PRICE COMPETITION IN A CAPACITY-CONSTRAINED DUOPOLY

by

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Abstract

We characterize what we show is the unique Nash equilibrium in a model of price-setting duopoly in which each firm has limited capacity, and demand may be nonlinear. We study the comparative statics of the equilibrium. In particular, we show that the equilibrium prices are lower, the smaller is demand relative to capacity. The equilibrium varies continuously with the capacities, so that when one firm is very small the solution approximates that of a monopoly.

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1. Introduction

A model of a price-setting duopoly is a natural starting point for a theory of the behavior of oligopolists. However, such a model has been completely solved only under a particularly unrealistic assumption--namely, that each firm can produce an unlimited quantity (or, at least as much as is demanded at the breakeven price) at constant unit cost. The unique equilibrium in this case involves each firm setting the breakeven price, so that (unless capacity can be bought and sold at will) at most half of the capacity of the duopolists is used in equilibrium. Given that the firms must at some point choose their capacities, it is difficult to see how such a situation could come about.

We study the case where the capacity of each firm is limited, and cannot be instantly changed. We describe, for each pair of capacities, what we show is the unique (Nash) equilibrium. We assume that the unit cost is the same for each firm, and is constant up to capacity, and we allow for nonlinear demand. A few special cases of the model have been considered before (for the details see Section 5 below). However, the previous analyses are quite limited. They are either concerned with existence of an equilibrium, or

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simply exhibit an equilibrium, without showing it is unique, for a small set of capacity pairs (e.g. those in which the capacities are equal). From the point of view of economic applications, these analyses tell us very little. Unless an equilibrium is unique, or it can be shown that all equilibria have certain features, the model does not yield any prediction about the way the firms will behave (there may be many other equilibria, involving all sorts of different behavior). Further, interesting economic implications are likely to emerge only when the variation of the equilibrium with the parameters (e.g. capacities) is studied. Here we briefly note two of these "comparative static" results which follow from our model (the details are in Section 4).

We find that the larger is capacity relative to demand, the lower are the equilibrium prices. This is, of course, intuitively plausible. It is noteworthy because it emerges, even though, up to capacity, the technology of both firms is the same. One can obtain a similar prediction in a competitive model only by assuming, for example, that there are more and less efficient firms; when demand is low, the latter are forced out of business and the price falls. In our model the result emerges from the fact that there is, in a precise sense, more competition when there is excess capacity. In contrast to the competitive outcome, profit is at the monopoly level if capacities are small, even though there is more than one firm.

We also find that as the capacity of one of the firms decreases to zero, the outcome approaches the monopoly outcome. Thus the Nash equilibrium does not display the sharp discontinuity found by Shitovitz [1973] in the core. In fact, it has precisely the characteristic which Shitovitz suggests a reasonable solution should possess (see pp. 497-498): when there is a large firm and a very small one, the large firm charges close to the monopoly price with high probability, while the small firm randomly chooses a slightly lower price.

Finally, we note that our results mean that the two-stage model of Kreps and Scheinkman [n.d.], in which firms first choose capacities and then choose prices, can be extended to a demand function which satisfies our assumptions, rather than being linear. It is easy to check that the qualitative outcome is preserved under the more general assumptions: Cournot quantities are chosen as capacities.

On the technical side, the game we analyze is a nonzerosum "game of timing" (i.e. a game in which the payoff functions are continuous, except possibly when both players use the same strategy). Previous work on such games is limited, and uniqueness has not previously been examined. For a class of zerosum games of timing, Karlin [1959] (pp. 293-295) gives a uniqueness proof which relies heavily on the fact that the equilibrium payoffs in such a game are unique. For the games we consider, a substantially more involved argument appears to be necessary.

In the next two sections we describe the model, and our results. In Sections 4 and 5 we discuss comparative static results, and the related literature. Finally, in Section 6 we provide proofs.

2. The Model

There are two firms, $i = 1, 2$. Firm i has capacity k_i; we assume throughout that $k_1 > k_2 > 0$. Each firm can produce the same good at the same, constant unit cost $c > 0$ up to its capacity. Given the prices of all other goods, the aggregate demand for the output of the firms as a function of price is $D\colon \mathbb{R}_+ \to \mathbb{R}_+$. Let p denote the excess of price over unit cost and let $S = [-c, \infty)$. We shall refer, somewhat loosely, to an element of S as a "price". Define $d\colon S \to \mathbb{R}_+$ by $d(p) = D(p + c)$; $d(p)$ is the aggregate demand for the good when its price exceeds the unit cost by p. We make the following assumption on the demand function.

(2.1) There exists $p_0 > 0$ such that $d(p) = 0$ if $p \geq p_0$ and $d(p) > 0$ if $p < p_0$, and d is smooth on $(-c, p_0)$, with $d'(p) < 0$ there.

For each $p \in S$, let $\pi(p) = pd(p)$. Given (2.1), π attains a maximum on S. To save on notation we choose the units in which price is measured so that the maximizer is 1, and the units for quantity so that the maximal value is also 1. We also assume that

(2.2) π is strictly concave on $[0, 1]$ and decreasing on $[1, p_0]$.

We now wish to define the payoffs of the firms at each pair of prices (p_1, p_2). We assume that if $p_i < p_j$ then consumers first try to buy from firm i; when its supply (k_i) is exhausted, they turn to firm j. (Whenever i and j appear in the same expression, we mean that i is not equal to j.) There is a large number of identical consumers, each with preferences which have no "income effect" (for details, see Section 2 of Osborne and Pitchik [1983]). The aggregate demand at the high price p_j then depends on the way the limited supply k_i is allocated among the consumers. It is natural to assume that the rationing scheme is chosen by firm i. However, since the payoff to firm i is independent of the scheme used (only the payoff of firm j is affected), this assumption does not define the one which is chosen. We solve this problem by assuming that firm i chooses the scheme which minimizes the payoff of firm j. In this scheme, each of the (identical) consumers is allowed to purchase the same fraction of k_i. (I.e. a rule like "limit two per customer" is imposed, rather than allowing those at the head of a queue to buy as much as they want).

