Details of Arguments in Appendix 1 of

"Equilibrium in Hotelling's Model of Spatial Competition",

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by

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Appendix 1: Proofs

Here we prove Propositions 1, 2, and 3, and establish some additional conditions which equilibria of $\Gamma(x_1, x_2)$ must satisfy. First we restrict the values a_i and b_i of the endpoints of the supports of any equilibrium strategies, by using domination arguments. For example, $K_i(a_j-z, p_j) > K_i(p_i, p_j)$ for any price $p_i < a_j - z$, for <u>all</u> $p_j \geq a_j$; hence $a_i \geq a_j - z$ (see (a) below). For those locations in P, these restrictions together imply that $a_i = b_i = 1 + (x_i-x_j)/3$ for i = 1, 2, proving Proposition 1. They also imply that $b_i - a_i \leq 2z$ for every (x_1, x_2) in S, so that the second sentence of Proposition 3 follows from the first.

To complete the proof of Proposition 3, we show that the prices at which the equilibrium strategies can have atoms lie in a restricted set. For example, if F_i has an atom at p, then F_j does not have an atom at p − z or at p + z (since j can do strictly better by charging slightly lower prices). In particular, we show that if $b_i - a_i \leq 2z$ then each equilibrium strategy can have an atom only at b_i (see (j) below); this leads fairly directly (see (k) through (o)) to the conclusion that every equilibrium is of type T. Finally, the straightforward proof of Proposition 2 (which uses domination arguments again) is given in (f).

In our proof, we repeatedly use the following properties of the payoff function K_i (see (1) and Figures 2 and 4):

for fixed p_j, K_i is linear in p_i (with slope 1) on $[0, p_j-z)$, jumps down at p_j-z (if $x_j > 0$), is strictly concave (with slope less than 1) on (p_j-z, p_j+z), jumps down at p_j+z (if $x_i > 0$), and is zero above p_j+z;

for fixed p_j, the maximum of K_i, if attained in (p_j-z, p_j+z), is attained at $p_i = (p_j + m_i)/2$;

for fixed p_i, K_i is zero on $[0, p_i - z)$, linear in p_j (with slope $p_i/2$) on $(p_i - z, p_i + z)$, and constant in p_j above $p_i + z$. (Most of our arguments are easy to follow when reference is made to a diagram like Figure 4; space constraints prohibit the inclusion of all the appropriate diagrams.)

For any set Q of prices of firm i, we say that p_i^{**} <u>strongly dominates</u> p_i^* <u>on</u> Q when $K_i(p_i^{**}, p_j) > K_i(p_i, p_j)$ for all $p_j \in Q$ and all p_i in an open neighborhood of p_i^*. We write $A_i = [a_i, b_i]$; if p_i^{**} strongly dominates p_i^* on A_j then p_i^* is not in the support of any equilibrium strategy of i.

If (F_1, F_2) is an equilibrium of $\Gamma(x_1, x_2)$ then $a_i \geq 0$ for $i = 1, 2$ since each firm can guarantee a profit of zero by setting a price of zero. We also have the following.

(a) $a_j - z \leq a_i \leq a_j + z$ <u>and</u> $b_j - z \leq b_i \leq b_j + z$ <u>for</u> $i = 1, 2$: Since $p_i = a_j - z$ strongly dominates any lower price on A_j we have $a_i \geq a_j - z$ and hence $a_j - z \leq a_i \leq a_j + z$. If $z > 0$, or $z = 0$ and $b_j > 0$, then $K_i(p, F_j) = 0$ if $p > b_j + z$, while $K_i(p, F_j) > 0$ for some $p \leq b_j + z$ (for example for $p = z/2$ if $z > 0$), so that $b_i \leq b_j + z$. If $z = 0$ and $b_j = 0$ then $K_j(F_j, F_i) = 0$, so that $b_i = 0$ (otherwise $K_j(p, F_i) > 0$ for some $p > 0$). Hence in both cases $b_j - z \leq b_i \leq b_j + z$.

(b) <u>If</u> $p > 0$ <u>is an atom of</u> F_i <u>and</u> $x_i > 0$ ($x_j > 0$) <u>then</u> $p - z$ ($p + z$) <u>is not an atom of</u> F_j: Under these conditions, the profit of j jumps down at $p - z$ ($p + z$), so this cannot be an atom of F_j.

(c) <u>If</u> $z = 0$ <u>and</u> b_i <u>exists for some</u> i, <u>or if every consumer has a finite reservation price, then</u> $a_i = b_i = 0$ <u>for</u> $i = 1, 2$ (i.e. <u>the only equilibrium is pure, each firm charging the price zero</u>): If b_i exists then

b_j exists and $b_j = b_i$ (by (a)). Let $b_i = b_j = b$. If $b > 0$ then it is not an atom of both F_i and F_j (by (b), since $z = 0$ means that $x_i > 0$ for some i). Suppose b is not an atom of F_j. Then $K_i(b, F_j)$ is equal to i's equilibrium profit (see (d) of Fact (B) in Osborne and Pitchik [1984]). But $K_i(b, F_j) = 0$, while $K_i(p, F_j) > 0$ for $0 < p < b$. Hence we must have $b = 0$. If every consumer has a finite reservation price then $K_i(p, q) = 0$ for all q if p is large enough, say if $p \geq \bar{p}$. Hence $b_i \leq \bar{p}$; the argument above establishes that $b_i = b_j = 0$.

This proves the second sentence of Proposition 1. From now on, we assume that $z > 0$.

(d) $a_i > 0$ for $i = 1, 2$, and the equilibrium profit of each firm is positive: This follows from the fact that firm i can guarantee a positive profit by setting the price $z/2$.

(e) If $x_j > 0$ and $b_i = b_j - z$ then b_i is an atom of F_i and b_j is not an atom of F_j: If b_i is not an atom of F_i then $K_j(b_j, F_i) = 0$ is the equilibrium profit of j (see (d) of Fact (B) in Osborne and Pitchik [1984]). This contradicts (d), so that b_i is an atom of F_i, and so b_j is not an atom of F_j (by (b)).

In the special cases in which $x_i = 0$ for some i, the proofs of some of the subsequent results require additional arguments (to avoid the use of (b) and (e), for example); since the length of these arguments is out of proportion to their significance, we omit them. Thus, in all the proofs below we assume that $x_i > 0$ for $i = 1, 2$.

The next result implies Proposition 2.

(f) $b_i \to 0$ <u>for</u> $i = 1, 2$ <u>as</u> $z \to 0$: If $K_j(b_i - z, b_i) < K_j(b_i - 3z, b_i)$, then every $p_j > b_i - z$ is strongly dominated (by $p_j - 2z$) on A_i, so that $b_j = b_i - z$. But then b_j is dominated (by $b_i - 3z$) on A_i, so that b_j is not an atom of F_j, contradicting (e). Hence $K_j(b_i - z, b_i) \geq K_j(b_i - 3z, b_i)$, or $(b_i - z)(b_i - (b_i - z) + m_j)/2 \geq b_i - 3z$, or $b_i \leq (2 + x_i)z/x_i$, from which the result follows (recall that we are assuming $x_i > 0$).

