Fair Division in the Presence of Indivisible Goods and Widespread Externalities*

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Abstract

We analyze an assignment problem of indivisible goods to a continuum of consumers in the presence of binding budget constraints and widespread externalities. We consider a benchmark model in which an indivisible good generates pollution. We introduce costly sharing to the model. In the presence or absence of sharing, we find the market permit equilibrium price and allocation for any pollution target and distribution of income. Except for a relatively small sharing cost parameter, the market allocation necessarily differs across one or more groups. We find that all individuals are indifferent among the allocations of the indivisible good, the lower the sharing cost, the less skewed the distribution of income is toward the poor, or the larger the middle class. over another. In the absence of any re-distribution of income the outsiders may be worse off in the market outcome relative to the laissez-faire outcome. Since a market-based policy favours the upper income group while compensating outsiders through a redistribution of income (based on ownership), we compare a market-based outcome (that targets a given pollution level) to the laissez-faire outcome. We provide necessary and sufficient conditions for the Pareto domination of the laissez-faire outcome by the targeted market equilibrium outcome. Pareto domination

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occurs the lower the sharing cost, the less skewed the distribution of income is toward the poor, or the larger the middle class.
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1. Introduction

The allocation of a set of indivisible goods to a set of individuals has been considered in the presence or absence of a fixed aggregate amount of a divisible good, say money.[1] In particular, in the presence of a divisible good, and common quasilinear preferences and in the absence of sellers, the optimal assignment in that in which the winners receive 1 unit of the indivisible good, the losers receive 0 units of the indivisible good, the winners may be in debt, the losers receive some amount of money and each is indifferent between receiving either allocation in the optimal assigment. In the presence of sellers, one mechanism that achieves the optimal assignment is an auction mechanism that allocates the entire surplus to the sellers. The discussion in the literature concerns fairness and efficiency[2] (in which case, the winners may be in debt), may include a seller and individual binding budget constraints[3] (in which case, all the surplus accrues to the seller), and, in the absence of a divisible good, may allocate the indivisible units in a probabilistic fashion[4]. We want to address the question of a fair division of indivisible goods in the context of redistribution of income and the presence of individual binding budget constraints. In this paper, we provide a model in which consumers have common quasilinear preferences but variable income constraints that may bind. In addition, the rationale for the paucity of indivisible objects relative to the size of the population is that the production of the goods causes widespread negative externalities. While the goal of the previous literature was to obtain a fair

[1] This literature begins with the seminal papers [13] and [18], and includes [6], [21], [2], and [22].

[2] See for example [21].

[3] See for example [22].

[4] See for example [3].

division of the goods, one of the goals of the assignment problem in this paper is to understand when it is possible that all consumers are better off after the reduction in the number of indivisible goods due to decreased pollution. As in the previous literature, we aim for a fair allocation in which no one envies the other (abstracting away from initial income), but, in contrast to the previous literature, there is no need for the entire surplus to accrue to the producer. Rather, the surplus is redistributed across consumers.

Unlike the previous literature, we have a continnum of consumers whose income I is distributed according to a continuously differentiable distribution function over some support. There is a widespread externality associated with the production of the good (not the means of production) and all prefer less pollution. The government mandates decreased sales. The question is how to allocate the relatively small set of indivisible goods to the set of consumers in a way that is fair and causes no envy. When it is not possible to obtain an allocation that is fair and envy-free, we consider ways to improve the properties of the policy. In particular, as in [5], we consider the implications of transforming the indivisibility of the good through costly sharing. It was shown in [5] that, unlike the case of divisible goods, the efficient allocation of an indivisible good need not be common across individuals with common preferences. In addition, since one group may be favoured over a group of outsiders, the efficient outcome need not Pareto dominate the laissez-faire outcome. In this paper, we are concerned with the implications of any given welfare-improving pollution target on populations that differ in income distribution.

We explore the market-based policy that targets a given pollution level through a distribution of transferable permits among the population. When the good is indivisible and sharing is infeasible, this targeted policy allocates the indivisible good efficiently. Associated with each policy outcome is an allocation of the indivisible good across individuals with a suitable transfer of money between consumers and firms and a transfer of money across individuals. We term a distinguished allocation to be one without the transfer of money across individuals.

We give necessary and sufficient conditions for any distinguished allocation to *khs*-dominate the laissez-faire outcome after a redistribution of income based on ownership of the indivisible good.[5] We show that the targeted policy outcome Pareto dominates the laissez-faire outcome if and only if (denoted by iff) its associated distinguished allocation *khs*-dominates the laissez-faire outcome. This results shows that the market-based permit policy possesses very good properties in the presence of binding budget constraints. It works whenever it is possible to achieve this outcome for any market based policy that essentially taxes the winners based on ownership.

We analyze the policy implications of a targeted pollution level in the case that individuals can share a unit of the good through joint ownership. Joint ownership is associated with an inconvenience cost. As in the case of no sharing, for each non-negative sharing cost, we find the equilibrium policy outcome for each pollution level. When the sharing cost is low enough relative to the targeted pollution level, all individuals receive a common allocation in the equilibrium outcome. Otherwise, in the equilibrium outcome, one or more groups are favoured over a group of outsiders and the size and allocation of this group varies with the cost of sharing. As in the case of no sharing, we give necessary and sufficient conditions for the equilibrium outcome to *khs*-dominate the laissez-faire outcome. Unlike the case of no sharing, however, the targeted policy may allocate the goods inefficiently relative to no redistribution of income but always achieves the best relative to an ownership tax that is redistributed among the non-owners. For any given pollution level that Pareto dominates the laissez-faire outcome given a uniform distribution of the indivisible good, we find that, in the market-based equilibrium, Pareto domination occurs the lower the sharing cost, the less skewed the distribution of income toward the poor, or the larger the middle class.

Our model builds upon the literate begun by [13], generalizes the example

[5] Outcome 1 *khs*-dominates outcome 2 if it is possible to reallocate income so that outcome 1, after reallocation, Pareto dominates outcome 2 but that the reverse is impossible. (See [7], [10], [11].) Outcome 1 *khs*-dominates outcome 2 if the transfers are based on ownership.

analyzed in [12] and [23] and is most closely related to [5]. Both [12] and [23] show that the use of transferable permits may result in an inefficient outcome. Given the functional forms and a given inefficient pollution target used in [12] and [23], their results indicate that the use of permits may make the poor worse off.[6] Though a permit scheme results in a transfer of money from the favoured rich to the poor outsiders, the transfer need not improve the outcome for the poor.[7] [5] extends their models in five directions. (1) the distribution of income and the utility function over consumption is generalized. (2) Sharing is introduced by using a general parametrized family of sharing cost functions.[8] (3) The efficient pollution level and allocation is obtained in the presence or absence of sharing. Necessary and sufficient conditions for Pareto domination of the laissez-faire outcome by the targeted policy outcome are given. Some policy recommendations are offered. In the current paper, we obtain the equilibrium outcome for all possible pollution targets and provide comparative statics with respect to changes in the cost of sharing as well as changes in the distribution of income. The model is also related to the literature on the allocation of indivisible goods and fair division begun by Koopmans and Beckman (1957).[9] This literature is concerned with the fair allocation of indivisible goods across a population. The models do not begin with an allocation of money across individuals. Instead, there is a pot of money available that can be used, in effect, to bribe those not allocated the indivisible good and to punish those allocated the good in such a way that no loser envies a winner and no winner envies a loser. If necessary, winners can be allocated a negative amount of money. In our model however, there is an initial distribution

[6] The efficient level of pollution in [12] and [23] is 0 so any positive target is inefficient.

[7] [9] assumes that an equal distribution of tradable permits ensures equitability.

[8] [4] adds a specific sharing function to the examples used in [12] and [23].

[9] See Koopmans and Beckman (1957), Gale and Shapley (1962), Shapley and Shubik (1971), Crawford and Knoer (1981), Leonard (1983), Demange and Gale (1985), Demange, Gale, and Sotomayer (1986), Tadenuma and Thomson (1995), Tadenuma and Thomson (1995), Azacis (2008)

of income across the population. Given a particular reduction of the indivisible good, we ask for a policy that may redistribute the income across the population in such a way that the winners and losers are better off relative to before the

We follow [12], [23] and [5] by making the simplifying assumption that the variability of the negative externalities generated by ownership and use of our indivisible good is attributed solely to aggregate ownership.[10] In addition, as in in [5], we consider sharing as a particular form of a change in the system that can improve social welfare. Though joint ownership may be feasible, it comes at a cost. Sharing succeeds when either the price of the indivisible good is high enough or the cost of sharing is low enough.[11]

When governments are pressured by international and domestic demands to reduce pollution, they may have an interest in using permit markets to reduce the consumption and the excess capacity associated with indivisible goods that generate negative externalities. We need to understand the ownership patterns and equity implications of such polices so that we can make predictions regarding negotiations over reduction in pollution levels. If a government wants to decrease the use of indivisible goods, then it may use a permit system that increases the price of the indivisible good. If, in addition, the cost of sharing is low, then individuals could enjoy the services of the indivisible good without the expense of sole ownership. If the cost of sharing is high, then outsiders may be compensated for their loss of services through lump sum transfers from the insiders. Our results suggest that additional policies aimed at reducing the inconvenience cost of sharing, increasing the wealth of the poorest individuals, or increasing the size of the middle class would improve the outcome of the permit system and thereby increase the set of feasible negotiated pollution levels. The permit system is essentially a system that provides an ownership tax that is used to compensate the non-owners for their loss in consumption.

The structure of the paper is as follows: Section 2 describes the model. Section

[10] [12] and [23] make this assumption with respect to their indivisible good, the automobile.

[11] See [19], [20] and (http://www.expatsingapore.com/once/cost.shtml))

3 analyzes the equilibrium of a market-based permit system that targets a given pollution level. Section 4 considers a simple bargaining model in which countries negotiate over pollution reduction. Section 5 summarizes the main results.

2. Model

Ownership of an indivisible good is tied to the generation of a widespread negative externality. Each individual is affected negatively as a result of the aggregate consumption of the indivisible good, but each individual's contribution to the aggregate is of measure zero. We assume that preferences over an indivisible good and a composite commodity are common across individuals and that there are no income effects so that utility is quasilinear over the indivisible good and the composite commodity. Since preferences are common and there are no income effects, income rather than taste differentiates choices, so that consumers are denoted by their income level I. Income is distributed according to the continuously differentiable distribution function G over $\left[G^{-1}(0), G^{-1}(1)\right] = \left[\underline{I}, \overline{I}\right]$. We assume that I denotes income available to spend on the indivisible good so that it is feasible to spend all on the indivisible good. Let $\alpha(I)$ denote the indivisible good consumption of an individual with income I and let x parameterize the disutility of sharing the indivisible good. The utility function of an individual who faces aggregate consumption captured by α, and who consumes A units of the indivisible good and m other goods is

$$\phi(A) - \kappa(A, x) + m - \pi\left(\int_{G^{-1}(0)}^{G^{-1}(1)} \alpha(I)\, G'(I)\, dI\right)$$

where ϕ denotes the positive consumption utility over the indivisible good, $\kappa(A, x)$ denotes the negative consumption utility or inconvenience cost of sharing A units of the indivisible good when the sharing cost parameter is $x \geq 0$, and π denotes the disutility associated with the widespread externality generated by aggregate consumption of the indivisible good.[12] Thus

$$\phi(A) - \kappa(A, x) \tag{2.1}$$

can be interpreted as the net consumption utility over the indivisible good and π is the cost of pollution generated by aggregate consumption.

We assume that $\phi'(A) > 0$, $\phi''(A) < 0$, $\phi(0) = 0$, $\phi(1) > 1$, and that π is increasing, convex and that $\pi(0) = 0$, $\pi'(0) + 1 < \phi'(0)$. We now discuss the disutility or cost of sharing, $\kappa(A, x) \geq 0$.[13] We introduce the following notation to make it easier to discuss the relationship between fractions and the family of sharing cost functions. Let $\lceil A \rceil$ denote the largest integer less than or equal to A and let $\lfloor A \rfloor$ denote the smallest integer greater than or equal to A. We assume when $x > 0$, that for every fraction $A \in (\lceil A \rceil, \lfloor A \rfloor)$, (i) $\kappa(A, x) > 0$ so that the cost of sharing is incurred if the consumer chooses to share, (ii) $\kappa'_x(A, x) > 0$ for $x \in \Re_+$ (where $\Re_+$ denotes the non-negative real numbers), (iii) $-\infty < \kappa'_A(A, x) \leq 0$ so that the cost of sharing decreases as the quantity of sharing decreases, and (iv) $\kappa(A, x)$ is twice continuously differentiable with $k''_{AA} \geq 0$ and $k''_{Ax} < 0$. We assume that when $A \in \mathbb{N}_+$ (where $\mathbb{N}_+$ denotes the non-negative integers) then $\kappa(A, x) = 0$ for all $x \geq 0$, (i.e., $\phi(A) - \kappa(A, x) + m = \phi(A) + m$ for $A \in \mathbb{N}_+$) so that the consumer incurs no cost of sharing whenever the consumer chooses not to share. We assume that $\kappa(A, 0) = \kappa'_A(A, 0) = 0$ for all $A \in \Re_+$ and that $\lim_{A \uparrow \lfloor A \rfloor} \kappa'_A(A, x) = -x$. Thus, κ decreases in A from something positive to 0 for $A \in (\lceil A \rceil, \lfloor A \rfloor]$ and then jumps up to something positive again immediately to the right of $A = \lfloor A \rfloor$ so that there is a discontinuity from the right at each integer. Thus, κ is left-continuous for every A but is right-discontinuous at each integer.[14] Therefore $\phi(A) - \kappa(A, x) + m$ is continuous in A for $A \in \Re \setminus \mathbb{N}_+$, but is discontinuous from the right for $A \in \mathbb{N}_+$.

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[12] One can easily extend the model to include differential taste and income

[13] These functions generalize ($\pi(z) = \sigma z, \sigma > 1$) used in [12] (with $\phi(A) = 2A$) and used in [23] (with $\phi(A) = 2A^{1/2}$). There is no counterpart to $\kappa(A, x)$.

[14] One example of such a function κ is $\kappa(A, x) = x(\lfloor A \rfloor - A)$. A second example is $\kappa(A, x) = x(1/(A - \lceil A \rceil) - 1)$ if $\lceil A \rceil < A < \lfloor A \rfloor$, 0 otherwise.

