State Dependent Preferences Can Explain the Equity Premium Puzzle

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Abstract

We introduce state dependent recursive preferences into the Mehra-Prescott economy. We show that such preferences can match the historical first two moments of the returns on equity and the risk free rate. Other authors have reported similar results using state dependent expected utility preferences. These authors have tended to emphasize the importance of countercyclical risk aversion in explaining the equity premium puzzle. We find that countercyclical risk aversion plays an important role but only when combined with modest cyclical variation in intertemporal substitution.

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1. Introduction

Representative agent models that embed the Lucas-Breeden (Lucas (1978), Breeden (1979)) paradigm for explaining asset returns are generally regarded as inconsistent with the empirical data. Difficulties such as the equity premium puzzle (Mehra and Prescott (1985)), and the risk free rate puzzle (Weil (1989)) are well documented and it has been shown that these puzzles are very robust (Kocherlakota (1996), Campbell (1996), and Cochrane (1997) provide good surveys). Recently, several authors (Campbell and Cochrane (1999), and Gordon and St-Amour (2000,2001), among others) have pointed to countercyclical risk aversion as a potential source of mis-specification that may account for these puzzles. However, risk aversion and intertemporal substitution are intertwined in these models, just as they are in the additive expected utility model, therefore it is difficult to interpret which feature of preferences varies over the cycle and the role played by each. Moreover, it is useful to investigate the role of state dependent preferences in a simpler setting, where it is easier to control the various ingredients in the mix, and where we can abstract from sampling error.

The preferences suggested by Epstein and Zin (1989) separate the coefficient of relative risk aversion ($CRRA$) from the elasticity of intertemporal substitution (EIS), and in doing so provide a partial separation of attitutes toward risk from preferences over deterministic consumption paths. This paper generalizes the model of Epstein and Zin (1989) by allowing the representative agent to display state dependent preferences and assesses if the resulting choices can add to the explanation of various empirical puzzles that have been investigated in finance and macroeconomics. For simplicity, we work within the simple environment introduced by Mehra and Prescott (1985). We investigate various combinations of state dependent $CRRA$ with state dependent EIS. In the case of constant elasticity of intertemporal substitution and time varying $CRRA$, our results look very similar to those generated without state dependence. However, we also investigate the same model but with time variation in other features of preference. We find that introducing an EIS that varies only slightly with the state while keeping the other features of preferences constant provides a good approximation to the first two moments of the returns on equity and the risk free rate. Combining a state dependent EIS with a state dependent $CRRA$ allows us to match perfectly the US historical data. For completeness, we also consider in our calculations the possibility that patience is state dependent, but find that adding state dependence to the discount factor adds little to the story described above.

There is a long tradition in finance of showing, in the absence of arbitrage, that asset prices can be rationalized by a representative agent with some utility function (for two recent examples, see Duffie (1992) or Kubler (2003a,2003b)). Although these results are for finite horizon economies and do not cover the state dependent recursive preferences that we investigate, it suggests that at some level the results in this paper should not be particularly surprising. However, we think it is useful to have an explicit example of a preference ordering that does the job. Our preferences introduce no new state variables beyond the exogenous state variables required to describe the endowment consumption process and they allow us to generate a stationary recursive equilibrium. If nothing else, our results should remind readers that statements such as "representative agent models are inconsistent with the data" or "consumption is too smooth to explain the equity premium" must be preceded by the qualifying phrase "given standard preferences". Our example also suggests that economists need to sharpen their intuition on the role played by state dependent risk aversion in explaining asset returns. We find that the received wisdom is supported by our findings, but this support only occurs when we also allow features of the preferences that govern choices over deterministic sequences to be state dependent as well.

Section 2 reviews the equity premium puzzle. Sections 3 and 4 provide new results that are useful in interpreting both the usefulness of the Mehra and Prescott economy as an approximation, and why standard preferences fail in this setting. In Section 3, we show how the first two moments of consumption growth, the return on equity, and the risk-free rate, combined with no arbitrage and a simple behavioural assumption can be used in the Mehra-Prescott economy to uniquely determine the points of support for these three processes. We also report summary statistics on the implied Sharpe ratios and predictability of asset returns that arise from this general setting. In Section 4, we go a bit further and compute the unique stochastic discount factor process that is required to match the data on asset returns in the Mehra and Prescott economy; we also report some properties of this process that can be compared to estimates obtained elsewhere. Some new intuition is provided for the inability of the class of isoelastic expected utility preferences to match the data and a heuristic argument is made to motivate state dependent preferences. Sections 5 and 6 provide a formal statement of the agent's problem and a characterization of the equilibrium. Technical results are relegated to an appendix. Our numerical results are reported in Section 7, and we end with a brief conclusion.

2. Equity Premium Puzzle

Mehra and Prescott (1985) report that the unconditional mean of the risk premium observed in the US stock market is much larger than that predicted by a standard version of the representative agent model. From 1898 to 1978 in the United States, the average annual real rate of return on short-term bills was 0.80 percent and the average annual real rate of return on stocks was 6.98 percent. Thus the average equity premium was 618 basis points. They calibrated an asset pricing model with time-additive isoelastic expected utility to see if the model could deliver unconditional rates of return close to the historical average rates of return on stocks and bills. They used a 2-point Markov process for consumption growth with mean $E(g_t) = 1.018$, standard deviation $\sigma(g_t) = 0.036$, and correlation $(g_t,\ g_{t-1}) = -0.14$. They found that the model cannot account for both the average level of the risk-free rate (0.8%) and the difference (6.18%) between the average rates of return on equity (6.98%) and on risk free securities during the period 1898-1978. With the $CRRA$ below 10, the model could not produce more than a 35 basis point equity premium. This result is called the equity premium puzzle. Although there is some debate over the moments used by Mehra and Prescott to calibrate their economy, we choose to proceed using their estimates in order to make our results as comparable as possible; at various points below, however, we comment briefly on the sensitivity of our results to modest variations in these moments.

In the time-additive isoelastic expected utility framework, the coefficient of relative risk aversion and the elasticity of intertemporal substitution are intertwined. The elasticity of intertemporal substitution is constrained to be the inverse of the coefficient of relative risk aversion. A very high coefficient of relative risk aversion (of the order of 40 or 50) makes it possible to replicate the large secular risk premium on equity, yet, the implied very low elasticity of intertemporal substitution also leads to the counter-factual prediction of an extremely high risk free rate. Conversely, a low coefficient of relative risk aversion leads to a counter-factually low equity premium, although it does imply a relatively low risk free rate.

Within the context of representative agent models, various authors have attempted to solve the equity premium puzzle by introducing more general preferences. One approach disentangles risk aversion attitudes from intertemporal substitution using a special class of recursive preferences (e.g. Epstein and Zin (1989), Weil (1989)). The intuition behind this approach is that given the risk free rate is mainly controlled by the magnitude of the elasticity of intertemporal substitution, while the risk premium is a reflection of the coefficient of relative risk aversion, a preference ordering that can parametrize the elasticity of intertemporal substitution and the coefficient of relative risk aversion independently should provide the additional degree of freedom required to replicate both the level of the risk free rate and the risk premium. The results reported in Epstein and Zin (1990) and Weil (1989), however, were not very encouraging (although Epstein and Melino (1995) do much better). A second approach modifies the preferences by introducing habit formation. Habit formation leads to a form of state dependent preferences that can generate substantial variation in the price of risk.

A very successful example is Campbell and Cochrane (1999). However, risk aversion and substitution are intertwined in their model; therefore it is hard to interpret which feature of preferences varies over the cycle, and, in particular, if changes in attitudes towards risk are all that is required. In a third approach, that is in many ways closest in spirit to this paper, Gordon and St. Amour (2000,2001) use expected-utility preferences but allow the $CRRA$ to be a latent process. They also introduce a scaling parameter and show that the ratio of consumption to their scaling parameter plays a crucial role in matching the data. Using Bayesian methods, they find that they can provide a good match to post WWII data even though the $CRRA$ process shows very little variation over the cycle. Again, it is not obvious which feature of preferences is varying over the cycle in their model. By working with an artificial economy we can simplify what's going on and provide an interesting perspective on their results.

