CHAPTER 9

The role of risk aversion in a simple bargaining model

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The purpose of this paper is to study the effect of a change in an individual's degree of risk aversion on the perfect Bayesian Nash equilibrium in a simple model of bargaining. I find that, contrary to the results in the axiomatic model with riskless outcomes due to Nash, an opponent may be made worse off by such a change. Further, an individual may want to take an action that identifies him as more, rather than less, risk averse than he really is. In the course of the analysis, I fully characterize the equilibria of a class of "wars of attrition" with incomplete information, and single out one as "perfect" in a certain sense; this result may be of independent interest.

9.1 Introduction

The role of risk aversion in bargaining has been widely studied within the axiomatic framework of Nash (1950) (see, for example, Roth (1979), Perles and Maschler (1981)). It has been found that if the negotiation concerns riskless outcomes, then the more risk averse an individual is, the higher the payoff of his opponent. Related results show that in this case it is to the advantage of an individual to "pretend" to be less risk averse than

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he really is (Kurz (1977, 1980), Thomson (1979), Sobel (1981)). These results have some intuitive appeal: Given any (probabilistic) beliefs about the behavior of his opponent, it seems that an individual should behave more cautiously, the more risk averse he is. However, this fact influences his opponent's behavior, and without a more detailed specification of the information possessed by both parties and of the precise structure of the negotiation, it is not clear how the *equilibrium* behavior changes. (In the case where the potential agreements involve lotteries, the axiomatic model predicts that an increase in an individual's risk aversion may reduce the payoff of his opponent (see Roth and Rothblum (1982)). Here, I restrict attention to the case in which agreements concern riskless outcomes.)

It is natural to investigate these issues by modeling the process of negotiation as a (noncooperative) strategic game, and by studying the effect of changes in the players' risk aversions on the characteristics of the Nash equilibria. For such a comparative static exercise to make sense, the game must have a unique equilibrium. It is clear that if the equilibrium strategies are pure, then a change in a player's risk aversion that preserves his preferences over certain outcomes has no effect on his opponent's payoff. (This is the case, for example, in Rubinstein's (1982) model.[1]) Thus, for the degree of risk aversion to influence the outcome, the equilibrium strategies must involve randomization.

The model that I analyze is designed with these facts in mind. It is a simple version of those formulated by Hicks (1932), Bishop (1964), and Cross (1965). At each time in $[0,1]$, two individuals can produce a flow of one unit of some good desirable to them both. Before production can begin, a contract must be negotiated that specifies how the flow of output will be divided between the two parties. At time 0, each party begins by demanding some fraction of the flow – say individual i demands $d_i(0)$. So long as the demands are incompatible (i.e., sum to more than the output available), no production takes place. In the most general version of the model, at each time, each individual may adjust his demand. If t is the first time at which the demands are compatible, and in fact $d_1(t) + d_2(t) = 1$, then at each time in $[t,1]$ each individual i receives the flow $d_i(t)$. This most general form of the model is unwieldy; in order to get some rather specific results, I assume that the allowable concession patterns are very special.

In the simplest case (considered in Sections 9.2 and 9.3), the demands of both individuals at time 0 are fixed, incompatible, and the same. At each time, each individual may leave his demand the same, or concede to that of his opponent. I model the interaction between the individuals as a strategic game[2] in which a pure strategy of an individual is an element t of

[0,1], with the interpretation that the individual will concede at t if his opponent has not done so by then. (Once his opponent has conceded, there is no cause for further action on his part.) In a slightly richer version of the model (considered in Section 9.4), each individual may choose how much to demand at time zero, but may subsequently only stand firm or concede. Though there are clearly many aspects of negotiation not included in this model, it does capture the tradeoff involved in the intuitive arguments concerning the effects of changes in risk aversion. That is, by delaying concession, an individual sacrifices payoff now in return for the chance that his opponent will concede in the future.

As regards the informational structure of the negotiation, I assume that each individual may be one of many types, which differ in their degrees of risk aversion. The solution is that of Bayesian Nash equilibrium, modified by "perfection" of a certain sort (see Section 9.3). This standard solution captures the idea that each player is uncertain of the type of his opponent. However, it may also be given a more concrete interpretation. Thus, suppose that there are two populations, each consisting of a continuum of individuals of different types. In any play of the game, each member of one population is randomly matched with a member of the other population. A Bayesian Nash equilibrium has the property that if each individual's beliefs about the distribution of concession times in the opponent population is correct, then his equilibrium strategy is optimal. Given this interpretation, it is natural to consider also the case where members of a single population are matched with each other. From the point of view of the Bayesian Nash equilibrium, this is, of course, simply a special case of the two-population model, in which the characteristics of both populations are the same, and attention is restricted to symmetric equilibria (i.e., equilibria in which the strategies used in both populations are the same). However, the comparative static question, which is my main focus, requires separate analysis in the two cases. Viewed as a special case of the two-population model, a change in risk aversion of a potential opponent in the one-population case is a change not only in the characteristics of the opponent population, but also in the characteristics of the player's own population. Given this, I analyze the two cases separately.

First, consider the case in which initial demands are fixed. In the one-population model, there is a unique equilibrium distribution of concession times[3]; in the two-population model, there is a set of equilibria (characterized in theorem 3), but only one is perfect in a certain sense (see proposition 5). In both cases, more risk averse individuals concede earlier in the (perfect) equilibrium. The comparative static results are as follows.

In the one-population case, an individual is made better off by an increase in the risk aversion of his potential opponents, whereas in the two-population case, the opposite is true. Thus, in the two-population model, the prediction of Nash's model is not supported. Unless one argues that the model does not capture some essential aspect of bargaining, or that Nash equilibrium is an inappropriate solution concept, the conclusion is that the effect of a change in an opponent's risk aversion on an individual's negotiated payoff can go in either direction, depending on the precise structure of the negotiation.

To address the issue of "distortion" of preferences in this simple model, I consider how an individual's payoff changes as the fraction of his own population that is less risk averse than him increases. This change causes his opponents to believe with smaller probability that he is risk averse, and so gives him an opportunity to "pretend" that he is not. However, such a change does not affect his equilibrium payoff, although it does reduce the payoff of his less risk averse colleagues.

Although the simple version of the model does not fit into Nash's framework (the set of payoffs to possible agreements may not be convex), it is clear that the solution does not obey appropriately modified versions of his axioms. Most conspicuously, the (perfect) equilibrium is not Pareto-efficient. This lack of efficiency does not derive from uncertainty about opponents' payoffs – even if everyone is identical, the solution is not efficient. Rather, it is the (inevitable) uncertainty about opponents' actions that prevents agreement at time zero. It seems that the continuous nature of the model contributes to this outcome: If disagreement is once-and-for-all (as in Nash's (1953) "demand game"), then it seems less likely that it will be the outcome of negotiation. If, on the other hand, demands may be adjusted continuously (or, in the simple case here, a concession can be made at any time), then it seems quite unlikely that an equilibrium will involve agreement from the very beginning.

My analysis of the case in which initial demands may be chosen is limited. I show that when there are two types in each population and two possible initial demands, there is no separating equilibrium in which all members of a given type choose the same demand and the two types in each population choose different demands. The reason for this is that the less risk averse individuals can benefit from pretending to be more risk averse (see Section 9.4). There is thus another sense in which the model works differently from the axiomatic one of Nash. I also show that there is a continuum of pooling equilibria, in which a positive fraction of each type in each population makes each initial demand. Given this non-uniqueness, it is not possible to perform the comparative static exercises discussed previously; it is an open question whether the model can be modified to produce a unique equilibrium. However, the analysis does show that the basic model does not degenerate when choice of initial demand is allowed.

Recently, a number of authors (e.g., Samuelson (1980), McLennan (1981), Crawford (1982), Rubinstein (1982), Chatterjee and Samuelson (1983), Fudenberg and Tirole (1983*a*), and Sobel and Takahashi (1983)) have modeled bargaining as a noncooperative strategic game. None of these focuses specifically on the role of risk aversion. In most cases, the bargaining parties are assumed to be risk neutral (although Samuelson (1980) and Chatterjee and Samuelson (1983) do contain some analysis of the effect of changes in the players' risk aversions). The model here, designed specifically to address the role of risk aversion, differs in several respects from these models. Most significantly, time runs continuously, so that the players have great flexibility in choosing their time of action. A player can always wait a short time (thereby losing at most a very small amount of payoff) in case his opponent will concede; if time is discrete, this is not possible. Note, however, that because the possibility for changing demands is so limited, players' actions (or lack thereof) transmit no useful information during the course of play (except for their choice of initial demand, when this is allowed), whereas this information transmission is central to some of the models just cited. Young (1983) analyzes a model that is in some respects similar to the simple model considered here. However, the structure of the payoffs in his model is not quite the same, and time is discrete; he does not consider the effect of changes in risk aversion.

The game associated with the simple version of my model is what is known in the literature as a "war of attrition" (see, for example, Riley (1980)). Nalebuff and Riley (1984) have (independently) found a class of equilibria in a model that is different in some respects from mine (e.g., the time horizon is infinite, and there is a continuum of types), but is similar in general structure. However, they do not show that they have found all of the equilibria; nor do they consider the issue of perfection, or the effect of a change in an individual's risk aversion. Also related is the work of Fudenberg and Tirole (1983*b*), who have (independently) shown that, in another version of a war of attrition, there is a unique Bayesian Nash equilibrium that is perfect in a certain sense.

9.2 Bargaining within a single population

The model

The population consists of a continuum of individuals. The environment of the negotiation between any two individuals is as follows. Time runs continuously in [0,1]. At each point in time, a flow of one unit of output can be produced, if the individuals can agree how to split it between them. The rules of negotiation are simple. At time 0, each individual demands

$\frac{1}{2} < a < 1$ units of output. At any subsequent time in [0,1], each may concede to the demand of the other. The outcome of negotiation for each individual is an output stream $\mathbf{x}$: $[0,1] \to [0,1]$ of the form

$$\mathbf{x}(s) = \begin{cases} 0 & \text{if } 0 \le s < t, \\ x & \text{if } t \le s \le 1, \end{cases}$$

where $0 \le t \le 1$ and $0 \le x \le 1$. Such an output stream is characterized by the pair $(x,t) \in [0,1]^2$. If an individual is first to concede, and does so at t, he receives the output stream $(1 - a,t)$; his opponent receives (a,t). If the individuals concede simultaneously at t, the equilibrium of the game that I study is independent of the output stream received by each individual, so long as that stream is of the form (c,t), where $1 - a < c < a$; for notational convenience, I assume that it is $(\frac{1}{2},t)$. There are m *types* of individuals. The fraction $\gamma_i > 0$ of the population is of type $i(= 1, \ldots, m)$. The preferences over lotteries on output streams of individuals of type i are represented by a von Neumann-Morgenstern *utility function* u_i: $[0,1]^2 \to \mathbb{R}_+$ with the following properties:

(P.1) For each (x,t), $u_i(x,1) = u_i(0,t) = 0$;
(P.2) For each $t < 1$, u_i is increasing in x;
(P.3) For each $x > 0$, u_i is continuous in t, and continuously differentiable and decreasing in t whenever $t < 1$.

