CARTELS, PROFITS, AND EXCESS CAPACITY

by

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Abstract

We study a model of a collusive duopoly in which each firm has limited capacity. The negotiated output quotas depend on the bargaining power of the firms, which derives from the damage they can do by cutting prices. If the capacities are fixed, then the unit profit of the small firm is at least as large as that of the large firm, and the relative position of the small firm is better when demand is low. If the capacities can be chosen once-and-for-all, then in equilibrium there is excess capacity so long as the cost of capacity is not too high. This is because a larger capacity permits more damaging threats, so that an extra unit of capacity may be valuable even if it is not used in production.

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1. Introduction

Our purpose is to study a model of the behavior of a cartel. There are a number of reasons why this is an interesting exercise. Whenever they have not been outlawed, cartels have existed, and frequently flourished, in a wide variety of industries. Their modes of operation have been diverse, and their longevity far from uniform, but they cannot be dismissed as transitory phenomena[1]. From a theoretical point of view, the fact that the firms in an industry can collectively benefit from colluding rather than competing means that there is an incentive to form a cartel. There is always the problem that an entrant may upset a collusive arrangement, but in any industry with a barrier to entry it is in the interests of all firms to reach a binding collusive agreement. Even if such an agreement is outlawed, there may be a collusive arrangement which is self-enforcing (as, for example, in the models of Radner [1980], Stigler [1964], and Green and Porter [1981]). Finally, in

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[1]For a discussion of the extent of cartelization, see Chapter 6 of Scherer [1980]; for detailed accounts of several major cartels, see Stocking and Watkins [1946].

order to evaluate the desirability of making cartels illegal, it is necessary to understand their behavior, and how it depends on the nature of the demand for output and the available technology.

We address two questions. What collusive agreement will a group of firms of possibly different sizes reach? What implications does the nature of the agreement have for the choice of size by each member? Our model is very simple. There are two firms, with possibly different capacities. Up to its capacity, each firm has the same, constant average cost of production. Capacity can be changed only with difficulty, so that, in particular, entry is not an issue (the market is "noncontestable").

First, we fix the capacities. The firms negotiate an agreement, which involves an output quota for each firm, and a price at which output is sold. The outcome of the negotiation is determined by the damage each firm can inflict on the other by undercutting the monopoly price. (We have examined elsewhere (Osborne and Pitchik [1983a]) the consequence, in this part of the model, of assuming that the firms threaten to expand output, rather than cut prices. The assumption of threatened price-wars seems more appropriate for most industries.)

We find that the profit per unit of capacity of the small firm is always at least equal to that of the large one, and if the joint capacity of the firms exceeds the monopoly output, then the inequality is strict. The reason is that each firm, regardless of size, can equally well disrupt the collusive outcome--in this respect each firm has the same power. The large firm can inflict more damage, but the net effect favors the small firm (per unit of capacity). (Stigler's [1964] model yields the same conclusion if information is imperfect; our result derives solely from the threat-potential of each firm.) We also find that the ratio of the unit profit of the small firm relative to that of the large firm is higher, the lower is demand relative to capacity. Thus the model predicts that, if capacities are fixed while demand varies cyclically, this ratio will vary procyclically.

There is some evidence that small firms do fare better than large ones in cartels, especially when demand is low. We have not examined the data systematically, but the agreement reached in the Addyston Pipe Cartel (see Stevens [1913], pp. 205-209 and Bittlingmayer [1982]), and the outcome of OPEC's negotiations (see, for example, Gately [1979], p. 311) seem to accord well with our results.

If we want to compare the outcome in our model with one which is "competitive", there are two alternatives. In the standard "perfectly competitive" outcome, both firms sell at the same price, so that their unit profits are the same. The outcome of price competition (between firms with limited capacities) predicted by the Bertrand model involves the same unit profits for both firms whenever the capacity of the large firm is less than the monopoly output (and even when it is larger, so long as the small firm is not too small; for the details, see Osborne and Pitchik [1983b]). Thus, whenever the monopoly output exceeds the capacity of the large firm, and is less than the industry capacity, the unit profit is the same for each firm in either of the "competitive" outcomes, while the small firm fares better in the cartel. If the cartel outcome can be achieved by an implicit agreement, this provides a criterion to distinguish between competitive and collusive industries.

