Candidate Positioning and Entry in a Political Competition

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I first show that if there are more than two potential candidates in the Hotelling–Downs model of the simultaneous choice of positions by politicians then an equilibrium fails to exist in a wide range of situations. Subsequently I study a temporal model in which candidates are free to act whenever they wish. For the case of three potential candidates I find that in every equilibrium exactly one candidate enters. There is always an equilibrium in which the position chosen by the entrant is the median; the only other possibility is that the position chosen is far from the median. *Journal of Economic Literature* Classification Numbers: C72, D72. © 1993 Academic Press, Inc.

1. INTRODUCTION

Hotelling's (1929) model of spatial competition, as interpreted by Downs (1957), allows us to see clearly why politicians in two-candidate competitions choose similar platforms. It is much less successful in yielding insights about the outcomes of competitions in which there are more than two candidates. I begin by studying its equilibria under plurality rule in this case. My results are negative. I show that if each potential candidate has the option of not entering the competition and prefers to stay out rather than to enter and lose then for almost *any* distribution of the voters' ideal points (most preferred positions) the game has no Nash equilibrium in pure strategies. If each potential candidate prefers to enter and lose

than to stay out of the competition and chooses a position to maximize her plurality then the game has no pure strategy Nash equilibrium for almost any single-peaked distribution of the voters' ideal points. In both cases Nash equilibria in mixed strategies appear to be intractable.

In the simultaneous-move model a configuration of positions fails to be a Nash equilibrium if a player can choose a different position and obtain more votes than any other, given that no player responds. The assumption that players who have chosen positions cannot change them may (or may not) be attractive, but it seems unreasonable for a candidate to assume that a different action on her part will never encourage the entry of a new candidate: potential candidates are not usually restricted to acting at a predetermined time. Thus a Nash equilibrium of the simultaneous-choice model does not seem to capture the strategic reasoning of candidates in the world; a model that appears to do better in this regard is one in which each player can choose a position whenever she wishes. (Delayed entry may entail a cost, but not one that overwhelms all potential benefits.) In such a model a player who considers deviating has to worry about the reaction of any player who has not yet chosen a position; this makes it more likely that an equilibrium exists.

In the second part of the paper I study such a model. Time is discrete and starts at period 1. In each period any player who has not yet chosen a position may do so. Once chosen, a position cannot be changed. When all the players who are ever going to enter the competition have done so an election is held. Each of a continuum of voters endorses the candidate whose position is closest to her ideal point. (That is, as in the standard Hotelling–Downs model, voting is "sincere.") The winner of the election is the candidate who receives the most votes. Potential candidates prefer to stay out of the competition rather than to enter and lose. When there are three potential candidates I show that the game has an equilibrium in which one candidate enters at the median ideal point in period 1 and the other two players stay out of the competition. If the distribution of the voters' ideal points is single-peaked and symmetric about its median then this is the only equilibrium. If not then the only other equilibria are ones in which exactly one candidate enters, and chooses a relatively extreme position.

Related Work[1]

Cox (1987) studies the simultaneous-choice model when the candidates are plurality-maximizers. He finds Nash equilibria when the distribution of ideal points is uniform on [0, 1], and shows that if for some distribution

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[1] Shepsle and Cohen (1990) and Shepsle (1991) survey the literature.

a Nash equilibrium exists then at least one candidate chooses a position outside the interval from the first to the third quartile. In (1990, p. 183) he conjectures that equilibria generically fail to exist. My Proposition 5 shows that among single-peaked densities this conjecture is correct, so that in this case Cox's second result is almost vacuous. Parts of my arguments are similar to those of Eaton and Lipsey (1975) for an economic location model.

Feddersen *et al.* (1990) remove the restriction in the simultaneous-move model that individuals vote "sincerely." There are then equilibria in which several candidates enter at the median ideal point, each obtaining the same number of votes; these are the only equilibria in which the voters use undominated strategies. These equilibria are supported as follows. If a potential candidate deviates and enters at a point different from the median then all the voters who strictly prefer the deviant's position vote for her; the remainder vote for one of the candidates—say the one with the lowest index—located at the median. The key point is that though the remaining voters are indifferent between all the candidates at the median, they all vote for *one* of these candidates in the event of a deviation. Thus, as Feddersen *et al.* say, "a disturbing feature of the equilibrium set is that it depends on implicit coordination (or cooperation) among voters" (p. 1014). (Eddie Dekel has pointed out to me that polls may serve as a coordination device, though to model them formally requires an expansion of the model.) The equilibria in my model have the advantage that they do not depend on this sort of coordination:[2] a deviation is deterred by the entry of another candidate that it induces.

Cox's results raise the issue of whether equilibria are always "centrist." On this, my model and that of Feddersen *et al.* give different answers: in my model there are, for some distributions of ideal points, noncentrist equilibria, while in FSW's there are not.

My temporal model (in Section 4) is related to that of Palfrey (1984). In his model there are three candidates; the game has two stages. In the first stage two of the candidates simultaneously choose positions; in the second stage the remaining candidate does so. Each candidate acts to maximize the number of votes she receives; she does not have the option of staying out of the competition. Relative to my model, the timing of decisions is rigid: Palfrey's model captures a situation in which there are two established parties, each of which has little flexibility in the time at which it must choose a position.

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[2] Another type of coordination is necessary in both my model and that of Feddersen *et al.*: in any equilibrium in which not all the potential candidates enter there is some "coordination" required to select those who do. When there is an equilibrium with a single entrant, for example, there is one such equilibrium for every potential candidate.