Let $MRS_x(A,m)$ denote the marginal rate of substitution given the sharing cost parameter x. Given $x > 0$, for every commodity bundle (A,m),

$$MRS_x(A,m) = \phi'(A) - \kappa'_A(A,x) > \phi'(A) = MRS_0(A,m). \tag{2.2}$$

For any fraction $A \in (\lceil A \rceil, \lfloor A \rfloor)$, indifference curves are decreasing and convex in A and $MRS_x(A,m)$ increases in x. In addition, $MRS_x(A,m)$ is continuous at any non-integer but is continuous only from the left at any integer.

Since the laissez-faire outcome (LFO) is not the focus of the paper, we formulate the model so that each consumer demands 1 unit of the indivisible good in the LFO. We assume that the measure of suppliers is greater than the measure of consumers, the supply of the indivisible good is perfectly elastic at $p = 1$, and $G^{-1}(0) \geq p = 1$. Since m is a composite commodity, its price is normalized to 1. Thus, the price ratio is 1. We assume that $\phi'(1) = 1$ so that $MRS_x(1,m) \geq \phi'(1) = 1$. Thus, if $x = 0$ (that is, the indivisible good is made divisible at no cost), then, independent of α, each consumer demands 1 unit of the indivisible good. Since 1 unit is preferred to any non-negative number when $x = 0$, then 1 unit is preferred to any other integer when sharing is not an option. If $x > 0$, then, since the cost of sharing is zero at an integer but positive everywhere else, each consumer still demands 1 unit at the price 1. Thus, in the LFO, $p = 1$, $\alpha(I) = 1\ \forall I \in [\underline{I}, \overline{I}]$, $\int \alpha(I)\, G'(I)\, dI = 1$ and the utility of individual I is given by $\phi(1) + I - 1 - \pi(1)$.

In the presence of sharing, the individual I who consumes A units spends $I - A$ on the composite good since the equilibrium money price of the indivisible unit is $p = 1$. Given A, α, and I, individual I's utility equals

$$\phi(A) - \kappa(A,x) + I - A - \pi\left(\int \alpha(I)\, dI\right) \tag{2.3}$$

in the presence of sharing.

Suppose that we use the market scheme $< g, h >$ in which g permits are distributed to each individual and the permit price of a unit of the indivisible good is h. In the presence of sharing and the market scheme $< g, h >$, each individual's utility-maximizing choice over $A \in [0,1]$ depend on the cost of sharing x and the permit price ρ. Given the permit parameters g, h, the permit price ρ, the allocation function α, the income I, and a choice $A \in [0,1]$ of the indivisible good, individual I's utility equals

$$\phi(A) - \kappa(A,x) + I + g\rho - (1+h\rho)A - \pi\left(\int \alpha\left(I'\right) dI'\right)$$

Let $AVG(A,x)$ denote the average net consumption utility and let $MAR(A,x)$ denote the marginal net consumption utility so that

$$AVG(A,x) = \frac{\phi(A) - \kappa(A,x)}{A} \tag{2.4}$$

and

$$MAR(A,x) = \phi'(A) - \kappa'_A(A,x) \tag{2.5}$$

Since utility over $A \in (0,1]$ is

$$\phi(A) - \kappa(A,x) + I + g\rho - (1+h\rho)A - \pi\left(\int \alpha(I)\,dI\right)$$

and utility at $A = 0$ is

$$I + g\rho - \pi\left(\int \alpha(I)\,dI\right)$$

we obtain the following. If $MAR(A,x) = 1 + h\rho$, then consumers prefer A to any alternative in $(0,1]$. If $AVG(A,x) \geq 1 + h\rho$, then consumers prefer A to 0 (strict inequality implies strict preference). Income then determines the constrained utility-maximing choice as follows. If $A \in (0,1]$ and

$$\begin{aligned} MAR(A,x) &= 1 + h\rho \\ AVG(A,x) &> 1 + h\rho \end{aligned} \tag{2.6}$$

then, while all individuals strictly prefer A to anything else in $[0,1]$, the budget-constrained, utility-maximizing choice of individual I (denoted by $C(I,\rho)$) is

$$C(I,\rho)) = \begin{array}{lll} A & \text{if} & I + g\rho \geq (1+h\rho)A \\ \frac{I+g\rho}{1+h\rho} & \text{if} & AVG\left(\frac{I+g\rho}{1+h\rho}, x\right) \geq 1 + h\rho \text{ and } I + g\rho < (1+h\rho)A \\ 0 & \text{if} & AVG\left(\frac{I+g\rho}{1+h\rho}, x\right) < 1 + h\rho \text{ and } I + g\rho < (1+h\rho)A \end{array} \tag{2.7}$$

If an equality replaces the strict inequality in (2.6) then consumers are indifferent between A and 0 and prefer either to any alternative in $[0,1]$. In this case, $A = Z_0(x)$ in (2.7) where Z_0 is defined implicitly as a function of x by[15]

$$AVG(Z_0, x) - MAR(Z_0, x) = 0 \tag{2.8}$$

(so that $Z_0(x)$ equates the average net consumption utility to the marginal net consumption utility and therefore the average net consumption utility is maximized at $Z_0(x)$ over all $A > 0$ and the Implicit Function Theorem implies that Z_0 increases in x). If, in (2.6), 1 replaces A and the equality is replaced by a weak inequality then consumers prefer 1 to any alternative in $[0,1]$ and $A = 1$ in (2.7). Thus, for any given ρ and distribution of income, there are 11 feasible aggregate demand forms. In aggregate demand form $[A]$ each individual demands a common fraction that is less than 1; in form [1], each individual demands 1 unit of the indivisible good; in form $[A0]$, some individuals demand a common fraction and the rest demand 0; in form [10], some individuals demand 1 and the rest demand 0; in form $[AE0]$, some individuals demand a common fraction, some spend all income on the indivisible good, some demand 0; in form $[1E0]$, some individuals demand 1, some spend all income, some demand 0; in form $[E0]$, some individuals spend all their income and some spend 0; in form $[1E]$, some buy 1 and the remainder spend all their income; in form $[AE]$, some demand a common fraction, the rest spend all income; in form $[E]$, all individuals spend all their income on the indivisible good; in form [0], all demand 0.

In equilibrium however, the demand of permits must equal its supply so that aggregate demand must equal g/h. In the presence of sharing, since all can buy $g/h < 1$ (by definition of the permit system), the set, Δ, of feasible equilibrium aggregate demand forms reduce to the following set Δ of 6 demand forms:

$$\Delta = \{[A], [A0], [10], [AE0], [1E0], [E0]\} \tag{2.9}$$

[15] For $x = 0$, (2.8) is $\phi(Z_0)/Z_0 - \phi'(Z_0) = 0$. By L'Hopital's rule, $\lim_{Z_0 \downarrow 0} \phi(Z_0)/Z_0 = \phi'(0)$. Since $\phi(0) = 0$, ϕ is concave, $\phi(z) > \phi'(z)z$ for $z > 0$, $Z_0(0) = 0$.

In demand form $[A]$, all buy g/h. For each other demand form in Δ, there exists a group that buys 0 and there exists one or more groups that buy more than g/h.[16] The groups are described as follows. In $[A0]$, each in a group of size $(g/h)/Z_0(x)$ buys the fraction $Z_0(x) \in [g/h, 1]$; in [10], each in a group of size g/h buys 1 and the rest buy 0; in $[AE0]$ (there exists an income range for each of the three budget-constrained utility maximizing choices available in (2.7)), there exists $\widehat{I}_a$ and $\widehat{I}_b$ for which each $I \geq \widehat{I}_b$ buys the fraction $\left(\widehat{I}_b + g\rho\right)/(1+h\rho) > g/h$, each $I \in \left(\widehat{I}_a, \widehat{I}_b\right)$ spends all on the indivisible good, and each $I < \widehat{I}_a$ buys 0; in $[1E0]$ there exists $\widehat{I}_a$ and $\widehat{I}_b$ as in $[AE0]$ for which $\left(\widehat{I}_b + g\rho\right)/(1+h\rho) = 1$; in $[E0]$, there exists $\widehat{I}_a$ for which $I > \widehat{I}_a$ spends all on the indivisible good and the rest buy 0. Which of these cases prevails in equilibrium depends on the value of x as well as the distribution of income. In the first three cases listed in (2.9), either all individuals obtain a common fraction g/h which is at least preferred to 0, or all individuals are indifferent between obtaining 0 and a common positive amount that is greater than g/h. When equilibrium demand takes one of the first three forms we deem it a natural equilibrium. The natural equilibrium permit price makes individuals indifferent among the natural equilibrium allocations of the indivisible good. Whether a natural equilibrium exists depends not only on x but also on the distribution of income as indicated by (2.7). When the equilibrium price is not the natural price, then, in equilibrium, there may be more than two groups, the allocation need not be homogenous within each group and individuals are not indifferent among the equilibrium allocations of the indivisible good. Instead, there is a most preferred option in $[0,1]$ that all desire and that most preferred option is at least as much or more than the natural positive allocation of the indivisible good but not all have enough income to buy this single most preferred option.

[16] By assumption $I > 1 > g/h$ so that, given a permit price ρ those who spend all their income buy $(I + g\rho)/(1+h\rho) > g/h$ units.

3. Equilibrium Outcome given $< g, h >$

We find the market equilibrium allocation of the indivisible good when the market system is $< g, h >$ under two different assumptions: no sharing and sharing (with cost parameter $x \geq 0$).

As indicated above and verified in Theorems 3.1, 3.3, and 3.8, there may be one or more favoured groups and a group of outsiders in equilibrium. Theorems 3.2, 3.4, 3.12 and their Corollaries, as well as Proposition 3.9 determine the conditions for which the equilibrium outcome Pareto dominates the LFO. Whether Pareto domination succeeds or not, the change in utility is never lower among the favoured group than among the outsiders relative to the LFO.

As we show below, the distribution of income affects the distribution of the benefits of policy $< g, h >$ across the population. The policy benefits are distributed equally across all individuals when the marginal income is high enough. When the marginal income is too low, the policy benefits are weighted toward those with higher income.

We begin with the case in which sharing is not an option.

3.1. No sharing

In the case that sharing is not an option, an individual may either buy an integral number of units at the per unit price 1 or buy 0 units so that $\kappa(A, x)$ disappears from (2.3). Suppose that the desired aggregate pollution level is $g/h \in (0, 1)$ and that this level is attained through the market policy $< g, h >$.

In order to state Theorem 3.1, we let I_f denote the bottom income level of the highest-income group of size $f \in [0, 1]$ so that $I_f = G^{-1}(1 - f)$. Unless otherwise stated, all proofs omitted from the text are in the Appendix.

Theorem 3.1. *Suppose that the market policy is $< g, h >$ where $g/h \in (0, 1)$. If there is no sharing, then the equilibrium per unit price of the indivisible good* *is*

$$\rho^*_{ns} = \begin{cases} \frac{\phi(1)-1}{h} & \textit{if} \quad I_{g/h} - 1 \geq \left(1 - \frac{g}{h}\right)(\phi(1) - 1) \\ \frac{I_{g/h}-1}{h-g} & \textit{if} \quad I_{g/h} - 1 \leq \left(1 - \frac{g}{h}\right)(\phi(1) - 1) \end{cases} \tag{3.1}$$

The equilibrium allocation $\alpha^*_{ns} : \left[\underline{I}, \overline{I}\right] \to \{0, 1\}$ *assigns* 1 *to individual* $I \geq I_{g/h}$ *and* 0 *to all else.*

Theorem 3.1 is intuitive.[17] Since the good is indivisible and $g/h \in (0, 1)$, the equilibrium allocation must assign 1 to a group of size g/h and 0 to all others. Thus, under policy $\langle g, h\rangle$, the fraction g/h determines the size of the group whose members receive 1 unit. If the top inequality in 3.1 holds, then there is at least one group of size g/h with income large enough to purchase 1 unit at the natural permit price. This price makes each individual indifferent between 1 and 0 units. If the second inequality in 3.1 holds then there is no group of size g/h with income large enough to purchase 1 unit at the natural permit price. As a result, the marginal income, $I_{g/h}$, determines the price. At this unatural permit price, indivisibility compels those with less than the marginal income to consume 0.

We next provide conditions under which the equilibrium outcome of policy $< g, h >$ Pareto dominates the laissez-faire outcome (LFO). Recall from the Introduction that in a distinguished allocation, if A units of the good are allocated to individual I, then $I - A$ of income is allocated to individual I. The equilibrium outcome of policy $< g, h >$ allocates 1 unit of the indivisible good and $I + gp^*_{ns} - (1 + hp^*_{ns})$ in income to individual $I > I_{g/h}$ and allocates 0 units of the indivisible good and $I + gp^*_{ns}$ in income to each individual $I < I_{g/h}$ where $g\rho^*_{ns}$ is the equilibrium value of the endowment.

Theorem 3.2. *In the absence of sharing, the equilibrium outcome of policy* $\langle g, h\rangle$ *Pareto dominates the LFO iff its associated distinguished allocation khs-dominates the LFO iff*

$$I_{g/h} - 1 \geq \frac{(1 - g/h)}{g/h}\left[\phi(1) - 1 - \left(\pi(1) - \pi\left(\frac{g}{h}\right)\right)\right] \tag{3.2}$$

[17] In the case that the types differ w.r.t. income I and tastes say, $\alpha\phi$ where α varies, the equilibrium price may be defined in three pieces rather than the two shown here.

and

$$\pi\left(1\right)-\pi\left(\frac{g}{h}\right)\geq\left(1-\frac{g}{h}\right)\left(\phi\left(1\right)-1\right) \tag{3.3}$$

Thus, if the absence of sharing, then, by Theorems 3.1 and 3.2, the equilibrium outcome of policy $\langle g,h\rangle$ Pareto dominates the LFO iff it is possible for any self-financing market policy that allocates a measure of g/h units of the indivisible good across the population to Pareto dominate the LFO using ownership based transfers. The results are intuitive. Inequality (3.3) ensures that the aggregate gain from decreased pollution outweighs the aggregate loss in consumption of the indivisible good. Inequality (3.2) ensures that the marginal income is large enough to finance the minimum ownerhip-based transfer required for *khs*-domination. By Theorem 3.2, the natural equilibrium Pareto dominates the LFO iff the aggregate gain from decreased pollution outweighs the aggregate loss from decreased consumption of the invisible good. The invisible hand ensures that whenever there exists a feasible ownership-based transfer that results in Pareto domination (that is, *khs*-domination holds), the market policy outcome will in fact Pareto dominates the LFO. In the absence of *khs*-domination, Pareto domination of the LFO by the market policy fails.

When the income level of the marginal consumer satisfies the top inequality of (3.1) resulting in the natural permit price, all individuals benefit equally from the permit policy with the uniform endowment of permits. When the income level of the marginal consumer does not satisfy (3.1), the resulting equilibrium permit price is less than the natural permit price, thereby increasing the benfits of the insiders, at the expense of the outsiders. While the unatural equilbirium may stil Pareto dominate the LFO, the policy is regressive.