In order to isolate the role of risk aversion, I assume that all of the types have the same preferences over sure outcomes; they differ only in their degrees of risk aversion, type i being more risk averse than type $i + 1$. Precisely, a utility function v is *more risk averse* than a utility function u if there is a strictly concave function f: $\mathbb{R}_+ \to \mathbb{R}_+$ such that $v = f \circ u$. I assume the following:

(P.4) For each $i = 1, \ldots, m - 1$, u_i is more risk averse than u_{i+1}.

It is easy to check that an example of a collection $\{u_i\}$ of utility functions that satisfies (P.1) through (P.4) is that for which $u_i(x,t) = (1 - t)^{\alpha_i} x^{\alpha_i}$, with $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < 1$.

The only choice an individual has is the time at which to concede. Thus, a (mixed) *strategy* of an individual is simply a cumulative probability distribution on [0,1]. Only the average strategy of individuals of type i is determined in equilibrium, not the strategy of any particular individual. I refer to this average strategy as the *strategy of* i, and denote it E_i. For each $0 \le t \le 1$, let

$$G(t) = \sum_{i=1}^{m} \gamma_i E_i(t), \tag{9.1}$$

so that $G(t)$ is the probability that a randomly selected individual concedes at or before t. I refer to G as the *distribution of concession times in the population.* The distribution of concession times relevant to an individual's choice is the one generated by all of the *other* individuals' strategies. However, since the population is nonatomic, this is the same as G. If an individual of type i uses the pure strategy t, his expected payoff in negotiations with a randomly selected opponent is

$$P_i(t,G) = \int_{[0,t)} u_i(a,s)\, dG(s) + u_i(\tfrac{1}{2},t)J_G(t) + u_i(1-a,t)(1-G(t)), \qquad \textbf{(9.2)}$$

where $J_G(t)$ is the size of the atom in G at t. The payoff to the mixed strategy E_i is $P_i(E_i,G) = \int_{[0,1]} P_i(t,G)\, dE_i(t)$, and $(E_1, \ldots, E_m)$ is a (Bayesian Nash) *equilibrium* if for $i = 1, \ldots, m$ we have

$$P_i(E_i,G) \geq P_i(E,G) \quad \text{for all strategies } E,$$

where G is defined in (9.1).

Equilibrium

There is a unique equilibrium $(E_1, \ldots, E_m)$, defined as follows. There exist numbers $0 = p_0 < \cdots < p_y = \cdots p_m = 1$ such that the support of E_i (denoted supp E_i) is equal to $[p_{i-1},p_i]$ for $i = 1, \ldots, m$. The strategies E_i are nonatomic on $[0,1)$, and such that G causes the payoff $P_i(t,G)$ of an individual of type i to be constant on $[p_{i-1},p_i]$. A distribution G with this property can be found by solving the differential equations obtained by setting equal to zero the derivative with respect to t of each $P_i(t,G)$. We find that, for some $A > 0$, for $p_{i-1} \leq t < p_i$,

$$G(t) = 1 - A \exp\left(\int_0^t U_i(a,a,s)\, ds\right),$$

where, for any $\frac{1}{2} < a < 1$, $\frac{1}{2} < b < 1$, and $0 \leq s < 1$, and any utility function u: $[0,1]^2 \rightarrow \mathbb{R}_+$, the function U is defined by

$$U(a,b,s) = \frac{-D_2 u(1-a,s)}{u(b,s) - u(1-a,s)}. \qquad \textbf{(9.3)}$$

(I have made the definition of U more general than necessary for the present purposes; it will be used also later.) Now, the fact that G is generated by the E_i's means that the equilibrium is as follows. For notational convenience, let $\Gamma(0) = 0$ and $\Gamma(k) = \Sigma_{i=1}^k \gamma_i$ for $k = 1, \ldots, m$. The p_i's are defined iteratively. First, $p_0 = 0$. Now, given $p_{i-1} < 1$, suppose that there exists $\bar{p} < 1$ such that

$$\frac{1 - \Gamma(i)}{1 - \Gamma(i-1)} = \exp\left(-\int_{p_{i-1}}^{\bar{p}} U_i(a,a,s)\, ds\right). \qquad \textbf{(9.4)}$$

Then, let $p_i = \bar{p}$, and continue the process. If there is no such $\bar{p}$, set $i = y$ and let $p_y = p_{y+1} = \cdots = p_m = 1$. For $i = 1, \ldots, y$, the equilibrium strategy E_i of type i has support $[p_{i-1}, p_i]$,

$$E_i(t) = [1 - \Gamma(i-1)] \frac{\left[1 - \exp\left(-\int_{p_{i-1}}^{t} U_i(a,a,s)\, ds\right)\right]}{\gamma_i} \quad \text{if } p_{i-1} \le t < p_i, \qquad \textbf{(9.5)}$$

and $E_i(t) = 1$ if $p_i \le t$. For $i = y + 1, \ldots, m$, the equilibrium strategy E_i is purely atomic, with mass at $t = 1$ (i.e., $E_i(t) = 0$ if $t < 1$ and $E_i(1) = 1$).

The fact that this defines an equilibrium, and that there is no other, follows from the results of Section 9.3 (see corollary 4). However, it is easy to check that each E_i is a strategy and that $P_i(t,G)$ is constant on $[p_{i-1}, p_i]$ (= supp E_i).

Note that if all individuals are identical, the equilibrium does not degenerate – in fact, all individuals then use mixed strategies with support [0,1]. The only efficient outcome is for all to concede at time 0, but this is not an equilibrium. If all individuals in a set S of positive measure concede at time 0, the distribution of concession times in the population contains an atom at 0. Hence, every individual, including those in S, can benefit from waiting a short period, and so it is not optimal for them to concede at time 0.

The effect of a change in risk aversion

Let k be such that $p_k < 1$ (i.e., in the unique equilibrium, individuals of type k concede with probability 1 before time 1). Now, suppose that individuals of type k become more risk averse, but not more so than individuals of type $k - 1$. That is, consider a new game in which the utility function of type k is $\hat{u}_k$, which is more risk averse than u_k, and less risk averse than u_{k-1}. (Throughout, I use a circumflex to denote the new value of an object.) This means that the order in which the types concede in equilibrium is preserved (since the ordering of risk aversions is preserved, and the unique equilibrium has the property that the most risk averse types concede first). In particular, the support of $\hat{E}_k$ lies between those of $\hat{E}_{k-1}$ and $\hat{E}_{k+1}$.

I first argue that this change has no effect on the equilibrium payoffs of types $1, \ldots, k-1$. To see this, note that from the definition of p_i (see (9.4)), we have $\hat{p}_i = p_i$ for $i = 1, \ldots, k-1$, and hence from (9.5) we have $\hat{E}_i = E_i$ for $i = 1, \ldots, k-1$. Thus, $\hat{G}(t) = G(t)$ for all $0 \leq t \leq p_{k-1}$. Now, $P_i(t,G)$ is constant on supp E_i, and so the equilibrium payoff of type i is equal to $P_i(p_{i-1},G)$, which depends on the form of G only on $[0,p_{i-1}]$ (see (9.2)). Hence, the equilibrium payoff of types $1, \ldots, k-1$ is unaffected by the change.

To analyze the changes in the payoffs of the remaining types, I need the following result (see (9.3) for the definition of U). (The result is more general than necessary for the present analysis; it will be used also in the next section.)

Lemma 1. Suppose that the utility function $\hat{u}$ is more risk averse than the utility function u. Then, for any $\frac{1}{2} < a < 1$ and $\frac{1}{2} < b < 1$, we have $\hat{U}(a,b,s) > U(a,b,s)$ for all $0 \leq s < 1$.

Proof. Let $\hat{u} = f \circ u$. The result follows from the fact that, since f is strictly concave and $1 - a < \frac{1}{2} < b$, if $0 \leq s < 1$, then

$$f'(u(1-a,s))(u(b,s) - u(1-a,s)) > f(u(b,s)) - f(u(1-a,s)).$$

This result implies that $\hat{U}_k(a,a,s) > U_k(a,a,s)$ for all $0 \leq s < 1$, and thus from (9.4) we have $\hat{p}_k < p_k$, and from (9.5) we have $\hat{E}_k(t) > E_k(t)$ for all $p_{k-1} = \hat{p}_{k-1} < t \leq \hat{p}_k$. Thus, $\hat{G}(t) > G(t)$ on $(p_{k-1},\hat{p}_k]$ (see Figure 9.1). Now, $\hat{u}_i = u_i$ for $i = k+1, \ldots, m$, so that $\hat{U}_i = U_i$; but since $\hat{p}_k < p_k$, we have $\hat{p}_{k+1} \leq p_{k+1}$, with strict inequality if $p_{k+1} < 1$ (see (9.4)), and so from (9.5), we have $\hat{E}_{k+1}(t) > E_{k+1}(t)$ for all $\hat{p}_k < t < \hat{p}_{k+1}$. Thus, $\hat{G}(t) > G(t)$ also on $(\hat{p}_k,\hat{p}_{k+1})$. Continuing this argument, we see that $\hat{G}(t) > G(t)$ on $(p_{k+1},1)$.

Now, as noted previously, the equilibrium payoff of type i is equal to $P_i(p_{i-1},G)$. If we integrate by parts in the expression for $P_i(p_{i-1},G)$ (see (9.2)), using the fact that G is nonatomic on $[0,1)$ and $u_i(x,1) = 0$ for all x (see (P.1)), then, given that each u_i is decreasing in t (see (P.3)) and $\hat{G}(t) > G(t)$ on $(p_{k-1},1)$, we see that $P_i(\hat{p}_{i-1},\hat{G}) > P_i(p_{i-1},G)$ for all $i = k+1, \ldots, m$. That is, the equilibrium payoffs of types $k+1, \ldots, m$ increase. We can summarize these results as follows.

Proposition 2. Let k be a type that in equilibrium concedes with probability 1 before time 1. Then, if individuals of type k become more risk averse (but not more so than those of type $k-1$), the equilibrium payoffs

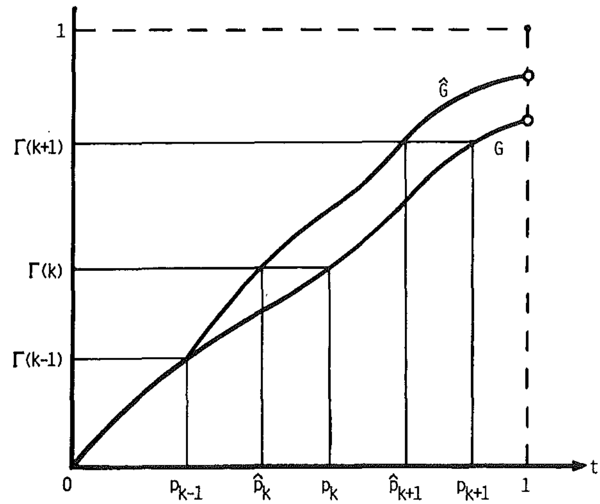

Figure 9.1 Change in the equilibrium distribution of concession times caused by an increase in the risk aversion of individuals of type k in the one-population model

of less risk averse individuals increase, whereas those of more risk averse individuals are unaffected.

