mind being active at price P. On the other hand, if this is negative, then bidder i shouldn't be active at price P, during round k. In equilibrium, this expected profit is positive when P is small and negative when P is large.[8] The price P at which he should quit the auction during round k, defined as his *bid function* for round k, is the price $\beta_i^k(X_i) \equiv P_k^*$ at which he will have a zero expected profit in round k if all other active bidders simultaneously quit at the same price:

$$\beta_i^k(X_i) = \Big\{P: \ E\big[V_i \mid X_i = (\beta_i^k)^{-1}(P), X_j = (\beta_j^k)^{-1}(P), j = 1, \dots, N-k, j \neq i; \\ X_j = (\beta_j^{N-j})^{-1}(P_{N-j}), j = N-k+1, \dots, N\big] - P = 0.\Big\} \tag{1.7}$$

The properties of the expected bid function is illustrated in figure 1; the expected profit function intersects the x-axis at the unique price P_k^*. Note that if we treat P as the parameter and treat the inverse bid functions $(\beta_i^k)^{-1}(P), i = 1, \dots, N-k$ as the unknown

is *far* from an assertion that in equilibrium, bidder i believes that his remaining competitors have the same private signal as he does. This is true only for the symmetric case, where $\beta_j^k(X_j) = \beta^k(x), \ \forall\, j = 1, \dots, N$. The conditioning event $\beta_j^k(X_j) = \beta_i^k(X_i)$ then simplifies to $X_j = X_i$.

[8]This relies only on two conditions (1) the bid functions $\beta_i^k(P)$ are strictly increasing in P, $i = 1, \dots, N$, $k = 0, \dots, N-2$; and (2) the conditional expectations for each bidder i, $E(V_i \mid \cdots)$ are strictly increasing in i's signal.

In equilibrium, in round k, bidder i bids a price P such that

$$P = E[V_i \mid (\beta_i^k)^{-1}(P), (\beta_j^k)^{-1}(P), \ j = 1, \dots, N-k, \ j \neq i; X_{N-k+1}, \dots, X_N] \tag{1.4}$$

(see equation 1.7 in main text). On the other hand, if bidder i wins in round k at a price of P, his *ex post* gain is

$$E[V_i \mid X_i, (\beta_j^k)^{-1}(P), \ j = 1, \dots, N-k, \ j \neq i; X_{N-k+1}, \dots, X_N]. \tag{1.5}$$

Given these two equations, we can rewrite expected profit, for a given P, as expression (1.5) less the right hand side of equation (1.4):

$$E\Pi_i^k(P) = E[V_i \mid X_i, (\beta_j^k)^{-1}(P), \ j = 1, \dots, N-k, \ j \neq i; X_{N-k+1}, \dots, X_N] \ - \\ E[V_i \mid (\beta_i^k)^{-1}(P), (\beta_j^k)^{-1}(P), \ j = 1, \dots, N-k, \ j \neq i; X_{N-k+1}, \dots, X_N]. \tag{1.6}$$

Define $P^*(X_i)$ as the price such that $X_i = (\beta_i^k)^{-1}(P)$. By assumption (2) above, the second term in (1.6) is increasing in $(\beta_i^k)^{-1}(P)$ (the first term after the $|$) which, in turn, is increasing in P, by assumption (1). Therefore, for each X_i, , $P^*(X_i)$ is unique, and $E\Pi_i^k(P)$=0 at $P^*(X_i)$, is > 0 for all $P < P^*(X_i)$, and < 0 for all $P > P^*(X_i)$.

variables, we can solve for the inverse bid functions directly from (1.7), which forms a system of $N-k$ equations in $N-k$ unknowns.

Although (1.7) provides a direct way of solving for the equilibrium bidding strategies, better intuition about how bidders update their information as the ascending auction proceeds can be gained by rewriting (1.7) as:

$$\begin{aligned}\beta_i^k(X_i) = E\big[V_i \mid X_i, \beta_j^k(X_j) = \beta_i^k(X_i), j = 1, \dots, N-k, j \neq i;\\ \beta_j^{N-j}(X_j) = P_{N-j}, j = N-k+1, \dots, N\big]\end{aligned} \tag{1.8}$$

Generalizing, then, the set of $N-k$ equilibrium bid functions for the bidders $i = 1, \dots, N-k$

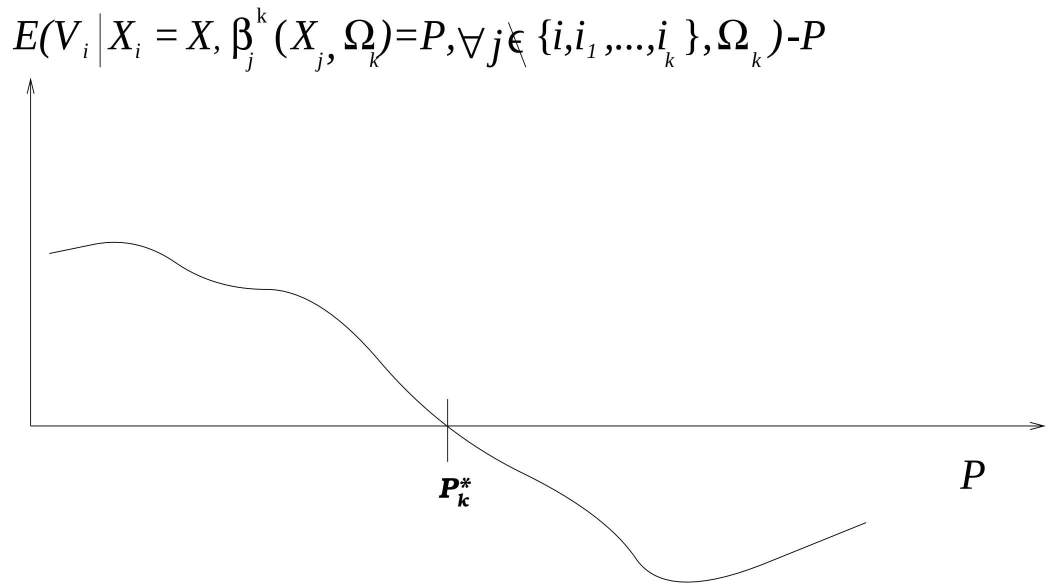


Figure 1: Bidder i's decision problem during round k

$\Omega_k \equiv \left\{X_j = (\beta_j^{N-j})^{-1}(P_{N-j}),\ j = N-k+1, \dots, N\right\}$

active in round k are implicitly defined via the following system of equations, for any $P > P_{k-1}$:

$$\begin{aligned}\beta_1^k(X_1) &= E\left[V_1 \mid X_1, \beta_j^k(X_j) = \beta_1^k(X_1), j = 2, \dots, N-k, \Omega_k\right]\\ \beta_2^k(X_2) &= E\left[V_2 \mid X_2, \beta_j^k(X_j) = \beta_2^k(X_2), j = 1, 3, \dots, N-k, \Omega_k\right]\\ &\cdots\\ \beta_{N-k}^k(X_{N-k}) &= E\left[V_{N-k} \mid X_{N-k}, \beta_j^k(X_j) = \beta_{N-k}^k(X_{N-k}), j = 1, \dots, N-k-1, \Omega_k\right]\end{aligned} \tag{1.9}$$

where $\Omega_k \equiv \left\{X_j = (\beta_j^{N-j})^{-1}(P_{N-j}), \text{ for } j = N-k+1, \dots, N\right\}$. The full set of equilibrium bid functions is analogously described by sets of $N-k$ equations for each round

$k = 0, \ldots, N-1$.[9]

From a computational point of view, the structure of these equilibrium bid functions is particularly attractive since the conditioning events are *points* rather than *sets*, which would have been the case for asymmetric second price auctions. In the case of the latter, the conditional expectations in (1.9) would involve multi-dimensional integrals, which can be difficult to compute.

2 Equilibrium in multiple-round auctions: formal characterization

In this section we will provide sufficient assumptions on the joint distribution of the signals X and valuations V to ensure a unique Bayesian Nash Equilibrium with strictly increasing bidding strategies in the ascending auctions. In the previous section, we labelled bidders by their dropout order, so that bidder i dropped out at the end of round $N-i$. In the following characterization, however, we need to make a distinction between the (arbitrary) bidder labels and the order in which they drop out, since the dropout order is random depending on the realization of bidders' private signals. Therefore in this section let $i = 1, \ldots, N$, be the fixed labels attached to the bidders and let $i_1, \ldots, i_N$ denote the dropout order, with i_1 indexing the first dropout bidder and i_N the winner. The random vector $(i_1, \ldots, i_N)$ can equal any of the possible permutations of $(1, \ldots, N)$.

Before stating the assumptions we introduce the some notation. Define $\mathcal{V}_i(X_1, \ldots, X_N) \equiv E(V_i \mid X_1, \ldots, X_N), i = 1, \ldots, N$, the N expectations of the values of the object to all bidders conditional on the vector of realized private signals of bidders $(X_1, \ldots, X_N)$. For any round k $(0 \leq k \leq N-2)$ and for any permutation $(i_1, \ldots, i_N)$ of $(1, \ldots, N)$, fix $X_{i_1}, \ldots, X_{i_k}$ (the private signals corresponding to the bidders who have already dropped out prior to round k) at any possible set of realized private signals. The $N-k$ conditional expectations for the $N-k$ bidders active in round k constitute a system of $N-k$ equations with $N-k$

[9] From a decision-theoretic point of view for each individual bidder, the structure of the equilibrium bid function shares some interesting similarities with the optimal policy in an *optimal stopping* problem, in which the "state variable" is the conditional expectation of the object's value (EV_i) and time is measured by the (deterministically rising) price level.

Note, however, that the "law of motion" for the state variable EV_i is not stationary, i.e., functionally independent of our time measure P: how EV_i evolves as P rises depends on the number (and identities) of the bidders who have already dropped out before price P, as well as the number and identities of the remaining bidders. Because of this, the optimal "stopping policy" would also be non-stationary, in the sense that the "critical value" of the state under which bidder i should drop out — this is his bid function — is not constant over the course of the auction.

unknowns

$$\begin{aligned} \mathcal{V}_{i_{k+1}}\left(X_{i_{k+1}},\dots,X_{i_N};X_{i_1},\dots,X_{i_k}\right) &= P \\ &\dots \\ \mathcal{V}_{i_n}\left(X_{i_{k+1}},\dots,X_{i_N};X_{i_1},\dots,X_{i_k}\right) &= P \end{aligned} \tag{2.10}$$

where $X_{i_{k+1}},\dots,X_{i_N}$ are the unknown variables and p is taken as the parameter.

Consider the following two assumptions,

E1 For all $1 \leq i,j \leq N$, $i \neq j$,

$$\mathcal{V}_i\left(\underline{X},\dots,\underline{X}\right) = \mathcal{V}_j\left(\underline{X},\dots,\underline{X}\right) \equiv \underline{\mathcal{V}}$$

and

$$\mathcal{V}_i\left(\overline{X},\dots,\overline{X}\right) = \mathcal{V}_j\left(\overline{X},\dots\overline{X}\right) \equiv \overline{\mathcal{V}}$$

We refer to $\left[\underline{\mathcal{V}},\overline{\mathcal{V}}\right]$ as the common range of bids.

E2 The solution of the $N-k$ unknown variables in equations (2.10) are unique and strictly increasing for all $P \in \left(\underline{\mathcal{V}},\overline{\mathcal{V}}\right)$, the common range of bids.

Conditions E1 and E2 lead to an equilibrium proposition for the English auction.

