Understanding Investment Irreversibility in General Equilibrium

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Abstract

In general equilibrium, irreversibility affects both the wealth of consumers and the return on assets. As long as the inter-temporal elasticity of substitution is realistically low, irreversibility not only prevents capital destruction, but it also induces capital creation. Furthermore, under certain conditions, irreversibility raises the risk premium by increasing the variability of consumption and market portfolio.These issues are dealt in a simple model of investment irreversibility with multiple types of capital. Its tractability allows for analytical results which explain the contrast, emphasized in the extant literature, between the consequences of irreversibility for individual markets and the consequences of irreversibility for the whole economy. JEL No: E22, G12. Keywords: Irreversible Investment, Stochastic Growth, Asset Pricing.

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1 Introduction

Many forms of investment are irreversible. A growing body of literature[1] has shown that irreversibility is important for firms' investment decisions under uncertainty. However, we still know little about the consequences of investment irreversibility for the economy as a whole, that is, in general equilibrium. Learning about these consequences is not only important for a better understanding of capital accumulation, it is also important for an understanding of the information that asset returns provide about the business cycle. In fact, recent papers by Boldrin, Christiano, and Fisher [1995], Beaudry and Guay [1996], and Jermann [1998] convincingly argue that impediments to the reallocation of capital are crucial to generate realistic dynamics for asset returns. Thus, understanding the general equilibrium consequences of investment irreversibility is important to the theories of economic growth, the business cycle, and asset pricing.

In this paper, I advance an analytically tractable model to explore the consequences of investment irreversibility in general equilibrium. The main contribution of this model to the existing literature is analytical tractability. Analytical results from the literature dealing with investment irreversibility in general equilibrium are scarce and mainly limited to general properties such as existence of solutions. (See Sargent [1980] and Olson [1989].) Fortunately, thanks to numerical simulations, we know some of the effects of investment irreversibility in general equilibrium. For example, using an extension to the neoclassical growth model, Coleman [1997] documents that irreversibility significantly affects interest rates, and it may increase aggregate investment even when no constraints are binding.[2] In contrast, firms' behavioral analyses, and the related competitive equilibrium model by Leahy [1993], show that irreversibility typically discourages capital creation. The model advanced in this paper helps to understand and to formalize why such contrast exists.

The model of this paper extends the endogenous growth model found in Barro [1990]. The four main assumptions of the model are the following. All production processes yield the same output which can be consumed or invested in many types of capital. All factors of production are a type of capital. The aggregate production function yields constant returns to scale. Preferences are homothetic and modeled along the lines of Epstein and Zin [1989]. These preferences include as a special case the standard isoelastic time additive preferences, but they allow to distinguish between the inter-temporal elasticity of substitution and the inverse of the coefficient of relative risk aversion. These two parameters play important and distinct roles in the analysis.

To attain strong analytical results, the model in this paper has to adopt the restrictive assumption that all factors of production are a form of capital. However, the model remains quite flexible to capture important elements of reality: The model allows for many types of capital, so some of them can be interpreted as human capital. Also, the model allows great flexibility on the stochastic shocks affecting the production possibilities of the economy. Finally, the model allows for the coexistence of irreversible and flexible types of capital with or without complementarities between them. This diversity of capital types is important to generate endogenous interest rates and endogenous marginal products of capital. Also, capital diversity allows for strongly binding irreversibility constraints even with positive gross investment (as it has always been).

The model reveals that in general equilibrium there are three effects of irreversibility on aggregate investment. The first is the obvious direct effect of preventing capital destruction in periods one may wish to consume part of the capital stock.[3] The second is the substitution effect induced by the option value lost as a result of constraining the future capital stock when investment takes place. This substitution effect, as stressed in much of the partial equilibrium literature, discourages capital creation. The third, unique to a general equilibrium context, is the wealth effect induced by reducing the set of feasible paths with the irreversibility constraints. This wealth effect promotes capital creation because with investment irreversibility the representative consumer is poorer, so it chooses to consume less and thus to save-invest more. In addition, this wealth effect dominates the substitution effect as long as the inter-temporal elasticity of substitution is, as all empirical studies find, less than one.[4] This result is related to, but distinct from, the demand for precautionary saving.[5] With precautionary saving, one considers a consumer faced with various portfolios with returns that have the same mean but distinct variances. In contrast, investment irreversibility affects not only the variance but also the mean of the returns of the portfolio of a representative consumer. In addition to affecting aggregate investment, irreversibility affects the capital mix in the economy. In a well-defined sense, investment is diverted from irreversible into flexible types of capital at the cost of lowering the aggregate return of capital. Consequently, even if irreversibility encourages investment, it still has an ambiguous effect on growth and the long-run capital stock.

Irreversibility affects asset returns, first, through a direct effect on the market return because it opens the possibility to capital gains and losses. Second, it affects asset returns through an indirect effect, because it changes the variability of consumption and the market return, and thus the price of risk. For analytical tractability, I study with more detail the effects of irreversibility on asset returns in a simple special case with independent and identically distributed (i.i.d.) shocks. For this case, as long as the relative risk aversion coefficient is not lower than one, I find that irreversibility raises both the price of risk and the risk premium. Moreover, it makes both the risk-free rate and the risk premium counter-cyclical. With persistent shocks, the model is not analytically tractable. However, the results with i.i.d. shocks give some intuition as to why the risk-free rate in the United States is less pro-cyclical than equilibrium models without investment irreversibility predict, and why, as Ferson and Harvey [1991] document, the risk premium is counter-cyclical.