(g) <u>If</u> p <u>is an atom of</u> F_i <u>then</u> $p \geq 2x_i$: If p is an atom of F_i then $K_j(\cdot, F_i)$ jumps down at $p - z$ (if $p > z$), and at $p + z$, so that supp F_j contains no point in $(p - z, p - z + \varepsilon)$ or in $(p + z, p + z + \varepsilon)$ for some $\varepsilon > 0$. But then $K_i(\cdot, F_j)$ is increasing on $(p, \min(p + \varepsilon, 2x_i))$ if $p < 2x_i$, contradicting the fact that p is an atom of F_i.

(h) <u>If</u> $p \in$ supp F_i <u>is not an atom of</u> F_i <u>then either</u> $p - z \in$ supp F_j <u>or</u> $p + z \in$ supp F_j: If neither $p - z$ nor $p + z$ is in supp F_j then supp F_j contains no point in $(p - z - \varepsilon, p - z + \varepsilon)$ or in $(p + z - \varepsilon, p + z + \varepsilon)$ for some $\varepsilon > 0$. Now, since $p \in$ supp F_i, we have $p + z \geq a_i + z \geq a_j$ and $p - z \leq b_i - z \leq b_j$ (by (a)), so that $a_j \leq p + z - \varepsilon$ and $b_j \geq p - z + \varepsilon$. Hence supp F_j intersects $(p - z + \varepsilon, p + z - \varepsilon)$, so that, given the other restrictions on supp F_j, $K_i(\cdot, F_j)$ is strictly concave on $(p - \varepsilon, p + \varepsilon)$. Hence p is isolated, and therefore an atom of F_i.

(i) $b_i \leq (b_j + m_i)/2$ <u>for</u> $i = 1, 2$, <u>and hence</u> $b_i \leq \gamma_i \equiv \min(1 + (x_i - x_j)/3, 2(1 - x_j), 3(1 - x_i) - x_j)$ <u>for</u> $i = 1, 2$: We first show that $(b_j + m_i)/2 \geq b_j - z$. If not, then $p_i = b_j - z$ strongly dominates any higher price on A_j, so that $b_i = b_j - z$ (by (a)). Further, $b_i - \varepsilon$ dominates b_i

(though not strongly) on A_j, for some $\varepsilon > 0$, so that b_i is not an atom of F_i, contradicting (e). Two cases remain. If $(b_j + m_i)/2 \geq b_j + z$ then the result follows from (a). If $b_j - z \leq (b_j + m_i)/2 \leq b_j + z$, then $(b_j + m_i)/2$ dominates any higher price on A_j, so that $b_i \leq (b_j + m_i)/2$ for $i = 1, 2$. Combining these two inequalities yields $b_i \leq 1 + (x_i - x_j)/3$; combining $b_i \leq (b_j + m_i)/2$ and $b_j \leq b_i + z$ (see (a)) yields $b_i \leq 2(1 - x_j)$ and $b_j \leq 3(1 - x_j) - x_i$.

Now, for each a_j, let $U_i(a_j)$ be the lowest price of firm i which is not strongly dominated on $[a_j, \gamma_j] \supset A_j$ (the inclusion from (i)). Obviously then we must have $a_i \geq U_i(a_j)$ for $i = 1, 2$; these restrictions are helpful below. The form of U_i can be found by using the fact that if p_i is less than $\min(2(a_j - x_i), 2(a_j + x_i/3))$ then the best potential dominator is $(p_i + z + m_i)/2$ (i.e. if any price dominates p_i, then this one does), while if p_i is between $2(a_j - x_i)$ and $2x_i$ then the best potential dominator is $a_j + z$, and if p_i exceeds $\max(2x_i, 2(a_j + x_i)/3)$ then the best potential dominator is $a_j + m_i - p$. The details are very messy, and we do not give them here. Obviously, U_i is nondecreasing; an example is shown in Figure 6.

By combining the conditions $a_i \geq U_i(a_j)$ and $b_i \leq \gamma_i$ for $i = 1, 2$, we can obtain two useful restrictions on the nature of equilibria of $\Gamma(x_1, x_2)$. Let a_i^* be the minimal value of a_i such that $a_i \geq U_i(a_j)$ and $a_j \geq U_i(a_i)$ for some a_j. Then $a_i \geq a_i^*$ in any equilibrium of $\Gamma(x_1, x_2)$. Thus if $a_i^* = \gamma_i$ for $i = 1, 2$ (as is the case in Figure 6) then the pure equilibrium $(p_1, p_2) = (\gamma_1, \gamma_2)$ is the only possible equilibrium of $\Gamma(x_1, x_2)$. A very tedious analysis of the functions U_i $(i = 1, 2)$ (the details of which we omit) shows that this is so for every (x_1, x_2) in P; this completes the proof of Proposition 1. Also, if $\gamma_i - a_i^* \leq 2z$ then we know that $b_i - a_i \leq 2z$. This is

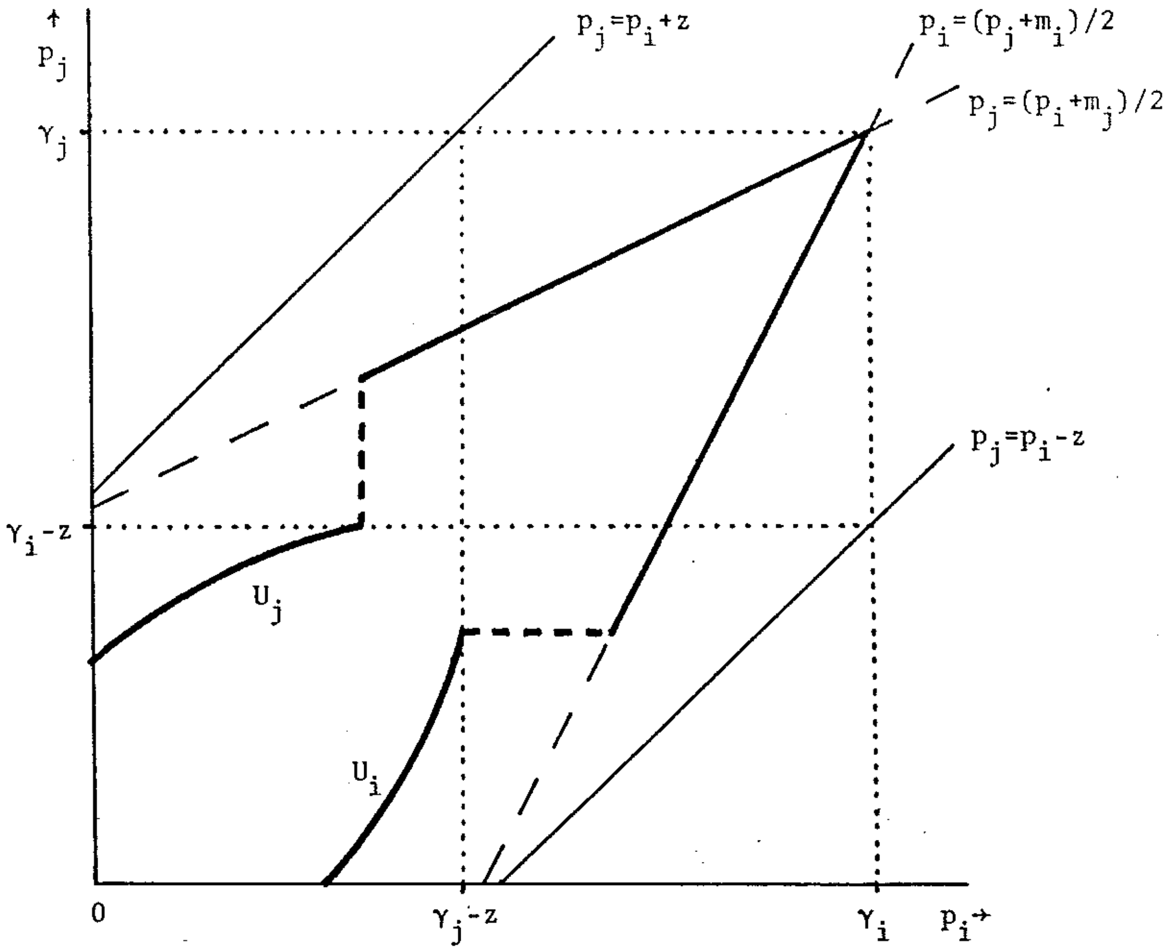


Figure 6: The functions U_1 and U_2.