PROPOSITION 1 *If both assumptions E1 and E2 are satisfied, there exists a Perfect Bayesian equilibrium for the English auction game. The equilibrium strategies are recursively defined. At round k, let* $\phi_{i_j,k}\left(P;P_{i_1},\dots,P_{i_k}\right), k+1 \leq j \leq N$ *solve the system of equations (2.10). Define the inverse functions of* $\phi_{i_j,k}, k+1 \leq j \leq N$ *with respect to their first arguments:* $\beta_{i_j,k}\left(x;P_{i_1},\dots,P_{i_k}\right) = \phi^{-1}_{i_j,k}\left(x;P_{i_1},\dots,P_{i_k}\right)$, *for* $k+1 \leq j \leq N$ *. The collection of bid functions* $\beta_{i_j}\left(\cdot\right), k+1 \leq j \leq N$ *for* $0 \leq k \leq N-2$ *constitute a Perfect Bayesian Nash equilibrium of the English auction game.*

The proof is essentially based on the intuitive arguments presented in the previous section and can be found in the appendix.

2.1 Separable asymmetric ascending auction models

As noted above, conditions under which the equilibrium bid functions in the ascending auction game are unique and strictly increasing are equivalently conditions which ensure uniqueness and monotonicity of the solutions to the system of equations (2.10). In contrast to symmetric models (such as the ascending auction model presented in MW [19]), general affiliation assumptions on the joint distribution of $(V_1,\dots,V_N,X_1,\dots,X_N)$ are not sufficient to ensure (E2) for the asymmetric case.

For the remainder of this paper, we restrict our attention to a class of joint distributions of $(V_1, \ldots, V_N, X_1, \ldots, X_N)$ in which the value of the object to each bidder is *separable* into common and private value components. For example, consider the *additive* case $V_i = A_i + V$ where A_i bidder i's private value and V is the component of bidder i's valuation which is common to all bidders.[10]. Define for $1 \leq i \leq N$:

$$\mathcal{A}_i (X_1, \ldots, X_N) = E(A_i \mid X_1, \ldots, X_N) \tag{2.11}$$

$$\mathcal{V}(X_1, \ldots, X_N) = E(V \mid X_1, \ldots, X_N) \tag{2.12}$$

$$\mathcal{V}_i (X_1, \ldots, X_N) = \mathcal{A}_i (X_1, \ldots, X_N) + \mathcal{V}(X_1, \ldots, X_N)$$

In particular,

The following two assumptions are imposed on this additive structure to ensure that (E2) is satisfied:

F1 $\partial \mathcal{V}/\partial X_i > 0$, $\partial \mathcal{A}_i/\partial X_i > 0$ for $1 \leq i \leq N$. $\partial \mathcal{A}_i/\partial X_j \leq 0, j \neq i$.

F2 *Diagonal dominance*; There exist numbers $d_j > 0, 1 \leq j \leq N$, such that $d_j \dfrac{\partial \mathcal{A}_j}{\partial X_j} > \sum_{i \neq j} d_i \left| \dfrac{\partial \mathcal{A}_i}{\partial X_j} \right|$ for $j = 1, \ldots, N$.

Assumption (F1) states that a bidder's signal is positively related to the common value component and to his own private value components. It also states that taking his signal as given, another bidder's signal is negatively related to his private value, $\partial \vec{\mathcal{A}}_i/\partial X_j \leq 0, j \neq i$. Loosely speaking, a signal level of X_i gives bidder i an average estimate of V_i. Given this estimate of V_i, a higher signal level of X_{-i} implies a higher level of the common value V, which in turn implies a lower level of the private value component A_i. A sufficient condition for F1 is given in the following lemma, the proof of which can be found in the appendix:

LEMMA 1 *Suppose A_i and V are independent, A_i is independent from A_j for $i \neq j$, the signals A_i is conditionally independent conditional on $A_i + V$. If the (log) conditional density of X_i given $A_i + V$ is concave in $A_i + V$, then $\partial \mathcal{A}_i/\partial X_j \leq 0, j \neq i$.*

In particular, if conditional on $A_i + V$, X_i is normally distributed with mean $A_i + V$, then $\log f_{X_i|A_i+V}(\cdot)$ is concave in $A_i + V$.

Assumption (F2) is a condition of diagonal dominance. If we let $\mathcal{D}$ be the diagonal matrix with $\mathcal{D}_{ii} = d_i$, and let $\mathcal{A}$ be the Jacobian matrix $[\partial \mathcal{A}_i/\partial X_j, 1 \leq i, j \leq n]$, then each column of DA sum to a positive number. We interpret this as saying that the effect of X_i on his own private evaluation A_i dominates its effect on other bidders' private evaluations A_j.[11] In particular, this assumption is satisfied by the log normal model, which is described in the next section.

[10] Alternatively, we could have a multiplicative model $V_i = A_i \times V$, with the following conditions adjusted correspondingly

[11] In the terminology of input-output analysis, $[\partial \mathcal{A}_i/\partial X_j, 1 \leq i, j \leq n]$ is a Leontief matrix.

PROPOSITION 2 *Assumptions F1 and F2 implies E2.*

Proof: See appendix.

3 Log normal asymmetric ascending auction model

An econometric ascending auction model would use observed dropout prices to recover the parameters of the equilibrium bid function, which in turn depend on the preferences for the objects being sold, which vary across bidders. Formulating such a model involves first and foremost specifying distributions on the latent signals and valuations $X_1, ..., X_N, V_1, \ldots, V_N$ such that the resulting mapping (i.e., the equilibrium bid function) from observed dropout prices to unobserved valuations is tractable enough to apply data to and estimate.

Difficulties arise in doing this because the updating process introduces a large amount of recursivity into the definition of the bid function. For example, assume four bidders (A,B,C,D) and assume the first three drop out in rounds 1,2, and 3, respectively. After bidder A drops out, the remaining bidders (B,C,D) invert the equilibrium bid function for bidder A, in order to obtain his private signal X_A. In round three, bidder C and D must invert bidder B's bid function during round two, which is her expected value of the object in round two, *conditional not only on her private signal but also on x_A which she inferred by inverting bidder A's conditional expectation function from the previous round.*

The recursive structure which results (involving conditional expectations — in computational terms, integrals — functions which have as arguments inversions of other conditional expectation functions which are themselves inversions of other conditional expectation functions) quickly becomes intractable if the conditional expectations derived during the updating process become too complicated. Thus the challenge in implementing this model econometrically lies in picking a conjugate family of distributions for the latent variables (V, X) such that the resulting conditional expectation functions have tractable functional forms which facilitate inversion.

Fortunately, if we assume that bidders' valuations are log-normally distributed, we can derive closed-form formulas for the conditional expectation functions. we use as the basis of our empirical model the analytic formulas for the equilibrium bid functions for an multiplicative log-normal model, which generalizes Wilson[26]'s noninformative prior model to an informative prior model. This section contains the main steps in the derivation of these dropout prices given the log-normality assumption. [12] Suppose N bidders are contending for an single object in an ascending auction. The value of the object to bidder i is assumed

[12] The notation below, however, differs slightly from that of Wilson's[26] original notation.

to be $V_i = A_i \times V$. A_i is a bidder-specific private value for bidder i. V is the common market value which is not known to any of the bidders. In other word, V_i is the product of a common value part and a private value part.

The V and the A_i's are independently log normally distributed. If we let $v \equiv \ln V$ denote the natural log of V, and similarly let $a_i \equiv \ln A_i$. Then

$$v = m + \epsilon_v \sim N(m,\ r_0^2)$$
$$a_i = \bar{a}_i + \epsilon_{a_i} \sim N(\bar{a}_i,\ t_i^2).$$

Each bidder is assumed to have a noisy signal of the value of the object, X_i, which has the form $X_i = A_i \times E_i$. Here E_i is a noisy estimate of the common market value V. $E_i = V \times \exp\{s_i \xi_i\}$ in which ξ_i is an (unobserved) error term that has a normal distribution with mean 0 and variance 1. If we let $v_i \equiv \ln V_i$[13] and $x_i \equiv \ln X_i$, then conditional on v_i, $x_i = v_i + \epsilon_{e_i} \sim N(v_i,\ s_i^2)$. Note that Bidder i observes X_i, which is also known to other bidders after bidder i drops out.

We can summarize the information structure by the following equations:

$$\begin{array}{lllll} v & & & = & m + \epsilon_v \\ v_i & = & a_i + v & = & \bar{a}_i + m + \epsilon_v + \epsilon_{a_i} \\ x_i & = & v_i + \epsilon_{e_i} & = & \bar{a}_i + m + \epsilon_v + \epsilon_{a_i} + \epsilon_{e_i} \\ v_j & = & a_j + v & = & \bar{a}_j + m + \epsilon_v + \epsilon_{a_j} \\ x_j & = & v_j + \epsilon_{e_j} & = & \bar{a}_j + m + \epsilon_v + \epsilon_{a_j} + \epsilon_{e_j} \end{array} \tag{3.13}$$

All the ϵ's are normally distributed with zero mean. The variance for the ϵ_{a_i}'s are denoted t_i. The variance for the ϵ_{e_i}'s are denoted s_i. Let $r_i = \sqrt{t_i + s_i}$. Denote the variance for ϵ_v by r_0.

The joint distribution of $(V_i, X_i, i = 1, \ldots, N) = \exp(v_i, x_i, i = 1, \ldots, N)$ is fully characterized by $\{m, \bar{a}, t, s, r\}$ where $\bar{a}$ is the collection of $\bar{a}_i$'s, similarly for t, s, and r. These are all common knowledge among the bidders. From bidder i's perspective, it is useful to know x_j because, from the structure above, $x_j - \bar{a}_j + \bar{a}_i$ is normally distributed with a mean of v_i, which is the unknown value which bidder i wants to estimate.

3.1 Deriving the equilibrium bid functions

Before describing the process whereby we derive the equilibrium bid function, note that given the log-normality assumptions on V_i and the X_i's, $i = 1, \ldots, N$, the conditional expectation functions for V_i take the following form:

$$E[V_i \mid X_1, \ldots, X_N] = \exp\left(E(v_i \mid x_1, \ldots, x_N) + \frac{1}{2} Var(v_i \mid x_1, \ldots, x_N)\right), \tag{3.14}$$

[13]Recall from above that V_i is the true value of the object for bidder i.

for $i = 1, \ldots, N$. Given that $(v_i, x_1, \ldots, x_N)$ are jointly normal, the conditional mean function $E(v_i \mid x_1, \ldots, x_N)$ is linear in $x_1, \ldots, x_N$ and $Var(v_i \mid x_1, \ldots, x_N)$, the variance of v_i conditional on $x_1, \ldots, x_N$, is constant and does not depend on $x_1, \ldots, x_N$. This follows from the familiar conditional mean and variance formulas for jointly normal random variables[14]. Therefore the conditional expectation functions for $V_i \equiv \exp(v_i)$ are log-linear.

In round k of the auction, k bidders have dropped out. Recall that in equilibrium (as discussed above), the system of $(N-k)$ bid functions (for the $N-k$ remaining bidders) are conditional expectation functions (reproduced from equations (1.9)):

$$
\begin{aligned}
\beta_1^k(X) &= E[V_1 \mid X_1 = X, \beta_2^k(X_2) = \beta_1^k(X_1), \ldots, \beta_{N-k}^k(X_{N-k}) = \beta_1^k(X_1), \Omega_k] \\
\beta_2^k(X) &= E[V_2 \mid \beta_1^k(X_1) = \beta_2^k(X_2), X_2 = X, \ldots, \beta_{N-k}^k(X_{N-k}) = \beta_2^k(X_2), \Omega_k] \\
&\cdots \\
\beta_{N-k}^k(X) &= E[V_{N-k} \mid \beta_1^k(X_1) = \beta_{N-k}^k(X), \beta_2^k(X_2) = \beta_{N-k}^k(X), \ldots, X_{N-k} = X, \Omega_k].
\end{aligned}
$$

In period k, bidders $N-k+1, \ldots, N$ have dropped out, revealing $X_{N-k+1}, \ldots, X_N$. The equilibrium conjectures stated above mean that, essentially, bidder i conjectures that $X_j = (\beta_j^k)^{-1}(\beta_i^k(X_i))$, $\forall\ j = 1, \ldots, N-k,\ j \neq i$. Let $(\beta_j^k)^{-1}(P)$ indicate the inverse of $\beta_j^k(\cdot)$ evaluated at an arbitrary price P, i.e., what bidder j's type $(\beta_j^k)^{-1}(P)$ would be if he chooses to drop out at price P.

