Price Competition in
a Capacity-Constrained Duopoly*

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This paper characterizes the set of Nash equilibria in a model of price-setting duopoly in which each firm has limited capacity, and demand is continuous and decreasing. In general there is a unique equilibrium, so that comparative static exercises are meaningful. The properties of the equilibrium conform with a number of stylized facts. The equilibrium prices are lower, the smaller is demand relative to capacity. When demand is in an intermediate range, the firms use mixed strategies—they randomly hold "sales." If capacities are chosen simultaneously, before prices, the set of equilibrium capacities coincides with the set of Cournot quantities. *Journal of Economic Literature* Classification Numbers: 022, 611.
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1. INTRODUCTION

A model of a price-setting duopoly is a natural starting point for a theory of the behavior of oligopolists. However, such a model has been completely solved only under a particularly unrealistic assumption—namely, that each firm can produce an unlimited quantity (or, at least as much as is demanded at the breakeven price) at constant unit cost. The

unique equilibrium in this case involves each firm setting the breakeven price, so that (unless capacity can be bought and sold at will) at most half of the available capacity is used in equilibrium. Given that the firms must at some point choose their capacities, it is difficult to see how such a situation could come about.

We study the case where the capacity of each firm is limited, and cannot be instantly changed. We describe, for each pair of capacities, the set of Nash equilibria, which is in general a singleton. We assume that the unit cost is the same for each firm, and is constant up to capacity, and demand is continuous and decreasing. Previous analyses of the model are of two types. Either strong assumptions (e.g., linearity) are imposed on demand, or relatively weak results are proved about the character of equilibria. (For a discussion see Sect. 6.) In particular, little attention has been devoted to the uniqueness of equilibrium (except when demand is linear). This causes problems for economic applications, since without a uniqueness result we do not know if the characteristics of a particular equilibrium are shared by others, and we cannot legitimately perform comparative static exercises. Some consequences of our characterization are as follows (more details and discussion are contained in Sect. 4).

The larger is capacity relative to demand, the lower are the equilibrium prices. This is, of course, intuitively plausible. It is noteworthy because it emerges, even though, up to capacity, the technologies of both firms are the same. One can obtain a similar prediction in a competitive model only by assuming, for example, that there are more and less efficient firms; when demand is low, the latter are forced out of business and the price falls. In our model the result emerges from the fact that there is, in a precise sense, more competition when there is excess capacity. In contrast to the competitive outcome, profit is at the monopoly level if capacities are small, even though there is more than one firm.

When industry capacity is in an intermediate range, the equilibrium strategies involve randomization. The nature of the equilibrium distributions depends on the shape of the market demand function. Under some conditions, the large firm is most likely to charge either a high or a relatively low price, while the small firm is most likely to charge a low price. (For details, see Sect. 4.) Thus in this case, the large firm randomly holds "sales," as in Varian [18].

As the size of the large firm increases, its equilibrium profit increases; as the size of the small firm increases, the greater competition induced may offset the direct effect on its profit from larger sales, and the net effect is uncertain. The force at work here is closely related to that studied by Gelman and Salop [9]; even if capacity is free, a small firm may not find it advantageous to expand indefinitely. As the size of the small firm converges to zero, the equilibrium approaches the monopoly outcome.

In the game in which capacities are chosen simultaneously before prices, the set of pure capacity pairs chosen in Nash equilibria coincides with the set of (pure) Cournot equilibria (i.e., a slightly weaker version of the main result of Kreps and Scheinkman [11] holds under our more general assumptions on demand). However, there may be (depending on the shape of the demand function) Cournot equilibria which are not associated with subgame perfect equilibria of the two stage game. If demand varies cyclically, and the firms choose constant capacities appropriate for some intermediate level of demand, then in booms prices will be high, and constant, while in slumps the firms will hold random "sales."

In the next two sections we describe the model, and our results. In Section 4 we discuss economic implications of our characterization, in Section 5 we consider possible extensions, and in Section 6 we discuss the related literature. Finally, in the Appendix we provide proofs.

2. THE MODEL

There are two firms. Firm i has capacity k_i; we assume throughout that $k_1 \geqslant k_2 > 0$. Each firm can produce the same good at the same, constant unit cost $c \geqslant 0$ up to its capacity. Given the prices of all other goods, the aggregate demand for the output of the firms as a function of price is $D: \mathbb{R}_+ \to \mathbb{R}_+$. Let p denote the excess of price over unit cost and let $S = [-c, \infty)$. We refer, somewhat loosely, to an element of S as a "price." Define $d: S \to \mathbb{R}_+$ by $d(p) = D(p + c)$; $d(p)$ is the aggregate demand for the good when its price exceeds the unit cost by p. We make the following assumption on the demand function (which implies that profit $p\,d(p)$ attains a maximum on S). (Possible relaxations are discussed in Sect. 5.)

There exists $p_0 > 0$ such that $d(p) = 0$ if $p \geqslant p_0$ and $d(p) > 0$ if $p < p_0$, and d is continuous and decreasing on $(-c, p_0)$.   (2.1)

We now wish to define the profits of the firms at each pair of prices (p_1, p_2). We assume that if $p_i < p_j$ then consumers first try to buy from firm i; when its supply (k_i) is exhausted, they turn to firm j. (Whenever i and j appear in the same expression, we mean that i is not equal to j.) There is a large number of identical consumers, each with preferences which have no "income effect" (for details, see Sect. 3 of Osborne and Pitchik [14]). The aggregate demand at the high price p_j then depends on the way the limited supply k_i is allocated among the consumers. It is natural to assume that the rationing scheme is chosen by firm i. However, this is not enough to determine which scheme is chosen, since i's profit is independent of the scheme used (only j's profit is affected). Given that this is so, we make an assumption which appears to be consonant with the

competitive nature of the model: firm i chooses the scheme which minimizes the profit of firm j. In this scheme, each of the (identical) consumers is allowed to purchase the same fraction of k_i. (I.e., a rule like "limit two per customer" is imposed, rather than allowing some fraction of the customers to buy as much as they want.)

Under these assumptions, the demand faced by firm j when $p_i < p_j$ and $d(p_i) > k_i$ is just $\max(0, d(p_j) - k_i)$. (Note that this means that the shape of the residual demand function does not depend on the value of p_i.) Levitan and Shubik [12] and Kreps and Scheinkman [11] assume that the residual demand has this form; it is one of two possibilities studied by Gelman and Salop [9]. (There is another assumption on the microeconomic structure of demand which generates the same result: there is a large number of consumers, each with a reservation price; those with the highest reservation prices are served first by the low price firm.) Under these assumptions, the profit of firm i when it sets the price $p_i \in S$ and firm j sets $p_j \in S$ is

$$h_i(p_i, p_j) = \begin{cases} L_i(p_i) \equiv p_i \min(k_i, d(p_i)) & \text{if } p_i < p_j, \\ \phi_i(p) \equiv p \min(k_i, k_i d(p)/k) & \text{if } p_i = p_j = p, \quad (2.2) \\ M_i(p_i) \equiv p_i \min(k_i, \max(0, d(p_i) - k_j)) & \text{if } p_i > p_j, \end{cases}$$
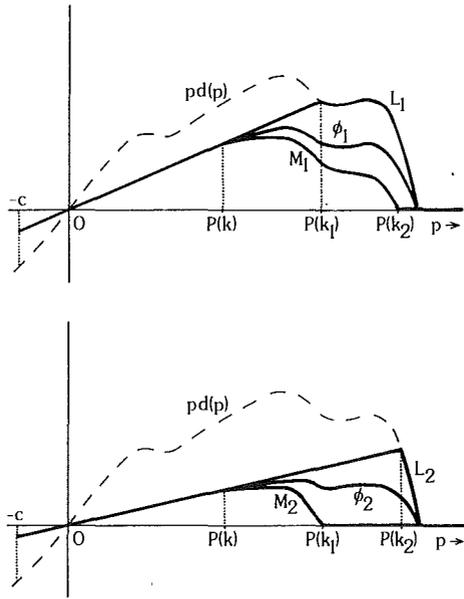

FIG. 1. The functions L_i, ϕ_i, and M_i for $i = 1, 2$. $L_i(p_i)$ is the profit of firm i when it sets the price $p_i < p_j$; $\phi_i(p_i)$ and $M_i(p_i)$ are its profits when $p_i = p_j$ and $p_i > p_j$, respectively.

where $k = k_1 + k_2$, and we assume that if $p_i = p_j$ then demand is allocated in proportion to capacities. Examples of the functions L_i, ϕ_i, and M_i are shown in Fig. 1. Let $\mathscr{S}$ be the set of mixed strategies (i.e., cumulative probability distribution functions on S). The domain of h_i can be extended to $\mathscr{S} \times \mathscr{S}$ in the natural way. In particular, if $F_j \in \mathscr{S}$ we have

$$h_i(p, F_j) = M_i(p)(F_j(p) - \alpha_j(p)) + \phi_i(p)\, \alpha_j(p)$$
$$+ L_i(p)(1 - F_j(p)), \qquad (2.3)$$

where $\alpha_j(p)$ is the size of the atom (if any) in F_j at p.