Under this rationing rule (which is the one implicitly adopted in Levitan and Shubik [1972] and Kreps and Scheinkman [n.d.]), the payoff to firm i when it sets the price $p_i \in S$ and firm j sets $p_j \in S$ is

$$(2.3)\quad h_i(p_i, p_j) = \begin{cases} L_i(p_i) \equiv p_i \min(k_i, d(p_i)) & \text{if } p_i < p_j \\ \phi_i(p) \equiv p \min(k_i, k_i d(p)/k) & \text{if } p_i = p_j = p \\ M_i(p_i) \equiv p_i \min(k_i, \max(0, d(p_i)-k_j)) & \text{if } p_i > p_j, \end{cases}$$

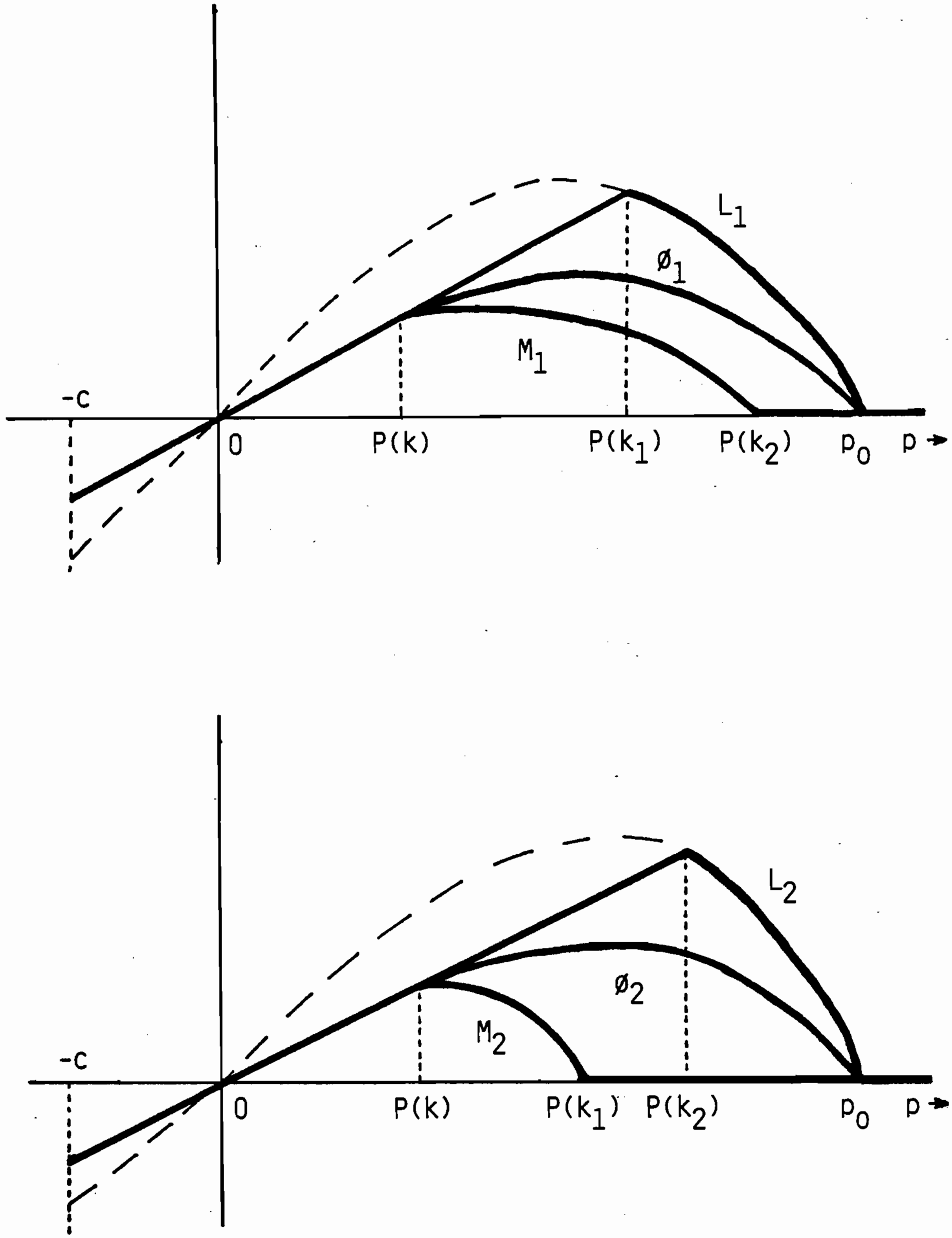
where $k = k_1 + k_2$, and we assume that if $p_i = p_j$ then demand is allocated in proportion to capacities. (For simplicity our notation does not incorporate the dependence of h_i on k_1 and k_2.) Examples of the functions L_i, ϕ_i, and M_i are shown in Figure 1. Let $\mathbf{S}$ be the set of mixed strategies (i.e. cumulative probability distribution functions on S). We extend the domain of h_i to $\mathbf{S} \times \mathbf{S}$ in the natural way. For each pair (k_1, k_2), we study the game $H(k_1, k_2)$ in which the strategy set of each firm is $\mathbf{S}$ and the payoff function of firm i (= 1, 2) is h_i.

3. The Nash Equilibrium of $H(k_1, k_2)$

The qualitative characteristics of the Nash equilibrium of $H(k_1, k_2)$ depend on the value of (k_1, k_2). To define the relevant "regions", suppose that, for each value of $x \in [0, d(0)]$,

$$(3.1)\quad b(x) \text{ maximizes } p(d(p) - x) \text{ over } p \in S,$$

and let $B(x) = b(x)(d(b(x)) - x)$. If firm i sets the price p_i and $b(k_i) > P(k)$, then the best price for firm j to charge out of all those in excess of p_i is $b(k_i)$, independently of p_i (i.e. $b(k_i)$ maximizes M_j in this case). Let $P\colon [0, \infty) \to S$ be the inverse demand function defined



<u>Figure 1</u>

by $P(q) = d^{-1}(q)$ if $0 < q \leq d(-c)$, $P(0) = p_o$ (see (2.1)), and $P(q) = -c$ otherwise. The regions are illustrated[1] in Figure 2. In regions I and II the unique equilibrium strategies are pure, while in the remaining regions they are mixed. In the latter case the equilibrium strategies (F_1, F_2) always have the form

$$(3.2) \quad F_i(x) = \begin{cases} 0 & \text{if} \quad -c \leq x < a \\ G_i(x) & \text{if} \quad a \leq x < b(k_2) \\ 1 & \text{if} \quad b(k_2) \leq x, \end{cases}$$