The effect of irreversibility on the risk premium is not a potential resolution to the equity premium puzzle. The equity premium puzzle is about fitting the Euler equation that relates consumption growth rates with asset returns. (See Kocherlakota [1996]). Irreversibility has nothing to add to the puzzle of why, with the preferences used in this paper, for measured aggregate consumption and measured asset returns, this Euler equation implies implausibly high degrees of risk aversion. Instead, irreversibility should be useful to construct general equilibrium models that replicate the consumption and the asset returns we measure in the United States and other countries. Of course, if these models are to imply a high equity premium, they will have to assume either a high degree of risk aversion, or they will have to incorporate other factors such as heterogenous consumers or habit persistence.

In summary, the general equilibrium effects of irreversibility transmitted through the wealth of consumers and through the return of assets are important both to understand investment and to understand the interaction between the business cycle and asset pricing. The rest of the paper is organized as follows: Section 2 describes the model in detail. Section 3 reports analytical results. Section 4 concludes the paper.

2 The Model

The economy has a representative consumer. In each period, the consumer's problem is to consume or invest the output just obtained from a stochastic production process. All output is homogeneous, but capital is differentiated into N multiple types. In the present, the vector of capital stocks inherited from the past is $k \in R^N$ and the realized value of the stochastic shocks is $z \in R^M$. The vector z follows a Markov process endowed with the Feller property. A *gross* production function G maps (k, z) onto *gross* output:

$$x = G(k, z). \tag{1}$$

To simplify notation, the *gross* output x includes both production and the capital stocks inherited from the past valued at cost. The function G is concave, linearly homogeneous, and continuously differentiable in k, and measurable in z.

In the allocation of x, the consumer faces a set of irreversibility constraints on each one of the capital stocks:

$$k_i' \geq \mu_i(1 - \delta_i)k_i, \quad \text{for } i = 1...N; \tag{2}$$

where δ_i is the depreciation rate of capital i, μ_i is the degree of irreversibility of capital i, and a prime denotes the value of a variable next period. This specification accommodates flexible capital stocks with a zero μ_i, irreversible capital stocks with a unit μ_i, and types of capital with other degrees of irreversibility. In addition to the irreversibility constraints, the consumer faces a standard resource constraint:

$$c + \sum_{i=1}^{N} k_i' \leq x; \tag{3}$$

where c is consumption.

The preferences of the consumer are recursive, homothetic, and independent across states for atemporal lotteries, but not necessarily time-additive. Specifically, the consumer is endowed with the parametric version of Kreps and Porteus preferences introduced by Epstein and Zin (1989)[6]:

$$u = \left\{ (1-\beta)\, c^{1-\sigma} + \beta \left[E\left({u'}^{1-\gamma} \right) \right]^{\frac{1-\sigma}{1-\gamma}} \right\}^{\frac{1}{1-\sigma}}; \tag{4}$$

where u is present utility; γ is the coefficient of relative risk aversion for atemporal lotteries; and σ is the inverse of the inter-temporal elasticity of substitution along a deterministic path. Both γ and σ are assumed positive. The expectation E is conditional on present information.

The representative consumer maximizes (4) subject to (2) and (3). A solution to this optimal growth problem exists under some restrictions on G (see Epstein and Zin [1989]) which are assumed to be satisfied throughout the paper. Similarly, an analogous proof to Epstein and Zin (1989) shows that there is a value function V which maps the vector (k, z) onto the maximized utility of the consumer. Standard recursive dynamic programming arguments imply that, with respect to k, V is continuously differentiable, concave, and linearly homogeneous.

Using ν_i and λ as the Lagrange multipliers for constraints (2) and (3), the first order conditions to the optimal growth path are:

$$(1-\beta)\, V^{\sigma} c^{-\sigma} = \lambda, \tag{5}$$

and

$$\lambda - V^{\sigma}\beta\left[E\left(V'^{1-\gamma}\right)\right]^{\frac{\gamma-\sigma}{1-\gamma}} E\left(V'^{-\gamma}\frac{\partial V'}{\partial k_i'}\right) = \nu_i \geq 0, \quad \text{for } i = 1, ...N. \tag{6}$$

Condition (6) holds with equality if $k_i' > \mu_i(1-\delta_i)k_i$. Despite the length of these expressions, the intuition behind them is simple. Condition (5) equates the marginal utility of consumption to the marginal value of wealth (λ). Condition (6) states that each constraint (2) penalizes the consumer for a marginal value ν_i, which is equal to the difference between the marginal value of wealth and the marginal value of k_i'. Moreover, ν_i is always nonnegative, and it is zero when the constraint is not binding.

Using the Envelope Theorem, we can derive from these first order conditions a useful consumption rule. The Envelope Theorem states:

$$\frac{\partial V}{\partial k_i} = \lambda\frac{\partial G}{\partial k_i} - \nu_i\mu_i(1-\delta_i), \quad \text{for } i = 1...N. \tag{7}$$

Making use of the fact that V and G are linearly homogeneous, we can multiply both sides of (7) by k_i and add for all i to obtain:

$$V = \lambda\widetilde{x}, \text{ where } \widetilde{x} = x - \sum_{i=1}^{N}(1-q_i)\,\mu_i(1-\delta_i)k_i, \text{ and } q_i = 1 - \frac{\nu_i}{\lambda}. \tag{8}$$

The variable q_i is the real price (in units of the consumption good) of capital i ("Tobin's q"). This variable nets out the (shadow) capital losses incurred due to the irreversibility constraint from to the cost of producing one unit of capital. Consequently, the variable $\widetilde{x}$ is the market value of the goods available after production, which includes the output just produced and the value of the capital stocks inherited from the past. Finally, using (5) and (8), we obtain the consumption rule:

$$c = (1-\beta)^{\frac{1}{\sigma}}\,\lambda^{1-\frac{1}{\sigma}}\widetilde{x}. \tag{9}$$

Because preferences are homothetic, consumption is proportional to the consumer's wealth, which in this model it corresponds with the market value of the goods available after production $\widetilde{x}$. Consumption also depends on the marginal value of wealth λ. An increase in λ induces a negative substitution effect and a positive wealth effect on consumption. If $\sigma > 1$, the wealth effect dominates. All this is familiar. The novelty here is that $\widetilde{x}$ is not predetermined but is endogenous to the present decisions of the representative consumer.