For each pair (k_1, k_2), we study the price-setting game $H(k_1, k_2)$ in which the strategy set of each firm is $\mathscr{S}$ and the payoff function of firm i is h_i $(i = 1, 2)$.

3. Description of the Nash Equilibria of $H(k_1, k_2)$

The qualitative characteristics of the Nash equilibria of $H(k_1, k_2)$ depend on the value of (k_1, k_2). For each (k_1, k_2) let

$$M_i^*(k_i, k_j) = \max_{p \in S} M_i(p). \qquad (3.1)$$

We frequently write M_i^* rather than $M_i^*(k_i, k_j)$ when this does not cause confusion. M_i^* is the maximal profit of firm i from charging a price in excess of that charged by firm j. Note that if $p \leqslant P(k)$ then $M_i(p) = pk_i$ (see Fig. 1), so that the maximizer of M_i is at least equal to $P(k)$, and hence $M_i^* \geqslant k_i P(k)$.

It is easy to put some restrictions on the range within which equilibrium prices must lie. No firm will charge a price below $P(k)$ (since it could then raise its price and still sell all its output), nor below 0 (since 0 guarantees a zero profit, while a negative price (i.e., a price below unit cost) yields at least one of the firms a negative profit). Further, we can show that no firm ever charges a price above $P(k_2)$. (The details of these, and of all subsequent arguments are given in the Appendix.)

It is also easy to check that if $k_2 \geqslant d(0)$ (region I of Fig. 2) then $(0, 0)$ is a pure strategy equilibrium, and that if M_1 is maximized at $P(k)$ (in which case the same is true for M_2) then $(P(k), P(k))$ is a pure strategy equilibrium (region III). These equilibria can be given clear interpretations. In region I each firm has more than enough capacity to meet the demand even at the breakeven price ($p = 0$ under our normalization), so the capacity limits are irrelevant, and we are back to the standard Bertrand model, where prices are driven down to unit costs. At the other extreme, in
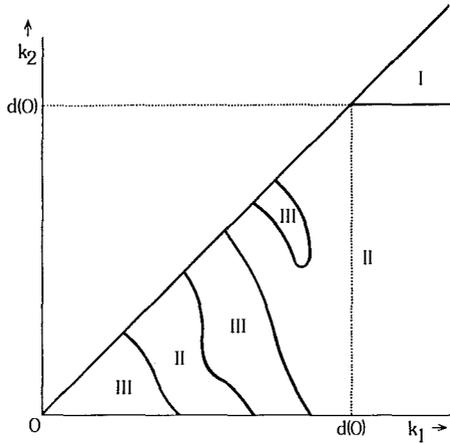
FIG 2. Type of equilibrium of $H(k_1, k_2)$ as a function of (k_1, k_2). The equilibria of the game $H(k_1, k_2)$ are of three types, as follows:

Region		Type of Equilibrium
I	$k_2 \geqslant d(0)$	Pure strategy equilibrium: $(0, 0)$
II	$k_2 < d(0)$ and $M_1^* > M_1(P(k))$	Mixed strategy equilibrium (or, in degenerate cases, equilibria)
III	$M_1^* = M_1(P(k))$	Pure strategy equilibrium: $(P(k), P(k))$ (and, in degenerate cases, possibly also mixed strategy equilibria)

region III there is undercapacity in the industry. In this case there is no incentive for "competition": each firm is producing at capacity, and so cannot benefit from undercutting its rival.

In the remaining case, where $k_2 < d(0)$ and $M_1^* > M_1(P(k))$ $(= k_1 P(k))$ (region II), no pure strategy equilibrium exists. We show that in this case there is, in general, a unique mixed strategy equilibrium. The basic idea behind the construction of an equilibrium is very simple. Let $(F_1, F_2) \in \mathscr{S} \times \mathscr{S}$ be an equilibrium and let $E_i = h_i(F_i, F_j)$, the equilibrium profit of i. Then $h_i(p, F_j) \leqslant E_i$ for all prices p. Using (2.3), this implies that if $L_i(p) > M_i(p)$ then $F_j(p) \geqslant Q_j(p; E_i)$, where

$$Q_j(p; E_i) = \frac{L_i(p) - E_i}{L_i(p) - M_i(p)}. \tag{3.2}$$

As argued above, we can show that the support of every equilibrium strategy is a subset of $\hat{S} = [X(k), X(k_2)]$, where $X(z) = \max(0, P(z))$; since $L_i(p) > M_i(p)$ if $X(k) < p \leqslant X(k_2)$, the only price not covered by this

argument is $X(k)$. Now, in order for (F_1, F_2) to be an equilibrium we also need $h_i(p, F_j) = E_i$ for all $p \in (\text{supp } F_i) \backslash Z_i$, where supp F_i is the support of F_i and Z_i is a set of F_i-measure zero. That is, almost all prices in the support must yield the equilibrium profit. (If some set of prices with positive F_i-measure yields less, it can profitably be eliminated from the support.) Using (2.3) again, this means that if $p \in (\text{supp } F_i) \backslash Z_i$ is not an atom of F_j and $L_i(p) > M_i(p)$ then $F_j(p) = Q_j(p; E_i)$. Now let $G_j(p) = \max(0, Q_j(p; E_i))$ for $i = 1, 2$. If each G_j is a strategy, $E_i = h_i(F_i, F_j)$ for $i = 1, 2$, and supp $G_1 = $ supp G_2, then (G_1, G_2) is an equilibrium: each G_j is nonatomic, since each Q_j is continuous in p, and hence each firm's profit is constant on the support of its strategy, and less outside. In general, under our assumptions, G_j is not a strategy; in particular it is not nondecreasing. However, the function Q_j remains the basis for the construction of equilibria. Let $\mathscr{I}Q_j$ be the nonnegative, nondecreasing cover of Q_j. That is,

$$\mathscr{I}Q_j(p; E_i) = \max\left\{0, \max_{x \leqslant p} Q_j(x; E_i)\right\}.$$

Then, since we need $F_j(p) \geqslant Q_j(p; E_i)$ for all p, and F_j must be nonnegative and nondecreasing, we certainly need $F_j(p) \geqslant \mathscr{I}Q_j(p; E_i)$ for all p.

So far, our argument has taken the equilibrium profits as given. Suppose firm i charges a price which maximizes M_i. If firm j charges a lower price, the profit of firm i is M_i^*, while if it charges the same or higher the profit of firm i exceeds M_i^*. Thus whatever firm j does, the profit of firm i is at least M_i^*. Hence i's equilibrium profit is at least M_i^*. Now let b_i be the highest price in the support of the equilibrium strategy of firm i. Then if $b_i \geqslant b_j$ and $b_i \notin J(F_j)$ (the set of atoms of F_j), firm i must get its equilibrium profit when it charges b_i. But this profit is just $M_i(b_i)$ (firm j charges a lower price with probability one, since $b_i \notin J(F_j)$), so that the equilibrium profit of firm i is M_i^*. Now, in this case F_i and F_j cannot both have an atom at the same price (since one firm could then always do better by charging a slightly lower price), so the argument establishes that the equilibrium profit of one of the firms is equal to M_i^*. It turns out that this must be firm 1; that is, $h_1(F_1, F_2) = M_1^*$. Given this, we can considerably restrict the nature of the equilibrium strategy of firm 2 (by arguments like those in the paragraphs above), and thus determine the equilibrium profit of firm 2, and hence restrict the nature of the equilibrium strategies of firm 1.