3. The Environment

To highlight the role played by state dependent preferences, we shall follow as closely as possible the environment introduced by Mehra and Prescott (1985). We work with a Lucas endowment economy in which consumption growth follows a Markov process that takes on two values. We assume both points of support for consumption growth are equally likely and choose $(g_{l,}\ g_h)$ to match the first two moments of consumption growth given for the US historical data: $E(g) = 1.018$ and $\sigma(g) = 0.036$. This involves solving the equation system:

$$\begin{aligned} 0.5g_l + 0.5g_h &= E(g), \\ 0.5\left(g_l - E(g)\right)^2 + 0.5\left(g_h - E(g)\right)^2 &= s^2(g), \end{aligned} \tag{3.1}$$

Because the first equation is linear in the unknowns $(g_{l,}\ g_h)$, and the second is quadratic, there are two solutions. Under the obvious requirement $g_l < g_h$, we obtain

$$\begin{bmatrix} g_l \\ g_h \end{bmatrix} = \begin{bmatrix} 0.982 \\ 1.054 \end{bmatrix}. \tag{3.2}$$

These parameters for the endowment process and its points of support are the same as those used by Mehra and Prescott (1985), along with Weil (1989), Epstein and Zin (1990) and Epstein and Melino (1995). Mehra and Prescott (1985) also assume that the probability of staying in the same state is 0.43 and the transition probability matrix for consumption is symmetric, that is[1],

$$\Pi = \begin{bmatrix} \pi_{ll} & \pi_{lh} \\ \pi_{hl} & \pi_{hh} \end{bmatrix} = \begin{bmatrix} 0.43 & 0.57 \\ 0.57 & 0.43 \end{bmatrix}. \tag{3.3}$$

Our objective in this environment is to find preferences that match the first two moments of the returns on equity and the risk-free rate. Mehra and Prescott (1985) estimate the historical average return and standard deviation of equity to be $E(M) = 1.07$ and $\sigma(M) = 0.165$, and estimate the historical average return and standard deviation of T-bills to be $E(r_f) = 1.008$ and $\sigma(r_f) = 0.056$.

Melino and Epstein (1995) point out that if we assume consumption growth is a sufficient statistic for asset returns, then we can use the first two moments of asset returns to solve for the points of support for the risk free rate and equity processes in the Mehra-Prescott economy. They restrict attention to economies with stationary asset returns and a state vector given by $S_t = (s_t, c_t)$, where c_t denotes the economy's endowment of consumption at time t and s_t is an exogenous process that determines the evolution of endowment consumption growth g_t. Stationarity of the risk free process and the assumption that behaviour doesn't add any additional endogenous state variables implies that the risk free process is measurable with respect to s_t. This means that, just as with consumption growth, the risk-free rate process can take on only two values, and fitting the first two moments exactly leads to a quadratic equation with two solutions, namely $(r_l, r_h) = (0.952, 1.064)$ or $(r_l, r_h) = (1.064, 0.952)$. In order for equity returns to be stationary, the price-earnings[2] ratio at time t must also be expressable as a function of s_t; then the first two moments of equity returns can be used to solve for the two values that the price-earnings ratio can realize in this economy. Again, this leads to a quadratic equation, so there are two solutions for the support of the price-earnings ratio: $(P_l, P_h) = (23.4671, 27.8385)$ and $(P_l, P_h) = (28.9511, 22.7302)$. At first glance it looks like there are four solutions for the points of support of the risk-free rate and price-earnings process that will match the first two moments of the asset returns. However, among these four 'solutions', three imply arbitrage opportunities and can be discarded. Therefore, in this simple Mehra-Prescott environment, rather than focus on the first two moments of asset returns, our objective can be phrased as matching the data $(P_l, P_h) = (23.467, 27.839)$ and $(r_l, r_h) = (1.064, 0.952)$. Note that the required price-earnings

[1] Cecchetti, Lam, and Mark (1993) use a slightly later sample and more carefully constructed consumption data and estimate this probability to be 0.47. Yang (2001) shows that the conclusions in this paper are robust to small increases in π_{ll}.

[2] Earnings and dividends are equal in the Lucas endowment economy, so the price-earnings ratio is also the price-dividend ratio. The return on the market can be written as $M_{t+1} = (P_{t+1} + 1)/P_t * g_{t+1}$ where P is the price-dividend ratio.

ratio is procyclical, but the required risk-free rate is countercyclical. These cyclical patterns for returns have been obtained simply by assuming that they are measurable with respect to the same process that determines consumption growth, matching their first two moments, and excluding arbitrage. If we also assume there are no other sources of uncertainty in the economy beyond S_t, then the price of all other assets can be determined in the Mehra-Prescott environment from the price-earnings and risk-free rate processes. In the next section, we develop a particularly convenient way to summarize the asset pricing implications of this environment. Before doing so, it is straightforward to calculate the returns to equity, M_{ij}, that obtain when consumption growth goes from state i at time t to state j at time $t+1$ and to summarize some of their important properties:

M	l	h
l	1.024	1.295
h	0.863	1.092

(3.4)

So when the low consumption growth state is realized, i.e. the economy is 'recession', the representative agent faces a random return to holding equity of 2.4% if the economy stays in the low growth state next period, but an almost 30% return if the economy draws a high consumption growth next period. Similarly, when the high consumption growth state is realized, she faces a potential loss of about 14% if the economy falls into recession and a gain of 9.2% if the expansion continues. By construction, the returns in this two state economy fit the historical Sharpe ratio, defined by $E(M-r_f)/\sigma(M)$, equal to 0.376. The conditional Sharpe ratio, equal to 0.852 and 0.083 in the low and high growth states respectively, is strongly countercyclical.

Several authors have reported that excess returns are forecastable by a variety of variables, such as the short-term interest rate, r_{ft}, (e.g. Campbell (1991)), the dividend price-ratio, $1/P_t$ (Fama and French (1988) and Campbell and Shiller (1987)), and the consumption-wealth ratio (Lettau and Ludvigson (2001)); we show below that in a Lucas endowment economy the consumption-wealth ratio is just $1/(1+P_t)$. To assess the forecastability of excess returns using these variables, as well as lagged excess returns, consider a regression of the form:

$$M_{t,t+1} - r_{ft} = \alpha + \beta X_t + u_t \tag{3.5}$$

We can easily calculate the population counterparts of β and the R^2 from (3.5) for the choices of X listed in the previous paragraph. The results are summarized in the following table:

X_t	β	R^2
$M_{t-1,t} - r_{f,t-1}$	-0.375	.141
r_{ft}	0.938	.152
$1/P_t$	15.706	.152
$1/(1+P_t)$	16.963	.152

(3.6)

Not surprisingly, the negative serial correlation in the consumption endowment process induces a negative serial correlation in the excess return to holding equity. The corresponding population R^2 is higher than what we would compute using historical samples based on annual data. It would be lowered if we reduced the negative serial correlation in the consumption growth process; a similar comment applies to the R^2 corresponding to the remaining regressors reported in (3.6). To anticipate our later results, reducing the serial correlation in consumption growth also makes it 'easier' to rationalize the asset return processes in that it increases the set of parameters within our preference ordering that does so. It also us to do so with somewhat lower coefficients of risk aversion.

4. Intuition on Why State Dependent Preferences will Work

Hansen and Richard (1987) show that in the absence of arbitrage opportunities, there exists a (positive) stochastic discount factor $Q_{t,t+1}$ that allows us to price assets via the fundamental equation

$$E_t(Q_{t,t+1}R^m_{t,t+1}) = 1 \tag{4.1}$$

where E_t denotes the conditional expectation operator, and $R^m_{t,t+1}$ denotes the return to holding asset m from time t to $t+1$.

In the simple Mehra-Prescott economy, we have two states and two assets (equity and the risk-free bond), so the stochastic discount factor process is uniquely determined from the asset return processes derived in the previous section. Let Q_{ij} denote the discount factor that is applied in state i to payoffs next period in state j. Using the results from the previous section[3] and eq(4.1), we can compute the 2x2 matrix of discount factors for this economy:

Q	l	h
l	1.862	0.244
h	1.127	0.949

(4.2)

[3]Recall that the return to equity realized if we go from state i at time t to state j at time $t+1$ is given by $M_{ij} = (P_j + 1)/P_i * g_j$ and the risk-free rate in state i is r_i.

Looking at the Mehra-Prescott economy from this new perspective introduces some useful insights. To match the data on the first two moments of asset returns, we require a stochastic discount factor that is very sensitive to the current state. When the representative agent experiences a low draw for consumption growth, she assigns a very high value to assets that will pay off one unit of consumption if the low state is realized next period (1.862) but a relatively low value to assets that will pays off one unit of consumption if the high state is realized next period (0.244). By contrast, when the representative agent draws a high value for consumption growth, she assigns approximately equal weights to the payoffs in the two states. Informally, the representative agents acts as if she is very risk averse during recessions, but only modestly risk averse during booms.

