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ABSTRACT

Central banks have consistently differentiated the return on the securities they have issued (money and national debt). In contrast, first best efficiency demands that these securities earn the same return: the return on capital. A self-financed central bank, without capital and taxes, cannot achieve this first best. The resulting gaps between the return on capital and the returns on public securities are implicit taxes. These taxes increase the opportunity costs of the commodities financed with the liquidation of these securities, so they are indirect taxes on these commodities. Because taxes on investment are less efficient than taxes on consumption, securities intensive in financing investment should be taxed at a lower rate than securities intensive in financing consumption. This is feasible if national debt is investment intensive. Then, this security should earn interest and be imposed artificial costs on second hand trading. In addition, because money is specialized in providing short term liquidity, raising the return on national debt delays expenditure toward the future. Hence, the payment of interest on national debt brings a windfall of resources during transitions across balanced paths in addition to the long term welfare gains of this policy. Similar arguments apply to short and long term maturities of the national debt.

1. INTRODUCTION

Governments provide a diverse offer of public liabilities with a diverse menu of returns. For example, they pay interest on national debt, especially to long term maturities, while they pay no interest on cash. In contrast, for first best efficiency, all public liabilities should earn the same return (adjusted for risk), and this should be the social return on capital. (See Friedman's [1969] optimum quantity of money.) To implement this first best, governments must fund their liabilities with capital or taxes, and they must be able to generate income from these funding sources efficiently[1]. Realistically, for a variety of reasons, including the same transaction costs and information imperfections that generate a demand for money, these funding sources are costly to administer. As a result, governments have never funded their liabilities to the extend that all their returns approach the return on capital. Could this funding limitations explain the prevalent policy of issuing a diverse offer of public liabilities? To answer this question, I analyze the optimal offer of public liabilities issued by a self-financed central bank, without capital or taxes, acting as the only branch of government. This central bank is imbedded in a model with overlapping production activities where both money and national debt provide liquidity to the economy.

The model of this paper develops the framework in Woodford (1990) in the following fashion. Production projects take time to mature, and the various types of projects in the economy overlap. So in every period, some projects yield a harvest of output, while other projects require

[1]In Faig [1998b], I elaborate on the ultimate implications of this proposition when the return on capital has idiosyncratic risk.

investment of real resources for future harvests to come. Individuals can participate at most in one project at a time, and credit among them is blocked because as in Lucas (1980) credit contracts cannot be enforced. In this environment, individuals with a harvest have no problem financing their consumption and investment expenditures, while individuals who are between harvests have a liquidity problem because they have no earnings on hand. Individuals solve this problem by selling a portion of their harvests to acquire money and national debt. Then, in periods without a harvest, individuals use these liquid assets to finance their expenditures.

In this model, the central bank may choose to keep the nominal supply of all public liabilities constant. If this baseline policy is pursued, in a balanced path all public liabilities earn the same real rate of return: the growth rate of the economy. In this instance, the national debt earns no (nominal) interest, and in practice all public liabilities are alike. The central bank may choose to depart from this baseline policy and pay interest on the national debt. To pay this interest permanently, the self-financed central bank must cross-subsidize the return on the national debt with money creation. With this policy, the existence of a demand for money in equilibrium requires that the central bank imposes artificial costs to second hand transactions of the national debt. (Or alternatively, the central bank could impose other inconveniences for using the national debt for short term holding horizons.) The end result of this policy is a specialization of liquid assets by holding period, money providing liquidity at the short end of the spectrum, and the various maturities of the national debt covering the rest. The main focus of this paper is to identify possible rationales for this diverse offer of public securities.

The first rationale I provide for diversifying the return on public securities is based on a diverse investment intensity for the private expenditures financed with them. A self-financed central bank cannot pay the return on capital to all public securities. The resulting gaps between the return on capital and the returns on public securities are implicit taxes.[2] These taxes increase the opportunity costs of the commodities financed with the help of the liquidity services the public securities provide.[3] Because taxes on investment are more inefficient than taxes on consumption, securities intensive in financing investment should be taxed at a lower rate than securities intensive in financing consumption. This is feasible if national debt is investment intensive. Then, this security should earn interest and be imposed artificial costs on second hand trading. A similar argument applies to the various maturities of the national debt. For example, an upward-sloping term structure of the return on the national debt, which is what we normally observe, is a second best policy for a self-financed central bank if long term maturities are more intensive financing investment than short term maturities.

A second rationale for paying interest on the national debt is based the fact that the national debt is specialized in relatively long holding periods. Because of this specialization, raising the return on the national debt delays expenditure toward the future. Hence, the payment of interest on the national debt, and especially to long term maturities of the national debt, brings a windfall of resources during transitions across balanced paths. With careful policy by the central

[2]These taxes finance the negative net worth of the central bank (the central bank issues liabilities but has no assets).

[3]This forward shifting does not depend on what securities are finally used to purchase goods, but what securities are held during the period preceding the purchase.

bank, these resources can lead to a Pareto improvement versus the baseline policy of paying zero interest on the national debt.

The efficiency of differentiating the returns of public liabilities by paying interest on national debt was previously questioned by Bryant and Wallace (1979), Bryant (1980a), and Romer (1993). Despite major differences in the modeling strategies, my results compare easily with these earlier contributions. My results agree with the inefficiency of paying interest on national debt if public securities do not help to finance future investment, and we limit welfare to comparisons across balanced paths. This inefficiency is stressed in the contributions by Bryant and Wallace. My results agree with Romer that there are conditions under which paying interest on national debt is optimal. Romer's results are based on how the liquidity services of money and national debt enter the utility function of a representative individual. Instead, my results follow from the specialization of liquid assets by holding period, and the diverse investment intensity of public securities that this specialization allows.