Theorem 3.2 argues in favour of a market permit policy in the presence of externalities generated by indivisible goods. Whenever there is a feasible lump sum transfer from the outsiders to the insiders that leaves everyone better off relative to the LFO given a distinguished allocation of the indivisible good, the market permit policy associated with the distinguished allocation will generate a Pareto improvement over the LFO. As we noted in [5], *khs*-domination fails at the chosen pollution level in [12] and in [23], so that no market policy could have achieved Pareto domination given the specific distribution of income used in these papers.[18]

We note that an adaption of the probabilistic mechanism used in [3] would allocate 1 car to a person with probability g/h so that in aggregate the expected sum of cars allocated is g/h and there are no transfers among individuals. In this case, ex ante, all are better off iff (3.3) but ex post, all are better off iff

$$\pi(1) - \pi\left(\frac{g}{h}\right) \geq \phi(1) - 1$$

Thus, if all are better off ex post under the probabilistic mechanism, then all are better off under the permit scheme. In contrast, under the above permit scheme, all are better off ex ante and ex post iff inequalities (3.2) and (3.3) hold.

3.1.1. Pareto Improving Non-uniform Distribution of Permits in the absence of sharing

In the previous sections we considered only policy $\langle g, h\rangle$ in which g permits are uniformly distributed among the individuals and h permits are required for the purchase of 1 unit of the indivisible good. Given that the uniform distribution of permits may not result in a Pareto improvement over the LFO, in this section we consider alternatives to the uniform distribution of permits across the population.

For example, $g/h > 1/2$ and there is no sharing, it may be possible to show that, under some further conditions, there exists a pareto improving distribution of permits . Suppose that the natural equilibrium price does not exist so that $I_{g/h} - 1 < (1 - g/h)(\phi(1) - 1)$. Now suppose that each in a group of size g/h receives h permits and the rest receive 0 permits. If it is the bottom g/h of income types who receive the permits, then, we could get the natural equilibrium price provided $I_{1-g/h} \geq 1 + h\rho$ where $\rho = (\phi(1) - 1)/h$ is the natural price so that $I_{1-g/h} - 1 \geq (\phi(1) - 1)$.

[18] In these two papers $\phi(1) - 1 < \pi(1) - \pi(0)$.

Thus, if $I_{g/h}-1 < (1-g/h)(\phi(1)-1) < (\phi(1)-1) < I_{1-g/h}-1$ then, there exists a non-uniform distribution of permits that results in the natural permit price and each is indifferent between 0 and 1. The poor person who has h permits compares $\phi(1)+I+hp-(1+hp) = \phi(1)+I-1$ to $\phi(0)+I+hp = I+h(\phi(1)-1)/h = \phi(1)+I-1$ and so is indifferent between 0 and 1. The rich person who has 0 permits compares $\phi(1) + I - (1 + hp) = \phi(1) + I - [1 + h(\phi(1) - 1)/h] = I$ to $\phi(0) + I = I$ so is indifferent between 0 and 1.

In the above case, the poor are definitely better off than at the LFO since their utility equals $\phi(1) + I - 1 - \pi(g/h) > \phi(1) + I - 1 - \pi(1)$. The rich are better off provided $I - \pi(g/h) > \phi(1) + I - 1 - \pi(1)$. Thus, Pareto domination results if $\pi(1) - \pi(g/h) > \phi(1) - 1$. Under the uniform distribution Pareto domination results provide inequalities (3.3) and (3.2) hold.

Thus, suppose that

$$\pi (1) - \pi \left(\frac{g}{h}\right) \geq (\phi (1) - 1)$$

so that (3.3) holds but in this case, so does (3.2) hold since the failure of (3.2) implies

$$I_{g/h} - 1 < \frac{(1 - g/h)}{g/h} \left[\phi (1) - 1 - \left(\pi (1) - \pi \left(\frac{g}{h}\right)\right)\right] < 0$$

which is never true since $0 < I_{g/h} - 1$. Thus, if this particular non-uniform distribution of permits Pareto dominates the LFO, then, so does the uniform distribution of permits.

Recall the statement "Thus, if the absence of sharing, then, by Theorems 3.1 and 3.2, the equilibrium outcome of policy $\langle g, h\rangle$ Pareto dominates the LFO iff it is possible for any self-financing market policy that allocates a measure of g/h units of the indivisible good across the population to Pareto dominate the LFO using ownership based transfers" given previously. Whenever it is possible for some self-financing market policy to Pareto dominate the LFO, then the uniform distribution of permits results in an outcome that Pareto dominates the LFO.

So, perhaps the above cannot be used to improve upon the Pareto results of the uniform distribution but they can clearly be used to change the allocation and the particular welfare outcomes of the individuals.

3.2. Costly Sharing and the natural equilibrium: $x \geq 0$

In the case that there is costly sharing, individuals may buy any fraction of a unit but need not buy a common fraction. We first consider the case in which the form of equilibrium demand is natural (and so is one of the first three forms in (2.9)). The natural equilibrium price varies with x and owes its existence to characteristics of the income distribution G.

Theorem 3.3 states the natural equilibrium price and allocation for any sharing cost parameter $x \geq 0$ and characterizes the set of income distributions for which it exists. We introduce the following notation to state Theorem 3.3 below. We define X_0 implicitly as a function of A by

$$AVG\left(A, x\right) - MAR\left(A, x\right) = 0 \tag{3.4}$$

so that X_0 is the inverse of Z_0 and X_0 increases in A; we define x_1 to be the solution in x to

$$\phi\left(1\right) = \phi'\left(1\right) - \kappa'_A\left(1, x\right) \tag{3.5}$$

(so that $Z_0\left(x_1\right) = 1$ or $X_0\left(1\right) = x_1$ since $\kappa\left(1, x\right) = 0$). When g/h is fixed, we may denote $X_0\left(g/h\right)$ by x_0. We note that if $0 \leq x < X_0\left(g/h\right)$ then $Z_0\left(x\right) < Z_0\left(X_0\left(g/h\right)\right) = g/h < 1$ and $AVG\left(g/h, x\right) > MAR\left(g/h, x\right)$; if $X_0\left(g/h\right) < x < x_1$ then $g/h = Z_0\left(X_0\left(g/h\right)\right) < Z_0\left(x\right) < Z_0\left(x_1\right) = 1$; if $x_1 < x$, $g/h < 1 = Z_0\left(x_1\right) < Z_0\left(x\right)$ and $MAR\left(1, x\right) > AVG\left(1, x\right)$. As seen in Theorem 3.3, the roles of $X_0\left(g/h\right)$ and x_1 are boundaries regarding the relationship between the sharing cost parameter and the form of aggregate demand in equilibrium when the natural equilibrium exists. Since the form of equilibrium aggregate demand varies, the form of equilibrium price varies as well. We define the following feasible natural equilibrium permit prices. The price $\rho_m\left(x\right)$ is the solution in ρ to

$$MAR\left(\frac{g}{h}, x\right) = 1 + h\rho, \tag{3.6}$$

$\rho_n(x)$ is the solution in ρ to

$$AVG(Z_0(x), x) = MAR(Z_0(x), x) = 1 + h\rho, \tag{3.7}$$

and $\rho_\eta(x)$ is the solution in ρ to

$$AVG(1, x) = 1 + h\rho_\eta \tag{3.8}$$

Theorem 3.3 gives the conditions under which the income distribution admits a natural equilibrium.

Theorem 3.3. *Suppose that the market policy is* $< g, h >$ *and that* $g/h \in (0, 1)$*. If there is sharing and* $x \geq 0$*, then the equilibrium price of one unit of the indivisible good is*

$$\rho^*(x, G) = \begin{cases} \rho_m(x) & \text{iff} \quad 0 \leq x \leq X_0(g/h) \\ \rho_n(x) & \text{iff} \quad X_0(g/h) \leq x \leq x_1 \text{ and } I_{g/hZ_0(x)} \geq Z_0(x) + \rho_n(x)(hZ_0(x) - g) \\ \rho_\eta(x) & \text{iff} \quad x_1 \leq x \text{ and } I_{g/h} \geq 1 + \rho_\eta(x)(h - g) \end{cases} \tag{3.9}$$

Under the first parameter range in (3.9), the equilibrium demand form is $[A]$*, all individuals consume* g/h *and* $\rho^*(x, G)$ *increases in* x*; under the second, the equilibrium form is* $[A0]$*,* $I \geq I_{g/hZ_0(x)}$ *consumes* $Z_0(x)$*,* $I \leq I_{g/hZ_0(x)}$ *consumes* 0*, and* $\rho^*(x, G)$ *decreases in* x*; under the third, the equilibrium form is* $[10]$*,* $I \geq I_{g/h}$ *consumes* 1*,* $I \leq I_{g/h}$ *consumes* 0*, and* $\rho^*(x, G)$ *is constant in* x *and equals* $(\phi(1) - 1)/h$*, the first alternative for* ρ^*_{ns} *in (3.1).*

Theorem 3.3 is intuitive. If x is small enough, the good is essentially divisible so that g/h units may be allocated homogeneously across all individuals. Since individuals are always able to afford the fraction g/h by definition of the permit scheme and since g/h is preferred to 0 at the permit price $\rho_m(x)$ when x is small enough, the equilibrium price is that which makes all individuals desire g/h and is independent of the income distribution. An increase in the sharing cost, x, increases the marginal rate of substitution at g/h, $MAR(g/h, x)$, since an increase in the cost of sharing increases the value of buying a larger fraction (which decreases the sharing). Thus, as x increases in this range, the equilibrium price must increase so as to maintain (3.6) and thereby maintain g/h as the optimal choice for each individual. At $x = X_0(g/h)$ and $\rho = \rho_m(X_0(g/h))$, individuals are indifferent between 0 and the fraction $g/h = Z_0(X_0(x))$ at the price $\rho_m(x)$. For any higher x, individuals strictly prefer 0 to g/h at the price $\rho_m(x)$ so that the equilibrium price is no longer $\rho_m(x)$ and the allocation is no longer homogeneous. The equilibrium allocation switches to one in which each individual in a favoured group of size $g/hZ_0(x)$ receives $Z_0(x) \in (g/h, 1)$, each outsider receives 0, and each individual is indifferent between $Z_0(x)$ and 0. As x increases in this mid-range, the average consumption utility decreases for every given fraction since the cost of sharing has gone up so that the maximum average consumption value, $AVG(Z_0(x), x)$, decreases. Thus, as x increases in this range, the equilibrium price must decrease so as to maintain (3.7) and thereby keep the optimal positive choice at $Z_0(x)$, the argument that maximizes the average. As x increases $Z_0(x)$ increases and eventually $Z_0(x) = 1$ at $x = x_1$. Past $x = x_1$, there is no longer any effect of an increase in x on the allocation or price since no one is sharing any longer. We note, for each sharing cost range in (3.9), the natural equilibrium permit price is constant across all income distribution that satisfy the necessary and sufficient conditions in the range.

We are now ready to consider the conditions under which the natural equilibrium outcome Pareto dominates the LFO. We first note that though the equilibrium price $\rho_m(x)$ increases in x when $x \leq X_0(g/h)$, the value of the endowment $g\rho_m(x)$ also increases, and since $h\rho_m(x)(g/h) = g\rho_m(x)$ the equilibrium value of permits required to buy g/h units exactly equals the value of the endowment. Thus, for fixed g/h, individual utility in equilibrium is independent of the permit price and varies with $x \leq X_0(g/h)$ only directly through the cost of sharing function. We also note that since $X_0(g/h)$ increases in g/h, the set of x for which all individuals share in equilibrium increases in g/h. For $x > X_0(g/h)$, each individual is indifferent between some postive quantity and 0 and prefer either to any alternative. Since utility is quasi-linear, and all are indifferent between the two equilibrium allocations of the indivisible good, all are better off iff

$$g\rho^* \geq \phi(1) - 1 - \left(\pi(1) - \pi\left(\frac{g}{h}\right)\right)$$

the equilibrium value of the endowment plus the gain from decreased pollution compensates for the loss from decreased consumption.

By Theorem 3.1 in the presence of sharing and the existence of the natural equilibrium, the natural equilibrium value of the permit endowment, $g\rho^*(x, G)$, equals

$$\begin{cases} \frac{g}{h}\left(MAR\left(\frac{g}{h}, x\right) - 1\right) & \text{iff} \quad 0 \leq x \leq X_0(g/h) \\ \frac{g}{h}\left(MAR(Z_0(x), x) - 1\right) = \frac{g}{h}\left(AVG(Z_0(x), x) - 1\right) & \text{iff} \quad X_0(g/h) \leq x \leq x_1 \\ \frac{g}{h}\left(\phi(1) - 1\right) & \text{iff} \quad x_1 \leq x \end{cases} \tag{3.10}$$

since $AVG(1, x) = \phi(1)$.

In the range, $0 < x < X_0(g/h)$, all consumers consume g/h so that consumers are worse off as x increases since consumption remains constant but the cost of sharing increases. In the range, $X_0(g/h) < x < x_1$, individuals are indifferent between consuming either $Z_0(x)$ and 0 in the natural equilibrium but, by (3.10), the value of the permit endowment (that is, the utility (net of original income) of consuming 0) decreases since the equilibrium price decreases. Thus, in the natural equilibrium, consumers are worse off as x increases whenever $X_0(x) < x < x_1$. In the range $x_1 \leq x$, consumers are indifferent between 1 and 0 in the natural equilibrium but the value of the permit endowment remains constant as x increases so that utility of consumers remains constant since no sharing takes place.

Theorem 3.4. *In the presence of sharing with $x \geq 0$ when the inequality constraints in (3.9) are satisfied, the natural equilibrium outcome of policy $\langle g, h\rangle$ Pareto dominates the LFO iff its associated distinguished allocation khs-dominates the LFO iff **either** $x < X_0(g/h)$ and*

$$\pi(1) - \pi\left(\frac{g}{h}\right) \geq \phi(1) - 1 - \left(\phi\left(\frac{g}{h}\right) - \frac{g}{h} - \kappa\left(\frac{g}{h}, x\right)\right) \tag{3.11}$$

or *$X_0\left(g/h\right) < x < x_1$,*

$$I_{\frac{g}{hZ_0(x)}} \geq \frac{\left(1-\frac{g}{hZ_0(x)}\right)}{\frac{g}{hZ_0(x)}}\left[\phi\left(1\right)-1-\left(\pi\left(1\right)-\pi\left(\frac{g}{h}\right)\right)\right]+Z_0\left(x\right). \tag{3.12}$$

and

$$\pi\left(1\right)-\pi\left(\frac{g}{h}\right) \geq \phi\left(1\right)-1-\left(\frac{g}{hZ_0\left(x\right)}\right)\left(\phi\left(Z_0\left(x\right)\right)-\kappa\left(Z_0\left(x\right),x\right)-Z_0\left(x\right)\right) \tag{3.13}$$

or *$x_1 < x$, and both (3.2) and (3.3) hold.*

Theorem 3.4 extends Theorem 3.2 in the presence of sharing in the case that the natural equilibrium exists. The results are intuitive. When the cost of sharing is small enough, no money is transferred in equilibrium, so that the equilibrium outcome is identical to its associated distinguished allocation and all are better off iff the gain from reduced pollution is larger than the loss from decreased consumption. When the cost of sharing is in the middle range, all are better off iff the marginal income is large enough to enable the required level of compensation and the gain from decreased pollution outweighs the loss from the decrease in consumption. Finally, when the cost of sharing is high enough, the natural equilibrium is identical to that when sharing is not an option so that the results are identical to those of Theorem 3.2.