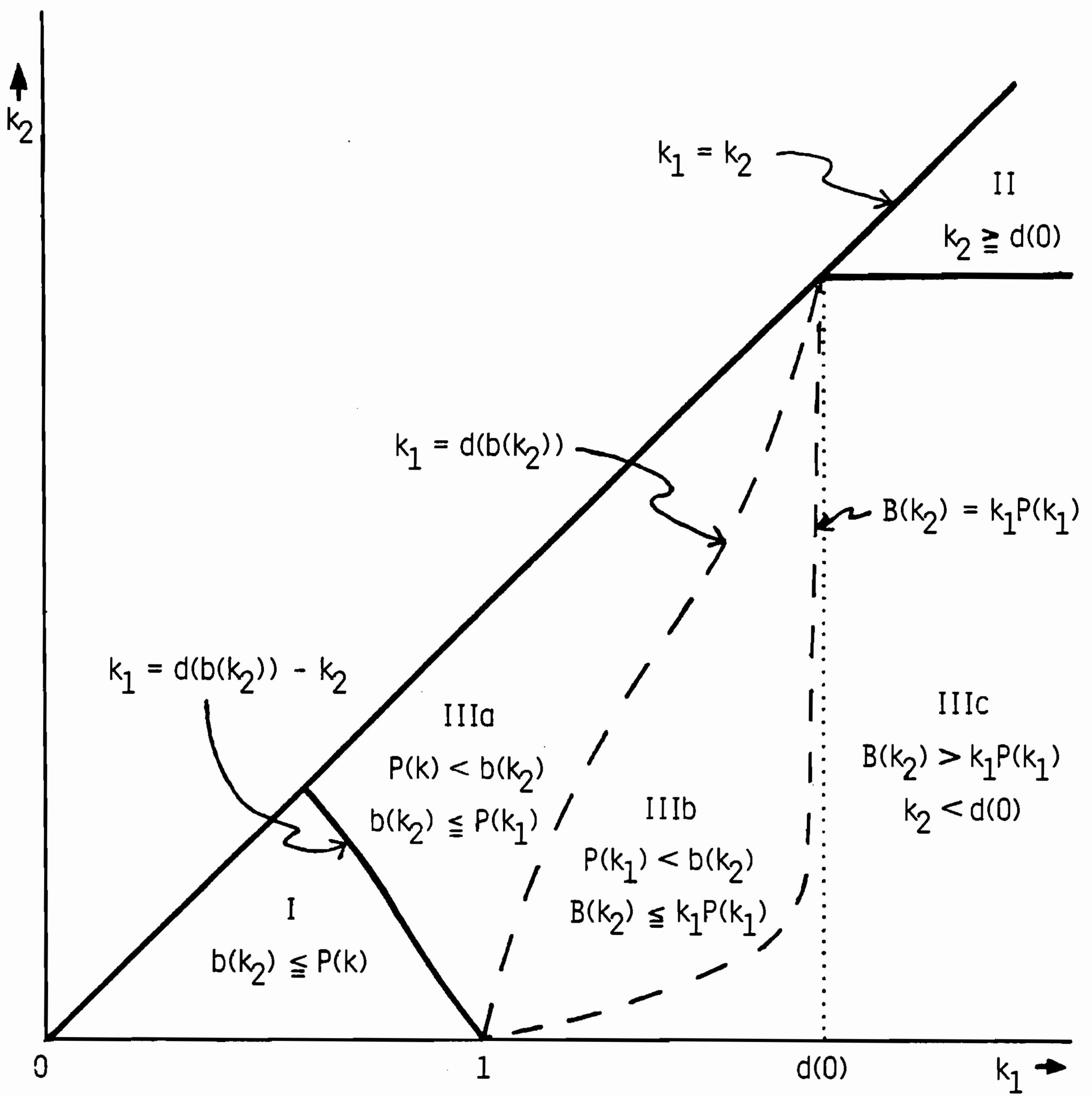
where a and G_i may depend on both k_1 and k_2, G_i: $[a, b(k_2)] \to [0, 1]$ is continuous, $G_i(a) = 0$ for $i = 1, 2$, and $G_2(b(k_2)) = 1$ (so that F_2 is continuous). (In particular, the support of F_1 is the same as that of F_2.) Our main result (which is proved in Section 6) is the following.

Theorem: For each pair (k_1, k_2) the game $H(k_1, k_2)$ has a unique Nash equilibrium. In regions I and II the equilibrium strategy pair is pure, equal to $(P(k), P(k))$ and $(0, 0)$ respectively; the equilibrium payoff to i $(= 1, 2)$ is $k_i P(k)$ and 0 respectively. In region III each equilibrium strategy is mixed, of type (3.2), with

$$(3.3) \quad G_i(p) = \frac{L_j(p) - L_j(a)}{L_j(p) - M_j(p)}, \qquad i = 1, 2,$$

where $a < b(k_2)$ is such that $L_1(a) = M_1(b(k_2))$. The equilibrium payoff of i $(= 1, 2)$ in this case is $L_i(a)$.

[1]Straightforward calculations show that region I contains all those points with $k_1 + k_2 \leq 1$ (and $k_1 \geq k_2$), and that for each value of k_2, there is at most one value of k_1 such that $b(k_2) = P(k)$. Also, the boundary between regions IIIa and IIIb is upward-sloping, and for every point (k_1, k_2) in region IIIb we have $k_1 \leq d(0)$.



k_2
$k_1 = k_2$
II
$k_2 \geqq d(0)$
$k_1 = d(b(k_2))$
$B(k_2) = k_1 P(k_1)$
$k_1 = d(b(k_2)) - k_2$
IIIa
$P(k) < b(k_2)$
$b(k_2) \leqq P(k_1)$
IIIb
$P(k_1) < b(k_2)$
$B(k_2) \leqq k_1 P(k_1)$
IIIc
$B(k_2) > k_1 P(k_1)$
$k_2 < d(0)$
I
$b(k_2) \leqq P(k)$
0
1
d(0)
k_1

Figure 2

By substituting in (3.3) the particular forms of L_j and M_j in the various regions we can obtain a more explicit description of the equilibrium strategies. Define functions $U\colon (P(k), \infty) \to \mathbf{R}$ and $V\colon (0, \infty) \to \mathbf{R}$ by

$$(3.3)\quad U(x) = \frac{k_1 x - B(k_2)}{x(k - d(x))} \quad \text{and} \quad V(x) = \frac{xd(x) - B(k_2)}{k_2 x} .$$

Then the equilibrium in each region is as given in Table 1.

The equilibrium strategies can be given natural interpretations. In region II each firm has more than enough capacity to meet the demand even at the breakeven price ($p = 0$ under our normalization), so the capacity limits are irrelevant, and we are back to the standard Bertrand model, where prices are driven down to unit costs. At the other extreme, in region I there is undercapacity in the industry: if $k < 1$ then the joint capacity of the firms is less than the output a monopolist without a capacity constraint would produce. In this case there is no incentive for "competition": each firm is producing at capacity, and so cannot benefit from undercutting its rival.

In the remaining regions, where there is neither under- nor over-capacity, the unique equilibrium strategies are mixed. In each case the large firm ($i = 1$) chooses the price $b(k_2)$ (the highest price in the support of the strategies) with positive probability, while the strategy of the small firm is continuous at $b(k_2)$. In regions IIIa and IIIc the continuous parts G_i of the equilibrium strategies are concave, so that prices close to a are chosen more frequently than ones near $b(k_2)$. In region IIIb, each "part" of each strategy is concave, but the slope of G_i $(i = 1, 2)$ to the left of $P(k_1)$ is less than that to the right (the strategies are illustrated in Figure 3).