As a check on the reasonableness of the required Q process given by (4.2), and indirectly on the reasonableness of using the two-state process underlying the Mehra-Prescott economy as an approximation, it is useful to point out that its mean and standard deviation, $E(Q) = .995$ and $\sigma Q) = .573$, satisfy the bounds computed by Hansen and Jaganathan (1991) (see their Figure 1). The first moment of Q varies somewhat across states, but most of its variation comes from the high conditional variance of Q in the low growth state. A further check on the required Q process can be obtained by computing the predicted return processes for various other assets. Yang (2001) shows that the required Q yields an average real yield on 10 year discount bonds of 3.5%; the conditional yields are 4.3% in the low growth state and 2.7% in the high growth state. We stress that these predictions, as well as the results in the previous section on the returns to holding the risk-free asset and equity, follow from a fairly small set of inputs: a) the two-state Markov process for consumption growth; b) the estimated first and second moments of the returns to the risk-free asset and equity; c) the assumption that consumption growth is a sufficient statistic for the risk-free rate and price-dividend ratio in each period; and d) no arbitrage. Any model consistent with the environment (a-b) and behaviour (c-d) inherits all of the other features described above.

Mehra and Prescott (1985) tried to approximate the required stochastic discount factor process using an equilibrium model with the environment above and a representative agent with isoelastic expected utility preferences. With such preferences, it is well known that the predicted stochastic factors turn out to be of the form

$$\widetilde{Q}_{ij} = \beta g_j^{-\gamma} \tag{4.3}$$

where β denotes the subjective discount rate and γ is the coefficient of relative risk aversion.

Notice that with isoelastic expected utility preferences, the stochastic discount factor varies over time only with the rate of consumption growth realized next period. Comparing to (4.2), this says that the predicted two rows in the Q matrix must be identical; obviously these preferences are not going to work. If we naively fit the parameters β and γ to the first row of the Q matrix (this would amount to assuming that the agent behaved myopically and did not realize her preferences were likely to change in the future), we obtain the estimates $\beta = 1.105$ and $\gamma = 28.722$. If we fit these parameters to the second row of the Q matrix, we obtain the estimates $\beta = 1.079$ and $\gamma = 2.415$. The two values of beta are fairly similar, although both indicate a preference for postponing consumption and would be ruled inadmissible. The implied value of the $EIS/CRRA$ parameter, however, varies sharply across the two states.

Epstein and Zin (2001) and Weil (1989) looked at this same economy but replaced the isoelastic expected utility preferences with a recursive preference ordering having a CES aggregator and an expected utility certainty equivalent. This leads to predicted stochastic discount factors of the form

$$\widetilde{Q}_{ij} = \beta g_j^{-\gamma} (\beta M_{ij}/g_j)^{\delta} \tag{4.4}$$

where M_{ij} denotes the return to holding equity from state i to state j, and δ is a function of preference parameters. The dependence of the predicted stochastic discount factor on M_{ij} removes the restriction that both rows of the predicted Q matrix must be identical. However, the failure of Epstein and Zin (2001) and Weil (1989) to find a significant improvement over results based on eq (4.3) indicates that the additional flexibility afforded by eq (4.4) is insufficient to match the data. Indeed, as we'll show below, once the stochastic discount factors of eq (4.4) are calibrated to match the equity return process, the predicted stochastic discount factors obtained with these preferences are virtually indistinguishable from those obtained using isoelastic expected utility preferences.

Various other preference orderings have appeared in the literature. Typically, they introduce additional state variables to the equilibrium asset pricing process and so do not yield the prediction that the price-dividend and risk free rate processes are measurable with respect to consumption growth. In some cases, a further drawback is that they do not lead to stationary asset returns. Two examples merit special attention.

Campbell and Cochrane (1999) consider a preference ordering with habits that leads to a stochastic discount factor of the form

$$\widetilde{Q}_{t,t+1} = \beta g_{t+1}^{-\gamma} (H_{t+1}/H_t)^{-\gamma} \tag{4.5}$$

where H_t denotes the proportion of consumption in excess of the inherited habit level at time t. The coefficient of relative risk aversion with these preferences is γ/H_t.

Notice that the stochastic discount factor of eq (4.5) varies explicitly with the current and the subsequent period. Campbell and Cochrane (1999) cleverly 'reverse engineer' a stochastic process for H and show that this allows them to match the historical behaviour of asset returns, but find that this requires that the representative agent has very high values of risk aversion during recessions. Introducing habit persistence augments the state space and in this paper we show that we can rationalize the first two moments of the equity return and risk free rate processes without introducing additional state variables. Notice, however, that with these preferences it is straightforward to obtain stationary returns as long as the model for habits yields a stationary process for H_{t+1}/H_t[4].

Gordon and St. Amour (2000) examine a version of state dependent expected utility preferences. Their preference ordering leads to a stochastic discount factor of the form

$$\widetilde{Q}_{t,t+1} = \beta g_{t+1}^{-\gamma(s_t)} \left(\frac{c_{t+1}}{\theta}\right)^{\gamma(s_t)-\gamma(s_{t+1})} \tag{4.6}$$

where the coefficient of relative risk aversion depends on a latent state variable, s_t (not necessarily the process that determines consumption growth, and c_{t+1}/θ is the ratio of next period's level of consumption to a scale parameter θ. Notice the stochastic discount factor depends upon both the level and growth rate of consumption. Adding the level of consumption as a state variable for asset returns augments the state space. It also leads to the result that in a growing economy asset returns will not be stationary, even if s_t is stationary. Our state dependent Epstein-Zin preferences will not include the Gordon-St. Amour state dependent expected utility preferences as a special case. However, our preferences have the benefit of leading to a stationary recursive equilibrium in a growing economy.

To match the first two moments of the returns on equity and the risk free rate, we require a stochastic discount factor process that varies sharply with the current state. Our naive calculation above suggests that state dependent preferences are one way to account for why the representative agent assigns such different values to next period's outcomes as the current state varies, and may be able to match the required stochastic discount factor process and therefore the first two moments of the returns on equity and the risk free rate. Of course, if preferences are state dependent, then the agent will take this account in forming her state contingent plans and the form of the equilibrium stochastic discount factor will not be a simple variation of (4.3). To proceed, we must describe the environment, the agent's preferences, and the equilibrium more formally.

[4]It is tempting to simply assume that H can only take on two values indexted by s_t and then treat the ratio H_i/H_j as a parameter to be determined (the individual terms H_i and H_j are not identifiable). However, our calculations show that this leads to an economy that cannot match the asset return data. We did not investigate richer specifications for the H process.

5. The Representative Agent's Problem

Consider an infinitely-lived representative agent who receives utility from the consumption of a single good in each period. At time t, current consumption is non-stochastic but future consumption levels are generally stochastic. It is convenient to discuss first the budget constraint and then the agent's preferences.

5.1. Budget Constraint Facing a Representative Agent

Assume there are n assets that pay dividends $d_t = (d_{1t}, d_{2t}, ..., d_{nt})$ and are traded competitively at prices $p_t = (p_{1t,} p_{2t}, ..., p_{nt})$. The agent starts each period with asset holdings denoted by $z_t = (z_{1t}, z_{2t}, ..., z_{nt})$.

Let x_t represent beginning-of-period wealth,

$$x_t = (d_t + p_t)z_t, \qquad t \geq 0. \tag{5.1}$$

Define the portfolio share vector $w = (w_1, w_{2,} ..., w_n)$ and the gross return vector $r_t = (r_{1t}, r_{2t}, ..., r_{nt})$ by

$$w_i = \frac{p_{it} z_{it}}{p_t z_t}, \tag{5.2}$$

and

$$r_{it} = \frac{d_{it} + p_{it}}{p_{i,t-1}}, \qquad t \geq 1. \tag{5.3}$$

Then we see that wealth evolves according to

$$x_{t+1} = (x_t - c_t) w_t r_{t+1} \tag{5.4}$$

5.2. Preferences of the Representative Agent

We assume the agent has preferences defined on random consumption sequences that may be constructed by means of the following recursive functional relation:

$$U_t = \left(c_t^{\rho(s_t)} + \beta(s_t) \left[E_t \left(U_{t+1}^{\alpha(s_t)} \right) \right]^{\frac{\rho(s_t)}{\alpha(s_t)}} \right)^{\frac{1}{\rho(s_t)}}, \ 0 \neq \alpha < 1, \ 0 \neq \rho < 1. \tag{5.5}$$

where E_t is the expectation conditional on period-t information, $\beta(s_t)$, $\alpha(s_t)$ and $\rho(s_t)$ are parameters that depend on an exogenous state variable s_t. The coefficient of relative risk aversion ($CRRA$) for 'timeless gambles' is $1-\alpha(s_t)$. The parameters $\beta(s_t)$ and $\rho(s_t)$ are harder to interpret. When $\rho(s_t) = \rho$, a constant, $\beta(s_t)$ is the discount parameter and measures time preference while the elasticity of intertemporal substitution (EIS) is $1/(1-\rho)$. For this reason, we will refer to $\beta(s_t)$ as the discount parameter and $\rho(s_t)$ as the EIS parameter throughout the paper. Some comments on the interpretation of these parameters when $\rho(s_t)$ varies with the state are provided below.

