EQUILIBRIUM IN HOTELLING'S MODEL OF SPATIAL COMPETITION*

by

Martin J. Osborne and Carolyn Pitchik

Department of Economics
Columbia University
New York
NY 10027

and

Department of Economics
McMaster University
Hamilton
Canada L8S 4M4

Department of Economics
University of Toronto
150 St. George Street
Toronto
Canada M5S 1A1

(Please address correspondence to Canadian addresses.)

Revised February 1985

*We are grateful to Andrew Muller, David Kreps, and two anonymous referees for helpful comments and encouragement, and to Shmuel Zamir for providing us with the details of some work of his (with Yuval Shilony) on this topic. Osborne's research was partially supported by grants from the Columbia University Council for Research in the Social Sciences, and from the National Science Foundation (SES-8318978). Most of Pitchik's research was done while she was at New York University, where it was partially supported by grants from the Office of Naval Research (contract N0014-78-C-0598), the National Science Foundation (SES-8207765), and the C.V. Starr Center for Applied Economics.

Abstract

Using a partly analytical, partly computational approach we find and study a mixed strategy equilibrium in Hotelling's model of spatial competition (in which each of two firms chooses a location in a line segment, and a price). In the equilibrium we find, the firms randomize only over prices. They choose locations close to the quartiles of the market. The support of the equilibrium price strategy of each firm is the union of two short intervals, and has an atom of approximate size 0.73 at the highest price. The equilibrium can be interpreted as one in which the firms charge a relatively high price most of the time, and occasionally hold a "sale".

1. Introduction

Hotelling [1929] formulated a natural model of the choice of location and price by firms. He assumed that consumers are distributed uniformly over a line segment, and travel, at a constant cost per unit, to a firm to buy one unit of a good. Each of two firms chooses a location and a price, and each consumer chooses to buy from the firm for which price plus travel cost is lowest. Hotelling's work has given rise to a large literature, partly because the "location" variable may be given a number of interpretations--for example, as the quantity of some "characteristic" possessed by a good. However, under Hotelling's original assumptions his model has not been completely solved.

Hotelling assumed that the firms know the outcome of price competition at each pair of locations; given the location of its opponent, each firm chooses its position to maximize its profit, taking into account the dependence of the equilibrium prices on locations. (That is, Hotelling sought a pure strategy subgame perfect equilibrium of the extensive game in which the firms first simultaneously choose locations, then simultaneously choose prices.) Intuition suggests that in equilibrium, some distance will separate the firms: if they locate at the same point then by the standard argument of Bertrand, price will be driven down to unit cost, while if they separate, the forces of competition should be weaker, and they should be able to earn positive profits. Hotelling made this argument (see his p. 54), but nevertheless claimed that there is an incentive, under his assumptions, for each firm to locate very close to the other.

However, Hotelling's analysis is flawed, as is pointed out by d'Aspremont, Gabszewicz, and Thisse [1979]. (Lerner and Singer [1937] and

Smithies [1941], in early analyses, questioned Hotelling's conclusion, and studied variations in the model and solution, but apparently did not perceive Hotelling's error.) The problem is that the pair of prices proposed by Hotelling is an equilibrium only if the firms are sufficiently far apart. In fact, when the firms are close, d'Aspremont et al. show that there is no price equilibrium in pure strategies.

There are a number of ways of avoiding this problem. For example, the model may be modified by making the space of location an infinite line, or a circle, rather than a line segment. Alternatively, the solution may be changed so that the firms choose their actions sequentially rather than simultaneously. There are contexts in which these modifications are reasonable, but there are also cases where Hotelling's assumptions seem to be appropriate. (The attribute of a product can frequently be associated with points in a line segment, and there is usually no natural sequence in which the firms should move.)

Our approach is to follow a standard game-theoretic route and allow the firms to randomize,[1] rather than restricting them to pure strategies. We retain all Hotelling's other assumptions, including the two-stage structure of the solution. In the equilibrium we find, the location choices are pure, while the firms randomize over prices. The equilibrium locations are consistent with the intuitive argument given above. On the line segment [0, 1], the firms locate approximately 0.27 from each endpoint, so that they are separated by nearly half the market. (Throughout, we measure the location of firm 1 by its distance from 0, and that of firm 2 by its distance from 1. Note that the equilibrium location pair (0.27, 0.27) is quite close to (0.25, 0.25), at which total transport cost is minimized.) If this is the only equilibrium, and the game is played a finite number of times, then the only subgame perfect equilibrium of the repeated game consists of independent repetitions of this equilibrium. Thus, in this case the model predicts that the firms will choose constant locations (or product charcteristics), while varying the price they charge from day to day.

The nature of this price variation is particularly interesting. Note that because of the assumption of constant unit travel cost, for each price p_i charged by firm i there is a critical price of firm j below which all consumers buy from j. We say that this critical price just "undercuts" p_i. The equilibrium price variation at a range of locations around the equilibrium location pair takes the following form. The firms randomize over two disjoint intervals of prices. Each firm chooses its price either from an interval immediately below a relatively high price (which it charges with positive probability), or from an interval immediately below the price which just undercuts its rival's highest price. Each of the prices in this lower interval is a best response to one of the prices in the rival's upper interval. Thus each firm either charges a relatively high price, or posts a "sale"[2] price which is a best response to one of its rival's high prices. At the location equilibrium the probability of holding such a sale is small (around 0.3%, or one day per year!); the equilibrium price strategy is given by the dashed line (labelled z=0.46) in Figure 1.

At location pairs far from the equilibrium, at least one firm randomizes over a single interval of prices, but high prices and low prices are still more likely than ones in between if the location pair is not too asymmetric. The equilibrium price strategies at a number of symmetric location pairs are shown in Figure 1. The equilibria vary smoothly with the

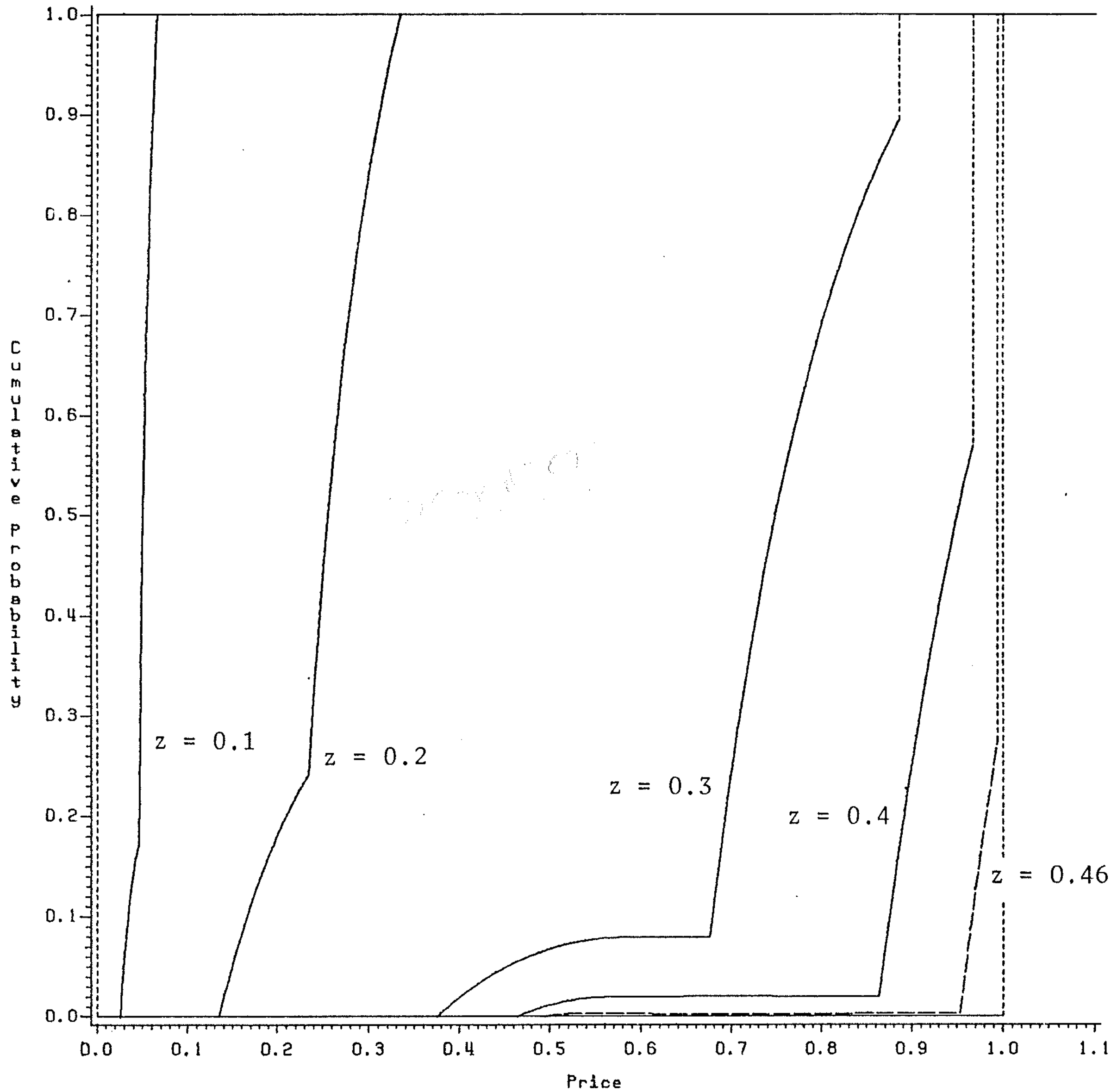


Figure 1: The equilibrium price strategies at a number of symmetric location pairs

The value of z is the distance between the firms. When z = 0 the equilibrium strategy for each firm is to set a price of 0 with probability one; when z = 0.5, it is to set a price of 1 with probability one. In the location equilibrium (0.27, 0.27), the value of z is 0.46.

location pair, and converge to the pure equilibria when the latter exist. As the firms get closer they first charge more variable prices, putting positive but decreasing weight on the highest price; then they charge less variable and lower prices, putting positive weight on no single price, until the highest price they charge approaches zero.

In order to analyze the equilibrium locations, it is necessary first to study the equilibrium price strategies at each location pair. It is known that for a range of location pairs there is a pure price equilibrium (the one found by Hotelling). We prove that for these location pairs there is no mixed equilibrium (with a minor qualification--see Proposition 1 in Section 3). The structure of the mixed equilibria at the remaining location pairs is quite complex, and a purely analytic approach appears to be impossible. For a subset of these location pairs (region S of Figure 3) we show that any equilibrium has a very specific form (see Proposition 3). We reduce the problem of calculating such an equilibrium to that of finding a solution to three (or, in some cases, two or one) highly nonlinear simultaneous equations in as many unknowns, and checking that the solution satisfies a number of inequalities.

We have computed an approximate solution to these equations (and have checked that it satisfies the inequalities), at a large number of location pairs, by using a grid search procedure. Although this procedure (the details of which are given in Appendix 2) is very (CPU-)time-consuming, it has the advantage that no solution is unreachable, in the sense that for each solution there is an initial grid such that the procedure converges to that solution. Since we used a variety of initial grids, we can thus be reasonably certain that the approximate solution we found is the only one. Further, there are two reasons why the approximate solutions, which constitute ε-equilibria for $\varepsilon < 10^{-7}$, are close to exact equilibria of the same type. First, in those cases in which the solution is a root of a single continuous function of one variable (region T2 of Figure 3), this follows (given the Intermediate Value Theorem) from checking that on each side of the approximate solution the function has different signs. When the simultaneous solution of more than one equation is involved, a complete analysis of this sort is more difficult (although an approximate solution which is not close to an exact solution is still, of course, a very "unlikely" occurence). Second, if, as seems likely, the approximate solution we found is the only one, then, given that an exact solution exists (which follows from the results of Dasgupta and Maskin [1982]), our approximate solution must be close to an exact one. In summary, at each location pair in S (which includes the locational equilibrium (0.27, 0.27)), there exists an equilibrium in prices of an specific form, and the evidence strongly suggests that this equilibrium is unique, and close to the ε-equilibrium we have found.

At location pairs which are outside S and at which there is no pure price equilibrium, our analysis is less complete. We show that any equilibrium satisfying a certain condition must have the same form as those in S (Proposition 3). We have found approximate equilibria of this type, using the techniques above. The first argument above again shows that (in region T2) the approximate equilibria are close to exact ones. Finally, although the evidence for a unique equilibrium is not as strong in this region as it is in S, we show that as the distance between the firms goes to zero, all equilibria approach the pure one in which each firm charges a price of zero (Proposition 2).

Having computed price equilibria at a large number of location pairs, we can construct the profit function of the location game. This game has a unique pure equilibrium (0.27, 0.27), as described above.

Many of our analytical results on the price equilibria (see Appendix 1) apply to a much more general model in which the distribution of consumers is nonuniform, the travel cost function is nonlinear, and the amount demanded by each consumer depends on price. Under assumptions close to those of Hotelling, presumably the equilibrium in locations is also similar; however, its precise form requires a separate computation.

2. The Model

Consumers are uniformly distributed on the line segment [0, 1]. We normalize the cost of travel to 1 per unit distance. Each of two firms chooses a location in [0, 1]. Let x_1 be the distance of firm 1 from 0, and let x_2 be the distance of firm 2 from 1. Since the problem is symmetric, we can assume that firm 1 locates to the left of firm 2 (i.e. $x_1 + x_2 \leq 1$). Let p_i be the price charged by firm i, and let $z = 1 - x_1 - x_2$, the distance between the firms. If $p_i < p_j - z$, all consumers buy from firm i, while if $p_j - z < p_i < p_j + z$, the fraction $x_i + (p_j - p_i + z)/2 = (p_j - p_i + 1 + x_i - x_j)/2$ does so. (Whenever the indices i and j appear in the same expression, $i \neq j$.) The division of consumers when $p_i = p_j - z$ is unimportant so long as firm i gets fewer customers than it does when p_i is slightly smaller than $p_j - z$ (and similarly when $p_i = p_j + z$); for convenience we assume that, given p_j, the profit of i is continuous in p_i from above. The cost of production is zero. For each pair of locations (x_1, x_2), let $\Gamma(x_1, x_2)$ be the game in which the strategies of the firms are prices. The payoff of firm i in $\Gamma(x_1, x_2)$ is

$$K_i(p_i, p_j) = \begin{cases} p_i & \text{if } p_i < p_j - z \\ p_i(p_j - p_i + m_i)/2 & \text{if } p_j - z \leq p_i < p_j + z \\ 0 & \text{if } p_j + z \leq p_i, \end{cases} \quad (1)$$

where $m_i = 1 + x_i - x_j$. (An example is shown in Figure 2.) Given the symmetry of the problem, we can use the equilibrium payoffs in $\Gamma(x_1, x_2)$ when $x_1 + x_2 \leq 1$ to define the equilibrium payoffs in $\Gamma(x_1, x_2)$ for any location pair (x_1, x_2). Let $E_i(x_i, x_j)$ be an equilibrium payoff of firm i thus defined. Then in the location game Γ, each firm i chooses x_i and receives the payoff $E_i(x_i, x_j)$. We seek a Nash equilibrium of Γ. That is, a pair of locations with the property that each firm, knowing how the equilibrium prices (and hence its profit) depend on locations, chooses a position which maximizes its profit given the position of its opponent. In order to find a Nash equilibrium of Γ, we first study the Nash equilibria of $\Gamma(x_1, x_2)$ for each pair (x_1, x_2) with $x_1 + x_2 \leq 1$.

3. Equilibrium in the Price-Setting Games

For those location pairs (x_1, x_2) at which $\Gamma(x_1, x_2)$ has a pure equilibrium, the following result provides a complete analysis.