Given this notation, we can rewrite the conditional expectations above as functions of the inverse bid functions $((\beta_1^k)^{-1}(P) \ldots (\beta_{N-k}^k)^{-1}(P))$, for an arbitrary P, rather than functions of $(X_1 \ldots X_{N-k})$:

$$
\begin{aligned}
P &= E[V_1 \mid X_1 = (\beta_1^k)^{-1}(P), (\beta_2^k)^{-1}(P), \ldots, (\beta_{N-k}^k)^{-1}(P), \Omega_k] \\
P &= E[V_2 \mid (\beta_1^k)^{-1}(P), X_2 = (\beta_2^k)^{-1}(P), \ldots, (\beta_{N-k}^k)^{-1}(P), \Omega_k] \\
&\cdots \\
P &= E[V_{N-k} \mid (\beta_1^k)^{-1}(P), (\beta_2^k)^{-1}(P), \ldots, X_{N-k} = (\beta_{N-k}^k)^{-1}(P), \Omega_k].
\end{aligned}
\tag{3.15}
$$

Given a value for P, (3.15) defines a set of $(N-k)$ nonlinear equations with $(\beta_1^k)^{-1}(P), (\beta_2^k)^{-1}(P), \ldots, (\beta_{N-k}^k)^{-1}(P)$ as the $(N-k)$ "unknowns". In this form, then, we can solve for the set of $(N-k)$ inverse bid functions for period k from that period's system of conditional expectation functions pointwise for each P, given Ω_k in the range of the bid.

Using the conditional mean formula for log-normal variates (3.14) and the conditional mean and variance formulas for jointly normal variates, the formulas in (3.15) take the

[14] see, for example, Amemiya [1], pg. 3

following form (where $p \equiv \log P$ and $b \equiv log\beta$):

$$
\begin{aligned}
p =& \frac{s_i^2/r_i^2 + t_i^2 h}{r_i^2 h} x_i + \frac{s_i^2}{r_i^2 h} \sum_{j=1, j\neq i}^{N-k} \frac{(b_j^k)^{-1}(p) - a_j + a_i}{r_j^2} \\
&+ \frac{s_i^2}{r_i^2 h}\left(\frac{m + a_i}{r_0^2} + \sum_{l=N-k+1}^{N} \frac{x_l - a_l + a_i}{r_l^2} \right) + \frac{1}{2}\frac{r_i^2 s_i^2 t_i^2 h + s_i^4}{r_i^4 h}
\end{aligned}
\tag{3.16}
$$

for each remaining bidder $i = 1, \ldots, N-k$.

If we invert this $(N-k)$ system of equations, we can solve for $(b_j^k)^{-1}(p)$ as a function of p and $x_{k+1} \ldots x_N$, for $j = 1 \ldots N-k$, in each round k of the auction. In fact, these equations are the log inverse bid functions $(b_j^k)^{-1}(p)$ for each of the remaining bidders:

$$
\left((b_l^k)^{-1}(p), \ldots, (b_{N-k}^k)^{-1}(p) \right)' =
$$

$$
\begin{pmatrix}
\frac{s_1^2/r_1^2 + t_1^2 h}{r_1^2 h} & \frac{s_1^2}{r_1^2 r_2^2 h} & \cdots & \frac{s_1^2}{r_1^2 r_{n-k}^2 h} \\
\frac{s_2^2}{r_1^2 r_2^2 h} & \frac{s_2^2/r_2^2 + t_2^2 h}{r_2^2 h} & \cdots & \frac{s_2^2}{r_2^2 r_{n-k}^2 h} \\
\vdots & \vdots & & \ddots \\
\frac{s_{N-k}^2}{r_1^2 r_{N-k}^2 h} & \frac{s_{N-k}^2}{r_2^2 r_{N-k}^2 h} & \cdots & \frac{s_{N-k}^2/r_{N-k}^2 + t_{N-k}^2 h}{r_{N-k}^2 h}
\end{pmatrix}^{-1}
\begin{bmatrix} p \\ p \\ \vdots \\ p \end{bmatrix}
+
\begin{bmatrix} f_1(x_{N-k+1}, \ldots, x_N) \\ f_2(x_{N-k+1}, \ldots, x_N) \\ \vdots \\ f_{N-k}(x_{N-k+1}, \ldots, x_N) \end{bmatrix}
\tag{3.17}
$$

where $f_1, f_2, \ldots, f_{N-k}$ are all linear functions of $(x_{N-k+1}, \ldots, x_N)$ which do not depend on p at all. Hence all the $((b_1^k)^{-1}(p), \ldots, (b_{N-k}^k)^{-1}(p))$ are linear functions of p. After deriving the inverse log-bid functions in this manner, we can easily invert them to get the bid functions.

In order for equilibrium to exist (and for the inverse bid functions to be well-defined), we require that the bid function are increasing in x. For the log-normal case considered in this paper, it turns out that we can directly verify (from analytically inverting the matrix in the above system of equations) that the inverse bid functions are increasing in p. This in turns implies that the bid functions themselves are increasing in x; therefore, the monotonicity conditions required for existence of equilibrium are satisfied.[15]

[15] In fact, this model satisfies all the conditions of lemma (1) in the last section for models with both private value and common value parts, after the log-transformation. Use Σ_{12} to denote the covariance matrix between $P = (A_1, \ldots, A_N)$ and $X = (X_1, \ldots, X_N)$, and use Σ_{22} to denote the variance-covariance matrix of $X = (X_1, \ldots, X_N)$. Then the jacobian matrix of the private values becomes $[\partial \mathcal{A}_i / \partial X_j]_{1 \leq i,j \leq n} = \Sigma_{12}\Sigma_{22}^{-1}$. By assumption A_i and X_{-i} are independent, so Σ_{12} is a diagonal matrix with t_i^2 as the diagonal elements. On the other hand, affiliation of $(X_1, \ldots, X_N)$ implies that Σ_{22}^{-1} has a dominant diagonal. It then follows easily that $\Sigma_{12}\Sigma_{22}^{-1}$ also has a dominant diagonal by simply taking d_i to be t_j^{-2} in condition F2. The existence of a unique equilibrium follows directly from Propositon 2.

3.2 Deriving the joint density of dropout price vector

The likelihood function for a given auction is the (multivariate) density of the observed set of dropout prices from that auction. Given distributional assumptions on the x's and the nature of equilibrium in the ascending auction game, the "regression equation" which we want for a given dropout price is

$$p_k = f(x_{N-k}; x_{N-k+1} \dots x_N); \tag{3.18}$$

in other words, the dropout price in round k as a function of bidder $N-k$'s valuation x_{N-k},[16] as well as all the information he has in round k, which are the valuations $(x_{N-k+1} \dots x_N)$ of the bidders who have dropped out before round k. This "regression equation" specifies the transformation from the unobserved latent variables (the x's) which we have made distributional assumptions about to the observed variables (the dropout prices).

However, the system of equations in 3.16 give p_i as a function of *all* of the x's. Therefore, in order to derive the regression equations in 3.18, we must go through two steps: first, we solve for x_k as a function of $(p, x_{k+1}, \dots, x_N)$, where p is a generic p, for rounds $k = 1, \dots, N-1$[17]; secondly, we invert these $(N-1)$ functions to get p_k as a function of $(x_k, \dots, x_N)$, as desired. We go into each step in more detail.

Step 1 In the first step, we loop over each round k of the auction in order to solve for x_k, bidder k's valuation, as a function of p_k and $x_{k+1} \dots x_N$, the valuations of the bidders who have already dropped out (here we continue the indexing convention that bidder N drops out first). This is done using the methodology described in the previous section.

In round 0, for example, none of the x's are known (since no one has dropped out yet) so using the entire system of equations in 3.16 (N equations in N unknowns) we can solve for $(b_j^1)^{-1}(p)$ as a function of just a generic p, for $j = 1 \dots N$. From these equations, we keep only the one for bidder N — $(b_N^1)^{-1}(p)$ — the bidder who drops out during round 0. This equation is the inverse bid function for bidder N during the round that he drops out. We keep only one equation from each round because we only observe dropout prices.

Similarly, in round k, we solve for $(b_j^k)^{-1}(p)$ as a function of generic p and $x_{k+1} \dots x_N$, for $j = 1 \dots N-k$, and keep the equation for bidder $N-k$. So after looping through all rounds $k = 0 \dots N-2$, we are left with the $(N-1)$ equations: namely, x_k as a function of p and $x_{k+1} \dots x_N$, for $k = 0, \dots, N-2$.

[16] Recall that by our indexing convention, bidder $N-k$ drops out in round k, at the (log) price of p_k.

[17] since we don't observe the "dropout price" for the winning bidder

Step 2 Secondly, we invert each of these equations singly in order to get p_k, the dropout price during round k, as a function of $x_k \ldots x_N$, for $k = 0 \ldots N-2$. Given our assumptions about the means and variances of the x's, we can analytically derive the vector of means $\vec{\mu}_p(\theta, z)$ and the covariance matrix $\Sigma_p(\theta, z)$ for these $N-1$ equations. θ is shorthand for the parameter vector and z are the included covariates (both described below). The sequence of (log) dropout prices is therefore a linear transformation of the sequence of (log) signals and has jointly a (truncated) multivariate normal distribution, where the truncation condition is further discussed below.

3.3 Equilibrium truncation conditions

Define the multivariate normal density with mean $\vec{\mu}_p(\theta, z)$ and covariance matrix $\Sigma_p(\theta, z)$:

$$f(\mathcal{P}, \theta, z) = (2\pi)^{-\frac{n}{2}} \left|\Sigma_p(\theta, z)\right|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\mathcal{P} - \mu_p(\theta, z)\right)' \Sigma_p(\theta, z)^{-1} \left(\mathcal{P} - \mu_p(\theta, z)\right)\right]. \tag{3.19}$$

The multivariate normal density in equation 3.19 does not fully impose all the equilibrium restrictions implied in our ascending auction model. More specifically, the joint density of the dropout prices in equation 3.19 conditions on the *observed dropout order*, i.e., the first element of $\mu_p(\theta, z)$ is derived under the assumption that bidder 1 is the observed winning bidder, the second element is derived under the assumption that bidder 2 is the observed second-highest bidder, etc. There are regions in the multivariate normal density in (3.19) from which vectors of prices could be drawn which imply a dropout order different than the one which is observed (e.g., bidder $N-k$ drops out before round k), which would be inconsistent with the structure of the multivariate normal density we have derived (e.g., the $N-k$th element of $\mu_p(\theta, z)$ is a function of bidder $N-k$'s covariates).

Therefore, the nature of equilibrium in this game imposes a set of truncation conditions on the joint distribution of the observed sequence of dropout prices in a given auction these conditions state that, in each round of the auction, given the parameters θ, the targeted dropout prices of the remaining bidders for that round must be higher than the dropout price at that round. This ensures, roughly speaking, that bidder $N-k$ will not drop out before round k.

To be more specific, the multivariate normal density (3.19) only imposes the condition that a bidder who drops out during round k, at a price $p_k(\theta, z)$, has an expected valuation of the object equal to $p_k(\theta, z)$:

$$\begin{aligned} E[v_{N-k} \mid x_{N-k}, (b_1^k)^{-1}(b_{N-k}^k(x_{N-k})), \ldots, (b_{N-k-1}^k)^{-1}(b_{N-k}^k(x_{N-k})); \\ x_{N-k+1}, \ldots, x_N] = \\ p_k(\theta, z) = b_{N-k}^k(x_{N-k}; \theta, z). \end{aligned}$$

In what follows, we suppress z as an argument in $p_k(\cdot)$ and $b_j^k(\cdot)$.