The first order conditions (5) and (6) can be transformed into useful asset pricing equations. For this transformation, the following preliminary results are needed.

The market value of the stock of capital after this period's investment is the difference between the market value of gross output and consumption:

$$qk' = \sum_{i=1}^{N} \left[k_i' - (1-q_i)\mu_i(1-\delta_i)k_i\right] = \widetilde{x} - c; \tag{10}$$

where q is the vector whose components are q_i, $i = 1...N$. For the first equality in (10), note that if constraint i is binding, $k_i' = \mu_i(1-\delta_i)k_i$, and if constraint i is not binding $q_i = 1$. The second equality in (10) follows from the resource constraint (6) and the definition of $\widetilde{x}$ in (8).

The first order condition (6) can be aggregated across capital types if we multiply by k_i' on both sides of (6), add the resulting equations for all i, and make use of the linear homogeneity of V:

$$\lambda\left(qk'\right) = V^{\sigma}\beta\left[E\left(V'^{1-\gamma}\right)\right]^{\frac{1-\sigma}{1-\gamma}}. \tag{11}$$

Finally, using (7), (8), (10), and (11), the first order condition (6) can be transformed into a pricing equation for capital i (see the Appendix for details in this derivation):

$$q_i = E\left\{\frac{V'^{-\gamma}\lambda'}{E\left(V'^{-\gamma}\lambda'\widetilde{R}\right)}\left[\frac{\partial G}{\partial k_i'} - (1-q_i')\mu_i\,(1-\delta_i)\right]\right\}; \tag{12}$$

where $\widetilde{R} \equiv \frac{\widetilde{x}}{qk'}$ is the *ex-post* (gross) market return. The price of one unit of installed capital is equal to the expected present value of its gross return next period. The gross return of capital i next period is the expression in square brackets. This expression is multiplied by a set of contingent prices denoting the relative value of output next period in terms of output this period. These contingent prices can be transformed using (8) to (11) to obtain the following standard Euler equation (see the Appendix for details):

$$q_i = E\left\{\left[\beta\left(\frac{c'}{c}\right)^{-\sigma}\widetilde{R}^{\frac{\sigma-\gamma}{1-\gamma}}\right]^{\frac{1-\gamma}{1-\sigma}}\left[\frac{\partial G}{\partial k_i'} - (1-q_i')\mu_i\,(1-\delta_i)\right]\right\}. \tag{13}$$

That is, the contingent prices depend on consumption growth and the *ex-post* market return in a fairly simple fashion. As a pricing formula, this Euler equation is just an application of Epstein and Zin (1989) and Weil (1989). The novelty here is how consumption and the *ex-post* market return are related to the irreversibility constraints. This novelty, as the next section shows, is important to understand how irreversibility affects the price of risk and the cyclical pattern of asset returns.

3 The Effects of Irreversibility

This section provides a comparison between an economy with the irreversibility constraints (2) and an economy where these constraints are relaxed. The aim of this comparison is to improve our understanding of the effects of irreversibility on capital accumulation and asset prices.

3.1 Capital Accumulation

Suppose there are two economies identical in all respects, except that in one the degrees of irreversibility for all types of capital are zero ($\mu_i = 0$ for all i), while in the other at least one of them is positive ($\mu_i > 0$ for some i). The first economy will be referred to as the flexible economy and the second one as the irreversible economy. Using (8) and (9), the consumption rule for both economies can be described as:

$$c = \left[(1-\beta)\, V^{\sigma-1}\tilde{x}\right]^{\frac{1}{\sigma}} \tag{14}$$

except, of course, that V and $\tilde{x}$ will differ in each economy. Let superscript F denote the flexible economy and the superscript I denote the irreversible economy. Because the constraints in (2) restrict the set of feasible paths for the irreversible economy versus those of the flexible economy, $V^F(k,z) \geq V^I(k,z)$. Moreover, $q_i = 1$ for all i in the flexible economy, while $q_i \leq 1$ in the irreversible economy, thus $\tilde{x}^F(k,z) = x(k,z) \geq \tilde{x}^I(k,z)$. Therefore, if $\sigma \geq 1$, the consumption rule (14) implies $c^F(k,z) \geq c^I(k,z)$. Using the resource constraint (3), this implies:

If the inter-temporal elasticity of substitution is between zero and one ($\sigma \geq 1$), investment in the flexible economy does not exceed investment in the irreversible economy. In addition, this inequality is strict if some irreversibility constraints are presently binding $\left[\tilde{x}^F(k,z) > \tilde{x}^I(k,z)\right]$, or, if $\sigma > 1$ and some constraints are expected to be binding sometime in the future $\left[V^F(k,z) > V^I(k,z)\right]$.

The elimination of the irreversibility constraints increases the marginal value of wealth (without the constraints wealth is more versatile). This induces a negative substitution effect and a positive wealth effect on consumption. If $\sigma \geq 1$, the wealth effect at least balances the substitution effect. Moreover, if some irreversibility constraints are presently binding, the direct effect of the elimination of these constraints revalues the capital stock, and so it raises wealth and consumption.

Proposition 1 is a comparison of policy functions. It is not a comparison about long-run capital stocks. Even under the conditions of Proposition 1 that ensure $c^F(k, z) \geq c^I(k, z)$ for all (k, z), and even if the irreversible and the flexible economies start in identical conditions and receive identical shocks, the flexible economy may end up with larger stocks of capital in the long-run.[7] Certainly, this paradoxical outcome never happens if $N = 1$, because then the marginal product of capital is identical in the two economies. However, in the more interesting case $N \geq 2$, irreversibility changes not only the amount of investment, but also its composition, typically sacrificing productivity to achieve flexibility (see subsection 3.5 below). Hence, the smaller amounts invested in the flexible economy may in average yield higher output than the larger amounts invested in the irreversible economy. Consequently, as one can check numerically,[8] the flexible economy may grow faster and have larger capital stocks in the long-run.