Even though it may be difficult to change capacities, they must ultimately be chosen by the firms. We incorporate this by assuming that capacities are chosen once-and-for-all, before negotiation over price and output quotas[2] (so that the structure is similar to that of the model of Kreps and Scheinkman [n.d.]). The capacities are thus neither objects of negotiation, nor strategic variables for the firms during negotiation. The idea is that, given the inflexibility of capacity, it is to the advantage of a firm to choose its capacity before entering negotiations. Of course, if the cartel lasts for a long time, there is scope for subsequent adjustments of capacity. Even so, in the absence of perfect enforcement, agreement on a capacity-reduction may be much less likely than agreement on a price-hike: if a firm cheats on the former, its opponent is in a very weak position, while any change in price is easily reversible.

We find that if the cost of capacity is relatively low then the sum of the capacities chosen by the firms exceeds the sum of the negotiated output quotas--i.e. the choices of the firms result in excess capacity in the industry. The reason for this is straightforward. The more capacity a firm has, the more potent the threats it can make, and hence the larger its payoff. If capacity is not too costly, it pays a firm to build more capacity, even if it is not all used in production (it is "used" to threaten the other firm and maintain the firm's negotiated payoff). If the firms choose their capacities in the expectation of competing (as in Kreps and Scheinkman [n.d.]), rather than colluding, then there is really no use for excess capacity, and the outcome always involves full utilization. The same is obviously true if there is a single firm in the industry.

We also find that a proportionate increase in demand at each price causes a proportionate rise in the capacities chosen. On the other hand, if the size and elasticity of demand at the monopoly price is fixed, while at

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[2]Rather than assuming that the industry is starting from scratch, we could suppose that it is currently competitive, and that adjustments in capacity are being made, in anticipation of subsequent collusion. Or, the industry may currently be a monopoly, which is changing its capacity in response to an entrant with which it will collude.

every other price demand becomes more elastic, then excess capacity increases. This makes sense: if demand is more elastic, a price-cut is more potent (it is less damaging to its perpetrator, and more damaging to the other firm), so that the marginal benefit of an extra unit of capacity is greater. Hence, given the unit cost of capacity, the equilibrium sizes chosen are larger.

In a slightly different context the idea that a firm will build excess capacity as a threat has been modeled before. Spence [1977] argues that the large capacity of an existing firm deters entry into an industry. However, the equilibrium of his model is not "perfect": if a firm actually enters, then it is not in the interest of the existing firm to carry out its threat. Dixit [1980] imposes credibility on the threat by assuming that after entry a Nash equilibrium (in quantities) is attained. Naturally, he finds that no excess capacity is chosen (the forces in his model are the same at those in Kreps and Scheinkman [n.d.]). In our model of a collusive industry, the threats are chosen according to the rules of Nash's [1953] "variable-threat" bargaining model. These rules incorporate a notion of credibility--the effect of a threat on a player's negotiated payoff depends on its cost to that player, as well as the damage it inflicts on his opponent. In the context of our model, in which payoff is transferable, Nash's solution has particularly strong support: it is the equilibrium of an explicit model of negotiation (see Binmore [1981]), and is the unique outcome of a system of attractive axioms (see Selten [1960] and [1964], particularly result E7 on p. 583 of the latter). Even though the concept of perfect equilibrium in a repeated game may be more "basic", it suffers from the disadvantage that the range of outcomes it predicts is typically very wide, unless more or less ad hoc restrictions are placed on the threats or agreements which are allowed (as, for example, in Green and Porter [1981]); Nash's model yields a single outcome.

2. The Economic Structure

There are two firms, $i = 1, 2$. The capacity of i is denoted k_i; when analyzing the outcome for fixed capacities, we assume that $k_1 \geq k_2 > 0$. Each firm can produce the same good at the same, constant unit cost $c \geq 0$ up to its capacity. Let p be the excess of price over unit cost; we frequently refer to p simply as a "price". The set of possible prices is $S = [-c, \infty)$. For each price p, let $d(p)$ be the aggregate demand for the output of the firms (given the prices of all other goods). We assume that

(2.1) there exists $p_o > 0$ such that $d(p) = 0$ if $p \geq p_o$ and $d(p) > 0$ if $p < p_o$, and d is smooth on $(-c, p_o)$, with $d'(p) < 0$ there.