To ensure that the "correct" dropout order occurs, we need to impose that, at the given parameter values, all remaining bidders $j = 1, \ldots, N-k-1$ have expected valuations greater than $p_k(\theta)$:

$$b_j^k(\theta) = E[v_j \mid x_j, (b_l^k)^{-1}(b_j^k(x_j)),\ l = 1, \ldots, N-k-1,\ l \neq j;\ x_{N-k+1}, \ldots, x_N] > p_k(\theta),$$

for all $j = 1, \ldots, N-k-1$, the bidders who are left in the auction after round j.

In other words, $b_{N-k}^k(x_{N-k};\theta) < b_j^k(x_j;\theta)$, for $j = 1, \ldots, N-k-1,\ k = 0, \ldots, N-2$, where $b_j^k(x_j;\theta)$ denotes the targeted (log) dropout price of bidder j in round k, at the current values of the parameters θ. These constraints can only be stated in terms of the model parameters; therefore they violate the standard regularity condition required for consistency and asymptotic normality of the maximum likelihood estimator.

The truncation probability is then equal to $P(\mathcal{A})$, where

$$P(\mathcal{A}) = \text{Prob(the truncation conditions hold)}.$$

Then the likelihood function is the sum of $NOBS$ terms, where $NOBS$ represents the number of auctions conducted. Each of the $NOBS$ terms is the multivariate normal density of the observed dropout prices divided by the truncation probability:

$$L(\mathcal{P} \mid \theta, z) = \begin{cases} \dfrac{f(\mathcal{P}, \theta, z)}{\int_{\mathcal{A}} f(\mathcal{P}, \theta, z)\, d\mathcal{P}} & \text{if truncation conditions hold.} \\ 0 & \text{otherwise.} \end{cases} \tag{3.20}$$

where θ denotes the parameter vector and z the covariates.

More detail on the likelihood function derivation, as well as formulae for $\vec{\mu}_p$ and Σ_p, is given in the appendix to this paper.

4 Estimation issues

4.1 Simulating the truncation probability

Using the density of the dropout prices that we have derived above (in equation 3.20) as the basis for estimation poses various problem. From a computational point of view, the integral in the denominator, which indicates the truncation probability, integrates the multivariate normal density over several dimensions and cannot be analytically evaluated. Numerical integration is also difficult. Therefore, in both the maximum and nonlinear least squares estimation methodologies considered below, we simulate the truncation probabilities.

For each fixed value of the parameter vector, we draw a normal random vector of the bidders' private valuations (the x's) for each auction, using the estimated mean and variance-covariance of the bidders' private valuations, and calculate all the targeted dropout prices at all rounds. Then we check all of the truncation inequalities. If all of them are satisfied, we set an indicator to be 1. If one of these conditions fails, we set an indicator to be zero. We repeat this random drawing for the number of simulations for each auction. The simulated truncation probability, is then the average of the indicators over the number of simulations.

The problem with the crude simulator is that the simulated probability is not smooth in the parameters. This can create severe problems for the optimization routines. To reduce non-smoothness we use an independent probit kernel smoother to smooth the indicator functions (see McFadden [16] for more details). For each auction t, and each simulated draw, we calculate the smoothed indicator function as:

$$I_t = \Pi_{k=0}^{N_t-2} \Pi_{j=1}^{N_t-k-1} \Phi \left(\frac{p_j^k - p_{N_t-k}^k}{h} \right)$$

where p_j^i is the targeted dropout price for bidder j in round i, and h is a small number chosen to be the bandwidth of the smoothing.

4.2 Maximum likelihood estimation: non-regularity problems

The equilibrium truncation conditions described above, however, involve model parameters. Since this implies that the support of the observables (the dropout prices) depends on model parameters,[18] the resulting maximum likelihood estimation problem is nonregular. Although the maximum likelihood estimator (i.e., that which maximizes the objective function 3.20) is still consistent, its asymptotic distribution is not clear, which makes statistical inference difficult in the maximum likelihood framework. For a detailed discussion, see Hong[11].

4.3 Simulated Nonlinear Least Squares Estimation

An alternative which avoids the nonregularity problem in estimating the model via maximum likelihood is simulated nonlinear least squares (SNLS) estimation, based on the methodology of LOV [13]. This estimator minimizes the usual nonlinear least squares (NLS) objective function

$$Q_T(\theta) = (1/T) \sum_{t=1}^{T} \sum_{k=0}^{N_t-2} \left(p_k^t - m_k^t(\theta) \right)^2 \tag{4.21}$$

[18] Usual truncated regression models assume that the truncation points are known constants. For example, the Tobit labor supply model assumes that an agent works only when his wage exceeds some reservation wage, normalized to be zero. This implies that the distribution of the observed wages is truncated at zero, a known constant.

where p_k^t is the kth observed log dropout price for auction t, and $m_k^t(\theta)$ is its corresponding expectation (or first moment, therefore denoted as m), taken with respect to its density as given in equation 3.20. Because $m_k^t(\theta)$, the mean of a multivariate truncated normal distribution, is difficult to compute analytically, we replace $m_k^t(\theta)$ in equation 4.21 by an unbiased simulation estimator $\tilde{m}_k^t(\theta) = \frac{1}{S}\sum_{s=1}^{S} p_{k,s}^t(\theta)$, where S is the number of simulation draws and $p_{k,s}^t$ is the simulated dropout price using the sth draw. The ensuing Simulated Nonlinear Least Squares (SNLS) objective function

$$Q_{S,T}(\theta) = \frac{1}{T}\sum_{t=1}^{T}\sum_{k=0}^{N_t-2}\left(p_k^t - \tilde{m}_k^t(\theta)\right)^2 \tag{4.22}$$

yields a consistent estimate of θ when $S \longrightarrow \infty$.[19]

Simulating $\mathbf{m_k^t}(\theta)$ To be more specific, the first moment $m_k^t(\theta)$ of the kth dropout price p_k^t, for $k = 0, \ldots, N_t - 2$, is

$$m_k^t(\theta) = \int_{\vec{x}} p_k^t(\vec{x};\theta)\,\mathbf{1}(\vec{x} \in A(\theta))\,\frac{f_t(\vec{x};\theta)}{Pr(A(\theta))}d\vec{x}. \tag{4.23}$$

where $\vec{x} \equiv \{x_1, \ldots, x_{N_t}\}$ denotes the vector of the signals of bidders in the order of dropping out, $p_k^t(\vec{x};\theta)$ specifies the kth dropout price as a function of the parameters and realized vector of bidder signals. $A(\theta)$ denotes the event that the observed order of dropping out is realized. The integration is over the N_t-dimensional vector of bidder signals.

Difficulties arise in minimizing an objective function which is calculated from random variables drawn directly from a truncated distribution where the truncation conditions (equivalently, the support) depend of the parameter values θ, as in our case. The SNLS objective function will not be very smooth in the parameters θ since, Using standard "acceptance-rejection" methods to sample from a truncated distribution, small changes in the estimates of θ can lead some formerly "rejected" draws to be "accepted".

In overcoming this problem, we employ *importance sampling.* Instead of drawing random variables directly from the multivariate truncated normal distribution of signals which depends on model parameters, we make random draws of $\vec{x}$ from a multivariate density function $g(\vec{x};\theta)$ which, while similar in shape to the truncated multivariate normal distribution which we wish to sample from, does not have a support which depends on the

[19]For a fixed value of S, the probability limit of the SNLS objective function $Q_{S,T}$ is not equal to Q_T, the NLS objective function. Therefore the estimates of θ will be inconsistent. Unfortunately, in our case, we cannot introduce a bias-correction term into the SNLS objective function, as in LOV ([13], pg. 959), because the corresponding bias-correction term in our case (i.e., the function $\Delta_{S,T}(\theta)$ which sets $\text{plim}_{T\to+\infty}\left(Q_{S,T}(\theta) + \Delta_{S,T}(\theta) - Q_T(\theta)\right) = 0$ for fixed S) is difficult to calculate.

parameter values. We can transform the expression in (4.23):

$$m_k^t(\theta) = \int_{\vec{x}} p_k^t(\vec{x};\theta)\,\mathbf{1}(\vec{x} \in A_t(\theta))\,\frac{f_t(\vec{x};\theta)}{g(\vec{x})\,Pr(A_t(\theta))} g(\vec{x})\,d\vec{x}. \tag{4.24}$$

Furthermore, $Pr(A(\theta))$ is also difficult to evaluate directly but can also be simulated by the same random draws:

$$Pr(A_t(\theta)) = \int_{\vec{x}} f_t(\vec{x};\theta)\,\mathbf{1}(\vec{x} \in A_t(\theta))\,d\vec{x} = \int_{\vec{x}} \frac{f_t(\vec{x};\theta)\,\mathbf{1}(\vec{x} \in A_t(\theta))}{g(\vec{x})} g(\vec{x})\,d\vec{x}$$

Therefore,

$$m_k^t(\theta) = \left[\int_{\vec{x}} p_k^t(\vec{x};\theta)\,\mathbf{1}(\vec{x} \in A_t(\theta))\,\frac{f_t(\vec{x};\theta)}{g(\vec{x})} g(\vec{x})\,d\vec{x}\right] \Big/ \left[\int_{\vec{x}} \mathbf{1}(\vec{x} \in A_t(\theta))\,\frac{f_t(\vec{x};\theta)}{g(\vec{x})} g(\vec{x})\,d\vec{x}\right].$$

This suggests estimating $m_k^t(\theta)$ by

$$\tilde{m}_{k,s}^t(\theta) = \frac{1}{S}\sum_{s=1}^{S} p_{k,s}^t(\theta)$$

where

$$p_{k,s}^t(\theta) = \left[p_k^t(\vec{x}_s;\theta)\,\frac{f_t(\vec{x}_s;\theta)\,\mathbf{1}(\vec{x}_s \in A_t(\theta))}{g(\vec{x}_s)}\right] \Big/ \left[\frac{1}{S}\sum_{s=1}^{S} \frac{f_t(\vec{x}_s;\theta)\,\mathbf{1}(\vec{x}_s \in A_t(\theta))}{g(\vec{x}_s)}\right].$$

In addition, to facilitate the optimization procedure, we smooth the indicator function in the above summations also using an independent probit kernel smoother. In particular, we estimate $m_k^t(\theta)$ by

$$p_{k,s}^t(\theta) = \left[p_k^t(\vec{x}_s;\theta)\frac{f_t(\vec{x}_s;\theta)\prod_{k=0}^{N_t-2}\prod_{j=1}^{N_t-k-1}\Phi\left(\frac{p_{k,j}^t(\vec{x}_s;\theta)-p_{k,N_t-k}^t(\vec{x}_s;\theta)}{h}\right)}{g(\vec{x}_s)}\right] \Big/ \left[\frac{1}{S}\sum_{s=1}^{S}\frac{f_t(\vec{x}_s;\theta)\prod_{k=0}^{N_t-2}\prod_{j=1}^{N_t-k-1}\Phi\left(\frac{p_{k,j}^t(\vec{x}_s;\theta)-p_{k,N_t-k}^t(\vec{x}_s;\theta)}{h}\right)}{g(\vec{x}_s)}\right] \tag{4.25}$$

where $p_{k,j}^t$ is the targeted dropout price for bidder j in round k, for auction t, as a function of $\vec{x}_s$ and θ.

In our case, we use the importance density $g(\vec{x};\theta) = f_t(\vec{x};\theta)$, the *untruncated* multivariate normal distribution for auction t. In this way, $f_t(\vec{x}_s;\theta)$ cancels out from both the numerator and denominator in equation 4.25.