3.2 Fixed Factors

This section modifies the model to explain why it is difficult to obtain strong analytical results on the effects of investment irreversibility in general equilibrium models where output is a strictly concave function of capital. These models, implicitly or explicitly, assume the existence of some factors, such as labor or land, which are inelastically supplied, so they either are fixed or grow at an exogenous rate. Thus, without loss of generality, let the capital stock 1 be fixed, in the sense that the constraint (2) must now hold with equality. Using analogous derivations to those in section 2, we obtain the consumption rule (14), except that now q_1 may either be above or be below unity. Let us compare this economy with a flexible economy with $\mu_i = 0$ for $i = 2, ..., N$. The removal of the constraints on all capital stocks except for 1,which now is a fixed factor, unambiguously raises the utility of the representative consumer V with the same type of substitution and wealth effects as in Proposition 1. However, this removal may induce capital losses on the fixed factor, so it may actually depress the market value of goods available $\tilde{x}$. An analogous result to Proposition 1 in this modified model requires that these potential capital losses are sufficiently low:

We can guarantee that investment in the flexible economy with a fixed factor does not exceed investment in the irreversible economy if, in addition to $\sigma \geq 1$, $\tilde{x}^F(k,z) \geq \tilde{x}^I(k,z)$.

3.3 Relocation Versus Irreversibility

A common motivation for irreversibility is the difficulty of relocating installed capital from a particular use to another. This strong form of irreversibility is captured in this model when the set of constraints (2) applies to each particular use of capital. A weaker constraint is to allow relocation but continue with an aggregate irreversibility constraint. This is achieved by replacing the set of constraints (2) with this unique constraint on the total capital stock:

$$\sum_{i=1}^{N} k_i' \geq \mu \sum_{i=1}^{N} (1 - \delta_i) k_i. \tag{15}$$

We can now evaluate the constraint on relocation by comparing an economy with complete irreversibility (I superscript) with another with relocation (L superscript). Suppose to facilitate the comparison that the degrees of irreversibility in the economy with irreversibility are all equal to the aggregate degree of irreversibility in the economy with relocation, that is $\mu_i = \mu$ for all $i = 1, ..., N$. With this assumption, the set of feasible paths with the constraint in (15) includes the feasible paths with the constraints in (2). For states in which the constraint (15) is not presently binding, Proposition 1 is valid for exactly the same reasons as before. For states in which the constraint (15) is binding, we have:

$$\sum_{i=1}^{N} k_i^{\prime L} = \mu \sum_{i=1}^{N} (1-\delta_i) k_i = \sum_{i=1}^{N} \mu_i (1-\delta_i) k_i \leq \sum_{i=1}^{N} k_i^{\prime I} \tag{16}$$

Consequently:

If $\sigma \geq 1$, investment in the economy with relocation does not exceed investment in the economy with complete capital irreversibility.[9]

This proposition implies that modeling investment irreversibility by imposing to the aggregate capital stock a degree of irreversibility which is an average of the degrees of irreversibility of the individual types of capital typically understates the effects of irreversibility. Conversely, when aggregation is feasible, the degree of irreversibility that yields accurate predictions for the aggregate model may be much higher than the average degree of irreversibility across capital stocks. The following examples illustrate this last point.

3.4 Aggregation Across Capital Types

This subsection provides simple examples to show that irreversibility constraints may be binding even though gross investment is positive, and it discusses the consequences for the degree of irreversibility when aggregating capital across distinct types. To allow for perfect commodity aggregation, it is assumed perfect complementarity across capital types.

Suppose an economy with two indispensable types of capital both with a low but positive depreciation rate. One type of capital is irreversible: $\mu_1 = 1$, while the other is flexible: $\mu_2 = 0$. Both types of capital are perfect complements. For example, refrigerators, k_1, and cows, k_2, are required at a fixed proportion (Leontieff coefficient) to produce milk. Refrigerators can never be consumed, but cows can be killed for meat. With catastrophic shocks, the representative consumer will choose to let some refrigerators idle, and eat the badly needed meat from cows. In this case, perfect aggregation will no longer apply. However, as long as shocks are not catastrophic, the representative consumer combines cows and refrigerators at the fixed proportion dictated by the technology. That is, killing a cow becomes more costly because it implies leaving some refrigerators idle, so it is avoided for shocks that are not catastrophic. Consequently, an economy without catastrophic shocks works as if the degree of irreversibility for the aggregate capital stock is 1, even though only one of the two types of capital is irreversible.

Suppose that the previous example is modified so cows are now irreversible, that is for some reason, religious or otherwise, meat from cows cannot be consumed, so $\mu_1 = \mu_2 = 1$. Also, even if the two types of capital are still perfect complements in production, the Leontieff coefficient now varies over time: The amount of refrigerators required to process the milk of one cow depends on the warmth of the weather, so the required ratio of refrigerators over cows alternates from a high in the summer to a low in the winter. Since investment takes place one period before the new capital becomes productive, investment, if unconstrained, is cow-intensive in the summer and refrigerator-intensive in the winter, in an attempt to have all resources fully employed. In the absence of shocks, this attempt is successful as long as the economy grows sufficiently fast or the rates of depreciation for the two types of capital are sufficiently high. Suppose that in an equilibrium of this type the production of milk is now subject to total factor productivity shocks. With adverse shocks, the representative consumer smooths consumption by investing less. When a severe shock comes in the winter, the irreversibility constraint on cows is the first to bind because investment in the winter is refrigerator-intensive. When this happens and as long as the shock is not catastrophic, the representative consumer continues to invest in refrigerators so that all the next summer's milk can be kept cool. Consequently, an economy without catastrophic shocks works as if the degree of irreversibility for the aggregate capital stock is higher than 1, that is the irreversibility of cows prevents lowering the investment of refrigerators to zero.