2. THE FRAMEWORK

A *policy* or *position* is a point on the real line X. Each of $n \geq 2$ players can, according to rules described below, choose to offer some policy, or to stay out of the competition. A player who enters the competition becomes a *candidate*. There is a continuum of voters, each of whom has a most preferred ("ideal") policy. The distribution function of the voters' ideal policies is F. I assume that F is nonatomic and that its support is an interval; I denote its density by f. I assume that voting is "sincere": each voter endorses a candidate whose position is closest to her ideal. If k candidates offer the same policy x then each receives the fraction $1/k$ of the votes of those individuals whose ideal points are closer to x than to the policy of any other candidate. (Since F is nonatomic it does not matter which of two equidistant candidates at different positions a voter endorses.)

Given a profile $(y_1, \ldots, y_k)$ of occupied positions with $y_i < y_{i+1}$ for all $i = 1, \ldots, k - 1$, the *constituency* of the position y_i is the fraction of the population that votes for one of the candidates at y_i (i.e. $F((y_{i+1} + y_i)/2) - F((y_i + y_{i-1})/2)$ if $2 \leq i \leq k - 1$, with appropriate modifications for $i = 1$ and $i = k$). This constituency consists of two *semiconstituencies:* the *left constituency*—the fraction of the population that votes for a candidate at y_i and has an ideal point less than y_i (i.e., $F(y_i) - F((y_i + y_{i-1})/2)$ if $i \geq 2$)—and the *right constituency,* defined symmetrically. (Throughout I think of the policy space as running from left to right, with negative numbers on the left, and use "x lies to the left of y" as a synonym for "$x < y$.")

Candidates' Preferences

In order to describe the game that I study I need to specify the structure of the players' choices and the players' preferences over outcomes (profiles of vote totals). In this section I discuss the latter.

The assumptions that I make about preferences model a situation in which the winner of the election is the candidate who receives the most votes. For any profile x of positions for the candidates let $v_i(x)$ be the fraction of the votes received by candidate i and let $M_i(x)$ be i's plurality:

$$M_i(x) = v_i(x) - \max_{j \neq i} v_j(x).$$

An essential feature of electoral competition is that the objective of each candidate is to win, so that it is natural to assume that each candidate prefers to win outright than to tie for first place with any number of other candidates:

$$x >_i y \qquad \text{whenever } M_i(x) > 0 \text{ and } M_i(y) = 0. \tag{1}$$

It is natural also to assume that each player prefers to tie for first place with any number of candidates rather than to lose, though this is an assumption that I do not always need:[3]

$$x >_i y \qquad \text{whenever } M_i(x) = 0 \text{ and } M_i(y) < 0. \tag{2}$$

As to additional restrictions on preferences, I consider two cases. In one case I assume that each Player i prefers to stay out of the competition rather than to enter and lose:

$$x >_i y \qquad \text{whenever } x_i = \text{OUT and } M_i(y) < 0, \tag{3}$$

where OUT is the action of staying out of the competition. When the players' preferences satisfy this assumption I sometimes make two additional assumptions. First, tying for first place with one other candidate is preferred to staying out of the competition:

$$x >_i y \qquad \text{whenever } M_i(x) = 0, w(x) = 2, \text{ and } y_i = \text{OUT}, \tag{4}$$

where $w(x)$ is the number of candidates who win (have a nonnegative plurality) in the profile x. Second, each Player i is indifferent between any two outcomes in which she wins outright:

$$x \sim_i y \qquad \text{whenever } M_i(x) > 0 \text{ and } M_i(y) > 0. \tag{5}$$

The other case that I study is that in which each player prefers to enter the competition rather than to stay out, even if she loses. (As a loser she may influence the policy carried out by the winner, especially if the winner's margin of victory is small, and entering the competition, even if she

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[3] Note that (1) and/or (2) are *not* satisfied by some of the objectives considered in the literature. An objective drawn from models of the location of firms is that candidates prefer to obtain more votes rather than fewer; this leads candidate 1 to prefer, for example, the profile of votes (4, 5, 0, 0), in which she loses, to the profile (3, 2, 2, 2), in which she wins. A case that Denzau *et al.* (1985) study is that in which each candidate i minimizes a weighted average of the numbers of candidates who obtain more votes than i and who are tied with i. If the weight on the second number is zero then candidate 1 is indifferent between (2, 1, 1), in which she wins outright, and (2, 2, 0), in which she is tied for first place. If the weight on the second number is positive and there are sufficiently many candidates then there is a pair of profiles v and v' with the property that in v candidate i is tied for first place while in v' she loses, and she prefers v' to v. (Note also that Denzau *et al.* assume throughout that the distribution of the voters' ideal points is uniform.)

loses, may be necessary for her to attain the credibility she needs to have a chance of winning future elections. These benefits may outweigh the cost of entry.) Under this assumption, no player chooses OUT in any equilibrium; for this reason I simply do not give the players the option of choosing OUT. Further, in this case I assume specifically that each player is a *plurality maximizer*:[4]

$$x >_i y \qquad \text{whenever } M_i(x) > M_i(y). \tag{6}$$

3. SIMULTANEOUS MOVE GAMES

I first consider the case in which the candidates take actions simultaneously. Denote by $G_n(X \cup \{\text{OUT}\})$ and $G_n(X)$ the game forms in which there are n players and the strategy set of each player is $X \cup \{\text{OUT}\}$ and X respectively. Unless otherwise stated I restrict attention to Nash equilibria in pure strategies.