Proposition 1: *If $(1 + (x_i - x_j)/3)^2 \geq 4(x_i + 2x_j)/3$ for $i = 1, 2$ (region P of Figure 3) then $\Gamma(x_1, x_2)$ has a unique equilibrium, which is pure, with $p_i = 1 + (x_i - x_j)/3$ and a profit for i of $(1 + (x_i - x_j)/3)^2$, for $i = 1, 2$. If $x_1 + x_2 = 1$ and every consumer has a finite reservation price then $\Gamma(x_1, x_2)$ has a unique equilibrium, which is pure, with $p_1 = p_2 = 0$ and profits of zero. For no other location pair is there a pure equilibrium.*

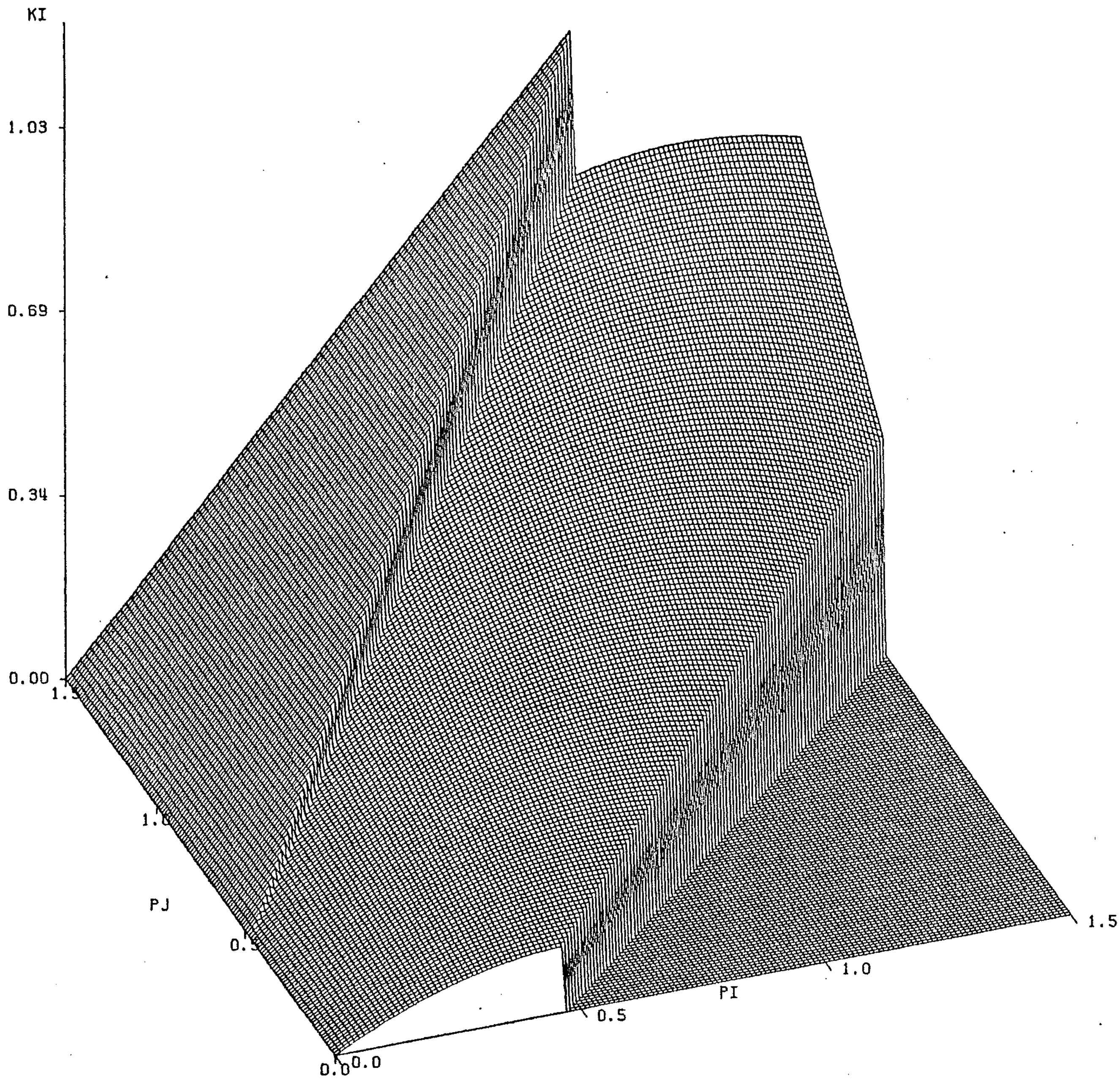


Figure 2: The payoff function in a price-setting game

The payoff K_i of firm i as a function of (p_i, p_j) at the location pair (0.27, 0.27).

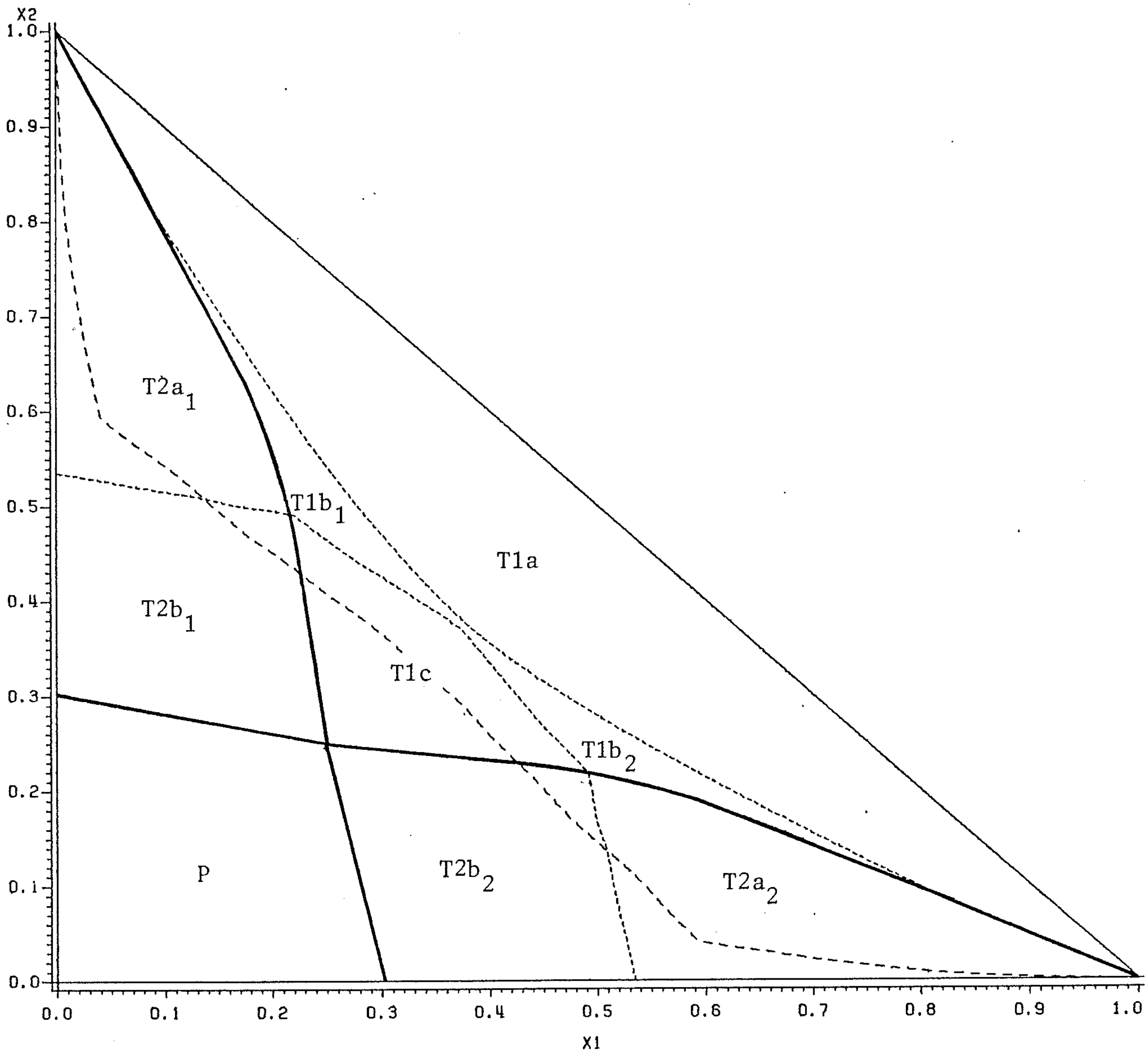


Figure 3: Types of equilibrium in the price-setting games $\Gamma(x_1, x_2)$

The solid lines separate the regions T1, T2, and P; the dotted lines subdivide T1 and T2. Region S is the area below the dashed line. If (x_1, x_2) is in P, or on the line $x_1 + x_2 = 1$, the unique equilibrium of $\Gamma(x_1, x_2)$ is pure. In region S an equilibrium must be of type T. In regions T1 and T2 we find equilibria of type T satisfying the following conditions.

T1a: $b_i - a_i = 2z$ for $i = 1, 2$

$T1b_i$: $b_i - a_i < 2z$ and $b_j - a_j = 2z$

T1c: $b_i - a_i < 2z$ for $i = 1, 2$

$T2a_i$: $b_i = b_j - z$

$T2b_i$: $b_i > b_j - z$

The equilibrium in region P is the one found by Hotelling. The extent of P is established by d'Aspremont et al. [1979], who also show that both equilibria are unique within the class of pure equilibria. We prove in Appendix 1 (see (c), and the discussion after (i)) that there are no mixed equilibria in these cases. The restriction of finite reservation prices is very weak. Without it, there are mixed equilibria when $x_1 + x_2 = 1$ in which each firm charges arbitrarily high prices with positive probability;[3] an equilibrium of this type does not exist for any other location pair (see (i) of Appendix 1). The next result (a consequence of (f) in Appendix 1) gives additional support to the equilibrium (0, 0).

<u>Proposition 2</u>: <u>Every equilibrium of</u> $\Gamma(x_1, x_2)$ <u>converges to the pure equilibrium</u> $(p_1, p_2) = (0, 0)$ <u>as</u> $x_1 + x_2 \to 1$.

If (x_1, x_2) does not satisfy the conditions in Proposition 1, then $\Gamma(x_1, x_2)$ has a mixed strategy equilibrium (by the results of Dasgupta and Maskin [1982]). Let (F_1, F_2) be an equilibrium (each F_i being a cumulative probability distribution function over prices), and let a_i and b_i be respectively the smallest and largest prices in the support of F_i, for i = 1, 2. We show that for a range of location pairs, (F_1, F_2) must take a specific form. Define an equilibrium to be of <u>type</u> T if $b_i - a_i \leq 2z$, each F_i is atomless except possibly at b_i, and either (i) the support of each F_i is $[a_i, b_j-z] \cup [a_j+z, b_i]$, and each F_i has an atom at b_i if and only if $b_i - a_i < 2z$ (<u>type</u> T1), or (ii) the support of F_j is $[a_j, b_j]$, that of F_i is $[a_j-z, b_j-z] \cup \{b_i\}$ with $b_i \geq b_j - z$ (i = 1 or 2), F_i has an atom at b_i, and F_j has an atom at b_j if and only if $b_i > b_j - z$ (<u>type</u> T2). The nature of the supports of F_1 and F_2 in a type T1 equilibrium with $b_i - a_i < 2z$ for $i = 1, 2$ is shown in Figure 4. Note that each price in the lower interval $[a_i, b_j-z]$ charged by i just undercuts (in the sense used in the Introduction) a price in j's upper interval $[a_i+z, b_j]$. Our result (proved in Appendix 1) is as follows.

Proposition 3: *Every equilibrium of* $\Gamma(x_1, x_2)$ *in which* $b_i - a_i \leq 2z$ *for* $i = 1, 2$ *is of type* T. *If* (x_1, x_2) *is in region* S *(see Figure 3) then* $b_i - a_i \leq 2z$ *for* $i = 1, 2$ *in every equilibrium of* $\Gamma(x_1, x_2)$, *so that every equilibrium of* $\Gamma(x_1, x_2)$ *is of type* T, *and there is at least one such equilibrium.*

(The arguments in Appendix 1 impose a number of additional restrictions on any equilibrium. See, in particular, (a) and (i).) Now, if (F_1, F_2) is an equilibrium of type T, then a standard argument[4] shows that each F_i is differentiable on the interior of its support. This means that F_j is such that the profit $K_i(p, F_j)$ of i is constant (say equal to E_i) on the interior of the support of F_i, and on the union of this with b_i if F_i has an atom at b_i. (Roughly, each firm must be indifferent between actions taken with positive probability.) Conversely, if F_j satisfies these conditions, and $K_i(p, F_j) \leq E_i$ for all p outside the support of F_i, then (F_1, F_2) is an equilibrium.

Now, the condition that $K_i(p, F_j)$ be constant on the interior of the support of F_i is equivalent, upon differentiation, to the condition that F_j satisfy an integral-differential equation. This equation is hard to work with, but may be differentiated again and, in the case of a type T1 equilibrium, solved, subject to the condition $F_j(a_j) = 0$, to give

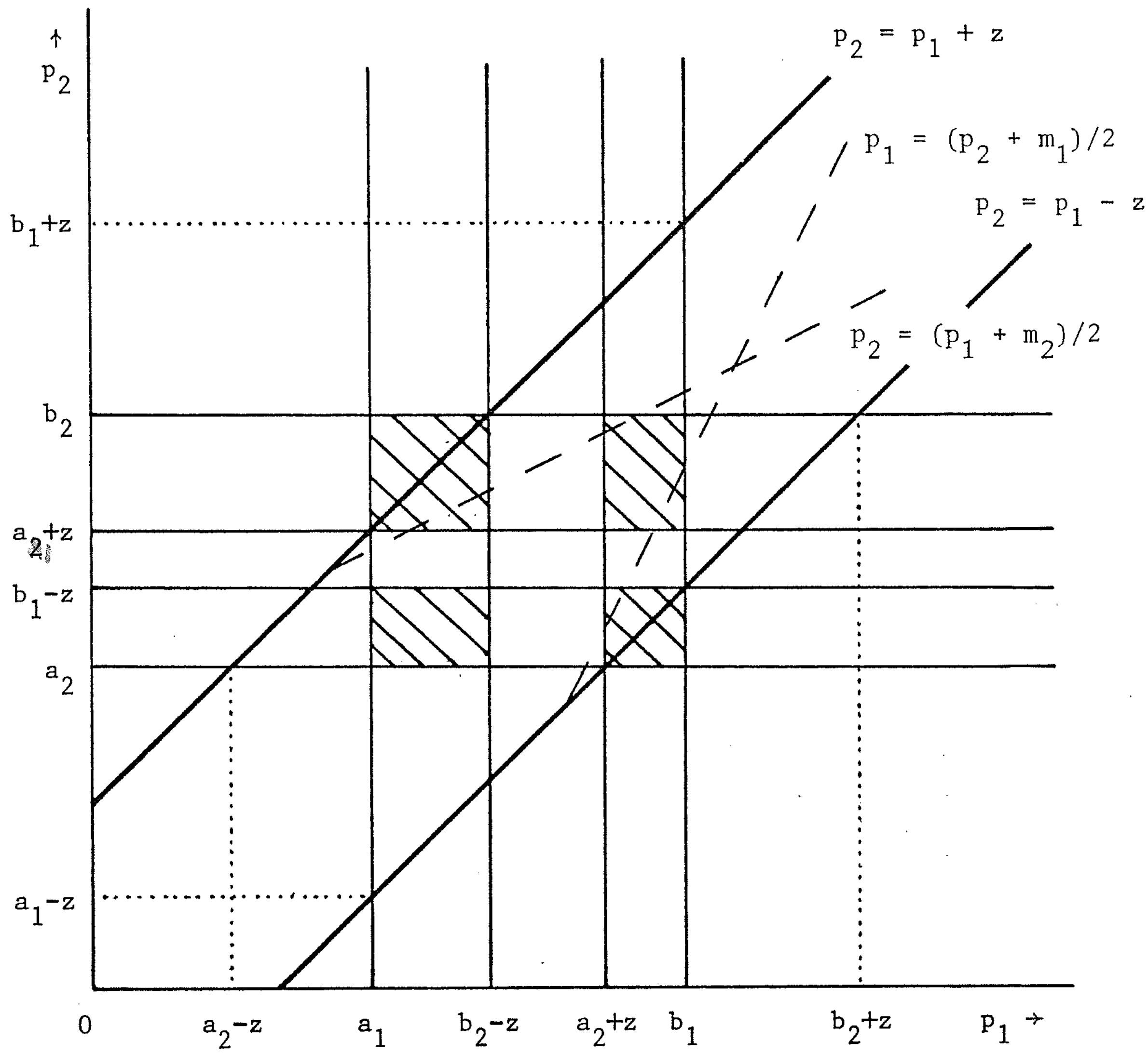


Figure 4: The supports of the equilibrium strategies in a type T1c equilibrium of $\Gamma(x_1, x_2)$

In region T1c, the supports of the equilibrium strategies in $\Gamma(x_1, x_2)$ take the form shown. In the other regions, the forms of the supports are indicated in Figure 3. (For each value of p_j, $p_i = (p_j + m_i)/2$ maximizes the payoff of firm i in $(p_j - z, p_j + z)$.)

$$F_j(p) = \begin{cases} 1 - \exp(\frac{p-a_j}{2x_i}) + A_j\exp(\frac{p+z}{2x_i}) \int_{-p}^{-a_j} h(s,x_i,z)ds & \text{if } a_j \le p \le b_i - z \\ (1-\delta_j)\exp(\frac{b_j-p}{2x_j}) - B_j\exp(-\frac{p-z}{2x_j}) \int_p^{b_j} h(s,x_j,z)ds & \text{if } a_i+z \le p < b_j, \end{cases} \qquad (2)$$

for some A_j and B_j, where δ_j is the size of the atom in F_j at b_j, and $h(s,x,z) = (s-z)^{-2}\exp((s-z)/2x)$. (The integrals can be expressed as infinite series by making the substitution $t = (s-z)/2x$, integrating by parts, and using the fact that $\int(e^t/t)dt = \ln|t| + \sum_{n=1}^{\infty} t^n/nn!$.) The case of a type T2 equilibrium can be dealt with in a similar fashion.

Thus if F_j is defined by (2), the derivative of $K_i(p, F_j)$ is constant on the interior of the support of F_i. By substituting F_j into the expression for $K_i(p, F_j)$ for i = 1, 2, we obtain conditions on (a_i, b_i, δ_i, A_i, B_i) for i = 1, 2 which ensure that this derivative is zero. A number of other conditions have to be satisfied for (F_1, F_2) to be an equilibrium: if $K_i(p, F_j) = E_i$ for $a_i \le p \le b_j - z$ then we must have $K_i(p, F_j) = E_i$ for $a_j + z \le p < b_i$; we need $F_j'(p) \ge 0$ for all p in the support of F_i, and $F_j(b_i-z) = F_j(a_i+z)$, so that F_j is a distribution function; and we need $K_i(p, F_j) \le E_i$ for all p outside the support of F_i. We obtain from these conditions ten equations and eight inequalities which the ten variables (a_i, b_i, δ_i, A_i, B_i) (i = 1, 2) must satisfy. Simple algebraic manipulations reduce this system to at most three equations in three variables (depending on the type of equilibrium), together with some inequalities.

These arguments establish that a solution of this system defines an equilibrium; Proposition 3 guarantees that if (x_1, x_2) is in S then every equilibrium of $\Gamma(x_1, x_2)$ is associated with a solution of the system.