Region	Equilibrium Strategies	Equilibrium Payoffs
I	Pure: $(P(k),\ P(k))$	$(k_1P(k),\ k_2P(k))$
II	Pure: $(0,\ 0)$	$(0,\ 0)$
IIIa	Mixed, of type (3.2), with $a = B(k_2)/k_1$ and $G_1(x) = k_2U(x)/k_1$, $G_2(x) = U(x)$	$(B(k_2),\ k_2B(k_2)/k_1)$
IIIb	Mixed, of type (3.2), with $a = B(k_2)/k_1$ and $G_1(x) = \begin{cases} k_2U(x)/k_1 & \text{if } a \leq x \leq P(k_1) \\ 1 - a/x & \text{if } P(k_1) \leq x < b(k_2) \end{cases}$ $G_2(x) = \begin{cases} U(x) & \text{if } a \leq x \leq P(k_1) \\ V(x) & \text{if } P(k_1) \leq x < b(k_2) \end{cases}$	$(B(k_2),\ k_2B(k_2)/k_1)$
IIIc	Mixed, of type (3.2), with $0 \leq a \leq 1$ such that $\pi(a) = B(k_2)$ and $G_1(x) = 1 - a/x$, $G_2(x) = V(x)$	$(B(k_2),\ k_2a)$

Table 1

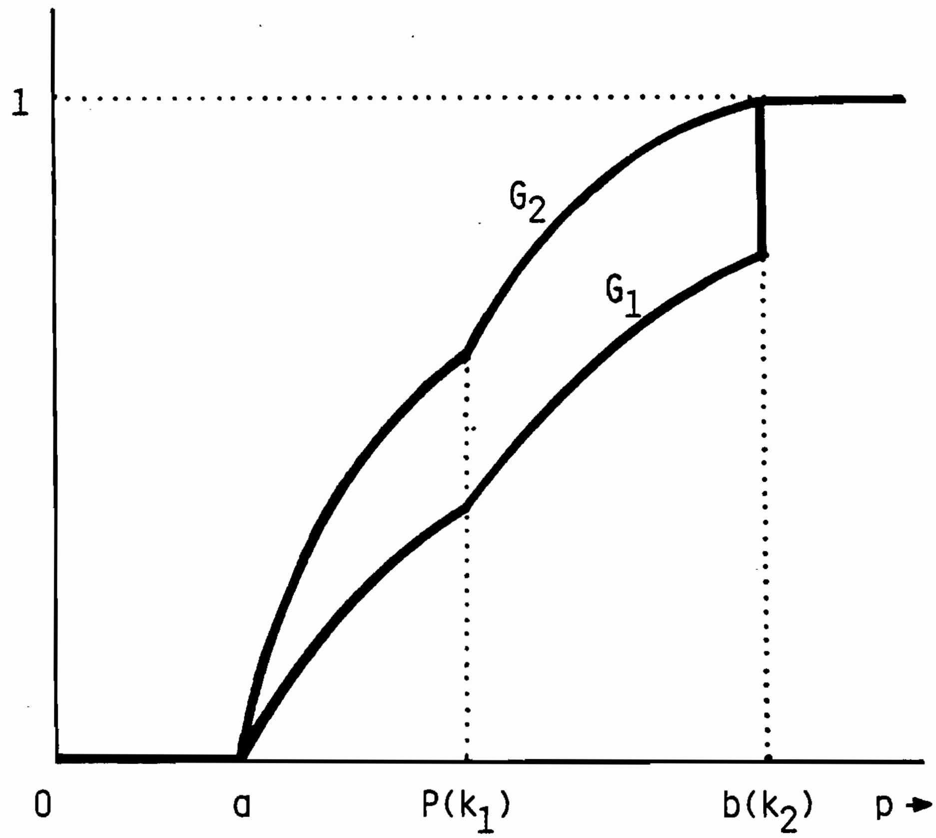


Figure 3

4. Properties of the Nash Equilibrium

In region III the unique Nash equilibrium involves mixed strategies. This means that if the duopoly lasts for more than one period, the model predicts variation in prices between periods (as the firms' random devices generate different realizations). The large firm chooses the high price $b(k_2)$ with positive probability, while the small one puts most of the probability weight on low prices, close to a (and, in region IIIb, on prices close to $P(k_1)$). Varian [1980] has interpreted a random choice of prices by firms as a policy of holding "sales". In our model, these sales emerge from the process of competition itself; they do not depend on the presence of imperfect information of any sort, as in Varian's model.

If both firms increase in size, while the ratio of their sizes is constant (or, equivalently, if demand decreases) then in region I (where the capacities are small), the equilibrium price falls, since it is just $P(k)$. In region II there is no change--price has already been driven down to unit cost. In regions IIIa and IIIb the support of the equilibrium strategies is $(B(k_2)/k_1, b(k_2))$. If $k_1/k_2 = u$, then this is $(B(k_2)/uk_2, b(k_2))$; since $b(k_2)$ and $B(k_2)$ both decrease with k_2, a proportionate increase in k_1 and k_2 decreases the minimal and maximal elements in the support. In region IIIc the fact that B is decreasing in k_2 also means that the support shifts in the same way. Thus in each case the support of the equilibrium strategies decreases with proportionate increases in k_1 and k_2. This means that the model predicts constant, high prices when demand is high, constant low ones when demand is low, and in between price variation at levels which decrease with demand.

In the regions of mixed strategy equilibrium, the degree of variation depends on the relative sizes of the firms. If the size k_2 of the small firm is fixed, while that of the large one increases, then in regions IIIa and IIIb the length of the support of the equilibrium strategies increases (the highest price is fixed at $b(k_2)$, while the lowest price is $B(k_2)/k_1$). Thus for capacities in these regions, the model predicts that the larger is the large firm relative to the small one, the more variable the price.

The unit profits of the firms are the same in all regions except IIIc. There, since $\pi(P(k_1)) = k_1P(k_1) < B(k_2)$ we have $a > P(k_1)$ or $k_1 >$ $d(a)$, so that $k_1a > \pi(a) = B(k_2)$ and hence $v_1^*(k_1, k_2)/k_1 < v_2^*(k_1, k_2)/k_1$ -- i.e. the unit profit of the large firm is less than that of the small firm. If the capacities increase, while their ratio is fixed, then the profit of each firm increases until $k = 1$, and then decreases (since B is decreasing).