A precise description of all possible equilibria is complex, since there is a number of possible degeneracies which must be taken into account. In the nondegenerate case there is a unique equilibrium and its structure is quite simple. Let $\hat{S}_0 = \hat{S} \backslash \{X(k)\} = (X(k), X(k_2)]$. Then the support of each equilibrium strategy is the same subset of $\hat{S}_0$. The equilibrium strategy F_2 of firm 2 is nonatomic, with $F_2(p) = \mathscr{I}Q_2(p; M_1^*)$ for all $p \in \hat{S}_0$. To describe

the equilibrium strategy F_1 of firm 1, let $\bar{p}$ be the highest price in the support of F_2 which is at most equal to $P(k_1)$ (with $\bar{p} = P(k_1)$ if no such price exists). If $p < \bar{p}$ then $F_1(p) = \mathscr{I}Q_1(p; E_2)$; if $p \geqslant \bar{p}$ then F_1 has an atom at the end of each interval in the support of F_2, while on the rest of each interval $F_1(p) = Q_1(p; E_2) = 1 - E_2/k_2 p$ (refer to Fig. 3). (If the lowest price in the common support of F_1 and F_2 is less than $P(k_1)$ then $E_2 = k_2 E_1/k_1$, so that (using (2.2) and (3.2)) we have $F_1(p) = k_2 F_2(p)/k_1$ if $p \in [P(k), \bar{p}]$.)

(If the profit function $p\, d(p)$ is concave, then Q_1 and Q_2 are increasing, so that the support of each equilibrium strategy is the same interval, and the only atom in F_1 occurs at the highest price in the common support.)

Degeneracies occur if $Q_2(p; M_1^*)$ is constant and equal to $\mathscr{I}Q_2(p; M_1^*)$ over some range, or if the set of prices such that $Q_2(p; M_1^*) = F_2(p) = 0$ is not a singleton. Figure 4 illustrates these possibilities, which account for the nature of the conditions on F_1 in part (b)(ii) of Theorem 1 below.

Theorem 1 also allows for a degeneracy which may occur when M_1 is maximized at $P(k)$ (so that $M_1^* = k_1 P(k)$). In this case there is always a pure equilibrium $(P(k), P(k))$ (as discussed above), which is included in part (b)(i) of Theorem 1 ($r = 1$, $a_0 = P(k_2)$). But if there exists $p > P(k)$ such that $M_1^* = M_1(p)$ (i.e., M_1 is not uniquely maximized at $P(k)$) then
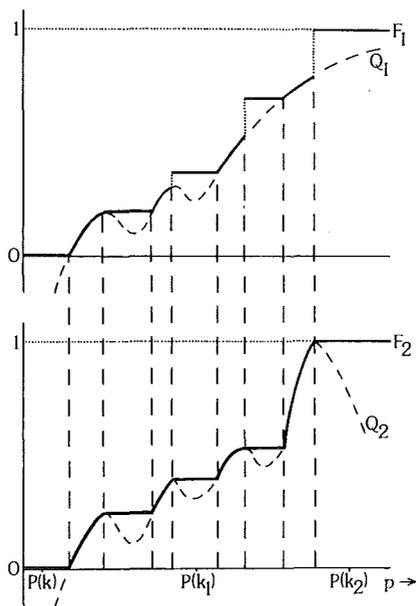

FIG. 3. An example of a nondegenerate mixed strategy equilibrium of $H(k_1, k_2)$. F_i $(i = 1, 2)$ is the equilibrium strategy of firm i; Q_i $(i = 1, 2)$ is the function defined in (3.2).
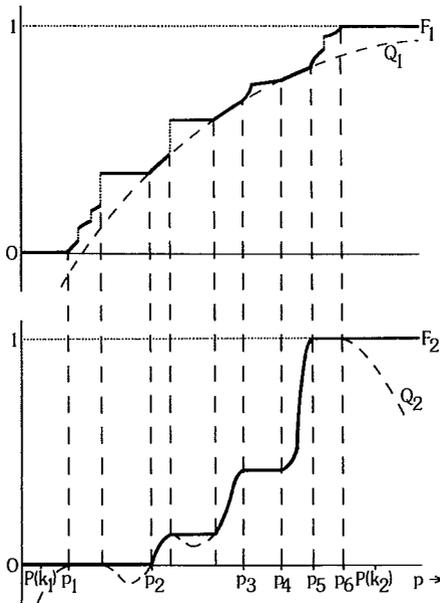
FIG. 4. An example of a mixed strategy equilibrium of $H(k_1, k_2)$ exhibiting some degeneracies. F_i $(i = 1, 2)$ is the equilibrium strategy of firm i; Q_i $(i = 1, 2)$ is the function defined in (3.2). The example is degenerate since (1) there are many prices p for which $Q_2(p) = 0$ and $Q_2(x) \leqslant 0$ for $x \leqslant p$, and (2) over the intervals $[p_3, p_4]$ and $[p_5, p_6]$, Q_2 is constant. The smallest member of A (see (b) of the theorem) is p_1, while the smallest member of the support of F_2 is p_2, so that there are many choices for a (see (b)(ii) of the theorem).

there may be other, mixed equilibria. This accounts for the complexity in part (b)(i) of Theorem 1.

THEOREM 1. (a) If $k_2 \geqslant d(0)$ (region I) the unique equilibrium of $H(k_1, k_2)$ is pure, equal to $(0, 0)$. The equilibrium profits are $(0, 0)$.

(b) If $k_2 < d(0)$ then $(F_1, F_2) \in \mathscr{S} \times \mathscr{S}$ is an equilibrium of $H(k_1, k_2)$ if and only if it satisfies the following conditions, where $A = \{p \in \hat{S}_0 : F_2(p) = Q_2(p; M_1^*)\}$.

(i) If $M_1^* = k_1 P(k)$ (region III) then supp $F_i \subset \hat{S}$ for $i = 1, 2$ and for some $0 \leqslant r \leqslant 1$ and $P(k) \leqslant a_0 \leqslant P(k_2)$ we have

$$F_2(p) = \begin{cases} r & \text{if} \quad X(k) \leqslant p \leqslant a_0, \\ \mathscr{I}Q_2(p; M_1^*) & \text{if} \quad a_0 < p \leqslant P(k_2), \end{cases} \tag{3.3}$$

$$\text{supp } F_1 \cap \hat{S}_0 \subset A, \tag{3.4}$$

and for $a = P(k)$ *we have*

$$F_1(p) \geqslant \mathscr{I} Q_1(p; k_2 a) \text{ if } p \in \hat{S}_0, \text{ and } F_1(p) = Q_1(p; k_2 a) \text{ if } F_2 \text{ is}$$
right-increasing at p. $\qquad$ (3.5)

(ii) *If* $M_1^* > k_1 P(k)$ *(region II) then* supp $F_i \subset \hat{S}_0$ *for* $i = 1, 2$, $F_2(p) = \mathscr{I} Q_2(p; M_1^*)$ *for all* $p \in \hat{S}_0$, *and* F_1 *satisfies* (3.4) *and* (3.5) *for some* $a \in$ [min A, min supp F_2].

In both cases (i) *and* (ii) *the equilibrium profits are* $(M_1^*(k_1, k_2), k_2 a)$.

4. Applications

Characteristics of Equilibrium

In region II all Nash equilibria involve mixed strategies. This means that if the duopoly lasts for more than one period, the model predicts variations in prices between periods (as the firms' random devices generate different realizations). Varian [18], in a model in which some consumers are imperfectly informed, also finds that firms randomize in equilibrium. In his model, the equilibrium density of prices is U-shaped; his interpretation is that the firms sometimes hold "sales." In our model, the randomization by firms emerges from the process of competition itself; it does not depend on imperfect information. The nature of the distributions of prices charged by the firms depends on the shape of the demand function. If demand is such that the profit function $p\,d(p)$ is concave then the support of the equilibrium strategy of each firm is an interval $[a, b]$. If $P(k_1)$ is outside this interval, then the equilibrium strategy of the small firm is atomless and concave on $[a, b]$, while that of the large firm is atomless and concave on $[a, b)$, and has an atom at b. Thus in this case the large firm either charges a high price or is likely to offer a substantial discount (as in Varian's model), while the small firm tends to charge low prices most of the time. In the case that $P(k_1)$ is between a and b, the equilibrium strategies are concave separately on $[a, P(k_1))$ and $(P(k_1), b)$, but are not concave on $[a, b]$. Thus the pattern of prices charged is similar to the one in the previous case, except that prices just above $P(k_1)$ are now relatively likely to be offered by both firms. If the profit function is not concave then the equilibrium strategies can be quite irregular, as shown in Figs. 3 and 4; there are "holes" in the supports of the strategies over those intervals of prices where demand decreases most rapidly.