When considering a particular Nash equilibrium I adopt the convention that k is the number of candidates who enter, the position of candidate i is x_i, with $x_1 \le x_2 \le \cdots \le x_k$, and the number of occupied positions is r. If $r = 1$ the occupied position is y_1, and if $r \ge 2$ the occupied positions are $y_1 < \cdots < y_r$; k_j is the number of candidates at y_j.

Players Who Prefer to Stay Out Rather than to Enter and Lose

LEMMA 1. *If each player's preferences satisfy* (1) *and* (3) *then any Nash equilibrium of* $G_n(X \cup \{\text{OUT}\})$ *in which at least two players enter satisfies the following.*

(a) $k_i \le 2$ *for all* i.
(b) $k_1 = k_r = 2$.
(c) *If* $k_i = 2$ *then the left and right constituencies of* y_i *are equal.*
(d) *The share of the vote of each of the* k *players who enters is* $1/k$.

Proof. If not all candidates obtain the share $1/k$ then one of them loses, and so by (3) prefers not to enter, establishing (d). To demonstrate the remaining properties of an equilibrium I show the following.

CLAIM. *Every semiconstituency is at most* $1/k$.

Proof. Suppose that a semiconstituency—without loss of generality the left constituency—of y_i exceeds $1/k$. If follows from (d) that $k_i \ge 2$. If one of the candidates currently at y_i moves to a point just to the left of

y_i then she obtains more than $1/k$ of the vote. Her move slightly reduces the share of each candidate at y_{i-1} (if $i \geq 2$) and reduces the share of the other candidates at y_i (using (d) again); it affects no other candidate. Hence she wins outright, an outcome she prefers (by (1)) to tying with the other candidates at y_i.

If follows that the constituency of any point is at most $2/k$, so that (a) follows from (d). If $k_i = 1$ for $i = 1$ or $i = r$ then the candidate at y_i can, by moving closer to her neighbor, increase her share without increasing any other candidate's share, and hence win outright, so that (b) follows from (1). Finally, if $k_i = 2$ then the constituency of y_i is $2/k$ by (d), so that (c) follows from the claim. ∎

It follows that if there are two players ($n = 2$) whose preferences satisfy (1) and (3) then the only possible equilibrium in which both players enter is that in which they both do so at the median of F. If their preferences satisfy also (4) then this is fact an equilibrium, and is the only one. If they both prefer to stay out of the competition rather than to tie for first place with one other candidate and their preferences satisfy (5) then there is an equilibrium, in every equilibrium just one of them enters, and the position chosen by the entrant is the median of F. I now consider the case $n \geq 3$.

LEMMA 2. *For no distribution F is there a profile of positions that satisfies the four conditions of Lemma 1 for $k = 3$. For almost no distribution F is there a profile of positions that satisfies the four conditions of Lemma 1 for $k \geq 4$.*

Proof. The case $k = 3$ follows immediately from (a) and (b) of Lemma 1. If $k \geq 4$ then by (a) of Lemma 1 we have $r \geq 2$. From (b), (c), and (d) we know that $y_1 = F^{-1}(1/k)$ and the boundary between the constituencies of y_1 and y_2 is $F^{-1}(2/k)$, so that $y_2 - F^{-1}(2/k) = F^{-1}(2/k) - y_1$, or $y_2 = 2F^{-1}(2/k) - F^{-1}(1/k)$. Hence by (a) and (c) there can be two candidates at y_2 only if $F^{-1}(3/k) = 2F^{-1}(2/k) - F^{-1}(1/k)$ ($= y_2$), which is true for almost no distribution F. Thus for almost all distributions any profile of positions that satisfies the conditions has exactly one candidate at y_2.

I now argue that for almost any distribution there is exactly one candidate at y_j for all $j \geq 2$. From the argument above we know that $y_2 = 2F^{-1}(2/k) - F^{-1}(1/k)$ and that there is one candidate at y_2. Hence by (d) of the lemma the boundary between the constituencies of y_2 and y_3 is $F^{-1}(3/k)$. This determines the position of y_3; for almost no distribution is the left constituency of y_3 equal to $1/k$, so that by (a) and (c) of Lemma 1 we conclude that there is one candidate at y_3. Continuing in the same way we conclude that there is one candidate at every occupied position to the right of y_1. For $j = r$ this contradicts (b) of the lemma, completing the proof. ∎

I can now establish my result on the generic nonexistence of a Nash equilibrium in $G_n(X \cup \{\text{OUT}\})$.

PROPOSITION 3. *Suppose that each player's preferences satisfy* (1), (3), *and* (4). *Then for any distribution F the game* $G_3(X \cup \{\text{OUT}\})$ *has no Nash equilibrium; if* $n \geq 4$ *then for almost any distribution F the game* $G_n(X \cup \{\text{OUT}\})$ *has no Nash equilibrium.*

Proof. If $n \geq 3$ then under (1) and (4) there is no equilibrium in which no player enters; under (4) there is no equilibrium in which one player enters. By Lemma 1 in any equilibrium in which two players enter they both do so at the median of F. But under (1) and (4) this is not an equilibrium, since a third player can enter and win outright. The result follows from Lemmas 1 and 2. ■

Suppose that the players' preferences satisfy (1) and (3), but violate (4). If each player prefers to win outright rather than to stay out of the competition and (5) is satisfied then it is an equilibrium for one player to enter at the median, and it follows from part (d) of Lemma 1 that if the violation of (4) is strict then there is no other equilibrium.