Maximizing U_t subject to the wealth accumulation constraint given in eq(5.4) yields the following first order conditions

$$E_t\left(IMRS_{t,t+1}M_{t,t+1}\right) = 1, \tag{5.6}$$
$$E_t\left(IMRS_{t,t+1}r_{j,t+1}\right) = 1, \quad j = 1, 2, ..., n \quad , \tag{5.7}$$

where $M_{t,t+1} = w_t^* r_{t+1} = \sum_{i=1}^{n} w_{it}^* r_{i,t+1}$ is the gross return to holding the optimal portfolio w_t^* from time t to $t+1$, and the intertemporal marginal rate substitution from time t to $t+1$ ($IMRS_{t,t+1}$) is given by

$$IMRS_{t,t+1} = \beta(s_t) g_{t+1}^{\alpha(s_t)-1} \left(\frac{\beta(s_t)M_{t,t+1}}{g_{t+1}}\right)^{\frac{\alpha(s_t)}{\rho(s_t)}-1} (a_{t+1})^{\frac{\alpha(s_t)}{\rho(s_t)}-\frac{\alpha(s_t)}{\rho(s_{t+1})}} \tag{5.8}$$

where $a_t \equiv c_t/x_t$ is the consumption wealth ratio. See Appendix A for a derivation of this expression. Note that eq (5.8) exploits only information about the agent's preferences and her budget constraint. In the Lucas endowment economy, the equilibrium conditions afford a further simplification, see eq(6.8) , that will be exploited below.

If $\alpha(s_t) = \alpha$ (a constant), $\rho(s_t) = \rho$ (a second constant), and $\beta(s_t) = \beta$ (a third constant), then eq(5.6) and eq(5.7) reduce to the model of Epstein and Zin (1989), namely

$$IMRS_{t,t+1}^{EZ} = \beta g_{t+1}^{\alpha-1} \left(\frac{\beta M_{t,t+1}}{g_{t+1}}\right)^{\frac{\alpha}{\rho}-1} \tag{5.9}$$

with $EIS = 1/(1-\rho)$ and $CRRA = 1-\alpha$.

If $\alpha(s_t) = \rho(s_t) = \rho$ (the same constant), and $\beta(s_t) = \beta$ (another constant), eq(5.6) and eq(5.7) reduce further to the familiar Euler equations of the expected utility model (see Hansen and Singleton (1983)), i.e.

$$IMRS_{t,t+1}^{eu} = \beta g_{t+1}^{\rho-1} \tag{5.10}$$

with $EIS = 1/(1-\rho)$ and $CRRA = 1-\rho$.

5.3. Some interpretation of the preference parameters

As we shall see below, having a state dependent EIS parameter plays a crucial role in rationalizing the Mehra-Prescott data, so some intuition as to its interpretation would indeed be welcome. Suppose the future states $\{s_t, s_{t+1}, ...\}$ are known and we consider only deterministic consumption sequences. In this case, the recursive definition of the preference ordering simplifies to

$$U_t^{\rho(s_t)} = c_t^{\rho(s_t)} + \beta(s_t) U_{t+1}^{\rho(s_t)} \tag{5.11}$$

Suppose we consider consumption paths that are small perturbations around a reference path $\{\overline{c}_t\}$. For example, we could have a reference path such as the endowment sequence $\overline{c}_t = g(s_t)\overline{c}_{t-1}$. Let $\overline{V}_t$ denote the value of utility associated with the reference path. (We show in the Appendix that if $\{\overline{c}\}_t$ is the endowment path, then the maximized value of $\overline{V}_t$ is proportional to $\overline{c}_t$). Without loss of generality, we can rewrite (5.11) as

$$\left(\frac{U_t}{\overline{V}_t}\right)^{\rho(s_t)} = \left(\frac{c_t}{\overline{V}_t}\right)^{\rho(s_t)} + \beta(s_t)\left(\frac{\overline{V}_{t+1}}{\overline{V}_t}\right)^{\rho(s_t)} \left(\frac{U_{t+1}}{\overline{V}_{t+1}}\right)^{\rho(s_t)} \tag{5.12}$$

Assume that the perturbations are such that we can use a first order approximation to the left hand side of (5.12) around $U_t = \overline{V}_t$, i.e.

$$\left(\frac{U_t}{\overline{V}_t}\right)^{\rho(s_t)} = 1 + \rho(s_t)\left(\frac{U_t}{\overline{V}_t} - 1\right) \tag{5.13}$$

Use the same approximation for the last term on the right hand side of (5.12). Simplifying yields a linear difference equation in U_t of the form

$$U_t = u(c_t; s_t) + \beta(s_t)\left(\frac{\overline{V}_{t+1}}{\overline{V}_t}\right)^{\rho(s_t)-1} U_{t+1} \tag{5.14}$$

where the felicity function has derivative

$$u'(c_t; s_t) = \left(\frac{c_t}{\overline{V}_t}\right)^{\rho(s_t)-1} \tag{5.15}$$

When $\overline{V}_t$ is a constant, as we might expect if the reference sequence is more or less stationary, and assuming that $\beta(s_t)$ is a constant, the approximation (5.14) leads to the same preferences over deterministic sequences as those used by Gordon and St-Amour (2001). In general, however, the parameter $\rho(s_t)$ influences the curvature of the felicity function, the reference level of utility, and the rate of time preference. The first effect is relatively straightforward. Let $Q(t, t+1)$ denote the (absolute value of the intertemporal) marginal rate of substitution between $c(t)$ and $c(t+1)$. The elasticity of intertemporal substution, $EIS(t, t+1)$, measures the elasticity of $c(t+1)/c(t)$ with respect to $Q(t, t+1)$ along an isoquant. Using the approximation (5.14), we obtain

$$Q(t, t+1) = \frac{\beta(s_t) \left(\frac{\overline{V}_{t+1}}{\overline{V}_t}\right)^{\rho(s_t)-1} \left(\frac{c_{t+1}}{\overline{V}_{t+1}}\right)^{\rho(s_{t+1})-1}}{\left(\frac{c_t}{\overline{V}_t}\right)^{\rho(s_t)-1}} \tag{5.16}$$

Therefore, the *inverse* of $EIS(t, t+1)$ is given by

$$\begin{aligned} \frac{d \ln Q(t, t+1)}{d \ln(c_{t+1}/c_t)} &= 1 - \rho(s_t) + [\rho(s_{t+1}) - \rho(s_t)] \frac{d \ln(c_{t+1})}{d \ln(c_{t+1}/c_t)} \\ &= 1 - \rho(s_t) + \frac{\rho(s_{t+1}) - \rho(s_t)}{1 + Q(t, t+1)\frac{c_{t+1}}{c_t}} \end{aligned} \tag{5.17}$$

where the second line in (5.17) follows from noting that along an isoquant we have $dc_t = -Q(t, t+1) dc_{t+1}$. In those adjacent periods for which the state doesn't change, we have $\rho(s_{t+1}) = \rho(s_t)$, so $EIS(t, t+1) = 1/\left(1 - \rho(s_t)\right)$. In general, we $EIS(t, t+1)$ lies between $1/\left(1 - \rho(s_t)\right)$ and $1/\left(1 - \rho(s_{t+1})\right)$, as $Q(t, t+1)c_{t+1}/c_t$ is strictly positive.