3.5 Capital Mix

In the aggregate, the intuition from partial equilibrium analyses that irreversibility depresses capital creation is misleading. With some qualifications, this intuition is nonetheless correct when applied to the composition of capital; that is, irreversibility drives investment away from irreversible types of capital and toward flexible types of capital.[10] Suppose capital F is flexible $(\mu_F = 0)$ while capital I is irreversible $(\mu_I > 0)$. Subtracting equations (7) applied to next period for $i = F$ and I, we have

$$\frac{\partial V'}{\partial k'_F} - \frac{\partial V'}{\partial k'_I} = \lambda' \left(\frac{\partial G}{\partial k'_F} - \frac{\partial G}{\partial k'_I} \right) + \nu'_I \mu_I (1 - \delta_I) \tag{17}$$

Multiplying both sides of (17) by $V'^{-\gamma}$ and taking the conditional expectation of the resulting expressions, we obtain

$$E\left[V'^{-\gamma}\left(\frac{\partial V'}{\partial k'_F}-\frac{\partial V'}{\partial k'_I}\right)\right]=E\left[V'^{-\gamma}\lambda'\left(\frac{\partial G}{\partial k'_F}-\frac{\partial G}{\partial k'_I}\right)\right]+E\left[V'^{-\gamma}\nu'_I\mu_I(1-\delta_I)\right] \tag{18}$$

If the irreversibility on capital I is not binding, the first order condition (6) implies that the left-hand side of (18) is zero. Also, (6) applied to the next period implies that $E\left[V'^{-\gamma}\nu'_I\mu_I(1-\delta_I)\right]$ is nonnegative. Consequently,

$$E\left[V'^{-\gamma}\lambda'\left(\frac{\partial G}{\partial k'_F}-\frac{\partial G}{\partial k'_I}\right)\right]\leq 0 \tag{19}$$

The expression $V'^{-\gamma}\lambda'$ stands for the contingent value to the representative consumer of one good next period, so:

When the representative consumer is investing in capital I, so the irreversibility constraint on capital I is not binding, the expected value of the contingent evaluation of the marginal product of capital I is at least as large as the expected value of the contingent evaluation of the marginal product of flexible capital F. Moreover, if there is a positive probability that the irreversibility constraint on I will be strictly binding next period, this inequality is strict.

Proposition 4 implies that investment in a particular type of capital is discouraged not only by the irreversibility of this particular type of capital, but also by a low co-movement of its return with the contingent value of goods. This is the central idea of portfolio allocation, and it is applied next to the choice between capital and bonds.

3.6 Asset Returns

The risk-free rate consistent with the representative consumer choosing neither to borrow nor to lend, that is choosing a zero net demand for a one period risk-free bond, is determined by a first order condition analogous to (6):

$$\lambda = V^{\sigma}\beta \left[E\left(V'^{1-\gamma}\right)\right]^{\frac{\gamma-\sigma}{1-\gamma}} E\left(V'^{-\gamma}\lambda' r\right); \tag{20}$$

where r is the gross risk-free rate of interest (one plus the net risk-free rate of interest), and $\lambda' r$ replaces $\frac{\partial V'}{\partial k_i'}$ as an application of the Envelope Theorem (7). Using (8), (10), and (11), this equation is transformed into:

$$r = E\left(\psi \widetilde{R}\right), \text{ where } \psi = \frac{V'^{-\gamma}\lambda'}{E\left(V'^{-\gamma}\lambda'\right)}. \tag{21}$$

The risk-free rate is equal to the expectation of the product between the *ex-post* market return, $\widetilde{R}$, and the contingent prices, ψ, that denote the relative value of future output at different states. These contingent prices depend on the relative scarcity of goods at different states and how this scarcity affects utility at the margin. Using (21) the risk premium is:

$$E\left(\widetilde{R}\right) - r = E\left[\left(1-\psi\right)\widetilde{R}\right]. \tag{22}$$

Irreversibility affects the risk-free rate and the risk premium both through the effect it has on the *ex-post* market return, because it opens the possibility to capital gains and losses, and through the contingent prices of future goods, because it affects the variability of the market value of output and consumption.

3.7 A Simple Special Case

Even in the highly tractable model advanced in this paper, the effects of irreversibility on asset returns are hard to describe beyond the general formulae (20) to (22). For further analytical tractability, I will specialize the model, to have one type of capital and a multiplicative i.i.d. stochastic shock, that is $x = zk$ where z is the gross return of capital (it includes the net return and the survival rate $(1-\delta)$). The realizations of z are assumed to be positive and to satisfy the restrictions studied in Epstein and Zin [1989] for the existence of an optimal growth path.

In a flexible economy, the utility of the representative consumer depends only on the total amount of goods available, x. Moreover, since the value function is linearly homogeneous of degree one, we have $V^F = v_0 x = v_0 zk$ for a positive number v_0. In an irreversible economy, the irreversibility constraint (2) may be binding, so the value function depends not only on x, but also on its composition. In this case, the linear homogeneity of the value function implies $V^I = v(z)x = v(z)zk$ for a positive function v. Also, the function v is weakly increasing because for a given total amount of goods x, the representative consumer is less constrained on future choices the lower is k and so the higher is z.