As discussed in the Introduction and in Appendix 2, we computed approximate solutions to the equations, and checked that they satisfied the inequalities, at a large number of location pairs (x_1, x_2). The equilibrum profit $E_1(x_1, x_2)$ of firm 1 associated with this collection of price equilibria is shown in Figure 5. An analysis of this profit function forms the basis for our study of the location game Γ.

4. Equilibrium in Locations

The information in Figure 5 allows us to find, for each value of x_2, an approximate best response x_1 for firm 1. For example, when $x_2 = 0$ this best response is between 0.3 and 0.4. An examination of the change in the best response with x_2 indicates that there is a unique pure equilibrium (x, x) with $0.266 < x < 0.274$ (the difference between the payoffs at adjacent points differs by enough to exceed possible computational errors). The equilibrium price strategy when $x = 0.27$ (and also when x takes several other values) is shown in Figure 1.

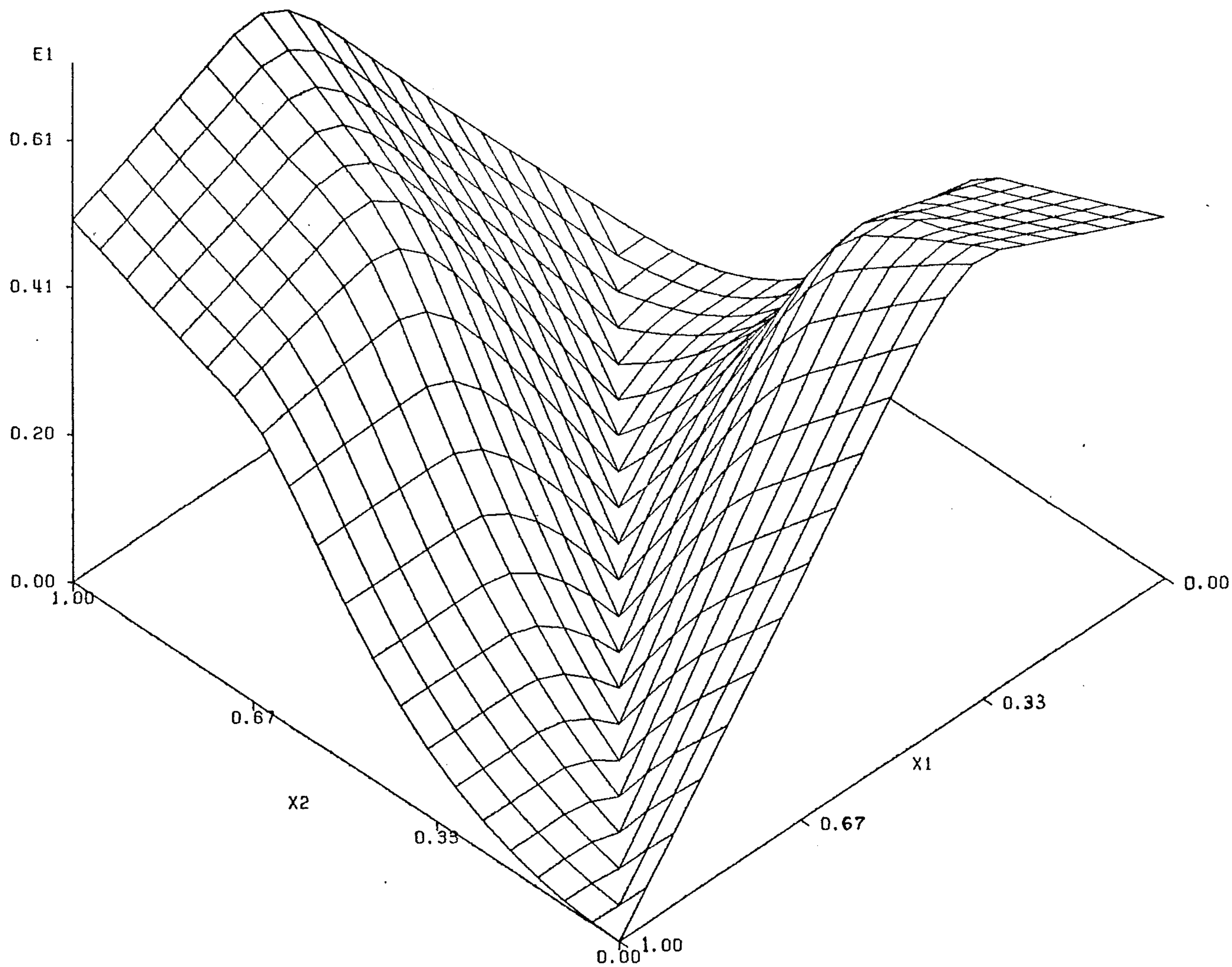


Figure 5: The payoff function in the location game Γ

For each pair (x_1, x_2) of locations, the equilibrium payoff $E_1(x_1, x_2)$ of firm 1 in the price setting game $\Gamma(x_1, x_2)$ is shown.

Appendix 1: Proofs

Here we prove Propositions 1, 2, and 3, and establish some additional conditions which equilibria of $\Gamma(x_1, x_2)$ must satisfy. First we restrict the values a_i and b_i of the endpoints of the supports of any equilibrium strategies, by using domination arguments. For example, $K_i(a_j-z, p_j) > K_i(p_i, p_j)$ for any price $p_i < a_j - z$, for all $p_j \geq a_j$; hence $a_i \geq a_j - z$ (see (a) below). For those locations in P, these restrictions together imply that $a_i = b_i = 1 + (x_i-x_j)/3$ for i = 1, 2, proving Proposition 1. They also imply that $b_i - a_i \leq 2z$ for every (x_1, x_2) in S, so that the second sentence of Proposition 3 follows from the first.

To complete the proof of Proposition 3, we show that the prices at which the equilibrium strategies can have atoms lie in a restricted set. For example, if F_i has an atom at p, then F_j does not have an atom at p - z or at p + z (since j can do strictly better by charging slightly lower prices). In particular, we show that if $b_i - a_i \leq 2z$ then each equilibrium strategy can have an atom only at b_i (see (j) below); this leads fairly directly (see (k) through (o)) to the conclusion that every equilibrium is of type T. Finally, the straightforward proof of Proposition 2 (which uses domination arguments again) is given in (f).

In our proof, we repeatedly use the following properties of the payoff function K_i (see (1) and Figures 2 and 4):

for fixed p_j, K_i is linear in p_i (with slope 1) on $[0, p_j-z)$, jumps down at p_j-z (if $x_j > 0$), is strictly concave (with slope less than 1) on (p_j-z, p_j+z), jumps down at p_j+z (if $x_i > 0$), and is zero above p_j+z;

for fixed p_j, the maximum of K_i, if attained in (p_j-z, p_j+z), is attained at $p_i = (p_j + m_i)/2$;

for fixed p_i, K_i is zero on $[0, p_i-z)$, linear in p_j (with slope $p_i/2$) on (p_i-z, p_i+z), and constant in p_j above $p_i + z$. (Most of our arguments are easy to follow when reference is made to a diagram like Figure 4; space constraints prohibit the inclusion of all the appropriate diagrams.)

For any set Q of prices of firm i, we say that p_i^{**} <u>strongly dominates</u> p_i^* <u>on</u> Q when $K_i(p_i^{**}, p_j) > K_i(p_i, p_j)$ for all $p_j \in Q$ and all p_i in an open neighborhood of p_i^*. We write $A_i = [a_i, b_i]$; if p_i^{**} strongly dominates p_i^* on A_j then p_i^* is not in the support of any equilibrium strategy of i.

If (F_1, F_2) is an equilibrium of $\Gamma(x_1, x_2)$ then $a_i \geq 0$ for i = 1, 2 since each firm can guarantee a profit of zero by setting a price of zero. We also have the following.

(a) $a_j - z \leq a_i \leq a_j + z$ <u>and</u> $b_j - z \leq b_i \leq b_j + z$ <u>for</u> i = 1, 2: Since $p_i = a_j - z$ strongly dominates any lower price on A_j we have $a_i \geq a_j - z$ and hence $a_j - z \leq a_i \leq a_j + z$. If $z > 0$, or $z = 0$ and $b_j > 0$, then $K_i(p, F_j) = 0$ if $p > b_j + z$, while $K_i(p, F_j) > 0$ for some $p \leq b_j + z$ (for example for $p = z/2$ if $z > 0$), so that $b_i \leq b_j + z$. If $z = 0$ and $b_j = 0$ then $K_j(F_j, F_i) = 0$, so that $b_i = 0$ (otherwise $K_j(p, F_i) > 0$ for some $p > 0$). Hence in both cases $b_j - z \leq b_i \leq b_j + z$.

(b) <u>If</u> $p > 0$ <u>is an atom of</u> F_i <u>and</u> $x_i > 0$ ($x_j > 0$) <u>then</u> $p - z$ ($p + z$) <u>is not an atom of</u> F_j: Under these conditions, the profit of j jumps down at $p - z$ ($p + z$), so this cannot be an atom of F_j.

(c) <u>If</u> $z = 0$ <u>and</u> b_i <u>exists for some</u> i, <u>or if every consumer has a finite reservation price, then</u> $a_i = b_i = 0$ <u>for</u> i = 1, 2 (i.e. <u>the only equilibrium is pure, each firm charging the price zero</u>): If b_i exists then

b_j exists and $b_j = b_i$ (by (a)). Let $b_i = b_j = b$. If $b > 0$ then it is not an atom of both F_i and F_j (by (b), since $z = 0$ means that $x_i > 0$ for some i). Suppose b is not an atom of F_j. Then $K_i(b, F_j)$ is equal to i's equilibrium profit (see (d) of Fact (B) in Osborne and Pitchik [1984]). But $K_i(b, F_j) = 0$, while $K_i(p, F_j) > 0$ for $0 < p < b$. Hence we must have $b = 0$. If every consumer has a finite reservation price then $K_i(p, q) = 0$ for all q if p is large enough, say if $p \geq \bar{p}$. Hence $b_i \leq \bar{p}$; the argument above establishes that $b_i = b_j = 0$.

This proves the second sentence of Proposition 1. <u>From now on, we assume that</u> $z > 0$.

(d) $a_i > 0$ <u>for</u> $i = 1, 2$, <u>and the equilibrium profit of each firm is positive</u>: This follows from the fact that firm i can guarantee a positive profit by setting the price $z/2$.

(e) <u>If</u> $x_j > 0$ <u>and</u> $b_i = b_j - z$ <u>then</u> b_i <u>is an atom of</u> F_i <u>and</u> b_j <u>is not an atom of</u> F_j: If b_i is not an atom of F_i then $K_j(b_j, F_i) = 0$ is the equilibrium profit of j (see (d) of Fact (B) in Osborne and Pitchik [1984]). This contradicts (d), so that b_i is an atom of F_i, and so b_j is not an atom of F_j (by (b)).

In the special cases in which $x_i = 0$ for some i, the proofs of some of the subsequent results require additional arguments (to avoid the use of (b) and (e), for example); since the length of these arguments is out of proportion to their significance, we omit them. Thus, in all the proofs below we assume that $x_i > 0$ for $i = 1, 2$.

The next result implies Proposition 2.

(f) $b_i \to 0$ <u>for</u> $i = 1, 2$ <u>as</u> $z \to 0$: If $K_j(b_i - z, b_i) < K_j(b_i - 3z, b_i)$, then every $p_j > b_i - z$ is strongly dominated (by $p_j - 2z$) on A_i, so that $b_j = b_i - z$. But then b_j is dominated (by $b_i - 3z$) on A_i, so that b_j is not an atom of F_j, contradicting (e). Hence $K_j(b_i - z, b_i) \geq K_j(b_i - 3z, b_i)$, or $(b_i - z)(b_i - (b_i - z) + m_j)/2 \geq b_i - 3z$, or $b_i \leq (2 + x_i)z/x_i$, from which the result follows (recall that we are assuming $x_i > 0$).

(g) <u>If</u> p <u>is an atom of</u> F_i <u>then</u> $p \geq 2x_i$: If p is an atom of F_i then $K_j(\cdot, F_i)$ jumps down at $p - z$ (if $p > z$), and at $p + z$, so that supp F_j contains no point in $(p - z, p - z + \varepsilon)$ or in $(p + z, p + z + \varepsilon)$ for some $\varepsilon > 0$. But then $K_i(\cdot, F_j)$ is increasing on $(p, \min(p + \varepsilon, 2x_i))$ if $p < 2x_i$, contradicting the fact that p is an atom of F_i.

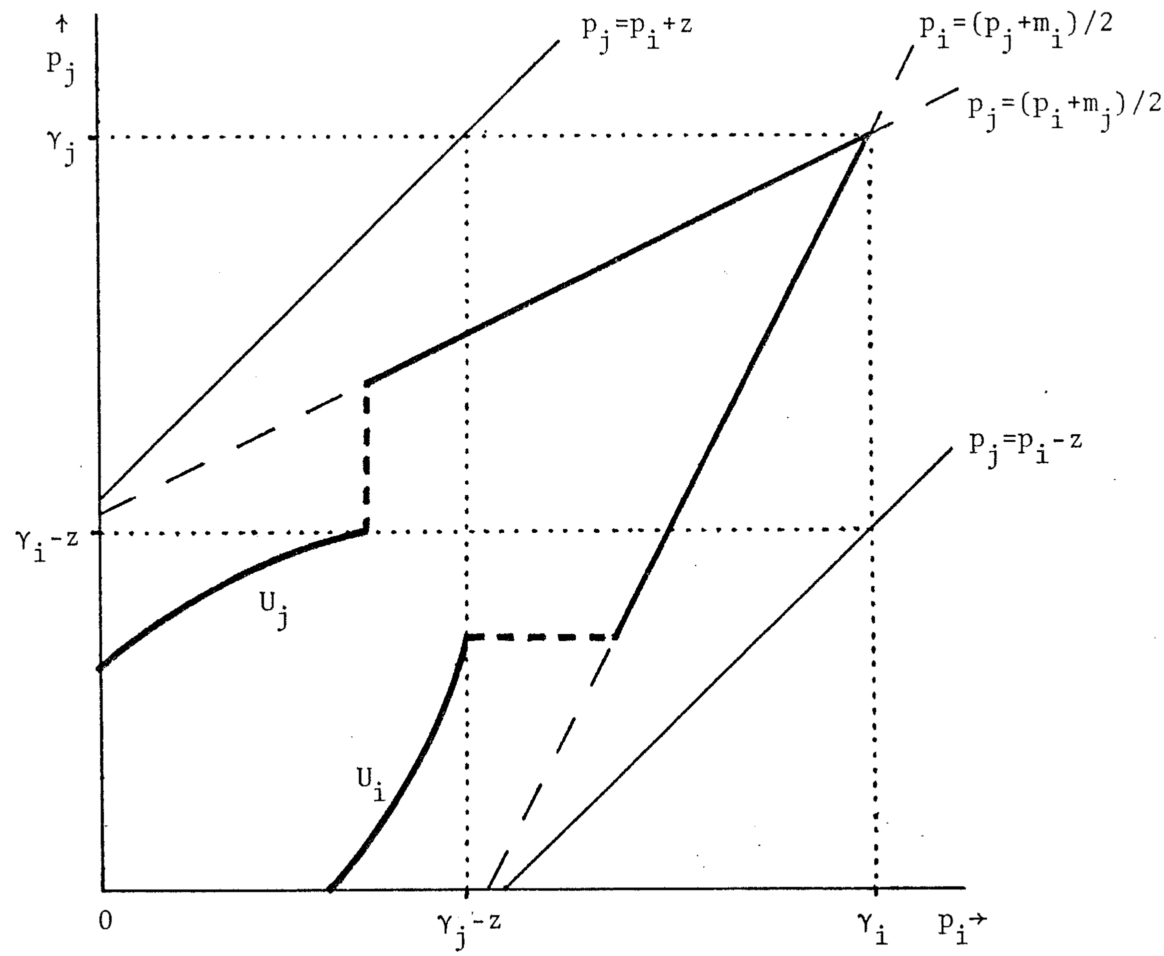
(h) <u>If</u> $p \in$ supp F_i <u>is not an atom of</u> F_i <u>then either</u> $p - z \in$ supp F_j <u>or</u> $p + z \in$ supp F_j: If neither $p - z$ nor $p + z$ is in supp F_j then supp F_j contains no point in $(p - z - \varepsilon, p - z + \varepsilon)$ or in $(p + z - \varepsilon, p + z + \varepsilon)$ for some $\varepsilon > 0$. Now, since $p \in$ supp F_i, we have $p + z \geq a_i + z \geq a_j$ and $p - z \leq b_i - z \leq b_j$ (by (a)), so that $a_j \leq p + z - \varepsilon$ and $b_j \geq p - z + \varepsilon$. Hence supp F_j intersects $(p - z + \varepsilon, p + z - \varepsilon)$, so that, given the other restrictions on supp F_j, $K_i(\cdot, F_j)$ is strictly concave on $(p - \varepsilon, p + \varepsilon)$. Hence p is isolated, and therefore an atom of F_i.