Finally, when one firm is very small, the solution possesses the characteristic suggested by Shitovitz [1973]. Thus, if k_2 is very small then (k_1, k_2) is either in region I, in which case the equilibrium price is the monopoly price, or it is in region IIIc, or k_1 is close to 1. If (k_1, k_2) is in region IIIc then since $b(k_2)$ and $B(k_2)$ are close to 1 (the monopoly price in this case), the value of a is also close to 1, so that the support of both strategies lies just below the monopoly price. Further, the size of the jump in F_1 at $b(k_2)$ is $a/b(k_2)$, which is close to 1. Thus the solution is close to that of a monopoly. In the remaining case, where k_1 is close to 1, it is easy to check that the same is true.

5. <u>Relation to the Literature</u>

Levitan and Shubik [1972] describe the Nash equilibrium of $H(k_1, k_2)$ in the case where d is linear and either $k_1 = k_2$ (the diagonal in Figure 2) or $k_1 = d(0)$. They do not study uniqueness. Kreps and Scheinkman [n.d.]

also consider the case where d is linear; they establish a result concerning the equilibrium payoffs, though they do not give a complete characterization (it is not needed in their model). Beckmann [1965] studies the Nash equilibrium under a different rationing scheme. He also assumes that d is linear and $k_1 = k_2$. He considers uniqueness, but his argument is flawed[2]. The results of Dasgupta and Maskin [1982a] guarantee that an equilibrium of $H(k_1, k_2)$ exists. Under the rationing scheme adopted by Beckmann, Dasgupta and Maskin say something about the supports of the equilibrium strategies when $k_1 = k_2$, though they do not characterize the strategies (see [1982b]). Finally, Shapley reports (in an abstract ([1957]) of a paper which was never written) a characterization of the equilibrium of a price-setting duopoly game, the qualitative features of which are similar to the ones we have found; it is not clear precisely what his model or assumptions were.

6. Proofs

Here we prove the Theorem of Section 3. First we check that the strategies described there do indeed constitute an equilibrium (Lemma 6.3); then, in a series of results, we establish that there is no other equilibrium.

If $F_j \in \mathbf{S}$ (i.e. F_j is a mixed strategy of j) then $h_i(p, F_j) = \int_S h_i(p, q)dF_j(q)$, so that (using the notation of (2.3))

$$(6.1) \quad h_i(p, F_j) = M_i(p)(F_j(p) - \alpha_j(p)) + \phi_i(p)\alpha_j(p) + L_i(p)(1 - F_j(p)),$$

where $\alpha_j(p)$ is the size of the jump (if any) in F_j at p. (Recall that the appearance of i and j as indices means that j is not equal to i.) A (Nash) equilibrium of $H(k_1, k_2)$ is a pair $(F_1, F_2) \in \mathbf{S} \times \mathbf{S}$ such that

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[2]For example, the inequality in his (15) should be reversed, and in any case the claimed inequality shows only that the equilibrium payoffs of the players are the same, not that they are unique.

for $i = 1, 2$,

(6.2) $h_i(F, F_j) \leq h_i(F_i, F_j)$ for all $F \in \mathbf{S}$.

If $F \in \mathbf{S}$, let supp F denote the support of F. It follows from (6.2) that

(A) (F_1, F_2) is an equilibrium if and only if for $i = 1, 2$ we have $h_i(p, F_j) = h_i(F_i, F_j)$ for all $p \in (\text{supp } F_i)\backslash Z_i$, and $h_i(p, F_j) \leq h_i(F_i, F_j)$ for all other p, where Z_i is a set of F_i-measure zero.

For any $F \in \mathbf{S}$, let $J(F)$ be the set of points of discontinuity (jumps) of F. It follows from fact (A) that if, for $i = 1, 2$, we have supp F_i = $[a, b]$ and $h_i(p, F_j) = c_i$ for all $p \in (a, b) \cup J(F_i)$, with $h_i(p, F_j) \leq c_i$ otherwise, then (F_1, F_2) is an equilibrium, with payoffs (c_1, c_2). The equilibrium we find is of this type.

<u>Lemma 6.3</u>: <u>In region I</u>, $(P(k), P(k))$ <u>is a pure strategy equilibrium; in region II</u>, $(0, 0)$ <u>is a pure strategy equilibrium. In region III there is a mixed strategy equilibrium</u> (F_1, F_2) <u>where</u> F_i <u>is of type (3.2)</u>, $a <$ $b(k_2)$ <u>is such that</u> $L_1(a) = M_1(b(k_2))$, <u>and</u> $G_i(p) = (L_j(p) - L_j(a))/(L_j(p) - M_j(p))$, <u>for</u> $i = 1, 2$.

<u>Proof</u>: It is easy to check the cases of pure strategy equilibrium. It is also easy to verify that G_i is nondecreasing, with $G_i(a) = 0$ and $G_i(b(k_2)) \leq 1$, so that $F_i \in \mathbf{S}$, $i = 1, 2$. Finally, $h_i(p, F_j)$ is constant on $(a, b) \cup J(F_i)$, and less elsewhere; hence (F_1, F_2) is an equilibrium.

This establishes the existence part of the Theorem. The uniqueness part is much more difficult to prove, since $\mathbf{S}$ is such a rich set. For an arbitrary game it is <u>not</u> true that if (F_1, F_2) is an equilibrium, then

$h_i(p, F_j)$ is equal to $h_i(F_i, F_j)$ for all $p \in$ supp F_i. However, if $p \in$ supp F_i then either $p \in J(F_i)$ (so that $h_i(p, F_j) = h_i(F_i, F_j)$ by fact (A)), or there is a sequence $\{p_n\}$ with $p_n \in$ supp F_i and $p_n < p$ for all n, $p_n \uparrow p$, and $h_i(p_n, F_j) = h_i(F_i, F_j)$ for all n, or there is a decreasing sequence with similar properties[3]. By taking limits, we have the following.

(B) If (F_1, F_2) is an equilibrium and $p \in$ supp F_i then

(a) if $p \in J(F_i)$ then

$$h_i(F_i, F_j) = h_i(p, F_j)$$

$$= M_i(p)(F_j(p) - \alpha_j(p)) + \phi_i(p)\alpha_j(p) + L_i(p)(1 - F_j(p));$$

(b) if there exists $\{p_n\}$ with $p_n \in$ supp F_i and $p_n <$ p for all n, and $p_n \uparrow p$, then

$$h_i(F_i, F_j) = M_i(p)(F_j(p) - \alpha_j(p)) + L_i(p)(\alpha_j(p) + 1 - F_j(p))$$

$$= h_i(p, F_j) + (L_i(p) - \phi_i(p))\alpha_j(p);$$

(c) if there exists $\{p_n\}$ with $p_n \in$ supp F_i and $p_n >$ p for all n, and $p_n \downarrow p$, then

$$h_i(F_i, F_j) = M_i(p)F_j(p) + L_i(p)(1 - F_j(p))$$

$$= h_i(p, F_j) + (M_i(p) - \phi_i(p))\alpha_j(p).$$

Moreover, at least one of (a), (b), and (c) holds.

