Notes on parts (a) and (i) of the proof in
"Equilibrium in Hotelling's model of spatial competition"

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1. Explanation for part (a)

The payoff to player i if player i chooses a price p that is less than $a_j - z$ equals p (see payoff function on page 915). This payoff increases in p. If $a_i < a_j - z$, then $K_i(p, F_j) = p$ for all p in $(a_i, a_j - z)$ but this increases in p which violates a_i being the minimum price in the support of F_i. Thus, for each player, a_i must be greater than or equal to $a_j - z$ for $\{i, j\} = 1, 2$. This implies that $a_1 \geq a_2 - z$ and $a_2 \geq a_1 - z$ which implies that $a_1 \leq a_2 + z$ and $a_2 \leq a_1 + z$, which implies (a). A similar argument holds for the inequality about the end points of the supports.

2. Explanation for the first part of part (i)

If player i chooses a price p that is greater than $p_j - z$ and less than $p_j + z$ then the the payoff to player i (see middle portion of payoff function on page 915) increases in p so long as p is less than $(p_j + m_i)/2$ and decreases thereafter. We note that from (d) equilibrium profit is positive so that $b_i \leq b_j + z$. Thus, in the case that (i) is false, i.e. if b_i is greater than $(b_j + m_i)/2$ then there exists p that lies between the two that results in a higher payoff (from the middle portion of the payoff function on page 915) for player i than when player i plays b_i, contradicting the fact that b_i is used in equilibrium. Thus, (i) must be true.

3. Arguments about V in paragraphs below (i)

To give an idea of the arguments about V used in the two paragraphs that begin below (i), consider the case in which $x_1 = \frac{1}{3}$, $x_2 = \frac{1}{3}$, $a_j = \frac{1}{3}$. In this case, $z = \frac{1}{3}$ and $m_i = 1$. The best dominator of p depends on p.

- If $p < \min(2(a_j - x_i), 2(a_j + x_i)/3) = 0$, then the best dominator is $(p + z + m_i)/2 = p/2 + \frac{2}{3}$.
- If $2(a_j - x_i) = 0 < p < \frac{2}{3} = 2x_i$, then the best dominator is $a_j + z = \frac{2}{3}$.
- If $\max(2x_i, \frac{2}{3}(a_j + x_i)) = \frac{2}{3} < p$, then the best dominator is $a_j + m_i - p = \frac{4}{3} - p$.

We can use these results to find the smallest undominated p, which also depends on a_j, x_i and x_j.

Since $2(1+x_j) - 4\sqrt{x_j} = \frac{8}{3} - 4/(\sqrt{3}) > 0 = \min(2(a_j - x_i),\, 2(a_j + x_i)/3)$, $(a_j + z)(m_i - a_j)/(2 - z - a_j) = \frac{2}{3} = 2x_i$, and $a_j = \frac{1}{3} < \frac{2}{3} = 2x_i$, the smallest undominated p is $(a_j + z)(m_i - a_j)/(2 - z - a_j) = \frac{2}{3}$. That is, $V_i(a_j) = \frac{2}{3}$ when $x_i = \frac{1}{3}$, $x_j = \frac{1}{2}$, $a_j = \frac{1}{3}$.

Now consider the case in which $x_i = \frac{1}{3}$, $x_j = \frac{1}{2}$, $a_j = \frac{2}{3}$. In this case, $z = \frac{1}{3}$, $m_i = 1$, as above, and the best dominator of p depends on p.

- If $p < \min(2(a_j - x_i), 2(a_j + x_i)/3) = \frac{2}{3}$, then the best dominator is $\frac{1}{2}(p + z + m_i) = \frac{1}{2}p + \frac{2}{3}$.
- If $2(a_j - x_i) = \frac{2}{3} < p < \frac{2}{3} = 2x_i$, then the best dominator is $a_j + z = 1$.
- If $\max(2x_i, 2(a_j + x_i)/3) = \frac{2}{3} < p$, then the best dominator is $a_j + m_i - p = \frac{5}{3} - p$.

We can use these results to find the smallest undominated p, which also depends on a_j, x_i and x_j.

Since $2(1+x_j) - 4\sqrt{x_j} = \frac{8}{3} - 4/(\sqrt{3}) < \frac{2}{3} = \min(2(a_j - x_i), 2(a_j + x_i)/3)$, the smallest undominated p is $2(1+x_j) - 4\sqrt{x_j} = \frac{8}{3} - 4/(\sqrt{3})$. That is, $V_i(a_j) = \frac{8}{3} - 4/(\sqrt{3})$ when $x_i = \frac{1}{3}, x_j = \frac{1}{2}, a_j = \frac{2}{3}$.

Thus,

$$V_i(a_j) = \begin{cases} \frac{2}{3} & \text{if } x_i = \frac{1}{3},\ x_j = \frac{1}{2}, \text{ and } a_j = \frac{1}{3} \\ \frac{8}{3} - 4/\sqrt{3} & \text{if } a_j = \frac{2}{3}. \end{cases}$$

Thus, if $a_i = \frac{2}{3}$ and $a_j = \frac{1}{3}$, then $\frac{2}{3} = a_i \geq V_i(a_j) = \frac{2}{3}$ but $\frac{1}{3} = a_j \leq V_j(a_i) = \frac{8}{3} - 4/(\sqrt{3})$. We need to find all a_j for which $a_i = \frac{2}{3} \geq V_i(a_j)$ and $a_j \geq V_j(a_i)$ for some a_j. So far, we have shown the if $a_i = \frac{2}{3}$, then $a_j = \frac{1}{3}$ doesn't work. We need to do this for all a_j for a given a_i. Then, we need to find the smallest a_i for which $a_i \geq V_i(a_j)$ and $a_j \geq V_j(a_i)$. We note that $\gamma_i = \min\{(1 + \frac{1}{3}(x_i - x_j), 2(1 - x_j), 3(1 - x_i) - x_j)\} = \min(1, \frac{4}{3}, \frac{5}{3}) = 1$.