Asymptotic distribution of $\theta_{\mathbf{SNLS}}$ The asymptotic distribution of θ_{SNLS} is given by:

$$\hat{\Sigma}^{-1/2}\sqrt{T}\left(\theta_{SNLS}-\theta_0\right) \xrightarrow{d} N\left(0,I\right)$$

where $\hat{\Sigma}=\hat{\mathcal{A}}^{-1}\hat{\mathcal{B}}\hat{\mathcal{A}}^{-1}$, and

$$\hat{\mathcal{A}}=\frac{1}{T}\sum_{t=1}^{T}\sum_{k=0}^{N_t-2}\left[\frac{\partial\tilde{m}_k^t\left(\theta\right)}{\partial\theta}\right]\left[\frac{\partial\tilde{m}_k^t\left(\theta\right)}{\partial\theta}\right]',$$

$$\hat{\mathcal{B}}=\frac{1}{T}\sum_{t=1}^{T}\left(\left[\sum_{k=0}^{N_t-2}\frac{\partial\bar{m}_k^t\left(\theta\right)}{\partial\theta}\left(p_k^t-\tilde{m}_k^t\left(\theta\right)\right)\right]\left[\sum_{k=0}^{N_t-2}\frac{\partial\tilde{m}_k^t\left(\theta\right)}{\partial\theta}\left(p_k^t-\tilde{m}_k^t\left(\theta\right)\right)\right]'\right).$$

5 Empirical application: the FCC Spectrum Auctions

In the past several years, the FCC has held auctions to allocate Personal Communications Services (PCS) licenses. Each license covers a particular slice of the radio spectrum over a particular geographic area. Licenses were offered both at the MTA and BTA level[20] The data used in this paper come from the most important spectrum auction, the **MTA broadband PCS auction**, which began on December 5, 1994 and ended on March 13, 1995, after 112 rounds of bidding. 99 MTA licenses were offered– two 30 MHz licenses in most of the 51 MTAs which comprise the US and its territories abroad.

In this paper, we analyze the auctions of 91 of these licenses. 30 firms participated in this auction, and 19 of them eventually won licenses, yielding over $7 billion in government revenue. We analyze the auctions for the A and B block spectra.

The licenses were allocated using a *simultaneous multiple round auction*, proposed by Paul Milgrom, Robert Wilson, and Preston McAfee. The main features of this auction format is the simultaneity and multiple rounds. Simultaneous auctions of many objects allows bidders to realize cross license synergies, if any exist. The multiple-round format, as explained above, "allows the bidders to react to information revealed in prior rounds, [thus] enabling the bidders to bid more aggressively" (Cramton [6], p. 2). This paper is concerned about measuring the amount of information revealed in this multiple round auction.

[20]Respectively, *major trading area* and *basic trading area*. This designations are from Rand McNally.

5.1 Data description

The data on the auction results from the MTA broadband auction is taken from the FCC's web site (`http://www.fcc.gov`). This data gives us information on the participants and the bids that they submitted during each round on the various licenses. We supplemented this data with market characteristics at the MTA level from the Rand-McNally guide. The cellular presence data came from the *Wireless Market Book* [5] from the Cellular Telephone Industry Association (CTIA). We discuss how we created the dependent variable and the regressors in turn.

5.1.1 The dropout prices

While the model presented above captures the multi-round aspect of the FCC auctions, it does not correspond as closly to other aspects of these auctions. The most serious of these is that bidders are not required to make irreversible dropout decisions. Therefore, in order to fit the model to the FCC data, we impose some assumptions about bidders' beliefs concerning the dropout behavior of the other bidders. We assign a "dropout price" to bidder j which is the last price at which he was "active" (in a sense to be clarified below). We assume that all remaining bidders also believe this assigned price to be bidder j's dropout price.

Next we define how we classify a bidder to be "active". The following example will be useful: suppose that there are four bidders (A, B, C, D) and we observe that the last submitted bids for A, B, and C were 10, 20, and 30, respectively. If the price goes up by increments of 5, then, D will win the object at a price of 35 (assuming that his valuation is greater than that).

One simple way would be to assign to each bidder a dropout price equal to his last submitted bid, and assume that the winner's dropout price was greater than or equal to the winning price, i.e., $P_A = 10, P_B = 20, P_C = 30, P_D \geq 35$. This method is inconsistent, because of the gap between the the second-highest dropout price P_C and the lower bound on the highest dropout price P_D. As Milgrom and Weber[19] note, their formulation of the ascending model model reduces to a second-price auction when there are only two bidders left– in this case, these would be C and D. The problem with the above assignment of dropout prices is that the winner – D – doesn't win the object at the "second-price", which is C's dropout price.

To deal with this problem, we assign a dropout price to a given bidder equal to the last submitted bid of the *next* bidder who drops out. In the example above: $P_A = 20, P_B = 30, P_C = 35, P_D \geq 35$. The reason this problem occurs is that the Milgrom and Weber[19] (and Wilson[26]) model assume continuously rising prices and instantaneous dropouts, whereas in the FCC auctions (and probably in most real-life situations) the price ascends by discrete intervals.[21]

5.1.2 Model parameterization

We estimate this model in order to recover bidders' valuations from their observed bids (or, in this case, their dropout prices).

We estimate three sets of parameters:

- We parameterize m, the mean of the log common value distribution for a given object, to be a function of object attributes (such as population or size of the spectrum band to be sold). These attributes are MTA-level demographic variables which should capture the across-license variation in values. We use (1) population (2) population change, and (3) average household income. Population measures the size of the market, while population change accounts for future market growth. Average income is used as a measure of demand for PCS services.

- $\bar{a}_i = Z_i' \alpha$– we parameterize the publicly-known mean of bidder i's private valuation of the object to be a linear function of firm-and-object specific covariates. The only regressor we use at this point is an indicator of cellular presence in the surrounding area. More precisely, this regressor is a tally of the total number of the BTA's surrounding[22] a particular MTA in which a given firm has cellular presence. For a given firm, these regressors will vary across licenses, and for a given license, these regressors will vary across firms.[23]

- $\theta \equiv (s, t, r_0)$, the standard deviations for the private and common value estimates, and for the common value distribution. Below, we report results from both models in which we assumed these parameters to be the same across all bidders, as well as models where we allow some heterogeneity in s across firms, using the same variables as we use to parameterize differences in m across licenses.

Figure 2 presents summary statistics for all the variables we use in the analysis.

[21] To our knowledge, no one has come up with a discrete version of the ascending auction model.

[22] Firms with substantial cellular coverage in a given market were barred from bidding for PCS spectra in that market.

[23] Previous work by Moreton and Spiller [24] indicated the existence of PCS-Cellular complementarities in these auction from winning price regressions.

Figure 2: Summary statistics for data variables

Variable	**N**	**mean**	**StdDev**	**min**	**max**
Winning prices ($mill)	91[1]	75.87	89.71	4.39	493.5
Population (millions)	91	5.15	4.14	1.15	26.78
Pop'n change (1990-95, %)	91	6.00	3.53	0.40	12.80
Per capita income ('000)	91	15.86	3.71	11.96	20.70
Dropout prices ($mill)	423	53.18	69.28	0.89	493.5
Cell. pres	423	0.61	1.28	0	8

[1]: We omitted the observations for: Puerto Rico, Guam, Samoa

6 Estimation results

Figure 3 shows the results for two specifications of the full model, estimated using the simulated nonlinear least squares (SNLS) methodology described above. Models **1a** and **1b** in figure3 differ in the extent to which log s — the log of the variance on the bidders' common value estimates — is parameterized. Simply comparing the (simulated) NLS objective function values, the extra covariates clearly improve the performance of the model.

Before discussing the results, we mention a few points about identification. In all the models, the constant term in $\bar{a}$, the mean of the log private value estimates, is not separately identified apart from the constant term in m, the mean of the log common value estimates. Therefore we do not estimate the former.

Only the various constant terms are estimated with any reasonable precision; the coefficients attached to the covariates are very imprecisely estimated, and small in magnitude. In particular, in the current specifications, asymmetries across the bidders is captured only by including the CELL PRES covariate in $\bar{a}$, the mean of the distribution from which bidders' private values are drawn. The small and insignificant estimated coefficient on this covariate (0.008 for model 1b) indicates that bidders are largely symmetric given this specifications and our results. This finding has implications on how the equilibrium bid functions for a given bidder changed during the course of the auction. We turn to these bid functions next.

Figure 3: Simulated Nonlinear Least-Squares Estimates

M (number of simulation draws): 100; H (bandwidth)=0.01

	Model 1a		**Model 1b**	
Coefficient	Estimate	Std. Err	Estimate	Std. Err
Components of log s:				
Constant	2.515	0.354	1.627	0.604
POP (mills)			1.124	0.824
INCOME (per cap.,$'000)			5.852	1.549
Components of log t:				
Constant	3.004	0.315	2.999	0.237
log $r_0{}^a$:				
Constant	0.793	0.318	0.813	0.183
Components of m:				
Constant	10.762	8.306	6.458	14.069
POP (mills)	-0.035	4.301	-0.036	0.081
POP CHANGE (%)	0.212	0.328	0.483	0.511
Components of $\bar{a}$:				
Constant[b]				
CELL PRES	0.009	0.069	0.008	0.019
NLS Obj.Fun.[c]	23002.29		21955.38	
Num of Obs	91		91	

[a] variance of common value distribution

[b] Not separately identified from constant in m

[c] simulated

6.1 Estimated bid functions

Figure 4 shows plots of the estimated (log) bid functions for the winning bidder ("bidder 1", using the indexing in this paper), in each of the rounds of selected auctions. His log signal x_1 is plotted on the x-axis, while $b_1^k(x_1)$, his log bid functions for rounds $k = 1, \ldots, N-1$, are plotted on the y-axis. The units on the y-axis are log($mills).

First note that the log bid functions are linear in the signals; this is due to the log-normality assumption. Secondly, note that the bid functions decrease in slope as the auction progresses, implying that for any given valuation x_1 in the range in which bidder 1 would have won the auction, the targeted dropout price falls as bidders drop out. For example, for auction #89 (Hawaii block B), if $x_1 = 1.5$, then we can read off the graph that bidder 1's targeted log dropout price falls from around $4.1 mills in round 0, to about $2.8 mills in the final round. Furthermore, it is easy to show that a bidder's equilibrium bid functions for any two successive rounds k and $k+1$ intersect at a log price equal to p_k, the log dropout price for round k.[24]

This monotonic change in the slope of the bid functions is characteristic of *symmetric* ascending auctions. As noted above, for our estimates, bidders are essentially symmetric given our specification. Changes in the slope of the bid function occur because he conditioning events change as the auction progresses, as bidder 1 learns the private signals of the bidders who have dropped out. In a symmetric ascending auction, where no differences exist among his competitors, bidder 1's expected valuation for the object is either increasing or decreasing in each and every of his competitors' private signals. Furthermore, when bidder j remains in the auction, bidder 1 assumes in equilibrium that bidder j's private signal is equal to x_1.[25] Once bidder j drops out, bidder 1 learn X_j and, given symmetry, $X_j < X_1$. Essentially, bidder 1 "plugs" a smaller number X_j into his bid function. This is true for each bidder $j \neq 1$, since bidder 1 wins the auction. This causes the slopes of the successive bid functions to change monotonically as the auction progresses. That the slopes monotonically *decrease* arises if bidder 1's expected valuation for the object is increasing in all his competitors' private signals. This is true for our additive log-normal model with a common value component, which induces positive correlation among all the bidders' signals. This

[24] Essentially, the first $N-k-1$ of the $N-k$ values $(b_1^k)^{-1}(p_k), \ldots, (b_{N-k}^k)^{-1}(p_k) = p_k$ which solve the system of $N-k$ equations which define the equilibrium bid functions in round k (see equation 3.16) also solve the system of $N-k-1$ equations for round $k-1$, given $p = p_k$.