In particular:

(d) $h_i(F_i, F_j) = h_i(p, F_j)$ if $p \in (\text{supp } F_i)\backslash J(F_j)$;

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[3]See p. 211 of Pitchik [1982].

(e) if $p \in \text{supp } F_i$ then $h_i(F_i, F_j)$ is a linear combination of $M_i(p)$, $\phi_i(p)$, and $L_i(p)$.

Using the first part of fact (A) for sequences which increase and decrease to p, and using the same limiting arguments, we have

(C) if (F_1, F_2) is an equilibrium and $p \in J(F_i)$ then

$$(\phi_i(p) - L_i(p))\alpha_j(p) \geq 0 \text{ and } (\phi_i(p) - M_i(p))\alpha_j(p) \geq 0.$$

(We can draw analagous conclusions if, rather than $p \in J(F_i)$, we have sequences as in (b) or (c) of fact (B).)

We now turn to the specific features of the games $H(k_1, k_2)$. The following properties of the payoffs are easy to establish for $i = 1, 2$. (Examples of the functions L_i, ϕ_i, and M_i are shown in Figure 1.)

(6.4) $L_i(p) \leq \phi_i(p) \leq M_i(p) \leq 0$ if $p \leq 0$, and $\phi_i(p) < 0$ if $p < 0$.

(6.5) $0 \leq M_i(p) \leq \phi_i(p) \leq L_i(p)$ if $p \geq 0$, and $0 < \phi_i(p)$ if $0 < p < d(0)$.

From (6.1) this gives, for $i = 1, 2$ and any $F \in S$,

$$(6.6) \quad \left.\begin{matrix} L_i(p) \\ M_i(p) \end{matrix}\right\} \leq h_i(p, F) \leq \begin{cases} M_i(p) & \text{if } p \leq 0 \\ L_i(p) & \text{if } p \geq 0. \end{cases}$$

Also, using (e) of fact (B) we have

(6.7) if (F_1, F_2) is an equilibrium and $p \in \text{supp } F_i$ with $p \geq 0$ then $M_i(p) \leq h_i(F_i, F_j) \leq L_i(p)$.

<u>From now on, F_i always denotes an equilibrium strategy of</u> i. In narrowing down the possible equilibrium strategies, we first establish bounds on their supports. For $i = 1, 2$, let $a_i = \min \text{supp } F_i$ and let $b_i =$

max supp F_i (i.e. a_i and b_i are the smallest and largest points in the support of F_i; since S is bounded below, a_i exists, but we have yet to show that b_i exists).

<u>Lemma 6.8</u>: <u>For</u> $i = 1, 2$ <u>we have</u> $a_i > 0$.

<u>Proof</u>: Since $M_i(0) = \phi_i(0) = L_i(0) = 0$, we have $h_i(0, F_j) = 0$ (by (6.1)). Hence by (6.2), $h_i(F_i, F_j) > 0$ for $i = 1, 2$. Let $a_i \leq a_j$. If $a_i \in J(F_j)$ (which is only of course possible if $a_i = a_j$) then we need, using (a) of fact (B), $0 < h_j(F_j, F_i) = h_j(a_i, F_i) = \phi_j(a_i)\alpha_i(a_i) + L_j(a_i)(1 - \alpha_i(a_i))$ and hence, using (6.4), $a_i > 0$. If $a_i \notin J(F_j)$ then $F_j(a_i) = 0$, and either (a) or (c) of fact (B) holds, so that we need $0 < h_i(F_i, F_j) = L_i(a_i)$, and hence, again using (6.4), $a_i > 0$.

<u>Lemma 6.9</u>: <u>For</u> $i = 1, 2$ <u>we have</u> $a_i \geq P(k)$.

<u>Proof</u>: If $p \leq P(k)$ then $M_i(p) = \phi_i(p) = L_i(p) = k_i p$, so that (by (6.1)) we have $h_i(p, F_j) = k_i p$, which is increasing in p. Thus by fact (A) we have $a_i \geq P(k)$.

<u>Lemma 6.10</u>: <u>If there exists</u> $p \in \text{supp } F_i$ <u>with</u> $p > 0$ <u>then</u> b_j <u>exists</u>, $0 < b_j < P(0)$, <u>and</u> $h_j(F_j, F_i) > 0$.

<u>Proof</u>: First note that since $M_i(p) = \phi_i(p) = L_i(p) = 0$ if $p = 0$ or $p \geq P(0)$, it follows from (e) of fact (B) that $h_i(F_i, F_j) = 0$ whenever $0 \in \text{supp } F_i$ or $p \in \text{supp } F_i$ with $p \geq P(0)$. Now, if $p \in \text{supp } F_i$ with $p > 0$ then there exists $0 < s < p$ such that $F_i(s) < 1$ and $L_j(s) > 0$ (see (6.5)), so that $h_j(s, F_i) > 0$ (see (6.1)). Hence $h_j(F_j, F_i) > 0$ (by (6.2)), so that b_j exists and $0 < b_j < P(0)$.

<u>Corollary 6.11</u>: b_i <u>exists for</u> $i = 1, 2$.

<u>Proof</u>: If $b_1 = b_2 = 0$, we are done, so suppose there exists $p > 0$ with $p \in \text{supp } F_i$, for some i. Then by the lemma, b_j exists, and $P(0) > b_j > 0$; applying the lemma now with the indices reversed, we conclude that b_i exists (and $P(0) > b_i > 0$).

We now need the following, which limits the points at which both F_1 and F_2 can have jumps.

<u>Lemma 6.12</u>: <u>If</u> $L_i(p) > \phi_i(p)$ <u>then</u> $p \notin J(F_1) \cap J(F_2)$.

<u>Proof</u>: Follows immediately from fact (C).

This allows us to establish the following useful result, which pins down the equilibrium payoff of one of the firms.

<u>Lemma 6.13</u>: <u>If</u> $b_i > b_j$ <u>then</u> b_i <u>maximizes</u> M_i <u>and</u> $h_i(F_i, F_j) = M_i(b_i)$. <u>If</u> $b_1 = b_2$ <u>then there exists</u> i <u>such that either</u> $b_i \in J(F_i)$ <u>or</u> $b_i \notin J(F_j)$, <u>and in both cases</u> b_i <u>maximizes</u> M_i <u>and</u> $h_i(F_i, F_j) = M_i(b_i)$.