Comparative Statics

The comparative statics of some features of the equilibria can be studied with the aid of Fig. 5 (cf. Fig. 2 of Kreps and Scheinkman [11]). This
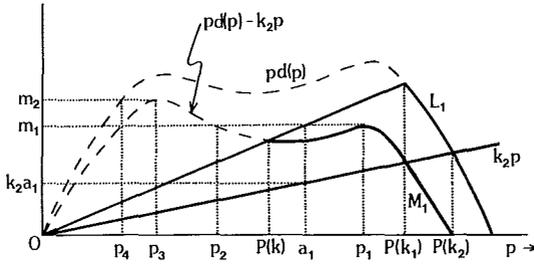
FIG. 5. An example of the construction of an equilibrium. The limits of the supports of the equilibrium strategies are a_1 and p_1; the equilibrium profits are m_1 and k_2a_1.

diagram shows a nondegenerate case for region II, in which there is a single price (a_1) at which $Q_2(\,\cdot\,; M_1^*)$ is zero, and a single maximizer (p_1) of M_1. Thus a_1 and p_1 are respectively the lowest and highest prices in the supports of the equilibrium strategies; the equilibrium profits are $E_1 = M_1^*(k_1, k_2) = m_1$ and $E_2 = k_2a_1$, as shown.

Consider first the effect of an increase in the size k_1 of the large firm, holding fixed the size k_2 of the small firm. The only functions in Fig. 5 affected by this increase are L_1 and M_1, the linear parts of which rotate counterclockwise about the origin. If the new value of $P(k)$ is between p_2 and p_3 then the equilibrium is pure (region III); if it is below p_3 then it is mixed again (region II). The equilibrium profit (M_1^*) of the large firm is nondecreasing in k_1, while that of the small firm is nonincreasing; the endpoints of the supports of the equilibrium strategies are also non-increasing. The paths these variables take for the example of Fig. 5 are shown in Fig. 6 $(i = 1, j = 2, k_i > k_j)$.
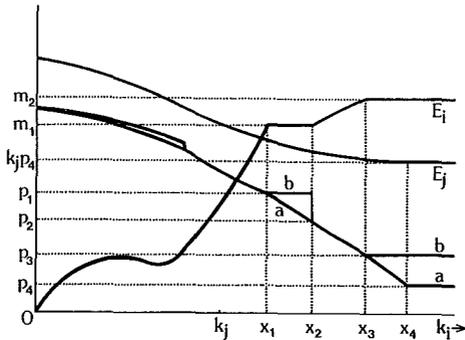
FIG. 6. An example of the dependence of the equilibrium strategies and profits on (k_1, k_2). For the profit function $p\,d(p)$ shown in Fig. 5, the equilibrium profits $(E_i$ and $E_j)$ of the firms, and the limits of the supports of the equilibrium strategies (a and b) are shown as functions of k_i, for a fixed value of k_j. The capacity x_h is such that $P(x_h + k_j) = p_h$ for $h = 1,..., 4$ (where p_h is given in Fig. 5).

If the size k_2 of the small firm decreases, given the size k_1 of the large firm, the curvilinear part of M_1 increases, and the function $k_2 p$ rotates clockwise about the origin. The impact on the equilibrium strategies and profits for the example of Fig. 5 is again shown in Fig. 6 ($i = 2$, $j = 1$, $k_i < k_j$). The equilibrium profit of the large firm and the endpoints of the supports of the equilibrium strategies increases as k_2 decreases. The equilibrium profit of the small firm may rise or fall as k_2 decreases: the fact that the firm is smaller reduces the competition with the large firm and raises equilibrium prices, possibly offsetting the direct effect on firm 2's profits of its smaller size. (The arguments of Gelman and Salop [9] are closely related to this point. They argue that even if capacity is free, a firm may not want to expand indefinitely, because of the lower equilibrium prices associated with larger industry capacity.) As k_2 converges to zero, the equilibrium approaches the monopoly outcome; when k_2 is very small, the large firm almost always charges a price close to the monopoly price, while the small firm charges slightly variable, slightly lower prices. (The solution thus possesses a characteristic that Shitovitz [17, pp. 497–8] suggests is reasonable.)

If there is more than one price p for which $L_1(p) = M_1^*$ and $L_1(x) \leqslant M_1^*$ if $x \leqslant p$, then there are many equilibria, yielding different payoffs to the small firm. The limits of this range of payoffs are nonincreasing in the size of the large firm.

The comparative statics of a shift in demand are more difficult to determine. Any increase in demand raises (or leaves constant) $M_1(p)$ at each price p, so that the equilibrium profit M_1^* of the large firm does not decrease. However, the effects on the equilibrium profit of the small firm and on the range of prices charged in equilibrium depend on the precise nature of the increase in demand. When demand is very small (relative to industry capacity), prices are equal to unit costs (region I). As demand increases, the firms begin to use mixed strategies and, roughly, prices increase, until region III is reached, where the equilibrium strategy is again pure, and prices are high.

Capacity Choice

Kreps and Scheinkman [11] study the game G in which the firms first simultaneously choose capacities, then simultaneously choose prices. They show, under the assumptions that the inverse demand function is concave and the cost of capacity is increasing and convex, that in the unique Nash equilibrium the capacities chosen are the (unique) Cournot quantities. Under our weaker assumptions on demand, there may be many Cournot equilibria[1]; we can show that the set of capacity pairs chosen in Nash

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[1] There is at least one Cournot equilibrium, by a result of McManus [13].

equilibria of G for which the capacity choices are pure coincides with the set of (pure) Cournot quantity pairs. (We have not investigated mixed-strategy Cournot equilibria.) That is, the result of Kreps and Scheinkman generalizes naturally. However, under our assumptions the set of subgame perfect equilibria of G can be a proper subset of the set of Nash equilibria, so that there can be Cournot equilibrium quantities which are not associated with subgame perfect equilibria of G.

Proofs of these results can be outlined as follows. First, let (k_1^*, k_2^*) be a (pure) Cournot equilibrium. Suppose that in G each firm i first chooses k_i^*, and then sets the price $P(k_1^* + k_2^*)$ if j chose k_j^* in the first stage, and the price 0 if j chose $k_j \neq k_j^*$. These strategies clearly constitute a Nash equilibrium of G, as argued in [11, p. 327]. Now let (k_1^*, k_2^*) be the capacity pair chosen in the first stage of a Nash equilibrium of G. The price strategies used in the second stage of this equilibrium must be an equilibrium of $H(k_1^*, k_2^*)$, so the profit of firm i at this equilibrium is $E_i(k_i^*, k_j^*) - u(k_i^*)$, where $E_i(k_i^*, k_j^*)$ is an equilibrium profit of i given in Theorem 1 and u is the cost function of capacity. Since i can guarantee a profit of $k_i P(k_i + k_j^*) - u(k_i)$ in G by choosing k_i and setting the price $P(k_i + k_j^*)$, we know that $E_i(k_i^*, k_j^*) - u(k_i^*) \geq k_i P(k_i + k_j^*) - u(k_i)$ for all k_i. We can complete the argument that (k_1^*, k_2^*) is a Cournot equilibrium by showing that $E_i(k_i^*, k_j^*) = k_i^* P(k_i^* + k_j^*)$. Without loss of generality, assume that $k_1^* \geq k_2^*$; then by Theorem 1, $E_1(k_1^*, k_2^*) = M_1^*(k_1^*, k_2^*)$. Now, $M_1^*(k_1, k_2^*)$ is the nondecreasing cover of $k_1 P(k_1 + k_2^*)$ (as a function of k_1); $M_1^*(k_1, k_2^*)$ is constant in k_1, at least equal to $k_1 P(k_1 + k_2^*)$, in regions I and II, and increasing, equal to $k_1 P(k_1 + k_2^*)$, in region III. Thus those values of k_1^* for which $M_1^*(k_1^*, k_2^*) - u(k_1^*) \geq k_1 P(k_1 + k_2^*) - u(k_1)$ for all k_1 are such that (k_1^*, k_2^*) is in region III, and hence $E_i(k_i^*, k_j^*) = k_i^* P(k_i^* + k_j^*)$ for $i = 1, 2$ (see (b)(i) of Theorem 1).