[25] As explained in the very beginning of the paper, this is because, in equilibrium, bidder 1 bids a price $\beta_1(X_1)$ in which his expected revenue from winning is just equal to $\beta_1(X_1)$. If he in fact wins at price $\beta_1(X_1)$, this must mean that bidder j has dropped out at that price, implying that bidder j's private signal $X_j = \beta_j^{-1}(\beta_1(X_1))$. For the symmetric case, $\beta_j(\cdot) = \beta(\cdot)$, $\forall\ j$, so that $X_j = \beta_j^{-1}(\beta_1(X_1)) = X_1$.

is clear from equation 3.16, in which all the coefficients associated with bidders' private signals (for bidder i, these would be $\frac{s_i^2}{r_i^2 h r_j^2}$ for all bidders $j \neq i$) are positive.

For the asymmetric case, this monotonicity need not hold, even assuming, as in the log-normal model, that bidder 1's expected valuation is increasing in each and every private signal. This is because in every round, bidder 1 not only learns the private signal of the dropout bidder during that round, but also revises his beliefs about the remaining bidders' signals knowing that these bidders are also revising their beliefs upon observing dropout behavior. While the coefficients on all the private signals in bidder 1's expected valuation of the object are still positive, it is not clear whether the new values for the signals "plugged in" during each round are larger or smaller than the old values; therefore, it is unclear how this change in information affects the slope of the bid function. Therefore it is not clear that bidder 1's sequence of targeted dropout prices will change monotonically, as we have found for the symmetric case.

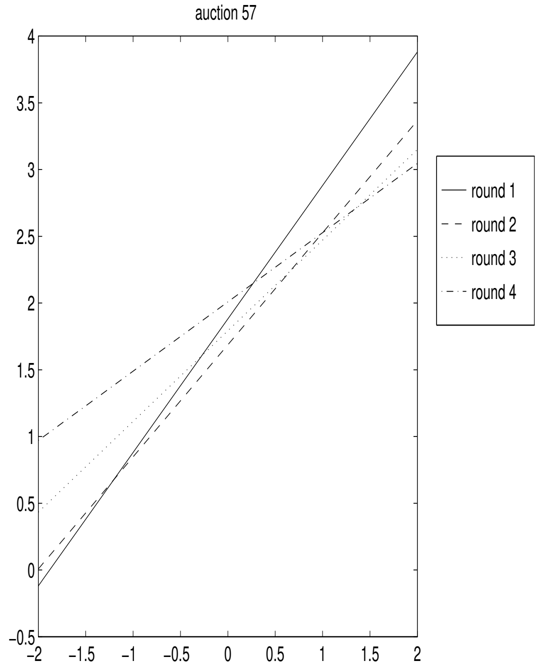
Nonetheless, there is some tension between our finding of bidders' "pessimism" (as evidenced by the fall in targeted dropout prices) as the auction proceeds with Milgrom and Weber's assertion that information raises *ex ante* expected revenues from an auction. But MW are making a statement about different auction forms (i.e., with common values, English auctions yields higher *ex ante* expected revenue), whereas our pessimism just occurs across the rounds of an English auction: despite it, what MW are saying is that the *ex ante* expected winning price in an English auction exceeds that of a second-price auction, for example.

7 Conclusions

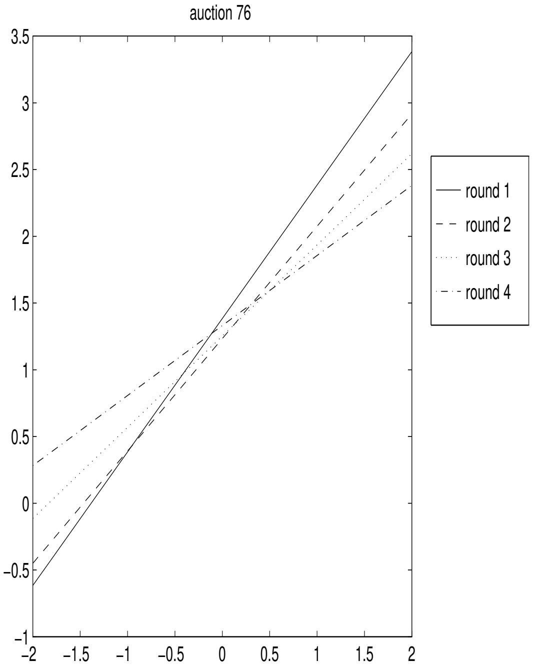
The goal in this paper has been twofold. Firstly, we have characterized increasing-strategy Bayesian Nash equilibria in asymmetric ascending (English) auctions. Secondly, we derived a structural econometric model based on the assumption that bidders follow these equilibrium bidding strategies. In the case that bidders' private signals are drawn from non-identical log-normal distributions, we show that the vector of log dropout prices observed in a given ascending auction is distributed multivariate truncated normal.

We estimate this model using data from the FCC spectrum auctions. Our model captures the multi-round aspect of these auctions. The current model does not incorporate the potential synerges that may result from complementarities across licenses. Recent work by Marshall and Raiff [14] has measured these synergies in the Georgia school milk market auctions. Furthermore, Haile [10] relaxed the "irrevocable dropout" assumption of the MW model in recent work on the USFS timber auctions. These are promising lines of research

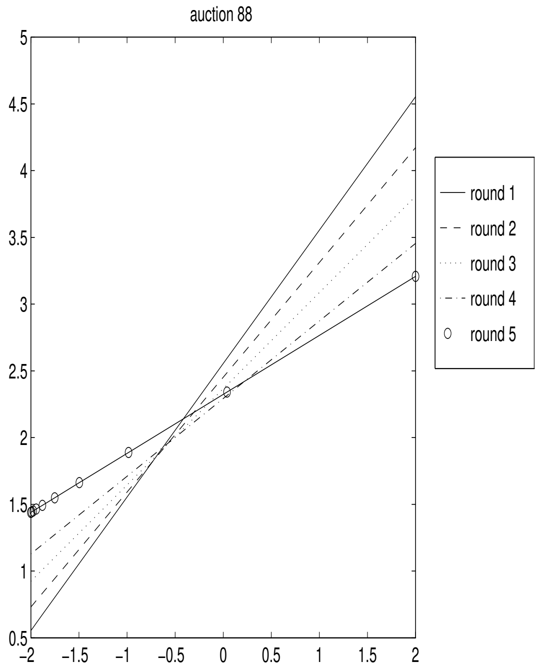
Figure 4: Estimated bid functions: a few examples



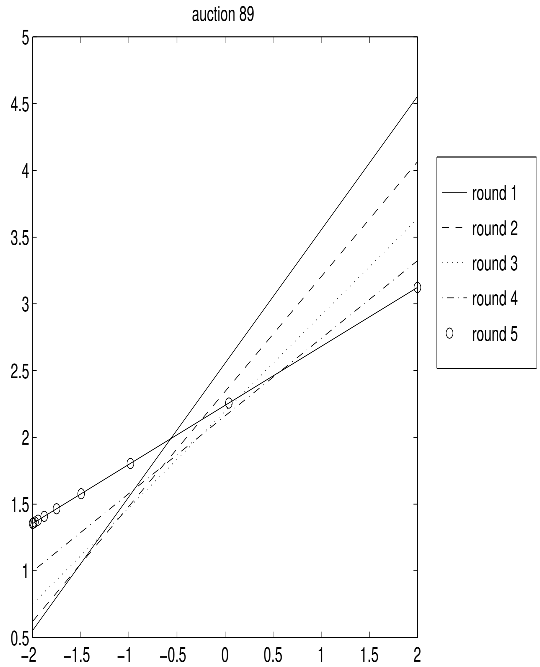
MTA 31, Block B: Indianapolis



MTA 41, Block A: Oklahoma City



MTA 47, Block A: Hawaii



MTA 47, Block B: Hawaii

along which we plan to extend our analysis of the FCC auctions.

A Proof of proposition (1)

Proof: It is shown below that if all bidders $j \neq i$ follow their equilibrium strategies, by following his own equilibrium strategy, bidder i will win the auction if and only if his expected net payoff is positive conditional on winning. Furthermore, if bidder i deviates at any round of the auction and stays over the bidding limit specified by his equilibrium strategy in that round, then he will never be able to gain a positive expected payoff should he ever win the auction in any of the latter rounds. As soon as he stays over his equilibrium bidding limit, his best strategy is to quit as soon as possible.

Consider bidder i, who observes his own signal to be X_i. At any round k of the auction, take $i = i_N$, bidder i will win the auction at price level P if and only if all the other remaining bidders $i_{k+1}, \ldots, i_{N-1}$ instantaneously quit. Bidder i will then infer that his competitors are of types $\left(\phi_{i_j,k}\left(P; P_{i_1}, \ldots, P_{i_k}\right), k+1 \leq j \leq N-1\right)$. Bidder i will expect to gain

$$\mathcal{V}_i\left(X_i, \phi_{i_j,k}\left(P; P_{i_1}, \ldots, P_{i_k}\right), k+1 \leq j \leq N-1, \phi_{i_l,l-1}\left(P_{i_l}; P_{i_1}, \ldots, P_{i_{l-1}}\right), 1 \leq l \leq k\right) \tag{A.26}$$

and has to pay P. By construction,

$$\begin{aligned}\mathcal{V}_i\Bigg(&\phi_{i,k}\left(P; P_{i_1}, \ldots, P_{i_k}\right), \phi_{i_j,k}\left(P; P_{i_1}, \ldots, P_{i_k}\right), k+1 \leq j \leq N-1, \\ &\qquad\qquad \phi_{i_l,l-1}\left(P_{i_l}; P_{i_1}, \ldots, P_{i_{l-1}}\right), 1 \leq l \leq k\Bigg) = P\end{aligned} \tag{A.27}$$

Since $\mathcal{V}_i(\cdot)$ is strictly increasing in its ith argument, (A.26) is greater than (A.27) if and only if $X_i > \phi_{i,k}\left(P; P_{i_1}, \ldots, P_{i_k}\right)$, or, if and only if $\beta_{i,k}\left(X_i; P_{i_1}, \ldots, P_{i_k}\right) > P$. Therefore if bidder i plays $\beta_{i,k}\left(X_i; P_{i_1}, \ldots, P_{i_k}\right)$ in round k, he will gain a positive expected payoff if and only if he wins the auction.

Next we observe that the equilibrium price path must be increasing, even if we allow for $P \leq P_{i_k}$ in solving the system of equations (2.10). Suppose bidder i_k with signal X_{i_k} drops out at price P_{i_k} in round $k-1$. We define $X'_{i_{k+1}}, \ldots, X'_{i_N}$ to be the solutions to equations (2.10) when $P = P_{i_k}$. We can think of $X'_{i_j}, j > k$ as what bidder i_j's type would be if he also choose to quit at price level P_{i_k} in round $k-1$. Now if we consider (2.10), we can see that $\left(X'_{i_{k+1}}, \ldots, X'_{i_N}\right)$ solves this round k system of equations at $P = P_{i_k}$. This is just a restatement that $X_{i_k}, X'_{i_{k+1}}, \ldots, X'_{i_N}$ solves the round $k-1$ system of equations at $P = P_{i_k}$. By E1 and E2, there must be $\beta_{i_{k+1},k}\left(X'_{i_{k+1}}; \cdots\right) = P_{i_k}, \ldots,$
$\beta_{i_N,k}\left(X'_{i_N}; \cdots\right) = P_{i_k}$. Since bidders $i_{k+1}, i_{k+2}, \ldots, i_N$ do not drop out in round $k-1$, we know that $X_{i_{k+1}} > X'_{i_{k+1}}, \ldots, X_{i_N} > X'_{i_N}$. Therefore $\beta_{i_{k+1},k}\left(X_{i_{k+1}}; \cdots\right) > P_{i_k}, \ldots,$

$\beta_{i_N,k}(X_{i_N};\cdots) > P_{i_k}$, which implies that the equilibrium price path must be increasing. It is essential to require that the price path does not have decreasing parts.
Now suppose that bidder i stays over his bidding limit in round k, and during round k bidder i_{k+1} drops out at price level $P_{i_{k+1}}$. If we define X_i'' by $\beta_{i,k}(X_i; P_{i_1},\ldots,P_{i_k}) = P_{i_{k+1}}$, then $X_i'' > X_i$. Applying this argument inductively, assume bidder i later wins the auction at round $k' > k$, because bidders $i_{k'+1},\ldots,i_{N-1}$ drop out instantaneously at price level $P_{i_{k'+1}}$. If we define X_i^* by $\beta_{i,k'}\left(X_i^*; P_{i_1},\ldots,P_{i'_k}\right) = P_{i_{k'+1}}$, then $X_i^* > X_i$. That is, bidder i plays as if he has observed a signal $X_i = X_i^*$. Let X_{-i}^* be the vector of revealed signals of other bidders, then bidder i will have to pay $P_{i'_k+1} = E\left(V_i \mid X_i = X_i^*, X_{-i} = X_{-i}^*\right)$, which is smaller than his expected gain, $E\left(V_i \mid X_i, X_{-i} = X_{-i}^*\right)$, since $X_i < X_i^*$. This also implies that once bidder i drops out in some round k, he will never be interested in being active in the auction again. Q.E.D.
Remark: It is clear from the proof that the crucial conditions for the equilibrium construction are: 1) $\mathcal{V}_i(\cdot)$ is strictly in its ith arguments (assumption A2); 2) There exists strictly increasing solutions to (2.10) (assumption E2). In particular, it is not neccessary that $\mathcal{V}_i(\cdot)$ be increasing in its jth arguments, $j \neq i$.

