Details of Derivation of Equation (1) and of Computation of Endpoints of Intervals in Support of Equilibrium Strategy for "Equilibrium in Hotelling's Model of Spatial Competition", (Econometrica 55 (1987) 911-922)

by

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We restrict attention to Region T1c (the others are similar).

For $a_i < p < b_j - z$ we have

$$2K_i(p, F_j) = \int_{a_j}^{p+z} p(q - p + m_i)dF_j(q) + \int_{p+z}^{b_j} pdF_j(q).$$

We need the first derivative of this with respect to p to be zero, so, assuming F_j to be differentiable, we need

$$-2px_jF_j'(p+z) - (2 - m_i + 2p)F_j(p+z) + 2 + \int_{a_j}^{p+z} qdF_j(q) = 0.$$

For this to hold, it is necessary that its derivative be everywhere zero, which is equivalent to

$$-2px_jF_j''(p+z) - (p + 4x_j)F_j'(p+z) - 2F_j(p+z) = 0.$$

A solution of this takes the form

$$F_j(q) = B_j\exp\left(-(q-z)/2x_j\right) \int_{A_j}^{q} h(s, x_j, z)ds \quad \text{for} \quad a_i+z < q < b_j,$$

where A_j and B_j are constants.

A similar argument for $a_j + z < p < b_i$ shows that

$$F_j(q) = 1 - C_j\exp\Big((q+z)/2x_i\Big)\int_{-D_j}^{-q} h(s,\ x_i,\ z)ds \quad \text{for} \quad a_j < q < b_i-z,$$

where C_j and D_j are constants.

Given that F_j takes these forms, we find that $K_i(p,\ F_j) = B_jx_j$ if $a_i < p < b_j - z$ and $K_i(p,\ F_j) = C_jx_i$ if $a_j + z < p < b_i$. Thus for equality of the payoffs of the two intervals (a necessary condition for equilibrium) we need

$$(1) \qquad B_jx_j = C_jx_i.$$

Now, the condition that the second derivative of the payoff on each interval in the support of the equilibrium strategy be zero means that the first derivative is constant. Using the form of F_j found above, the condition that the derivative of the payoff on $(a_i,\ b_j-z)$ be zero is

$$(2) \quad 2C_jx_i\Big[1/(a_j+z) - 1/b_i - 1/a_i\Big] + 2 + a_j - b_i + z + (b_i-a_i-2)F_j(a_i+z) = 0,$$

and the condition that the derivative of the payoff on $(a_j+z,\ b_i)$ be zero is

$$(3) \quad 2C_jx_i\Big[1/(b_j-z) - 1/b_i - 1/a_i\Big] + 2 + b_j - b_i - z - 2\delta_jx_j + (b_i-a_i-2)F_j(a_i+z) = 0,$$

where δ_j is the size of the atom in F_j at b_j. Further, in order for F_j to be a distribution function we need

$$(4) \qquad F_j(a_j) = 0,$$

$$(5) \qquad F_j(b_j) = 1 - \delta_j,$$

$$(6) \qquad F_j(b_i-z) = F_j(a_i+z).$$

Finally, in order that the payoff to i be nonincreasing to the right of b_i

we need $G_j'(b_i-z) \le 0$. Hence we must have

$$(7) \qquad G_j'(b_i-z) = 0.$$

We can carry out an entirely analagous argument for F_i, yielding seven more conditions analagous to (1) through (7). Thus we have 14 equations in the 14 variables $(a_k, b_k, A_k, B_k, C_k, D_k, \delta_k)$ $(k = i, j)$. Using (1) we can eliminate B_k for $k = 1, 2$, and using (4) and (5) we can eliminate A_k and D_k for $k = i, j$. We now have the following 8 equations in the 8 unknowns $(a_k, b_k, C_k, \delta_k)$ $(k = i, j)$: (2), (3),

$$(8) \qquad 2C_jx_i = b_i^2\left\{\exp\left[(b_i-z-a_j)/2x_i\right] - C_j\exp\left[b_i/2x_i\right]\int_{-(b_i-z)}^{-a_j} h(s, x_i, z)ds\right\}$$

(condition (7) after substitution),

$$(9) \qquad \begin{aligned} &1 - \exp\left[(b_i-z-a_j)/2x_i\right] + C_j\exp\left[b_i/2x_i\right]\int_{-(b_i-z)}^{-a_j} h(s, x_i, z)ds \\ &\quad = (1 - \delta_j)\exp\left[(b_j-a_i-z)/2x_j\right] - (C_jx_i/x_j)\exp\left[-a_i/2x_j\right]\int_{a_i+z}^{b_j} h(s, x_j, z)ds \end{aligned}$$

(condition (6) after substitution), and the analogs with i and j reversed.

Now (8) can be solved for C_j in terms of the other variables:

$$(10) \quad C_j = \left[b_i^2\exp\left[(b_i-z-a_j)/2x_i\right]\right]/\left[2x_i + b_i^2\exp\left[b_i/2x_i\right]\int_{-(b_i-z)}^{-a_j} h(s, x_i, z)ds\right].$$

Let

$$(11) \quad G_j = F_j(a_i+z) = 1 - \exp\left[(b_i-z-a_j)/2x_i\right] + C_j\exp\left[b_i/2x_i\right]\int_{-(b_i-z)}^{-a_j} h(s, x_i, z)ds,$$

and similarly let $G_i = F_i(a_j+z)$. Now we can solve (3) for δ_j in terms of the other variables:

$$\delta_j = \left[1/2x_j\right]\left[2C_jx_j\left(1/(b_j-z) - 1/b_i - 1/a_i\right) + 2 + b_j - b_i - z + (b_i-a_i-2)G_j\right].$$

This reduces the system to the four equations (2), (9), and their analogs with i and j reversed in the four variables (a_k, b_k) $(k = i, j)$. Finally, if we multiply (2) by a_i we have a quadratic in a_i, which can be solved to yield

$$a_i = \left[X \pm (X^2 - 8C_jG_jx_i)^{1/2}\right]/2G_j,$$

where

$$X = 2C_jx_i\left(1/(a_j+z) - 1/b_i\right) + 2 + a_j - b_i + z + (b_i-2)G_j,$$

and C_j and G_j are defined in terms of a_j, b_i, and b_j by (10) and (11). Thus we are left with equation (9) and its analog with i and j reversed, and the analog of equation (2) with i and j reversed--a total of three equations in the three variables a_j, b_i, and b_j. This is the system to which we computed an approximate solution.

In the other regions, similar arguments can be made; in each case the system can be reduced to at most two equations in two unknowns.