In ranking random consumption sequences, the roles played by the various parameters is more involved. The intuition provided by Epstein and Zin (1989) that the certainty equivalent function describes risk preferences over timeless gambles is still valid. Intuitively, timeless gambles can be thought of those risks where the uncertainty is realized in an atemporal setting such as an experiment. Epstein and Zin (1989) showed that if the preference parameters are not state dependent, then risk aversion is strictly increasing in α for fixed values of (β, ρ). They also showed that some random consumption sequences could be ranked having only knowledge of the certainty equivalent function. These two properties constituted what they called a 'partial separation' of attitudes toward risk and intertemporal substitution. We have been unable to verify either of these two properties for our state-dependent version of the Epstein-Zin preferences. In fact, it appears that all three parameters play an important role in ranking growing random consumption sequences that is difficult to separate. To illustrate, suppose we consider random consumpton sequences of the form $\{c_t, \widetilde{g}c_{t+1}, \widetilde{g}^2 c_{t+1}, ...\}$, so that next period's growth rate is random but once realized it will continue forever thereafter so that $\widetilde{g} = g_{t+1} = g_{t+2} = ...$, etc. Notice that all uncertainty is eliminated at time $t+1$, so from then the preference parameters can be treated as constants. For such random sequences, the utility function (5.5) reduces to

$$U_t^{\rho(s_t)} = c_t^{\rho(s_t)} + \beta(s_t) c_t^{\rho(s_t)} \left[E_t \left(g_{t+1}^{\alpha(s_t)} \left(1 - \beta(s_{t+1}) g_{t+1}^{\rho(s_{t+1})} \right)^{\frac{-\alpha(s_t)}{\rho(s_{t+1})}} \right) \right]^{\frac{\rho(s_t)}{\alpha(s_t)}}$$

6. Equilibrium

We consider a Lucas-style endowment economy. There is a single perishable consumption good, the total supply of which is described by the endowment process $c = \{c_t\}$. Equity, interpreted as a claim to the endowment process, is the only asset in non-zero net supply. Without loss of generality, we take equity to be the first asset in the vector z and we normalize the number of shares of equity in this economy to 1. The *equilibrium* is described by the price process p such that the goods market and asset markets clear. That is, the solution to the utility maximization above with price process p has $c_t = d_{1t}$ and $z_t = (1, 0, 0, ..., 0)$ for $t \geq 1$.

The return to the market portfolio $M_{t,t+1}$, under the previous assumption, equals the return to wealth

$$M_{t,t+1} = \frac{p_{1,t+1} + c_{t+1}}{p_{1,t}}. \tag{6.1}$$

For convenience, we will drop the first subscript and subsequently will refer to the price of equity and its dividend at time t as p_t and d_t.

Following Mehra and Prescott (1985), we assume that the endowment process for consumption is such that the growth rate $g_{t+1} = c_{t+1}/c_t$ follows a first-order, two-state Markov process. Further we assume the only component of the exogenous state variable s_t that indexes the parameters of the agent's preferences is the consumption growth rate. Now we will determine the supporting equilibrium price process.

We consider only equilibria where the ex-dividend price of equity is described by the time-invariant and positive function, $p\left(g_t, c_t\right)$ of the variables g_t and c_t. It follows, because of the homogeneity of preferences, that the price is linearly homogeneous in consumption; that is,

$$p\left(g, c\right) = p\left(g, 1\right) c = P\left(g\right) c. \tag{6.2}$$

Thus eq(6.1) simplifies to

$$M_{t,t+1} = \frac{p(g_{t+1}, c_{t+1}) + c_{t+1}}{p(g_t, c_t)} = \frac{P(g_{t+1}) + 1}{P(g_t)} g_{t+1}. \tag{6.3}$$

We also assume that the exogenous variable g_t takes only two values and is distributed as

$$g_{t+1} = \left\{ \begin{array}{llll} g_h & \text{w.p.} & \pi_h & \text{high state,} \\ g_l & \text{w.p.} & \pi_l & \text{low state.} \end{array} \right. \tag{6.4}$$

with transition probability matrix

$$\Pi = \begin{bmatrix} \pi_{ll} & \pi_{lh} \\ \pi_{hl} & \pi_{hh} \end{bmatrix} \tag{6.5}$$

where π_{ij} is the probability of going from state i to state j and its value is given by (3.3). Note that this transition probability matrix yields the marginal probabilities $\pi_h = \pi_l = 0.5$.

The conditional expected return on equity (which is the same as the return to the market portfolio given there is only one outside asset) if today's (time t) state is $i \in \{l, h\}$ and $t+1$ state is $j \in \{l, h\}$ will be:

$$E(M_{t,t+1} | g_t = g_i) = \sum_{j \in \{l,h\}} \pi_{ij} M_{ij}, \tag{6.6}$$

where π_{ij} is transition probability from eq(6.5) and M_{ij} is given by eq(6.3). For convenience, we use P_j and r_j to denote the price-dividend ratio and risk-free rate that obtain when the realized state is g_j.

In the Lucas endowment equilibrium, the agent's wealth, given by (5.1), reduces to

$$x_t = (c_t + p_t) \cdot 1 \tag{6.7}$$

so the consumption-wealth ratio in equilibrium can be written as

$$a_t \equiv \frac{c_t}{x_t} = \frac{1}{1 + p_t/c_t} = \frac{1}{1 + P_t} \tag{6.8}$$

where $P_t \equiv p_t/c_t$ is equity price-earnings ratio. Using eq(6.3) to replace $M_{t,t+1}$, and eq(6.8) to replace a_{t+1} with $1/(1 + P_{t+1})$, we see that in the equilibrium for this economy, the representative agent's intertemporal marginal rate substitution[5] from time

[5] We use the term 'stochastic discount factor' to describe the $IMRS$ after substituting terms involving market equilibrium.

t to $t+1$, given by eq(5.8) , simplifies to give us a stochastic discount factor for this economy of the form

$$Q_{t,t+1} = \beta(s_t) g_{t+1}^{\alpha(s_t)-1} \left[P_t/\beta(s_t)\right]^{1-\frac{\alpha(s_t)}{\rho(s_t)}} \left[(1+P_{t+1})\right]^{\frac{\alpha(s_t)}{\rho(s_{t+1})}-1} \tag{6.9}$$

For future reference, notice that $Q_{t,t+1}$ varies explicitly with the current period's coefficient of relative risk aversion, $\alpha(s_t)$, but only implicitly on $\alpha(s_{t+1})$ through the endogenous price-earnings process. By contrast, $Q_{t,t+1}$ in equilibrium varies explicitly with both current and next period's elasticity of intertemporal substitution, indexed by $\rho(s_t)$ and $\rho(s_{t+1})$.

7. Results

In this section we report our results for various combinations of state dependent $CRRA$, EIS and discount parameters. We begin with the case in which the coefficient of relative risk aversion ($CRRA$) varies with the state, but the elasticity of intertemporal substitution (EIS) is a constant. We show that this generalization does little to improve the match between the historical data and the predicted asset returns in the Mehra-Prescott economy. We then consider the case of a constant $CRRA$ but a state dependent EIS. Somewhat unexpectedly, we find that this is a more promising direction. Finally, we allow both features of preference to vary with the state. This allows us to match the historical first two moments exactly. This is done with very modest procyclical variation in the EIS but with strong countercyclical risk aversion on the part of the representative agent. We also investigate the potential contribution of allowing the discount parameter to also vary with the state and find that not much changes.

In what follows, it is convenient to denote by α_l the value of $\alpha(s_t)$ in the low growth consumption state. The parameters α_h, ρ_l, and ρ_h are defined analogously.

7.1. State Dependent $CRRA$, Constant EIS, and Constant Discount Parameters

With $\rho(s_t) = \rho$ (a constant) and $\beta(s_t) = \beta$ (a second constant), but allowing for cyclical variation in attitudes towards risk, eq(6.9)simplifies to

$$Q_{t,t+1} = \beta g_{t+1}^{\alpha(s_t)-1} \left(\frac{\beta(1+P_{t+1})}{P_t}\right)^{\frac{\alpha(s_t)}{\rho}-1} \tag{7.1}$$

Although many authors have stressed the role of countercyclical risk aversion in accounting for asset pricing puzzles, in this section we show that the Q given by eq (7.1) cannot match the observed first two moments of asset returns. Following Mehra and Prescott (1985), we fix $\beta = .95$. For a grid of values for ρ, we use the pricing equation for equity, eq (5.6), to solve for the values of α_l and α_h that match exactly the equity return process[6]. We then use these parameter values to compute the risk-free rate using (5.7). The results are reported in Table 1.

For $\rho > 0$, we see that matching the equity return process leads to inadmissible values for α_h. Taken at face value, this says that to match the equity process we need a representative agent who is risk-loving during expansions. These preference parameters lead to a risk free rate that is a) too high and b) procyclical rather than countercyclical as shown to be required in Section 4.