For each value of α_j, $U_i(\alpha_j)$ is the smallest price of i which is not strongly dominated.

useful because our subsequent results use the assumption that $b_i - a_i \leq 2z$; we show that the only equilibria satisfying this assumption are of type T. A computation shows that $\gamma_i - a_i^* \leq 2z$ for $i = 1, 2$ whenever (x_1, x_2) is in S (see Figure 3). Thus the second sentence of Proposition 3 follows from the first (given the existence result of Dasgupta and Maskin [1982]), which remains to be proved. <u>From now on, we assume that</u> $b_i - a_i \leq 2z$ <u>for</u> $i = 1, 2$.

(j) <u>If</u> p <u>is an atom of</u> F_i <u>then</u> $p = b_i$: Suppose $\bar{p}$ is an atom of F_i. Then $K_i(\bar{p}, F_j)$ is equal to the equilibrium profit of i. We argue first that $K_i(\cdot, F_j)$ is decreasing on $(\bar{p}, \bar{p} + \varepsilon)$ for some $\varepsilon > 0$. Since $K_j(\cdot, F_i)$ jumps down at $\bar{p} - z$ and $\bar{p} + z$, F_j has no support in $(\bar{p} - z, \bar{p} - z + \varepsilon)$ or in $(\bar{p} + z, \bar{p} + z + \varepsilon)$ for some $\varepsilon > 0$. Since $a_i \leq \bar{p} \leq b_i$, this means that $a_j \leq \bar{p} + z$ and $b_j \geq \bar{p} - z + \varepsilon$ (using (a)). Hence F_j has some support in $[\bar{p} - z + \varepsilon, \bar{p} + z]$. But then $K_i(\cdot, F_j)$ is strictly concave on $(\bar{p}, \bar{p} + \varepsilon)$; it is continuous at $\bar{p}$ (since neither $\bar{p} - z$ nor $\bar{p} + z$ are atoms of F_j (by (b))), so it is decreasing on $(\bar{p}, \bar{p} + \varepsilon)$ (since $K_i(\bar{p}, F_j)$ is equal to i's equilibrium profit).

Now, since $K_i(\cdot, F_j)$ is decreasing on $(\bar{p}, \bar{p} + \varepsilon)$, F_i cannot have any support in this interval. Assume that $\bar{p} < b_i$, and let $\hat{p}$ be the smallest price above $\bar{p}$ which is in supp F_i. Since $\bar{p} \geq 2x_i$ (by (g)) and $b_i \leq 2(1-x_j)$ (by (i)), we have $b_i \leq \bar{p} + 2z$; since $b_i \geq \hat{p}$ and $b_i - a_i \leq 2z$, we have $a_i \geq \hat{p} - 2z$. Therefore $K_j(\cdot, F_i)$ is strictly concave on $(\bar{p} - z, \hat{p} - z)$ and on $(\bar{p} + z, \hat{p} + z)$ (since $b_i \geq \hat{p}$), so that the support of F_j in these intervals can consist of at most a single isolated point in each interval, at which F_j has an atom. Let these points be q_1 and q_2, let the size of the atom in F_j at q_k be $J(q_k)$, and let

$$f(p) = [p(q_1 - p + m_i)/2]J(q_1) + \int_{\hat{p}-z}^{\bar{p}+z} (p(q - p + m_i)/2)dF_j(q)$$

$$+ pJ(q_2) + \int_{\hat{p}+z}^{b_j} pdF_j(q).$$

It is easy to check that f is concave. Also, it is immediate that $K_i(p, F_j) = f(p)$ if $p < p < \min(q + z, q - z)$, so that, by the argument above, f is decreasing in this range. The concavity of f implies, therefore, that it is decreasing for all $p > \bar{p}$.

We now argue that $K_i(p, F_j) \leq f(p)$ for all $\bar{p} \leq p \leq \hat{p}$. This implies that $\hat{p}$ is not in the support of F_i, contrary to our assumption, so that we have $\bar{p} = b_i$, completing the proof. First, note tht $K_i(\cdot, F_j)$ jumps down at $q_1 + z$ and at $q_2 - z$. Second, observe that the expression for $K_i(p, F_j)$ is similar to that for $f(p)$, except that if $q_1 + z < p \leq \hat{p}$ then the term in square brackets is zero, while if $q_2 - z < p \leq \hat{p}$ then the multiplier of $J(q_2)$ is $p(q_2 - p + m_i)/2$ (rather than p). Now, F_j can have an atom at q_1 only if $q_1 \geq 2x_j$ (see (g)), in which case $q_1 - p + m_i \geq 2x_j - (1 + (x_i - x_j)/3) + m_i$ $= 2x_i/3 + 4x_j/3$ if $p \leq \hat{p} \leq 1 + (x_i - x_j)/3$ (see (i)). Hence $p(q_1 - p + m_i) \geq 0$ for all $p \leq \hat{p}$. Finally, if $q_2 - z < p$ then $p(q_2 - p + m_i)/2 < p(1 - x_j) < p$. So $K_i(p, F_j) \leq f(p)$ for all $\bar{p} \leq p \leq \hat{p}$.

(k) <u>If</u> $a_i < b_j - z$ <u>then</u> $[a_i, b_j - z] \subset \text{supp } F_i$: Suppose $a_i < p < b_j - z$ with $p \notin \text{supp } F_i$. By (a) we know that $p < b_i$, so there exist smallest numbers $\varepsilon > 0$ and $\delta > 0$ such that $p - \varepsilon \in \text{supp } F_i$ and $p + \delta \in \text{supp } F_i$. Now, since $p - z - \varepsilon < b_j - 2z$ we have $p - z - \varepsilon < a_j$ (given that $b_j - a_j \leq 2z$); since $p - \varepsilon$ is not an atom of F_i (by (j)) we have $p + z - \varepsilon \in \text{supp } F_j$ (by (h)). Also, if $p + \delta < b_j - z$ then $p + \delta < b_i$ by (a) and hence (again using $b_j - a_j \leq 2z$) we have $p + z + \delta \in \text{supp } F_j$ by (h). Since $b_j \in \text{supp } F_j$ by definition, we have r_j

$\equiv \min(b_j, p + z + \delta) \in \text{supp } F_j$. Now, since $b_i \leq a_i + 2z$ and $b_i \geq b_j - z$ (by (a)) we know that $K_j(\cdot, F_i)$ is strictly concave on $(p + z - \epsilon, r_j)$. But then j's profit on some subset of $(p + z - \epsilon, p + z + \delta)$ exceeds its profit at one of the endpoints of this interval. Since the latter must equal its equilibrium profit, the gap in supp F_i is not compatible with equilibrium.