Several earlier contributions have provided empirical support for a quasi-monetary role for the national debt, especially for its short-term maturities. For example, Fried and Howitt (1983) argue that this role is a good explanation that at low frequencies the real interest rates earned on Treasury Bills decline as the rate of inflation rises. Similarly, Fried (1995) argues that this role explains the premium of one period returns of long-term Treasury Bills over short-term Treasury Bills, and it explains the correlations of this premium with other macroeconomic variables. Finally, Bansal and Coleman (1996) argue that this role can also explain the equity premium puzzle.

Although, the main emphasis of the present paper is normative, it reinforces this empirical support in providing a plausible normative rationale for the prevalent qualitative structure of the returns on public liabilities.

This paper extends my earlier work on the overlapping production activities model (Faig [1998a]) by prolonging the production time to more than two periods and introducing multiple liquid assets. In both papers, I borrow heavily from earlier contributions. The roles of money and the national debt in completing markets are similar to the roles they play in Foley and Helwig (1975) and Bewley (1980), and the rapidly growing literature that has followed these seminal contributions. In particular, the role of publicly supplied liquid assets in financing investment expenditures is similar to Woodford (1990), except for the fact that in here the returns on money and national debt affect no only the size of investment but also its composition. The demand for liquidity based on a costly or slow liquidation of alternative assets is found in classical contributions such as Baumol (1952), Tobin (1956), and Diamond and Dybvig (1983)[4]. Finally, the endogenous growth features of the model have much in common with the 'Ak' model in Barro (1990).

In summary, this paper advances a new perspective on the normative analysis of the returns of outside public liabilities such as: money and national debt. This new perspective emphasizes the liquidity roles of these securities to finance private expenditures, the specialization

[4]For recent contributions using this modeling of liquidity and dealing with issues related to this paper see Aiyagari and Gertler (1991) Schreft and Smith (1997), and Holmström and Tirole (1998).

of these securities by holding period, and their relative intensities to finance diverse types of private expenditures. In particular, if the expenditures financed with national debt are more investment intensive than those financed with money, it is second best for a self-financed central bank to cross-subsidize the payment of interest on national debt with money creation. Similarly, an upward sloping term structure of the return on national debt is second best if the investment intensity of the national debt increases with term to maturity. Thus, this paper provides a plausible normative rationale for the prevalent qualitative features of the structure of public liabilities. The rest of the paper is organized as follows. Section 2 describes the overlapping production activities model used in this paper. Section 3 analyzes the behavior of individuals in the framework of this model. Section 4 constructs a balanced path equilibrium. Section 5 compares the welfare of a representative individual across balanced paths. Section 6 analyzes the transitional effects across balanced paths for a tractable special case. Finally, section 7 concludes with a brief summary of the findings and some directions for future research.

2. THE MODEL

Consider an economy populated with a large number of individuals. In this economy, individuals possess a non transferable production technology which is the only vehicle to employ the capital they own. The output of these technologies is homogeneous, and it can either be consumed or invested. Invested capital is the only input these technologies require. Production takes time. To obtain a harvest of output, individuals have to invest in their technologies for several consecutive periods. Also, the various production activities overlap. So in every period, some individuals have a harvest, while others are investing for the harvests to come. This overlapping structure of production implies that individuals must trade, so they can consume and invest in the periods without a harvest of their own. But trading is limited by the fact that individuals can hide their identity and their capital if it is in their best interest to do so, precluding credit contracts in their multiple forms (mortgaging capital, future markets, joint firms, ...). (See Bryant [1980b] for a related discussion of the moral hazard problems behind the demand for money). These trading constraints are relaxed with the existence of two public liabilities: money and national debt. Individuals sell a portion of their harvest to acquire these assets, which are spent to buy output in periods when they do not receive a harvest.

The imperfect functioning of capital markets is crucial for public securities to be valued in this model. Following Lucas (1980), I preclude the existence of credit. At the cost of complicating the model, this extreme assumption could be replaced with weaker alternatives. For example, it could be replaced with transaction costs in establishing or enforcing credit contracts. (See for example Aiyagari and Gertler (1991) for a related model with these costs.) Also, it could be relaxed with a limitation of credit to a portion of capital easy to identify and thus suitable as collateral. For simplicity, these alternative assumptions are not pursued here.

Money and national debt are differentiated by their liquidity. Money is perfectly liquid, while national debt is composed of discount bonds with second hand trading costs. By making these costs sufficiently high, the government can create a demand for money for short term holding horizons even if money yields a lower return than national debt. Likewise, different maturities of national debt can coexist even if their return increases with their term as long as the cost on second hand trading is high enough to discourage using long term debt for short term holding horizons. The cost to second hand trades of the national debt is a matter of policy. Its imposition is a device to differentiate the returns on money and national debt of various maturities. The government can eliminate this cost and equate the return on all liquid assets.

Individuals with a harvest allocate their wealth into consumption, capital, and liquid assets. The flow budget constraint for a representative individual at this stage, to be denoted stage 0, is the following equation:

(1) $$c_{0t} + k_{0t} + \sum_{s=1}^{S} m_{st} = x_t \ ;$$

where c_{0t} = consumption at stage 0 in period t;

k_{0t} = capital at stage 0 in period t;

x_t = real wealth at the beginning of period t; and

m_{st} = real demand for liquid asset (of term) s in period t.

Liquid asset 1 is money. The other liquid assets are the various pure discount bonds composing the national debt. The relationship between the real quantities of liquid assets and their nominal face value is

(2) $$m_{st} = \frac{M_{st} q_{st}}{p_t} \ .$$

where M_{st} = nominal face value invested in liquid asset s in period t;

q_{st} = discount price of liquid asset s in period t (q_{1t} = 1 for all t) ; and

p_t = price of output in period t.

