Modifications in argument when $x_i = 0$ for some i for "Equilibrium in Hotelling's model of spatial competition"

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(see top of page 921)

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These notes relate to the February 1985 version of the paper.

(i) The first argument uses $x_j > 0$ (via (e)). Addition: Suppose $x_j = 0$. Then $m_j = 1 - x_i = z$, so that $(\beta_i + m_j)/2 < \beta_i + z$ [Figure (i3)]. Hence by the arguments above with the indices reversed we have $\beta_j \leq (\beta_i + m_j)/2$ (if $\beta_i - z \leq (\beta_i + m_j)/2$) or $\beta_j \leq \beta_i - z$ (if $(\beta_i + m_j)/2 < \beta_i - z$), both of which contradict $\beta_i \leq \beta_j - z$ (given that $\beta_j \geq \alpha_j > 0$ (or alternatively $\beta_j > 0$ because the equilibrium is not pure)).

(f) If $x_i = 0$ then $\beta_i \leq (\beta_j + m_i)/2$ (see (g)) and $\beta_j \leq \beta_i + z$ (see (a)) imply that $\beta_i \leq m_i + z = 2z$, so the result follows.

(g) Begin: "If $x_i = 0$ there is nothing to prove. If $x_i > 0$ and ...". Conclude F_j has no support in $(p - z, p - z + \varepsilon)$. Then: "If $x_j = 0$ then $K_i(\cdot, F_j)$ is increasing on $(p, \min(p + \varepsilon, 2x_i))$ if $p < 2x_i$, contradicting the fact that p is an atom of F_i (which implies that $K_i(p, F_j)$ is equal to the equilibrium profit of i)."

Then: if $x_j > 0$, F_j has no support in $(p + x, p + z + \delta)$ either ... so K_i is increasing on $(p, \min(p + \varepsilon, p + \delta, 2x_i))$ if $p < 2x_i$.

(j) After third sentence: "First consider the case $x_i > 0$ for $i = 1$, 2. Then ...".

After first paragraph, insert:

Second, consider the case in which $x_i = 0$ and $x_j > 0$. Then, as above, F_j has no support in $(\overline{p} + z, \overline{p} + z + \varepsilon)$ for some $\varepsilon > 0$, though it may have support in $(\overline{p} - z, \overline{p} - z + \varepsilon)$. However, an explicit calculation (see below), using the fact that $x_i = 0$, shows that if F_j has support in $(\overline{p} - z, \overline{p} + z)$ then $K_i(\cdot, F_j)$ is still strictly concave on $(\overline{p}, \overline{p} + \delta)$ for some $\delta > 0$, and the argument follows the lines of the previous paragraph.

Last, if $x_i = 0$ and $x_j = 0$, or if $x_i = 0$ for $i = 1$, 2, then the required strict concavity also follows from an explicit calculation.

(o) Discussion of $x_j = 0$ case omitted.

Explicit calculation to show strict concavity of $K_i(\cdot, F_j)$

If $\overline{p} < p < \overline{p} + \varepsilon$,

$$K_i(p, F_j) = \tfrac{1}{2} \int_{p-z}^{\overline{p}+z} p(q - p + m_i)\, dF_j(q) + \int_{\overline{p}+z+\varepsilon}^{\beta_j} p\, dF_j(q)$$

$$\begin{aligned}
\frac{d}{dp} K_i(p, F_j) &= -\tfrac{1}{2} p(-z + m_i) F_j'(p - z) + \tfrac{1}{2} \int_{p-z}^{\overline{p}+z} (q - 2p + m_i)\, dF_j(q) + \int_{\overline{p}+z+\varepsilon}^{\beta_j} dF_j(q) \\
&= \tfrac{1}{2} \int_{p-z}^{\overline{p}+z} (q - 2p + m_i)\, dF_j(q) + \int_{\overline{p}+z+\varepsilon}^{\beta_j} dF_j(q) \qquad \text{(using } x_i = 0\text{)} \\
\frac{d^2}{dp^2} K_i(p, F_j) &= \tfrac{1}{2}(p - z - 2p + m_i) F_j'(p - z) + \tfrac{1}{2} \int_{p-z}^{\overline{p}+z} (-2)\, dF_j(q) \\
&= -\tfrac{1}{2} p F_j'(p - z) - \int_{p-z}^{\overline{p}+z} dF_j(q).
\end{aligned}$$