If the density f of F is single-peaked (i.e., increasing up to its maximum and decreasing thereafter) then we can draw a conclusion stronger than that of Proposition 3: if $n \geq 5$ there is no Nash equilibrium for *any* distribution F. If $n = 4$ then there is a Nash equilibrium in this case only if f is symmetric and its maximum is not too large. A case that has received some attention in the literature is that in which F is uniform on $[0, 1]$. In this case $G_n(X \cup \{\text{OUT}\})$ has a Nash equilibrium whenever n is even; in any equilibrium all the candidates enter, $k_i = 2$ for all i, and $y_i = (2i - 1)/n$ for all i. (Cox (1987, Theorem 2) shows that these are the only equilibria when the players' preferences satisfy conditions more restrictive than (1) and (2).) Proposition 3 shows that this result depends critically on the uniformity of F.

Plurality Maximizers

I now turn to the case in which the players' preferences satisfy (6) (i.e., the players are plurality maximizers).

LEMMA 4. *If* $n \geq 3$ *and each player's preferences satisfy* (6) *then for almost any distribution F at least one candidate loses in any Nash equilibrium of* $G_n(X)$.

Proof. Cox (1987, Lemma 1) shows that if each player is a plurality maximizer then any Nash equilibrium of $G_n(X)$ satisfies (a), (b), and (c) of Lemma 1. Thus by Lemma 2 for almost any F any Nash equilibrium violates (d), and hence at least one candidate loses. ■

PROPOSITION 5. *If $n \geq 3$ and each player's preferences satisfy (6) then for almost every single-peaked density f the game $G_n(X)$ has no Nash equilibrium.*

Proof. First note that by Cox (1987, Lemma 1), (a), (b), and (c) of Lemma 1 hold. Fix a Nash equilibrium and denote the set of candidates for whom $M_i(x)$ is largest—the winners—by W. Suppose without loss of generality that there is a candidate in W who is located at or to the right of the maximizer of f. Let y_i be the position that is furthest to the right among those positions at which there is a candidate in W. I consider two cases separately.

$i < r$: Suppose that a candidate at y_{i+1} moves to the left by a distance small enough that W either remains the same or shrinks. If $k_{i+1} = 1$ then this move increases the share of the candidate who moves (since y_i lies to the right of the maximizer of f) and either reduces the share of the winning candidate (if only the candidate(s) at y_i are winners) or leaves this share the same. Hence the candidate who moves is better off. If $k_{i+1} = 2$ (the only other possibility) the same is true since the two semi-constituencies of y_{i+1} are equal.

$i = r$: We have $k_i = 2$ and the left and right constituencies of y_i are equal. Hence $f(y_{i-1}) < f(\frac{1}{2}(y_{i-1} + y_i))$ (otherwise the candidate(s) at y_{i-1} obtain more than the candidates at y_i, contradicting the fact that the latter are winners). Thus y_{i-1} lies to the left of the median. If there is a candidate not in W then let y_j be the first position to the left of y_i at which there is such a candidate. If a candidate at y_j moves slightly to the right then, as for the case of a move to the left by a candidate at y_{i+1} in the case $i < r$, she can increase her share and either leave W unchanged (if only those at y_i are winners) or shrink this set. Hence she is better off. The remaining possibility is that all candidates are in W. The result follows from Lemma 4. ■

The games that I have studied may possess Nash equilibria in mixed strategies when $n \geq 3$. However, the problem of finding any such equilibria seems to be intractable. Further, voters may have an aversion to candidates who choose their positions randomly (because, for example, they doubt the "sincerity" of such candidates); if so, a richer model is required to study equilibria in which candidates randomize.

4. A TEMPORAL GAME IN WHICH EACH PLAYER MAY ACT WHEN SHE WISHES

I now study a game in which each of the n players may enter the competition whenever she wishes. In each of the infinite sequence of periods 1, 2, . . . every player who has not yet chosen a position either

does so or chooses to wait until the next period. As in the previous section, I restrict attention, unless otherwise stated, to pure strategies. Denoting the option to wait by w, each candidate who has chosen w in every previous period chooses, in period t, a member of the set of *actions* $X \cup \{w\}$. The choices of the players in any given period are simultaneous. Once a player has chosen a position she can take no further action. One option of a player is to choose w in *every* period—i.e., to stay out of the race entirely. Given a strategy profile for the players, there is some date after which no further players enter. At this date an election is held, and, as before, the winner is the candidate who receives the most votes. *Throughout I assume that the players' preferences satisfy* (1) *through* (5). This defines an extensive game, which I denote $\Gamma(n)$.

Equilibrium

As in many games in which play may proceed over many periods, a strategy in $\Gamma(n)$ is complex. Moreover, the notion of subgame perfect equilibrium, which requires that the players' strategies define a Nash equilibrium in every subgame, requires that each player's strategy be specified fully. However, in $\Gamma(n)$, as in other sequential games in which some choices are made simultaneously, the spirit of subgame perfect equilibrium is captured by a notion that requires only a *partial* specification of the players' strategies. The idea is that no player should be able to increase her payoff by changing her action in any period, given that the behavior of the players in the subgame to which the deviation leads is optimal, in the sense that it satisfies the same condition. To check that a strategy profile σ meets this condition, no information is needed about the behavior that σ prescribes in subgames that are reached when more than one player deviates from σ in some period. This leads to the following definition of equilibrium.