The value of installed capital in (13) specializes to:

$$q = E\left\{\left[\beta\left(\frac{c'}{c}\right)^{-\sigma}\widetilde{R}^{\frac{\sigma-\gamma}{1-\gamma}}\right]^{\frac{1-\gamma}{1-\sigma}}\widetilde{z}'\right\}, \quad \text{where } \widetilde{z}' = z' + (q'-1)\,\mu\,(1-\delta)\,. \tag{23}$$

Since k' is a uni-dimensional vector, $\widetilde{z}' = \frac{\widetilde{x}'}{k'}$(see the definition of $\widetilde{x}$ in (8)). Also, the *ex-post* market return is

$$\widetilde{R} \equiv \frac{\widetilde{x}}{qk'} = \frac{\widetilde{z}'}{q}. \tag{24}$$

Hence, with the help of the consumption rule (14), equation (23) is transformed into

$$q^{\frac{1-\gamma}{1-\sigma}} = E\left\{\left[\frac{\beta}{1-\beta}\frac{c^{\sigma}}{\left[v(z')z'k'\right]^{\sigma-1}\widetilde{x}'}\right]^{\frac{1-\gamma}{1-\sigma}}\widetilde{z}'^{\frac{1-\gamma}{1-\sigma}}\right\}. \tag{25}$$

Taking the variables presently known out of the expectation and simplifying, we have

$$q = \left(\frac{c}{k'}\right)^{\sigma} \frac{\beta}{1-\beta} \left(E\left\{[v(z')z']^{1-\gamma}\right\}\right)^{\frac{1-\sigma}{1-\gamma}}. \tag{26}$$

When the irreversibility constraint is binding $k' = \mu(1-\delta)$ and $c = z - \mu(1-\delta)$, so we have

$$q = \min\left\{\left[\frac{z-\mu(1-\delta)}{\mu(1-\delta)}\right]^{\sigma} \frac{\beta}{1-\beta} \left(E\left\{[v(z')z']^{1-\gamma}\right\}\right)^{\frac{1-\sigma}{1-\gamma}}, 1\right\}. \tag{27}$$

Consequently, q increases with z as long as the irreversibility constraint is binding, and it ceases to bind when $q = 1$.

In the flexible economy, $\tilde{x} = x = zk$ and $V^F = v_0 x$, so (8) implies v_0 is equal to the marginal value of wealth λ. Also, the *ex-post* market return in this economy is $\widetilde{R} = \tilde{z} = z$, so equation (21) determining the risk-free rate simplifies to:

$$r^F = E\left[\psi^F z'\right], \text{ where } \psi^F = \frac{z'^{-\gamma}}{E(z'^{-\gamma})}. \tag{28}$$

The contingent prices of future goods decrease with z' reflecting that in states of nature that z' is low, goods are more scarce and more valuable (see Weil [1989] for further discussion of this equation). Equation (28) implies that r^F is constant because z' is i.i.d. In the irreversible economy, $\tilde{x} = \tilde{z}k$ and $V^I = v(z)zk$, so (8) implies $\lambda = [v(z)z]\,\tilde{z}^{-1}$. Therefore, equation (21) simplifies now to

$$r^I = E\left(\psi^I \frac{\tilde{z}'}{q}\right), \quad \text{where } \psi^I = \frac{[v(z')z']^{1-\gamma}\,\tilde{z}'^{-1}}{E\left\{[v(z')z']^{1-\gamma}\,\tilde{z}'^{-1}\right\}}. \tag{29}$$

In this case, the risk-free rate r^I is not constant because q depends on z. Moreover, since q weakly increases with z, r^I weakly falls with z. Once the risk-free rates are determined, the risk premium for the irreversible and the flexible economies follow from equation

(22). The following proposition establishes the effect of irreversibility on the spread of the contingent prices of future goods, ψ^F and ψ^I , which determines the price of risk and the risk premium.

In this special case, if $\gamma \geq 1$, the contingent prices of future goods for the irreversible economy, ψ^I, are a mean preserving spread of the analogous prices for the flexible economy, ψ^F. As a consequence, the risk premium in the irreversible economy is not lower than the risk premium in the flexible economy. (See the Appendix for the proof).

This proposition implies that irreversibility raises the value of future goods when z' is low and depresses them when z' is high, so the variability of returns (risk) is penalized more heavily in the irreversible economy than in the flexible economy. This reflects the extra scarcity of goods induced by binding irreversibility constraints associated with low values of z'. The assumption $\gamma \geq 1$ is necessary because with irreversibility, even if the shock z is i.i.d., the market return is not: Capital losses today are expected to be reversed with capital gains tomorrow. For this reason, $\gamma \geq 1$ is necessary to ensure that the representative consumer values capital highly when output is scarce (wealth effect) as opposed to when the expected market return is high (substitution effect).

Proposition 5 has also strong implications for the cyclical pattern of the risk-free rate and the risk premium. In good times, when z is high enough for the irreversibility constraint not to bind, q is one, and thus the market return in the irreversible economy is $\tilde{z}'$. Then, if $\gamma \geq 1$, $r^I \leq r^F$ for two reasons. First, the possibility of capital losses implies that $\tilde{z}' \leq z$ for all states of nature. Second, as seen, the contingent prices of future goods penalize risk more heavily in the irreversible economy than in the flexible economy. In contrast, in bad times, when z is low enough so $q < 1$, the *ex-post* market return is

$\frac{\tilde{z}'}{q} > \tilde{z}'$. Then, the possible capital gains from an increase in q raises the expected market return. With i.i.d. shocks, the contingent prices of future goods do not depend on z, and $\tilde{z}'$ is i.i.d., so r^I is counter-cyclical. The constraint irreversibility imposes on the supply of goods raises their inter-temporal price: the risk-free rate. Equation (22) implies that a similar argument applies to the risk premium. Consequently, as z falls a portion of the higher expected market return goes to raise the risk-free rate while another portion goes to raise the risk premium.

Proposition 5 does not imply that, by comparison with a flexible economy, irreversibility depresses the average risk-free rate. When the irreversibility constraint is binding, r^I may well exceed r^F. Furthermore, as one can check numerically,[11] the average risk-free rate in the irreversible economy may be above or below the average risk-free rate in the flexible economy. In conclusion, even tough, irreversibility may lead to lower average risk-free rates as in Coleman [1997], this is not a general result even in this simple economy.