(i) $b_i \leq (b_j + m_i)/2$ <u>for</u> $i = 1, 2$, <u>and hence</u> $b_i \leq \gamma_i \equiv \min(1 + (x_i - x_j)/3, 2(1 - x_j), 3(1 - x_i) - x_j)$ <u>for</u> $i = 1, 2$: We first show that $(b_j + m_i)/2 \geq b_j - z$. If not, then $p_i = b_j - z$ strongly dominates any higher price on A_j, so that $b_i = b_j - z$ (by (a)). Further, $b_i - \varepsilon$ dominates b_i

(though not strongly) on A_j, for some $\epsilon > 0$, so that b_i is not an atom of F_i, contradicting (e). Two cases remain. If $(b_j + m_i)/2 \geq b_j + z$ then the result follows from (a). If $b_j - z \leq (b_j + m_i)/2 \leq b_j + z$, then $(b_j + m_i)/2$ dominates any higher price on A_j, so that $b_i \leq (b_j + m_i)/2$ for i = 1, 2. Combining these two inequalities yields $b_i \leq 1 + (x_i - x_j)/3$; combining $b_i \leq (b_j + m_i)/2$ and $b_j \leq b_i + z$ (see (a)) yields $b_i \leq 2(1 - x_j)$ and $b_j \leq 3(1 - x_j) - x_i$.

Now, for each a_j, let $U_i(a_j)$ be the lowest price of firm i which is not strongly dominated on $[a_j, \gamma_j] \supset A_j$ (the inclusion from (i)). Obviously then we must have $a_i \geq U_i(a_j)$ for i = 1, 2; these restrictions are helpful below. The form of U_i can be found by using the fact that if p_i is less than $\min(2(a_j - x_i), 2(a_j + x_i/3))$ then the best potential dominator is $(p_i + z + m_i)/2$ (i.e. if any price dominates p_i, then this one does), while if p_i is between $2(a_j - x_i)$ and $2x_i$ then the best potential dominator is $a_j + z$, and if p_i exceeds $\max(2x_i, 2(a_j + x_i)/3)$ then the best potential dominator is $a_j + m_i - p$. The details are very messy, and we do not give them here. Obviously, U_i is nondecreasing; an example is shown in Figure 6.

By combining the conditions $a_i \geq U_i(a_j)$ and $b_i \leq \gamma_i$ for i = 1, 2, we can obtain two useful restrictions on the nature of equilibria of $\Gamma(x_1, x_2)$. Let a_i^* be the minimal value of a_i such that $a_i \geq U_i(a_j)$ and $a_j \geq U_i(a_i)$ for some a_j. Then $a_i \geq a_i^*$ in any equilibrium of $\Gamma(x_1, x_2)$. Thus if $a_i^* = \gamma_i$ for i = 1, 2 (as is the case in Figure 6) then the pure equilibrium $(p_1, p_2) = (\gamma_1, \gamma_2)$ is the only possible equilibrium of $\Gamma(x_1, x_2)$. A very tedious analysis of the functions U_i (i = 1, 2) (the details of which we omit) shows that this is so for every (x_1, x_2) in P; this completes the proof of Proposition 1. Also, if $\gamma_i - a_i^* \leq 2z$ then we know that $b_i - a_i \leq 2z$. This is



<u>Figure 6</u>: <u>The functions</u> U_1 <u>and</u> U_2.

For each value of α_j, $U_i(\alpha_j)$ is the smallest price of i which is not strongly dominated.

useful because our subsequent results use the assumption that $b_i - a_i \le 2z$; we show that the only equilibria satisfying this assumption are of type T. A computation shows that $\gamma_i - a_i^* \le 2z$ for i = 1, 2 whenever (x_1, x_2) is in S (see Figure 3). Thus the second sentence of Proposition 3 follows from the first (given the existence result of Dasgupta and Maskin [1982]), which remains to be proved. From now on, we assume that $b_i - a_i \le 2z$ for $i = 1, 2$.

(j) If p is an atom of F_i then $p = b_i$: Suppose $\bar{p}$ is an atom of F_i. Then $K_i(\bar{p}, F_j)$ is equal to the equilibrium profit of i. We argue first that $K_i(., F_j)$ is decreasing on $(\bar{p}, \bar{p} + \varepsilon)$ for some $\varepsilon > 0$. Since $K_j(., F_i)$ jumps down at $\bar{p} - z$ and $\bar{p} + z$, F_j has no support in $(\bar{p} - z, \bar{p} - z + \varepsilon)$ or in $(\bar{p} + z, \bar{p} + z + \varepsilon)$ for some $\varepsilon > 0$. Since $a_i \le \bar{p} \le b_i$, this means that $a_j \le \bar{p} + z$ and $b_j \ge \bar{p} - z + \varepsilon$ (using (a)). Hence F_j has some support in $[\bar{p} - z + \varepsilon, \bar{p} + z]$. But then $K_i(., F_j)$ is strictly concave on $(\bar{p}, \bar{p} + \varepsilon)$; it is continuous at $\bar{p}$ (since neither $\bar{p} - z$ nor $\bar{p} + z$ are atoms of F_j (by (b))), so it is decreasing on $(\bar{p}, \bar{p} + \varepsilon)$ (since $K_i(\bar{p}, F_j)$ is equal to i's equilibrium profit).

Now, since $K_i(., F_j)$ is decreasing on $(\bar{p}, \bar{p} + \varepsilon)$, F_i cannot have any support in this interval. Assume that $\bar{p} < b_i$, and let $\hat{p}$ be the smallest price above $\bar{p}$ which is in supp F_i. Since $\bar{p} \ge 2x_i$ (by (g)) and $b_i \le 2(1-x_j)$ (by (i)), we have $b_i \le \bar{p} + 2z$; since $b_i \ge \hat{p}$ and $b_i - a_i \le 2z$, we have $a_i \ge \hat{p} - 2z$. Therefore $K_j(., F_i)$ is strictly concave on $(\bar{p} - z, \hat{p} - z)$ and on $(\bar{p} + z, \hat{p} + z)$ (since $b_i \ge \hat{p}$), so that the support of F_j in these intervals can consist of at most a single isolated point in each interval, at which F_j has an atom. Let these points be q_1 and q_2, let the size of the atom in F_j at q_k be $J(q_k)$, and let

$$f(p) = [p(q_1 - p + m_i)/2]J(q_1) + \int_{\hat{p}-z}^{\bar{p}+z} (p(q - p + m_i)/2)dF_j(q)$$

$$+ pJ(q_2) + \int_{\hat{p}+z}^{b_j} pdF_j(q).$$

It is easy to check that f is concave. Also, it is immediate that $K_i(p, F_j) = f(p)$ if $p < p < \min(q + z, q - z)$, so that, by the argument above, f is decreasing in this range. The concavity of f implies, therefore, that it is decreasing for all $p > \bar{p}$.

We now argue that $K_i(p, F_j) \leq f(p)$ for all $\bar{p} \leq p \leq \hat{p}$. This implies that $\hat{p}$ is not in the support of F_i, contrary to our assumption, so that we have $\bar{p} = b_i$, completing the proof. First, note tht $K_i(\cdot, F_j)$ jumps down at $q_1 + z$ and at $q_2 - z$. Second, observe that the expression for $K_i(p, F_j)$ is similar to that for f(p), except that if $q_1 + z < p \leq \hat{p}$ then the term in square brackets is zero, while if $q_2 - z < p \leq \hat{p}$ then the multiplier of $J(q_2)$ is $p(q_2 - p + m_i)/2$ (rather than p). Now, F_j can have an atom at q_1 only if $q_1 \geq 2x_j$ (see (g)), in which case $q_1 - p + m_i \geq 2x_j - (1 + (x_i - x_j)/3) + m_i = 2x_i/3 + 4x_j/3$ if $p \leq \hat{p} \leq 1 + (x_i - x_j)/3$ (see (i)). Hence $p(q_1 - p + m_i) \geq 0$ for all $p \leq \hat{p}$. Finally, if $q_2 - z < p$ then $p(q_2 - p + m_i)/2 < p(1 - x_j) < p$. So $K_i(p, F_j) \leq f(p)$ for all $\bar{p} \leq p \leq \hat{p}$.

(k) <u>If</u> $a_i < b_j - z$ <u>then</u> $[a_i, b_j - z] \subset \text{supp } F_i$: Suppose $a_i < p < b_j - z$ with $p \notin \text{supp } F_i$. By (a) we know that $p < b_i$, so there exist smallest numbers $\varepsilon > 0$ and $\delta > 0$ such that $p - \varepsilon \in \text{supp } F_i$ and $p + \delta \in \text{supp } F_i$. Now, since $p - z - \varepsilon < b_j - 2z$ we have $p - z - \varepsilon < a_j$ (given that $b_j - a_j \leq 2z$); since $p - \varepsilon$ is not an atom of F_i (by (j)) we have $p + z - \varepsilon \in \text{supp } F_j$ (by (h)). Also, if $p + \delta < b_j - z$ then $p + \delta < b_i$ by (a) and hence (again using $b_j - a_j \leq 2z$) we have $p + z + \delta \in \text{supp } F_j$ by (h). Since $b_j \in \text{supp } F_j$ by definition, we have r_j

$\equiv \min(b_j, p + z + \delta) \in \text{supp } F_j$. Now, since $b_i \leq a_i + 2z$ and $b_i \geq b_j - z$ (by (a)) we know that $K_j(., F_i)$ is strictly concave on $(p + z - \varepsilon, r_j)$. But then j's profit on some subset of $(p + z - \varepsilon, p + z + \delta)$ exceeds its profit at one of the endpoints of this interval. Since the latter must equal its equilibrium profit, the gap in supp F_i is not compatible with equilibrium.

(l) If $a_j + z < b_i$ then $[a_j + z, b_i] \subset \text{supp } F_i$: This follows from an argument similar to that in (k).

(m) If $b_j - z < a_j + z$ (i.e. if $b_j - a_j < 2z$) then supp $F_i \cap (b_j - z, a_j + z) = \emptyset$ or $\{b_i\}$: If $p \in \text{supp } F_i$ and $b_j - z < p < a_j + z$ then p is an atom of F_i by (h), so that $p = b_i$ by (j).

(n) If $a_i > a_j - z$ for i = 1, 2 then $b_j > a_i + z$: Since a_i is not an atom of F_i (by (j)) we have $a_i + z \in \text{supp } F_j$ (by (h)). Hence $b_j \geq a_i + z$. If $b_j = a_i + z$ then $b_j - a_j < 2z$ (since $a_j > a_i - z$), so that a_i is an isolated member of supp F_i (by (m), using $b_j - z = a_i$), contradicting (j).

(o) If $a_i > a_j - z$ for i = 1, 2 then b_j is an atom of F_j if and only if $b_j - a_j < 2z$: If $b_j - a_j < 2z$ and b_j is not an atom of F_j then $b_j - z \in \text{supp } F_i$ by (h) (since $b_j + z > a_i + 2z \geq b_i$). Since a_j is not an atom of F_j, we also have $a_j + z \in \text{supp } F_i$. But then $K_i(., F_j)$ is continuous and strictly concave on $[b_j - z, a_j + z]$, which means that i's profit cannot be maximized at both endpoints, where it must attain its equilibrium profit. Hence b_j is an atom of F_j. Now assume that $b_j - a_j = 2z$. Then $a_i < a_j + z$ $(= b_j - z) < b_i$ (the second inequality by (n)), so that by (k) and (l) we have supp $F_i = [a_i, b_i]$. If F_j has an atom at b_j then $K_i(., F_j)$ jumps down at

$a_j + z$, contradicting the (a.e.) constancy of $K_i(\cdot, F_j)$ on supp F_i. So b_j is not an atom of F_j.

We can now show that every equilibrium of $\Gamma(x_1, x_2)$ in which $b_i - a_i \leq 2z$ is of type T.

<u>Type T1</u>: If $a_i > a_j - z$ for i = 1, 2 then (n), (k), (l), and (m) imply that supp $F_i = [a_i, b_j - z] \cup [a_j + z, b_i]$ for i = 1, 2. By (o), b_i is an atom of F_i if and only if $b_i - a_i < 2z$.

<u>Type T2</u>: If $a_i = a_j - z$ then (k), (l), and (m) imply that supp $F_j = [a_j, b_j]$, and supp $F_i = [a_j - z, b_j - z]$ or $[a_j - z, b_j - z] \cup \{b_i\}$. In the first case $b_j - z$ is an atom of F_i and b_j is not an atom of F_j by (e); in the second case b_j is an atom of F_j (otherwise i's payoff in $(b_j - z, b_i)$ exceeds that at $b_j - z$ and at b_i, as in the proof of (o)), and b_i is an atom of F_i.

(We can further refine these results by using the constraints on a_i and b_i to rule out some sorts of equilibria for particular ranges of (x_1, x_2). For example, if (x_1, x_2) is in some range around (0.27, 0.27) then any equilibrium is either of type T1 with $b_i - a_i < 2z$ for i = 1, 2 or of type T2 with $b_i > b_j - z$.

Appendix 2: Notes on Computational Techniques and Accuracy

Techniques

As discussed in section 3 above, the problem of finding an equilibrium can be reduced to that of simultaneously solving one, two, or three equations (depending on the value of (x_1, x_2)) in as many unknowns, and checking that the solution satisfies a number of inequalities. To find an approximate solution of the equations for a particular pair (x_1, x_2) of locations, we evaluated their left-hand sides at each point in a grid, found the point in the grid which generates the lowest sum of absolute values of these left-hand sides, and then repeated the procedure on a smaller grid. We stopped this iterative procedure when we obtained an absolute value for the sum of the left-hand sides less than 10^{-7}. We then used the resulting parameter values to calculate equilibrium payoffs and equilibrium strategies, and to check that the inequalities are satisfied. We carried out this procedure for about 350 pairs (x_1, x_2) (which involved computations at 175 points, given the symmetry of the problem). All calculations were performed by a DEC 20 computer, programmed in APL, with an internal precision of about 18 decimal digits.

Accuracy

1. The integrals in (2) can be expressed only as infinite series. Let $I(t) = -e^t/t + \ln|t| + \sum_{n=1}^{\infty} t^n/nn!$. Then $\int_a^b h(s,x,z) = (I((b-z)/2x) - I((a-z)/2x))/2x$ (integrating as discussed in the text). We used the first 25 terms to approximate the infinite series in I. This approximation is better, the smaller is the absolute value of the argument of I. For $x_1 = x_2 = 0.27$ (the approximate locational equilibrium) we have $a = -p$ and $b = -a_i$ for the integral in $F_i(p)$ on $[a_i, b_j - z]$, and $a = p$ and $b = b_i$ for the integral in

$F_i(p)$ on $[a_j + z, b_i]$ (see (2)). Given that $a_i \approx 0.5$ and $b_i \approx 1$ for $i = 1, 2$ in this case, $(b - z)/2x \approx -1.78$ and $(a - z)/2x$ ranges between -1.78 and -1.85 in the first integral, and $(b - z)/2x \approx 1$ and $(a - z)/2x$ ranges between 0.93 and 1 in the second integral. This means that the omitted terms in the infinite series are of the form $t^n/nn!$, with $-1.85 < t < 1$ and $n \geq 26$; the absolute value of the sum of all such terms is at most $(1/26)[r^{26}/26! + r^{27}/27! + \ldots]$, where $r = |t|$, which is at most $e^r r^{26}/26 \times 26!$ (using an upper bound for the Lagrange form of the remainder term in the expansion for e^r). Given that $-1.85 < t < 1$, this is less than 10^{-20}, and hence the approximation error is less than the computational error. As x_1 and x_2 vary, this error changes. However, an analysis of the various cases shows that the error does not exceed 10^{-14} at any of the points (x_1, x_2) we studied.

2. The solution we find is also only approximate because we find parameter values which solve the nonlinear equations only to within 10^{-7}. Since the length of the support of every equilibrium strategy is at most 1, this means that the payoff of each firm varies by at most 10^{-7} on the support (our solution guarantees that the derivative of the payoff is at most 10^{-7}). For prices outside the support, our computations (and in some cases analytical arguments) show that the payoff is less than the equilibrium payoff. Hence the equilibria we find are ε-equilibria for $\varepsilon < 10^{-7}$.

Footnotes

[1]If the idea of firms randomly choosing a price is unappealing, reference can be made to one of the other interpretations of mixed strategies. The work on "purification" (see Aumann et al. [1983] and Milgrom and Weber [1981]) shows that, under certain conditions, a mixed strategy equilibrium can be interpreted as a pure strategy equilibrium of a game of imperfect information. Firms may not know precisely where the consumers are. If they receive independent signals on this variable then there may be an equilibrium in which they each choose an optimal pure strategy for every signal they receive, in such a way that the prices induced by the variation in the signal have the distribution given by the mixed strategy. (Because of discontinuities in the payoff functions, the game we analyze is not covered by the results of Milgrom and Weber. However, if we approximate our game by one in which each player has finitely many pure strategies, then the results of Aumann et al. apply.)

Gal-Or [1982] and Shilony and Zamir (unpublished work reported to us in private correspondence) have previously obtained some preliminary results on the outcome of allowing firms to randomize in Hotelling's model.

[2]Varian [1980] interprets a mixed strategy as a policy of holding "sales"; this designation seems particularly appropriate here.

[3]Shmuel Zamir pointed this out to us in private correspondence.

[4]See, for example, Solution to Problem 17 on p. 294 of Karlin [1959].

References

Aumann, R.J., Y. Katznelson, R. Radner, R.W. Rosenthal, and B. Weiss [1983], "Approximate Purification of Mixed Strategies", Mathematics of Operations Research, 8, pp. 327-341.

Dasgupta, P. and E. Maskin [1982], "The Existence of Equilibrium in Discontinuous Economic Games", Part 1 ("Theory") and Part 2 ("Applications"), Discussion Papers 82/54 and 82/55, International Centre for Economics and Related Disciplines, London School of Economics.

d'Aspremont, C., J. Jaskold Gabszewicz, and J.-F. Thisse [1979], "On Hotelling's "Stability in Competition"," Econometrica, 47, pp. 1145-1150.

Gal-Or, E. [1982], "Hotelling's Spatial Competition as a Model of Sales", Economic Letters, 9, pp. 1-6.