In the interim range of sharing costs, inequalities (3.13) and (3.12) are the analogues of inequalities (3.3) and (3.2) in Theorem 3.2. In the case of sharing in the interim range, in both the policy $\langle g,h\rangle$ equilibrium and its distinguished allocation, each favoured group member receives $Z_0\left(x\right)$ units of the indivisible good and each outsider receives 0. The intuition is analogous to that given under Theorem 3.2. As in Theorem 3.2, the invisible hand is at work to ensure that whenever there is a feasible ownership-based transfer in the distinguished allocation that results in Pareto domination, the market permit outcome will Pareto dominates the LFO.

3.3. Costly Sharing and the income dependent equilibrium: $x \geq X_0\,(g/h)$

In the case that $0 \leq x \leq X_0\,(g/h)$, we know by Theorem 3.3 that the natural equilibrium price is $\rho_m\,(x)$ independent of the distribution of income but that if $x > X_0\,(g/h)$, the existence of the natural equilibrium price depends on the income distribution. However, given $x > X_0\,(g/h)$, the natural equilibrium price is constant across all income distributions G for which the pair (x, G) satisfies the associated condition in (3.9). In this case, two groups (insiders and outsiders) are distinguished in the natural equilibrium. Though the allocation of the indivisible good differs across the two groups, the allocation is constant among all individuals in a given group and each individual is indifferent between the two natural allocations ($Z_0\,(x)$ or 1 and 0) at the natural equilibrium price (ρ_n or ρ_m).

3.3.1. Demand systems corresponding to demand forms

In order to derive the equilibrium price when $X_0\,(g/h) < x$ and the pair (x, G) violates (3.9) we first use (2.7) and subsequent discussion to determine the set of aggregate equilibrium demand systems $\{A, A0, 10, AE0, 1E0, E0\}$ that correspond to the last 5 feasible aggregate equilibrium demand forms in Δ. given in (2.9). The third line in each system below corresponds to equality between permit demand and supply.

We note that the equilibrium systems associated with equilibrium demand forms [$A0$] and [10] are given implicitly by Theorem 3.3. Specfically, System $A0$ is

$$\begin{array}{l} MAR\,(Z_0\,(x)\,, x) = 1 + h\rho \\ AVG\,(Z_0\,(x)\,, x) = 1 + h\rho \\ \int_{I_{(g/h)/Z_0(x)}}^{\overline{I}} Z_0\,(x)\,G'\,(I)\,dI = \frac{g}{h} \\ \frac{g}{h} \leq Z_0\,(x) \leq \min\left\{1, \frac{I_{g/hZ_0(x)} + g\rho}{1+h\rho}\right\} \end{array} \tag{3.14}$$

and System 10 is identical to system $A0$ except $Z_0\,(x) = 1$ in all four lines of

(3.14) and the first equality becomes weakly greater. System $AE0$ is

$$\begin{array}{l} MAR\left(\frac{\widehat{I}_b+g\rho}{1+h\rho},x\right)=1+h\rho \quad (1)\\ AVG\left(\frac{\widehat{I}_a+g\rho}{1+h\rho},x\right)=1+h\rho \quad (2)\\ \int_{\widehat{I}_a}^{\widehat{I}_b}\left(\frac{I+g\rho}{1+h\rho}\right)G'(I)\,dI+\left(\frac{\widehat{I}_b+g\rho}{1+h\rho}\right)\left(1-G\left(\widehat{I}_b\right)\right)=\frac{g}{h} \quad (3)\\ \underline{I}\leq\widehat{I}_a\leq I_{(g/h)/\left(\left(\widehat{I}_b+g\rho\right)/(1+h\rho)\right)}\leq\widehat{I}_b\leq\overline{I} \quad (4)\\ \frac{\widehat{I}_a+g\rho}{1+h\rho}\leq Z_0(x)\leq\frac{\widehat{I}_b+g\rho}{1+h\rho}\leq 1 \quad (5)\\ 0\leq\rho\leq\rho_n \quad (6) \end{array} \tag{3.15}$$

so that individual $I<\widehat{I}_a$ buys 0, $\widehat{I}_a<I<\widehat{I}_b$ spends all, $I>\widehat{I}_b$ buys $\left(\widehat{I}_b+g\rho)/1+h\rho\right)$. System $1E0$ is identical to system $AE0$ except that $\left(\widehat{I}_b+g\rho\right)/(1+h\rho)$ is replaced by 1 in lines (1), (3), and (4), the equality in line (1) becomes weakly greater, and line (5) becomes

$$\frac{\widehat{I}_a+g\rho}{1+h\rho}\leq\min\left\{Z_0(x),\frac{\widehat{I}_b+g\rho}{1+h\rho}=1\right\} \tag{3.16}$$

so that individual $I<\widehat{I}_a$ buys 0, $\widehat{I}_a<I<\widehat{I}_b$ spends all, $I>\widehat{I}_b$ buys 1. System $E0$ is identical to system $AE0$ except that $\left(\widehat{I}_b+g\rho\right)/(1+h\rho)$ is replaced by $\left(\overline{I}+g\rho\right)/(1+h\rho)$ in lines (1), (3), and (4), $\widehat{I}_b$ is replaced by $\overline{I}$ in line (3), the equality in line (1) becomes weakly greater, and line (5) becomes

$$\left(\widehat{I}_a+g\rho\right)/(1+h\rho)\leq\min\{Z_0(x),1\}\ \text{ and }\ \left(\overline{I}+g\rho\right)/(1+h\rho)\leq 1 \tag{3.17}$$

so that individual $I<\widehat{I}_a$ buys 0, $I>\widehat{I}_a$ spends all.

3.3.2. The nature of equilibrium when the conditions in (3.9) are violated

While aggregate demand varies with the permit price ρ, the structure of aggregate demand depends on ϕ, κ, x and G. Since the pair (x,G) violates (3.9), individuals are no longer indifferent between two "natural" options (0 and either $Z_0(x)$ or 1) in equilibrium. Since demand is smaller than supply at the natural price the equilibrium price must lie below the natural price. If the permit price is smaller than the natural price, there exists a single most preferred option (which is positive and greater than or equal to the natural positive option) among options in $[0, 1]$ but the income distribution imposes limitations on who can buy the most preferred option.

Proposition 3.5. *If $x > x_1$, then the feasible equilibrium systems of policy $< g, h >$ are 10, $E0$, or $1E0$. If $X_0\,(g/h) < x < x_1$, then the feasible equilibrium systems are $A0$, $AE0$, $E0$, or $1E0$.*

In order to understand the effect of variable distributions on the equilibrium demand form and price, suppose that there is a uniformly continuous family of distributions $\{G_t\}_{t\in[0,\tau]}$ (if necessary, $t \in [0, \infty)$) for which G_s stochastically dominates G_t whenever $s < t$. Since the left-hand side of line (3) in systems $AE0$, $1E0$, and $E0$ decreases in t, and since the systems $AE0$, $1E0$, and $E0$ do not intersect over any open interval of distributions G_t for $t \in [0, \tau]$, we make the following observations in Proposition 3.6.

Proposition 3.6. *Let $\{G_t\}_{t\in[0,\tau]}$ be a uniformly continuous family of feasible income distributions for which G_s stochastically dominates G_t whenever $s < t$. If $x > X_0\,(g/h)$, then, for fixed x, as we pass from the region of (x, t) for which (x, G_t) satisfies (3.9) to the region of (x, t) for which (x, G_t) violates (3.9), the natural equilibrium may morph from $A0$ to $AE0$ or from 10 to $1E0$. For all $(x, t\ )$ for which $x > X_0\,(g/h)$ and the pair (x, G) violates (3.9), fix x and let t increase. The equilibrium of policy $< g, h >$ solves system $AE0$, $1E0$, or $E0$. As t increases, the equilibrium price $\rho^*\,(x, G_t)$ decreases and the equilibrium systems may morph from system $AE0$ to $1E0$ or from system $1E0$ to $E0$.*

The equilibrium system and price varies continuously among the finite set of feasible alternatives if the distribution of income remains fixed and x varies continuously or if x remains fixed and the distribution of income varies continuously.

Given x and given that the pair (x, G) violates (3.9) for a specific distribution G, there are many ways in which one can construct a uniformly continuous family of distributions[19] $\{G_t\}_{t\in[0,\tau]}$ (denoted by Γ_G) that includes G and has the following properties. If $s < t$, then G^s stochastically dominates G^t; G_0 gives rise to the natural equilibrium; for t large enough, G^t eventually gives rise to an equilibrium in which the equilibrium system is of the form $[E0]$. For example, a family could be constructed as follows. Suppose that the initial distribution is G with support $\left[\underline{I}, \overline{I}\right]$ with $I_{g/hA_0(x)} < A_0(x) + \rho_0(x)(h-g)$ where A_0 denotes the natural positive allocation (which may equal $Z_0(x)$ or 1) and ρ_0 is the natural price (which may equal $\rho_n(x)$ or $\rho_\eta(x)$). If $\overline{I} < A_0(x) + \rho_0(x)(h-g)$ we can first begin the family by constructing distributions of the type

$$G^t(I) = G^s\left(\left(\frac{\overline{I}^s - \underline{I}}{\overline{I}^t - \underline{I}}\right)(I - \underline{I}) + \underline{I}\right)$$

for $s < t$ and $\overline{I}^s > \overline{I}^t$ with $\overline{I}^s \in \left[\overline{I}, \infty\right)$ so that, in this case,

$$\begin{aligned} G^t(I) - G^s(I) &= G^s\left(\left(\frac{\overline{I}^s - \underline{I}}{\overline{I}^t - \underline{I}}\right)(I - \underline{I}) + \underline{I}\right) - G^s(I) \\ &= G^s\left(I + \left(\frac{\overline{I}^s - \overline{I}^t}{\overline{I}^t - \underline{I}}\right)(I - \underline{I})\right) - G^s(I) > 0 \text{ for } I > \underline{I} \end{aligned}$$

and for such a family, $\{G^t\}$, for all $\varepsilon > 0$, there exists $\delta_1, \delta_2, \delta_3 > 0$ such that $t > s$, $\overline{I}^t < \overline{I}^s$, $t - s = |t - s| < \delta_1$ implies $\overline{I}^s - \overline{I}^t = \left|\overline{I}^s - \overline{I}^t\right| < \delta_2$ implies

$$\left|\left(\frac{\overline{I}^s - \underline{I}}{\overline{I}^t - \underline{I}}\right) - 1\right| = \left(\frac{\overline{I}^s - \underline{I}}{\overline{I}^t - \underline{I}}\right) - 1 < \delta_3$$

implies

$$\left|\left(\frac{\overline{I}^s - \underline{I}}{\overline{I}^t - \underline{I}}\right)(I - \underline{I}) + \underline{I} - I\right| = \left(\frac{\overline{I}^s - \underline{I}}{\overline{I}^t - \underline{I}}\right)(I - \underline{I}) + \underline{I} - I < \delta_3$$

[19] If necessary, the family may be $\{G_t\}_{t\in[0,\infty)}$.

implies

$$\left|G^s\left(\left(\frac{\overline{I}^s-\underline{I}}{\overline{I}^t-\underline{I}}\right)(I-\underline{I})+\underline{I}\right)-G^s(I)\right|=G^s\left(\left(\frac{\overline{I}^s-\underline{I}}{\overline{I}^t-\underline{I}}\right)(I-\underline{I})+\underline{I}\right)-G^s(I)<\varepsilon$$

for all $I\in\left[\underline{I},\overline{I}^t\right]$. In terms of finding G_0, we may stop the process as soon as $I^s_{g/hA_0(x)}>A_0(x)+\rho_0(x)(h-g)$ for all x. If this inequality fails to hold, take the limit as $\overline{I}^s$ goes to infinity. If $\lim_{t\uparrow\infty}I^t_{g/hZ_0(x)}<Z_0(x)+\rho_n(x)(h-g)$ then further expand the family by constructing distributions of the type

$$G^t(I)=G^s\left(\left(\frac{\overline{I}-\underline{I}^s}{\overline{I}-\underline{I}^t}\right)\left(I-\overline{I}\right)+\overline{I}\right)$$

for $s<t$ and $\underline{I}^s>\underline{I}^t$, so that, in this case,

$$\begin{aligned}G^t(I)-G^s(I) &= G^s\left(\left(\frac{\overline{I}-\underline{I}^s}{\overline{I}-\underline{I}^t}\right)\left(I-\overline{I}\right)+\overline{I}\right)-G^s(I)\\ &= G^s\left(I+\left(\frac{\underline{I}^s-\underline{I}^t}{\overline{I}-\underline{I}^t}\right)\left(\overline{I}-I\right)\right)-G^s(I)>0\text{ for }\overline{I}>I\end{aligned}$$

In terms of finding G_0, we may stop the process as soon as $I^s_{g/hZ_0(x)}>Z_0(x)+\rho_n(x)(h-g)$ for all x. In particular, in the case that $\underline{I}^s=Z_0(x)+\rho_n(x)(h-g)$ it must be the case $I^s_{g/hZ_0(x)}>Z_0(x)+\rho_n(x)(h-g)$. For such a family $\{G^t\}$, for all $\varepsilon>0$, there exists $\delta_1,\delta_2>0$ such that $t>s$, $\overline{I}^t<\overline{I}^s$, $\overline{I}^s-\overline{I}^t=\left|\overline{I}^s-\overline{I}^t\right|<\delta_1$ implies

$$\left|\left(\frac{\overline{I}^s-\underline{I}}{\overline{I}^t-\underline{I}}\right)-1\right|=\left(\frac{\overline{I}^s-\underline{I}}{\overline{I}^t-\underline{I}}\right)-1<\delta_2$$

implies

$$\left|\left(\frac{\overline{I}^s-\underline{I}}{\overline{I}^t-\underline{I}}\right)(I-\underline{I})+\underline{I}-I\right|=\left(\frac{\overline{I}^s-\underline{I}}{\overline{I}^t-\underline{I}}\right)(I-\underline{I})+\underline{I}-I<\delta_2$$

implies

$$\left|G^s\left(\left(\frac{\overline{I}^s-\underline{I}}{\overline{I}^t-\underline{I}}\right)(I-\underline{I})+\underline{I}\right)-G^s(I)\right|=G^s\left(\left(\frac{\overline{I}^s-\underline{I}}{\overline{I}^t-\underline{I}}\right)(I-\underline{I})+\underline{I}\right)-G^s(I)<\varepsilon$$

for all $I \in \left[\underline{I}, \overline{I}^t\right]$. In terms of finding G_∞, the process will certainly stop as $\overline{I}^s \geq \underline{I}^s$ tend toward 1.