When ρ is negative but greater than -9, there are no values of α_l and α_h that match exactly the equity return process. For very small values of $EIS = 1/(1-\rho)$, solutions exist and lead to admissible values for the $CRRA$ parameters, but the representative agent now turns out to be very risk averse in both states. The implied risk free rate is too low but is now countercyclical as required. However, as we vary ρ, not much happens to the implied risk-free rate process; it improves in some directions but deteriorates in others. Weil (1989) examined a special case of our economy. His preferences were not state dependent, so he set $\alpha(s_t) = \alpha$ (a constant). Weil concluded that the Epstein-Zin preferences had very little to contribute to an explanation of the equity premium puzzle[7]. He found no substantive improvement over using expected utility preferences. Although he could come close to matching the first moments of equity and the equity premium with a $CRRA = 45$ and an $EIS = 1/10$ (whereas with $CRRA = 45$ and $EIS = 1/45$ the match was horrible), he concluded that the need to rely on such high risk aversion made the results very nonrobust and moreover the match in second moments was poor. Our results in Table 1, and an enormous variety of additional experiments reported in Yang (2001) lead to an even stronger statement. Epstein-Zin preferences with state dependent risk aversion have very little to contribute to an explanation of the equity premium puzzle: We can do about as well Weil (1989).

Some insight is gained by looking at the stochastic discount factors implied by the

[6] The values of $\alpha_l = 0$ and $\alpha_h = 0$ will always solve eq (5.6), but these values have been ruled out by assumption to simplify our expressions. Care must be taken in the numerical work to avoid these spurious solutions.

[7] Epstein and Melino (1995) give a stronger result. Using a revealed preference argument, they show that no member of the class of state independent recursive preferences (with a CES aggregator) can rationalize the Mehra-Prescott data.

preference parameters in Table 1. These values when they exist do not vary much with ρ, so we can make the point by focusing on one example. When $\rho = -44$, fitting the risk aversion parameters leads to $Q_{ll} = 2.11$ and $Q_{lh} = 0.12$. These are fairly close to the required values given in eq (4.2). But we also obtain $Q_{hl} = 1.89$ and $Qhh = 0.14$, whereas the required values in the high growth state are much closer to equality.[8] We conclude that if we calibrate these preference parameters to the equity process, we simply don't move away much from the isoelastic expected utility prediction that the stochastic discount factors will not vary with the initial state.

7.2. Constant $CRRA$, State Dependent EIS, and Constant Discount Parameters

With $\alpha(s_t) = a$ (a constant), and $\beta(s_t) = \beta$ (a second constant), but allowing for cyclical variation in the elasticity of intertemporal substitution, eq(6.9) simplifies to

$$Q_{t,t+1} = \beta g_{t+1}^{\alpha-1} \frac{[(1 + P_{t+1})]^{\frac{\alpha}{\rho(s_{t+1})}-1}}{[P_t/\beta]^{\frac{\alpha}{\rho(s_t)}-1}} \tag{7.2}$$

Note that the predicted stochastic discount factor for this economy varies with the current state only through the denominator of the third term in eq(7.2).

Again, we fix $\beta = .95$. For a grid of values for α, we use the pricing equation for equity, eq(5.6), to solve for the values of ρ_l and ρ_h that match exactly the equity return process. We then use these parameter values to compute the risk-free rate using (5.7). The results are reported in the first panel of Table 2.

For low values of risk aversion, the implied risk-free rates are too high on average, and the implied equity premium is too low. For example, at $\alpha = 1$ (equivalently, $CRRA = 0$), we obtain a risk free rate in the low growth state, r_l, that is too high (1.17 vs. 1.064), but a value in the high growth state, r_h, that is about right (0.95). As we raise $CRRA$, both risk free rates fall but in a way such that the equity premium rises. At the value of $\alpha = -14$, we match the historical equity premium to within 15 basis points. As we are matching the equity process by construction, this means that at these parameter values, we replicate the first moment of the return on equity and the risk free rate. We also match the second moment of equity returns, although the risk free rate process is too volatile. Unfortunately, the values of the EIS parameters

[8] When $\rho > 0$, the stochastic discount factors in the low state are similar to those reported for $\rho = -44$, but in the high state they are *reversed*.

obtained with $\beta = .95$ and $\alpha = -14$ are inadmissible: They satisfy the agent's first order conditions only.

If we increase the value of β and repeat the exercise, however, we obtain admissible parameter values. For a given $CRRA$, varying β changes the values of ρ_l and ρ_h that are required to match exactly the equity return process, but the implied stochastic discount factor does not change. As a result the predicted risk free rates in Table 2 depend only on the value of α. The second through fifth panels of Table 2 give the values of ρ_l and ρ_h that are required to match the equity return process for some values of β ranging from .96 to .99. Increasing β lowers the required values of ρ_l and ρ_h so that at these higher values of β, we can match the first moment of the return on equity and the risk free rate with values of ρ_l and ρ_h less than one. Note also that the EIS parameters is procyclical in panels 3-5.

Although the EIS parameter varies only a little across the two states, this is enough to have a major effect on the predicted asset returns. We know from the work of Weil (1989) and others, that the Epstein-Zin preferences without state dependence cannot match the the first two moments on asset returns. Indeed, if we keep $\alpha = -14$, but set $\beta = .98$, and $\rho_l = \rho_h = 1.85$ (the average of the two values from Table 2), we get $E(M) = .96$ and $E(r) = 1.05$, and an equity premium that is negative.

Although we do much better by allowing the EIS parameter to vary while holding the $CRRA$ constant than vice-versa, the excessive volatility in the risk free rate is bothersome. A look at the implied stochastic discount factors from Table 2 obtained setting $\alpha = -14$ is also troubling: the implied stochastic discount factors do not vary enough with the initial state to match the required pattern given in eq(4.2).

7.3. Cyclical $CRRA$, Constant EIS, and Cyclical Discount Parameters

Table 3 gives the results with $\rho(s_t) = \rho$ (a constant), but with cyclical variation in risk aversion and time preference. For each value of ρ, we solve for the values of $\{\beta_l, \beta_h, \alpha_l, \alpha_h\}$ that fit the required stochastic discount factor process and exactly matches the first two moments of the return to equity and the risk free asset. Note that as ρ falls, the values of β_l and β_h in Table 3 both increase. The value of β_h, however, always exceeds 1. This example shows that having a flexible preference ordering by itself is not enough to rationalize the asset return data. Although we have four parameters to choose and exactly four moments to fit, we can't do so within the set of admissible parameters[9].

7.4. Cyclical *CRRA*, Cyclical *EIS*, and Constant Discount Parameters

We now consider the case where $\beta(s_t) = \beta$ (a constant), but both α and ρ vary with the state. Given β, eq (5.6) and eq (5.7) that price equity and the risk free asset reduce to four equations in four unknowns in the Mehra-Prescott economy. The results from choosing the preference parameters to match exactly the first two moments of asset returns are given in Table 4.

For $\beta = .95$, the representative agent's first order conditions price the two assets exactly with values risk aversion parameters of $\alpha_l = -23.25$ and $\alpha_h = 0.83$. As anticipated in Section 4, this means the agent is very risk averse during recessions (low growth states) but almost risk neutral during booms (high growth states). The agent's implied values for the elasticity of intertemporal substitution parameters are $\rho_l = 1.00$ and $\rho_h = 1.75$. Unfortunately the latter value is inadmissible.

Increasing β has almost no effect on the risk aversion parameters needed to rationalize the first two moments of asset returns. But, as we increase β , the values for ρ_l and ρ_h decrease. For values of $\beta \geq .97$, both elasticity of intertemporal substitution parameters fall in the admissible range. Although the values of ρ_l and ρ_h are very similar in the two states, we can see comparing to Table 1 that this modest procyclical variation in the *EIS* is crucial to fitting the moments of asset returns.

7.5. State Dependent *CRRA*, *EIS* and Discount Parameters

What happens if we allow all three of our preference parameters to vary with the state? The stochastic discount factor, reproduced again for convenience, is given by

$$Q_{t,t+1} = \beta(s_t) g_{t+1}^{\alpha(s_t)-1} \frac{[(1+P_{t+1})]^{\frac{\alpha(s_t)}{\rho(s_{t+1})}-1}}{[P_t/\beta(s_t)]^{\frac{\alpha(s_t)}{\rho(s_t)}-1}} \tag{7.3}$$

[9] As pointed out earlier, Epstein and Melino (1995) searched over an infinite dimensional parameter space (the space of certainty equivalent functions) without finding an admissible set of parameters that rationalized the asset returns in this setting.