9.3 The case of two populations

The model

There are two populations like that described in the previous section. Each individual in population 1 bargains with an individual in population 2. The name of an object in population 2 is the alphabetic successor of the name of the corresponding object in population 1. Thus, in population 2 there are n types. I refer to type i in population ℓ as "type ℓi". The fraction δ_j of population 2 is of type $j(= 1, \ldots, n)$; the sum $\Sigma_{j=1}^{k} \delta_j$ is denoted $\Delta(k)$. At time 0, individuals in population 2 demand $\frac{1}{2} < b < 1$ units of output. Individuals of type $2j$ have a utility function v_j: $[0,1]^2 \to \mathbb{R}$ satisfying (P.1) through (P.3). The function v_j is more risk averse than v_{j+1}, as in (P.4). A strategy of type $2j$ is denoted F_j, and the distribution of concession times in population 2 is H. If an individual of type $1i$ uses the pure strategy t, then his payoff is

$$P_i(t,H) = \int_{[0,t)} u_i(a,s)\, dH(s) + u_i(\tfrac{1}{2},t)J_H(t) + u_i(1-b,t)(1-H(t)); \quad \textbf{(9.6)}$$

if an individual of type $2j$ uses the pure strategy t, then his payoff is

$$Q_j(t,G) = \int_{[0,t)} v_j(b,s)\, dG(s) + v_j(\tfrac{1}{2},t)J_G(t) + v_j(1-a,t)(1-G(t)). \quad (9.7)$$

(Once again, for convenience I assume that simultaneous concessions give a payoff of $\frac{1}{2}$ to each individual.)

Equilibrium

In this model, there are many equilibria; they are fully characterized in theorem 3, to be given later. However, I will argue (in the next subsection) that only one equilibrium is perfect in a certain sense. It is this equilibrium that I describe first. Although the details of its definition are somewhat complex, its structure is easy to outline. Within each population, the pattern of concessions is similar to the equilibrium pattern in the one-population model. That is, there exist numbers $0 = p_0 < \cdots < p_y = \cdots = p_m = 1$ and $0 = q_0 < \cdots < q_z = \cdots = q_n = 1$ such that the support of the equilibrium strategy E_i of type $1i$ is $[p_{i-1}, p_i]$ and that of the equilibrium strategy F_j of type $2j$ is $[q_{j-1}, q_j]$. Informally, the p_i's and q_j's can be defined as follows. First, find the distributions of concession times G_1 and H_1 that make types 11 and 21, respectively, indifferent between conceding at any point in $[0,1]$. Now, the equilibrium distributions G and H have to be generated by the actions of the individuals in the two populations. Since type 11 constitutes the fraction γ_1 of population 1, this means that only that part of G_1 up to the point s_1 where $G_1(s_1) = \gamma_1$ can be generated by the actions of individuals of type 11. After that point, the actions of type-12 individuals have to generate G_1. However, in order for the strategy of type-12 individuals to have support commencing at s_1, from this point H has to be such that these individuals, not those of type 11, are indifferent. Similarly, if we try to generate H_1 by the actions of individuals in population 2, we run out of individuals of type 21 at the point t_1 where $H(t_1) = \delta_1$. After this point, G has to be such that type-22 individuals are indifferent. Thus, the equilibrium distributions G and H can be constructed as follows. Start at $t = 0$ with $G = G_1$ and $H = H_1$. Increase t to the point where either $G_1(t) = \gamma_1$ or $H_1(t) = \delta_1$ (i.e., s_1 or t_1 in the preceding discussion), whichever comes first. Suppose that s_1 comes first. Then, starting from s_1, H has to be modified so that type-12 individuals are indifferent. Then, H no longer reaches δ_1 at t_1, but at some other point, say t_1'. After t_1', G must be modified so that type-22 individuals are indifferent; a new point, s_2, for which $G(s_2) = \gamma_1 + \gamma_2 (= \Gamma(2))$, is defined, and the process of building G and H can continue.

Formally, G and H, and hence the equilibrium strategies $E_1, \ldots, E_m$

and $F_1, \ldots, F_n$, can be defined iteratively. The iterative procedure that I describe is slightly more general than necessary to define the present equilibrium, because I will use it later to define other equilibria. For any $0 \leq \alpha < 1$ and $0 \leq \beta < 1$, the procedure $\Pi(\alpha,\beta)$ is as follows.

Procedure $\Pi(\alpha,\beta)$. Let w and x be such that $\Gamma(w) \leq \alpha < \Gamma(w+1)$ and $\Delta(x) \leq \beta < \Delta(x+1)$ (possibly $w=0$ and/or $x=0$), and let $0=p_0=\cdots=p_w$ and $0=q_0=\cdots=q_x$. Suppose that the numbers $0<p_{w+1}<\cdots<p_k<1$ and $0<q_{x+1}<\cdots<q_\ell<1$, where $0 \leq k \leq m-1$, $0 \leq \ell \leq n-1$, and, say, $q_\ell \leq p_k$, satisfy the following properties. First, let $G(0)=\alpha$, and define G on $(q_{j-1},q_j]$ for $j=w+1, \ldots, \ell$ and on $(q_{j-1},p_k]$ for $j=\ell+1$ by

$$G(t)=1-(1-G(q_{j-1}))\exp\left(-\int_{q_{j-1}}^{t} V_j(a,b,s)\,ds\right); \tag{9.8}$$

let $H(0)=\beta$, and define H on $(p_{i-1},p_i]$ for $i=x+1, \ldots, k$ by

$$H(t)=1-(1-H(p_{i-1}))\exp\left(-\int_{p_{i-1}}^{t} U_i(b,a,s)\,ds\right). \tag{9.9}$$

Now, assume that the p_i's and q_j's are such that $G(p_i)=\Gamma(i)$ for $i=w+1, \ldots, k$, $H(q_j)=\Delta(j)$ for $j=x+1, \ldots, \ell$, and $H(p_k)<\Delta(\ell+1)$. (Refer to Figure 9.2.) Note that G and H are continuous and increasing, and $G(t)<1$ and $H(t)<1$ for all $0 \leq t \leq p_k$. Now, as noted before, for any H the payoff $P_i(t,H)$ of type $1i$ depends on the form of H only on $[0,t]$ (see (9.6)), and similarly for $Q_j(t,G)$. Thus, even though G and H are not yet defined on the whole of $[0,1]$, we can calculate $P_i(t,H)$ on $[p_{i-1},p_i]$ for $i=w+1, \ldots, k$ (i.e., $P_i(t,H)$ is independent of the way in which H is extended to $[0,1]$); it is easy to check that it is constant there. Similarly, G is designed so that $Q_j(t,G)$ (see (9.7)) is constant on $[q_{j-1},q_j]$ for $j=x+1, \ldots, \ell$ and on $[q_{j-1},p_k]$ for $j=\ell+1$.

We now extend G and H to the next p_i or q_j, whichever comes first. To do so, for $p_k \leq t \leq 1$, let

$$G_{k+1}(t)=1-(1-\Gamma(k))\exp\left(-\int_{p_k}^{t} V_{\ell+1}(a,b,s)\,ds\right)$$

and let

$$H_{\ell+1}(t)=1-(1-H(p_k))\exp\left(-\int_{q_\ell}^{t} U_{k+1}(b,a,s)\,ds\right).$$

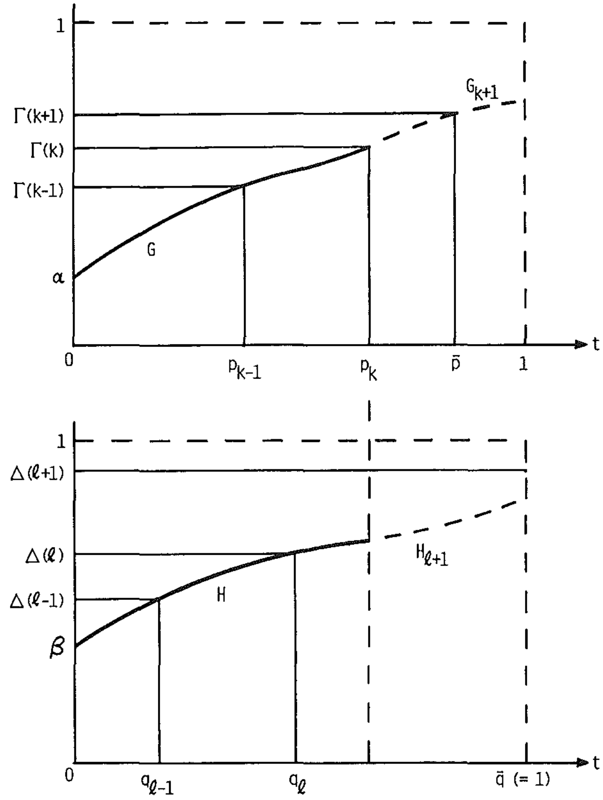

Figure 9.2 Construction of the functions G and H in procedure $\Pi(\alpha,\beta)$

Let $\bar{G}(t) = G(t)$ for $0 \le t \le p_k$ and $\bar{G}(t) = G_{k+1}(t)$ for $p_k < t \le 1$, and define $\bar{H}$ similarly. Then, $P_{k+1}(t,\bar{H})$ and $Q_{\ell+1}(t,\bar{G})$ are constant on $[p_k,1]$. Now, define numbers $\bar{p}$ and $\bar{q}$ as follows. If $\bar{G}(1) \le \Gamma(k+1)$, let $\bar{p} = 1$; otherwise, let $\bar{p}$ be the unique number in $(p_k,1)$ such that $\bar{G}(\bar{p}) = \Gamma(k+1)$. If $\bar{H}(1) \le \Delta(\ell+1)$, let $\bar{q} = 1$; otherwise, let $\bar{q}$ be the unique number in $(p_k,1)$ such that $\bar{H}(\bar{q}) = \Delta(\ell+1)$. (Such numbers exist since $\bar{G}$ and $\bar{H}$ are continuous and increasing.) Now, if $\min(\bar{p},\bar{q}) = 1$, let $p_{k+1} = \cdots = p_m = 1$ and $q_{\ell+1} = \cdots = q_n = 1$; if $\min(\bar{p},\bar{q}) = \bar{p} < 1$, let $p_{k+1} = \bar{p}$; if $\min(\bar{p},\bar{q}) = \bar{q} < 1$, let $q_{\ell+1} = \bar{q}$. In each case, extend G and H to $[0,\min(\bar{p},\bar{q})]$ by letting $G(t) = \bar{G}(t)$ and $H(t) = \bar{H}(t)$ if $0 \le t < 1$, and $G(1) = H(1) = 1$.