(l) <u>If</u> $a_j + z < b_i$ <u>then</u> $[a_j + z, b_i] \subset \text{supp } F_i$: This follows from an argument similar to that in (k).

(m) <u>If</u> $b_j - z < a_j + z$ (i.e. <u>if</u> $b_j - a_j < 2z$) <u>then</u> supp $F_i \cap (b_j - z, a_j + z) = \emptyset$ <u>or</u> $\{b_i\}$: If $p \in \text{supp } F_i$ and $b_j - z < p < a_j + z$ then p is an atom of F_i by (h), so that $p = b_i$ by (j).

(n) <u>If</u> $a_i > a_j - z$ <u>for</u> $i = 1, 2$ <u>then</u> $b_j > a_i + z$: Since a_i is not an atom of F_i (by (j)) we have $a_i + z \in \text{supp } F_j$ (by (h)). Hence $b_j \geq a_i + z$. If $b_j = a_i + z$ then $b_j - a_j < 2z$ (since $a_j > a_i - z$), so that a_i is an isolated member of supp F_i (by (m), using $b_j - z = a_i$), contradicting (j).

(o) <u>If</u> $a_i > a_j - z$ <u>for</u> $i = 1, 2$ <u>then</u> b_j <u>is an atom of</u> F_j <u>if and only if</u> $b_j - a_j < 2z$: If $b_j - a_j < 2z$ and b_j is not an atom of F_j then $b_j - z \in \text{supp } F_i$ by (h) (since $b_j + z > a_i + 2z \geq b_i$). Since a_j is not an atom of F_j, we also have $a_j + z \in \text{supp } F_i$. But then $K_i(\cdot, F_j)$ is continuous and strictly concave on $[b_j - z, a_j + z]$, which means that i's profit cannot be maximized at both endpoints, where it must attain its equilibrium profit. Hence b_j is an atom of F_j. Now assume that $b_j - a_j = 2z$. Then $a_i < a_j + z$ $(= b_j - z) < b_i$ (the second inequality by (n)), so that by (k) and (l) we have supp $F_i = [a_i, b_i]$. If F_j has an atom at b_j then $K_i(\cdot, F_j)$ jumps down at

$a_j + z$, contradicting the (a.e.) constancy of $K_i(\cdot, F_j)$ on supp F_i. So b_j is not an atom of F_j.

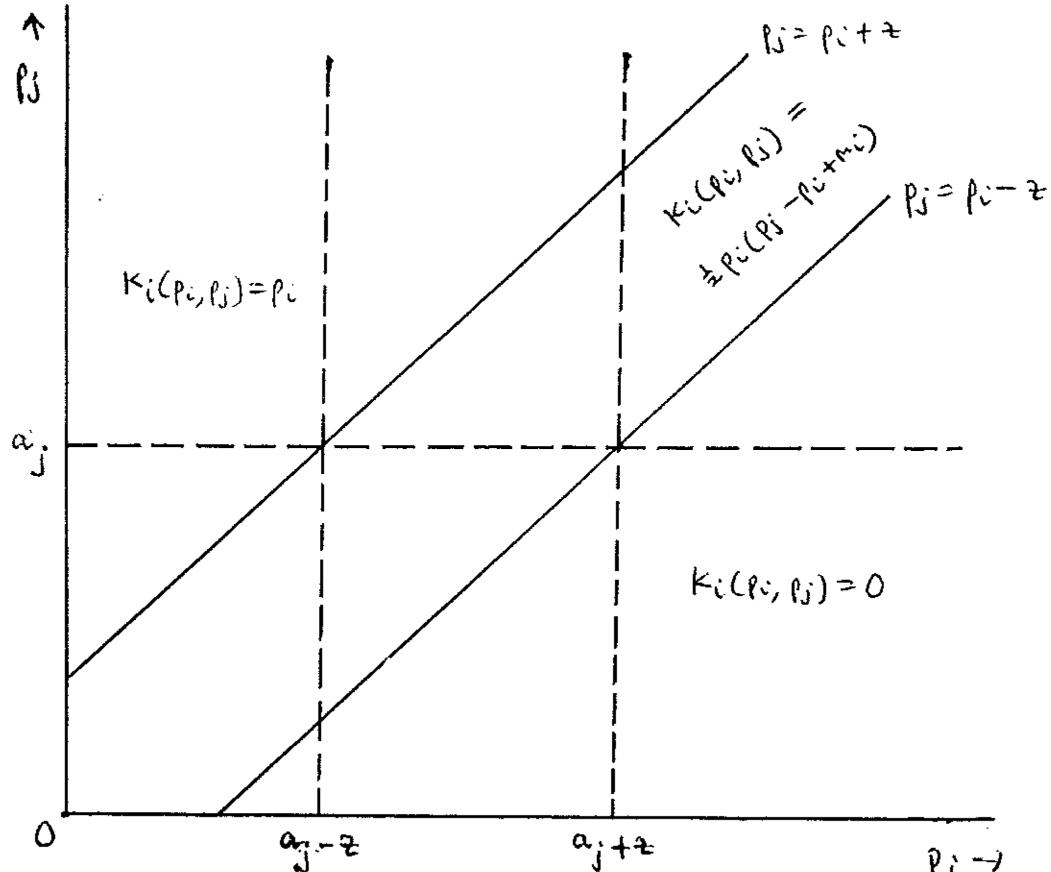
We can now show that every equilibrium of $\Gamma(x_1, x_2)$ in which $b_i - a_i \leq 2z$ is of type T.

<u>Type T1</u>: If $a_i > a_j - z$ for $i = 1, 2$ then (n), (k), (l), and (m) imply that supp $F_i = [a_i, b_j - z] \cup [a_j + z, b_i]$ for $i = 1, 2$. By (o), b_i is an atom of F_i if and only if $b_i - a_i < 2z$.