Hotelling, H. [1929], "Stability in Competition", Economic Journal, 39, pp. 41-57.

Karlin, S. [1959], Mathematical Methods and Theory in Games, Programming, and Economics, Volume II, Addison-Wesley, Reading, MA.

Lerner, A.P. and H.W. Singer [1937], "Some Notes on Duopoly and Spatial Competition", Journal of Political Economy, 45, pp. 145-186.

Milgrom, P.R. and R.J. Weber [1981], "Distributional Strategies for Games with Incomplete Information", Discussion Paper 428R, J.L. Kellogg Graduate School of Management, Northwestern University.

Osborne, M.J. and C. Pitchik [1984], "Price Competition in a Capacity-Constrained Duopoly," Discussion Paper 185, Department of Economics, Columbia University, April.

Smithies, A. [1941], "Optimum Location in Spatial Competition", Journal of Political Economy, 49, pp. 423-439.

Varian, H.R. [1980], "A Model of Sales", American Economic Review, 70, pp. 651-659.

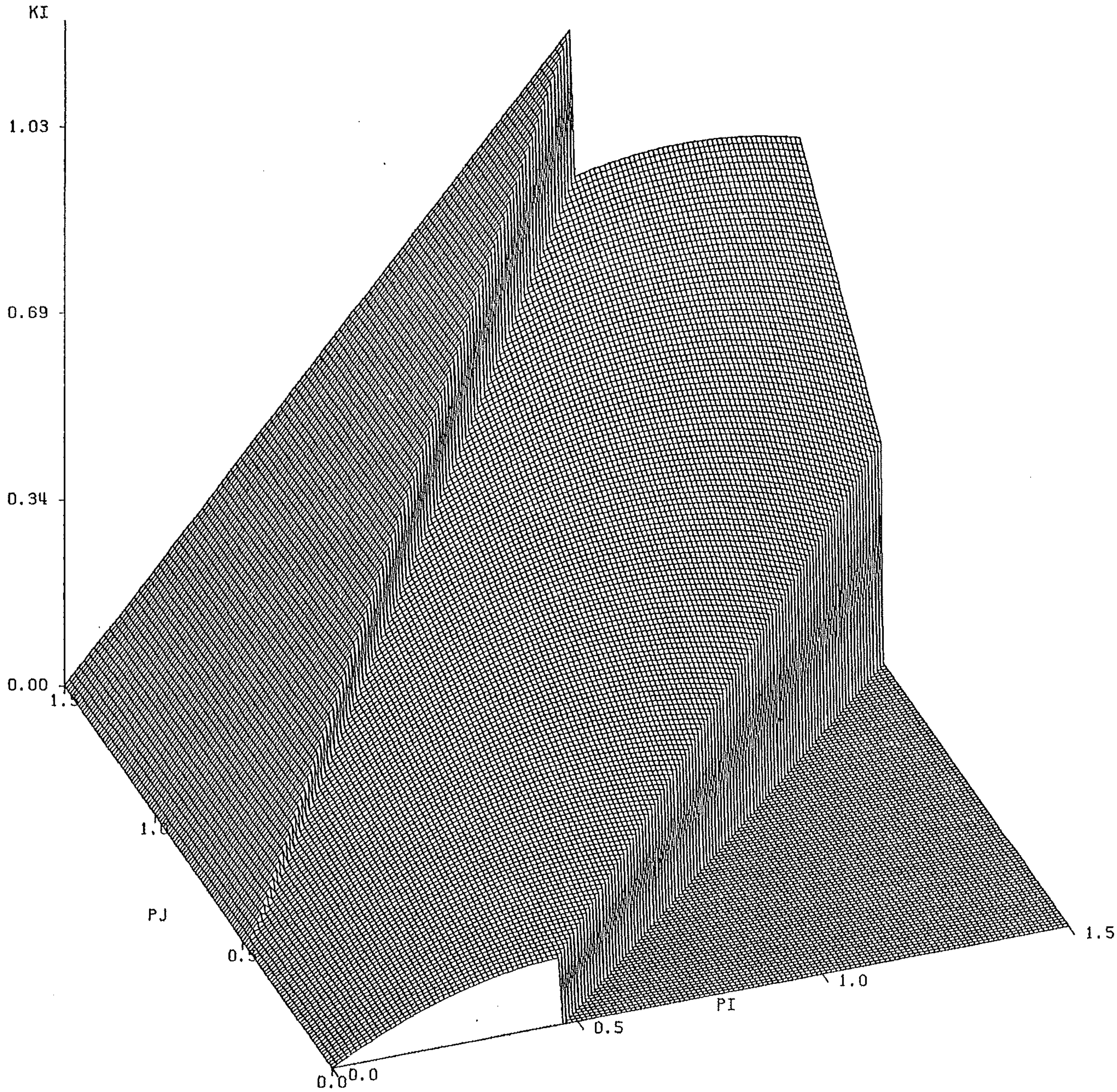


Figure 2: The payoff function in a price-setting game

The payoff K_i of firm i as a function of (p_i, p_j) at the location pair (0.27, 0.27).

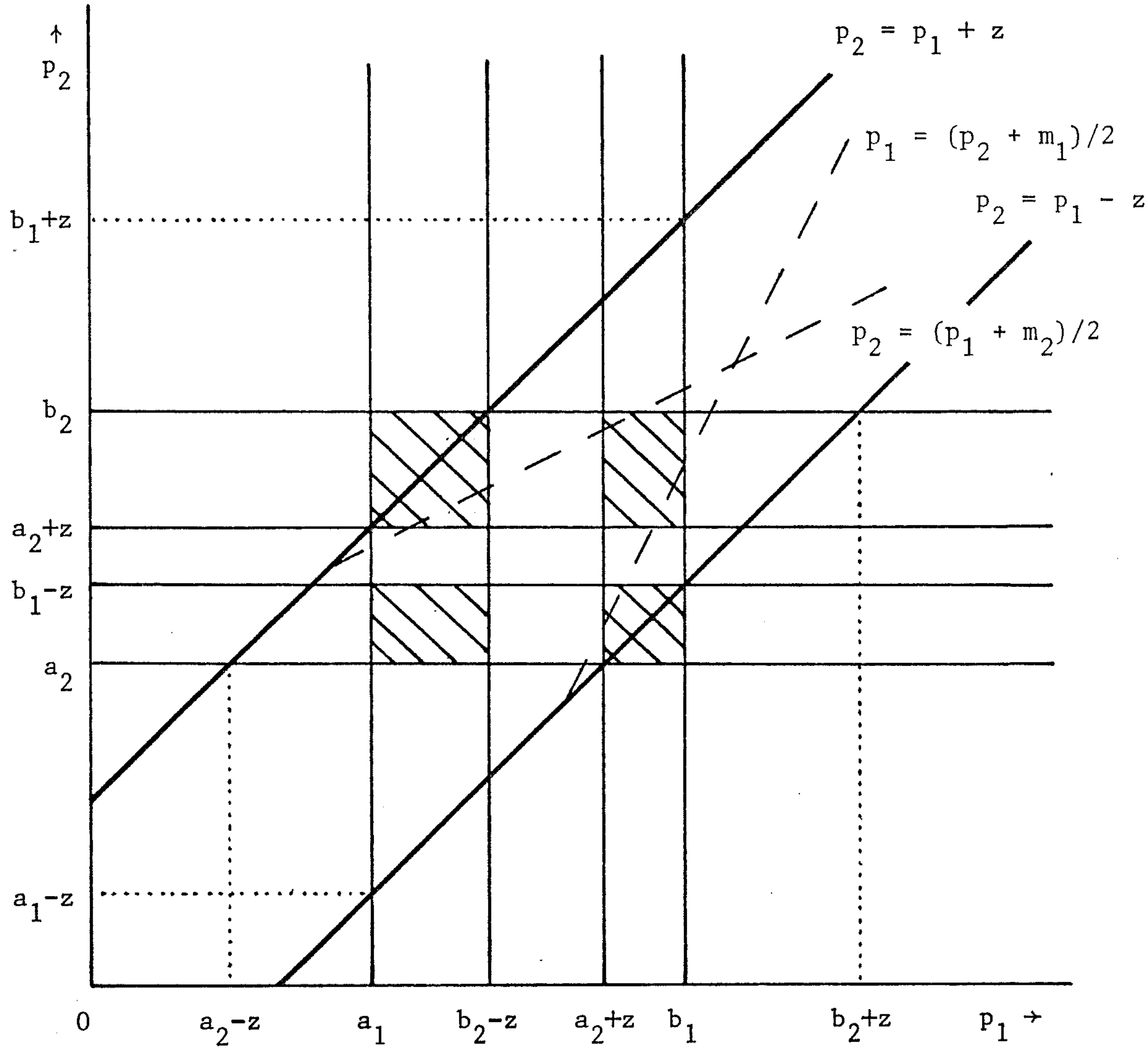


Figure 4: The supports of the equilibrium strategies in a type T1c equilibrium of $\Gamma(x_1, x_2)$

In region T1c, the supports of the equilibrium strategies in $\Gamma(x_1, x_2)$ take the form shown. In the other regions, the forms of the supports are indicated in Figure 3. (For each value of p_j, $p_i = (p_j + m_i)/2$ maximizes the payoff of firm i in $(p_j - z, p_j + z)$.)

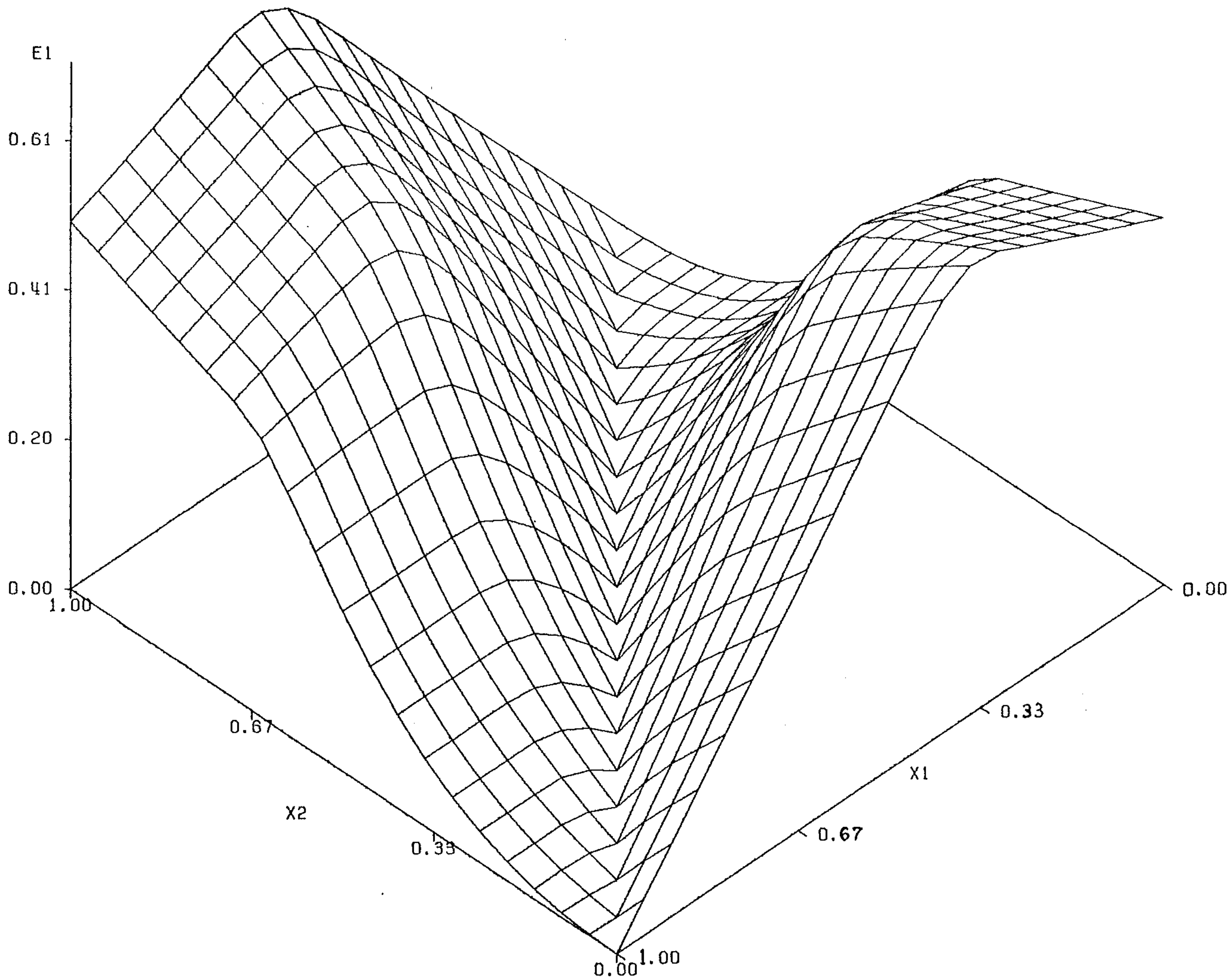


Figure 5: The payoff function in the location game Γ

For each pair (x_1, x_2) of locations, the equilibrium payoff $E_1(x_1, x_2)$ of firm 1 in the price setting game $\Gamma(x_1, x_2)$ is shown.