Individuals without a harvest use their liquid wealth to finance their consumption and investment expenditures. In principle, they could also use their liquid wealth to purchase new liquid assets. However, in all the deterministic equilibria considered in this paper this possibility is not carried out because of a combination of three facts: The return on physical capital exceeds that of liquid assets. The transaction costs for second hand trading of the national debt are high enough to prevent using long term debt for short term holding horizons. And the expected one period return on liquid assets is non decreasing with their term. Consequently, individuals not having a harvest spend the value of their liquid assets maturing in that period:

(3) $$c_{st} + k_{st} = \frac{M_{s(t-s)}}{p_t} \equiv m_{s(t-s)} r_{st} \, ;$$

where r_{st} = real *gross* return of liquid asset s that matures in period t; that is $r_{st} = \dfrac{p_{t-s}}{q_{s(t-s)} p_t}$.

In all periods, consumption, capital, and the demand for liquid assets must be nonnegative:

(4) $$c_{st} \geq 0, \quad k_{st} \geq 0, \ \text{ and } m_{st} \geq 0, \text{ for all } s \text{ and all } t \, .$$

In the equilibria considered where the return on liquid assets is dominated by the return on capital, an individual holds no liquid assets at the beginning of a period with a harvest. The harvest itself depends on the capital invested in the preceding $S + 1$ periods. Therefore, the wealth of individuals with a harvest in $t + S + 1$ is

(5) $$x_{t+S+1} = F(k_t)$$

where k_t = the vector $(k_{0t}, k_{1t+1}, \ldots, k_{S(t+S)})$; and

F = *gross* production function.

For ease of notation, the undepreciated capital is incorporated in the function *F*. The function *F* is positive, twice continuously differentiable, increasing, concave, and homogenous of degree one. Also, it satisfies standard Inada conditions for an interior solution.

The individuals' horizon is infinite. In recursive form, the utility of an individual having a harvest in period *t* is

(6) $$U_t = \sum_{s=0}^{S} \beta^s \frac{c_{s(t+s)}^{1-\sigma}}{1-\sigma} + \beta^{S+1} U_{t+S+1};$$

where U_t = utility in period *t*;

ß = discount factor (ß > 0); and

s = inverse of the inter-temporal elasticity of substitution (positive, and if s = 1, the instantaneous utility should be understood as logarithmic).

The only government agency is a central bank that provides and manages the supply of liquid assets in the economy. The central bank can choose to change the return on liquid assets by changing their supply through open market operations, and revising if necessary the transaction costs that hinder second hand trading of national debt. These activities, though, must be self-financed. Thus, the current revenue from selling bonds and issuing money must equal the current value of maturing bonds. Equivalently, the net seigniorage measured in a cash-flow basis must be zero in all periods:

(7) $$\sum_{s=2}^{S} M_{st}q_{st} + \Delta M_{1t} - \sum_{s=2}^{S} M_{s(t-s)} = 0 .$$

The equilibrium concept to be applied is that of a recursive liquidity dominated competitive equilibrium (henceforth equilibrium) where the sequence of prices and returns, the allocation of harvests, and the quantities of liquid assets satisfy the following conditions:

(i) individuals have perfect foresight about prices and returns and take them as given;

(ii) individuals maximize utility subject to constraints (1) to (5);

(iii) all liquid assets are valued, their one period return is non decreasing with term to maturity and dominated by the return on capital; and

(iv) all markets clear:

(8) $$\sum_{s=0}^{S} (c_{st} + k_{st}) = x_t, \text{ for all } t \text{, and}$$

(9) $$M_{st} = M_{s(t-s)} + \Delta M_{st} \text{ for all } s \text{ and all } t .$$

Among the set of equilibria, special attention is given to balanced path equilibria where all real variables grow at the same rate and the real rates of return on liquid assets are constant over time.

3. INDIVIDUALS' BEHAVIOR

This section characterizes an optimal plan for a representative individual. In this optimal plan, investment must be intertemporally allocated to maximize the size of the next harvest. That is, if the individual gets a harvest in period *t*, then the vector of capital stocks relevant to the next harvest, k_t, must solve the following program:

$$\max_{k_t} F(k_t) \text{ subject to } \sum_{s=0}^{S} k_{s(t+s)} r_{s(t+s)}^{-1} = \kappa_t x_t\,; \tag{10}$$

where $?_t$ = proportion of wealth to be used in financing capital up to the next harvest. The first order conditions for an interior solution to this problem are

$$F_{0t} = F_{s(t+s)} r_{s(t+s)} \text{ for } s = 1, \ldots S; \text{ where } F_{st} \equiv \frac{\partial F}{\partial k_{s(t+s)}}. \tag{11}$$

This condition states that at stage 0, the marginal return of investing a unit of output immediately must be equal to the marginal compounded return of purchasing a liquid asset to finance investment at a later stage of the project.

In the optimal plan, consumption must be allocated to satisfy standard Euler equations. In periods with a harvest, the marginal rate of substitution between immediate consumption and consumption at a period before the next harvest must be equal to the gross real return on the liquid asset connecting these two periods:

(12) $$\frac{c_{0t}^{-\sigma}}{\beta^{s} c_{s(t+s)}^{-\sigma}} = r_{s(t+s)}, \text{ for } s = 1,\ldots,S \,.$$

Also, in all periods the marginal rate of substitution between immediate consumption and consumption at the next harvest must be equal to the gross marginal product of immediately invested capital:

(13) $$\frac{c_{s(t+s)}^{-\sigma}}{\beta^{S+1-s} c_{0t+S+1}^{-\sigma}} = F_{s(t+s)}, \text{ for } s = 0,\ldots S \,.$$

To ensure the existence of an optimal path, we must restrict the parameters describing preferences and technology. These restrictions must ensure that there is at most one feasible path that gives infinite utility to the individual. Also, they must ensure that there is at least one feasible path that gives utility higher than minus infinity. The following assumption achieves both objectives for the equilibrium paths to be analyzed here (see Alvarez and Stokey [1998])[5] :

ASSUMPTION 1: *Let $F^*(\mu)$ be the solution to problem (10) for $r_s = \mu^s$. Preferences and technology must satisfy:*

(i) If s < 1, then › $\hat{\mu}$ > 0 such that $F^*(\hat{\mu})$ # $\hat{\mu}^{S+1}$ and ß $\hat{\mu}^{1-s}$ < 1.