<u>Proof</u>: If $b_i > b_j$ or $b_i = b_j$ and $b_i \notin J(F_j)$ then by (a) or (b) of fact (B) we have $h_i(F_i, F_j) = M_i(b_i)$. If $b_i = b_j$ and $b_i \in J(F_1) \cap J(F_2)$ then by Lemma 6.12 we have $L_i(b_i) = \phi_i(b_i)$, $i = 1, 2$. But then since $b_i > 0$ by Lemma 6.8, we have $M_i(b_i) = L_i(b_i) = \phi_i(b_i)$, $i = 1, 2$, and so $h_i(F_i, F_j) = h_i(b_i, F_j) = M_i(b_i)$.

Now, in each case, if $p > b_i$ then $h_i(p, F_j) = M_i(p)$, while for all $p > 0$ we have $h_i(p, F_j) \geq M_i(p)$ (by (6.6)), and M_i is nondecreasing if $p < 0$. So $h_i(F_i, F_j) \geq M_i(p)$ for all p, by (6.2). Hence b_i maximizes M_i.

We can now immediately dispense with regions I and II.

<u>Lemma 6.14: In region I the unique equilibrium strategy pair is pure, equal to</u> $(P(k), P(k))$.

<u>Proof</u>: In region I, $P(k)$ maximizes M_i for $i = 1, 2$. Since $b_i \geq P(k)$ by Lemma 6.9, the result follows immediately from Lemmas 6.13 and 6.3.

<u>Lemma 6.15: In region II the unique equilibrium strategy pair is pure, equal to</u> $(0, 0)$.

<u>Proof</u>: In region II we have $M_i(p) = 0$ for all $p \geq 0$, $i = 1, 2$. By Lemma 6.8 we have $b_i \geq a_i \geq 0$, $i = 1, 2$, so $h_i(F_i, F_j) = 0$ for $i = 1, 2$ by Lemma 6.13. Hence by Lemma 6.10 we have $b_i = b_j = 0$.

We now turn to region III, where $b(k_2) > P(k)$ and $P(k_2) > 0$, so that $b(k_2)$ maximizes M_1 and $M_1(b(k_2)) > 0$. <u>In all the subsequent lemmas, we assume that</u> (k_1, k_2) <u>is in region III, so that</u> $b(k_2) > P(k)$ <u>and</u> $P(k_2) > 0$. First we restrict the support of F_i and the equilibrium payoffs as follows.

<u>Lemma 6.16: For</u> $i = 1, 2$, <u>we have</u> $a_i > \max(0, P(k))$ <u>and</u> $h_i(F_i, F_j) > L_i(\max(0, P(k)))$.

<u>Proof</u>: Let $\max(0, P(k)) = x$. Then $h_1(b(k_2), F_2) \geq M_1(b(k_2)) > \max(0, M_1(P(k))) = M_1(x) = L_1(x)$ (the first inequality from (6.6)), so $h_1(F_1, F_2) > L_1(x)$ by fact (A). Since $M_1(x) = \phi_1(x) = L_1(x)$ this means $x \notin \text{supp } F_1$ by (e) of fact (B). But L_2 is increasing on (x, a_1) and $h_2(p, F_1) = L_2(p)$ there. Hence, again by fact (A), $x \notin \text{supp } F_2$ and $h_2(F_2, F_1) > L_2(x)$.

<u>Corollary 6.17: For</u> $i = 1, 2$, <u>we have</u> $\text{supp } F_i \subset$

$(\max(0, P(k)), b(k_2)]$.

Proof: Since both equilibrium payoffs are positive by the lemma, we must have $M_i(b_i) > 0$ for the i such that b_i maximizes M_i (see Lemma 6.13). But if b_i is positive, the maximizer of M_2 is less than that of M_1, namely $b(k_2)$. The result follows from Lemma 6.13.

Corollary 6.18: $J(F_1) \cap J(F_2) = \emptyset$.

Proof: Follows from Corollary 6.17, Lemma 6.12, and the fact that $L_i(p) > \phi_i(p)$ if $\max(0, P(k)) < p \leq b(k_2)$.

We can now restrict the smallest points in the supports of F_1 and F_2, and further characterize the equilibrium payoffs.

Lemma 6.19: $a_1 = a_2 = a$, say, and $h_i(F_i, F_j) = L_i(a)$ for $i = 1, 2$.

Proof: For $i = 1, 2$, L_i is increasing on $(\max(0, P(k)), b(k_2))$, so by Corollary 6.17 it is increasing on supp F_i. Suppose $a_i \leq a_j$. Then by (6.7) we have $h_i(F_i, F_j) \leq L_i(a_i)$. But we need $h_i(F_i, F_j) \geq h_i(p, F_j) = L_i(p)$ if $p < a_j$. Hence $a_i \geq a_j$, so that $a_i = a_j = a$, say, and $h_i(F_i, F_j) = L_i(a)$.

We shall now refine Lemma 6.13. From now on, F_i^* denotes the equilibrium strategy of i defined in Lemma 6.3 for region III, and a^* denotes the value of a specified there

Lemma 6.20: b_1 maximizes M_1 (so that $b_1 = b(k_2)$) and $h_1(F_1, F_2) = M_1(b_1)$.

Proof: Let m maximize M_2 and let $w < m$ be such that $L_2(w) = M_2(m)$. We first show that $w < a^*$. If $a^* < m$ then $m \in$ supp F_i^*, $i = 1, 2$, so that $h_2(F_2^*, F_1^*) = L_2(a^*) = h_2(m, F_1^*) = M_2(m)F_1^*(m) + L_2(m)(1 - F_1^*(m)) > M_2(m)$. Hence $w < a^*$, because L_2 is increasing to a^*. If, on the other hand, $m < a^*$ then certainly $w < a^*$. Hence $L_1(w) < L_1(a^*) = M_1(b(k_2))$. Now, $h_1(b(k_2), F) > M_1(b(k_2))$ for all $F \in S$ by (6.6), so any equilibrium payoff of firm 1 exceeds $L_1(w)$ (by (6.2)). Hence by Lemma 6.19, $a > w$. But then any equilibrium payoff of firm 2 exceeds $L_2(w) = M_2(m)$. Thus by Lemma 6.13 we must have $b_1 = b(k_2)$ and $h_1(F_1, F_2) = M_1(b_1)$.

We now show that, except possibly for a single point, the supports of the equilibrium strategies coincide.

Lemma 6.21: supp F_1 = supp $F_2 \cup \{b(k_2)\}$.

Proof: Suppose $a < p < b(k_2)$ and $p \notin$ supp F_2. Since L_1, ϕ_1, and M_1 are all increasing at p, so is $h_1(p, F_2)$. Hence $p \notin$ supp F_1.