We have more parameters than equations, so there are lots of ways of combining the parameters to match the data perfectly. To get some idea of what is possible, we fix the values of β_l and β_h, keeping the average value at $\beta = .98$, then we use eq (5.6) and eq (5.7) that price equity and the risk free asset to solve for the remaining preference parameters so as to match exactly the first two moments of asset returns. The results are given in Table 5. If β is procyclical, so that the agent is more impatient in recessions than in booms, the coefficient of relative risk aversion required to match the asset returns falls slightly in the recession and increases a bit during booms relative to the values reported in Table 5. The opposite happens if β is countercyclical. While the induced changes in the α parameters are interesting, what is most striking from Table 5 is that there does not appear to be much value added in letting β vary with the state, in addition to α and ρ.

8. Conclusion

State dependent preferences have a long history in economics (see Gordon and St. Amour (2000,2001) and the references therein). Many authors have concluded recently that countercyclical risk aversion plays an important role in explaining the behaviour of asset prices. However, no distinction is made between attitudes toward risk and intertemporal substitution. In this paper, we disentangle these features of preferences to assess the role played by each in accounting for the first two moments of asset returns. We conclude that countercyclical risk aversion and a constant elasticity of intertemporal substitution adds very little to the predictions made using preferences that are state independent and cannot rationalize the observed data. However, we can match the average risk free rate and equity premium by keeping risk preferences constant and allowing the elasticity of intertemporal substitution to vary. To match both moments of the return on equity and the risk free rate, we require a strong countercyclical variation in risk aversion, and a very modest procyclical movement in the elasticity of intertemporal substitution.

Economists have developed a good deal of intuition about the role that countercyclical risk aversion can play in explaining the pattern of asset returns. Much less is known about the role played by state dependent intertemporal substitution or time preference. We hope our paper stimulates further research.

Table 1: Cyclical CRRA

ρ	EIS	α_l	α_h	r_l	r_h	$E(r)$	$E(M-r)$
1	∞	-30.40	22.90	1.02	1.09	1.06	.013
0.5	2	-22.63	40.99	1.02	1.09	1.06	.012
-4	0.2	-10.40	*n.a.*				
-9	0.1	-40.95	*n.a*				
-14	1/15	-42.38	-106.2	1.05	.86	.95	.116
-19	0.05	-42.74	-63.12	1.05	.87	.95	.116
-44	1/45	-43.05	-35.01	1.04	.88	.96	.113
-99	.01	-43.10	-28.11	1.04	.88	.96	.111
-999	.001	-43.11	-24.04	1.04	.89	.96	.108

Notes: We fix $\beta = .95$. For each value of ρ, the parameters α_l and α_h are chosen to price the required equity process exactly, and then the risk free rates and equity premium are calculated from the implied stochastic discount factor process. The entry 'n.a.' means no solution is available.

Table 2: Cyclical EIS

β	α	ρ_l	ρ_h	r_l	r_h	$E(r)$	$E(M-r)$
.95	1	.70	.74	1.17	.95	1.06	.01
	0.5	.71	.75	1.16	.95	1.06	.01
	−1	.73	.77	1.16	.95	1.05	.02
	−3	.77	.81	1.15	.94	1.05	.02
	−6	.83	.88	1.14	.93	1.04	.03
	−10	.93	.98	1.12	.92	1.02	.05
	−14	1.06	1.12	1.11	.91	1.01	.06
	−19	1.30	1.37	1.09	.90	1.00	.07
.96	−10	−.01	-.01	1.15	.94	1.04	.03
	−14	.13	.14	1.14	.93	1.04	.03
	−19	.13	.14	1.13	.93	1.03	.04
.97	−10	−.71	−.74	1.12	.92	1.02	.05
	−14	−.86	−.90	1.10	.91	1.00	.07
	−19	−1.13	−1.19	1.08	.90	.99	.08
.98	−10	−1.51	−1.58	1.12	.92	1.02	.05
	−14	−1.80	−1.89	1.10	.91	1.01	.06
	−19	−2.32	−2.43	1.09	.90	.99	.08
.99	−10	−2.31	−2.42	1.12	.92	1.02	.05
	−14	−2.74	−2.86	1.10	.91	1.01	.06
	−19	−3.50	−3.66	1.09	.90	.99	.08

Notes: We fix β within each panel of Table 2 at the indicated value. Within a panel, for each value of α, the parameters ρ_l and ρ_h are chosen to price the required equity process exactly, and then the risk free rates and equity premium are calculated from the implied stochastic discount factor process. For $\beta = .96$, there are two solutions at each value of α. The solutions not reported resemble those reported in Table 1, with $\rho_l \approx \rho_h \approx -.015$; at these values, the stochastic discount factor assigns almost zero value to consumption in the high growth state, and the equity premium is too high at 0.125.

Table 3: Cyclical CRRA and Cyclical Time Preference

ρ	β_l	β_h	α_l	α_h
1.0	.90	1.05	−7.64	.27
0.5	.90	1.05	−4.50	.16
0.1	.90	1.06	−1.05	.04
−0.1	.90	1.06	1.14	−.04
−1.0	.91	1.07	19.20	−.69
−2.0	.92	1.09	157.19	−5.6
−3.0	.92	1.10	−112.6	4.0
−4.0	.93	1.11	−60.6	2.16
−9.0	.97	1.18	−34.2	1.22
−14.0	1.00	1.27	−30.5	1.09
−19.0	1.04	1.35	−28.94	1.03

Notes: For each value of ρ, the preference parameters β_l, β_h, α_l and α_h are chosen to price the required equity and risk free rate processes exactly. There is a discontinuity in the solution values for α_l and α_h that occurs for ρ just below -2.

Table 4: Cyclical CRRA and Cyclical EIS

β	α_l	α_h	ρ_l	ρ_h
.95	-25.00	.89	1.25	1.31
.96	-51.89	1.85	.16	.17
.965	-18.91	.67	$-.38$	$-.40$
.97	-21.21	.76	$-.92$	$-.97$
.98	-22.25	.79	-1.98	-2.10
.99	-22.57	.81	-3.04	-3.22

Notes: For each value of β, the preference parameters α_l, α_h, ρ_l and ρ_h are chosen to price the required equity and risk free rate processes exactly. There is a discontinuity in the solution values of α_l and α_h that occurs around β=.961.

Table 5: Cyclical Time Preference, Cyclical CRRA and Cyclical EIS

β_l	β_h	α_l	α_h	ρ_l	ρ_h
.961	.999	-33.13	1.18	-1.49	-1.55
.965	.995	-29.34	1.05	-1.59	-1.67
.970	.990	-26.13	.93	-1.73	-1.81
.975	.985	-23.90	.85	-1.86	-1.96
.980	.980	-22.25	.79	-1.98	-2.10
.985	.975	-20.98	.75	-2.11	-2.24
.990	.970	-19.98	.71	-2.24	-2.38
.995	.965	-19.16	.68	-2.37	-2.53
.999	.961	-18.61	.66	-2.47	-2.65

Notes: For each value of β_l and β_h, the preference parameters α_l, α_h, ρ_l and ρ_h are chosen to price the required equity and risk free rate processes exactly.

Appendix A
First Order Conditions

Here we solve for the first order conditions of the following problem; for further details, please refer to Epstein (1988). The agent's objective is to choose feasible consumption and portfolio shares to maximize utility. The Bellman equation for this problem is

$$J\left(x_t,I_t\right) = \max_{\{c_t,w_t\}} \left(c_t^{\rho(s_t)} + \beta(s_t)\mu_t\left(J\left(x_{t+1},I_{t+1}\right)\right)^{\rho(s_t)}\right)^{\frac{1}{\rho(s_t)}}, \tag{.1}$$

s.t:

$$x_{t+1} = (x_t - c_t)w_t r_{t+1}. \tag{.2}$$

where I_t denotes "information variables" and $\mu_t\left(\widetilde{z}\right) \equiv [E_t(\widetilde{z}^{\alpha(s_t)})]^{\frac{1}{\alpha(s_t)}}$ for all random variables $\widetilde{z}$.

The homogeneity of $\mu_t(\cdot)$ and the linearity of x_{t+1} in (x_t,c_t), by eq(.2) , implies that the solution has the form:

$$J\left(x_t,I_t\right) = A\left(I_t\right)x_t. \tag{.3}$$

where $A\left(I_t\right)$ denotes an I_t-measurable random variable.