A *substrategy* σ_i of Player i is a function that assigns an action of Player i to every member of a subset of the set of histories at which Player i has not already chosen a position. Denote the set of histories after which σ_i specifies an action by $H(\sigma_i)$. A profile σ of substrategies is an *equilibrium* if (1) for every Player i, $H(\sigma_i)$ includes all histories that result when at most one player deviates from σ in any given period and (2) after any such history, no player can increase her payoff by a unilateral change of strategy, given that the other players continue to adhere to σ.

The advantage of working with this notion of equilibrium rather than subgame perfect equilibrium in the game $\Gamma(n)$ is that it is not necessary to specify the players' behavior, or, indeed, worry about the existence of an equilibrium, in "irrelevant" subgames—subgames that cannot be reached by a sequence of deviations from σ in which at most one player

deviates in each period. This advantage is considerable: relative to the equilibria that I consider, there is a huge number of irrelevant subgames. To illustrate briefly, suppose that there are three players and consider a strategy profile in which Players 1 and 2 enter in period 1 at x_1^* and x_2^* respectively and Player 3 chooses w in every period. A strategy for Player 1 must specify an action in period 2 for *every* first-period profile of actions (w, s_2, s_3), where s_2 and s_3 are members of $X \cup \{w\}$. However, there is just *one* relevant subgame in which Player 1 has to take an action: the one that follows the first-period action profile (w, x_2^*, w).

The relation between an equilibrium in this sense and a subgame perfect equilibrium is close: a subgame perfect equilibrium is an equilibrium, and if every subgame has a subgame perfect equilibrium then an equilibrium is associated with at least one subgame perfect equilibrium.

Results: Three Players

I first study the game $\Gamma(3)$. *Subsequently I normalize the median of F to be 0.* The following result says that in every equilibrium of $\Gamma(3)$ exactly one player enters; the position at which she does so is either the median or far from the median. Let α^* be the length of the shortest interval containing the median that contains $\frac{1}{3}$ of the ideal points (i.e., α^* is the smallest value of α for which $a + \alpha \geq 0$ and $F(a + \alpha) - F(a) = \frac{1}{3}$ for some $a \leq 0$).

PROPOSITION 6. *The game $\Gamma(3)$ has an equilibrium in which one player enters at the median of F in period 1 and the other two players stay out of the competition. In any other equilibrium one player enters at a position to the left of $-\alpha^*$ or to the right of α^* in period 1 and the other two players stay out of the competition.*

To prove this I first establish the following.

LEMMA 7. *Any subgame of $\Gamma(3)$ that follows a history in which one player has chosen a position and the other two players have not yet entered has an equilibrium.*

Proof. Let the position that has been chosen be x_1. Suppose that there is one candidate at x_1 and one at some point x_2. I claim that there then exists an optimal action for the third player. If there is a point at which the third player can win then any such point is optimal (using (5)); if there is no such point but there is a point at which the third player ties with at least one other player for first place then there is a finite number of such points, and one of them is optimal; if the third player loses at any point at which she enters then OUT is optimal. Let $\xi(x_1, x_2)$ be an optimal action for the third player, given x_1 and x_2. The form of an equilibrium of the subgame depends on which of the following two conditions is satisfied.

● For all values of x_2 the action $\xi(x_1, x_2)$ is such that the player at x_2 loses.

● There is a value of x_2 such that given $\xi(x_1, x_2)$ the player at x_2 either wins outright or is tied for first place.

In the first case there is an equilibrium of the subgame in which neither of the remaining players enters. In the second case there is an equilibrium in which one of the remaining players enters and the other player chooses a best response (which may involve entering or not). ■

Proof of Proposition 6. I separate the argument into steps.

STEP 1. $\Gamma(3)$ *has an equilibrium in which one player enters at the median of F in period* 1 *and the other two players stay out of the competition.*

Proof. The following is an equilibrium of $\Gamma(3)$. After any history in which no player has entered, Player 1 enters at the median of F in the next period and the other two players stay out of the competition. If Player 1 enters at some point other than the median then the other players act as in one of the equilibria shown to exist by Lemma 7. If, after a history in which Player 1 has entered at the median, either Player 2 or Player 3—say Player 2—enters at a point x_2 for which $x_2/2$ is in the support of F then Player 3 enters in the following period at a point close enough to the median that she wins outright (such a point exists because F is continuous). If Player 2 enters at some other point then Player 3 enters at the median. In each case Player 2 loses.

Player 1 can do no better than enter at the median since this leads her to win outright; the other players' actions are optimal by construction.

STEP 2. *There is no equilibrium of* $\Gamma(3)$ *in which two players enter in period* 1 *and the third stays out of the competition.*

Proof. In an equilibrium of this type the entrants must locate either at the same point s or at different points $-t$ and t symmetrically about the median (so that they receive the same number of votes). In the first case the third player can enter either at $s - \varepsilon$ or at $s + \varepsilon$ and win outright, so there is no equilibrium of this type. In the second case either the third player can enter (between the other two) and win outright, or there is no position at which the third player can win outright, in which case one of the first two entrants could have entered slightly closer to the median, at a position at which the third player could not even tie for first place, and could thereby have won outright rather than tying with another player. Thus there is no such equilibrium.