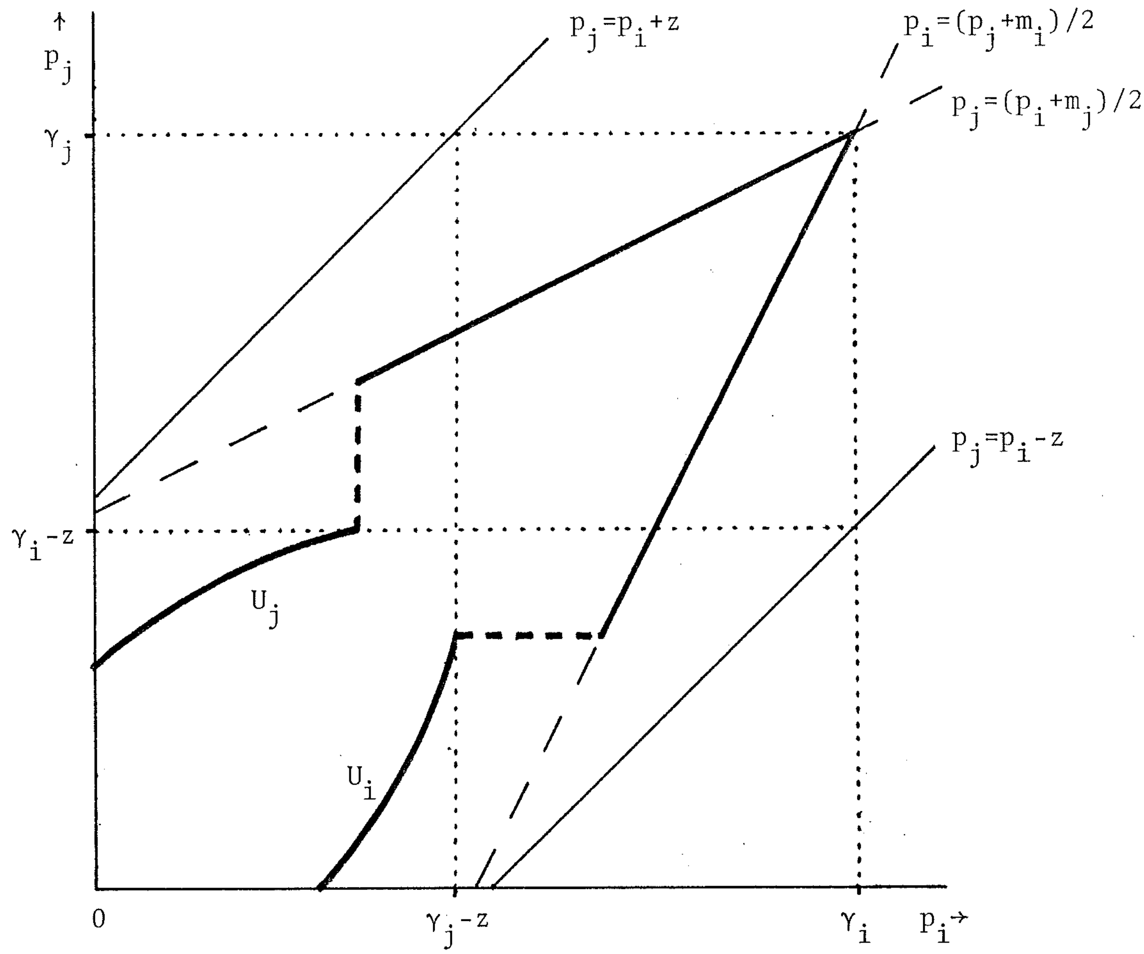
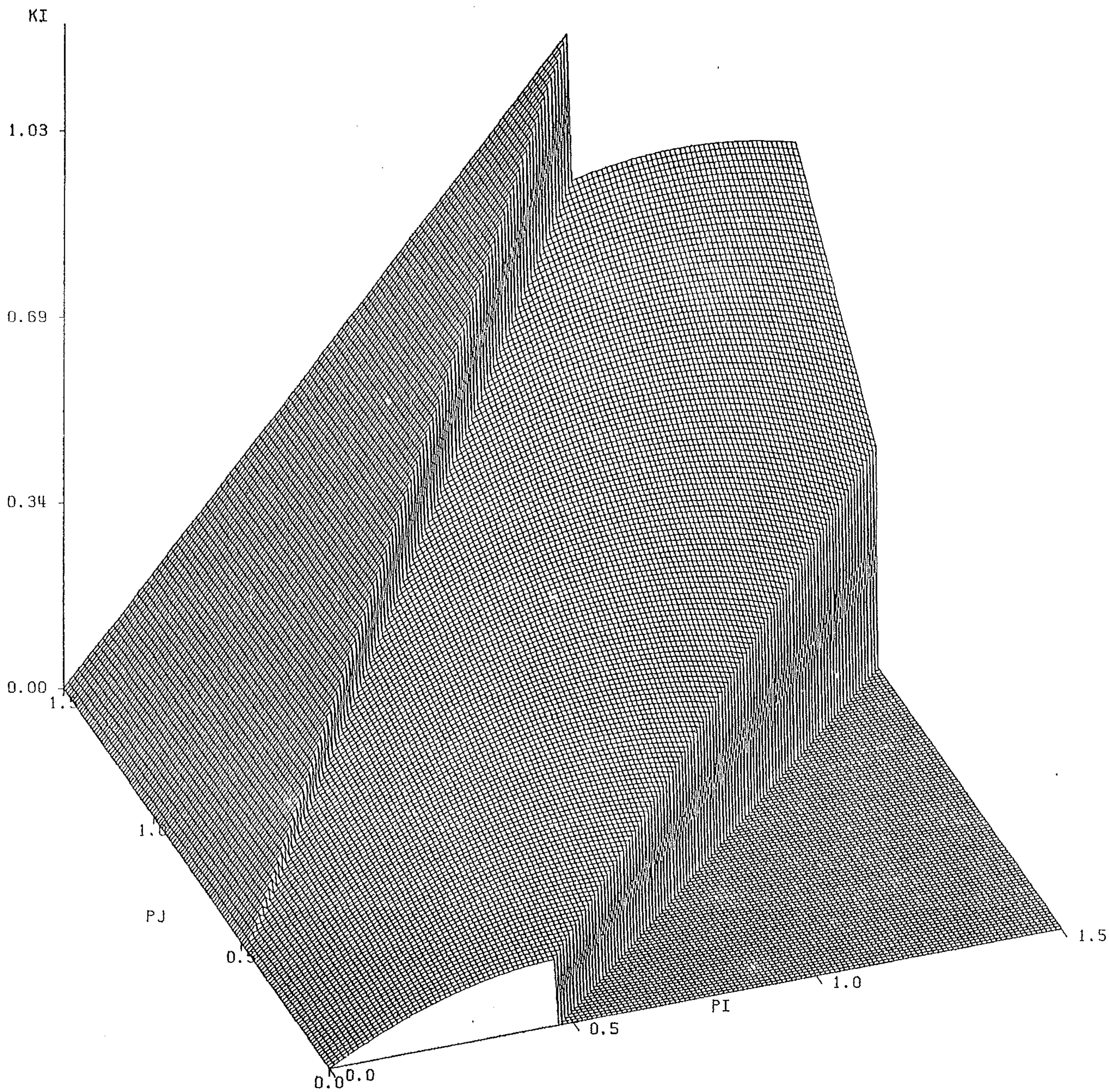
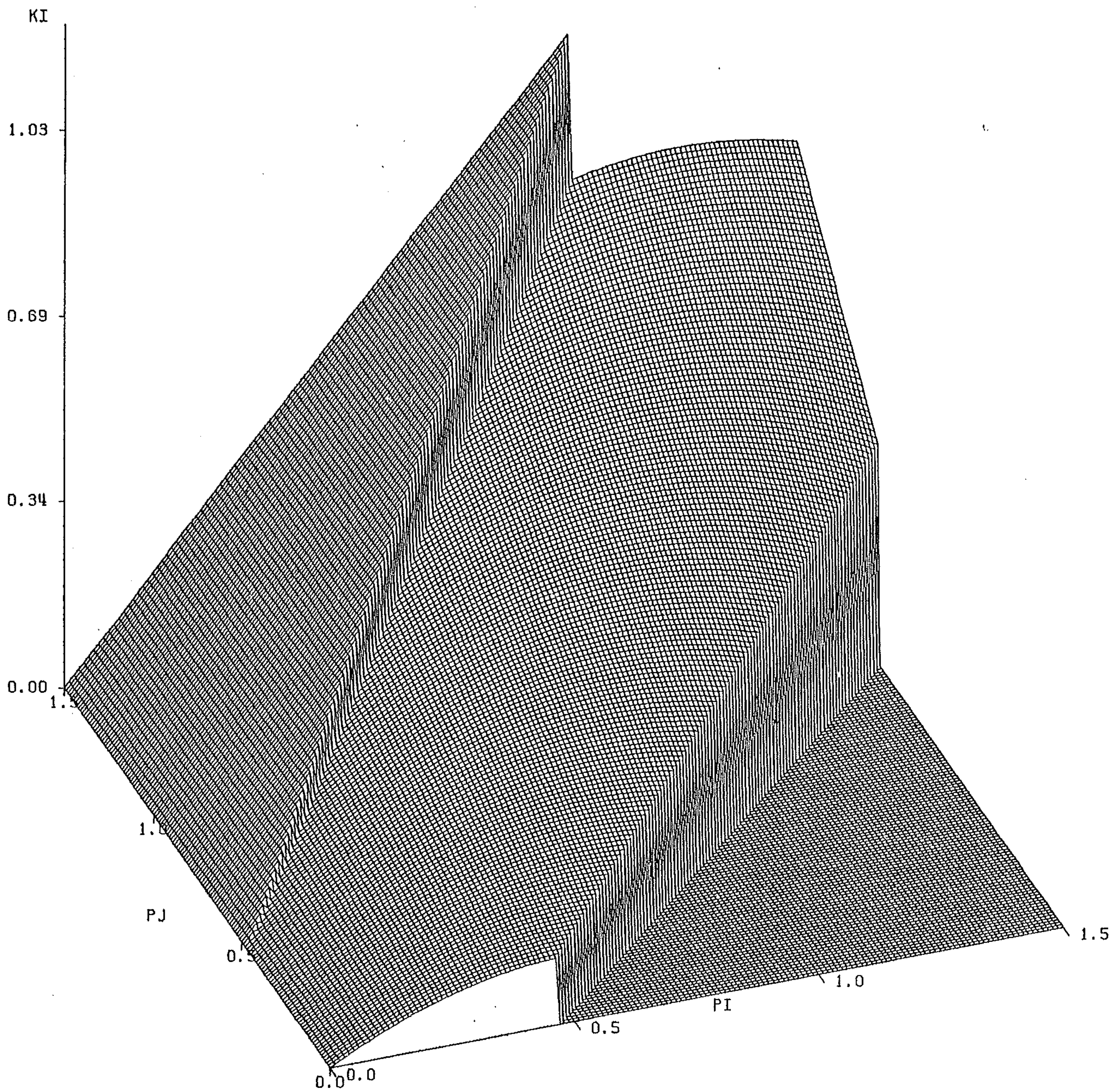
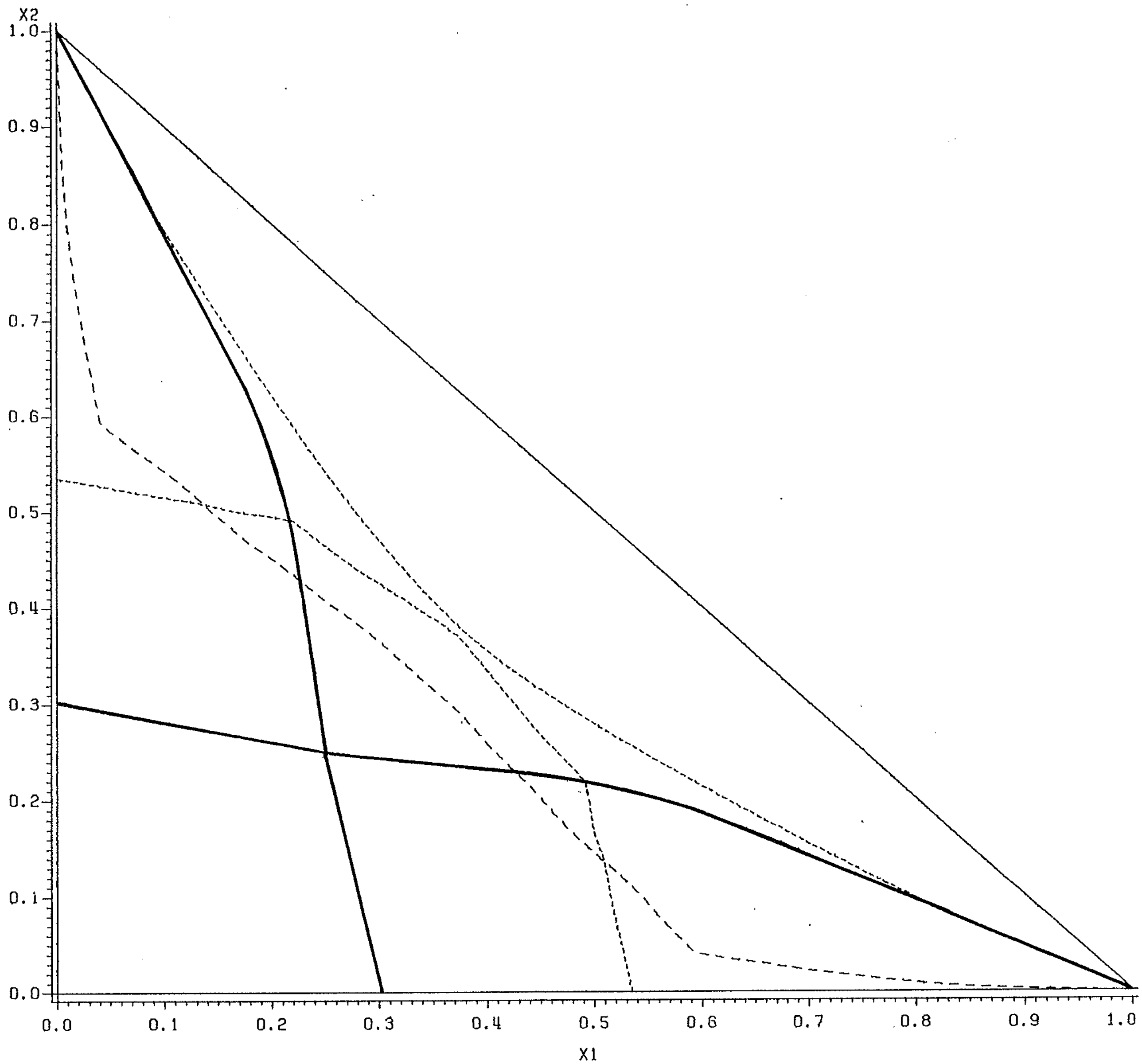


Figure 6: The functions U_1 and U_2.

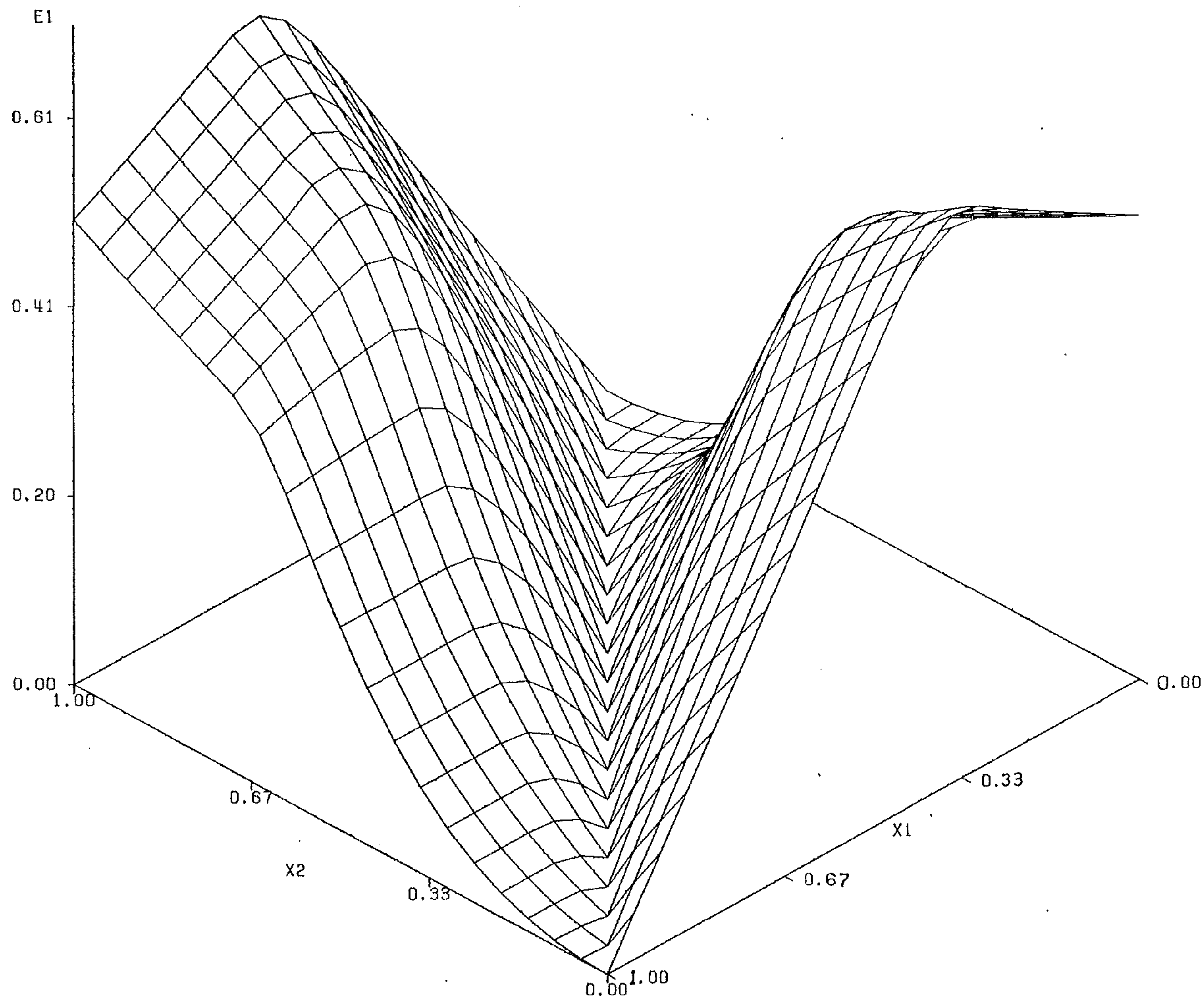
For each value of α_j, $U_i(\alpha_j)$ is the smallest price of i which is not strongly dominated.