Let S_σ be the set of $x \in [0, \infty)$ for which the equilibrium satisfies system σ with associated demand form $[\sigma] \in \Delta$. Proposition 3.7 characterizes some features of S_σ for $[\sigma] \in \Delta$.

Proposition 3.7. *Let G be a fixed income distribution and let $x \geq 0$. Given policy $< g, h >$, the set S_σ is a union of closed and bounded intervals for any equilibrium system σ not equal to systems $1E0$ or 10. $S_{[A]} = [0, X_0\,(g/h)]$. If S_σ is of positive Lebesque measure for $\sigma \in \{[10]\,, [1E0]\}$, then there exists x^e for which $S_\sigma = [x^e, \infty)$.*

Proposition 3.7 allows for equilibrium systems that have measure zero (as would happen, for example, if S_σ is a point) and also allows for equilibrium systems to be a union of two or more disjoint closed and bounded intervals of positive measure.[20]

By Theorem 3.3, if the pair (x, G) satisfies (3.9), the equilibrium price strictly decreases in x for $X_0\,(x) < x < x_1$ and then remains constant for $x \geq x_1$. We now show that the equilibrium price strictly decreases in x when $x > X_0\,(x)$ and the pair (x, G) violates (3.9).

[20] Suppose that

$$\phi\,(A) - \kappa\,(A, x) = 2\,(A)^{\frac{1}{2}} - x\,(1 - A)$$

and the family is $\{G_t\}_{t \in [0,7]}$ where

$$G^t\,(I) = \begin{cases} \frac{10}{3}\left(\frac{3+(t)}{10}\right)(I-1) & \text{if} \quad 1 \leq I \leq \frac{13}{10} \\ \frac{1}{10}\,(t) + \frac{19}{162}I + \frac{239}{1620} & \text{if} \quad \frac{13}{10} \leq I \leq \left(\frac{488-81(t)}{95}\right) + \frac{\frac{13}{10} - \left(\frac{488-81(t)}{95}\right)}{\left(4\left(\frac{3+(t)}{10}\right) - 3\right)} \end{cases}$$

If $t = 3.95$, then the equilibrium systems morph from A to $A0$ to $AE0$ to $A0$ to 10 and the set $S_{A0} = [0.5, 0.875081476] \cup [0.966966162, 1]$ is a union of two disjoint sets of positive measure. If $t = 6.489801294$, then the equilibrium system morphs from A to $A0$ to $AE0$ to $1E0$ to $E0$ to $1E0$. The sets $S_{AE0} = [0.84974525, 0.980015372]$, $S_{1E0} = [0.980015372, 0.980015372] \cup [0.980015372, \infty)$ have positive measure but the set $S_{E0} = [0.980015372]$ has measure zero.

Theorem 3.8. *If $x > X_0(g/h)$ and the pair (x, G) violates (3.9) in Theorem 3.3, then the equilibrium price $\rho^*(x, G)$ of policy $< g, h >$ decreases in x and satisfies system $AE0$, $1E0$, or $E0$. For $x \geq x_1$, either (x_1, G) satisfies (3.9) in which case $\rho^*(x, G)$ is constant in x or (x_1, G) does not satisfy (3.9) in which case $\rho^*(x, G)$ strictly decreases in x. In either case, $\lim_{x\to\infty} \rho^*(x, G) = \rho^*_{ns}$.*

Theorem 3.8 compares the equilibrium price in the presence of sharing to that in the absence. As $x > X_0(g/h)$ increases, the equilibrium price decreases to the no sharing equilibrium price.[21] In the case that (3.9) is satisfied, the insiders and outsiders in the policy outcome become worse off as x increases since the welfare of each equals income plus the value of the endowment which decreases in the equilibrium price. In the case that (3.9) is violated, there are insiders, outsiders and a middle group. As x increases, the insiders become better off while the outsiders become worse off. Those in the top of the middle group become better off; those in the bottom become worse off. We now derive conditions under which the equilibrium outcome Pareto dominates the LFO.

3.3.3. Pareto Domination

Proposition 3.9. *If $x > X_0(g/h)$, then the equilibrium outcome of policy $<g, h>$ Pareto dominates the LFO iff*

$$g\rho^*(x, G) \geq \phi(1) - 1 - (\pi(1) - \pi(g/h)) \tag{3.18}$$

Corollary 3.10. *The equilibrium outcome of policy $\langle g, h\rangle$ in the presence of sharing Pareto dominates the LFO if inequalities (3.3) and (3.2) hold.*

Corollary 3.11. *Sharing improves the efficiency properties of policy $\langle g, h\rangle$.*

[21] So, for example, let $\phi(A) = 2A^{1/2}$, $\kappa(A, x) = x(1 - A)$, $g = 1$, $h = 2$, and G is uniform on $[1, 1 + 2d]$. If $d = \left(I_{g/h} - 1\right) \geq (1 - g/h)(\phi(1) - 1)) = 0.5$ the equilibrium price decreases to $(\phi(1) - 1)/h = 1/2$ at $x = x_1 = 1$ where it remains. If $d \leq 0.5$ the equilibrium price decreases to $\left(I_{g/h} - 1\right)/(h - g) = d$ as x increases to ∞ past x_1.

Proposition 3.9 provides a necessary and sufficient condition for Pareto domination that we use below. Corollary 3.10 provides a sufficient condition for Pareto domination that is easy to check. Corollary 3.11 highlights the fact that sharing provides a benefit over no sharing.

Theorem 3.4 shows that the Pareto domination of the LFO by the equilibrium policy is equivalent to *khs*-domination of the LFO by its associated designated allocation when the equilibrium is natural. Theorem 3.4 also gives necessary and sufficient conditions for *khs*-domination. Its Corollary **??** gives necessary and sufficient conditions for the natural equilibrium outcome of policy $< g, h >$ to Pareto dominate the LFO. Though Corollary 3.10 provides sufficient conditions, it remains to provide tighter conditions for Pareto domination in the case that $x > X_0\,(g/h)$ and the distribution G does not satisfy (3.9).

As shown in Theorem 3.12 below, the *khs*-domination of the LFO by the distinguished allocation associated with the natural equilibrium implies Pareto domination of the LFO by the equilibrium outcome even when the natural equilibrium does not exist.

Theorem 3.12. *In the presence of sharing with $x \geq 0$, the equilibrium outcome of policy $\langle g, h \rangle$ Pareto dominates the LFO if the distinguished allocation associated with the natural equilibrium khs-dominates the LFO even when the inequalities in (3.9) fail.*

In Theorem 3.13, for a fixed income distribution we characterize the set of sharing costs for which Pareto domination of the LFO by the equilibrium outcome may fail and we give conditions for the existence of a sharing cost that results in the failure of Pareto domination. In addition, given a sharing cost, we give conditions for the existence of a distribution within a given family of distribution that results in the failure of Pareto domination and also characterize the set of distributions within a family for which Pareto domination fails.

For any given income distribution, G, with support $\left[\underline{I}, \overline{I}\right]$, we introduce a set, F_G, of parametrized families of distributions that are derived from G. Each family of distributions $\{G^t\}_{t\in[o,\infty)}$ is parametrized by t and contains the fixed distribution G. If $t_1 < t_2$, then G^{t_1} first order stochastically dominates G^{t_2}. For any family $\{G^t\}_{t\in[o,\infty)}$ the support of G^t is $\left[\underline{I}(t), \overline{I}\,(t)\right]$ and $\underline{I}(t) \geq 1$. Each family is continuous in t. Let $\underline{I}_{\lim}$ denote the limit of the lower end of the supports so that $\lim_{t\uparrow\infty} \underline{I}(t) = \underline{I}_{\lim} \geq 1$. Let $\overline{I}_{\lim}$ denote the limit of the upper end of the supports so that $\lim_{t\uparrow\infty} \overline{I}(t) = \overline{I}_{\lim} \geq \underline{I}_{\lim}$.

Theorem 3.13. *Suppose $x \geq 0$. Let ϕ, κ, π, and G be given and let $g/h \in (0,1)$ be fixed. (i) Consider $X_0\,(g/h) \leq x$. Either there exists $\widehat{x} \geq X_0\,(g/h)$ for which Pareto domination fails for $x > \widehat{x}$ or Pareto domination succeeds for all $x \geq X_0\,(g/h)$. There exists $\widehat{x} \geq X_0\,(g/h)$ for which Pareto domination fails for $x > \widehat{x}$ iff either (3.2) or (3.3) fails. (ii) If $X_0\,(g/h) \leq x$ and (3.9) fails for G, then suppose that the family $\{G^t\} \in F_G$.* ***Either*** *there exists $\widehat{t} < \infty$ for which Pareto domination fails given G^t iff $t > \widehat{t}$* ***or*** *Pareto domination succeeds given G^t for all $t \geq 0$. Suppose that $\overline{I}_{\lim} = \underline{I}_{\lim}$. There exists $\widehat{t}$ for which Pareto domination occurs iff $0 \leq t \leq \widehat{t}$ IFF $\rho = \underline{\rho}_{\lim}$ satisfies*

$$g\rho < \phi\,(1) - 1 - \left(\pi\,(1) - \pi\left(\frac{g}{h}\right)\right) \tag{3.19}$$

where $\underline{\rho}_{\lim}$ solves

$$AVG\left(\frac{I + g\rho}{1 + h\rho}, x\right) = 1 + h\rho \tag{3.20}$$

for $I = \underline{I}_{\lim}$.

We note that in (i) of Theorem 3.13 G is fixed and one can vary x. Whether any given x leads to a failure of Pareto domination in the presence of sharing depends only on whether or not Pareto domination fails in the absence of sharing which in turn depends on the fraction g/h and on the marginal individual's income given G relative to ϕ, κ, π, and x. In (ii) of Theorem 3.13 $x > x_s$ is fixed and one can transform any given distribution into a family of distributions in F in a variety of ways. For example one can decrease the upper limit of the support, decrease the lower limit of support or skew the distribution towards the poor with no change in support. Whether or not any particular family of distributions will lead to a failure of Pareto domination depends on the marginal income in the limiting distribution in the family relative to ϕ, κ, π, and x. In the extreme case that the upper and lower limits of the support tend toward each other, the marginal income tends toward the limiting smallest income.

Theorem 3.13 gives conditions under which one can find a distribution or a sharing cost for which Pareto domination fails. The permit price, $\underline{\rho}_{\lim}$, defined by (3.20) for $I = \underline{I}_{\lim}$, is the price that leaves the limiting poorest individual indifferent between trading all permits or keeping all permits. Given a distribution G, there exists a stochastically dominated transformation of G for which Pareto domination fails iff the value of the permit endowment at $\underline{\rho}_{\lim}$ is not enough to compensate an outsider for their loss in consumption despite the reduction in pollution. Given a distribution G, there exists $x > x_1$ for which Pareto domination of the LFO by the policy equilibrium fails iff *khs* domination by its associated designated allocation fails.

Corollary 3.14. *Given ϕ, κ, π, $x > x_s$ and G, there exists a family $\{G^t\}_{t \in [1,\infty)} \in F$, for which there exists $\widehat{t} < \infty$ for which Pareto domination fails for $t > \widehat{t}$ iff $\underline{\rho}$ satisfies (3.19) where $\underline{\rho}$ solves (3.20) for $I = 1$.*

For any given distribution and $x > X_0(g/h)$, Corollary 3.14 tells us if there exists a stochastically dominated transformation of G for which Pareto domination fails at x. Theorem 3.13 states when there exists such a transformation in any given family in F. Relative to ϕ, κ and π, if either the marginal individual is poor enough or the lower bound of the support is low enough, then either increasing the cost of sharing with G fixed,[22] or transforming the distribution by decreasing the income of the richest individual,[23] will eventually lead to the absence of Pareto domination. However, even if the poorest individual has the lowest possible income

[22] $4 < \widehat{x} < 5$ if $G(I)$ is uniform on $[1, 1.1]$, $\phi(A) = 2A^{1/2}$, $\kappa(A, x) = x(\lfloor A \rfloor - A)$, $\pi(z) = z^2$.

[23] $3/2 < \widehat{t} < 7/3$, $x_s < x = 0.95 < x_1$, $\overleftarrow{Z}(0.95) \simeq 0.503$, $\underline{\rho}^0 \simeq 0.09$ satisfies (3.19) if $G^t(I)$ uniform on $[1, 1 + 1/(t+1)]$, $\phi(A) = 2A^{\frac{1}{2}}$, $\kappa(A, x) = x(1/A - 1)$, $\pi(z) = 0.974\,339\,5111 z^{1.1}$.

(which equals 1), Pareto domination may result no matter the upper income. This occurs when $\underline{\rho}^0$ is large enough to compensate the outsiders.[24] Note also that, if the distribution of income is relatively skewed enough toward lower income levels, then Pareto domination may fail even if the highest income level is infinity.[25] Lastly, if the gain from the decrease in pollution is higher than the loss from the decrease in consumption then Pareto domination occurs regardless of the income distribution.[26]

4. Conclusion

When there is a reduction of an indivisible good that generates negative externalities, the resulting aggregate consumption patterns necessarily allocates this good asymmetrically across consumers ex post. Given the ex post presence of insiders and outsiders, an outcome may not Pareto dominate the laissez-faire outcome even when, in aggregate, consumers are better off. We give necessary and sufficient conditions for the market based permit outcome to Pareto dominate the laissez-faire outcome and show that it is superior to other methods of allocation. The targeted policy increases the full price of the indivisible good via the permit price by enacting a 'tax on ownership' that is transfered to outsiders to compensate them for their loss in consumption in the presence of Pareto domination. While it is difficult to improve upon the permit policy when restricting to policies that essentially tax ownership, there are methods that may improve the properties of the policy. In particular, increasing the size of the middle class or increasing the wealth of the poorest individuals may improve the outcome as it

[24] $\widehat{t} = \infty$, $x = 0.89$, $\overleftarrow{Z}(0.89) \simeq 0.507$, $g\underline{\rho}^0 \simeq 0.5133$ violates (3.19) if $G^t(I)$ uniform on $[1, 1/(t+1)+1]$, $\phi(A) = 2A^{1/2}$, $\kappa(A,x) = x(\lfloor A \rfloor - A)$, $\pi(z) = z^2$.