If $\min(\bar{p},\bar{q}) = 1$, then the process ends and G and H are defined on the whole of $[0,1]$. If this is not so, then either the collection of p_i's or the collection of q_j's has been augmented and G and H have been extended in a way that satisfies the conditions necessary to repeat the process. Thus, this procedure defines uniquely numbers $0 = p_0 = \cdots = p_w < \cdots < p_y = \cdots = p_m = 1$ and $0 = q_0 = \cdots = q_x < \cdots < q_z = \cdots = q_n = 1$ and continuous and increasing functions G and H on $[0,1)$ with $G(p_i) = \Gamma(i)$ for $i = w + 1, \ldots, y$, $H(q_j) = \Delta(j)$ for $j = x + 1, \ldots, z$, and $G(1) = H(1) = 1$.

Define strategies $E_1, \ldots, E_m$ and $F_1, \ldots, F_n$ as follows:

$$E_i(t) = \begin{cases} 0 & \text{if } 0 \le t < p_{i-1}, \\ \dfrac{G(t) - \Gamma(i-1)}{\gamma_i} & \text{if } p_{i-1} \le t < p_i, \\ 1 & \text{if } p_i \le t \le 1; \end{cases} \tag{9.10}$$

$$F_j(t) = \begin{cases} 0 & \text{if } 0 \le t < q_{j-1}, \\ \dfrac{H(t) - \Delta(j-1)}{\delta_j} & \text{if } q_{j-1} \le t < q_j, \\ 1 & \text{if } q_j \le t \le 1. \end{cases} \tag{9.11}$$

(Note that this means, for example, that $E_1, \ldots, E_w$ are pure strategies involving concession at $t = 0$, E_{y+1} may have an atom at $t = 1$, and $E_{y+2}, \ldots, E_m$ are pure strategies involving concession at $t = 1$.) This completes the description of the procedure $\Pi(\alpha,\beta)$.

Now, I claim that the strategies defined by $\Pi(0,0)$ constitute an equilibrium of the game. Note that all of the strategies thus defined are nonatomic on $[0,1)$. As noted in the construction, G and H are such that $Q_j(t,G)$ is constant on $[q_{j-1},q_j]$ for $j = 1, \ldots, n$, and $P_i(t,H)$ is constant on $[p_{i-1},p_i]$ for $i = 1, \ldots, m$. To show that the E_i's and F_j's constitute an equilibrium, it suffices to show that $Q_j(t,G)$ is increasing on $(0,q_{j-1})$ and decreasing on $(q_j,1)$, and similarly for $P_i(t,H)$ (since the nonatomicity of G and H on $[0,1)$ implies that P_i and Q_j are continuous in t).

Consider $Q_j(t,G)$ on (q_{h-1},q_h), with $h \le j - 1$. Using the definition of G (see (9.8)), the derivative of $Q_j(t,G)$ with respect to t on (q_{h-1},q_h) is

$$(1 - G(t))(v_j(b,t) - v_j(1 - a,t))(V_h(a,b,t) - V_j(a,b,t)).$$

However, from lemma 1 we have $V_h(a,b,t) > V_j(a,b,t)$ (since h is more risk averse than j), so that the derivative is positive, as required. A similar argument establishes that the derivative on $(q_j,1)$ is negative, and a like argument can be made for P_i. Hence, the E_i's and F_j's defined by $\Pi(0,0)$ constitute an equilibrium.

The remaining equilibria are of two types. One type is closely related to the equilibrium just defined. In fact, it should be clear (by arguments similar to the preceding) that for any $0 < \alpha < 1$ and any $0 < \beta < 1$, the strategies defined by $\Pi(\alpha,0)$ and those defined by $\Pi(0,\beta)$ are equilibria of the game.

The final type of equilibria involves all individuals in one population conceding with probability 1 at time 0. The members of the other population use any strategies that generate a distribution of concession times that puts enough weight near $t = 1$ to make the members of the first population concede at $t = 0$. (Such strategies clearly exist – for example, all individuals can concede at $t = 1$ with probability 1.) This defines an equilibrium: Since all members of the first population concede at time 0, all members of the second population are indifferent between all concession times in (0,1] (they always receive a payoff stream equal to their demand from time 0).

It is much more difficult to argue that every equilibrium of the game is of one of these types; a proof is given in the Appendix. We can summarize the results as follows.

Theorem 3. $(E_1, \ldots, E_m; F_1, \ldots, F_n)$ is an equilibrium of the two-population model if and only if it is one of the following:

1. $E_i (i = 1, \ldots, m)$ and $F_j (j = 1, \ldots, n)$ are defined by $\Pi(0,0)$.
2. For some $0 < \alpha < 1$ and $0 < \beta < 1$, $E_i (i = 1, \ldots, m)$ and $F_j (j = 1, \ldots, n)$ are defined by either $\Pi(\alpha,0)$ or $\Pi(0,\beta)$.
3. Either (a) $E_i(t) = 1$ for all $0 \leq t \leq 1$, for all $i = 1, \ldots, m$, and $F_1, \ldots, F_n$ are any strategies that generate a distribution H of concession times for which $P_i(0,H) \geq P_i(t,H)$ for any $0 \leq t \leq 1$, for $i = 1, \ldots, m$; or (b) the equilibrium is similar to this, with the roles of populations 1 and 2 reversed.

An immediate consequence of this result is the following.

Corollary 4. If the characteristics of populations 1 and 2 are the same, then the only symmetric equilibrium is the one defined by $\Pi(0,0)$. That is, the only equilibrium in the one-population model is the one defined in the previous section.

Perfect equilibrium

Selten (1975) argues that equilibria in games with finite pure strategy sets should possess a certain robustness. Suppose that a game is perturbed by insisting that each player devote at least some small probability to some completely mixed strategy. An equilibrium is *perfect* if it is close to an equilibrium of such a perturbed game. Okada (1981) suggests that one should insist that the equilibrium be close to an equilibrium of *every* such perturbed game; he calls such an equilibrium *strictly perfect.* Kohlberg and Mertens (1982) study a related notion (a strictly perfect equilibrium is a stable component in their sense), and show that the equilibria it generates have a number of attractive properties.

In the game here, each player has a continuum of pure strategies. In such a game, it is not clear how to formulate these notions of perfection. I do not attack this problem. Rather, I consider a small collection of perturbed games, in which the perturbing strategy is concession at time 0 with probability 1. (Note that this is an equilibrium strategy – of type (3)). The following result shows that the only equilibrium that is robust with respect to small perturbations of the strategy sets in the direction of this strategy is the one of type (1). That is, in the game in which each individual thinks that there is a positive probability that his opponent will concede at time 0, the only equilibrium is close to the one of type (1). It seems likely that this equilibrium is the only one that satisfies an appropriately modified version of strict perfection – that is, it is robust with respect to *all* small perturbations of the strategy sets. However, a precise argument to this effect is beyond the scope of this paper.

Proposition 5. For each $\epsilon > 0$, let Γ^ϵ be the perturbed game in which the strategy space of each player is the set of cumulative probability distributions F on $[0,1]$ such that $F(0) \geq \epsilon$. Then, for each $\epsilon > 0$, the game Γ^ϵ has a unique equilibrium, which converges to the one defined in (1) of theorem 3 as $\epsilon \rightarrow 0$.

Proof. Let $E_i (i = 1, \ldots, m)$ and $F_j (j = 1, \ldots, n)$ be the equilibrium strategies given in (1) of theorem 3, and let G and H be the corresponding distributions of concession times in the two populations. For each $\epsilon > 0$, let $E_i^\epsilon(t) = \epsilon + (1 - \epsilon)E_i(t)$ and $F_j^\epsilon(t) = \epsilon + (1 - \epsilon)F_j(t)$ for all $0 \leq t \leq 1$, $i = 1, \ldots, m$, and $j = 1, \ldots, n$. Now, $(E_1^\epsilon, \ldots, E_m^\epsilon; F_1^\epsilon, \ldots, F_n^\epsilon)$ is an equilibrium of Γ^ϵ, because the derivative with respect to t on $(0,1)$ of the payoff $P_i(t,H^\epsilon)$, where $H^\epsilon(t) = \Sigma_{j=1}^n \delta_j F_j^\epsilon(t) = \epsilon + (1 - \epsilon)H(t)$ for all $0 \leq t \leq 1$, is precisely $(1 - \epsilon)$ times the derivative of $P_i(t,H)$. Hence, $P_i(t,H^\epsilon)$ is increasing on $(0,p_{i-1})$, constant on (p_{i-1},p_i), and decreasing on $(p_i,1)$. Also, $P_i(0,H^\epsilon)$ is less than $P_i(\eta,H^\epsilon)$ for some $\eta > 0$, since H^ϵ has an atom at 0 (compare lemma A3 of the Appendix). Similar arguments can obviously be made for the payoffs of individuals in population 2. Further, since there is no other nonatomic equilibrium of the original game (Γ^0), and no equilibrium in which the strategies of players in both populations have atoms at 0, it is clear that this is the only equilibrium of Γ^ϵ. In addition, the E_i^ϵ's and F_j^ϵ's converge (pointwise) to the E_i's and F_j's.

The effect of a change in risk aversion

Here, I investigate the effect of an increase in the risk aversion of individuals of type 2ℓ on the equilibrium singled out as perfect by the preceding arguments. Suppose that the utility function $\hat{v}_\ell$ is more risk averse than v_ℓ, and less risk averse than $v_{\ell-1}$ (so that the ordering of risk aversions in population 2 is preserved).

Suppose that $q_\ell < 1$ and $q_{\ell-1} \in \operatorname{supp} E_k$. Then, as in the one-population case, $\hat{p}_i = p_i$ for $i = 1, \ldots, k-1$, $\hat{q}_j = q_j$ for $j = 1, \ldots, \ell - 1$, and $\hat{G}(t) = G(t)$ and $\hat{H}(t) = H(t)$ for $0 \le t \le q_{\ell-1}$, so that the equilibrium payoffs of types $11, \ldots, 1k$ and $21, \ldots, 2(\ell - 1)$ are unaffected.