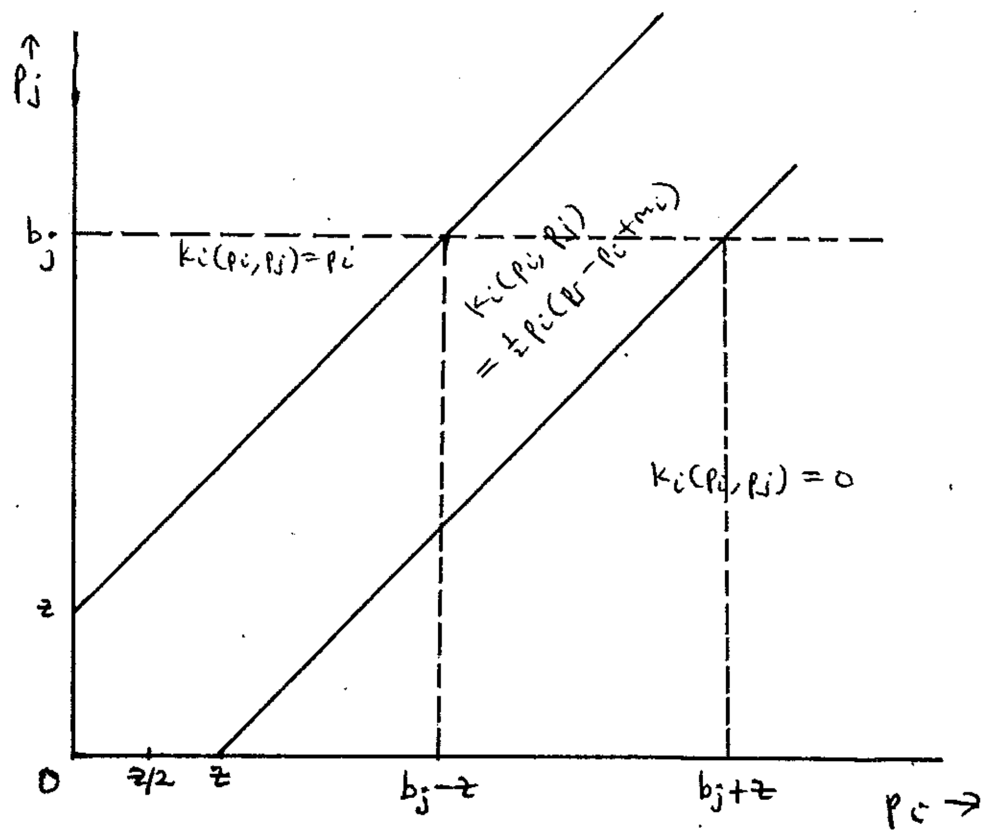
<u>Type T2</u>: If $a_i = a_j - z$ then (k), (l), and (m) imply that supp $F_j = [a_j, b_j]$, and supp $F_i = [a_j - z, b_j - z]$ or $[a_j - z, b_j - z] \cup \{b_i\}$. In the first case $b_j - z$ is an atom of F_i and b_j is not an atom of F_j by (e); in the second case b_j is an atom of F_j (otherwise i's payoff in $(b_j - z, b_i)$ exceeds that at $b_j - z$ and at b_i, as in the proof of (o)), and b_i is an atom of F_i.

(We can further refine these results by using the constraints on a_i and b_i to rule out some sorts of equilibria for particular ranges of (x_1, x_2). For example, if (x_1, x_2) is in some range around (0.27, 0.27) then any equilibrium is either of type T1 with $b_i - a_i < 2z$ for $i = 1, 2$ or of type T2 with $b_i > b_j - z$.