STEP 3. *There is no equilibrium of* $\Gamma(3)$ *in which two players enter in period* 1 *and the third enters in a later period.*

Proof. In this case all three candidates must obtain the same number of votes (otherwise one of them loses). There are three possibilities. In each case I describe a deviation that leads to an outcome that the deviant prefers.

First, the candidates are all at the same point. In this case the last entrant can deviate and enter slightly to one side and win outright.

Second, they are at three different points. If the last player to enter is at one of the extreme positions then she can move slightly closer to the middle player and win outright. If the last player to enter takes the middle of the three positions then the player who obtains more than $\frac{1}{2}$ of the votes in the absence of the last player can move slightly closer to the middle, thereby inducing the last player not to enter, and winning outright.

Third, two players (say Players 1 and 2) are at the same position s, while the third (Player 3) is at a different position t. Without loss of generality assume that $t > s$ (so that s and t are equidistant from $F^{-1}(\frac{2}{3})$). If Player 3 is the last to enter then she can move slightly closer to the others and win outright. If one of the other players is the last to enter then there are three cases. First, $s > F^{-1}(\frac{1}{3})$, in which case the last player can enter slightly to the left of s and win outright. Second, $s < F^{-1}(\frac{1}{3})$, in which case the last player can enter slightly to the right of s and win outright. Third, $s = F^{-1}(\frac{1}{3})$, in which case the player at t can enter instead just to the left of $-s$, induce the last player not to enter, and win outright.

STEP 4. *There is no equilibrium in which all three players enter in the same period.*

Proof. The argument is the same as for the nonexistence of an equilibrium in the game in which the players are restricted to act simultaneously.

STEP 5. *The subgame following the entry of one player at the median in period 1 has a unique equilibrium, in which no more players enter.*

Proof. As argued in the proof of Step 1, for any position chosen by one of the remaining players there is a position of the other remaining player that wins outright. Hence the only equilibrium of the subgame is that in which no further players enter.

STEP 6. *In every equilibrium one player enters; she does so in period 1 either at the median or at a position to the left of $-\alpha^*$ or to the right of α^*.*

Proof. By Steps 2, 3, and 4 either one player enters in period 1 or none do so. By Steps 1 and 5 there is no equilibrium in which no player enters in period 1, since in any equilibrium at least one player does not win outright, and by entering at the median in period 1 a player can guarantee that she wins outright. Thus in every equilibrium one player
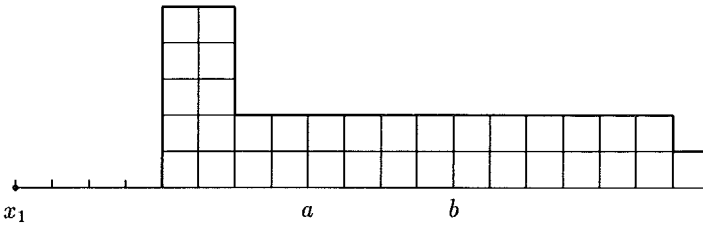
FIG. 1. An example of a distribution of ideal points for which there is an equilibrium in which there is a single entrant at an extreme position. Each square is of the same size. If one candidate enters at x_1 then no further candidate enters.

enters in period 1; again by Steps 1 and 5 no more players enter. Finally suppose that a player enters in period 1 at a position $s \in [-\alpha^*, 0)$. Then one of the remaining players can enter slightly to the left of $-s$, induce the last player not to enter, and win outright. ∎

Proposition 6 does not give conditions for the existence of an equilibrium in which the single entrant chooses an "extreme" position. However, it is not hard to give examples of such equilibria that do not depend sensitively on the nature of F. The single entrant may even occupy a position that is outside the support of F—i.e., that is more extreme that the ideal point of *any* voter. An example is the following. The distribution of ideal points is that given in Fig. 1. Suppose that a single entrant takes the position x_1. If an additional player enters at or to the left of b then a third player who enters slightly to the right of the entrant wins outright (and hence causes the second entrant to lose); if an additional player enters at or to the right of b then a third player who enters at a wins outright. Thus if one player locates at x_1 then in every equilibrium of the subsequent subgame there is no further entry.

Equilibria in which the single entrant chooses a position different from the median of F appear to depend sensitively on the restriction that there are just three potential candidates, so whether they can shed any light on political systems in which there is one extreme party is unclear.

For some distributions F we can rule out equilibria with extremist candidates, as follows.

PROPOSITION 8.   *If the density f of the distribution F of ideal points is single-peaked and symmetric about its median then in every equilibrium of $\Gamma(3)$ one player enters at the median of F and the other two players stay out of the competition.*

Proof.   To prove this I need to show only that in no equilibrium of the subgame that starts in period 2 after Player 1 (say) has located at $x_1 \neq 0$

in period 1 does Player 1 win outright. Throughout I consider only the case $x_1 > 0$; the case $x_1 < 0$ is symmetric.

Given the symmetry and single-peakedness of f, we have $\alpha^* = 2F^{-1}(\frac{2}{3})$; let $\beta^* = \frac{3}{2}\alpha^*$. Note that if the three candidates locate at $-\alpha^*$, 0, and α^*, or if one locates at β^* and the other two locate at $F^{-1}(\frac{1}{3})$, then each receives a third of the votes.