Now, consider the changes in G and H on $(q_{\ell-1}, 1]$. From lemma 1, we have $\hat{V}_\ell(a,b,s) > V_\ell(a,b,s)$ for all $0 \le s < 1$, so that from (9.8) (with $j = \ell$), we have $\hat{G}(t) > G(t)$ on $(q_{\ell-1}, \min(\hat{q}_\ell, q_\ell)]$ (since on this interval, both G and $\hat{G}$ have to keep type 2ℓ indifferent). (Refer to Figure 9.3.) This means that $q_{\ell-1} < \hat{p}_k < p_k$. On $(q_{\ell-1}, \hat{p}_k)$, H is unchanged (since u_k is unchanged). However, on $[\hat{p}_k, \min(p_k, \hat{p}_{k+1})]$, $\hat{H}$ has to keep type $k+1$ indifferent, whereas H keeps type k indifferent. Since $H(\hat{p}_k) = \hat{H}(\hat{p}_k)$ and, from lemma 1, $U_{k+1}(b,a,s) < U_k(b,a,s)$ for all $0 \le s < 1$, we have, from (9.9), $\hat{H}(t) < H(t)$ for $\hat{p}_k < t \le \min(p_k, \hat{p}_{k+1})$. Now, there are several cases to consider, but the arguments are very similar. Suppose that $p_k < q_\ell$ and $\hat{G}(q_\ell) < \Gamma(k+1)$, so that $q_\ell < \hat{p}_{k+1}$. Then, on $[p_k, \min(p_{k+1}, \hat{p}_{k+1})]$, $\hat{H}$ has to keep type $1(k+1)$ indifferent, as H did. However, since $\hat{H}(p_k) < H(p_k)$, we deduce, from (9.9), that $\hat{H}(t) < H(t)$ on $[p_k, \min(p_{k+1}, \hat{p}_{k+1})]$. Now, suppose that $\hat{H}(\hat{p}_{k+1}) > \Delta(\ell)$. Then, $\hat{q}_\ell > q_\ell$, and we can consider the behavior of $\hat{G}$ on $[q_\ell, \hat{q}_\ell]$. Now, $\hat{G}$ has to keep type 2ℓ indifferent whereas G keeps type $2(\ell+1)$ indifferent; also, $G(q_\ell) < \hat{G}(q_\ell)$, and so from (9.8), $\hat{G}(t) > G(t)$ on $[q_\ell, \hat{q}_\ell]$. On $[\hat{q}_\ell, q_{\ell+1}]$, both G and $\hat{G}$ keep type $2(\ell+1)$ indifferent; since $\hat{G}(\hat{q}_\ell) > G(\hat{q}_\ell)$, we have $\hat{G}(t) > G(t)$ on this interval. Continuing in the same way, we see that $\hat{G}(t) > G(t)$ for all $q_{\ell-1} < t < 1$ and $\hat{H}(t) < H(t)$ for all $\hat{p}_k < t < 1$.

Then, arguing exactly as in the one-population case (integrating by parts in the expressions for $P_i(p_{i-1}, H)$ and $Q_j(q_{j-1}, G)$ for $i = k+1, \ldots, m$ and $j = \ell + 1, \ldots, n$), we find that the equilibrium payoffs of types $1(k+1), \ldots, 1m$ *decrease* and those of types $2(\ell+1), \ldots, n$ *increase*. That is, an increase in risk aversion in population 2 causes those individuals in population 1 who in (perfect) equilibrium concede later to be made worse off, whereas the members of population 2 who concede later are better off. Summarizing, we have the following.[4]

Proposition 6. Suppose that individuals of type ℓ in population 2 become more risk averse (but not more so than those of type $\ell - 1$). Suppose

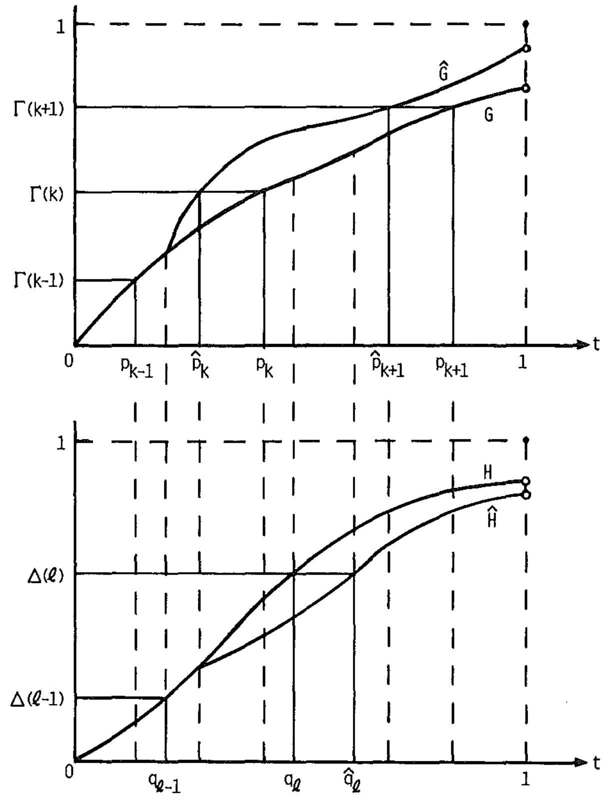

Figure 9.3 Changes in the perfect equilibrium distributions of concession times caused by an increase in the risk aversion of individuals of type ℓ in population 2 in the two-population model

also that in the old (perfect) equilibrium, type 2ℓ concedes with probability 1 before time 1, and the smallest point in the support of the strategy of type 2ℓ is a member of the support of the strategy of type $1k$. Then, the (perfect) equilibrium payoffs of the types at least as risk averse as type k in population 1 are unaffected, whereas those of the less risk averse types decrease; the equilibrium payoffs of the types more risk averse than type ℓ in population 2 are unaffected, whereas those of the less risk averse types increase.

Thus, in the two-population model the effect of a change in risk aversion is exactly the opposite of that predicted by the axiomatic models. The reason is that in equilibrium, the concession pattern in population 1 must make the actions of the individuals in population 2 optimal, and vice versa. Thus, if some members of population 2 become more risk averse, the individuals in population 1 have to concede on average earlier in order to keep the members of population 2 indifferent over some interval of concession times. However, if concessions in population 1 are on average earlier, the optimal concessions in population 2 are later; hence, the payoffs of individuals in population 1 decrease. (This type of argument is a standard one concerning mixed-strategy Nash equilibria; it is not made possible by some peculiar feature of the model.)

The effect of a change in the size of a type

Finally, I consider the effect of a change in the fraction of a population that is of a given type. I do so to see if any meaning can be given to the claim that in equilibrium, individuals will pretend to be less risk averse than they really are. Suppose that the fraction of the population taken up by relatively risk neutral individuals increases. Then, one might imagine that since this causes an opponent to ascribe a lower probability to an individual being risk averse, those who are risk averse can do better – they can "hide" among the mass of relatively risk neutral individuals. It turns out that this is not the case, although it *is* true that the ratio of the payoffs of the more risk averse to those of the less risk averse increases; the former are constant, whereas the latter decrease.

To see why this is true, we can use the previous result concerning a change in the degree of risk aversion. Suppose that the fraction of population 2 occupied by individuals of type ℓ increases from δ_ℓ to $\hat{\delta}_\ell = \delta_\ell + \epsilon$, and the fraction of type $\ell - 1$ decreases from $\delta_{\ell-1}$ to $\hat{\delta}_{\ell-1} = \delta_{\ell-1} - \epsilon$ (so that the population becomes on average less risk averse). This change is equivalent to one of the types considered in the previous subsection. Thus, break types $\ell - 1$ and ℓ into three types, which constitute the fractions $\delta_{\ell-1} - \epsilon$, ϵ, and δ_ℓ of the population (the first two types having the same utility function before the change,[5] and the second two having the same utility function after the change). Then, the change defined previously is a decrease in the risk aversion of the middle type. Hence, by proposition 6 the equilibrium payoffs of the types more risk averse than ℓ are unaffected, whereas those of type ℓ and the less risk averse types decrease. That is, we have the following.

Corollary 7. Suppose that the fraction of individuals of type ℓ in some population increases, whereas the fraction of those of type $\ell - 1$ decreases by the same amount, and those of type $\ell - 1$ concede with probability 1 before time 1. Then, the equilibrium payoff of every individual more risk averse than type ℓ in that population is unaffected, whereas the equilibrium payoff of every individual at most as risk averse as type ℓ decreases.

9.4 The case in which there is a choice of initial demand

Here, I elaborate on the previous model by allowing individuals to choose a "demand" at time 0. I assume that there are only two possible demands, a and b ($0 \leq a < b < 1$). I study the perfect (Bayesian Nash) equilibria of the two-population model in which each individual simultaneously first chooses a demand, and then negotiates as in the previous model (i.e., subsequently simply chooses a time at which to concede). Throughout, I consider only the perfect equilibrium described in the previous section. To keep the analysis relatively simple, I assume that there are only two types in each population ($m = n = 2$). Note that in this model, the actions of an individual do, in general, convey useful information to his opponent. Unless the same fraction of each type demands a, the demand that an individual makes at time 0 allows his opponent to revise the probability that the individual is of a given type. Note, however, that an individual's subsequent behavior does not convey any additional useful information to his opponent.

I show that if $a > \frac{1}{2}$, there is no separating equilibrium in which the two types in each population choose different demands. I also show that in this case there is a continuum of pooling equilibria, in which a positive fraction of each type in each population chooses each possible demand.

Given the results in the axiomatic models that an individual can benefit from pretending to be less risk averse than he really is, one might imagine that the reason no separating equilibrium exists is that a more risk averse individual can benefit from changing his demand to that of a less risk averse individual. However, given the result of the previous section (proposition 6), it should come as no surprise that the opposite is true. That is, the less risk averse can benefit from pretending to be more risk averse (see the arguments that follow). Thus, there is another sense (in addition to the one considered in the previous subsection) in which the result derived in the axiomatic framework does not hold in my model.

If $a < \frac{1}{2}$, then there is no equilibrium of either type. Later, under the heading "Discussion" I consider briefly what may happen when there are more than two types in each population. However, I do not consider another natural extension, in which there are many, even a continuum, of possible demands. My analysis is not comprehensive, but is simply intended to establish that the basic model considered in the previous sections does not degenerate when at least some choice of initial demand is allowed.

Nonexistence of separating equilibrium

Suppose that $a > \frac{1}{2}$. Consider a situation in which all individuals of type 1 in each population demand a, whereas all those of type 2 demand b. When two individuals meet and reveal their demands, they know immediately each other's type, and the only perfect equilibrium distributions of concession times are given by the two-population model described previously in which each population contains one type. For example, if two type 1's meet, then their perfect equilibrium concession strategies are given by

$$E_{11}(t) = 1 - \exp\left(-\int_0^t V_1(a,a,s)\,ds\right)$$

and

$$F_{11}(t) = 1 - \exp\left(-\int_0^t U_1'(a,a,s)\,ds\right)$$

for $0 \leq t < 1$ (see (9.10) and (9.11)). (That is, F_{11} keeps type 11 indifferent over all points in [0,1], and E_{11} keeps type 21 indifferent over all such points.) The payoffs to these strategies are $u_1(1-a,0)$ and $v_1(1-a,0)$ (i.e., the payoffs obtained by immediate concession).