B Proof of lemma (1)

Note that the additive structure of the model (specifically, the presence of the common component V) implies that the private signals $(X_1,\ldots,X_N)$ will be affiliated. For conditionally independent signals, affiliation means that $f_{X_i|V}(X_i \mid v)$ satisfies the *monotone likelihood ratio property*, in that $f(X_i \mid V_1)/f(X_i \mid V_2)$ is increasing in X_i whenever $V_1 > V_2$, and is decreasing whenever $V_1 < V_2$. The most important implication of MLRP is that for any increasing function $g(V)$, $E(g(V) \mid X_i)$ is increasing in X_i. See Theorem 2.1 of Milgrom[18].

Without loss of generality consider $i = 1$. We can write the conditional expectation $E(A_1 \mid X_1,\ldots,X_N)$ by first conditioning on, and integrating out with respect to, V, the common value:

$$E(A_1 \mid X_1,\ldots,X_N) = \int E(A_1 \mid V, X_1)\, f_{V|X_1,\ldots,X_N}(V \mid X_1,\ldots,X_N)\, dV$$

Note that $E(A_1 \mid V, X_1,\ldots,X_N) = E(A_1 \mid V, X_1)$ because of our assumptions of conditional independence. Lemma (1) will hold if we can show that $E(A_1 \mid V, X_1)$ is a decreasing function of V. This will be true if, conditional on X_1, $f(V \mid A_1, X)$ satisfies the reverse

MLRP in that for $A_1^1 > A_1^2$ and $V_1 > V_2$:

$$\frac{f(V_1 \mid A_1, X_1)}{f(V_1 \mid A_2, X_1)} < \frac{f(V_2 \mid A_1, X_1)}{f(V_2 \mid A_2, X_1)}$$

By writing $f(V \mid A_1, X_1)$ as $f(V) f(A_1) f(X_1 \mid V + A_1) / f(A_1, X_1)$ and after some simplication, this is equivalent to requiring:

$$\frac{f(X_1 \mid A_1 + V_1)}{f(X_1 \mid A_2 + V_1)} < \frac{f(X_1 \mid A_1 + V_2)}{f(X_1 \mid A_2 + V_2)}$$

This will be true if $\log f(X_1 \mid A_1 + V)$ is concave in $A_1 + V$.

C Proof of proposition (2)

Proof: Suppose $(X_1^0, \ldots, X_n^0)$ solves (2.10) for $P = P_0$, then we can linearize (2.10) around P_0 and write it as

$$P - P_0 = \sum_{j=1}^{n} \partial \vec{\mathcal{V}}_i / \partial X_j \left(\phi_j(P) - X_j^0\right), i = 1, \ldots, n$$

Solve this system of linear equations for $\left(\phi_j(P) - X_j^0\right), j = 1, \ldots, n,$

$$\left(\phi_i(P) - X_i^0\right) = \left(\sum_{j=1}^{n} \left[\left(\frac{\partial V}{\partial X}\right)^{-1}\right]_{i,j}\right)(P - P_0), i = 1, \ldots, n$$

where we use $\left[\left(\frac{\partial V}{\partial X}\right)^{-1}\right]_{i,j}$ to denote the i,jth element of the inverse matrix of $\frac{\partial V_i}{\partial X_j}, i = 1, \ldots, N, j = 1, \ldots, N$. The condition that $\phi_i(P), 1 \leq i \leq n$ are increasing in P is equivalent to requiring that:

$$\sum_{j=1}^{n} \left[\left(\frac{\partial V}{\partial X}\right)^{-1}\right]_{i,j} > 0, \forall 1 \leq i \leq n$$

i.e., Each row of the inverted Jacobian matrix must sum to a positive number.
The Jacobian matrix of (2.10) is the sum of the jacobian matrix for the corresponding private value and common value parts:

$$[\partial V_i / \partial X_j] = [\partial \mathcal{A}_i / \partial X_j] + [\partial V / \partial X_j], 1 \leq i, j \leq N.$$

Any $N - k, 0 \leq k \leq N - 2$ principal submatrix of the left hand side $\left[\partial V_{i_j} / \partial X_{i_k}\right], k + 1 \leq i_j, i_k \leq N$ can be rewritten as $D^* + xy'$, where $D^* = \mathcal{D}[i_{k+1}, \ldots, i_N; i_{k+1}, \ldots, i_N]$ is the corresponding principal submatrix of $D = [\partial \mathcal{A}_i / \partial X_j]$, x is a $N - k$ column vector of 1's, and y is the $N - k$ vector of

$\left[\partial V/\partial X_{i_{k+1}}, \ldots, \partial V/\partial X_{i_N}\right]$. The dominant diagonal assumption implies that D^* is a nonsingular matrix. Therefore $\left[\partial V_{i_j}/\partial X_{i_k}\right], k+1 \leq i_j, i_k \leq N$ can be uniquely inverted because it is the sum of a nonsigular matrix and a rank one matrix. The following matrix inversion formula [26]

$$(D + xy')^{-1} = D^{-1} - \frac{1}{1 + y'D^{-1}x} D^{-1}xy'D^{-1} = D^{-1}\left(I - \frac{1}{1 + y'D^{-1}x} xy'D^{-1}\right). \tag{C.28}$$

is applicable here, with D^* replacing the role of D. Because D^* has a dominant diagonal with positive diagonal elements and non-positive off-diagonal elements, D^{*-1} is a nonnegative matrix.(Theorem 4, McKenzie[17]) Therefore $y^{*'} = y'D^{*-1}$ is a nonpositive vector. We can follow the same arguments as those after (C.28) to show that the row sums of $\left(I - \frac{1}{1+y'D^{*-1}x} xy'D^{*-1}\right)$ are positive. Since D^{*-1}'s elements are also nonnegative, it follows that the row sums of $D^{*-1}\left(I - \frac{1}{1+y'D^{*-1}x} xy'D^{*-1}\right)$ must also be positive. Assumption E2 is satisfied. A unique equilibrium can be constructed through the arguments used in Proposition 1.

D Details of dropout price density derivation

In this appendix we provide a detailed derivation of the joint density of the dropout prices, given in equation 3.20 in the main text.

Let $(V_1, \ldots, V_N, X_1, \ldots, X_N)$ be lognormally distributed. Let $v_i \equiv \ln V_i$ and $x_i \equiv \ln X_i$, $\forall i = 1, \ldots, N$. Assume that $(v_1, \ldots, v_N, x_1, \ldots, x_N)$ are distributed as jointly normal with mean vector $\mathcal{U} = (u_1, \ldots, u_N, u_1^*, \ldots, u_N^*) \equiv (\mu, \mu^*)$ and variance-covariance matrix $\hat{\Sigma} = ((\Sigma, \Sigma_{12}), (\Sigma_{12}', \Sigma^*))'$. Explicit formulas for the elements of the vector $\mathcal{U}$ and the matrix $\hat{\Sigma}$ can be derived from the equations in (3.13).

As before, we have

$$E[V_i \mid X_1, \ldots, X_N] = \exp\left(E(v_i \mid x_1, \ldots, x_N) + \frac{1}{2}V(v_i \mid x_1, \ldots, x_N)\right)$$

If we denote the marginal mean-vector and marginal variance-covariance matrix of $(v_i, x_1, \ldots, x_N)$ by

$$\mu_i \equiv (u_i, \mu^*) \quad \text{and} \quad \Sigma_i \equiv \begin{pmatrix} \sigma_i^2 & \sigma_i^{*\prime} \\ \sigma_i^* & \Sigma^* \end{pmatrix} \quad \text{also let} \quad x' \equiv (x_1, \ldots, x_N)$$

[26] See pp39 Dhrymes[8].

Then, using the conditional mean and variance of joint normal random variables (see, for example, Amemiya [1], pg. 3):

$$E(v_i \mid x) = \left(u_i - \mu^{*\prime} \Sigma^{*-1} \sigma_i^*\right) + x' \Sigma^{*-1} \sigma_i^*$$

and

$$V(v_i \mid x) = \sigma_i^2 - \sigma_i^{*\prime} \Sigma^{*-1} \sigma_i^*.$$

At round k,[27] let $x_d^k \equiv (x_{N-k+1}, \dots, x_N)$ denote the vector of k valuations for the bidders who have dropped out prior to round k, and $x_r^k \equiv (x_1, \dots, x_{N-k})$ denote the vector of $(N-k)$ valuations for the bidders who have not yet dropped out as of round k. Analogously, partition Σ^{*-1} into $(\Sigma_{k,1}^{*-1}, \Sigma_{k,2}^{*-1})'$ where $\Sigma_{k,1}^{*-1}$ is a $((N-k) \times N)$ matrix and $\Sigma_{k,2}^{*-1}$ is a $(k \times N)$ matrix. Then the conditional mean function can be re-written as:

$$E(v_i \mid x) = \left(u_i - \mu^* \Sigma^{*-1} \sigma_i^*\right) + x_r^{k\prime} \Sigma_{k,1}^{*-1} \sigma_i^* + x_d^{k\prime} \Sigma_{k,2}^{*-1} \sigma_i^*.$$

After substituting the conditional mean and variance formulas into the equations in (3.14) and taking the log of both sides, we get the following set of $(N-k)$ linear equations for the $N-k$ bidders active in round k:

$$p = \left(u_i - \mu^* \Sigma^{*-1} \sigma_i^*\right) + \sigma_i^{*\prime} \Sigma_{k,2}^{*-1\prime} x_d^k + \sigma_i^{*\prime} \Sigma_{k,1}^{*-1\prime} x_r^k + \frac{1}{2}\left(\sigma_i^2 - \sigma_i^{*\prime} \Sigma^{*-1} \sigma_i^*\right) \tag{D.29}$$

for $i = 1, 2, \cdots, N-k$. Equations (3.16) in the main text are derived from (D.29) by substituting in the explicit formulas for $(u_i,\ \mu^*,\ \Sigma^{*-1},\ \sigma_i^*,\ \Sigma_{k,1}^{*-1},\ \Sigma_{k,2}^{*-1})$ which we can derive from the equations in (3.13).