An example (the details of which we omit) shows that a Cournot equilibrium may not be associated with a subgame perfect equilibrium of G. The example works as follows. There is a symmetric Cournot equilibrium (x, x) in region III. If $k_i > x$ then firm i's equilibrium profit in the subgame following (k_i, x) exceeds its Cournot profit only when the former is decreasing (this follows from the properties of M_1^* mentioned in the previous paragraph), so that i cannot benefit from increasing capacity. However, for some values of k_i with $k_i < x$ and (k_i, x) in region II, i's equilibrium profit in the subgame following (k_i, x) exceeds its Cournot profit even when the former is increasing. Thus it pays firm i to decrease its capacity from the Cournot level x. (Under the assumptions of Kreps and Scheinkman [11], this cannot happen, since if (x, x) is in region III then any pair (k_i, x) with $k_i < x$ is also in region III, where the equilibrium profits are equal to the Cournot levels.) Thus (x, x) is not associated with a subgame perfect equilibrium. (In our example there is another Cournot

equilibrium which *is* associated with a subgame perfect equilibrium of G; we do not know if there always exists such an equilibrium.)

5. GENERALIZATIONS AND EXTENSIONS

Our assumptions can be relaxed in a number of ways. Since the functions of central importance in the construction of the equilibria are $\mathscr{I}Q_i$ for $i = 1, 2$, variations in the assumptions which preserve their character can be made. For example, demand may have discontinuities, so long as it is left-continuous (since Q_i can then jump only down, preserving the continuity of $\mathscr{I}Q_i$). Almost all our arguments apply also when the unit costs of the firms differ and when demand may increase over some range, though the description of the equilibria is then somewhat more complex.

There are two generalizations which appear to require more significant changes in our arguments: the existence of more than two firms, and costs which are not linear up to capacity. (In the latter case each firm should announce both a price and the maximum amount it is willing to sell at that price. Dixon [7] shows, using the results of Dasgupta and Maskin [4], that an equilibrium exists in the case that both firms have the same, convex cost function.)

Perhaps the most significant way in which our arguments are limited concerns the form of the residual demand. An important feature is that the demand faced by a high-price firm depends only on its price, not on the price of the other firm. That is, M_i is a function of p_i alone. Many of our arguments can probably be extended to the case where M_i depends on both p_i and p_j, though the outcome may then be sensitive to the form of the dependence. As argued previously, it is natural to assume that the low-price firm chooses the method of rationing, but if it is chosen at the same time as price, the outcome is indeterminate (since the payoff of the low-price firm is independent of the rule used). It is possible that the rationing rule could be chosen first, though this would require an analysis of the price-setting game for each choice of rationing rules. Even given the rationing rule we assume, the nature of the residual demand depends on consumers' preferences, and our assumption of the absence of an income effect is quite restrictive.

6. RELATION TO THE LITERATURE

The game we analyze is a nonzerosum "noisy" game of timing (i.e., a game in which the payoff functions are continuous, except possibly when both players use the same strategy, and within each region of continuity each player's payoff depends only on his own action). Previous work on

such games is limited, and uniqueness has not previously been examined. For a class of zerosum games of timing, Karlin [10, pp. 293–295] gives a uniqueness proof which relies heavily on the fact that the equilibrium payoffs in such a game are unique. For the games we consider, a substantially more involved argument appears to be necessary.

In the economics literature, Levitan and Shubik [12] describe the Nash equilibrium of $H(k_1, k_2)$ in the case where d is linear and either $k_1 = k_2$ or $k_1 = d(0)$; they do not study uniqueness. Kreps and Scheinkman [11] assume that inverse demand is decreasing and concave; they establish a result concerning the equilibrium payoffs, though they do not give a complete characterization and do not study the uniqueness of the equilibrium strategies.

Several authors have worked with the rationing scheme in which the low-price firm allows some fraction of the customers to buy all they wish (rather than allowing all customers to buy some fraction of their demand)[2]. Edgeworth [8] argues that, for a range of capacity pairs, there is no equilibrium in pure strategies. Beckmann [3] assumes that d is linear and $k_1 = k_2$. He considers uniqueness, but his argument is flawed[3]. Dasgupta and Maskin [5] show that if d satisfies (2.1) then an equilibrium exists, and if $k_1 = k_2$ then the supports of the equilibrium strategies satisfy a certain property. Allen and Hellwig prove more detailed results on the nature of these supports when the demand function is not necessarily decreasing and there are two (see [2]) or more (see [1]) firms.

It is easy to argue that if this second rationing scheme is used, then the result of Kreps and Scheinkman [11] on the coincidence of the Nash equilibrium capacity pair in the two-stage game and the Cournot quantity pair no longer holds in general. Davidson and Deneckere [6] verify this in an example with linear demand.

Finally, Shapley [16] reports (in an abstract of a paper which seems to be unobtainable) a characterization of the equilibrium of a price-setting duopoly game; it is not clear precisely what his model or assumptions were.

APPENDIX: PROOFS

Here we prove Theorem 1 of Section 3. First we check that any pair of strategies satisfying the stated conditions is an equilibrium (Proposition 1). Then, in a series of results, we establish that there are no other equilibria.

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[2] Equivalently, there is a large number of customers, each with a reservation price, and the low-price firm serves a random sample of them.

[3] For example, the inequality in his (15) should be reversed. As a result, the argument which follows (15) does not establish uniqueness of the equilibrium payoffs.

We begin by stating some basic properties of equilibrium strategies which we shall use repeatedly. A (Nash) *equilibrium* of $H(k_1, k_2)$ is a pair $(F_1, F_2) \in \mathscr{S} \times \mathscr{S}$ such that for $i = 1, 2$,

$$h_i(F, F_j) \leqslant h_i(F_i, F_j) \qquad \text{for all } F \in \mathscr{S}. \tag{A.1}$$

(Recall that whenever we use the indices i and j in an expresssion, we mean that j is not equal to i.) For $F \in \mathscr{S}$, let supp F be the support of F. It follows from (A.1) that

(A) (F_1, F_2) is an equilibrium if and only if for $i = 1, 2$ we have $h_i(p, F_j) \leqslant h_i(F_i, F_j)$ for all $p \in S$, and $h_i(p, F_j) = h_i(F_i, F_j)$ for all $p \in (\text{supp } F_i) \backslash Z_i$, where Z_i is a set of F_i-measure zero.

For $F \in \mathscr{S}$, let $J(F)$ be the set of points of discontinuity (i.e., jumps or atoms) of F. If $p \in \text{supp } F_i$ then either $p \in J(F_i)$, or F_i is left-increasing at p (i.e., there is a sequence $\{p_n\}$ with $p_n \in \text{supp } F_i$, $p_n < p$ for all n, and $p_n \uparrow p$), or F_i is right-increasing at p (i.e., there is a decreasing sequence with similar properties). If (F_1, F_2) is an equilibrium then in the first case fact (A) implies that $h_i(p, F_j) = h_i(F_i, F_j)$; in the second case, there exists a sequence $\{q_n\}$ with $q_n \in \text{supp } F_i$ and $q_n < p$ for all n, $q_n \uparrow p$, and $h_i(q_n, F_j) = h_i(F_i, F_j)$ for all n; in the third case there is a decreasing sequence with similar properties (see p. 211 of [15]). By taking limits in the second and third cases we have the following:

(B) If (F_1, F_2) is an equilibrium and $p \in \text{supp } F_i$ then *either*

(a) $p \in J(F_i)$, in which case

$$h_i(F_i, F_j) = h_i(p, F_j) = M_i(p)(F_j(p) - \alpha_j(p))$$
$$+ \phi_i(p) \alpha_j(p) + L_i(p)(1 - F_j(p))$$

or

(b) F_i is left-increasing at p, in which case

$$h_i(F_i, F_j) = M_i(p)(F_j(p) - \alpha_j(p)) + L_i(p)(\alpha_j(p) + 1 - F_j(p))$$
$$= h_i(p, F_j) + (L_i(p) - \phi_i(p)) \alpha_j(p);$$

or

(c) F_i is right-increasing at p, in which case

$$h_i(F_i, F_j) = M_i(p) F_j(p) + L_i(p)(1 - F_j(p))$$
$$= h_i(p, F_j) + (M_i(p) - \phi_i(p)) \alpha_j(p).$$

In particular:

(d) $h_i(F_i, F_j) = h_i(p, F_j) = M_i(p) F_j(p) + L_i(p)(1 - F_j(p))$ if $p \in$ (supp F_i)\$J(F_j)$;

(e) if $p \in$ supp F_i then $h_i(F_i, F_j)$ is a weighted average of $M_i(p)$, $\phi_i(p)$, and $L_i(p)$.