Next, consider the consequence of an individual of type 2 in population 1 demanding a rather than b. If he demands b, then his expected payoff is $\delta_1 u_2(1-a,0) + \delta_2 u_2(1-b,0)$ (since with probability δ_i, he encounters an individual of type $2i$, who correctly infers that he is of type 2, and uses a strategy that makes such individuals indifferent over [0,1]). If he demands a, then any opponent incorrectly identifies him as being of type 11, so that the opponent uses a strategy that makes such individuals indifferent over [0,1]. That is, if his opponent is of type 21, his payoff if he concedes at t is

$$P_2(t,F_{11}) \equiv \int_{(0,t]} u_2(a,s)F_{11}(s) + u_2(1-a,t)(1-F_{11}(t)); \qquad \textbf{(9.12)}$$

whereas if his opponent is of type 22, it is

$$P_2(t,F_{21}) \equiv \int_{(0,t]} u_2(a,s)F_{21}(s) + u_2(1-b,t)(1-F_{21}(t)), \qquad \textbf{(9.13)}$$

where

$$F_{21}(t) = 1 - \exp\left(-\int_0^t U_1(b,a,s)\,ds\right)$$

for all $0 \le t < 1$. Differentiating with respect to t in (9.12), we obtain

$$(1 - F_{11}(t))(u_2(a,t) - u_2(1 - a,t))(U_1(a,a,t) - U_2(a,a,t)).$$

From lemma 1, this is positive if $t < 1$, so that an optimal action for the individual in this case is to concede at time 1. His payoff is then $P_2(1,F_{11})$, which, since $P_2(t,F_{11})$ is increasing in t, exceeds $P_2(0,F_{11}) = u_2(1 - a,0)$, the payoff he obtains when he demands a in this case. Similarly, differentiating in (9.13), we obtain

$$(1 - F_{21}(t))(u_2(a,t) - u_2(1 - b,t))(U_1(b,a,t) - U_2(b,a,t)).$$

So by the same argument as before, the individual can obtain $P_2(1,F_{21})$, which exceeds $P_2(0,F_{21}) = u_2(1 - b,0)$, the payoff when he demands a in this case. Thus, his expected payoff against a random opponent exceeds $\delta_1 u_2(1 - a,0) + \delta_2 u_2(1 - b,0)$, and therefore he is better off demanding a, pretending to be more risk averse than he really is.

Thus, no separating equilibrium of this type exists. It is easy to see that the same argument also rules out any separating equilibrium in which all individuals of type 1 in one population demand b, whereas all individuals of type 2 in that population demand a, and the members of the other population act either similarly, or as they did before. So there is no separating equilibrium in which within each population the two types choose different demands.

If $a < \frac{1}{2}$, then when two individuals who demand a meet, they can reach agreement immediately; if $a + b < 1$, this is also true if individuals demanding a and b meet. It matters precisely what the payoffs are in these cases. I assume that if individual i demands d_i, $i = 1, 2$, and $d_1 + d_2 < 1$, then individual i receives the output stream $((1 + d_i - d_j)/2,0)$ (i.e., the individuals split equally the amount left over after their demands are satisfied; the precise method of splitting the excess does not matter for my argument, so long as the amount that i receives increases with d_i). First, consider the possibility of a separating equilibrium when $a < \frac{1}{2}$ and $a + b \ge 1$. In such an equilibrium, there are individuals in both populations who demand a. If the opponent of such an individual demands a, then the individual receives the output stream $(\frac{1}{2},0)$, whereas if the opponent demands b, the equilibrium payoff of this individual is the utility of the output stream $(1 - b,0)$. If the individual switches to a demand of b, the output stream he receives if his opponent demands a is $((1 + b - a)/2,0)$, and if his opponent demands b, he again receives the utility of $(1 - b,0)$. Since $(1 + b - a)/2 > \frac{1}{2}$, the individual will switch from a to b. A similar argument can obviously be made if $a + b < 1$. Thus, no separating equilibrium exists for any values of a and b.

Pooling equilibria

Now, consider the possibility of an equilibrium in which the fractions $0 < \pi_i < 1$ of type $1i$ and $0 < \rho_j < 1$ of type $2j$ demand a, whereas all other individuals in both populations demand b. Then, if for example a type-11 individual demands a and bargains with a type-21 individual who demands a, the equilibrium concession times are those given by the two-population model in which the fractions of the types in population 1 are $\pi_1/(\pi_1 + \pi_2)$ and $\pi_2/(\pi_1 + \pi_2)$ and those in population 2 are $\rho_1/(\rho_1 + \rho_2)$ and $\rho_2/(\rho_1 + \rho_2)$. For $(\pi_1,\pi_2,\rho_1,\rho_2)$ to constitute an equilibrium, each individual in each population must be indifferent between demanding a and b.

First, consider the case $a > \frac{1}{2}$. Note than an individual of type 1 in either population is always indifferent between demanding a and b. This is the case because 0 is always an element of the support of the equilibrium strategy of such an individual, whoever is his opponent (since type 1 is the most risk averse). Thus, the equilibrium payoff when the opponent demands a is $u_1(1 - a,0)$ (or $v_1(1 - a,0)$), and when the opponent demands b it is $u_1(1 - b,0)$ (or $v_1(1 - b,0)$), independent of $(\pi_1,\pi_2,\rho_1,\rho_2)$.

Consider the behavior of individuals of type 2 in population 1. Fix $0 < \pi_2 < 1$ and $0 < \rho_2 < 1$, and let $\mathbf{P}_i(c;\pi_1,\rho_1)$ be the equilibrium payoff of an individual of type i in population 1 who demands c when the fractions of those who demand a among types 11 and 21 are π_1 and ρ_1; similarly define $\mathbf{Q}_j(c;\pi_1,\rho_1)$ for type j in population 2. Suppose that $\pi_1 = 1$ (i.e., that all type-11 individuals demand a). Then, if a type-12 individual demands b, he identifies himself to be of type 2, and the equilibrium reached is the one in a two-population model where there is only one type in population 1. Hence, the support of the individual's strategy is $[0,1]$, and his equilibrium payoff is $u_2(1 - a,0)$ or $u_2(1 - b,0)$, depending on the demand of his opponent. Thus, for each $0 \leq \rho_1 \leq 1$,

$$\mathbf{P}_2(b;1,\rho_1) = (\rho_1 + \rho_2)u_2(1 - a,0) + (1 - \rho_1 - \rho_2)u_2(1 - b,0). \qquad \textbf{(9.14)}$$

Now, fix $0 \leq \rho_1 \leq 1$ and reduce π_1. As π_1 falls, the fraction of type 2's among those demanding b in population 1 decreases. Hence, by corollary 7, the equilibrium payoff of type 2, whether the opponent demands a or b, increases. That is, for each $0 \leq \rho_1 \leq 1$,

$$\mathbf{P}_2(b;\pi_1,\rho_1) \text{ is decreasing in } \pi_1. \qquad \textbf{(9.15)}$$

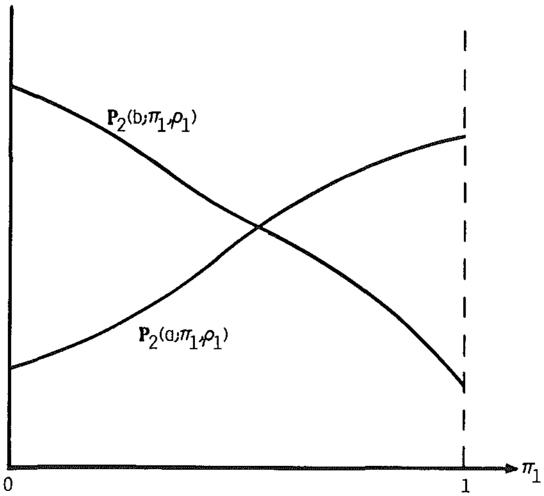

Figure 9.4 Functions $\mathbf{P}_2(a;\cdot,\rho_1)$ and $\mathbf{P}_2(b;\cdot,\rho_1)$

Next, consider what happens when a type-12 individual demands a. If $\pi_1 = 1$, then the equilibrium distributions of concession times are those for a two-population model in which the fraction $0 < 1/(1+\pi_2) < 1$ of population 1 is of type 1, and the fraction $0 < \pi_2/(1+\pi_2) < 1$ is of type 2 (since $\pi_1 = 1$ and $0 < \pi_2 < 1$). Hence, whether the opponent demands a or b, the smallest element in the support of the equilibrium strategy of the type-12 individual exceeds zero, and hence his equilibrium payoff exceeds $u_2(1-a,0)$ if his opponent demands a, and exceeds $u_2(1-b,0)$ if his opponent demands b. That is, for each $0 \le \rho_1 \le 1$, $\mathbf{P}_2(a;1,\rho_1) > (\rho_1 + \rho_2)u_2(1-a,0) + (1-\rho_1-\rho_2)u_2(1-b,0)$, and so by (9.14) we have, for each $0 \le \rho_1 \le 1$,

$$\mathbf{P}_2(a;1,\rho_1) > \mathbf{P}_2(b;1,\rho_1) \quad \textbf{(9.16)}$$

(see Figure 9.4). Now, suppose that π_1 decreases. This means that the fraction of type 2's among those demanding a in population 1 increases. Hence, again by corollary 7, for each $0 \le \rho_1 \le 1$,

$$\mathbf{P}_2(a;\pi_1,\rho_1) \text{ is increasing in } \pi_1. \quad \textbf{(9.17)}$$

Finally, suppose that $\pi_1 = 0$. Then, a symmetric argument establishes that for all $0 \le \rho_1 \le 1$,

$$\mathbf{P}_2(a;0,\rho_1) < \mathbf{P}_2(b;0,\rho_1). \quad \textbf{(9.18)}$$

It is also clear that for each $0 \le \rho_1 \le 1$, the equilibrium payoffs are continuous in π_1. (For example, if π_1 is close to zero, so that almost all type-11 individuals demand b, then the equilibrium payoff of a type-12 individual who demands a is close to $u_2(1 - a,0)$ or $u_2(1 - b,0)$ [depending on the opponent's demand], since the fraction of type 11's in the concession game after the demands are revealed is close to zero, so that the smallest element in the support of the equilibrium strategy of type 12 is close to zero.)

Combining (9.15) through (9.18), we conclude that for each $0 \leq \rho_1 \leq 1$, there exists a unique π_1 such that $\mathbf{P}_2(a;\pi_1,\rho_1) = \mathbf{P}_2(b;\pi_1,\rho_1)$. Denote this π_1 by $\Upsilon(\rho_1)$. Since all of the functions involved are continuous, so is Υ. Symmetric arguments can obviously be made for population 2. That is, for each $0 \leq \pi_1 \leq 1$, there exists a unique ρ_1 such that $\mathbf{Q}_2(a;\pi_1,\rho_1) = \mathbf{Q}_2(b;\pi_1,\rho_1)$. Denote this ρ_1 by $\Phi(\pi_1)$; Φ is continuous.