For each $p \in S$, let $\pi(p) = pd(p)$. Given (2.1), π attains a maximum on S. To save on notation we choose the units in which price is measured so that the maximizer is 1, and the units for quantity so that the maximum is also 1. We assume that

(2.2) π is strictly concave on $[0, 1]$, and decreasing on $[1, p_o]$.

We now define the profits of the firms if they noncooperatively choose prices p_1 and p_2. Suppose they set different prices, say $p_i < p_j$. Depending on the capacity of firm i, there may be some demand left over for firm j. Precisely how much remains depends on the preferences of the consumers and the way the available quantity is rationed, not just on the aggregate demand function d. We assume that there is a large number of identical consumers with preferences which do not have any "income effect"[3].

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[3]Precisely, for each quantity of the good which the firms produce, the marginal rate of substitution between that good and any other good is independent of the quantities of the other goods. We also need to assume [cont. on next page]

It is natural to assume that the rationing scheme is chosen by firm i. If firm i is concerned solely with its own payoff, this assumption does not generate a determinate outcome, since i's payoff is independent of the scheme chosen. However, in the bargaining model we use, it is to the advantage of a firm to choose a threat which reduces the payoff of firm j as much as possible. A rationing scheme which does this, independently of the action of firm j, is the following: each consumer is allowed to buy the same fraction of k_i (rather, for example, than some consumers being allowed to buy as much as they want, while others are allowed to buy nothing). Given that this is a dominant strategy, a firm will always adopt it when issuing a threat, and we can focus on the choice of prices. Under these assumptions, the demand facing firm j when $p_i < p_j$ is $d(p_j) - k_i$. If $p_i = p_j = p$, we assume that demand is allocated in proportion to capacities[4] (if these are large enough to serve that demand). Thus the profit of firm i at any price pair (p_i, p_j) is

$$(2.3)\quad h_i(p_i, p_j) = \begin{cases} L_i(p_i) \equiv p_i \min(k_i, d(p_i)) & \text{if } p_i < p_j \\ \phi_i(p) \equiv p \min(k_i, k_i d(p)/k) & \text{if } p_i = p_j = p \\ M_i(p_i) \equiv p_i \min(k_i, \max(0, d(p_i) - k_j)) & \text{if } p_i > p_j, \end{cases}$$

where $k = k_1 + k_2$. Examples of the functions L_i, ϕ_i, and M_i are shown in Figure 1.

By colluding, the firms can obtain the monopoly profit. It is easy to check that under our assumptions this is achieved by both firms selling at the