(ii) If s $ 1, then › μ > 0 such that $F^*(\mu)$ $ μ^{S+1} and ß μ^{1-s} < 1.

[5]The theorems in Alvarez and Stokey (1997) require that the ratio of consumption to wealth is bounded away from zero for s $ 1. Therefore, their application requires finding positive lower bounds for c_{st} / x_t that are never binding. This can be easily achieved along balanced paths and the transitional paths in Section 6 with an explicit solution.

Intuitively, the first part of the assumption ensures that technology is not too productive when $s < 1$ so the utility of feasible paths does not diverge to infinity. The second part of the assumption ensures that technology is sufficiently productive when s $ 1 so there is at least a feasible path with a utility higher than minus infinity.

4. The Balanced Path Equilibrium

In this section, I will construct a balanced path equilibrium along which both the rates of return on assets and the ratios among real variables are constant. First, I will construct the allocation along a balanced path equilibrium for given rates of return on liquid assets. Second, I will show how the rates of return on liquid assets are related to monetary policy. And third, I will characterize the constraints on monetary policy imposed by the equilibrium concept and the budget constraint of the central bank. Throughout this section, I will drop the time subscripts when unnecessary.

For a representative individual, let *t* be a period with a harvest. Denote $\tilde{k}_s$, $\tilde{c}_s$, and $\tilde{m}_s$ the relative demands $k_{s(t+s)}/x_t$, $c_{s(t+s)}/x_t$, and m_{st}/x_t respectively. Using (11), the relative allocation of capital $\{\tilde{k}_s/\tilde{k}_0\}_{s=1}^{S}$ is determined by the following S equations:

(14) $$F_0 = F_s r_s, \text{ for } s = 1, \ldots, S .$$

Because of constant returns to scale, the marginal products of capital depend only on the set of ratios $\{\tilde{k}_s/\tilde{k}_0\}_{s=1}^{S}$. Using (13), the *gross* rate of growth of consumption is then

(15) $$g = \beta^{\frac{1}{\sigma}} F_0^{\frac{1}{\sigma(S+1)}}.$$

Because preferences are homothetic and the rates of return are constant, wealth must grow at the same rate as consumption. Hence, using (5), the proportion of a harvest immediately allocated to capital is determined by the following condition:

(16) $$F\left(1, \frac{\tilde{k}_1}{\tilde{k}_0}, \ldots, \frac{\tilde{k}_S}{\tilde{k}_0}\right)\tilde{k}_0 = g^{S+1}.$$

Using (12), the set $\left\{\tilde{c}_s / \tilde{c}_0\right\}_{s=1}^{S}$ is determined by the following S equations:

(17) $$\frac{\tilde{c}_s}{\tilde{c}_0} = \left(\beta^s r_s\right)^{\frac{1}{\sigma}}, \text{ for } s = 1, \ldots S \; .$$

Finally, to complete a description of the allocation along a balanced path, the proportion of consumption consumed at stage 0 follows then from the intertemporal budget constraint that results combining (1), (3) and (5) with (14) to (17):

(18) $$\tilde{c}_0 = \frac{1 - \beta^{\frac{S+1}{\sigma}} F_0^{\frac{1}{\sigma}-1}}{\sum_{s=0}^{S} \beta^{\frac{s}{\sigma}} r_s^{\frac{1}{\sigma}-1}}.$$

I describe next how the rates of return on liquid assets are related to monetary policy. By definition, the discount price of asset s must satisfy the Fisher's equation:

(19) $$r_s \equiv \frac{p_{t-s}}{q_s p_t} = \frac{r_1^s}{q_s}\ .$$

Note that the *gross* real return on money, r_1, is the inverse of the *gross* rate of inflation. Also, the inverse of the discount price q_s is the nominal return on security *s* for the *s* periods this security takes to mature. The real quantities of liquid assets are implied by the constraint (3):

(20) $$\tilde{m}_s = (\tilde{c}_s + \tilde{k}_s) / r_s,\ \text{for } s = 1,\ldots S\ .$$

For the rates of return to be constant, this equation implies that the supplies of all liquid assets must grow at a common constant rate ? along a balanced path equilibrium. Market clearing in the money market then implies that ? must satisfy the standard relation:

(21) $$\gamma = \frac{g}{r_1}\ .$$

For a set of returns $\{r_s\}_{s=1}^{S}$ to be an equilibrium all markets must clear. The money clears if $M_{1t} / p_t = \tilde{m}_1 x_t$, so the standard neutrality result applies. The markets for the other liquid assets clear if their initial supplies satisfy the following equations:

(22) $$M_{st} = \frac{\tilde{m}_s}{\tilde{m}_1 q_s} M_{1t},\ \text{for } s = 2,\ldots,S\ .$$

Finally, by Walras' Law, the output market clears if the net seigniorage collected by the central bank is zero. Using (2) and (19), equation (7) is transformed into:

(23) $$R \equiv \sum_{s=1}^{S}\left(1-\frac{r_s}{g^s}\right)\tilde{m}_s = 0;$$

where $R =$ ratio of net seigniorage over gross output. A set of returns $\{r_s\}_{s=1}^{S}$ to be consistent with the equilibrium concept defined in Section 2 must satisfy not only equation (23) but also the following two incentive constraints. First, liquid returns must be weakly dominated by the return on capital, that is $F_s \geq r_{S+1-s}$. Second, the central bank must be able to make each liquid asset the best instrument to transfer wealth for a period equal to its maturity. For example, given that transaction costs on the national debt cannot be negative, this rules out money having a higher return than national debt. If this were the case, the demand for national debt would be zero and money would be the only public liability. Thus, this would be equivalent to a policy with zero interest on the national debt. These two incentive constraints on the set of returns are satisfied with the baseline policy of keeping a constant supply of all liquid assets. In this case, the return of liquid assets along a balanced path satisfies $r_s = g^s$ for $s = 1, ..., S$. So all assets have the same one period rate of return. This one period rate of return is dominated by the rate of return on capital as the following argument demonstrates. Multiplying by $k_{s(t+s)}$ both sides of (11) and aggregating over all s implies $F(k_t) = F_s r_s \kappa x_t$. Given that $F(k_t) = x_{t+S+1} = g^{S+1}x_t$ and ? # 1, the dominance of liquid assets by capital follows:

(24) $$F_s \geq \frac{g^{S+1}}{r_s} = g^{S+1-s} = r_{S+1-s}.$$

5. The Silvered Rule

This section characterizes the monetary policy that maximizes the utility of a representative individual in a balanced path. If the government could use lump-sum taxes to subsidize monetary assets, then, as it is well known, efficiency requires to raise the return on all monetary assets to equate the social return on capital. This first best policy, known as the optimum quantity of money after Friedman (1969), is precluded in this paper by the lack of funds of the central bank and the constraint that current seigniorage cannot be negative. The second best problem to be solved in this section imposes constraint (23) on the set of returns on liquid assets as well as the other constraints consistent with a balanced path equilibrium. With these constraints, the problem to be solved consists in finding the set of returns that maximizes the utility of a representative individual having a harvest if the inherited capital stocks of the individuals not having a harvest were automatically adjusted to the balanced path levels. The solution to this problem is a second best modified golden rule, which for short I will name the Silvered Rule. The transitional effects abstracted in this rule and the distributional consequences they entail are dealt in the next section.

Let V be the indirect utility function of a representative individual in a balanced path equilibrium. Consider a marginal reform in which the return on liquid asset i ($i > 1$) is raised and this increase is financed with a reduction on the return from money. Using Roy's identity, the total effect on V from this reform is the following expression:

(25) $$\frac{dV}{d\ln r_i} = \frac{V_x}{1-g^{S+1}F_0}\left(\tilde{m}_i + \tilde{m}_1 \frac{d\ln r_1}{d\ln r_i}\right).$$

Applying the Implicit Function Theorem on constraint (23), we obtain

(26) $$\frac{d\ln r_1}{d\ln r_i} = -\left(\frac{\partial R}{\partial \ln r_i}\right)\left(\frac{\partial \ln R}{\partial \ln r_1}\right)^{-1}.$$

Differentiating (23),

(27) $$\frac{\partial R}{\partial \ln r_i} = \frac{\tilde{m}_i r_i}{g^i} - \frac{\partial \ln g}{\partial \ln r_i}\left(\sum_{s=1}^{S} \frac{\tilde{m}_s r_s}{g^s} s\right) + \sum_{s=1}^{S}\left(\frac{r_s}{g^s} - 1\right)\frac{\partial \tilde{m}_s}{\partial \ln r_i} \text{ for } i = 1,\ldots,S.$$

Therefore, combining (25) to (27), the effect of the reform on V is

(28) $$\frac{dV}{d\ln r_i} = \frac{V_x \tilde{m}_i}{1-g^{S+1}F_0}\left[1 - \frac{\frac{r_i}{g^i} - \frac{\partial \ln g}{\partial \ln r_i}\left(\sum_{s=1}^{S} \frac{\tilde{m}_s r_s}{\tilde{m}_i g^s} s\right) + \sum_{s=1}^{S}\left(\frac{r_s}{g^s} - 1\right)\frac{\partial \ln \tilde{m}_s}{\partial \ln r_i}\frac{\tilde{m}_s}{\tilde{m}_i}}{\frac{r_1}{g} - \frac{\partial \ln g}{\partial \ln r_1}\left(\sum_{s=1}^{S} \frac{\tilde{m}_s r_s}{\tilde{m}_1 g^s} s\right) + \sum_{s=1}^{S}\left(\frac{r_s}{g^s} - 1\right)\frac{\partial \ln \tilde{m}_s}{\partial \ln r_1}\frac{\tilde{m}_s}{\tilde{m}_1}}\right].$$

The Silvered Rule implies the baseline policy of zero nominal interest on the national debt, or equivalently $r_s = g^s$ for all s, when expression (28) is non positive for all liquid assets. To see what this implies we can simplify (28) considerably. Applying the Envelop Theorem to problem

(10), the elasticity of g with respect to r_i is proportional to the discounted expense $\tilde{k}_i r_i^{-1}$:

(29) $$\frac{\partial \ln g}{\partial \ln r_i} = \frac{\tilde{k}_i r_i^{-1}}{\sigma(S+1)\kappa} \text{ for } i = 1,\ldots,S.$$

Using (29) and evaluating (28) at the point where $r_s = g^s$ for all s, expression (28) simplifies to