Now, the function $\Phi \circ \Upsilon$: $[0,1] \to [0,1]$ is continuous and hence has a fixed point, say ρ_1^*. Let $\pi_1^* = \Upsilon(\rho_1^*)$. Then, (π_1^*,ρ_1^*) is such that $\mathbf{P}_2(a;\pi_1^*,\rho_1^*) = \mathbf{P}_2(b;\pi_1^*,\rho_1^*)$ and $\mathbf{Q}_2(a;\pi_1^*,\rho_1^*) = \mathbf{Q}_2(b;\pi_1^*,\rho_1^*)$. By (9.16) and (9.18), we have $0 < \pi_1^* < 1$ and $0 < \rho_1^* < 1$. That is, given the fixed (π_2,ρ_2), and the fact that type 1 is indifferent between a and b for any $(\pi_1,\pi_2,\rho_1,\rho_2)$, $(\pi_1^*,\pi_2,\rho_1^*,\rho_2)$ is a pooling equilibrium.

Proposition 8. If $a > \frac{1}{2}$, then for each $(\pi_2,\rho_2) \in (0,1)^2$, there exists a pooling equilibrium in which a positive fraction of each type in each population make each demand.

When $a < \frac{1}{2}$, it is easy to show that there is no pooling equilibrium. The reason is that, exactly as in the case of a separating equilibrium, an individual of type 1 in each population can increase his payoff by demanding b.

Discussion

Given the continuum of pooling equilibria in proposition 8, we cannot perform the comparative static exercises of Section 9.3. It is not clear whether there are assumptions under which a unique equilibrium is selected. One possibility is to increase the number of types in each population. The arguments presented establish that the most risk averse individuals in both populations are always indifferent between demanding a and b. All other types are indifferent only in particular cases. This suggests that however many types there are, there is always a one-dimensional continuum of equilibria; as the size of the most risk averse type shrinks, the range of the equilibria may contract. Thus, in the limit, when there is a continuum of types, there is a possibility that an essentially determinate equilibrium is defined.

APPENDIX
The Characterization of All Equilibria in the Two-Population Model

I repeatedly use the following expression for the difference between the payoffs of conceding at r and at t. If $r \le t$, then

$$P_i(t,H) - P_i(r,H) = J_H(r)(u_i(a,r) - u_i(\tfrac{1}{2},r)) + \int_{(r,t)} (u_i(a,s) - u_i(1-b,r))\, dH(s) + J_H(t)(u_i(\tfrac{1}{2},t) - u_i(1-b,r)) + (1 - H(t))(u_i(1-b,t) - u_i(1-b,r)). \quad \textbf{(A.1)}$$

Throughout, E_i ($i = 1, \ldots, m$) is an equilibrium strategy of $1i$, and F_j ($j = 1, \ldots, n$) is an equilibrium strategy of $2j$; G and H are the equilibrium distributions of concession times in populations 1 and 2, respectively. Let $J(G)$ and $J(H)$ be the set of atoms (jumps) of G and H, respectively. Note that

$$\text{If } t \notin J(H) \text{ and } t \in \operatorname{supp} E_i, \text{ then } P_i(t,H) = P_i(E_i,H). \quad \textbf{(A.2)}$$

The following gives conditions on a distribution of concession times under which more risk averse individuals concede earlier.

Lemma A1. If $[0,s_0] \subset \operatorname{supp} H$, H is atomless on $(0,s_0] \cap (0,1)$, $r \in [0,s_0]$, $t \in [0,s_0]$, $r \in \operatorname{supp} E_i$, and $t \in \operatorname{supp} E_{i-1}$, then $t \le r$.

Proof. If $r = s_0 = 1$, the result is immediate. So suppose that $r < 1$. Let $r \in \operatorname{supp} E_i$ and suppose that $0 \le r < t \le s_0$. Then, $P_i(t,H) - P_i(r,H) \le 0$. Now, $u_{i-1} = f \circ u_i$, where f is strictly concave, so that $f(w) - f(z) \le f'(z)(w - z)$, with strict inequality if $w \ne z$. Hence,

$$u_{i-1}(a,s) - u_{i-1}(1-b,r) < f'(u_i(1-b,r))(u_i(a,s) - u_i(1-b,r))$$

unless s is such that $u_i(a,s) = u_i(1-b,r)$; by (P.2) and (P.3), there is only one such s. Hence, given that $[0,s_0] \subset \operatorname{supp} H$,

$$\int_{(r,t)} (u_{i-1}(a,s) - u_{i-1}(1-b,r))\, dH(s) < f'(u_i(1-b,r)) \int_{(r,t)} (u_i(a,s) - u_i(1-b,r))\, dH(s).$$

Also,

$$u_{i-1}(1-b,t) - u_{i-1}(1-b,r) < f'(u_i(1-b,r))(u_i(1-b,t) - u_i(1-b,r)).$$

However, since $t \in \text{supp}\, H$, then either $H(t) - H(r) > 0$ or $H(t) < 1$. Hence, given that r is not an atom of H, and either t is not an atom of H, or $t = s_0 = 1$ and hence $u_i(\frac{1}{2},t) = 0 = u_i(1-b,t)$, the preceding inequalities imply, using (A.1), that

$$P_{i-1}(t,H) - P_{i-1}(r,H) < f'(u_i(1-b,r))(P_i(t,H) - P_i(r,H)) \leq 0.$$

Hence, $t \notin \text{supp}\, E_{i-1}$. So if $t \in \text{supp}\, E_{i-1}$, then $t \leq r$.

Corollary A2. If $[0,s_0] \subset$ *supp* G, $[0,s_0] \subset \text{supp}\, H$, G is atomless on $(0,s_0] \cap (0,1)$, and H is atomless on $[0,s_0] \cap (0,1)$, then there exist $0 = p_0 = p_1 = \cdots = p_\ell < p_{\ell+1} < \cdots < p_{k-1} < s_0$ and $0 = q_0 < q_1 < \cdots < q_{h-1} < s_0$ such that

$[0,s_0] \cap \text{supp}\, E_i = \{0\}$ for $i = 1, \ldots, \ell$;

$[0,s_0] \cap \text{supp}\, E_i = [p_{i-1},p_i]$ for $i = \ell + 1, \ldots, k-1$,
and $[0,s_0] \cap \text{supp}\, E_k = [p_{k-1},s_0]$;

$[0,s_0] \cap \text{supp}\, F_j = [q_{j-1},q_j]$ for $j = 1, \ldots, h-1$,
and $[0,s_0] \cap \text{supp}\, F_h = [p_{h-1},s_0]$.

Proof. Immediate from lemma A1 (using (9.1), and the analogous relation between H and the F_j's).

Now, I show that G and H cannot have atoms at the same point, except possibly at 1. The reason is simple: If, for example, G has an atom at t_0, then all members of population 2 obtain a higher payoff by conceding just after t_0, rather than at t_0.

Lemma A3. If $t_0 \in J(G)$ and $t_0 < 1$, then $t_0 \notin J(H)$.

Proof. Let $t_0 \in J(H)$, $t_0 < 1$. Then, for each $\delta > 0$ there exists $0 < \epsilon < \delta$ such that $t_0 + \epsilon \notin J(H)$. Next, consider $P_i(t_0 + \epsilon,\, H) - P_i(t_0,H)$ (see (A.1)). The first term is positive, independent of ϵ; the second term is nonnegative for small ϵ; the third is zero; and the fourth can be made as small as necessary for choosing ϵ small enough. Hence, for ϵ small enough, we have $P_i(t_0 + \epsilon,H) > P_i(t_0,H)$.

The following is a very useful result, which says that if G has an atom at $t_0 \in (0,1)$, then no member of population 2 concedes in some open interval before t_0 (when t_0 is imminent, it is better to wait until afterward, since there is a positive probability of a concession occurring at t_0).

Lemma A4. If $t_0 \in J(G)$ and $0 < t_0 < 1$, then there exists $\epsilon > 0$ such that $(t_0 - \epsilon, t_0) \cap \text{supp } H = \emptyset$.

Proof. Let $\delta > 0$. For any $j = 1, \ldots, n$, we have

$$Q_j(t_0,G) - Q_j(t_0 - \delta,G) = J_G(t_0 - \delta)(v_j(b,t_0 - \delta) - v_j(\tfrac{1}{2},t_0 - \delta))$$
$$+ \int_{(t_0-\delta,t_0)} (v_j(b,s) - v_j(1 - a,t_0 - \delta))\, dG(s)$$
$$+ J_G(t_0)(v_j(\tfrac{1}{2},t_0) - v_j(1 - a,t_0 - \delta)) + (1 - G(t_0))(v_j(1 - a,t_0) - v_j(1 - a,t_0 - \delta))$$

(see (A.1)). However, $v_j(b,t_0 - \delta) - v_j(\tfrac{1}{2},t_0 - \delta) \geq 0$ and, since $t_0 < 1$, we can find $\epsilon_1 > 0$ and $\alpha > 0$ such that for all $0 < \delta < \epsilon_1$, we have

$$v_j(\tfrac{1}{2},t_0) > v_j(1 - a,t_0 - \delta) + \alpha,$$

and for all $t_0 - \delta < s < t_0$, we have

$$v_j(b,s) > v_j(b,t_0) > v_j(\tfrac{1}{2},t_0) > v_j(1 - a,t_0 - \delta).$$

Hence, for all $0 < \delta < \epsilon_1$,

$$Q_j(t_0,G) - Q_j(t_0 - \delta,G) \geq \alpha J_G(t_0) + (1 - G(t_0))(v_j(1 - a,t_0) - v_j(1 - a,t_0 - \delta)).$$

However, there also exists $\epsilon_2 > 0$ such that for all $\delta < \epsilon_2$, we have $v_j(1 - a,t_0) - v_j(1 - a,t_0 - \delta) > -\alpha J_G(t_0)/2$. But then for all $0 < \delta < \epsilon = \min(\epsilon_1,\epsilon_2)$, we have

$$Q_j(t_0,G) - Q_j(t_0 - \delta,G) > \frac{\alpha J_G(t_0)}{2} > 0,$$

and so $Q_j(t,G) < Q_j(t_0,G)$ for all $t \in (t_0 - \epsilon,t_0)$. Hence, $(t_0 - \epsilon,t_0) \cap \text{supp } H = \emptyset$.

The following states that if there is an interval not in the support of H, at the endpoints of which G does not have atoms, then the largest point in the interval can be in the support of G only if H is already 1 at that point. The reason is that the payoff to any member of population 1 is decreasing in the interval whenever there is a positive probability of a future concession by an opponent.