[25] $\widehat{t} = 16k$, $\overline{z} = 0.5$, $x = 1.11916055 > x_1, \overline{z} = 0.5$ if $\phi(A) = 2A^{\frac{1}{2}}$, $\kappa(A,x) = x(1-A)$ and $\pi(z) = 0.9743395111 z^{1.1}$, $G^t(I) = 1 - k/t(I-1)$ on $[k/t+1, \infty)$, for $k > 1/16$. If $t > \widehat{t}$, Pareto domination fails though income increases to ∞. This family of distribution functions includes that used in [12] and [23].

[26] If $\phi(A) = 2A^{\frac{1}{2}}$, $\kappa(A,x) = x(1-A)$, $\pi(z) = (4/3)z^3$, then $Z_c^*(x_s) \simeq 0.53$ so that (**??**) is violated for all $x \geq 0$. Outsiders are better off under the targeted market policy regardless of G.

may allow the permit price to make all indifferent (abstracting away from initial income) between obtaining the indivisible good by buying the requisite permits and retaining the value of the permit endowment. One way to achieve the desired outcome may be through a progressive tax as one could transfer more money from the very rich to those in the middle. Another alternative is to improve upon the technology for sharing to make sharing less costly. Intuitively, when the cost of sharing is low, indivisibilities are transformed into divisibilities so that the targeted policy equilibrium may Pareto dominate the laissez-faire outcome. As the cost of sharing increases to infinity, the targeted policy equilibrium converge to the case of no sharing.

Our results therefore suggest that the properties of the targeted policy would improve if combined with policies aimed at reducing the inconvenience cost of sharing, increasing the wealth of the poorest individuals, and increasing the size of the middle class. Our model also provides a simple setting in which to analyze the policy implications of negative externalities in the presence of indivisibilities and could be used as a second stage in a two stage bargaining model in which countries with differing income distributions bargain over pollution standards between countries.

A. Appendix

Proof. (THEOREM 3.1): In the absence of sharing, the only feasible individual demands are 1 and 0. If $I_{g/h} \geq u(1)(1-g/h)+g/h$ then, when $\rho = (u(1)-1)/h$, each individual is indifferent between 1 and 0 and each $I \geq I_{g/h}$ has enough income to purchase 1 unit at this price so that in equilibrium, the fraction g/h demand 1 and the rest demand 0. If $I_{g/h} < u(1)(1-g/h)+g/h$ then, when $\rho = \left(I_{g/h}-1\right)/(h-g)$ those $I > I_{g/h}$ demand 1 and those $I < I_{g/h}$ cannot afford to buy 1 and so must demand 0. ■

Proof. (THEOREM 3.2): We show that the distinguished outcome *khs*-dominates the LFO iff (3.2) and (3.3) hold. If $g/h \in (0,1)$, then all are better off at the distinguished outcome after an ownership-based transfer of income iff there exists M for which

$$\begin{aligned}
\phi(1) + I - 1 - \pi\left(\frac{g}{h}\right) - M &\geq \phi(1) + I - 1 - \pi(1) \quad (1) \\
\phi(0) + I - \pi\left(\frac{g}{h}\right) + \frac{\left(\frac{g}{h}\right) M}{1 - \frac{g}{h}} &\geq \phi(1) + I - 1 - \pi(1) \quad (2) \\
I_{g/h} - 1 &\geq M \quad (3)
\end{aligned}$$

iff

$$\begin{aligned}
\pi(1) - \pi\left(\frac{g}{h}\right) &\geq M \quad (1) \\
\left(\frac{1 - \frac{g}{h}}{\frac{g}{h}}\right)\left(\phi(1) - 1 - \left(\pi(1) - \pi\left(\frac{g}{h}\right)\right)\right) &= M \quad (2) \\
I_{g/h} - 1 &\geq M \quad (3)
\end{aligned}$$

where line (1) indicates that each insider remains better off relative to the LFO after transferring M; line (2), no outsider is worse off after receiving a transfer $gM/h\,(1 - g/h)$; line (3), each insider has adequate income to transfer M. Lines (2) and (3) are satisfied whenever (3.2) holds as required. It remains to show that line (2) implies line (1). When line (2) holds, line (1) is satisfied iff (3.3) and (3.2) hold.

We now show that the equilibrium outcome of policy $< g, h >$ Pareto dominates the LFO iff (3.3) and (3.2) hold. The "only if" direction is immediate. We now prove the "if" direction. Suppose that (3.3) and (3.2) hold. We need to show that (3.3) and (3.2) imply that

$$\begin{aligned}
\phi(1) + g\rho_{ns}^{*} + I - (1 + h\rho_{ns}^{*}) - \pi\left(\frac{g}{h}\right) &\geq \phi(1) + I - 1 - \pi(1) \text{ for all } I \geq I_{g/h} \\
I + g\rho_{ns}^{*} - \pi\left(\frac{g}{h}\right) &\geq \phi(1) + I - 1 - \pi(1) \text{ for all } I \leq I_{g/h}
\end{aligned}$$

The above two inequalities hold if and only if

$$\begin{array}{l}
\pi(1) - \pi\left(\frac{g}{h}\right) \geq (h - g)\,\rho_{ns}^{*} \\
g\rho_{ns}^{*} \geq \left(\phi(1) - 1 - \left(\pi(1) - \pi\left(\frac{g}{h}\right)\right)\right)
\end{array} \tag{A.1}$$

We note that if the top inequality in (3.1) holds then (A.1) is equivalent to

$$\begin{aligned}
\pi(1) - \pi\left(\frac{g}{h}\right) &\geq \left(1 - \frac{g}{h}\right)(\phi(1) - 1) \\
g\left(\frac{\phi(1) - 1}{h}\right) &\geq \left(\phi(1) - 1 - \left(\pi(1) - \pi\left(\frac{g}{h}\right)\right)\right)
\end{aligned}$$

which are each implied by (3.3). If instead, the bottom inequality in (3.1) holds then (A.1) is equivalent to

$$\begin{aligned}
\pi(1) - \pi\left(\frac{g}{h}\right) &\geq (h - g)\left(\frac{I_{g/h} - 1}{h - g}\right) \\
g\left(\frac{I_{g/h} - 1}{h - g}\right) &\geq \left(\phi(1) - 1 - \left(\pi(1) - \pi\left(\frac{g}{h}\right)\right)\right)
\end{aligned}$$

which is implied by (3.2) and (3.3) given the bottom inequality in (3.1). The result follows from Theorem 3.1 and (3.1). ■

Proof. (THEOREM 3.3): By (3.6), at the permit price $\rho_m(x)$, individuals prefer consuming $A = g/h$ to any alternative in $(0, 1]$. In addition, when $0 \leq x \leq X_0(g/h)$, $AVG(g/h, x) \geq MAR(g/h, x)$ so that individuals also prefer $A = g/h$ to 0 at the price $\rho_m(x)$. In this case, at $\rho_m(x)$ each individual exchanges the entire endowment of g permits for the fraction g/h units of the indivisible good. Thus, from the discussion following (2.6) and (2.7), if $0 \leq x \leq X_0(g/h)$, then since aggregate demand is g/h at the price $\rho_m(x)$, we conclude that $\rho_m(x)$ is the equilibrium price. The Implicit Function Theorem implies ρ_m increases in x. If instead, $X_0(g/h) < x$ then $g/h < \min\{Z_0(x), 1\}$ implies $AVG(g/h, x) < MAR(g/h, x)$ so that the role of $\rho_m(x)$ is irrelevant when $X_0(g/h) \leq x \leq x_1$ since individuals prefer 0 to g/h whenever g/h is preferred among alternatives in $(0, 1]$.

By 2.8, 3.4, and 3.5, if $X_0(x) \leq x \leq x_1$ then $Z_0(x) \in [g/h, 1]$. At the permit price $\rho_n(x)$, each individual is indifferent between consuming $Z_0(x)$ and 0 and prefers either to any alternative in $[0, 1]$. If $I_{g/hZ_0}(x) + g\rho_n \geq (1 + h\rho_n) Z_0(x)$ then any $I \geq I_{g/hZ_0}(x)$ can afford to buy $Z_0(x)$ units. Thus, if $X_0(g/h) \leq x \leq x_1$ and $I_{g/hZ_0}(x) + g\rho_n \geq (1 + h\rho_n) Z_0(x)$, then aggregate demand is g/h so that $\rho_n(x)$

is the equilibrium price. The role of $\rho_n(x)$ is irrelevant if either $x_1 < x$, (since then $Z_0(x) > 1$) or if $I_{g/hZ_0}(x) + g\rho_n < (1 + h\rho_n) Z_0(x)$ (since then not enough individuals can afford to buy $Z_0(x)$). By 3.7, 2.8, and $AVG'_x < 0$ we obtain that, when $0 < x \leq x_1$, $\rho_n(x)$ decreases in x since

$$\begin{aligned} h\rho'_n(x) &= AVG'_A(Z_0(x), x) Z'_0(x) + AVG'_x(Z_0(x), x) \\ &= \left(\frac{MAR(Z_0(x), x) - AVG(Z_0(x), x)}{Z_0(x)}\right) Z'_0(x) + AVG'_x(Z_0(x), x) \\ &= AVG'_x(Z_0(x), x) < 0 \end{aligned}$$

By (3.8), at the permit price $\rho_\eta(x)$, individuals are indifferent between consuming 1 and 0. In addition, when $x_1 < x$, $1 < Z_0(x)$ and $MAR(1, x) > AVG(1, x)$ so that individuals prefer 1 to any alternative in $(0, 1)$. Thus, if $x_1 < x$ and $I_{g/h} + g\rho_\eta \geq 1 + h\rho_\eta$, then at least the fraction g/h of individuals can afford to buy 1 and therefore aggregate demand is g/h and $\rho_\eta(x)$ is the equilibrium price. By (3.8) and $AVG'_x < 0$ we obtain that when $x_1 < x$, ρ_η decreases in x since

$$h\rho'_\eta(x) = AVG'_x(1, x) < 0$$

■

Proof. (THEOREM 3.4): By Theorem 3.3, if $x \leq X_0(g/h)$, then the natural equilibrium allocates g/h to each individual so that there is no transfer of income across individuals and the equilibrium outcome is identical to its associated distinguished allocation. Each is better off relative to the LFO iff (3.11) holds.

If $X_0(g/h) < x < x_1$, then, in the distinguished outcome associated with the policy equilibrium, $Z_0(x)$ units of the indivisible good are allocated to a group of size g/h and the rest receive 0. The proof of the result is analogous to the proof of Theorem 3.2. We first show that the distinguished outcome *khs*-dominates the LFO iff (3.12) and (3.13) hold. If $g/h \in (0, 1)$, then all are better off at the distinguished outcome after an ownership-based transfer of income iff there exists

M for which

$$
\begin{aligned}
\phi\left(Z_0\left(x\right)\right)+I-Z_0\left(x\right)-\kappa\left(Z_0\left(x\right),x\right)-\pi\left(\frac{g}{h}\right)-M &\geq \phi\left(1\right)+I-1-\pi\left(1\right)\\
\phi\left(0\right)+I-\pi\left(\frac{g}{h}\right)+\frac{\left(\frac{g/h}{Z_0(x)}\right)M}{1-\frac{g/h}{Z_0(x)}} &\geq \phi\left(1\right)+I-1-\pi\left(1\right)\\
I_{(g/h)/Z_0(x)}-Z_0\left(x\right) &\geq M
\end{aligned}
$$

iff

$$
\begin{aligned}
\pi\left(1\right)-\pi\left(\frac{g}{h}\right)-\left(\phi\left(1\right)-1-\left(\phi\left(Z_0\left(x\right)\right)-Z_0\left(x\right)\right)\right) &\geq M \quad (1)\\
\left(\frac{1-\frac{g/h}{Z_0(x)}}{\frac{g/h}{Z_0(x)}}\right)\left(\phi\left(1\right)-1-\left(\pi\left(1\right)-\pi\left(\frac{g}{h}\right)\right)\right) &= M \quad (2)\\
I_{(g/h)/Z_0(x)}-Z_0\left(x\right) &\geq M \quad (3)
\end{aligned}
$$

where line (1) indicates that insiders are better off; line (2), outsiders no worse off; line (3) insiders have adequate income. Lines (2) and (3) are satisfied whenever (3.12) holds as required. It remains to show that line (2) implies line (1) iff (3.13) holds. When line (2) holds, line (1) is satisfied iff

$$
\begin{aligned}
&\pi\left(1\right)-\pi\left(\frac{g}{h}\right)-\left(\phi\left(1\right)-1-\left(\phi\left(Z_0\left(x\right)\right)-Z_0\left(x\right)\right)\right)\\
\geq\ &\left(\frac{1-\frac{g/h}{Z_0(x)}}{\frac{g/h}{Z_0(x)}}\right)\left(\phi\left(1\right)-1-\left(\pi\left(1\right)-\pi\left(\frac{g}{h}\right)\right)\right)
\end{aligned}
$$

iff (3.13) holds.