Substituting eq(.3) into eq(.1) gives:

$$A\left(I_t\right)x_t = \max_{\{c_t,w_t\}} \left(c_t^{\rho(s_t)} + \beta(s_t)\mu_t^{\rho(s_t)}\right)^{\frac{1}{\rho(s_t)}}$$

This maximization problem can be decomposed as the following two problems:

- Consumption is chosen by:

$$A(I_t)x_t = \max_{c_t\in[0,x_t]} \{c_t^{\rho(s_t)} + \beta(s_t)\left[(x_t - c_t)\mu_t^*\right]^{\rho(s_t)}\}^{\frac{1}{\rho(s_t)}} \tag{.4}$$

- Portfolio choice can be described by:

$$\mu_t^* = \max_{\{w\in R_+^n, w\iota=1\}} E_t^{\frac{1}{\alpha(s_t)}}\left[\left(A(I_{t+1})wr_{t+1}\right)^{\alpha(s_t)}\right] \tag{.5}$$

where $\iota = (1,1,...,1)'$.

We first solve the consumption problem for arbitrary μ_t^*. Again, the homogeneity of eq(.4) implies that the optimal consumption policy can be written as

$$c_t^* = a(I_t)x_t \tag{.6}$$

where $a(I_t)$ denotes an I_t-measurable random variable. For convenience we use $a_t = a(I_t)$ below. Note that a_t can be thought of as the consumption wealth ratio.

Substitute eq(.6) into eq(.4):

$$A(I_t)^{\rho(s_t)}/\rho(s_t) = \max_{a_t \in [0,1]} \{a_t^{\rho(s_t)} + \beta(s_t)\left[(1-a_t)\mu_t^*\right]^{\rho(s_t)}\}/\rho(s_t)$$

The FOC of this problem is:

$$a_t^{\rho(s_t)-1} = \beta(s_t)(1-a_t)^{\rho(s_t)-1}\mu_t^{*\rho(s_t)} \tag{.7}$$

These last two equations combine to yield:

$$A(I_t) = a_t^{\frac{\rho(s_t)-1}{\rho(s_t)}} \tag{.8}$$

Thus:

$$A(I_{t+1}) = a_{t+1}^{\frac{\rho(s_{t+1})-1}{\rho(s_{t+1})}} \tag{.9}$$

Equation eq(.8) shows that the optimal consumption wealth ratio is an invertible function of $A(I_t)$. We exploit this feature in our solution of the portfolio choice problem below.

Now substitute eq(.9) into eq(.5); the portfolio choice problem becomes

$$\mu_t^* = \max_{\{w \in R_+^n, w\iota = 1\}} E_t^{\frac{1}{\alpha(s_t)}}\left(\left[a_{t+1}^{\frac{\rho(s_{t+1})-1}{\rho(s_{t+1})}} w r_{t+1}\right]^{\alpha(s_t)}\right) \tag{.10}$$

Substituting eq(.10) into eq(.7), we get

$$a_t^{\rho(s_t)-1} = \beta(s_t)(1-a_t)^{\rho(s_t)-1}\left(E_t^{\frac{1}{\alpha(s_t)}}\left(\left[a_{t+1}^{\frac{\rho(s_{t+1})-1}{\rho(s_{t+1})}} M_{t,t+1}\right]^{\alpha(s_t)}\right)\right)^{\rho(s_t)} \tag{.11}$$

where $M_{t,t+1} = w_t^* r_{t+1} = \sum_{i=1}^n w_{it}^* r_{i,t+1}$ is the gross return of the optimal portfolio w_t^*. Eq(.11) can be rearranged to get:

$$E_t\left[\beta(s_t)^{\frac{\alpha(s_t)}{\rho(s_t)}}\left(\frac{1-a_t}{a_t}\right)^{\alpha(s_t)\frac{\rho(s_t)-1}{\rho(s_t)}} a_{t+1}^{\alpha(s_t)\frac{\rho(s_{t+1})-1}{\rho(s_{t+1})}} M_{t+1}^{\alpha(s_t)}\right]=1 \qquad (.12)$$

Note that eq(.12) can be written in the form

$$E_t\left[IMRS_{t,t+1} * M_{t,t+1}\right]=1 \qquad (.13)$$

where

$$IMRS_{t,t+1}=\beta(s_t)^{\frac{\alpha(s_t)}{\rho(s_t)}}\left(\frac{1-a_t}{a_t}\right)^{\alpha(s_t)\frac{\rho(s_t)-1}{\rho(s_t)}} a_{t+1}^{\alpha(s_t)\frac{\rho(s_{t+1})-1}{\rho(s_{t+1})}} M_{t,t+1}^{\alpha(s_t)-1} \qquad (.14)$$

Also note that the budget constraint, eq(.2), implies $x_{t+1}=(x_t-c_t)\,M_{t,t+1}=(1-a_t)\,M_{t,t+1}x_t$; therefore $a_{t+1}=c_{t+1}/x_{t+1}=g_{t+1}a_t/\left[(1-a_t)\,M_{t,t+1}\right]$. Using this identity, we can rewrite eq(.14) into a variety of equivalent expressions.

The FOC of eq(.10) with respect to w_i gives

$$E_t\left(a_{t+1}^{\frac{\rho(s_{t+1})-1}{\rho(s_{t+1})}\alpha(s_t)} M_{t,t+1}^{\alpha(s_t)-1}\left(r_{i,t+1}-r_{j,t+1}\right)\right)=0, \qquad i\in(1,...,n)$$

Multiple by w_i for all i, and sum over $i=1,...,n$, we get:

$$E_t\left(a_{t+1}^{\frac{\rho(s_{t+1})-1}{\rho(s_{t+1})}\alpha(s_t)} M_{t,t+1}^{\alpha(s_t)-1}\left(M_{t,t+1}-r_{j,t+1}\right)\right)=0 \qquad (.16)$$

Selecting $r_{j,t+1}$ to be the risk free rate, r_{ft}, which is time t measurable (note we do not use the notation $r_{f,t+1}$), eq(.16) can also be written using eq(.13) as:

$$E_t\left(IMRS_{t,t+1}r_{ft}\right)=1 \qquad (.17)$$

where $IMRS_{t,t+1}$ is given by eq(.14).

Numerical Solution

Given two states l and h, eq(.13) and eq(.17) can be written as:

$$\sum_{j=\{l,h\}} \pi_{ij} IMRS_{i,j} M_{i,j} = 1 \quad (.18)$$

$$\sum_{j=\{l,h\}} \pi_{ij} IMRS_{i,j} r_i = 1 \quad (.19)$$

where $i \in \{l,h\}$. Using eq(6.3) and equilibrium condition eq(6.8) , then eq(.14) can be written as

$$\begin{aligned} IMRS_{t,t+1} &= \beta_t^{\frac{\alpha_t}{\rho_t}} P_t^{\alpha_t \frac{\rho_t - 1}{\rho_t}} \left(\frac{1}{1+P_{t+1}}\right)^{\alpha_t \frac{\rho_{t+1}-1}{\rho_{t+1}}} \left(\frac{1+P_{t+1}}{P_t} g_{t+1}\right)^{\alpha_t - 1} \\ &= \beta_t^{\frac{\alpha_t}{\rho_t}} P_t^{1-\frac{\alpha_t}{\rho_t}} (1+P_{t+1})^{\frac{\alpha_t}{\rho_{t+1}}-1} g_{t+1}^{\alpha_t - 1} \quad (.20) \end{aligned}$$

Thus eq(.18) and eq(.19) can be written as

$$\sum_{j=\{l,h\}} \pi_{ij} \beta_i^{\frac{\alpha_i}{\rho_i}} P_i^{1-\frac{\alpha_i}{\rho_i}} (1+P_j)^{\frac{\alpha_i}{\rho_j}-1} g_j^{\alpha_i - 1} M_{i,j} = 1$$

$$\sum_{j=\{l,h\}} \pi_{ij} \beta_i^{\frac{\alpha_i}{\rho_i}} P_i^{1-\frac{\alpha_i}{\rho_i}} (1+P_j)^{\frac{\alpha_i}{\rho_j}-1} g_j^{\alpha_i - 1} = \frac{1}{r_i}$$

Equivalently,

$$\sum_{j=\{l,h\}} \pi_{ij} \beta_i^{\frac{\alpha_i}{\rho_i}} \frac{(1+P_j)^{\frac{\alpha_i}{\rho_j}}}{P_i^{\frac{\alpha_i}{\rho_i}}} g_j^{\alpha_i} = 1 \quad (.21)$$

$$\sum_{j=\{l,h\}} \pi_{ij} \beta_i^{\frac{\alpha_i}{\rho_i}} \frac{(1+P_j)^{\frac{\alpha_i}{\rho_j}-1}}{P_i^{\frac{\alpha_i}{\rho_i}-1}} g_j^{\alpha_i - 1} = \frac{1}{r_i} \quad (.22)$$

where $i \in \{l,h\}$.

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