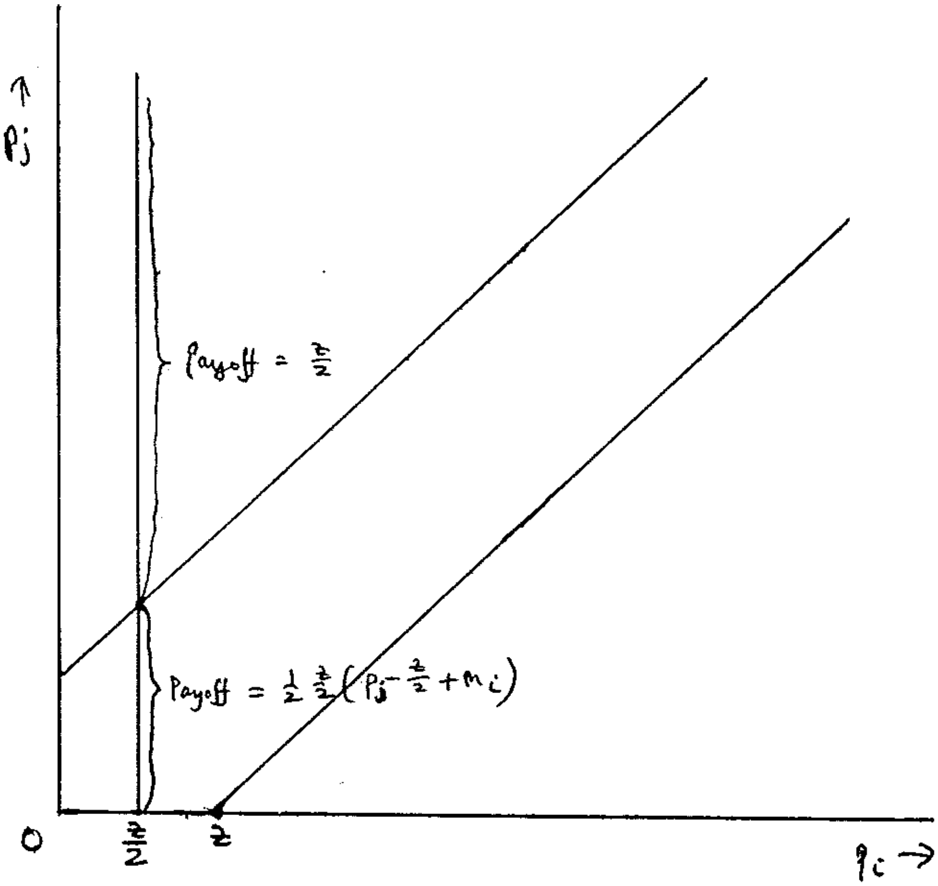
DIAGRAMS FOR PROOFS IN APPENDIX 1



(a) — First sentence

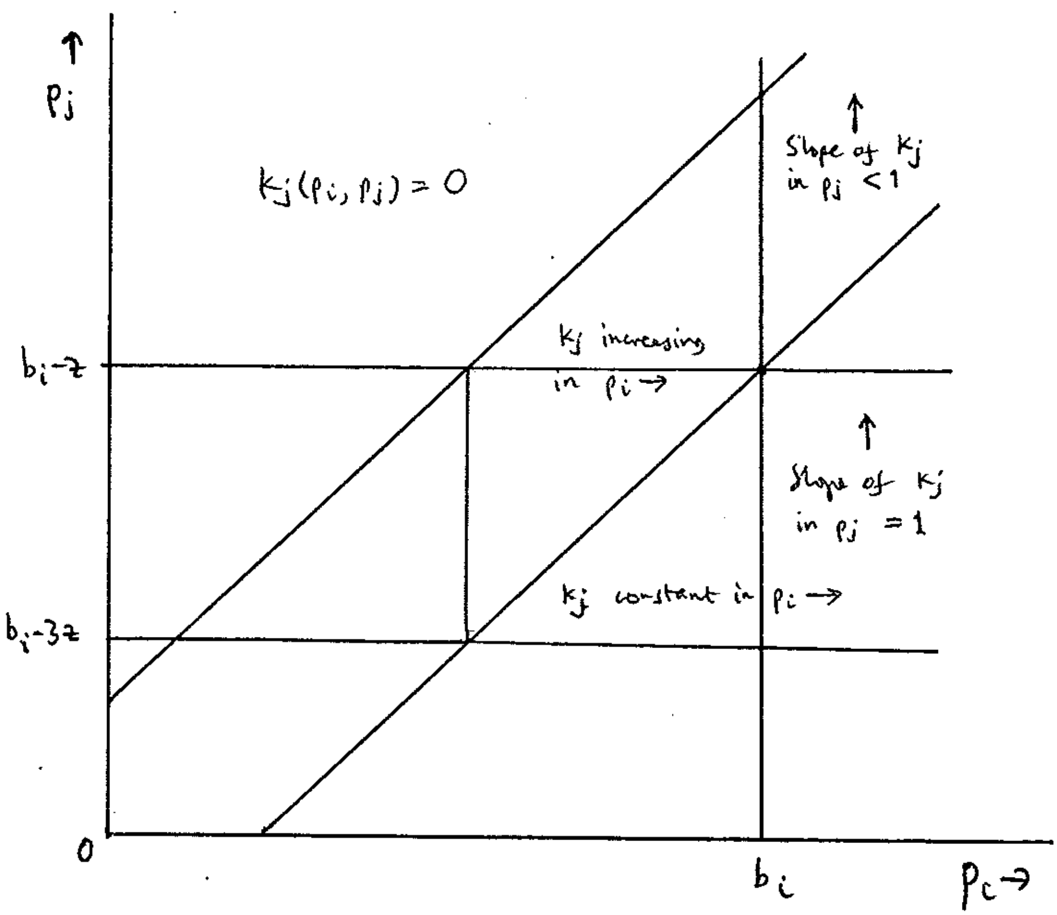


(a) — Second sentence



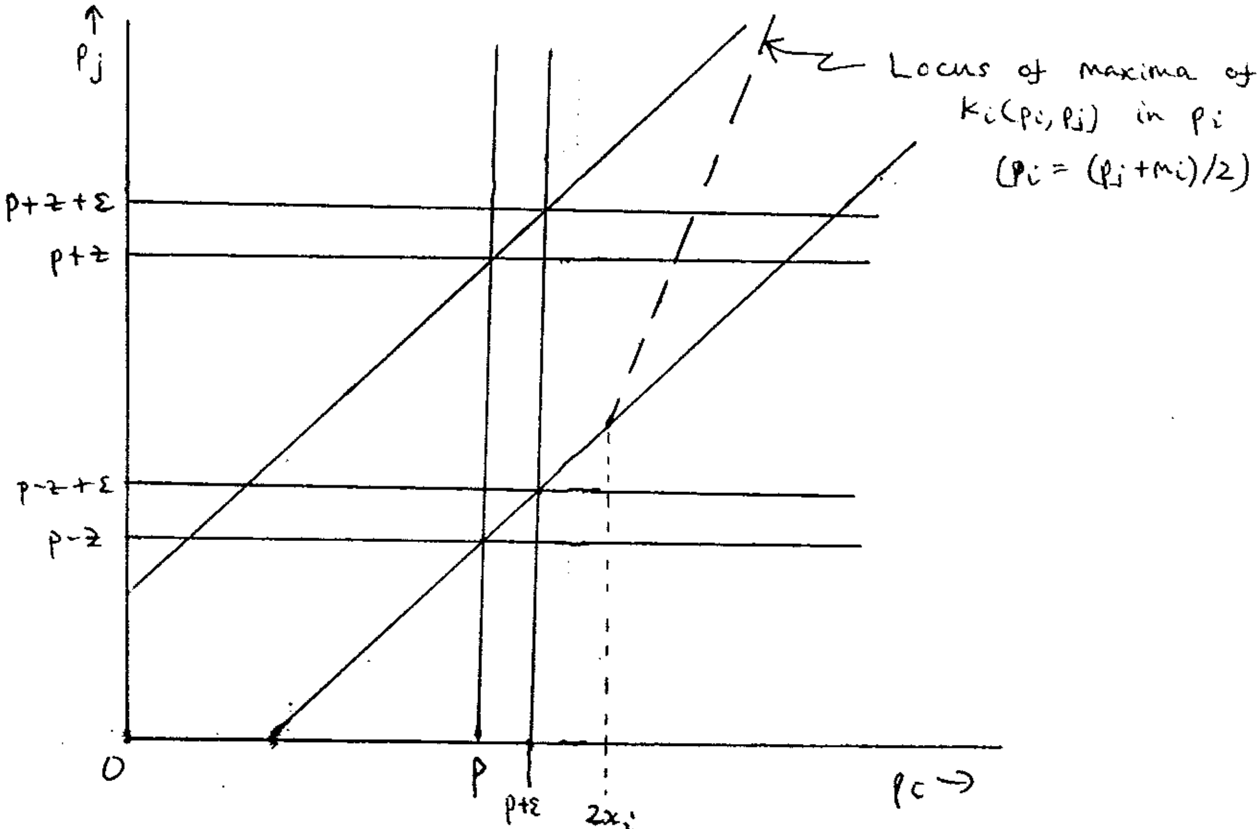
p_j
Payoff = $\frac{z}{2}$
Payoff = $\frac{1}{2}\frac{z}{2}(p_j - \frac{z}{2} + m_i)$
0
$\frac{z}{2}$
z
p_i

(d)

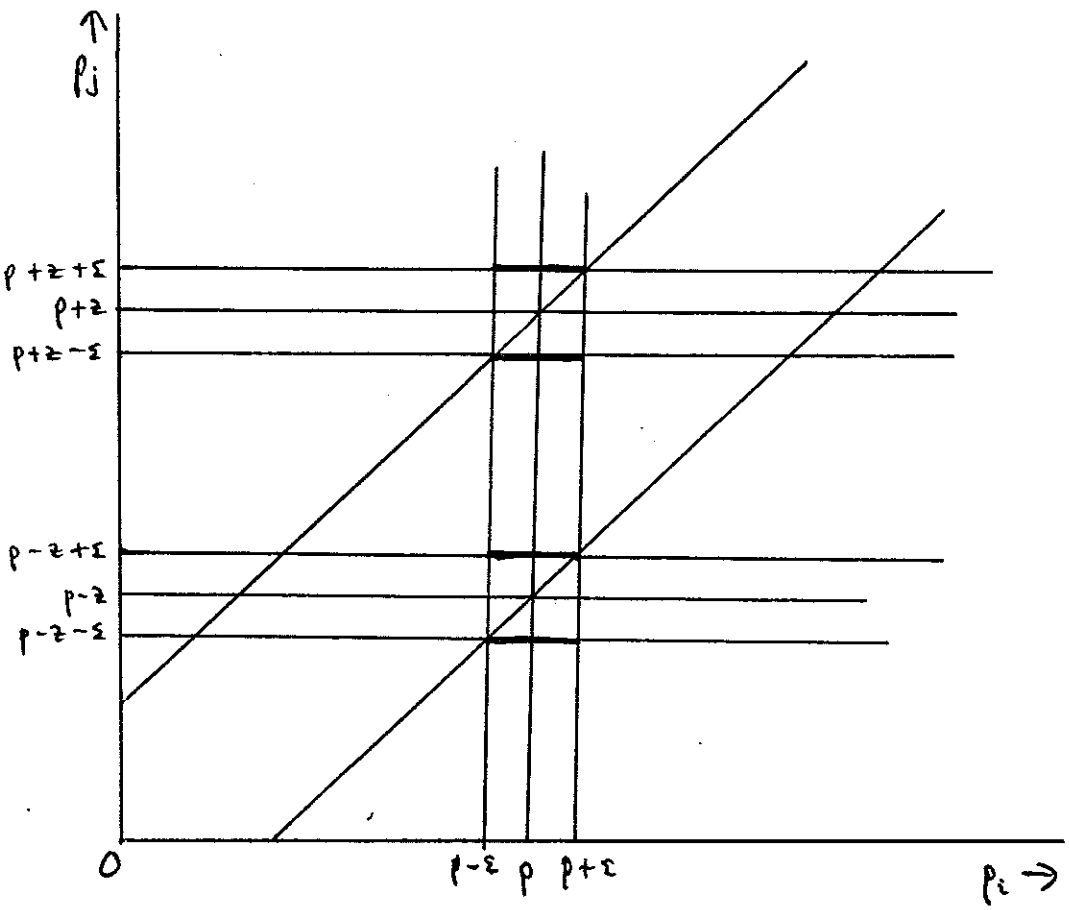


p_j
$k_j(p_i, p_j) = 0$
Slope of k_j in p_j < 1
k_j increasing in p_i
Slope of k_j in p_j = 1
k_j constant in p_i
$b_i - z$
$b_i - 3z$
0
b_i
p_i