We now turn to the specific features of our game. Recall that we write $X(z) = \max(0, P(z))$. The following properties of the functions L_i, ϕ_i, and M_i (see (2.2) and Fig. 1) are easy to establish.

$$M_i(p) = \phi_i(p) = L_i(p) = \begin{cases} k_i p & \text{if } -c \leqslant p \leqslant P(k) \\ 0 & \text{if } p = 0 \text{ or } p \geqslant p_0 \end{cases} \tag{A.2}$$

$$L_i(p) < \phi_i(p) < M_i(p) \leqslant 0 \qquad \text{if } P(k) < p < 0 \tag{A.3}$$

$$0 \leqslant M_i(p) < \phi_i(p) < L_i(p) \qquad \text{if } X(k) < p < p_0 \tag{A.4}$$

We can now prove Theorem 1. Our first few arguments lead to Lemma 4, which shows that we can work with the restricted game $\hat{H}(k_1, k_2)$ in which the (pure) strategy set of each firm is $\hat{S} = [X(k), X(k_2)]$. If $F_i \in \mathscr{S}$ we write $a_i = \min \operatorname{supp} F_i$ and $b_i = \max \operatorname{supp} F_i$.

LEMMA 1. *If (F_1, F_2) is an equilibrium then (a) $a_i \geqslant X(k)$ for $i = 1, 2$ and (b) either $b_i = 0$ and $h_i(F_i, F_j) = 0$ for $i = 1, 2$ or $b_i > 0$ and $h_i(F_i, F_j) > 0$ for $i = 1, 2$.*

Proof. (i) $a_i \geqslant P(k)$. If $p \leqslant P(k)$ then $h_i(p, F_j) = k_i p$ for $i = 1, 2$ by (A.2); since this is increasing in p, we have $a_i \geqslant P(k)$ by fact (A).

(ii) $a_i \geqslant 0$. By (A.2) we have $h_i(0, F_j) = 0$, so $h_i(F_i, F_j) \geqslant 0$ for $i = 1, 2$ by fact (A). Let $a_i \leqslant a_j$. If $a_i \in J(F_j)$ then, using (a) of fact (B), we need $0 \leqslant h_j(F_j, F_i) = h_j(a_i, F_i) = \phi_j(a_i) \alpha_i(a_i) + L_j(a_i)(1 - \alpha_i(a_i))$ and hence, using (A.3), $a_i \geqslant 0$. If $a_i \notin J(F_j)$ then $F_j(a_i) = 0$, and either (a) or (c) of fact (B) holds, so that we need $0 \leqslant h_i(F_i, F_j) = L_i(a_i)$, and hence, again using (A.3), $a_i \geqslant 0$.

(iii) Suppose $b_j > 0$. Then $h_i(p, F_j) > 0$ if $0 < p < \min(b_j, p_0)$ by (A.4), so that $h_i(F_i, F_j) > 0$. Also since $a_i = b_i = 0$ if $b_i = 0$ (by (ii)), and $h_i(0, F_j) = 0$, we have $b_i > 0$ and hence $h_j(F_j, F_i) > 0$. ∎

This allows us to restrict the prices at which both equilibrium strategies can have atoms.

LEMMA 2. *If (F_1, F_2) is an equilibrium and $p \in J(F_1) \cap J(F_2)$ then $L_i(p) = \phi_i(p) = M_i(p)$ for $i = 1, 2$.*

Proof. Consider a sequence $\{p_n\}$ with $p_n \uparrow p$. By fact (A) we need $h_i(p_n, F_j) \leqslant h_i(F_i, F_j) = h_i(p, F_j)$ for all n. Taking the limit of the left-hand side gives $(L_i(p) - \phi_i(p)) \alpha_j(p) \leqslant 0$. So since $p \in J(F_j)$ and $L_i(p) \geqslant \phi_i(p)$ (by Lemma 1, (A.2), and (A.4)) we have $L_i(p) = \phi_i(p) = M_i(p)$. ∎

We can now establish the following useful result, which pins down the equilibrium payoff of one of the firms.

LEMMA 3. *Let* (F_1, F_2) *be an equilibrium. Then* $h_i(F_i, F_j) \geqslant M_i^*$ *for* $i = 1, 2$. *If* $b_i > b_j$ *then* $h_i(F_i, F_j) = M_i(b_i) = M_i^*$. *If* $b_1 = b_2 = b$ *then there exists* i *such that either* $b \in J(F_i)$ *or* $b \notin J(F_j)$, *and in both cases* $h_i(F_i, F_j) = M_i(b) = M_i^*$.

Proof. We have $0 \leqslant M_i(p) \leqslant h_i(p, F_j) \leqslant h_i(F_i, F_j)$ if $p \geqslant 0$ (the first two inequalities by (A.2) and (A.4)), and $M_i(p) \leqslant 0$ if $p \leqslant 0$. Hence $h_i(F_i, F_j) \geqslant M_i^*$ for $i = 1, 2$.

Now, if $b_i > b_j$, or $b_i = b_j$ and $b_i \notin J(F_j)$, then by (a) or (b) of fact (B) we have $h_i(F_i, F_j) = M_i(b_i)$, and hence $M_i(b_i) = M_i^*$. If $b_i = b_j = b$ and $b \in J(F_1) \cap J(F_2)$ then by Lemma 2 we have $L_i(b) = \phi_i(b) = M_i(b)$ for $i = 1, 2$, and so $h_i(F_i, F_j) = h_i(b, F_j) = M_i(b)$, and hence $M_i(b) = M_i^*$. ∎

LEMMA 4. $(F_1, F_2) \in \mathscr{S} \times \mathscr{S}$ *is an equilibrium of* $H(k_1, k_2)$ *if and only if it is an equilibrium of* $\hat{H}(k_1, k_2)$.

Proof. First suppose that (F_1, F_2) is an equilibrium of $\hat{H}(k_1, k_2)$. We need to show that $h_i(p, F_j) \leqslant h_i(F_i, F_j)$ whenever $p < X(k)$ or $p > X(k_2)$. Now, by (A.2) and (e) of fact (B) we have $h_i(X(k), F) = k_i X(k)$ for any $F \in \mathscr{S}$, so by fact (A) we have $h_i(F_i, F_j) \geqslant k_i X(k) \geqslant 0$. But by (A.2) and (A.3) we have $h_i(p, F_j) \leqslant k_i X(k)$ if $p < X(k)$ and $h_i(p, F_j) = M_i(p) = 0$ if $p > X(k_2)$, so (F_1, F_2) is an equilibrium of $H(k_1, k_2)$.

Now suppose that (F_1, F_2) is an equilibrium of $H(k_1, k_2)$. We need to show that $F_i \in \mathscr{S}$ for $i = 1, 2$. By Lemma 1 we have $a_i \geqslant X(k)$, so we need to show that $b_i \leqslant X(k_2)$. If $b_1 = b_2 = 0$ this is certainly true, so by Lemma 1(b) assume $b_i > 0$ for $i = 1, 2$, so that $h_i(F_i, F_j) > 0$ for $i = 1, 2$. Since $M_i(p) = 0$ for $i = 1, 2$ if $p \geqslant P(k_2)$, it follows from Lemma 3 that $b_i < P(k_2)$. ∎

From now on we use Lemma 4 to restrict attention to the game $\hat{H}(k_1, k_2)$ (that is, we restrict the strategy space of each player to $\hat{\mathscr{S}}$). We can immediately prove part (a) of Theorem 1.

PROPOSITION 1. *If* $k_2 \geqslant d(0)$ *then the unique equilibrium strategy pair is pure, equal to* $(0, 0)$.

Proof. It is easy to check that $(0, 0)$ is an equilibrium; by Lemma 4 there can be no other, since $\hat{S} = \{0\}$ in this case. ∎

We now address part (b) of Theorem 1. The following preliminary result allows us to prove, in Proposition 2, that any pair of strategies satisfying the conditions of Theorem 1 is an equilibrium. Recall that $\hat{S}_0 = S_0 \setminus \{X(k)\} = (X(k), X(k_2)]$.

LEMMA 5. *If* $E_i \geqslant k_i X(k)$ *and* $F_j(p) \geqslant Q_j(p; E_i)$ *for all* $p \in \hat{S}_0$ *then* $h_i(p, F_j) \leqslant E_i$ *for all* $p \in \hat{S}$.