First suppose that $0 < x_1 < \beta^*$. Then there is a point at which one of the remaining players—say Player 2—can enter and be sure to win outright. The point is $g(x_1) + \varepsilon(x_1)$, where $g: (0, b) \to \mathbb{R}$ is given by

$$g(x) = \begin{cases} -x & \text{if } 0 < x \le \alpha^* \\ x - 2\alpha^* & \text{if } \alpha^* < x < \beta^*, \end{cases}$$

and $\varepsilon: (0, \beta^*) \to \mathbb{R}$ is any function that satisfies $0 < \varepsilon(x) < 2x/3$ if $0 < x \le \alpha^*$, and $0 < \varepsilon(x) < \beta^* - x$ if $\alpha^* < x < \beta^*$.

If Player 2 enters at $g(x_1) + \varepsilon(x_1)$ I claim that Player 3 loses wherever she enters. If $x_3 \ge x_1$ then Player 2 obtains more than half of the votes, so that Player 3 loss. If $x_2 \le x_3 < x_1$ then Player 3 obtains less than a third of the votes and hence loses. If $x_3 < x_2$ and $0 < x_1 \le \alpha^*$ then Player 3 obtains fewer votes than Player 1, and hence loses. Finally, if $x_3 < x_2$ and $\alpha^* < x_1 < \beta^*$ then Player 3 obtains less than a third of the votes and hence loses.

Given that by locating at $g(x_1) + \varepsilon(x_1)$ Player 2 is sure of winning outright there is no equilibrium of the subgame in which no player enters after Player 1. In any equilibrium that exists, therefore, Player 1 does not win outright.

Now suppose that $x_1 \ge \beta^*$. In this case there is a point at which an additional entrant ensures that at worst she ties for first place, and there is no point at which she can do better. The point is $h(x_1)$ defined as follows. Fix $x \ge \beta^*$ and consider the function ζ defined by $\zeta(z) = 2F(z) - F((x + z)/2)$ for $z \in [F^{-1}(\frac{1}{3}), 0]$. We have $\zeta(0) = 1 - F(x/2) \ge 0$ and $\zeta(F^{-1}(\frac{1}{3})) = \frac{2}{3} - F(\alpha)$ where $\alpha \ge F^{-1}(\frac{2}{3})$, so that $\zeta(F^{-1}(\frac{1}{3})) \le 0$. Further, ζ is increasing on $[F^{-1}(\frac{1}{3}), 0]$. Hence ζ has a unique zero on this interval. That is, for each $x \ge \beta^*$ there is a unique point $h(x)$ in $[F^{-1}(\frac{1}{3}), 0]$ for which

$$F(h(x)) = F\left(\frac{x + h(x)}{2}\right) - F(h(x)).$$

We have $h(\beta^*) = F^{-1}(\frac{1}{3})$, so that if $x_1 = \beta^*$ and Player 2 enters at $h(x_1)$ then the only point at which Player 3 does not lose is $h(x_1)$; at this point she ties for first place with the other two players. If Player 2 enters at

any other point then there is a point at which Player 3 can win outright. Hence if both Player 2 and Player 3 prefer to tie with two others for first place than to stay out of the competition then in every equilibrium of the subgame these players enter at $h(x_1)$, with the result that all three players tie for first place. If either Player 2 or Player 3 (or both) prefer to stay out of the competition than to tie with two others for first place then in every equilibrium of the subgame one of them enters at $h(x_1)$ and the other stays out of the competition; the one who enters wins outright. In each case there is no equilibrium in which Player 1 wins outright.

If $x_1 > \beta^*$ then $h(x_1) > F^{-1}(\frac{1}{3})$, so if Player 2 enters at $h(x_1)$ then it is optimal for Player 3 to enter at the same point, in which case Players 2 and 3 tie for first place and Player 1 loses. Further, if Player 2 enters at any other point then Player 3 can enter and win outright. Hence in every equilibrium of the subgame Players 2 and 3 enter and tie for first place and Player 1 loses. ■

It seems that the conditions on f stated in Proposition 8 are stronger than necessary to prove the result. They are used to ensure that if Player 1 locates at $x_1 > \beta^*$ then there is a position for Player 2 such that, given the optimal response of Player 3, Player 2 at worst ties for first place and Player 1 loses. The problem is that if Player 2 locates at $h(x_1)$ then for an arbitrary distribution F there may be a point in $(h(x_1), \beta^*)$ at which Player 3 can beat Player 2.

Results: Many Players

For the case in which there are more than three players my results are limited. For convenience, I strengthen (4) to the assumption that each Player i prefers to tie for first place with *any* number of candidates than to stay out of the competition:

$$x \succ_i y \qquad \text{whenever } M_i(x) = 0 \text{ and } y_i = \text{OUT}. \tag{7}$$

In addition I assume that each Player i prefers to tie for first place with as few candidates as possible and is indifferent between all outcomes in which she ties with the same number of candidates:

$$\text{if } M_i(x) = M_i(y) = 0 \text{ then } x \succ_i y \text{ if and only if } w(x) < w(y), \tag{8}$$

where $w(x)$ is the number of candidates who win in the profile x. Then if n is sufficiently small there is an equilibrium in which $n - 2$ players enter at the median of F and the remaining two players stay out of the

competition. How small n has to be depends on the distribution F; for *any* distribution $n \leq 5$ is enough, as the following results shows.

PROPOSITION 9. *For $n = 4$ or $n = 5$ the game $\Gamma(n)$ has an equilibrium in which $n - 2$ players enter at the median of F in period 1 and the remaining two players stay out of the competition.*

To prove this I use the following lemma.