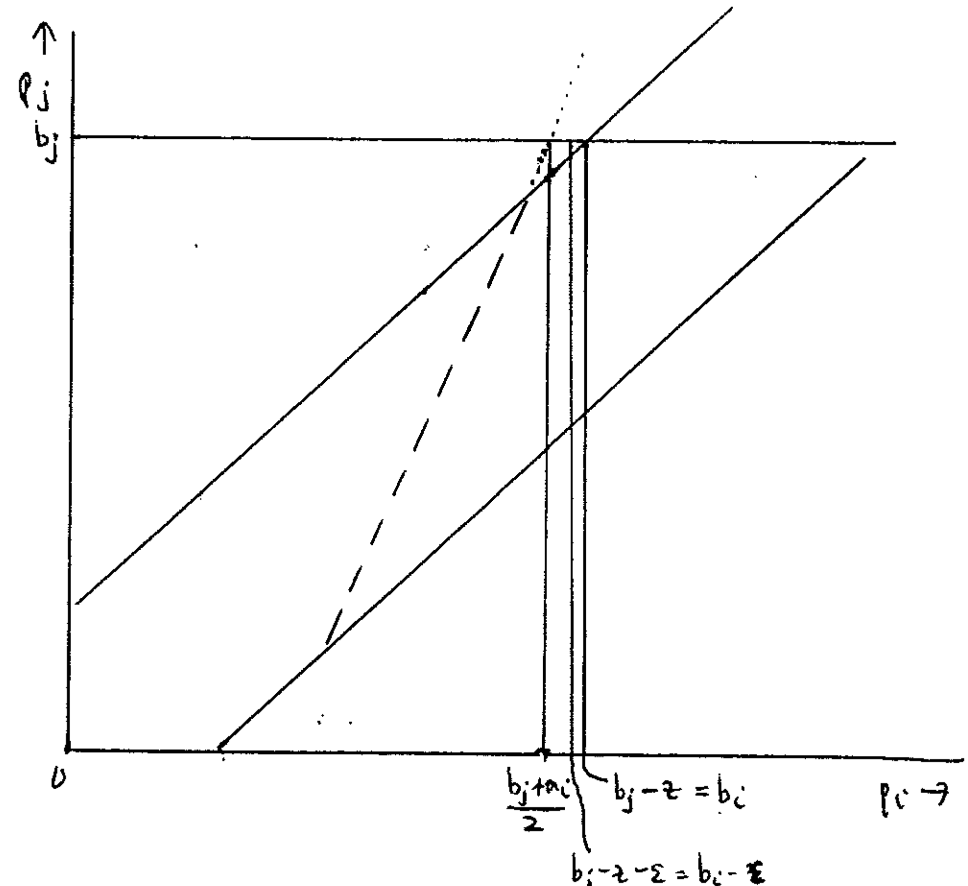
(f)


p_j
Locus of maxima of $K_i(p_i, p_j)$ in p_i $(p_i = (p_j + m_i)/2)$
$p+z+\varepsilon$
$p+z$
$p-z+\varepsilon$
$p-z$
0
p
$p+\varepsilon$
$2x_i$
p_i

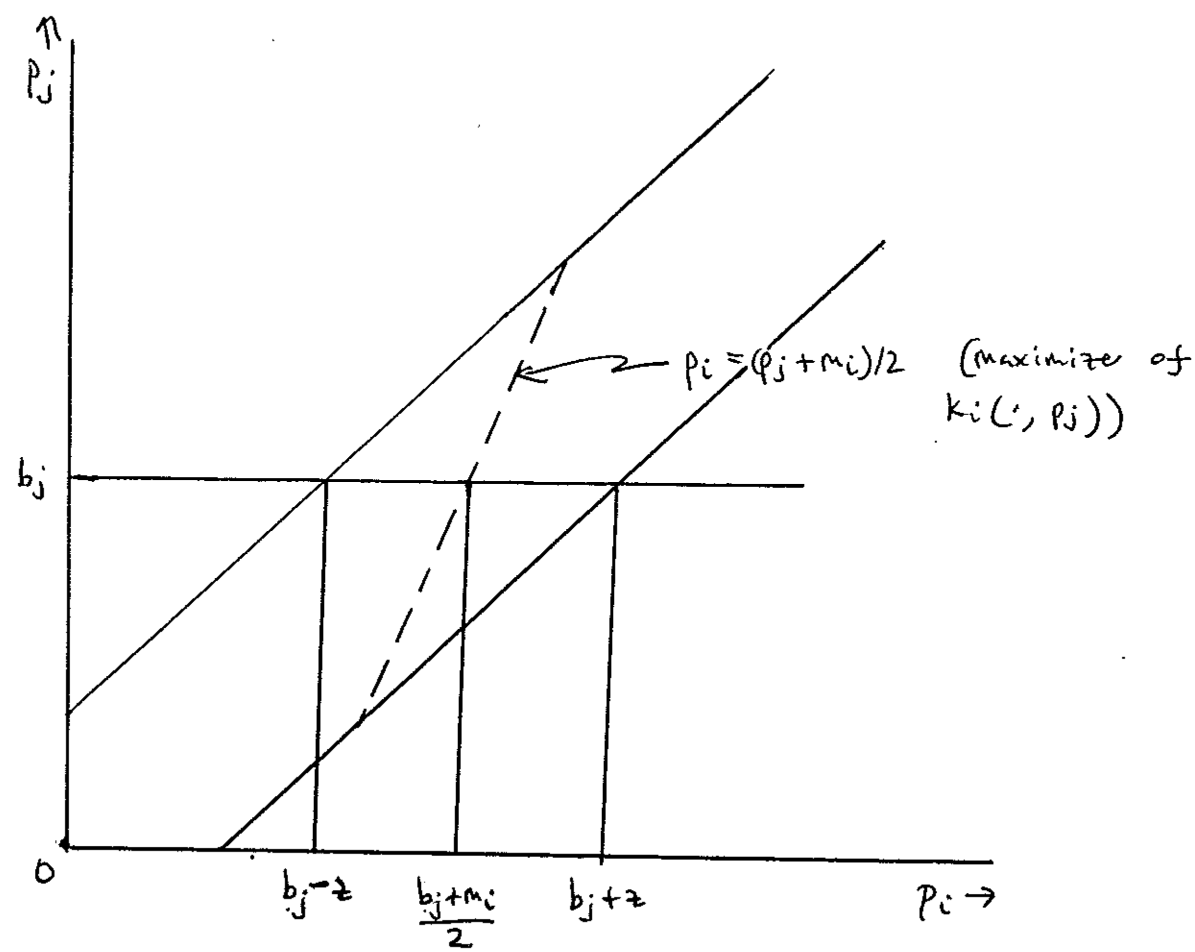
(g)


p_j
$p+z+\varepsilon$
$p+z$
$p+z-\varepsilon$
$p-z+\varepsilon$
$p-z$
$p-z-\varepsilon$
0
$p-\varepsilon$ p $p+\varepsilon$
p_i