Proof. If $F_j(p) \geqslant Q_j(p; E_i)$ for all $p \in \hat{S}_0$ then in fact $F_j(p) \geqslant Q_j(p; E_i) + \alpha_j(p)$ for all $p \in \hat{S}_0$ (consider a sequence $\{p_n\}$ with $p_n \uparrow p$). But then from (2.3) and (3.2) we have $h_i(p, F_j) \leqslant E_i$ for all $p \in \hat{S}_0$. Finally, $h_i(X(k), F_j) = k_i X(k) \leqslant E_i$ (see (A.2)). ∎

PROPOSITION 2. *Any pair of strategies satisfying the conditions in part* (b) *of Theorem* 1 *is an equilibrium of* $H(k_1, k_2)$.

Proof. Let (F_1, F_2) be a pair of strategies satisfying the conditions of part (b) of Theorem 1. We need to check that the conditions of fact (A) are satisfied. First consider the payoff of firm 1. Since $M_1^* \geqslant k_1 P(k)$ and $F_2(p) \geqslant \mathscr{I} Q_2(p; M_1^*)$ for all $p \in \hat{S}_0$, we have $h_1(p, F_2) \leqslant M_1^*$ for all $p \in \hat{S}$ by Lemma 5. Also $h_1(P(k), F_2) = k_1 P(k)$, and if $p \in \operatorname{supp} F_1 \cap \hat{S}_0$ then $F_2(p) = Q_2(p; M_1^*)$ (see (3.4)), so that, since F_2 is nonatomic on $\hat{S}_0$, we have $h_1(p, F_2) = M_1^*$. Hence $h_1(p, F_2) = M_1^*$ for all $p \in \operatorname{supp} F_1$, and the conditions of fact (A) are satisfied for $i = 1$.

Now consider the payoff of firm 2. Since $k_2 a \geqslant k_2 P(k)$ we have $h_2(p, F_1) \leqslant k_2 a$ for all $p \in \hat{S}$ by (3.5) and Lemma 5. Also $h_2(P(k), F_1) = k_2 P(k)$. Now, since F_2 is nonatomic on $\hat{S}_0$, the set $Z_2 = \{p \in \hat{S}_0 : F_2 \text{ is not right-increasing at } p\}$ has F_2-measure zero. If F_2 is right-increasing at p then $F_1(p) = Q_1(p; k_2 a)$ by (3.5), so $p \notin J(F_1)$ (by (3.5) and the continuity of Q_1). Hence $h_2(p, F_1) = k_2 a$ for all $p \in (\operatorname{supp} F_2) \setminus Z_2$. So the conditions of fact (A) are satisfied for $i = 2$. ∎

It remains to show that there is no other equilibrium. *From now on we assume that* $k_2 < d(0)$, *and* (F_1, F_2) *always denotes an equilibrium.* Note that if $k_2 < d(0)$ then $P(k_2) > 0$, so that $M_1(p) > 0$ if $0 < p < P(k_2)$ (see (2.2)), and hence $M_1^* > 0$. First we pin down the equilibrium payoff of firm 1. This is done in Lemma 7, after a preliminary result (Lemma 6) which gives a relation between the equilibrium payoffs and the smallest prices in the supports of any equilibrium strategies.

LEMMA 6. $a_1 \leqslant a_2$ *and* $h_1(F_1, F_2) = L_1(a_1)$; *if* $a_1 = a_2 = a$ *then* $h_i(F_i, F_j) = L_i(a)$ *for* $i = 1, 2$; *if* $a_2 \leqslant P(k_1)$ *then* $a_1 = a_2 = a$ *and* $h_i(F_i, F_j) = k_i a$ *for* $i = 1, 2$.

Proof. If $p < a_j \leqslant P(k_i)$ then $h_i(p, F_j) = L_i(p) = k_i p$ by (2.2), which is increasing in p, so $a_j \leqslant a_i$ by fact (A). Since $a_1 \leqslant P(k_2)$ by Lemma 4, this means that $a_1 \leqslant a_2$, and thus if $a_2 \leqslant P(k_1)$ then $a_1 = a_2$. If $a_1 < a_2$ then $h_1(F_1, F_2) = L_1(a_1)$ by (d) of fact (B).

If $a_1 = a_2 = a = X(k)$ then $h_i(F_i, F_j) = L_i(a)$ for $i = 1, 2$ by (A.2) and (e) of fact (B). Now we show that if $a_1 = a_2 = a > X(k)$ then $a \notin J(F_i)$. so that $h_i(F_i, F_j) = L_i(a)$ for $i = 1, 2$ (by (d) of fact (B)). By Lemma 2 we have $a \notin J(F_i)$ for some i. This means that there exists a sequence as in (c) of fact (B), so that $h_i(F_i, F_j) = M_i(a) F_j(a) + L_i(a)(1 - F_j(a))$. But if $p < a$ we have $h_i(p, F_j) = L_i(p)$ so from fact (A) and the fact that $M_i(a) < L_i(a)$ (see (A.4)) we have $F_j(a) = 0$, or $a \notin J(F_j)$. ∎

LEMMA 7. $h_1(F_1, F_2) = M_1^*$.

Proof. By Lemma 3 we have $h_i(F_i, F_j) = M_i(b_i) = M_i^*$ for some i. Suppose $h_2(F_2, F_1) = M_2(b_2) = M_2^*$. If $p \geqslant P(k_1)$ then $M_2(p) = 0$, so since $h_2(F_2, F_1) > 0$ (by Lemma 1(b) and the fact that $M_1^* > 0$), we have $b_2 < P(k_1)$. Then by Lemma 6 we have $a_1 = a_2 = a$ and $k_2 a = h_2(F_2, F_1) = M_2(b_2) = b_2(d(b_2) - k_1)$, so that $a = b_2(d(b_2) - k_1)/k_2$ and hence $h_1(F_1, F_2) = k_1 a = k_1 b_2(d(b_2) - k_1)/k_2$. Now, if $k_1 = k_2$ or $d(b_2) = k$ (so that $b_2 = P(k)$) this implies that $h_1(F_1, F_2) = M_1(b_2)$ and hence (by the first part of Lemma 3), $h_1(F_1, F_2) = M_1^*$. We now need to deal with the case $k_1 > k_2$ and $d(b_2) < k$. We know that

$$k_2 M_1^* \geqslant k_2 M_1(b_2) = k_2 b_2(d(b_2) - k_2)$$
$$= k_1 b_2(d(b_2) - k_1) - (k_1 - k_2) b_2(d(b_2) - k).$$

So if $k_1 > k_2$ and $d(b_2) < k$ we have $M_1^* > k_1 b_2(d(b_2) - k_1)/k_2 = h_1(F_1, F_2)$, contradicting the first part of Lemma 3. Hence we must have $h_1(F_1, F_2) = M_1(b_1) = M_1^*$. ∎

The following general result (parts (a), (b)(ii) and (iii) do not depend on our previous results) gives some relations between the equilibrium strategies and equilibrium payoffs.

LEMMA 8. *Let $E_i = h_i(F_i, F_j)$ and $p \in \hat{S}_0$. Then*

(a) $F_i(p) \geqslant \mathscr{I}Q_i(p; E_j)$;

(b) *if* (i) $p \in J(F_j)$, *or* (ii) F_j *is right-increasing at p, or* (iii) $p \in \text{supp } F_j$ *and $p \notin J(F_i)$, then* $F_i(p) = Q_i(p; E_j)$;

(c) *if* $F_i(p) > \mathscr{I}Q_i(p; E_j)$ *and there exists $x \in \text{supp } F_i$ with* $X(k) < x \leqslant p$ *then* $F_j(p) = \mathscr{I}Q_j(p; E_i)$.

Proof. (a) follows immediately from fact (A), (A.4), and the fact that F_i

must be nonnegative and nondecreasing. (b) follows from (a), (b), and (c) of fact (B) and Lemma 2. To prove (c), suppose that $F_i(p) > \mathscr{I}Q_i(p; E_j)$, so that $p > a_i$ (since $\mathscr{I}Q_i(p; E_j) \geqslant 0$), and let $x_k = \max\{x \in \operatorname{supp} F_k : x \leqslant p\}$ for $k = 1, 2$, so that $x_i > X(k)$ (since $a_i \geqslant X(k)$). If $x_i < x_j$ then $x_j \notin J(F_i)$, so $F_i(x_j) = Q_i(x_j; E_j)$ by (b)(iii) and hence $F_i(p) = F_i(x_j) = Q_i(x_j; E_j) \leqslant \mathscr{I}Q_i(p; E_j)$, contradicting $F_i(p) > \mathscr{I}Q_i(p; E_j)$. Hence $x_i \geqslant x_j$. But then $x_i \notin J(F_i)$ (if $x_i = x_j$ this follows from (b)(i), since $F_i(x_i) = F_i(p) > \mathscr{I}Q_i(p; E_j) \geqslant Q_i(x_i; E_j)$), so by (b)(iii) we have $F_j(x_i) = Q_j(x_i; E_i)$ and hence $F_j(p) = F_j(x_i) = Q_j(x_i; E_i) = \mathscr{I}Q_j(x_i; E_i) \leqslant \mathscr{I}Q_j(p; E_i)$ so that $F_j(p) = \mathscr{I}Q_j(p; E_i)$ (using (a)). ∎

The next two results allow us (in Proposition 3) to deal with the case where M_1 is maximized at $P(k)$.