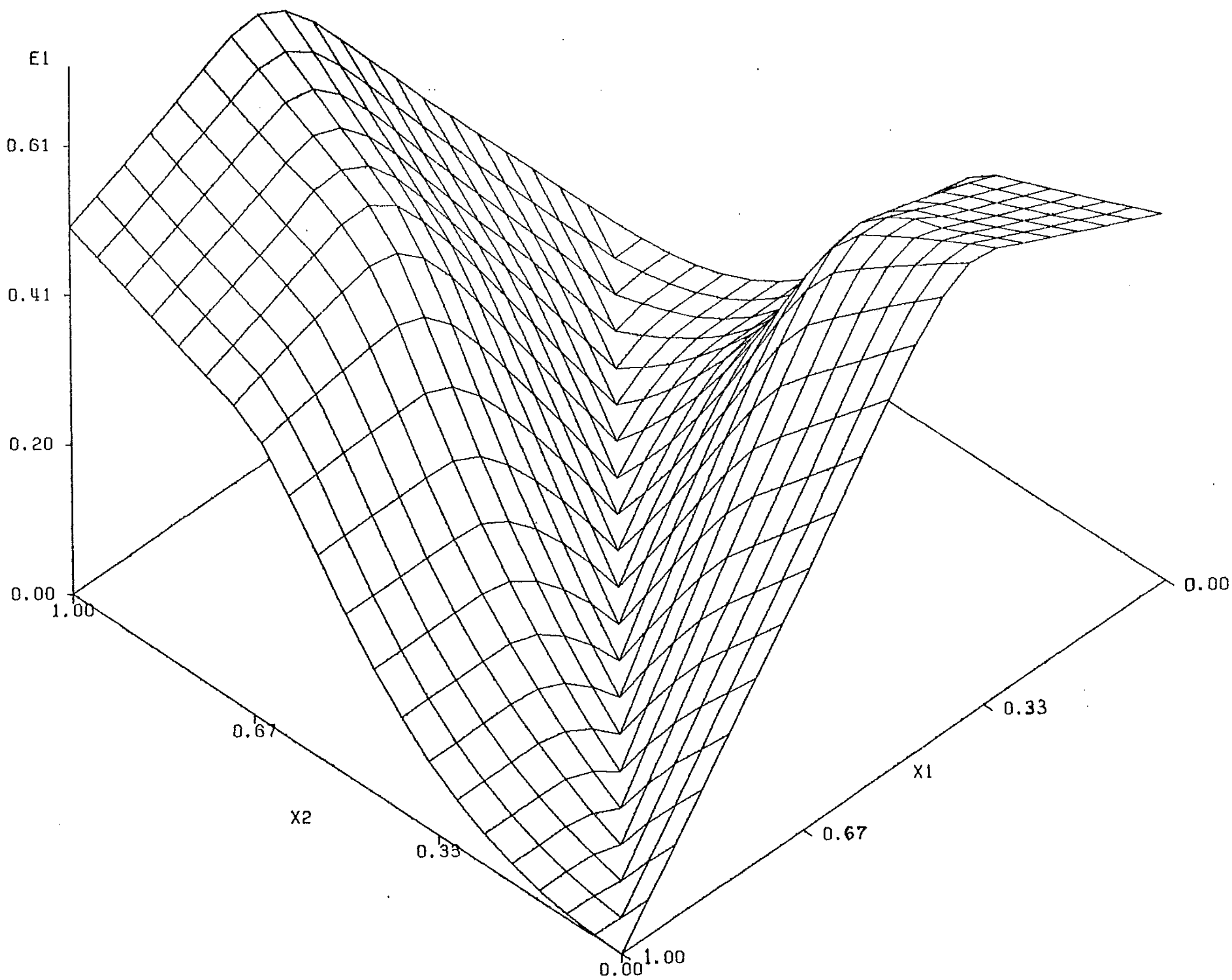

KI
1.03
0.69
0.34
0.00
PJ
PI
0.5
1.0
1.5
0.0
0.0

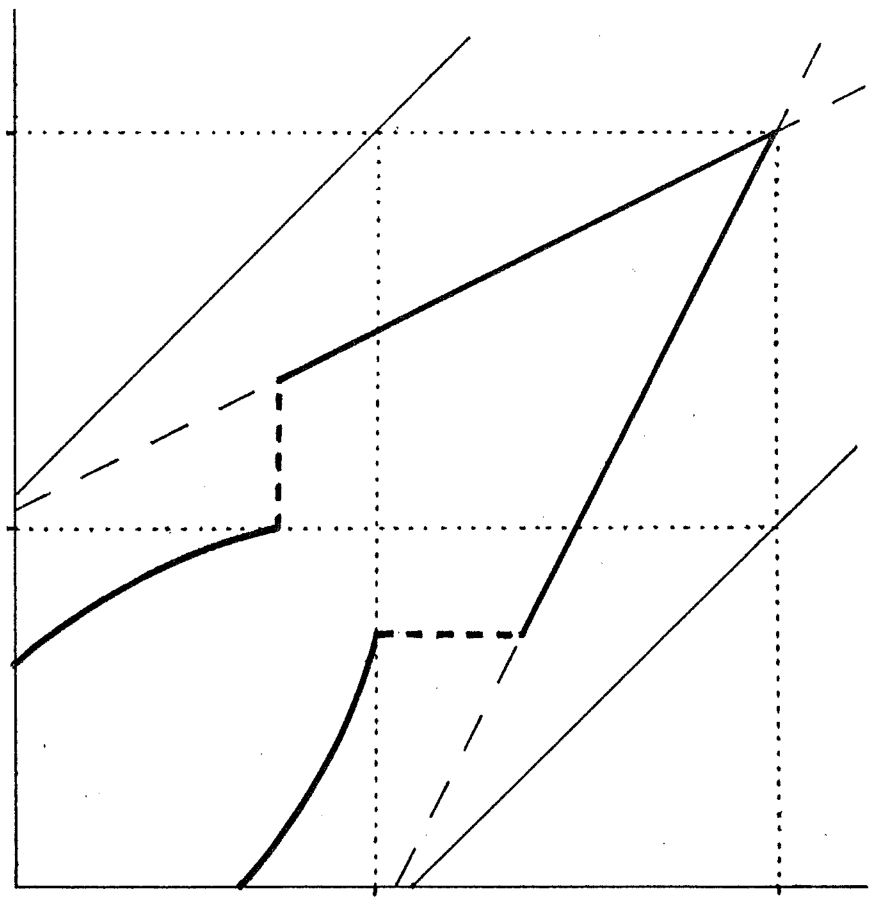

KI
1.03
0.69
0.34
0.00
1.0
PJ
0.0
0.0
0.5
PI
1.0
1.5



X2
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
X1

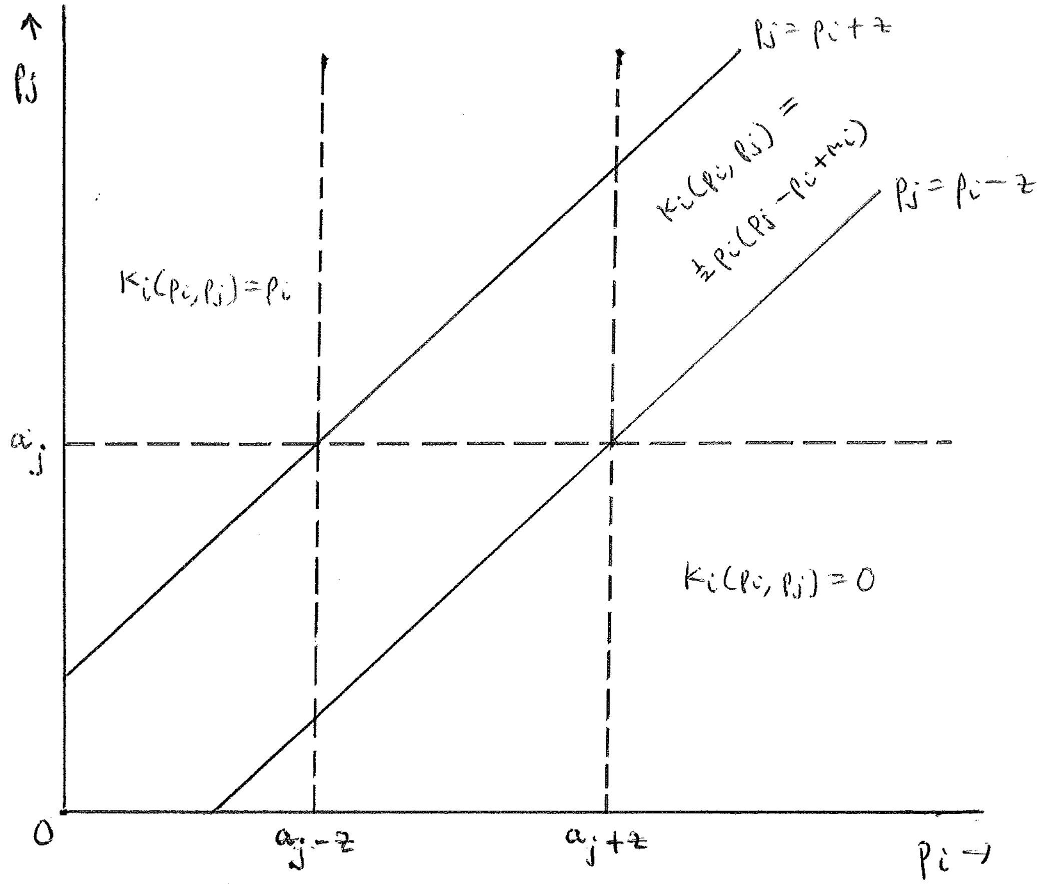

E1
0.61
0.41
0.20
0.00
1.00
0.67
X2
0.33
0.00
1.00
0.67
X1
0.33
0.00


E1
0.61
0.41
0.20
0.00
1.00
0.67
X2
0.33
0.00
1.00
0.67
X1
0.33
0.00

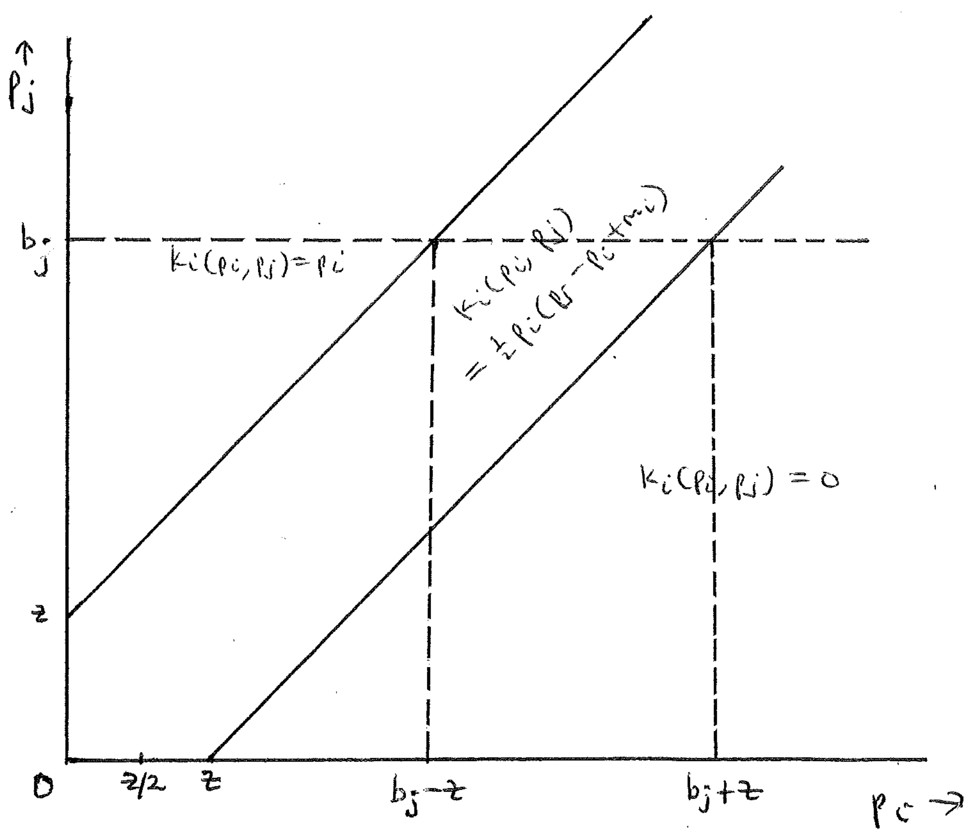


DIAGRAMS FOR PROOFS IN APPENDIX 1

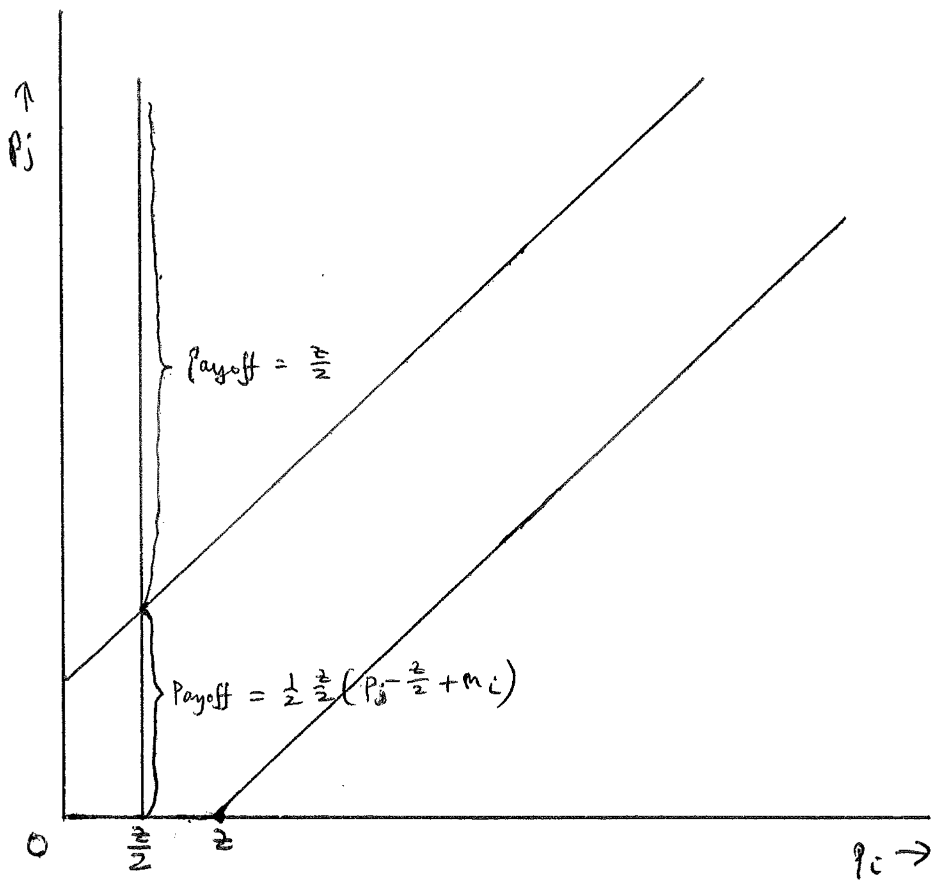
(For referees -- not intended for publication)



(a) — First sentence

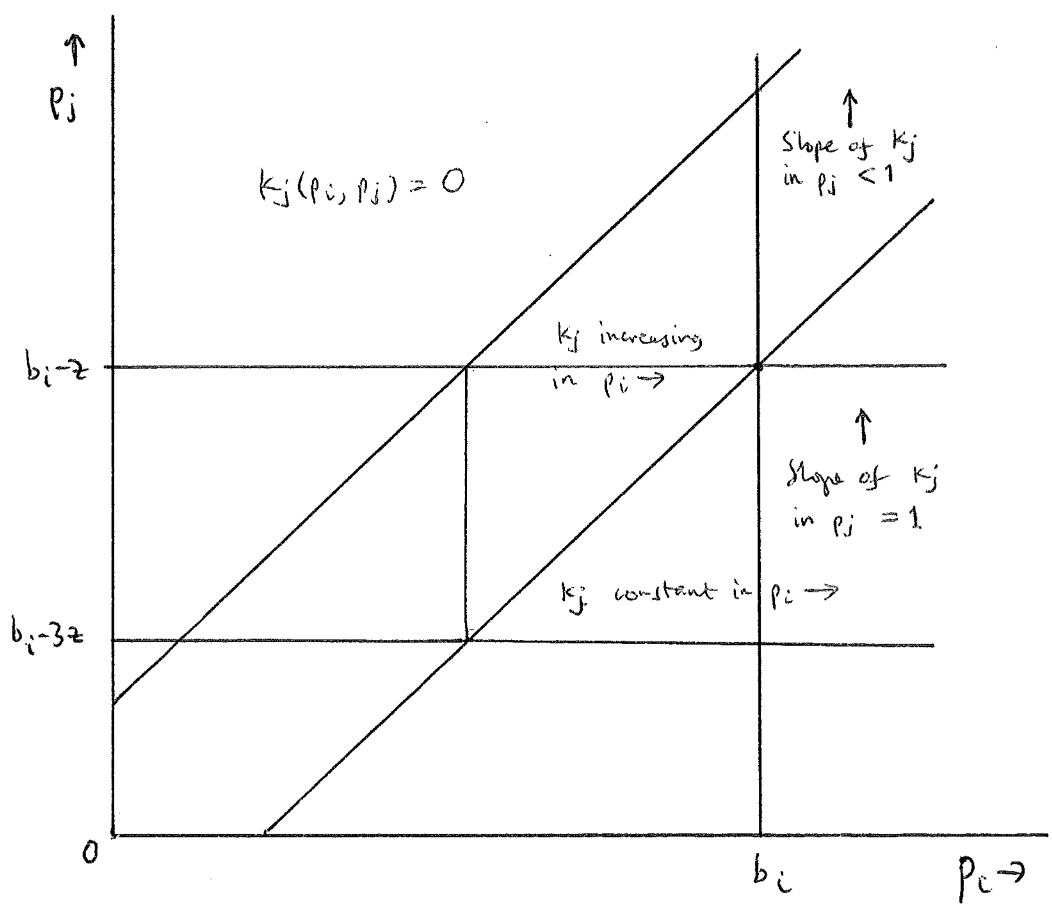


(a) — Second sentence



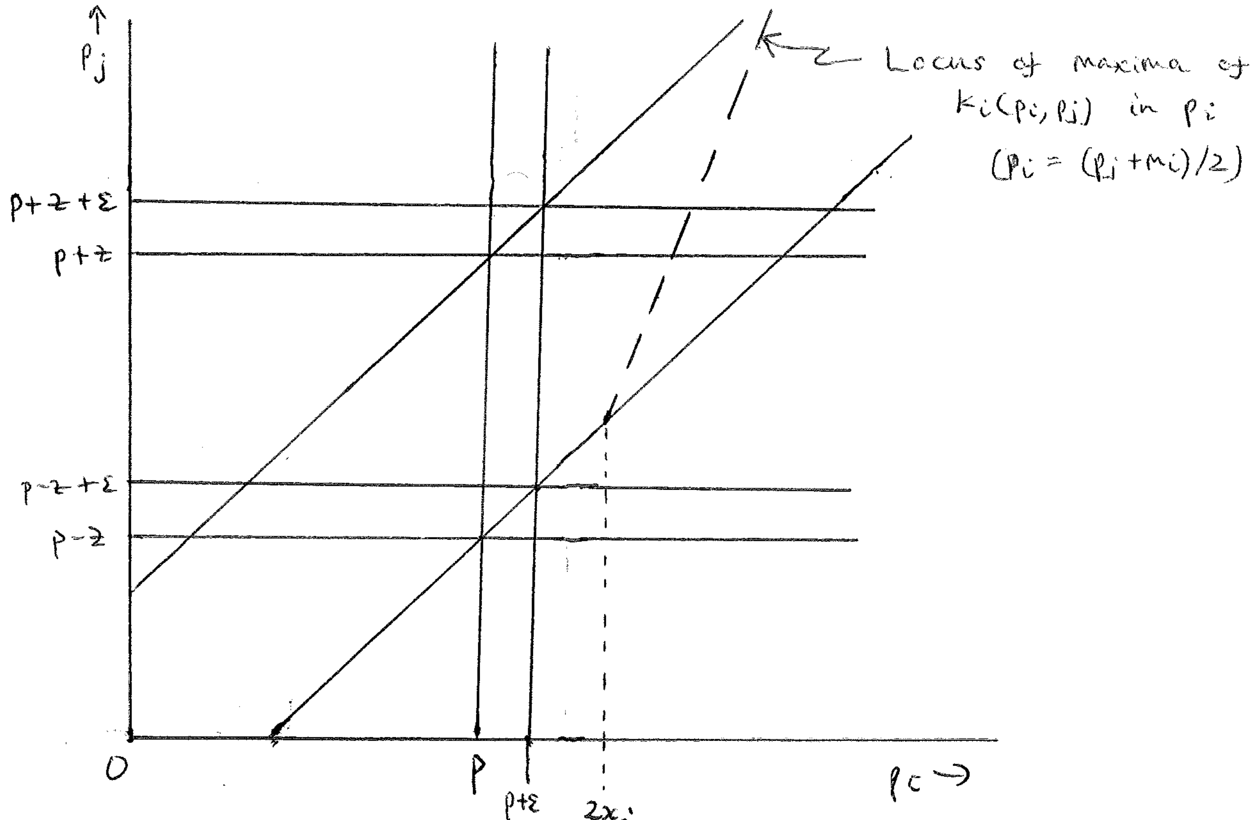
pj
Payoff = z/2
Payoff = ½ z/2 (pj − z/2 + mi)
0
z/2
z
pi