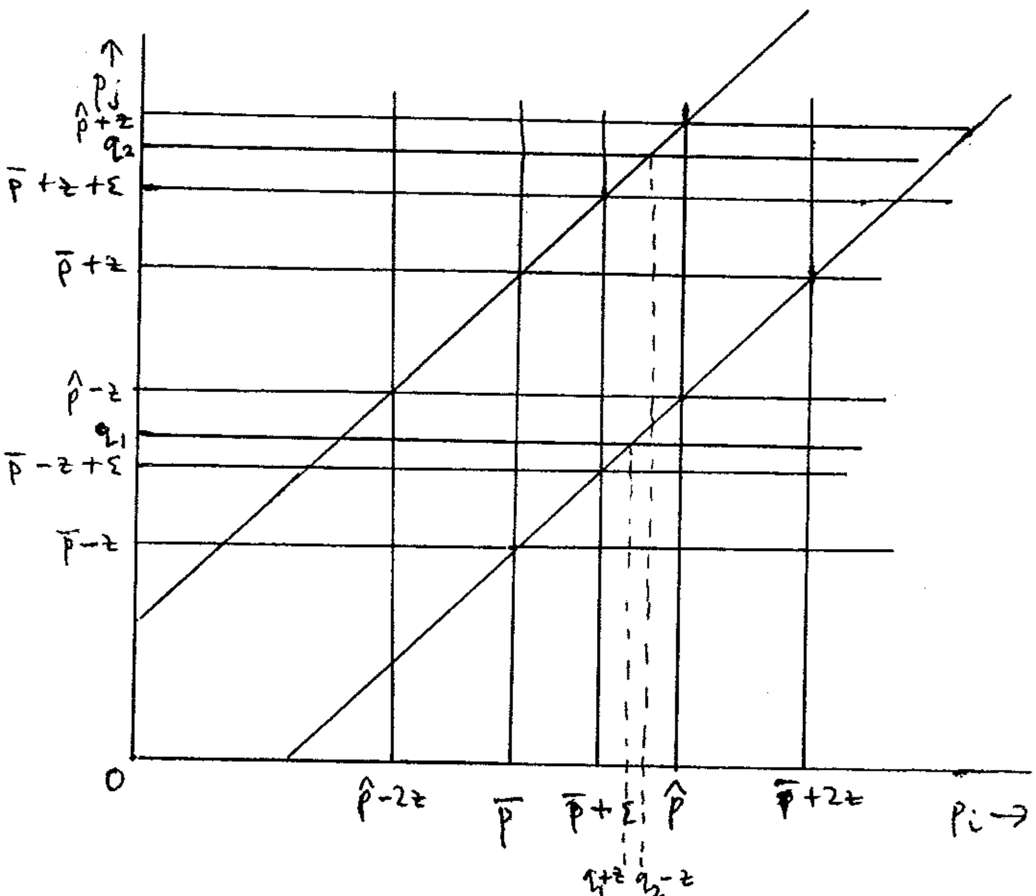
(h)



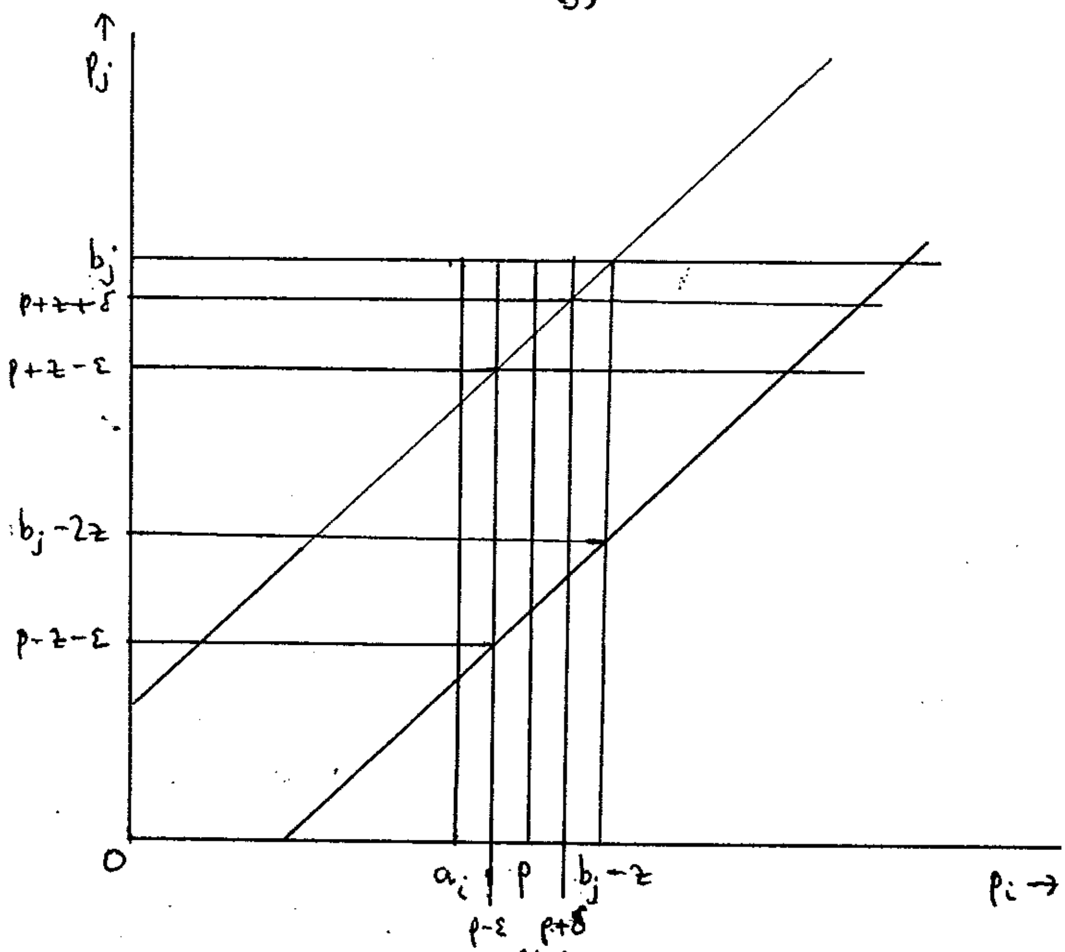
(i) — Second sentence



(i) — Sixth sentence


p_j
$\hat{p}+z$
q_2
$\bar{p}+z+\varepsilon$
$\bar{p}+z$
$\hat{p}-z$
q_1
$\bar{p}-z+\varepsilon$
$\bar{p}-z$
0
$\hat{p}-2z$
$\bar{p}$
$\bar{p}+\varepsilon$
$\hat{p}$
$\bar{p}+2z$
p_i
q_1+z
q_2-z

(j)


p_j
b_j
$p+z+\delta$
$p+z-\varepsilon$
b_j-2z
$p-z-\varepsilon$
0
a_i
p
b_j-z
p_i
$p-\varepsilon$
$p+\delta$

(k)