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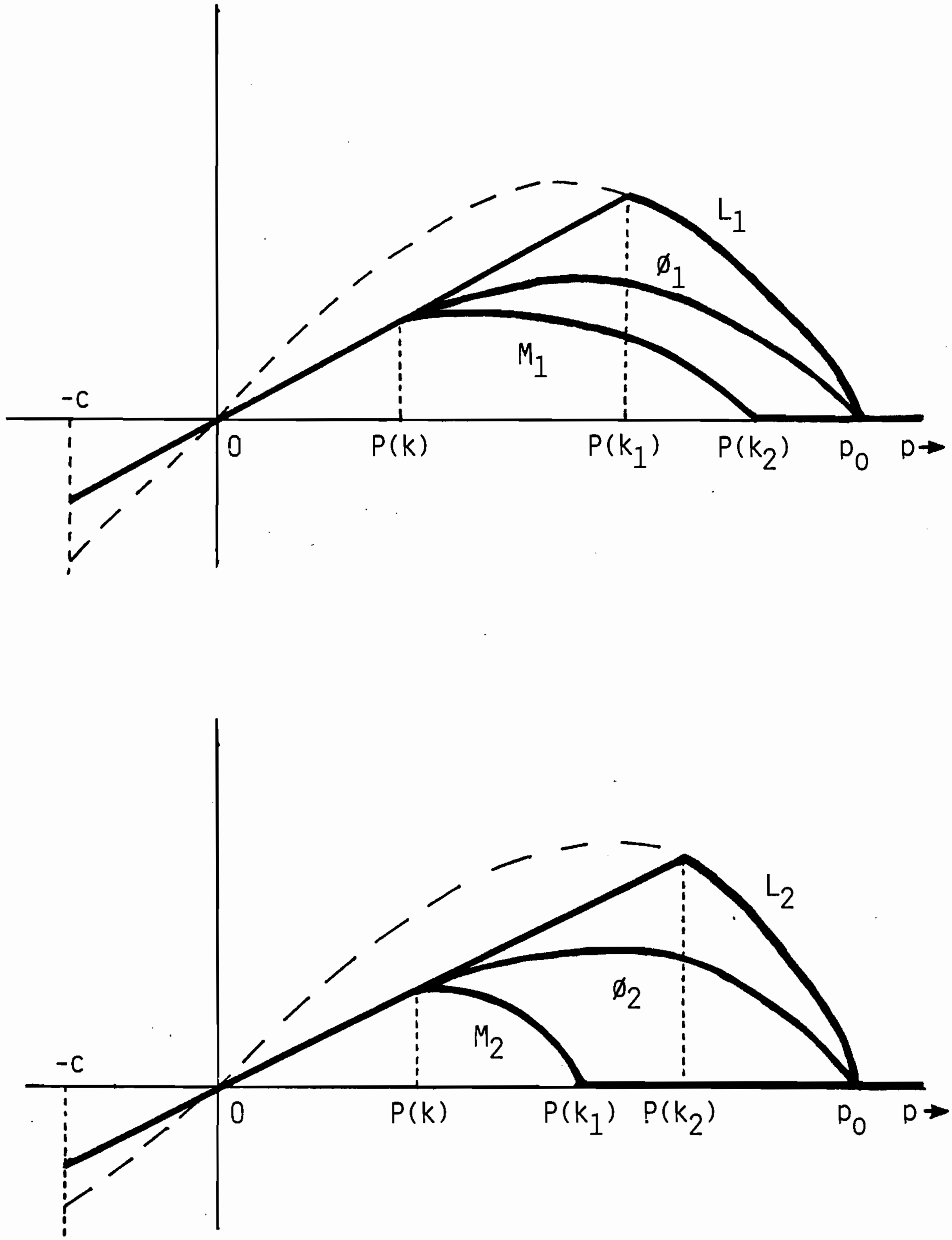
that, given the income of a consumer, and the prices of all other goods, his demand for all other goods is positive for every price of the good produced by the firms.

[4]If one adopts a rule in this case which is more favorable to the small firm (for example, the demand is split equally), then the negotiated payoff of the small firm is higher, and that of the large firm is lower.