Now suppose $a < p < b(k_2)$ and $p \notin$ supp F_1. Let $x = \max([a, p) \cap$ supp $F_1)$ and $y = \min((p, b_1] \cap$ supp $F_1)$; for $i = 1, 2$, let $Q_i(s) = M_i(s)F_j(x) + L_i(s)(1 - F_j(x))$. If $s \in (x, y)$ then $h_2(s, F_1) = Q_2(s)$. Since $p < b_1$, we have $F_1(x) < 1$. Also, since L_2 is increasing and M_2 is strictly concave on $(\max(0, P(k)), P(k_1))$, while L_2 is increasing, and M_2 is constant on $(P(k_1), b(k_2))$, Q_2 is increasing and/or strictly concave on $(\max(0, P(k)), P(k_1))$, and increasing on $(P(k_1), b(k_2))$ (since $F_1(x) < 1$). So if $p \in$ supp F_2 then $(x, y) \cap$ supp $F_2 = \{p\}$, so that $p \in J(F_2)$ and p maximizes Q_2 on (x, y). If $x \notin J(F_1)$ then $h_2(x, F_1) = Q_2(x) < Q_2(p)$, so $x \notin J(F_2)$ by (a) of fact (B); if $x \in J(F_1)$ then $x \notin J(F_2)$ by Corollary 6.18. Hence, either way $x \notin J(F_2)$, so that, by (d) of fact (B) we must have $h_1(F_1, F_2) = h_1(x, F_2) = Q_1(x)$. But $h_1(s, F_2) = Q_1(s)$ for $x < s < p$; since M_1 and L_1 are increasing this means that there exists $r \in (x, p)$ such that $h_1(r, F_2) > h_1(x, F_2) = h_1(F_1, F_2)$, contradicting the fact that (F_1, F_2) is an equilibrium. Hence $p \notin \text{supp } F_2$.

Next, it is easy to show that under certain conditions, the value of $F_i(p)$ for any equilibrium strategy F_i has to be $F_i^*(p)$.

<u>Lemma 6.22</u>: <u>If</u> $p \in (\text{supp } F_i)\backslash J(F_j)$ <u>then</u> $F_j(p) = F_j^*(p)$, $i = 1, 2$.

<u>Proof</u>: Since $p \in \text{supp } F_i\backslash J(F_j)$ we need $h_i(p, F_j) = h_i(F_i, F_j)$ by (d) of fact (B). But by Lemma 6.19 we have $h_i(F_i, F_j) = L_i(a)$, and by definition we have $h_i(p, F_j) = M_i(p)F_j(p) + L_i(p)(1 - F_j(p))$.

The next result, which states that the support of F_2 is an interval, allows us, with the aid of the following two straightforward results, to complete the proof of uniqueness.

<u>Lemma 6.23</u>: $\text{supp } F_2 = [a, b_2]$.

<u>Proof</u>: Suppose $a < p < b_2$ with $p \notin \text{supp } F_2$, so that $p \notin \text{supp } F_1$ (by Lemma 6.22). Let $x = \max([a, p) \cap \text{supp } F_2)$ and $y = \min((p, b_2] \cap \text{supp } F_2)$. Since L_1, ϕ_1, and M_1 are increasing on (a, b_2), $h_1(\cdot, F_2)$ is increasing on (x, y), and so $h_1(F_1, F_2) > h_1(x, F_2)$. Hence by (b) of fact (B) we have $x \notin J(F_1)$, so by Lemma 6.22 we have $F_1(x) = F_1^*(x)$. Since F_1^* is increasing on (x, y) this means (using Lemma 6.22 again) that $y \in J(F_1)$. But then $h_2(s, F_1) = M_2(s)F_1(x) + L_2(s)(1 - F_1(x))$ if $x < s < y$, so that we need $h_2(F_2, F_1) \geq M_2(y)F_1(x) + L_2(y)(1 - F_1(x))$. But by (c) of fact (B) we have $h_2(F_2, F_1) = M_2(y)F_1(y) + L_2(y)(1 - F_1(y)) = M_2(y)F_1(x) + L_2(y)(1 - F_1(x)) - \alpha_1(y)(L_2(y) - M_2(y)) < M_2(y)F_1(x) + L_2(y)(1 - F_1(x))$, a contradiction. Hence $p \in \text{supp } F_2$ for all $a < p < b_2$.

<u>Lemma 6.24</u>: <u>If</u> $p \in [a, b_2)$ <u>then</u> $p \notin J(F_i)$ <u>for</u> $i = 1, 2$.

Proof: By Lemmas 6.21 and 6.23 we have $[a, b_2] \subset \text{supp } F_i$, $i = 1, 2$. Hence by (c) of fact (B) we have $h_i(F_i, F_j) = M_i(a)F_j(a) + L_i(a)(1 - F_j(a))$. But by Lemma 6.19 we have $h_i(F_i, F_j) = L_i(a)$, so since $L_i(a) > M_i(a)$ (because $a > \max(0, P(k))$) we have $F_j(a) = 0$ —i.e. $a \notin J(F_j)$. Now let $a < p < b_2$. Then by Lemma 6.23 and (b) and (c) of fact (B) we need $(L_i(p) - \phi_i(p))\alpha_j(p) = (M_i(p) - \phi_i(p))\alpha_j(p)$, or $(L_i(p) - M_i(p))\alpha_j(p) = 0$, or $\alpha_j(p) = 0$, since $L_i(p) > M_i(p)$.

It remains to show that $b_2 = b(k_2)$.

Lemma 6.25: $b_2 = b(k_2)$, so that $\text{supp } F_1 = \text{supp } F_2 = [a^*, b(k_2)]$ and $F_i = F_i^*$, $i = 1, 2$.

Proof: First, from Lemmas 6.24, 6.23, and 6.22 we have $a = a^*$ and $F_i(p) = F_i^*(p)$ if $a^* < p < b_2$. Now, if $b(k_2) > b_2$ then $b_2 \in J(F_2)$ (otherwise, by (b) of fact (B) and Lemma 6.20, we have $h_1(F_1, F_2) = M_1(b_2) = M_1(b(k_2))$, which contradicts the fact that M_1 is increasing on $(b_2, b(k_2))$), and so by Corollary 6.18, $b_2 \notin J(F_1)$, and hence, by Lemma 6.22, $F_1(b_2) = F_1^*(b_2)$. But if $b_2 < s < b(k_2)$ then $h_2(F_2, F_1) = L_2(a) = L_2(a^*) = h_2(s, F_1^*) = M_2(s)F_1^*(s) + L_2(s)(1 - F_1^*(s)) < M_2(s)F_1^*(b_2) + L_2(s)(1 - F_1^*(b_2)) = h_2(s, F_1)$ (the first equality by Lemma 6.19, the fourth by Lemma 6.3, and the inequality because F_1^* is increasing and $M_2(s) < L_2(s)$). This contradicts the fact that (F_1, F_2) is an equilibrium. Hence $b_2 = b(k_2)$.

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