4 Concluding Remarks

The model advanced in this paper provides analytical results which explain the contrast, emphasized by Coleman [1997], between the consequences of investment irreversibility for individual firms and the consequences of irreversibility for the whole economy. The source of this contrast lies on the effect of irreversibility on the effective wealth of consumers and the return on assets. In the framework of the model, as long as the inter-temporal elasticity of substitution is less than one, investment irreversibility not only prevents capital destruction, but it also induces capital creation. Furthermore, irreversibility affects the price of risk by making both consumption and the market portfolio more variable.

For ease of exposition and analysis, the model presented in this paper has been set as an optimal growth problem. However, as it is well known, the solution path for this type of problems can be decentralized as a competitive equilibrium. Likewise, the irreversibility constraints are expressed as a simple inequality. However, costly reversion of investment should yield similar results (see Abel and Eberly [1996]). Finally, the irreversibility constraints have been modeled as physical constraints. However, to a certain degree, these constraints can also handle an extreme form of the "lemon's problem" which would lead to the disappearance of some markets. In this case, the interpretation of the irreversibility constraints would be that buying a car constitutes an irreversible commitment to a particular use for the car, for example being the means of transportation of the buyer. The buyer can still fully diversify the risk involved in this purchase, for example by selling shares on the ownership of the car. But at the end of the day, the only way the shareholders can get any dividend from the car is if the original buyer uses it. If this buyer does not need the car anymore, this is a social loss no matter how much diversification there was. If in addition to investment irreversibility capital markets were imperfect, the "lemon's problem" would lead to stronger wealth effects than the ones modeled in this paper. Then, aggregation does not hold in general, and I conjecture that the effects of irreversibility on capital accumulation and asset returns would, in general, be stronger.

Appendix

Derivation of equations (12) and (13)

Both (12) and (13) are derived from the first order condition (6). Rearranging terms, this condition can be re-stated as follows:

$$1 - \frac{\nu_i}{\lambda} = \lambda^{-1} V^{\sigma} \beta \left[E\left(V'^{1-\gamma} \right) \right]^{\frac{1-\sigma}{1-\gamma}} E\left\{ \frac{V'^{-\gamma}}{E\left(V'^{1-\gamma}\right)} \frac{\partial V'}{\partial k_i'} \right\}. \tag{30}$$

Using the Envelope Theorem (7), condition (11), and the definition of q_i in (8), this equation is transformed into:

$$q_i = (qk')\, E\left\{ \frac{V'^{-\gamma}}{E\left(V'^{1-\gamma}\right)} \left[\lambda' \frac{\partial G}{\partial k_i'} - \nu_i' \mu_i (1-\delta_i) \right] \right\}. \tag{31}$$

Rearranging terms and using the definition of q_i in (8), we have:

$$q_i = E\left\{ \frac{V'^{-\gamma}\lambda'}{E\left(V'^{-\gamma} \frac{V'}{qk'} \right)} \left[\frac{\partial G}{\partial k_i'} - (1-q_i')\mu_i (1-\delta_i) \right] \right\}. \tag{32}$$

Finally, using (8) and the definition of $\widetilde{R}$, we obtain equation (12).

The derivation of equation (13) uses the following two expressions:

$$V'^{1-\gamma} = \left[(1-\beta)\, c'^{-\sigma} \widetilde{x}' \right]^{\frac{1-\gamma}{1-\sigma}}, \tag{33}$$

and

$$E\left(V'^{1-\gamma} \right) = \left[\lambda V^{-\sigma} (qk')\, \beta^{-1} \right]^{\frac{1-\gamma}{1-\sigma}} = \left[\lambda^{1-\sigma} \widetilde{x}^{-\sigma} (qk')\, \beta^{-1} \right]^{\frac{1-\gamma}{1-\sigma}} = \left[(1-\beta)\, c^{-\sigma} (qk')\, \beta^{-1} \right]^{\frac{1-\gamma}{1-\sigma}}. \tag{34}$$

The first of these expressions is derived from (8) and (9). In the second expression, the first equality is a restatement of (11). The second equality uses (8). And the third equality uses (9). Using (8), the contingent prices in (12) can be written as follows:

$$\frac{V'^{-\gamma}\lambda'}{E\left(V'^{-\gamma} \lambda' \widetilde{R} \right)} = \frac{V'^{1-\gamma}}{E\left(V'^{1-\gamma}\right)} \frac{qk'}{\widetilde{x}'}. \tag{35}$$

Finally, equation (13) follows from substituting (33) and (34) into (35), using the definition of $\widetilde{R}$, and substituting the resulting expression into (12).

Proof of Proposition 5

The function $v(z)$ is positive and weakly increasing, so the expression $[v(z')]^{1-\gamma}$ is a positive and weakly decreasing with respect to z' if $\gamma \geq 1$. Likewise, q is a function of z that is positive, bounded above by one, and weakly increasing, so the expression $\widetilde{z}'^{-1}z'$, which using the definition of $\widetilde{z}$ in (24) is equal to $\left[1-\mu(1-\delta)\frac{1-q'}{z'}\right]^{-1}$, is a positive and weakly decreasing function of z'. Therefore, the product $[v(z')z']^{1-\gamma}\,\widetilde{z}'^{-1}z'^{\gamma}$, which is equal to $[v(z')]^{1-\gamma}\,\widetilde{z}'^{-1}z'$, is a positive an weakly decreasing function of z' if $\gamma \geq 1$.