LEMMA 9. *If $\mathscr{I}Q_2(\,\cdot\,; M_1^*)$ is right-increasing at p then so is $\mathscr{I}Q_1(\,\cdot\,; E_2)$.*

Proof. First note that

$$Q_1(p; E_2) = \begin{cases} (k_2 p - E_2)/p(k - d(p)) & \text{if } X(k) < p \leqslant P(k_1), \\ 1 - E_2/k_2 p & \text{if } P(k_1) \leqslant p \leqslant P(k_2), \end{cases} \quad (A.5)$$

so that if $E_2/k_2 \geqslant P(k_1)$ then

$$\mathscr{I}Q_1(p; E_2) = \begin{cases} 0 & \text{if } X(k) < p \leqslant E_2/k_2, \\ 1 - E_2/k_2 p & \text{if } E_2/k_2 \leqslant p \leqslant P(k_2). \end{cases} \quad (A.6)$$

Now suppose that $a_2 > P(k_1)$. Since either $a_2 \in J(F_2)$ or F_2 is right-increasing at a_2, Lemma 8(b) implies that $0 \leqslant F_1(a_2) = Q_1(a_2; E_2) = 1 - E_2/k_2 a_2$. Hence $a_2 \geqslant E_2/k_2$, so that $F_2(p) = 0$ if $p < E_2/k_2$, and hence $\mathscr{I}Q_2(p; M_1^*) = 0$ if $p \leqslant E_2/k_2$ (using Lemma 8(a)). Hence $\mathscr{I}Q_2(\,\cdot\,; M_1^*)$ is right-increasing on a subset of $[E_2/k_2, P(k_2)]$; since $\mathscr{I}Q_1(\,\cdot\,; E_2)$ is right-increasing on this set (see (A.6)), the result follows in this case.

If $a_2 \leqslant P(k_1)$ then $a_1 = a_2 = a$ and $E_2 = k_2 E_1/k_1$ (see Lemma 6), so that $Q_1(p; E_2) = k_2 Q_2(p; E_1)/k_1$ if $X(k) < p \leqslant P(k_1)$ (see (2.2) and (3.2)), and hence in this interval $\mathscr{I}Q_1(\,\cdot\,; E_2)$ is right-increasing whenever $\mathscr{I}Q_2(\,\cdot\,; E_1)$ is. Finally consider the case $P(k_1) < p \leqslant P(k_2)$. Let $p_i = \min\{p \geqslant P(k_1) : Q_i(p; E_j) = \mathscr{I}Q_i(P(k_1); E_j)\}$ for $i = 1, 2$, so that $\mathscr{I}Q_i(p; E_j)$ is constant on $[P(k_1), p_i]$. Since $\mathscr{I}Q_2(P(k_1); E_1) = k_1 \mathscr{I}Q_1(P(k_1); E_2)/k_2$ by the argument above, and $Q_2(p; E_1) = (p\, d(p) - E_1)/pk_2 \leqslant k_1(p - E_2/k_2)/pk_2 = k_1 Q_1(p; E_2)/k_2$ if $P(k_1) \leqslant p \leqslant P(k_2)$, we have $p_1 \leqslant p_2$. If $p \geqslant p_1$ then $\mathscr{I}Q_1(p; E_2) = Q_1(p; E_2) = 1 - E_2/k_2 p$ (see (A.5)), so that $\mathscr{I}Q_1(\,\cdot\,; E_2)$ is right-increasing on $[p, P(k_2)]$. Since $\mathscr{I}Q_2(p; E_1)$ is constant on $[P(k_1), p_2]$, the result follows. ∎

LEMMA 10. *If $p \in \hat{S}_0$ and $F_2(p) > \mathscr{I}Q_2(p; M_1^*)$ then supp $F_2 \subset \{X(k)\} \cup (p, P(k_2))$.*

Proof. If $F_2(p) > \mathscr{I}Q_2(p; M_1^*)$ then there exists $y \geqslant p$ such that $F_2(z) > \mathscr{I}Q_2(z; M_1^*)$ for all $z \in [p, y]$ and $\mathscr{I}Q_2(\cdot; M_1^*)$ is right-increasing at y, so that $\mathscr{I}Q_1(\cdot; E_2)$ is also right-increasing at y (by Lemma 9). So if there exists $x \in \text{supp } F_2$ with $X(k) < x \leqslant p$ then $F_1(z) = \mathscr{I}Q_1(z; E_2)$ for all $z \in [p, y]$ by Lemma 8(c). But then F_1 is right-increasing at y, contradicting Lemma 8(b)(ii) (since $F_2(y) > \mathscr{I}Q_2(y; M_1^*)$). ∎

PROPOSITION 3. *If $k_2 < d(0)$ and $M_1^* = k_1 P(k)$ then (F_1, F_2) is an equilibrium only if it satisfies the conditions in (b)(i) of Theorem 1.*

Proof. First note that since $M_1^* > 0$ we have $P(k) > 0$ and hence $X(k) = P(k)$. Since $M_1(P(k)) = M_1^*$ we have $Q_2(p; M_1^*) > 0$ if $p \in \hat{S}_0$, so that $a_2 = P(k)$ by Lemma 8(a), and hence $a_1 = P(k)$ and $h_2(F_2, F_1) = k_2 P(k)$ by Lemma 6. Now, Lemma 10 implies that F_2 satisfies (3.3) and hence is nonatomic on $\hat{S}_0$. Thus (3.4) follows from Lemma 8(b)(iii) and (3.5) from Lemma 8(a) and (b)(ii). ∎

To complete the proof of Theorem 1, we need the following:

LEMMA 11. *If $M_1^* > k_1 P(k)$ then (a) supp $F_i \subset \hat{S}_0$ for $i = 1, 2$, (b) $F_2(p) = \mathscr{I}Q_2(p; M_1^*)$ if $p \in \hat{S}_0$, and (c) $h_2(F_2, F_1) = k_2 a$ for some $a \in [\min A, a_2]$.*

Proof. (a) We have $X(k) \notin \text{supp } F_1$ by (e) of fact (B) and (A.2), so that $a_1 > X(k)$ and hence $a_2 > X(k)$ by Lemma 6.

(b) This is immediate from (a) and Lemma 10.

(c) If $a_1 = a_2 = a$ then $h_2(F_2, F_1) = L_2(a_2) = k_2 a_2$ by Lemma 6. Suppose that $a_1 < a_2$. Then since F_2 is right-increasing at a_2 (by (b) and the continuity of $\mathscr{I}Q_2(\cdot; M_1^*)$) we have $F_1(a_2) = Q_1(a_2; E_2)$ by Lemma 8(b)(ii), so since $F_1(a_2) \geqslant 0$ we have $L_2(a_2) = k_2 a_2 \geqslant E_2$. Since either $a_1 \in J(F_1)$ or F_1 is right-increasing at a_1 we have $F_2(a_1) = Q_2(a_1; M_1^*)$ by Lemma 8(b), so $a_1 \in A$. But now $L_2(a_1) = k_2 a_1 \leqslant E_2$ (if $p_n \uparrow a_1$ we need $L_2(p_n) = h_2(p_n, F_1) \leqslant E_2$), so $h_2(F_2, F_1) \geqslant k_2 \min A$. ∎

PROPOSITION 4. *If $k_2 < d(0)$ and $M_1^* > k_1 P(k)$ then (F_1, F_2) is an equilibrium only if it satisfies the conditions in (b)(ii) of Theorem 1.*

Proof. The conditions on F_2 follow from (a) and (b) of Lemma 11. The condition (3.4) follows from Lemmas 11(a) and 8(b)(iii), and (3.5) follows from Lemmas 11(c) and 8(b)(ii). ∎

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