LEMMA 10. *Suppose that $n = 4$ or $n = 5$, Players $1, \ldots, n - 3$ are located at the median of F, and Player $n - 2$ is located at $s > 2F^{-1}(\frac{3}{4})$. Then the two-player subgame that follows has an equilibrium, and in every equilibrium Player $n - 2$ either loses or ties with two other players.*

Proof. The argument for existence is the same as in the proof of Lemma 7. If there is an equilibrium in which no more players enter then Player $n - 2$ loses (since she obtains fewer votes than each of the players at the median). If there is an equilibrium in which exactly one more player, say $n - 1$, enters then this player cannot lose, and hence must enter to the left of the median (otherwise Player n could subsequently enter to the left of the median and win outright); if Player $n - 2$ does not lose then $n - 1$ and $n - 2$ must tie. Thus Player $n - 1$ must enter at a point to the left of $2F^{-1}(\frac{1}{4})$. But then Player n can win outright by entering at $-\varepsilon$ for ε small enough (in which case she wins more than $\frac{1}{4}$ of the votes). Thus in every equilibrium in which exactly one additional player enters Player $n - 2$ loses. Finally consider an equilibrium in which both Player $n - 1$ and Player n enter. By the same argument as for the previous case if Player $n - 2$ does not lose then Players $n - 2, n - 1$, and n must all tie, so that Player $n - 2$ ties with two other players. ∎

Proof of Proposition 9. Consider the following substrategy profile. In period 1 players 1 through $n - 2$ enter at the median of F and the remaining two players stay out of the competition. After a history in which $n - 2$ players have entered at the median and one player has entered at some other point the remaining player enters at a point on the other side of the median, sufficiently close to it that she wins outright. After a history in which $n - 3$ of the players in $\{1, \ldots, n - 2\}$ have entered at the median the remaining player in this set enters at the median in the next period and the other two players stay out of the competition. After a history in which $n - 3$ of the players in $\{1, \ldots, n - 2\}$ have entered at the median and the remaining player in this set has entered at some other point s the actions of Players $n - 1$ and n depend on s. If $0 < s \leq 2F^{-1}(\frac{3}{4})$ then Player $n - 1$ enters slightly to the right of $2F^{-1}[1 - F(s/2)]$ and Player n stays out; if Player $n - 1$ enters at some other point then Player n chooses a best response. If $s > 2F^{-1}(\frac{3}{4})$ then the players behave as in an

arbitrary equilibrium of the subgame. (The subgame has an equilibrium by Lemma 10.)

To see that this substrategy profile is an equilibrium, note that if a player in $\{1, \ldots, n - 2\}$ deviates and enters at a point different from the median then she either loses or ties with two other players for first place (if the point at which she enters is to the right of $2F^{-1}(\frac{3}{4})$, refer to Lemma 10). By (2) and (8) these two outcomes are no better than the outcome if she enters at the median (and ties with either one or two other players); by (7) such a player prefers to tie with $n - 3$ other players than to stay out of the competition. If, after $n - 2$ players have entered at the median, either $n - 1$ or n deviates and enters then the other enters and the original deviant loses. After a history in which $n - 3$ of the players in $\{1, \ldots, n - 2\}$ have entered at the median, the remaining player in this set has entered at the point s with $0 < s \leq 2F^{-1}(\frac{3}{4})$, and Player $n - 1$ has entered slightly to the right of $2F^{-1}[1 - F(s/2)]$, there is no point at which Player n can win or tie for first place, and Player $n - 1$ wins outright. The remaining actions of the players are optimal by construction. ∎

Mixed Strategy Equilibria

In addition to the pure strategy equilibria that I have described the game $\Gamma(n)$ has mixed strategy equilibria. For $n = 3$, for example, there is a symmetric equilibrium of the following form (as pointed out to me by Dan Bernhardt). All three players enter at the median of F with positive probability in period 1. If the realization is that no player enters in period 1 then all three enter with the same probabilities in period 2; if one player enters in period 1 then no more players enter; if two players enter in period 1 then the remaining player enters to one side of the median and wins outright.

I argue that the existence of such mixed strategy equilibria is an artifact of the presence of simultaneous moves in the game $\Gamma(n)$. Consider the alternate game $\Gamma'(n)$ in which, within each period, any players who have not yet chosen a position move sequentially, the order of their moves being determined at the beginning of the period by chance. This game is no less appealing than $\Gamma(n)$ as a model of candidate entry: the simultaneous moves in $\Gamma(n)$ do not appear to capture any essential feature of political competition. The set of pure strategy equilibria of the game $\Gamma'(3)$ correspond exactly to those of $\Gamma(3)$; in any equilibrium the first player to move is the one who enters. Further, $\Gamma'(n)$ is a game of perfect information with chance moves, and thus has no mixed strategy equilibria that differ substantively from its pure strategy equilibria, in the sense that every terminal history that occurs with positive probability in any mixed strategy equilibrium is the terminal history of a pure strategy equilibrium.

Concluding Remarks

It appears that the model has equilibria in circumstances more general than those I have considered—for example, in some cases, at least, when the policy space is two-dimensional.

My results depend on the assumption that the value of n—the number of potential candidates—is known. The nature of the equilibria when n is not known with certainty is unclear.

There are many features of real-world political competitions that are absent from the model. To the extent that these features are inessential the model is virtuous in excluding them. One aspect that may not be inessential is uncertainty. (In a world of certainty there is no need to hold an election!) It is not clear how the equilibria change when the candidates are unsure of the voters' preferences or the voters are unsure of the candidates' positions.

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