(d)



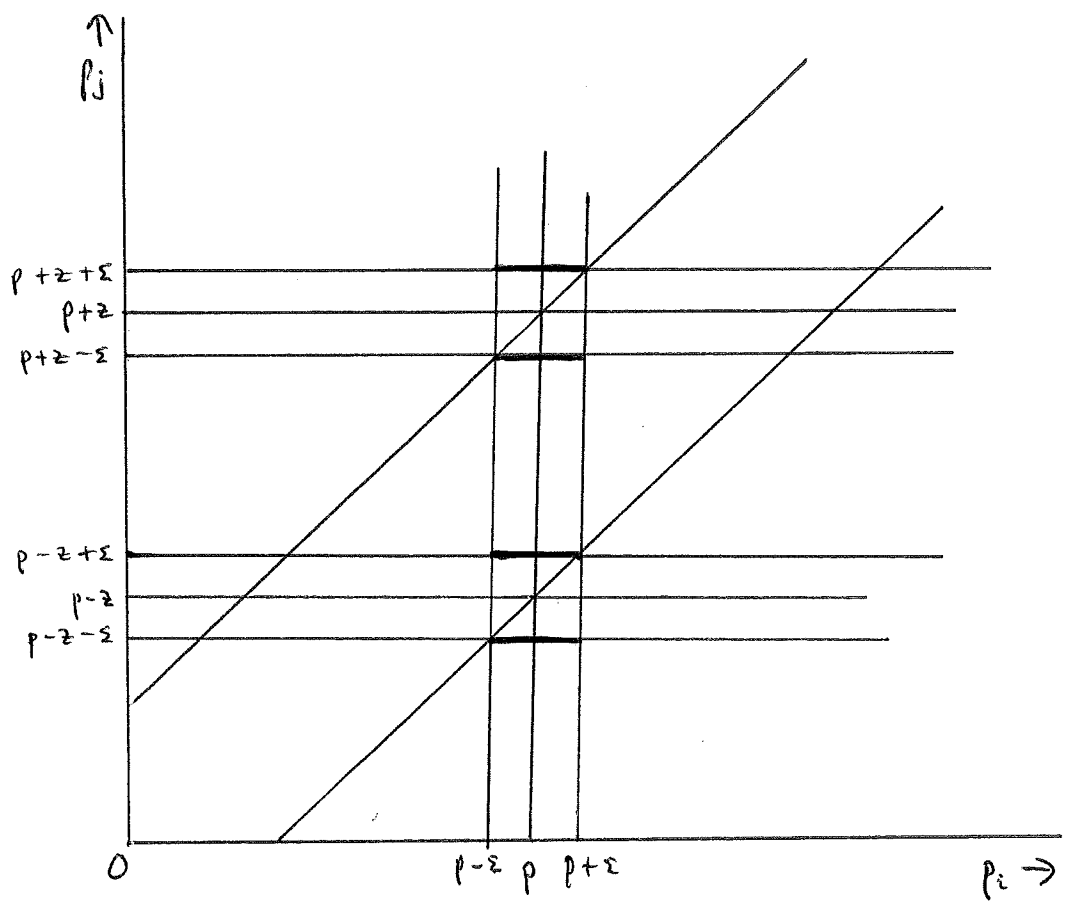
pj
kj(pi, pj) = 0
Slope of kj in pj < 1
kj increasing in pi
bi−z
Slope of kj in pj = 1
kj constant in pi
bi−3z
0
bi
pi

(f)



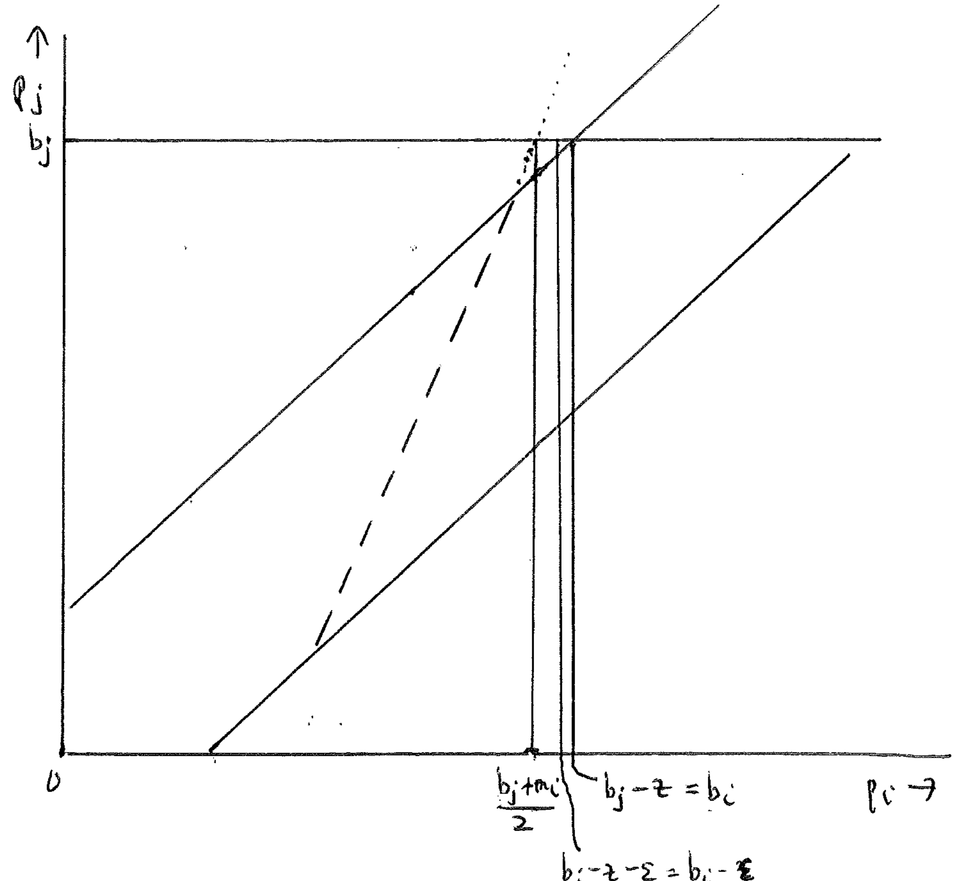
pj
Locus of maxima of
ki(pi, pj) in pi
(pi = (pj + mi)/2)
p+z+ε
p+z
p-z+ε
p-z
O
p
p+ε
2xi
pi

(g)

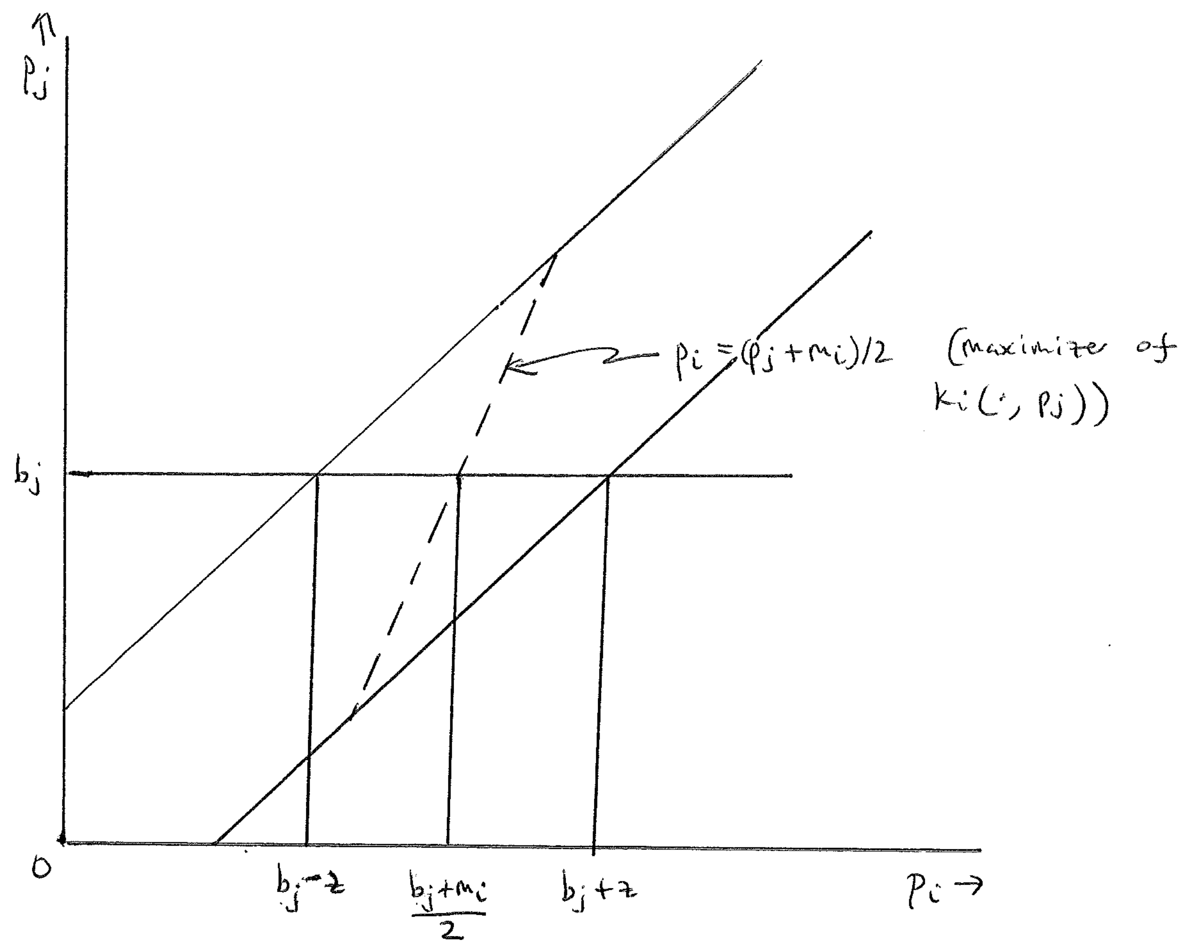


pj
p+z+ε
p+z
p+z-ε
p-z+ε
p-z
p-z-ε
O
p-ε
p
p+ε
pi

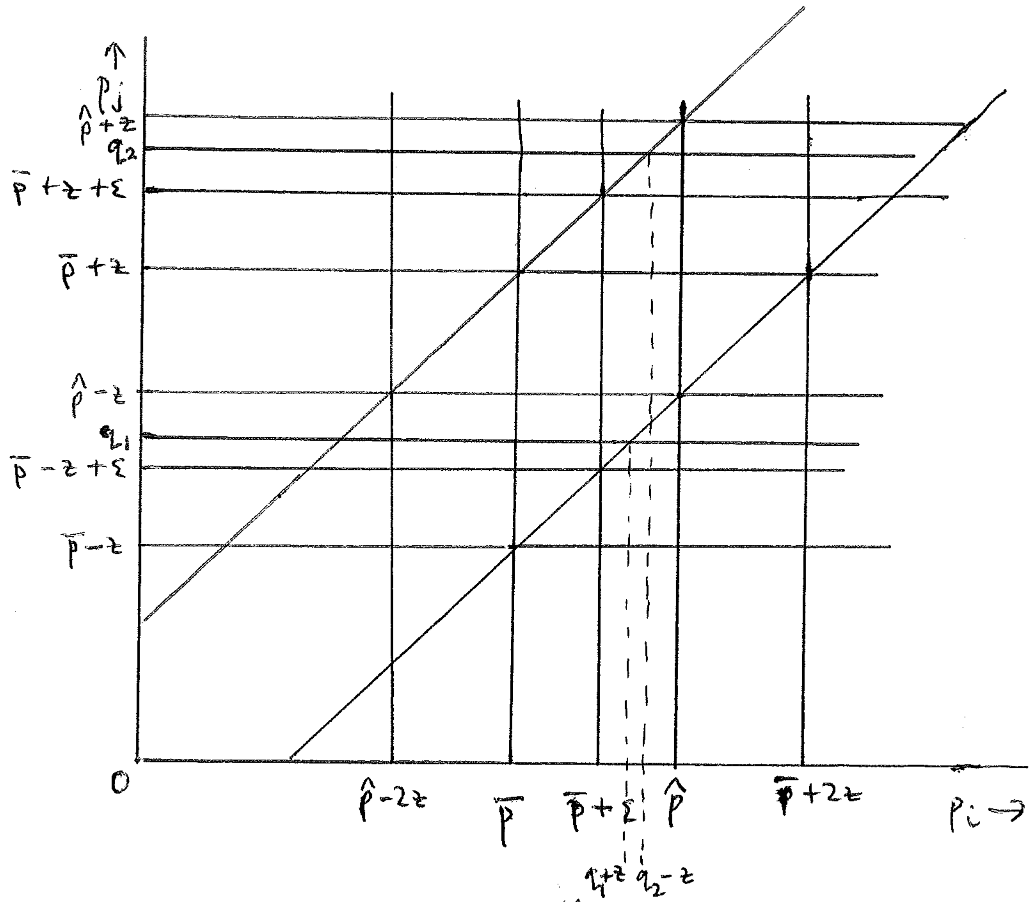
(h)



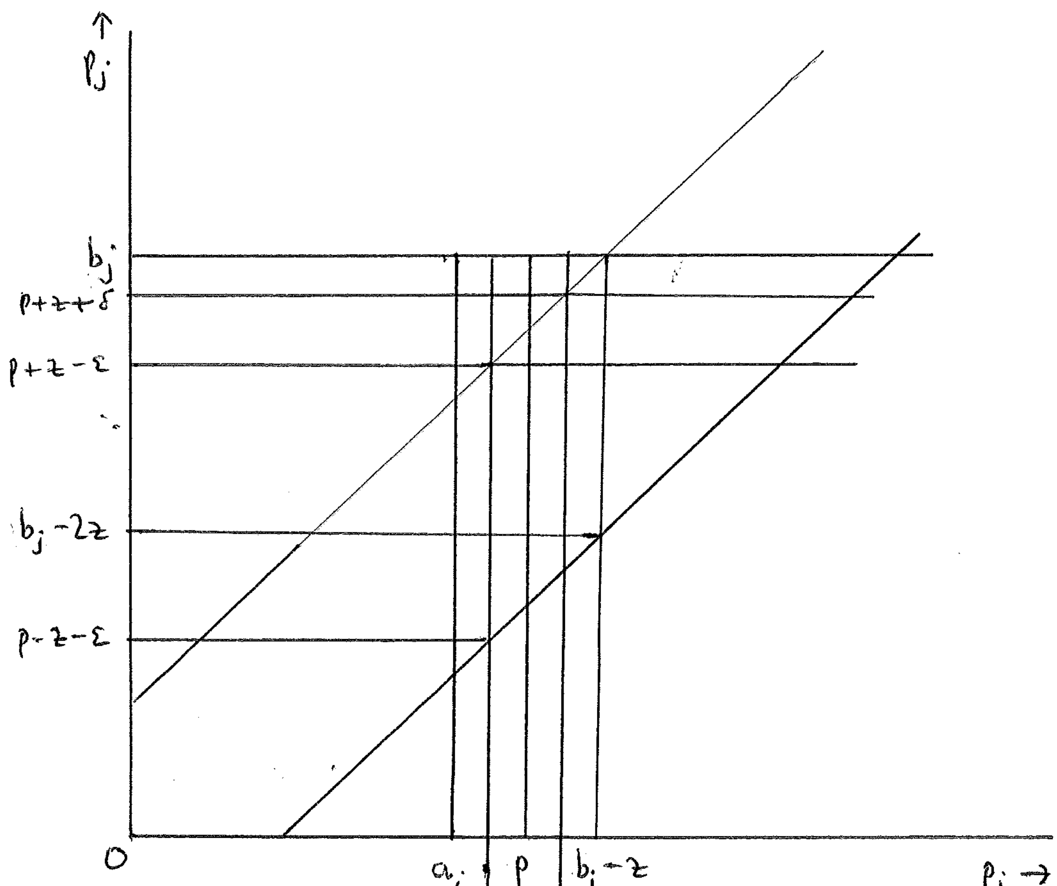
(i) — Second sentence



(i) — Sixth sentence


p_j
$\hat{p}+z$
q_2
$\bar{p}+z+\varepsilon$
$\bar{p}+z$
$\hat{p}-z$
q_1
$\bar{p}-z+\varepsilon$
$\bar{p}-z$
0
$\hat{p}-2z$
$\bar{p}$
$\bar{p}+\varepsilon$
$\hat{p}$
$\bar{p}+2z$
$p_i \rightarrow$
q_1+z
q_2-z

(j)


p_j
b_j
$p+z+\delta$
$p+z-\varepsilon$
b_j-2z
$p-z-\varepsilon$
0
a_i
p
b_j-z
$p-\varepsilon$
$p+\delta$
$p_i \rightarrow$

(k)