Let $[\check{z}, \hat{z}]$ be the space of the stochastic variable z and $F(z)$ its distribution function (z is i.i.d. in this special case). The contingent prices ψ^F and ψ^I, respectively defined in (28) and (29), are functions of z' and $E\left[\psi^F(z')\right] = E\left[\psi^I(z')\right] = 1$. Therefore, to prove that ψ^I is a mean preserving spread of ψ^F, we must show that

$$\int_{\check{z}}^{\bar{z}} \left[\psi^I(z') - \psi^F(z')\right] dF(z') \geq 0, \text{ for all } \bar{z} \in [\check{z}, \hat{z}]. \tag{36}$$

The weak inequality in (36) holds trivially for $\bar{z} = \check{z}$ and $\bar{z} = \hat{z}$. Suppose there is a $\bar{z}$ $\in [\check{z}, \hat{z}]$ for which condition (36) does not hold. This supposition implies the existence of a pair z_0 and z_1, $z_0 \leq \bar{z} < z_1$, that satisfies $\psi^I(z_0) < \psi^F(z_0)$ and $\psi^I(z_1) > \psi^F(z_1)$. Using the definitions of ψ^F and ψ^I in (28) and (29), these inequalities imply that the value of $[v(z')z']^{1-\gamma}\,\widetilde{z}'^{-1}z'^{\gamma}$ for $z' = z_0$ is smaller than its value for $z' = z_1$. This implication contradicts that $[v(z')z']^{1-\gamma}\,\widetilde{z}'^{-1}z'^{\gamma}$ is a weakly decreasing function of z'.

Using (22) and (24), the difference between the risk premia in an irreversible and in a flexible economy is:

$$E\left[\left(1-\psi^{I}\right)\frac{\tilde{z}'}{q}\right]-E\left[\left(1-\psi^{F}\right)z'\right]=$$
$$E\left[\left(1-\psi^{I}\right)\frac{z'}{q}\right]-E\left[\left(1-\psi^{F}\right)z'\right]+\frac{\mu(1-\delta)}{q}E\left[\left(1-\psi^{I}\right)(q'-1)\right]. \quad (37)$$

The term $E\left[\left(1-\psi^{I}\right)\frac{z'}{q}\right]-E\left[\left(1-\psi^{F}\right)z'\right]$ is nonnegative, because $q\leq 1$, both $\left(1-\psi^{I}\right)$ and $\left(1-\psi^{F}\right)$ are increasing functions of z', and $\left(1-\psi^{I}\right)$ is a mean preserving spread of $\left(1-\psi^{F}\right)$. Likewise, the final term $\frac{\mu(1-\delta)}{q}E\left[\left(1-\psi^{I}\right)(q'-1)\right]$ in (37) is nonnegative, because $\left(1-\psi^{I}\right)$ has conditional mean 0 and is nonnegatively correlated with $(q'-1)$ (both are non decreasing with z'). Consequently, the right-hand side of (37) is nonnegative.

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Notes

1. See Pindyck [1991], Dixit and Pindyck [1994] and Hubbard [1994] for an introduction and survey to this literature.

2. Other examples are: Dow and Olson [1992], Caplin and Leahy [1993], Bertola [1994], Boldrin, Christiano and Fisher [1995], Christiano and Fisher [1995], Ricketts and McCurdy [1995], Veracierto [1997], Huffman and Wynne [1995], and Christiano and Fisher [2000].

3. See Abel and Eberly [1999] for the importance of this effect on the long-run distribution of capital stocks in a partial equilibrium framework.

4. This result does not depend on constant marginal products of capital. With several types of capital, as the model allows, the marginal products of capital vary depending on the set of binding irreversibility constraints. Consequently, irreversibility discourages capital creation if the inter-temporal elasticity of substitution is higher than one. (This elasticity is implicitly infinite in analyses that maximize the expected present value of profits). However, irreversibility encourages capital creation when no constraints are binding, if the inter-temporal elasticity of substitution is less than one.

5. See Ejarque [1998] for an study on how increased uncertainty may increase investment in the presence of irreversibilities due to an increase in precautionary savings.

6. This formula is valid for $\sigma \neq 1$ and $\gamma \neq 1$. For $\sigma = 1$, use a Cobb-Douglas aggregator. For $\gamma = 1$, use $\exp\{E[\ln(u')]\}$ instead of $[E(u'^{1-\gamma})]^{\frac{1}{1-\gamma}}$.

7. In the model of this paper, invariant distributions of the capital stocks in general do not exist because these stocks are not bounded. However, invariant distributions of the rate of growth of the capital stocks exist as long as z is bounded.

8. A simple numerical example in which the flexible economy grows faster the irreversible economy even if $c^F(k,z) \geq c^I(k,z)$ for all (k,z) is: In both economies, $N = M = 2$, $G(k,z) = z(k_1 + 0.8k_2) + k1 + k2$, $\delta_1 = \delta_2 = 0$, z is iid, $z_1 = 0.08$, $z_2 = 0.04$, $\Pr(z_1) = \Pr(z_2) = 0.5$, $\beta = 0.98$, $\sigma = \gamma = 2$. In the flexible economy, $\mu_1 = \mu_2 = 0$. In the irreversible economy, $\mu_1 = 1$, and $\mu_2 = 0$. Using numerical methods to compute invariant distributions, we find that the average continuously compounded rates of growth of the capital stocks are: 0.019 for the flexible economy and 0.013 for the irreversible economy.

9. The condition $\mu_i = \mu$ for all i can be easily relaxed to $\mu_i \geq \mu$ for all i. The conditions for strict inequality between investment in these two economies are those at the end of Proposition 1.

10. See Eberly and Mieghem [1997] for a similar result in a partial equilibrium context.

11. For example, $\beta = 0.99$, $\sigma = 0.2$, $\gamma = 2$, $\mu = 0.99$, and z i,i.d. with two equally likely values 1.1 and 1.04 implies that in an invariant distribution $E\left(r^I\right) = 1.076$ and $r^F = 1.068$. In contrast, with these parameters except for $\gamma = 4$, it implies $E\left(r^I\right) = 1.060$, and $r^F = 1.067$.