We now show that if (3.9) holds, then the equilibrium outcome of policy $<g,h>$ Pareto dominates the LFO iff (3.12) and (3.13) hold. The "only if" direction is immediate since *khs*-domination is equivalent to (3.12) and (3.13). We now prove the "if" direction. Suppose that (3.9), (3.12) and (3.13) hold. By Theorem 3.3 and line (2) above, we need to show that

$$
g\rho_n\left(x\right)\geq\left(\phi\left(1\right)-1-\left(\pi\left(1\right)-\pi\left(\frac{g}{h}\right)\right)\right)
$$

which is equivalent to

$$\frac{g/h}{Z_0(x)}\left(\phi\left(Z_0(x)-\kappa\left(Z_0(x)\right)\right)\right)-\frac{g}{h}\geq\left(\phi(1)-1-\left(\pi(1)-\pi\left(\frac{g}{h}\right)\right)\right)$$

which is equivalent to (3.13) since $\rho_n(x)$ satisfies (3.7). ■

Proof. (PROPOSITION 3.5): By Theorem 3.3, type A occurs only iff $x \in [0, X_0(g/h)]$, type $A0$ occurs iff $x \in [X_0(g/h), x_1]$ and $I_{g/hZ_0(x)} + g\rho_n(x) \geq (1 + h\rho_n(x)) Z_0(x)$ and type 10 occurs iff $x \geq x_1$ and $I_{g/h}+g\rho_\eta(x) \geq \left(1 + h\rho_\eta(x)\right)$. Thus, if $x > x_1$, the natural positive option equals 1 so that the single most preferred option in $[0, 1]$ in equilibrium is 1 for any price that is less than $\rho_\eta(x)$ but not all may be able to buy 1. In this case, the feasible equilibrium aggregate demand forms among those in Δ are $[1E0]$ or $[E0]$. If $X_0(g/h) < x < x_1$, the natural positive option is $Z_0(x) \in (0, 1)$ so that, as the equilibrium price decreases below the natural price, the single most preferred option in $[0, 1]$ in equilibrium is larger than $Z_0(x)$ and may be less than or equal to 1, but again not all may be able to buy this most preferred option. In this case, the feasible equilibrium aggregate demand forms among those in Δ are $[AE0]$, $[1E0]$, or $[E0]$. Whichever system is in place depends on x and the distribution of income. ■

Proof. (PROPOSITION 3.6): Since (3.9) is violated, by 2.9, and by Theorem 3.3, the equilibrium systems that remain feasible are $AE0$, $1E0$, or $E0$. For all $(x, t\ )$ for which $x > X_0(g/h)$ and (3.9) is violated, the equilibrium price decreases in t and for each such (x, t), given x, the set in $[0, \infty)$ over which one of the systems $AE0$, $1E0$, or $E0$ is an equilibrium system is closed. In addition, for all such $(x, t\ )$, if x is fixed, then, as t increases, system $AE0$ morphs into system $1E0$ when $(I_b + g\rho) / (1 + h\rho) = 1$ in equilibrium; system $1E0$ morphs into system $E0$ when $I_b = \overline{I}$ in equilibrium; system $AE0$ can morph into System $1E0$ and $E0$ if, at the intersection of systems $AE0$ and $1E0$, $(I_b + g\rho) / (1 + h\rho) = 1$ and $I_b = \overline{I}$. Note that since the equilibrium price decreases in t, system $1E0$ cannot morph back into system $AE0$ since the most preferred option in $[0, 1]$ is 1 at the equilibrium price in system $1E0$ and will remain so as the price decreases. Note also that since the equilibrium price decreases in t, system $E0$ cannot morph back into system $1E0$ since a decrease in price implies I_b in system $1E0$ increases and so, once system $1E0$ admits $I_b = \overline{I}$, $\left(\overline{I} + g\rho\right) / (1 + h\rho) = 1$ and any further decrease in price will imply $\left(\overline{I} + g\rho\right) / (1 + h\rho) < 1$ so that system $E0$ takes over and remains the equilibrium system for any larger value of t. Similar reasoning shows that, if $x > X_0(g/h)$ and the equilibrium form is $[AE0]$, $[1E0]$ or $[E0]$ at t_0, then the equilibrium form is never $[A0]$ or $[10]$ for any $t > t_0$. ■

Proof. (PROPOSITION 3.7): By 2.9, we need to prove that S_σ is a union of closed and bounded sets for σ associated with demand forms $[A]$, $[A0]$, $[AE0]$ or $[E0]$. By Theorem 3.3 $S_A = [0, X_0(g/h)]$ so that the result is proven for demand form $[A]$. We now prove boundedness for systems $A0$, $AE0$ and $E0$. By Theorem 3.3

$$S_{A0} = \left\{x \in [X_0(g/h), x_1] : I_{g/hZ_0(x)} + g\rho_n(x) - (1 + h\rho_n(x)) Z_0(x) \geq 0\right\}$$

and by Proposition 3.5, systems $A0$ and $AE0$ may be equilibrium systems only on $[X_0(g/h), x_1]$ so that systems $A0$ and $AE0$ are bounded. We now consider system $E0$. If $S_{E0} \cap [x_1, \infty) = \varnothing$, then $S_{E0} \subset [X_0(g/h), x_1]$ so that S_{E0} is bounded. If instead, $S_{E0} \cap [x_1, \infty) \neq \varnothing$, then we show that the equilibrium demand form $[E0]$ must change into $[1E0]$ for some finite x. If $x_1 < x$, then $1 < Z_0(x)$ and therefore $MAR(1, x) > AVG(1, x) = 1 + h\rho_\eta(x)$. If $x \in S_{E0} \cap [x_1, \infty)$, then, the equilibrium solves system $E0$. Given the equilibrium solves system $E0$, then as x increases in S_{E0}, the equilibrium price $\rho^*(x, G)$ must decrease as shown in Step 1 of Theorem 3.8. Thus, since $\left(\overline{I} + g\rho^*(x, G)\right) / (1 + h\rho^*(x, G))$ increases as $\rho^*(x, G)$ decreases, eventually, the second inequality in (3.17) is replaced with an equality and then for larger x, system $1E0$ takes over as the equilibrium system.

Closedness follows from continuity. Since systems change continuously from one to the other as x increases, each S_σ consists of a union of closed intervals for all systems σ with associated demand forms $\sigma \in \Delta$.

Lastly, we show that there exists x^e for which $[x^e, \infty) \subset S_\sigma$ for some $x^e > X_0(g/h)$ if $\sigma = [10]$ or $[1E0]$. The result follows from Theorem 3.3 in the case that (3.9) is satisfied since then system 10 is the equilibrium system on $[x_1, \infty)$.

From now on suppose that $x \geq x_1$ and that (3.9) is violated so that, by Theorem 3.3 and Proposition 3.5, the equilibrium system is either system $E0$ or system $1E0$. ■

Proof. (THEOREM 3.8): By Proposition 3.6, we know that $\rho^* (x, G)$ satisfies system $AE0$, $1E0$, or $E0$. We now assume that $x \geq X_0 (g/h)$ and income distribution G violates (3.9). We construct the family of distributions $\Gamma_G = \{G_t \text{ for } t \in [0, \infty)\}$ as indicated in the text. We know, by Theorem 3.3, that the natural equilibrium price strictly decreases in $x \geq X_0 (g/h)$ when $t = 0$ since, by construction G_0 satisfies (3.9). Since G violates (3.9) the feasible equilibrium demand forms are $[AE0]$, $[1E0]$, or $[E0]$. The proof proceeds in three steps. **Step 1**: We show that, if the equilibrium system is type $1E0$ or $E0$ then the first three lines of either system can be used to show that the equilibrium price decreases in x. Suppose that the income distribution is G and that $(\rho^* (x_\alpha, G), I_a^\alpha (x_\alpha, G), I_b^\alpha (x_\alpha, G))$ solves system S which stands for either system $1E0$ or $E0$ when $x = x_\alpha$. Now suppose that x increases from x_α to x_β in the range of system S. In this case, if $\rho = \rho^* (x_\alpha, G)$ and $x = x_\beta$ in system S then the equation in the second line of system S implies that $I_a (x_\beta, \rho^* (x_\alpha, G)) > I_a (x_\alpha, \rho^* (x_\alpha, G))$ so that the left-hand side of the equation in the third line of system S shifts down which implies that demand is less than supply when $\rho = \rho^* (x_\alpha, G_t)$ and $x = x_\beta$. Thus, the equilibrium price $\rho^* (x_\beta, G) < \rho^* (x_\alpha, G)$. That is, the equilibrium price must decrease as x increases in the range of either system $1E0$ or $E0$. Step 1 proves the result for demand forms $[1E0]$ or $[E0]$. It remains to show that the equilibrium price decreases in x when the equilibrium system is type $AE0$. By Proposition 3.7, we know that the range of x for which the equilibrium system is $AE0$ is a union of closed and bounded intervals. Since only intervals of positive measure are important to the discussion, we prove that the equilibrium price decreases whenever the equilibrium system $AE0$ holds on a closed and bounded interval, say, $[x_I, x_S]$ where $x_I < x_S$ is the initial point of transition from system, $\sigma_1 \neq AE0$, to system $AE0$ and x_S is the point of transition from system $AE0$ to system $\sigma_2 \neq AE0$. **Step 2**: In this step, we use the constructed family Γ_G. As stated above, we know that at

$t = 0$, the equilibrium price decreases in x. By Theorem 3.3, and Step 1 above, we know that the equilibrium price decreases in x for all $x \geq X_0(x)$ for which the equilibrium system is not $AE0$ and for all $x \geq X_0(g/h)$ and all $t > 0$ for which G_t satisfies (3.9) We now show that whenever the equilibrium price decreases in x on $[x_I, x_S]$ for some t, then there exists a neighborhood of t for which the equilibrium price decreases in x on $[x_I, x_S]$ for G_s for all s in the neighborhood of t. Suppose that $\rho^*(x, G_t)$ strictly decreases in x on $[x_I, x_S]$ for some $t \geq 0$. By continuity of the equilibrium price in t and x, for each $x \in [x_I, x_S]$, we know that there exists a neighborhood $M(x,t)$ of t, and a neighborhood $N(x, t)$ of x, for which $s \in M(x,t)$, implies that $\rho^*(x, G_s)$ strictly decreases in x for $x \in N(x,t)$. Since the interval $[x_I, x_S]$ is compact, there exists a finite number of the neighborhoods $\{N(x, \varepsilon) : x \in [x_I, x_S]\}$ that cover $[x_I, x_S]$. Let's relabel the sets so that the finite cover is the set $\{N(x_i, \varepsilon), i = 1, ..., n\}$. Then, for each $x \in [x_I, x_S]$, there exists a neighborhood $N(x_i, \varepsilon)$ that contains x for which $s \in M(x_i, t)$ implies $\rho^*(x, G_s)$ strictly decreases in x. Let $M(t) = \cap_{i=1}^{n} M(x_i, t)$ so that $M(t)$ is a neighborhod of t since it is the intersection of a finite number of non-empty neighborhoods of t. Thus, $M(t)$ is a non-empty neighborhood of t for which $s \in M$ implies $\rho^*(x, G_s)$ decreases in x for all $x \in [x_I, x_S]$. Thus, whenever ρ strictly decreases in x for some t, then there exists a neighborhood of t for which ρ strictly decreases in x for all s in this neighborhood. This argument is a generic argument that uses continuity and compact sets. **Step 3**: In this step, we again use the family Γ_G. We show that if the equilibrium permit price decreases in x on $[x_I, x_S]$ for all $s < t$, then the equilibrium permit price decreases in x for t. We know, by Proposition 3.6, that the equilibrium price must decrease in t for fixed x when (3.9) is violated. We know that demand is non-increasing in t for any permit price ρ and that demand is strictly decreasing in t for any permit price close enough to the equilibrium permit price in (x_I, x_S). Fix $t = \widehat{t}$ and suppose that the equilibrium price $\rho^s(x)$ is strictly decreasing in x on $[x_I, x_S]$ for all $s < \widehat{t}$. Given $\widehat{t}$, $x_\sigma < x_\tau$, $(x_\sigma, x_\tau) \subset [x_I, x_S]$, we let ρ^i_γ denote the equilibrium price when the distribution is G^i, $i = s, t$ and the sharing cost is $x = x_\gamma$, $\gamma = \sigma, \tau$. By continuity (in the equilibrium price as a function of t in the family of distributions $\{G^t\}$) and by the assumption that $\rho^s(x)$ is decreasing in x for $s < \widehat{t}$, we infer that there exists $\widehat{s} < \widehat{t}$ for which

$$\rho_\sigma^{\widehat{s}} > \rho_\sigma^{\widehat{t}} > \rho_\tau^{\widehat{s}}$$

given that

$$\rho_\sigma^{\widehat{s}} > \rho_\tau^{\widehat{s}}$$

by assumption and

$$\rho_\sigma^{\widehat{s}} > \rho_\sigma^{\widehat{t}}$$

by Proposition 3.6 given the construction of the family $\{G^t\}$. Let $D(\rho, x, t)$ denote demand under distribution G_t at price ρ when the cost parameter is x. Given that the equilibrium system is $AE0$ at x given t, by continuity, we can always find $\rho_\sigma^{\widehat{s}} > \rho_\sigma^{\widehat{t}} > \rho_\tau^{\widehat{s}}$ for which the demand form is $[AE0]$ at each of these prices and those inbetween, given either of the x_σ or x_τ. In this case demand for the indivisible unit equals

$$D(\rho, x, t) = \int_{I_a(\rho)}^{I_b(\rho)} \left(\frac{I + g\rho}{1 + h\rho}\right) G_t'(I)\, dI + \left(\frac{I_b(\rho) + g\rho}{1 + h\rho}\right)(1 - G_t(I_b)) \qquad \text{(A.2)}$$

where

$$MAR\left(\frac{I_b(\rho) + g\rho}{1 + h\rho}, x\right) = 1 + h\rho \qquad \text{(A.3)}$$

$$\begin{aligned} AVG\left(\frac{I_a(\rho) + g\rho}{1 + h\rho}, x\right) &= 1 + h\rho \qquad \text{(A.4)} \\ \frac{I_a(\rho) + g\rho}{1 + h\rho} &\leq Z_0(x) \end{aligned}$$

As ρ increases, the facts that $MAR_A' < 0$ and $AVG_A' > 0$ (since $(I_a + g\rho)/(1 + h\rho) < Z_0(x)$) in the associated range of feasibility, along with equation (A.3) and (A.4) can be used to show that I_a and

$$\left(\frac{I_b + g\rho}{1 + h\rho}\right)$$

decrease in ρ. We can then use (A.2) and the assumption that $g/h < 1 \leq I$ to show that $D(\rho, x, t)$ decreases in ρ since $\partial D(\rho, x, t)/\partial\rho =$

$$
\begin{aligned}
&\left(\frac{I_b + g\rho}{1 + h\rho}\right) G'_t(I_b)\, I'_b(\rho) - \left(\frac{I_a + g\rho}{1 + h\rho}\right) G'_t(I_a)\, I'_a(\rho) + \int_{I_a}^{I_b} \left(\frac{g - hI}{(1 + h\rho)^2}\right) G'_t(I)\, dI \\
&+ (1 - G_t(I_b)) \frac{d}{d\rho}\left(\frac{I_b + g\rho}{1 + h\rho}\right) - \left(\frac{I_b + g\rho}{1 + h\rho}\right) G'_t(I_b)\, I'_b(\rho) \\
= \;& (1 - G_t(I_b)) \frac{d}{d\rho}\left(\frac{I_b + g\rho}{1 + h\rho}\right) - \left(\frac{I_a + g\rho}{1 + h\rho}\right) G'_t(I_a)\, I'_a(\rho) + \int_{I_a}^{I_b} \left(\frac{g - hI}{(1 + h\rho)^2}\right) G'_t(I)\, dI
\end{aligned}
$$

is less than 0. Now consider demand under distribution $\widehat{t}$ at price $\rho_\sigma^{\widehat{t}}$ when $x = x_\tau$. Since $D\left(\rho_\tau^{\widehat{s}}, x_\tau, \widehat{s}\right) = g/h$ (by definition of equilibrium), $\widehat{s} < \widehat{t}$, $\rho_\sigma^{\widehat{t}} > \rho_\tau^{\widehat{s}}$, and $\partial D(\rho, x, t)/\partial\rho < 0$, for all t, x for which demand is of form $[AE0]$, we know that $D\left(\rho_\sigma^{\widehat{t}}, x_\tau, \widehat{t}\right) < D\left(\rho_\sigma^{\widehat{t}}, x_\tau, \widehat{s}\right) < D\left(\rho_\tau^{\widehat{s}}, x_\tau, \widehat{s}\right) = g/h$. Thus, the equilibium price $\rho_\tau^{\widehat{t}}$ when $x = x_\tau$ and $t = \widehat{t}$ must satisfy $\rho_\tau^{\widehat{t}} < \rho_\sigma^{\widehat{t}}$. Thus, we have shown that whenever the equilibrium price $\rho^s(x)$ strictly decreasing in $x \in (x_I, x_S)$ for all $s < \widehat{t}$ then $\rho^{\widehat{t}}$ strictly decreasing in $x \in (x_I, x_S)$. Generally, when a function f^s is strictly decreasing in x for all $s < t$, then f^t is weakly decreasing in x. So, argument 1 is a special argument that uses the structure of the family and the details of the problem.