(30) $$\frac{dV}{d\ln r_i} = \frac{V_x \tilde{m}_i}{1-g^{S+1}F_0}\left[1 - \frac{1 - \dfrac{\tilde{k}_i}{\tilde{c}_i + \tilde{k}_i}\left(\dfrac{\sum_{s=1}^{S} r_s g^{-s}\tilde{m}_s s}{\sigma(S+1)\kappa}\right)}{1 - \dfrac{\tilde{k}_1}{\tilde{c}_1 + \tilde{k}_1}\left(\dfrac{\sum_{s=1}^{S} r_s g^{-s}\tilde{m}_s s}{\sigma(S+1)\kappa}\right)}\right].$$

Therefore, $\dfrac{dV}{d\ln r_i} \le 0$ if and only if $\dfrac{\tilde{k}_i}{\tilde{c}_i + \tilde{k}_i} \le \dfrac{\tilde{k}_1}{\tilde{c}_1 + \tilde{k}_1}$. That is, the Silvered Rule implies zero nominal interest on national debt when money is weakly specialized in financing investment expenditures. Conversely, the Silvered Rule implies paying interest on national debt when, realistically, money is specialized in financing consumption expenditures.

An interior Silvered Rule where the interest paid on national debt is positive and increasing with maturity can be easily characterized using (28) and (29). If the Silvered Rule is interior, all marginal reforms of the sort considered in deriving (28) must have a zero effect on *V*. This implies that the numerator of the fraction inside the square brackets in (28) must be equal to a constant ? for all *i*. Consequently,

(31) $$\frac{r_i}{g^i} = \lambda + \frac{\tilde{k}_i}{\tilde{c}_i + \tilde{k}_i}\left[\frac{\sum_{s=1}^{S} r_s g^{-s}\tilde{m}_s s}{\sigma(S+1)\kappa}\right] - \sum_{s=1}^{S}\left(\frac{r_s}{g^s} - 1\right)\frac{\tilde{m}_s}{\tilde{m}_i}\frac{\partial \ln \tilde{m}_s}{\partial \ln r_i} \text{ for all } i.$$

Thus the proportional premium of r_i over g^i is increasing with the specialization of asset i on investment expenses (middle term in RHS) and decreasing with the seigniorage cost that a change on r_i induces through the demands for liquid assets (last term in RHS). This last term vanishes when no assets are cross-subsidized. The constant ? is inversely related to the Lagrange multiplier of the central bank's budget constraint (23), and its value ensures that this constraint is satisfied. This constraint implies that if $r_i > g^i$ for some i, then there must be another security for which $r_i < g^i$. Hence if money is the public liability with the lowest return, the gross rate of growth of the money supply ? = g / r_1 must be larger than one.

As implied by (28), the Silvered Rule differs from a zero nominal interest policy only if the return on liquid assets affects growth. However, in general the Silvered Rule does not maximize the growth rate. The effect on the growth rate of a marginal reform in which r_i (i > 1) is raised at the expense r_1 is

$$\frac{d\ln g}{d\ln r_i} = \frac{1}{\sigma(S+1)\kappa}\left(\tilde{k}_i r_i^{-1} + \tilde{k}_1 r_1^{-1}\frac{d\ln r_1}{d\ln r_i}\right) \tag{32}$$

Comparison of (32) with (25) using (3) implies that a marginal reform with a non negative effect on V raises the growth rate if $\frac{\tilde{k}_i}{\tilde{c}_i + \tilde{k}_i} > \frac{\tilde{k}_1}{\tilde{c}_1 + \tilde{k}_1}$. Thus, at the Silvered Rule the marginal reform considered deriving (32) increases the growth rate if asset i is specialized in investment expenses in relation to money.

6. An Example With Transitional Dynamics

The Silvered Rule of Section 5 maximizes an incomplete measure of welfare because it abstracts from the transitional effects of long-term policy changes. Unfortunately, as it is common in most growth models, transitional dynamics have no explicit solution in general. For this reason, this section addresses these transitional effects for an interesting special case for which transitional dynamics have an explicit solution.

Out of the balanced path, an equilibrium can be explicitly solved with unit elasticities of substitution in consumption and gross production. For ease of exposition, I will further assume that harvests come three periods apart, so there are only two liquid assets money and a two period bond. Consequently, $F(k_{0t}, k_{1t}, k_{2t}) = Ak_{0t}^{\alpha_0} k_{1t}^{\alpha_1} k_{2t}^{\alpha_2}$. A short coming of this specialization is that it rules out partial depreciation of capital with the standard assumption that undepreciated capital is additive in the gross production function. However, this specialization does not rule out a high value for a_0 which mimics the fact that a large portion of gross output remains invested for the next harvest. With this specialization of functional forms, individuals allocate their harvests in shares independent of the rates of return on all assets:

(33) $\tilde{c}_0 = \theta_0(1-\beta^3)$, $\tilde{k}_0 = \alpha_0\beta^3$, and $\tilde{m}_i = \theta_i(1-\beta^3) + \alpha_i\beta^3$ for $i = 1$ and 2;

where $\theta_i = \beta^i(1+\beta+\beta^2)^{-1}$ for $i =$ 0, 1, and 2. Hence, the demand functions for liquid assets constitute a *generalized system of quantity equations* with constant velocity. In the periods when individuals do not have a harvest, they allocate their liquid wealth in proportions also independent of rates of the rates of return on all assets:

$$(34)\ c_{it} = \frac{\theta_i(1-\beta^3)}{\theta_i(1-\beta^3)+\alpha_i\beta^3}\frac{M_{it-i}}{p_t} \text{ and } k_{it} = \frac{\alpha_i\beta^3}{\theta_i(1-\beta^3)+\alpha_i\beta^3}\frac{M_{it-i}}{p_t} \text{ for } i = 1 \text{ and } 2.$$