Lemma A5. If $H(r) = H(t)$, $0 \leq r < t \leq 1$, $r \notin J(H)$, $t \notin J(H)$, and $t \in \text{supp } G$, then $H(t) = 1$.

Proof. For all $i = 1, \ldots, m$,

$$P_i(t,H) - P_i(r,H) = (1 - H(t))(u_i(1-b,t) - u_i(1-b,r))$$

(using (A.1)). Hence, $P_i(t,H) < P_i(r,H)$ unless $H(t) = 1$. Since $t \notin J(H)$, this means (from (A.2)) that $t \notin \text{supp } G$.

We can now restrict quite substantially the nature of the supports of the equilibrium distributions of concession times.

Lemma A6. If $t_0 \in J(G)$ and $0 < t_0 < 1$, then there exists $s_0 \in [0,t_0)$ such that $\text{supp } H = [0,s_0]$, $[0,s_0] \subset \text{supp } G$, and G and H are atomless on $(0,s_0]$.

Proof. From lemma A4, there exists $\epsilon > 0$ such that $(t_0 - \epsilon, t_0) \cap \text{supp } H = \varnothing$. Now, let $r = t_0 - \epsilon/2$ and $t = t_0$ in lemma A5. Since $t_0 \in J(G)$, $t_0 \notin J(H)$ by lemma A3, and so $H(r) = H(t)$. Hence, by lemma A5, $H(t_0) = 1$. Let $s_0 = \max \text{supp } H$. Then, $s_0 < t_0$. Now, if there is an atom of G in $(0,s_0]$, say at t_1, the same argument establishes that $H(t_2) = 1$ for some $t_2 < t_1$, contradicting the fact that $s_0 \in \text{supp } H$. Similarly, H can have no atom in $(0,s_0]$.

Now, suppose that there exists $y \in \text{supp } H$, and $0 < x < y$ such that $H(x) = H(y)$. Then, $y < s_0$, and so $H(y) < 1$, and $x \notin J(H)$, $y \notin J(H)$ by the preceding argument, and thus by lemma A5 we know that $y \notin \text{supp } G$. But then we know that there exists $w < y$ such that $G(w) = G(y) < 1$; reversing the roles of G and H in lemma A5, and letting $r = w$ and $t = y$, we conclude that $y \notin \text{supp } H$, a contradiction. Hence, $\text{supp } H = [0,s_0]$ and, similarly, $\text{supp } G \supset [0,s_0]$.

Now, we can conclude that if one distribution of concession times has an atom in the interior of its support, then all individuals in the opponent population must concede at time 0 with probability 1.

Lemma A7. If $t_0 \in J(G)$ and $0 < t_0 < 1$, then $\text{supp } H = \{0\}$.

Proof. By lemma A3, G and H cannot both have atoms at 0. Let s_0 come from lemma A6, and assume that $s_0 > 0$. Then, from corollary A2 and lemma A6, we know that there exist k and $0 \le p_{k-1} < s_0$ such that $[0,s_0] \cap \text{supp } E_k = [p_{k-1},s_0]$. Hence, H is such that $P_k(t,H)$ is constant, equal to $P_k(E_k,H)$, for all $t \in (p_{k-1},s_0]$. That is,

$$\int_{[0,t]} u_k(a,s)\, dH(s) + (1 - H(t))u_k(1-b,t) = P_k(E_k,H)$$

(since H is atomless on $(p_{k-1},s_0])$ for all $p_{k-1} < t \leq s_0$. Because the integral is (as a function of t) absolutely continuous, H is absolutely continuous. But then we can integrate by parts to obtain

$$u_k(a,t)H(t) - u_k(a,0)H(0) - \int_{[0,t]} H(s)\, D_2 u_k(a,s)\, ds + (1 - H(t))u_k(1-b,t) = P_k(E_k H).$$

Since $D_2 u_k$ is continuous in s, and H is atomless on $(0,s_0]$, this shows that H is differentiable. Differentiating, we find that every solution of the resulting differential equation is of the form

$$H(t) = 1 - A \exp\left(-\int_0^t U_k(b,a,s)\, ds\right) \tag{A.3}$$

for some $A > 0$. Now, we need $H(s_0) = 1$. Since $s_0 < t_0 < 1$, the integral with $t = s_0$ is always finite, and so we must have $A = 0$. But then $H(u_{k-1}) = 1$, and thus $s_0 \notin \text{supp } H$. So the only possibility is $s_0 = 0$.

As noted in the text, there exists an equilibrium of this type. For example, if supp $E_i = \{1\}$ for $i = 1, \ldots, m$, and supp $F_j = \{0\}$ for $j = 1, \ldots, n$, then $(E_1, \ldots, E_m; F_1, \ldots, F_n)$ is an equilibrium.

We can now use lemma A5 to restrict the nature of the supports of G and H when G and H are atomless on $(0,1)$.

Lemma A8. If G and H are atomless on $(0,1)$ and there exists $0 < t < 1$ such that $G(t) < 1$ and $H(t) < 1$, then $[0,t] \subset \text{supp } H$.

Proof. Suppose, to the contrary, that $0 < x < t$ and $x \notin \text{supp } H$. Let $y_G = \min\{z \geq x \colon z \in \text{supp } G\}$ and define y_H similarly. Since $x \notin \text{supp } H$, we have $y_H > x$. Also, $y_G > x$ (since if $y_G = x$, then there exists $y > x$, $y \in \text{supp } G$ such that $H(x) = H(y)$ and $H(y) < 1$, contradicting lemma A5). Let $y = \min\{y_G, y_H\} > x$. First, suppose that $y < 1$. If $y = y_G$, then $G(x) = G(y) < 1$, $x \notin J(G)$ and $y \notin J(G)$, and so by lemma A5, $y \notin \text{supp } H$. Hence, there exists $w < y$ such that $H(w) = H(y)$, $w \notin J(H)$, $y \notin J(H)$, and $H(y) < 1$, and so by lemma A5 again, $y \notin \text{supp } G$, contradicting the definition of y $(= y_G)$. If $y = y_H$, a similar argument can be made. Hence, the only possibility is $y = 1$. But then

$$P_i(y,H) - P_i(x,H) = J_H(y)(u_i(\tfrac{1}{2},y) - u_i(1-b,x)).$$

However, $u_i(\frac{1}{2},y) = 0$ if $y = 1$, and so this is negative unless $J_H(y) = 0$. But this is impossible, since we assumed that $G(t) = G(y) - J_H(y) = 1 -$

$J_H(y) < 1$. Hence, $[0,t] \subset$ supp H. A symmetric argument implies that $[0,t] \subset$ supp G.

Lemma A9. If neither supp $G = \{0\}$ nor supp $H = \{0\}$, then neither supp $G = \{1\}$ nor supp $H = \{1\}$.

Proof. If, for example, supp $G = \{1\}$, then clearly $Q_j(0,G) > Q_j(t,G)$ for all $t > 0$, so that supp $H = \{0\}$.

Corollary A10. If neither supp $G = \{0\}$ nor supp $H = \{0\}$, then there exists $0 < t < 1$ such that $G(t) < 1$ and $H(t) < 1$.

Lemma A11. If G and H are atomless on $(0,1)$ and neither supp $G = \{0\}$ nor supp $H = \{0\}$, then there exists $s_0 > 0$ such that either supp $G = [0,s_0]$ and supp $H \supset [0,s_0]$, or supp $H = [0,s_0]$ and supp $G \supset [0,s_0]$.

Proof. Let $z_G =$ max supp $G > 0$ and $z_H =$ max supp $H > 0$, and let $s_0 = \min\{z_G, z_H\} > 0$. The result then follows by letting $t = s_0 - \epsilon$ for any $\epsilon > 0$ in lemma A8, and using corollary A10.

Lemma A12. If G and H are atomless on $(0,1)$ and neither supp $G = \{0\}$ nor supp $H = \{0\}$, then there exist $0 \le \alpha < 1$ and $0 \le \beta < 1$ such that G and H are as specified either by the procedure $\Pi(\alpha,0)$ or by the procedure $\Pi(0,\beta)$.

Proof. Suppose that supp $H = [0,s_0]$ (see lemma A11). Then, from corollary A2, there exist k and $p_{k-1} < s_0$ such that $[0,s_0] \cap$ supp $E_k = [p_{k-1},s_0]$. But then we can argue, as in the proof of lemma A7, that H has the form given in (A.3) on $[p_{k-1},s_0)$. Now, if $s_0 < 1$, then by assumption, s_0 is not an atom of H, and so we need $H(s_0) = 1$; but this is possible only if $A = 0$, in which case $s_0 = 0$, contradicting the assumption that supp $H \neq \{0\}$. Hence, $s_0 = 1$, and so supp $G =$ supp $H = [0,1]$. By lemma A3, at most one of G and H have an atom at 0. But then using corollary A2, and solving the differential equation on each $[p_{i-1},p_i]$ and $[q_{j-1},q_j]$, as before, we find that the only solutions are those defined in the procedure $\Pi(\alpha,0)$ or $\Pi(0,\beta)$ by (9.8) and (9.9).

We have now proved that the only equilibria are those characterized in theorem 3. Lemma A12 states that the only equilibria in which G and H are atomless on $(0,1)$ and neither supp $G = \{0\}$ nor supp $H = \{0\}$ are those described in (1) and (2) of theorem 3; lemma A7 states that whenever G or H has an atom in $(0,1)$, then either supp $G = \{0\}$ or supp $H = \{0\}$, when the equilibrium is of the type specified in (3) of theorem 3.

NOTES

1. The change in risk aversion considered in Roth (1985) does not preserve a player's preferences over certain outcomes.
2. A continuous extensive game fits more directly with the preceding description. By analogy with discrete extensive games, a pure strategy in such a game specifies, for each time t, whether or not to concede. That is, it is a function from $[0,1]$ to $\{C(\text{oncede}), S(\text{tand firm})\}$. However, for each such strategy of an opponent, every strategy of an individual that has the same time of first concession yields the same outcome. Thus, there is a reduced strategic form for the game in which the pure strategies of each individual are those functions f from $[0,1]$ to $\{C,S\}$ such that $f(t) = S$ if $0 \leq t < t_0$ and $f(t) = C$ if $t_0 \leq t \leq 1$, for some t_0. This reduced strategic form is isomorphic to the strategic form specified in the text, and its Nash equilibria are of course outcome equivalent to the Nash equilibria of the extensive form.
3. Given the discussion of the previous paragraph, this is the same as saying that there is a unique *symmetric* equilibrium in the two-population model when the characteristics of the two populations are identical.
4. If a type that, in equilibrium, concedes with probability 1 at time 1 becomes more risk averse, then no individual's equilibrium payoff is affected.
5. This violates (P.4) (the function relating the utility functions of the two identical types is not *strictly* concave), but it should be clear that this assumption affects only some of the finer details of the results.

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