Theorem 3.3 and step 1 shows that the equilibrium price strictly decreases in x for $t = 0$ and strictly decreases in x for any t whenever the equilibrium system is $\sigma \in \Delta, \sigma \neq AE0$. It remains to prove the result for $x \in S_{AE0}$ when $\sigma = AE0$. By Proposition 3.7 we know that S_{AE0} is a disjoint union of closed and bounded intervals. Step 2 uses compactness to show that whenever the equilibrium price decreases in $x \in [x_I, x_S]$, one of the disjoint components of S_{AE0} for some $t \geq 0$, that it decreases in $x \in [x_I, x_S]$ for an open neighborhood around t. Steps 1 and 2 imply that for any x, there exists $\widehat{t} > 0$ for which the equilibrium price decreases in x on $\left[0, \widehat{t}\right)$ independent of the equilibrium system. Step 3 uses continuity and the properties of the model to show that whenever the equilibrium price decreases in x on one of the disjoint components $[x_I, x_S]$ for all $s < \widehat{t}$, then the equilibrium price decreases in x on $[x_I, x_S]$ for $t = \widehat{t}$. Thus, the equilibrium price decreases in

x on any of the closed intervals which are disjoint components of S_{AE0}.

The reason for the limiting result follows. If (x_1, G) satisfies (3.9) then, by Theorems 3.1 and 3.3, for $x \geq x_1$, $\rho^*(x, G)$ is constant in x and equal to the first alternative for ρ^*_{ns} in (3.1). If (x_1, G) does not satisfy (3.9) then by the above steps and by Proposition 3.5 $\rho^*(x, G)$ strictly decreases in x and the limiting demand system is $[1E0]$ by the proof of Proposition 3.7. Since $\rho^*(x, G)$ is strictly decreasing in x, the three equalities in system $1E0$ can then be used to show that, as x increases, $I_a \leq I_{g/h}$ increases and $I_b \geq I_{g/h}$ decreases and that I_a and I_b are converging to $I_{g/h}$. As $\rho^*(x, G)$ decreases on $[x_1, \infty)$, the ultimate line in system $1E0$ implies that $I^*_b(x) \geq I_{g/h}$ decreases so that the third line then implies that $I^*_a(x) \leq I_{g/h}$ increases and that $(I_a + g\rho)/(1 + h\rho)$ increases. Thus, as x increases for $x \in S_{1E0} \cap [x_1, \infty)$, we know that the equilibrium price decreases, the set $\left[I_b, \overline{I}\right]$ of individuals who buy 1 unit increases, the set $[I_a, I_b]$ of individuals who spend all decreases, and the set $[\underline{I}, I_a]$ of individuals who buy 0 decreases. If we substitute $I_{g/h}$ for I_b in the third equality in system $1E0$ we obtain that the equilibrium price converges to the second alternative for ρ^*_{ns} in (3.1). In either case, $\lim_{x\to\infty} \rho^*(x, G) = \rho^*_{ns} = \left(I_{g/h} - 1\right)/(h - g)$. ■

Proof. (PROPOSITION 3.9): By Theorem 3.3, we know that, if $x > X_0$ and (3.9) is satisfied, then, in equilibrium, all individuals are indifferent between consuming a specific positive quantity of the indivisible good and consuming 0 units. Thus, the equilibrium outcome Pareto dominates the LFO iff

$$\phi(0) + I + g\rho^*(x, G) - \pi(g/h) \geq \phi(1) + I - 1 - \pi(1) \triangleq D_I$$

iff (3.18) holds where $\triangleq$ denotes "is defined as". In the case that $x > X_0$ and (3.9) fails, the equilibrium form is either $[AE0]$, $[1E0]$ or $[E0]$ so that Pareto domination occurs iff

$$\begin{aligned}
\phi\left(\frac{I_b + g\rho}{1 + h\rho}\right) - \kappa\left(\frac{I_b + g\rho}{1 + h\rho}, x\right) + I + g\rho - (1 + h\rho)\frac{I_b + g\rho}{1 + h\rho} - \pi\left(\frac{g}{h}\right) &\geq D_I\,|_{I \geq I_b} \\
\phi\left(\frac{I + g\rho}{1 + h\rho}\right) - \kappa\left(\frac{I + g\rho}{1 + h\rho}, x\right) + I + g\rho - (1 + h\rho)\frac{I + g\rho}{1 + h\rho} - \pi\left(\frac{g}{h}\right) &\geq D_I\,|_{I_a < I < I_b} \\
I + g\rho - \pi\left(\frac{g}{h}\right) &\geq D_I\,|_{I \leq I_a}
\end{aligned}$$

where

$$\phi\left(\frac{I_a+g\rho}{1+h\rho}\right)-\kappa\left(\frac{I_a+g\rho}{1+h\rho},x\right)+I_a+g\rho-(1+h\rho)\frac{I_a+g\rho}{1+h\rho}=I_a+g\rho$$

or equivalently

$$AVG\left(\frac{I_a+g\rho}{1+h\rho},x\right)=1+h\rho$$

at $\rho=\rho^*(x,G)$. Since $MAR>AVG\geq 1+h\rho^*(x,G)$ at any of the positive quantities that are consumed when an individual spends all income, Pareto domination occurs iff the outsiders are better off iff

$$I_a+g\rho^*(x,G)-\pi\left(\frac{g}{h}\right)\geq\phi(1)+I_a-1-\pi(1)$$

iff (3.18) holds. ■

In the presence of sharing with $x\geq X_0(g/h)$ when the inequalities in (3.9) fail, the equilibrium outcome of policy $\langle g,h\rangle$ Pareto dominates the LFO if its associated distinguished allocation *khs*-dominates the LFO iff **either** $X_0(g/h)<x<x_1$, (3.12), and (3.13) **or** $x_1<x$, and both (3.2) and (3.3) hold.

Proof. (COROLLARY 3.10): By Proposition 3.9, inequality (3.18) is a necessary and sufficient condition for Pareto domination. By Theorem 3.8, the equilibrium value of the endowment under sharing, $g\rho^*$, decreases to $g\rho^*_{ns}$, the equilibrium value of the endowment in the absence of sharing. Thus, if $g\rho^*_{ns}$ satisfies inequality (3.18), then so does $g\rho^*$. By Theorem 3.2 $g\rho^*_{ns}$ satisfies inequality (3.18) iff inequalities (3.3) and (3.2) hold. ■

Proof. (COROLLARY3.11): Follows immediately from Corollary 3.10. ■

Proof. (THEOREM 3.12): By Theorem 3.4, if the natural equilibrium's associated distinguished allocation *khs*-dominates the LFO then **either** $X_0(g/h)<x<x_1$, and both (3.12) and (3.13) hold **or** $x_1<x$, and both (3.2) and (3.3) hold.

Suppose that $X_0(g/h)<x<x_1$, the inequality in (3.9) fails and that both (3.12) and (3.13) hold. By Proposition 3.5, at the equilibrium price ρ^* there exists an individual $\widehat{I}_a(\rho^*)$ who is indifferent between spending all income and spending 0 on the indivisible good. We also know that expenditure increases in I for all

individuals who spend all income and that there exists $\widehat{I}_b\left(\rho^*\right) \leq \overline{I}$ for which (i) $\widehat{I}_b\left(\rho^*\right)$ is the last individual who spends all on the indivisible good (ii) expenditure by $I \geq \widehat{I}_b\left(\rho^*\right)$ is constant and greater than or equal to $Z_0\left(x\right)$. Let ρ_0 denote the price that makes $I_{g/hZ_0(x)}$ indifferent between spending all and spending 0 on the indivisible good. If the equilibrium price $\rho^* = \rho_0$, then $\widehat{I}_a = I_{g/hZ_0(x)}$, and, by Proposition 3.9 and (3.12) the value of the endowment satisfies (3.18) so that Pareto domination results. If instead, $\rho^* > \rho_0$ then $g\rho^* > g\rho_0$ so that again, each outsider is better off than at the LFO and Pareto domination results by Proposition 3.9. If $\rho^* < \rho_0$, then, in this case, we argue that $I_{g/hZ_0(x)}$ spends all income on the indivisible good. Since the value of the endowment increases in the price and by definition of ρ_0, when $\rho^* < \rho_0$ we obtain that individual $I_{(g/h)/Z_0(x)}$ must prefer spending something to spending nothing so that $\widehat{I}_a\left(\rho^*\right) \leq I_{(g/h)/Z_0(x)}$. If the quantity bought by $I_{(g/h)/Z_0(x)}$ is greater than $Z_0\left(x\right)$ then the demand for permits is greater than supply. Thus, $\widehat{I}_a\left(\rho^*\right) \leq I_{\overleftarrow{Z}(x)/Z_0(x)} < \widehat{I}_b\left(\rho^*\right)$ so that $I_{Z_c^*/A_c^*(x)}$ spends all income on the indivisible good. Since expenditure is non-decreasing in income this implies that each individual $I > I_{Z_c^*/A_c^*(x)}$ spends at least as much as does $I_{Z_c^*/A_c^*(x)}$. By Proposition 3.9 and (3.12) the value of the endowment satisfies (3.18) and Pareto domination results.

Suppose that $x_1 \leq x$, the inequality in (3.9) fails and that both (3.2) and (3.3) hold. In this case, by Theorems 3.1 and 3.8, the equilibrium value of the endowment is greater than or equal to

$$\frac{\frac{g}{h}\left(I_{g/h} - 1\right)}{\left(1 - \frac{g}{h}\right)}$$

which is greater than or equal to the right-hand side of (3.18) by (3.2). Pareto domination of the LFO by the equilibrium outcome then follows from Proposition 3.9. ■

Proof. (THEOREM 3.13): (i) By Theorems 3.4 and 3.8 we know that Pareto domination fails only if $x > x_0\left(g/h\right)$ and that the equilibrium price decreases strictly in x on $\left[X_0\left(g/h\right), x_1\right]$. As x increases on $[x_1, \infty)$, either the equilibrium price is constant on $[x_1, \infty)$ or decreases strictly on $[x_1, \infty)$ but in either case, it converges to ρ_{ns}. By Proposition 3.9, Pareto domination succeeds iff (3.18) holds. Since the left-hand side of (3.18) decreases in ρ, and since the equilibrium price decreases in x, the set on which Pareto domination fails is either empty or equal to $[\widehat{x},\infty)$ where $\widehat{x} > X_0\,(g/h)$. Since $\rho^*\,(x,G)$ is continuous in x and converges to ρ_{ns}, it must be that ρ_{ns} satisfies (3.18) in the former case and that ρ_{ns} fails (3.18) in the latter case. By Theorems 3.1, 3.2, 3.3, and 3.4, this failure is equivalent to the failure ofeither (3.2) or (3.3). (ii) By Proposition 3.6, the equilibrium price decreases strictly in t for any $x > X_0\,(g/h)$ and G^t that do not satisfy (3.9). By Theorems 3.3 and 3.6, the equilibrium price is constant in t for any $x > X_0\,(g/h)$ and G^t that does satisfy 3.9. As above, we can conclude that, as t increases, the set on which Pareto domination fails is either empty or equal to $\left[\widehat{t},\infty\right)$ where $\widehat{t}$ and G do not satisfy (3.9). Since, in the limit, $\overline{I}_{\lim} = \underline{I}_{\lim}$, the limiting income of the marginal person must equal $\overline{I}_{\lim} = \underline{I}_{\lim}$ so that either the above set is empty and the limiting equilibrium price satisfies (3.18) or the set is non-empty and the limiting equilibrium price does not satisfy (3.18) in which case (3.19) and (3.20) must hold. ■

Proof. (COROLLARY 3.14): If $\underline{\rho}$ satisfies (3.19) where $\underline{\rho}$ solves (3.20) for $I = 1$, then construct the family $\{G^t\}_{t\in[1,\infty)} \in F_G$ as follow. If the lower limit of the support of G is greater than 1, then decrease the lower limit of the support of distribution continuously until it hits 1. Once the lower limit equals 1, decrease the upper limit of the support of the distribution continuously until it hits 1. The result follows from Theorem 3.13. Now suppose that there exists a family $\{G^t\}_{t\in[1,\infty)} \in F_G$ for which Pareto domination fails for $t > \widehat{t}$. Let $G^{\widehat{t}}$ be denoted by $\widehat{G}$. In this case, given the transformation $\widehat{G}$ of G, the equilibrium price $\widehat{\rho}$ satisfies (3.19) where $\widehat{\rho}$ solves (3.20) for a marginal individual $I_m > 1$ who is indifferent between buying 0 and buying some positive amount at the equilibrium price $\widehat{\rho}$. Since by Proposition 3.6, for any family $\{G^t\}_{t\in[1,\infty)} \in F_{\widehat{G}}$ the equilibrium price is non-increasing in t, we know that we can, by decreasing the upper and lower limits of the support of $\widehat{G}$, construct a family $\{G^t\}_{t\in[1,\infty)} \in F_{\widehat{G}}$ for which all Pareto domination fails for all distributions in the family $\{G^t\}_{t\in[1,\infty)} \in F_{\widehat{G}}$. The result follow by Theorem 3.13 since $\overline{I}_{\lim} = \underline{I}_{\lim} = 1$. ■

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