The market clearing conditions are

$$(35) \qquad \frac{M_{1(t-1)}}{p_t} = \frac{m_t}{\gamma_t} \text{ and } \frac{M_{2(t-2)}}{p_t} = m_{2t} + \left(1-\gamma_t^{-1}\right)m_{1t};$$

where ?$_t$ is the growth factor of the money supply in period t. The system of equation (33) to (35) implies that the allocation of x_t depends on ?$_t$. In particular,

$$(36) \qquad \tilde{c}_{1t} = \theta_1(1-\beta^3)\gamma_t^{-1},\ \tilde{k}_{1t} = \alpha_1\beta^3\gamma_t^{-1},\ \tilde{c}_{2t} = \theta_2(1-\beta^3)\varphi_t, \text{ and } \tilde{k}_{2t} = \alpha_2\beta^3\varphi_t \ ;$$

where $\varphi_t = 1 + (1-\gamma_t^{-1})(\tilde{m}_1/\tilde{m}_2)$. With a constant ?$_t$, the allocation of output remains constant over time. Thus, using (33) and (36), the law of motion of x_t is

$$(37) \qquad x_{t+3} = \zeta A\beta^3\prod_{s=0}^{2}\left(\alpha_s x_{t+s}\right)^{\alpha_s};\ \text{where } \zeta = \gamma^{-\alpha_1}\varphi^{\alpha_2}.$$

(The subscripts on ? and n have been dropped because these variables are held constant). The return on liquid assets affects the allocation of capital, and so it affects total factor productivity. In this special case, total factor productivity is proportional to the policy factor ?. Furthermore, because growth is increasing with the return on capital, the growth factor in a balanced path equilibrium is also increasing with ?:

$$g = \left(\zeta A \beta^3 \prod_{s=0}^{2} \alpha_s^{\alpha_s} \right)^{\frac{1}{3-\alpha_1-2\alpha_2}}.$$

In this special case, condition (31) characterizing an interior Silvered Rule drops the last term because the demand for assets is independent from rates of return. The simplified equations can then be easily solved together with (21) and (23) to obtain explicit formulae for the growth factor of money and the growth rate of return on the national debt along the Silvered Rule path:

(38) $$\gamma^* = \frac{\tilde{m}_1}{(\tilde{m}_1 + \tilde{m}_2 - R)} \left[1 + \frac{3\tilde{m}_2 + (1-\beta^3)(\alpha_2\theta_1 - \alpha_1\theta_2)}{3\tilde{m}_1 - 2(1-\beta^3)(\alpha_2\theta_1 - \alpha_1\theta_2)} \right]; \text{ and}$$

(39) $$r_2^* = \frac{(\tilde{m}_1 + \tilde{m}_2 - R)}{\tilde{m}_2} \left[\frac{3\tilde{m}_2 + (1-\beta^3)(\alpha_2\theta_1 - \alpha_1\theta_2)}{3(\tilde{m}_1 + \tilde{m}_2) - (1-\beta^3)(\alpha_2\theta_1 - \alpha_1\theta_2)} \right] g^2.$$

(Note $\tilde{m}_1$ and $\tilde{m}_2$ are independent of ? in this example). Thus, as long as R = 0, the Silvered Rule calls for $?^*$ > 1 if $a_1?_2 < a_2?_1$. This condition is equivalent to national debt being investment intensive. The definition of $?_i$ implies $?_1 > ?_2$, so $a_1?_2 < a_2?_1$ if a_1 # $a_{2.}$

To conclude this section, I will show that, at least in this example, 0transitional effects reinforce the case for paying positive interest on the national debt. Suppose an economy where money is not investment intensive: $a_1?_2$ # $a_2?_1$. In this economy, the Silvered Rule either calls for positive nominal interest rates ($a_1?_2 < a_2?_1$) or is at an interior solution with zero nominal interest rates ($a_1?_2 = a_2?_1$). The following argument shows that if this economy is in a balanced path with a zero nominal interest rate, there is a simple monetary reform leading to positive nominal interest rates that accomplishes a Pareto improvement. This reform consists in a marginal increase in the supply of liquid assets with the increase in the supply of national debt starting in period 0 and the increase in the supply of money starting in period 2. Because the demand for liquid assets is insensitive to interest rates, the price level and the allocation of output in the periods 0 and 1 is unchanged by the reform. The discounts on public bonds though become positive as their nominal supply increases and their real demand is unchanged. Hence, even if the harvest in period 2 is unchanged because it depends on previous investments, its allocation changes. With the reform, real resources shift from individuals at stage 1 of production (spending money) to individuals at stage 2 of production (spending bonds). Consequently, the harvest in period 3 is increased because its production has used the same capital at stages 0 and 1, but more capital at stage 2. All harvests after period 3 are also increased because of the law of motion (37) and the fact that at ? = 1, the policy factor ? locally increases with ?. (This follows immediately from the Implication Function Theorem applied to the expressions defining ? and n in [36] and [37]).

To show that all individuals are better off with the new policy, consider first the individuals with a harvest at the moment the money supply starts growing in period 2. Before period 2, the reform has no effect on these individuals. After period 2, the discount factors of future consumption goods are given by

(40) $$\rho_{0t} = \frac{\beta^t x_2}{x_t}, \ \rho_{1t} = \frac{\rho_{0t} x_{t-1}\gamma}{x_t}, \text{ and } \rho_{2t} = \frac{\rho_{0t} x_{t-2}}{x_t \varphi};$$

where ?$_{st}$ is the discount factor from period 0 to period *t* for individuals at stage of production *s*. At period 2, the marginal change in the indirect utility function as ? increases is