Deriving $p_k = f(x_{N-k}; x_{N-k+1} \dots x_N)$, the "regression equation" for round k dropout price p_k If we let l_k be the $(N-k) \times 1$ vector of 1's, $\mu_k = (u_1, \dots, u_{N-k})'$, $\Gamma_k = (\sigma_1^2, \dots, \sigma_{N-k}^2)'$, $\Lambda_k = (\sigma_1^*, \dots, \sigma_{N-k}^*)'$, then we could rewrite the above system of linear equations (D.29) as

$$p \times l_k = \frac{1}{2}\left(\Gamma_k - diag(\Lambda_k \Sigma^{*-1} \Lambda_k') + \Lambda_k \Sigma_{k,2}^{*-1\prime} x_d^k\right) + \mu_k - \Lambda_k \Sigma^{*-1} \mu^* + \Lambda_k \Sigma_{k,1}^{*-1\prime} x_r^k. \tag{D.30}$$

[27] $k = 0, \dots, N-2$. $x_d^0 = \emptyset$.

Solving out for the x_r^k, we obtain the set of $(N-k)$ log-inverse bid functions at round k:

$$x_r^k = \left[\left(\Lambda_k \Sigma_{k,1}^{*-1\prime} \right)^{-1} l_k \right] \times p$$
$$- \frac{1}{2} \left(\Lambda_k \Sigma_{k,1}^{*-1\prime} \right)^{-1} \left(\Gamma_k - diag(\Lambda_k \Sigma^{*-1} \Lambda_k') + \Lambda_k \Sigma_{k,2}^{*-1\prime} x_d^k + 2\mu_k - 2\Lambda_k \Sigma_k^{*-1} \mu^* \right). \tag{D.31}$$

Next let us introduce some shorthand notation:

$$\mathcal{A}^k \equiv \left(\Lambda_k \Sigma_{k,1}^{*-1\prime} \right)^{-1} l_k$$

and

$$\mathcal{B}^k \equiv \frac{1}{2} \left(\Lambda_k \Sigma_{k,1}^{*-1\prime} \right)^{-1} \left(\Gamma_k - diag(\Lambda_k \Sigma^{*-1} \Lambda_k') + \Lambda_k \Sigma_{k,2}^{*-1\prime} x_d^k + 2\mu_k - 2\Lambda_k \Sigma^{*-1} \mu^* \right),$$

both of them $(N-k)$-dimensional vectors. Then the inverse bid functions in (D.31) can be rewritten:

$$x_r^k = \mathcal{A}^k \times p - \mathcal{B}^k,$$

or, each equation singly:

$$x_{r\,i}^k = \mathcal{A}_i^k \times p - \mathcal{B}_i^k, \quad \text{for } i = 1, \ldots, N-k, \tag{D.32}$$

where $\mathcal{A}_i^k$ and $\mathcal{B}_i^k$ are the ith elements of the vectors $\mathcal{A}^k$ and $\mathcal{B}^k$.[28]

As described in the main text, in round k we keep the $(N-k)$-th equation from the system in (D.32); this is the equation $x_{r\,N-k}^k$ in the notation of (D.32). This equation gives us x_{N-k} as a function of p and $x_{N-k+1}, \ldots, x_N$. As described before, we do this for rounds $k = 0, \ldots, N-2$, and end up with $N-1$ analogous equations for $x_2, \ldots, x_N$: we can denote these by $x_{r\,N}^0, \ldots, x_{r\,N-k}^k, \ldots, x_{r\;\;2}^{N-2}$.

From the data, we observe the vector of dropout prices $\mathcal{P} \equiv (p_0, p_1, \ldots, p_{N-2})$. We can invert each of these equations described in the previous paragraph singly in order to derive an equation for p_k, the log dropout price for round k, as a linear function of $x_{N-k}, \ldots, x_N$, for $k = 0, \ldots, N-2$. We will get

$$p_k = \frac{x_{N-k} + \mathcal{B}_{N-k}^k}{\mathcal{A}_{N-k}^k}, \quad \text{for } k = 0, \ldots, N-2. \tag{D.33}$$

[28]The original bid function in its level can then be written as $b_i(X_i) = \exp\left(\frac{x_i + \mathcal{B}_i^k}{\mathcal{A}_i^k}\right) = \exp\left(\frac{\mathcal{B}_i^k}{\mathcal{A}_i^k}\right) X_i^{\frac{1}{\mathcal{A}_i^k}}$.

Since the $\mathcal{P}$ vector is a linear transformation of the x vector, it is also multivariate normal.

For purposes of deriving the joint density, it is useful to look at these equations describing the dropout prices in some more detail. To simplify notation, label bidders with $N, N-1, \ldots, 1$ in the order they drop out. Further decompose $\mathcal{B}^k$ into

$$\mathcal{C}^k \equiv \frac{1}{2}\left(\Lambda_k \Sigma_{k,1}^{*-1\prime}\right)^{-1}\left(\Gamma_k - diag(\Lambda_k \Sigma^{*-1}\Lambda_k') + 2\mu_k - 2\Lambda_k \Sigma^{*-1}\mu^*\right)$$

and

$$\mathcal{D}^k \equiv \frac{1}{2}\left(\Lambda_k \Sigma_{k,1}^{*-1\prime}\right)^{-1}\left(\Lambda_k \Sigma_{k,2}^{*-1\prime}\right).$$

Then the equations for $\mathcal{P}$ in terms of the x's in (D.33) can be rewritten as

$$\begin{aligned} p^0 &= \frac{\mathcal{C}_N^0}{\mathcal{A}_N^0} + \frac{x_N}{\mathcal{A}_N^0} \\ p^i &= \frac{\mathcal{C}_{N-i}^i}{\mathcal{A}_{N-i}^i} + \frac{x_{N-i}}{\mathcal{A}_{N-i}^i} + \frac{\mathcal{D}_{N-i}^i (x_{N-i+1}, \ldots, x_N)'}{\mathcal{A}_{N-i}^i} \qquad i = 1, \ldots, N-2 \end{aligned} \tag{D.34}$$

Again, introduce more shorthand notation:

$$\mathcal{F} = \left(\frac{\mathcal{C}_N^0}{\mathcal{A}_N^0}, \ldots, \frac{\mathcal{C}_{N-2}^2}{\mathcal{A}_{N-2}^2}\right)', \quad \mathcal{G}_i = \left(\underbrace{0, \ldots, 0}_{N-i-1}, \frac{1}{\mathcal{A}_{N-i}^i}, \frac{\mathcal{D}_{N-i}^i}{\mathcal{A}_{N-i}^i}\right), \ i = 0, 1, \ldots, N-2$$

Let $\mathcal{G} = (\mathcal{G}_0', \ldots, \mathcal{G}_{N-2}')'$

Given this notation, the system of equations describing the dropout prices (D.34) can be very succinctly written as:

$$\mathcal{P} = \mathcal{F} + \mathcal{G}(x_2, \ldots, x_N)'.$$

We denote the model parameters by θ. Suppose we also have a set of covariates z. The above mean and variance of observed dropout prices, together with the coefficients and constant terms in the truncation conditions, are all functions of the parameter vector θ and the covariates z.

Mean of the bidder evaluation x	$\mu^* = \mu^*(\theta; z)$
Var-Cov of the bidder evaluation x	$\Sigma^* = \Sigma^*(\theta; z)$
Jacobian of transforming x into $\mathcal{P}$	$\mathcal{G} = \mathcal{G}(\theta; z)$
Constant terms in the transformation of x into $\mathcal{P}$	$\mathcal{F} = \mathcal{F}(\theta; z)$

The mean vector μ_p and variance-covariance matrix Σ_p of the vector of dropout prices are:

$$\mu_p(\theta, z) = \mathcal{F}(\theta, z) - \mathcal{G}(\theta, z)\mu^*(\theta, z)$$
$$\Sigma_p(\theta, z) = \mathcal{G}(\theta, z)\Sigma^*(\theta, z)\mathcal{G}(\theta, z)'$$

and therefore the joint density of these prices without incorporating the truncation conditions is

$$f(\mathcal{P}, \theta, z) \equiv (2\pi)^{-\frac{n}{2}} |\Sigma_p(\theta, z)|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(\mathcal{P} - \mu_p(\theta, z))' \Sigma_p(\theta, z)^{-1}(\mathcal{P} - \mu_p(\theta, z)).\right]$$

Equilibrium truncation conditions The equilibrium truncation conditions can also be explicitly expressed in terms of the shorthand notation introduced above.

At each round k of the auction, the targeted dropout prices of the remaining bidders for that round must be higher than the dropout price at that round, in other words, $p^k \equiv p^k_{N-k} < p^k_i$, for $i = 1, \ldots, N-k-1$, $k = 0, \ldots, N-2$, where p^k_i denotes the targeted dropout price of bidder i in round k. Note that $x = \mathcal{G}^{-1}(\mathcal{P} - \mathcal{F})$. Let e^N_i be the ith column of a $N \times N$ identity matrix, and let $\mathcal{E}^N_k = (e^N_{N-k+1}, \ldots, e^N_N)'$, then $x_i = e^{N\prime}_i \mathcal{G}^{-1}(\mathcal{P} - \mathcal{F})$, and $(x_{N-k+1}, \ldots, x_N)' = \mathcal{E}^N_k \mathcal{G}^{-1}(\mathcal{P} - \mathcal{F})$. Since $p^k_i = \dfrac{\mathcal{C}^i_k}{\mathcal{A}^i_k} + \dfrac{x_i}{\mathcal{A}^k_i} + \dfrac{\mathcal{D}^k_i(x_{N-k+1}, \ldots, x_N)}{\mathcal{A}^k_i}$, then $p^k_i = \dfrac{\mathcal{C}^k_i}{\mathcal{A}^k_i} + \dfrac{e^{N\prime}_i \mathcal{G}^{-1}(\mathcal{P} - \mathcal{F})}{\mathcal{A}^k_i} + \dfrac{\mathcal{D}^k_i \mathcal{E}^k_i \mathcal{G}^{-1}(\mathcal{P} - \mathcal{F})}{\mathcal{A}^k_i}$, the constraint that $p^k \equiv p^k_{N-k} < p^k_i$ can thus be rewritten as

$$\begin{aligned} &\left(e^{N\prime}_{N-k} - \frac{e^{N\prime}_i \mathcal{G}^{-1}}{\mathcal{A}^k_i} - \frac{\mathcal{D}^k_i \mathcal{E}^k_i \mathcal{G}^{-1}}{\mathcal{A}^k_i}\right)\mathcal{P} < \frac{\mathcal{C}^k_i}{\mathcal{A}^k_i} - \frac{e^{N\prime}_i \mathcal{G}^{-1}\mathcal{F}}{\mathcal{A}^k_i} - \frac{\mathcal{D}^k_i \mathcal{E}^k_i \mathcal{G}^{-1}\mathcal{F}}{\mathcal{A}^k_i}, \\ &\text{For} \quad \begin{pmatrix} i = 1, \ldots, N-k-1 \\ k = 0, \ldots, N-2 \end{pmatrix} \end{aligned} \tag{D.35}$$

These truncation conditions, expressed in equation (D.35), involve model parameters. Both the coefficients and the constant terms in equation (D.35) are functions of parameters and covariates, i.e.,

$$\mathcal{A}^k_i = \mathcal{A}^k_i(\theta, z) \quad \mathcal{D}^k_i = \mathcal{D}^k_i(\theta, z) \quad \mathcal{C}^k_i = \mathcal{C}^k_i(\theta, z).$$

This violates regularity conditions for maximum likelihood estimation.

Joint density Given these truncation conditions, the joint density of the observed vector of dropout prices $\mathcal{P}$, given the parameter vector θ and the covariates z, is given by

$$L(\mathcal{P} \mid \theta, z) = \begin{cases} \dfrac{f(\mathcal{P}, \theta, z)}{\int_{\mathcal{A}} f(\mathcal{P}, \theta, z)\, d\mathcal{P}} & \text{if truncation conditions hold.} \\ 0 & \text{otherwise.} \end{cases}$$

where $\mathcal{A}$ denotes the event indicated by conditions (D.35) when $\mathcal{P}$ is treated as a random vector generated from the multivariate normal density $f(\mathcal{P}, \theta)$.

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