(41) $$\frac{dV}{d\ln\gamma} = \sum_{t=0}^{\infty} \frac{dV}{d\ln\rho_t}\frac{d\ln\rho_t}{d\ln\gamma}$$

$$= V_x \sum_{i=0}^{\infty} \rho_{3i}\left(-m_{1(3i+1)} + m_{2(3i+2)}\frac{d\ln\varphi}{d\ln\gamma} + \sum_{s=0}^{2} m_{s(3i+s)} r_{s(3i+s)}\frac{d\ln x_{3i+s}}{d\ln\gamma}\right)$$

$$> V_x \sum_{i=0}^{\infty} \rho_{3i}\left(-m_{1(3i+1)} + m_{2(3i+2)}\frac{m_{1(3i+2)}\gamma^{-1}}{m_{2(3i+2)}\varphi}\right) = 0\,.$$

In this expression, the second line uses Roy's identity and (40). The third line uses the Implicit Function Theorem applied to (36) and the positive effect on output induced by the reform. Finally, the last equality follows from n = 1 when ? = 1. Identical argument applies to the individuals with a harvest in period 1 changing subscripts appropriately. The argument is also applicable to the individuals with a harvest in period 0 from period 3 onwards. These individuals, though, get an extra bonus in period 2 when they receive a higher return on the debt they purchased at 0 without having to endure a lower return on money in period 1. Consequently, the policy reform leading to positive interest rates accomplishes a Pareto improvement.

7. CONCLUSION

The optimal structure of outside public liabilities when these securities provide liquidity services cannot disregard their specialization by holding period, and their relative intensities to finance investment and consumption expenditures. When these features are taken into account, there are plausible and empirically testable conditions to rationalize a diverse offer of public liabilities, with the national debt earning positive interest ,and with the return on the national debt increasing with term to maturity. This finding does not contradict the optimum quantity of money rule. If the central bank of my model could be efficiently funded with capital or taxes, then it would be efficient to equalize the return on all liquid assets to the social return on capital. In this paper, I do not discuss how efficiently the government can manage capital or raise taxes to finance the central bank operations. Instead, in this paper I take as a given the self-financing constraint of the central bank, and then I inquire about the possible rationales for diversifying the offer of public liabilities by paying interest on the national debt. A first rationale for this diversification is that the national debt is investment intensive compared to money. Because taxes on investment are less efficient than taxes on consumption, securities intensive in financing investment should be implicitly taxed at a lower rate than securities intensive in financing consumption. Hence the gap between the rates of return of capital and the national debt should be narrower than the gap between the rates of return of capital and money. A second rationale for paying interest on the national debt is based the fact that the national debt is specialized in relatively long holding periods. Because of this specialization, raising the return on the national debt delays expenditure toward the future. Hence, the payment of interest on the national debt, and especially to long term maturities of the national debt, brings a windfall of resources during transitions across balanced paths. With careful policy by the central bank, these resources can lead to a Pareto improvement versus the baseline policy of paying zero interest on the national debt. Similar rationales can justify an upward-sloping term structure for the return on the national debt.

The self-financing constraint on the central bank is convenient but not essential to the analysis of this paper. As long as public liabilities are not fully funded, the central bank will not be able to equalize the return on the liabilities it issues to the social return on capital, and a similar analysis to the one offered here characterizes the tradeoff between implicit taxes on money and national debt. If the net worth of the public sector changes over time, perhaps due to protracted government budget deficits, the implicit taxes on money and the national debt necessary to finance them must change as well. Typically, when a government relies heavily on seigniorage to finance the public sector, inflation will be high and the real interest rate on the national debt will be low.[6] Therefore, if long-run changes in inflation are driven by varying reliance on seigniorage, inflation and real interest rates on Treasury Bills will be negatively correlated at low frequencies. In contrast, if long-run changes in inflation are driven by arbitrary changes in the rate of growth of the money supply with a constant reliance on seigniorage, inflation and real interest rates on Treasury Bills will be positively correlated. The negative correlation between inflation and real interest rates on Treasury Bills at low frequencies documented by Summers (1983) was interpret

[6]This proposition is easily established using (38) and (39) and remarking that around the Silvered path increases in the rate of growth of the money supply are associated with a faster rate of growth. In general, however, this proposition requires that the investment intensity of money and national debt is not too sensitive to the return on liquid assets.

by Fried and Howitt (1983) as an indication of the national debt rendering liquidity services. This paper brings this interpretation one step further. This negative correlation is not only an indication that the national debt renders liquidity services, it is also an indication that policy makers are sensitive to the inefficiencies caused by the low returns earned on liquid securities . Thus, when the seigniorage to be financed by implicit taxes on money and national debt rises, both of these taxes are simultaneously raised to minimize their overall welfare costs.

Future research must address how the results of this paper are modified when there is uncertainty on future expenditure needs. An extension along these lines will have to determine the exact costs to second hand trading. In the present deterministic framework, the exact shape and size of these costs is irrelevant as they are never incurred in equilibrium. However, with uncertain expenditure needs individuals revise their plans over time, so there is an ex-post welfare cost of making it difficult for them to liquidate their national debt earlier than anticipated. In such an extension, the optimal policy will have to balance the tradeoff between making the national debt difficult to trade as a prerequisite to raise its return, and the ex-post welfare cost arising when individuals are prevented to revise their plans.

Future research must also address how the results of this paper are modified when outside moneys (cash and national debt) coexist with inside moneys (deposits). The existence of inside moneys can easily be incorporated to the present model by assuming that some types of capital are suitable as collateral. If these types are relatively scarce, inside and outside moneys may coexist . In this economy, the optimal structure of the monetary sector must address not only the efficiency issues raised in this paper, but also the efficiency issues raised by the likely disparity between the equilibrium rates of return earned by the types of capital suitable as collateral and the types of capital unsuitable as collateral. The solution to this problem will provide not only a normative theory of inflation and nominal interest rates, but also a normative theory of required reserves and other banking regulations.

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