<u>Figure 1</u>

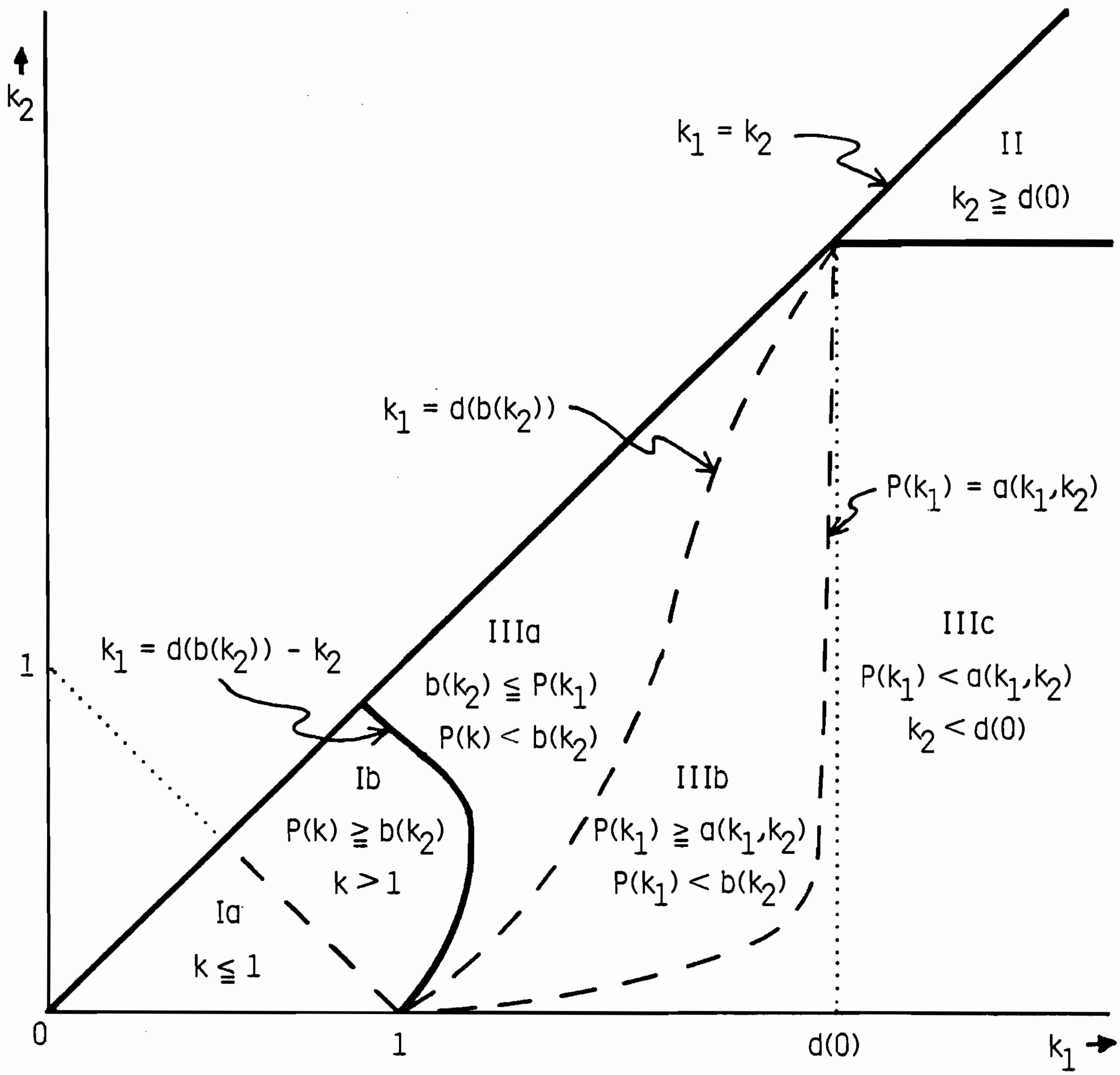

k2
k1 = k2
II
k2 ≧ d(0)
k1 = d(b(k2))
P(k1) = a(k1,k2)
IIIa
b(k2) ≦ P(k1)
P(k) < b(k2)
IIIc
P(k1) < a(k1,k2)
k2 < d(0)
1
k1 = d(b(k2)) - k2
Ib
P(k) ≧ b(k2)
k > 1
IIIb
P(k1) ≧ a(k1,k2)
P(k1) < b(k2)
Ia
k ≦ 1
0
1
d(0)
k1

Figure 2

same price[5]. Let $P:[0, \infty) \to S$ be the inverse demand function defined by $P(q) = d^{-1}(q)$ if $0 < q \leq d(-c)$, $P(0) = p_o$ (see (2.1)), and $P(q) = -c$ otherwise. Then, if the capacity of the industry is k, the monopoly profit is $Z(k) \equiv \max_q\{qP(q): q \leq k\}$. Given our normalization, and our assumptions on demand, we have

$$(2.4) \quad Z(k) = \begin{cases} kP(k) & \text{if } 0 \leq k \leq 1 \\ 1 & \text{if } 1 \leq k. \end{cases}$$

3. Negotiation between the Firms

The division of the monopoly profit between the firms is determined by negotiation. We use the bargaining model due to Nash [1953]. It has the following structure. First, the firms simultaneously announce threats; then a compromise is agreed upon. For each pair of threats, the firms know precisely which compromise will be reached. Each firm chooses its threat so as to maximize its compromise payoff, given the threat of the other firm. This model is supported both by the axiomatization of Selten [1960] and by the explicit concession model of Nash [1953] (see also Binmore [1981]). We assume that payoff is transferable--i.e. any division of the monopoly profit between the firms is possible. However, we show that the negotiated profits can be achieved without any transfers. Thus, we can think of the firms agreeing on output quotas.

Our assumption of the transferability of payoffs means that the relationship between the negotiated payoffs and the payoffs which would be received if the threats were carried out is quite simple. Suppose (p_1, p_2) is a pair of threats. Then the negotiated payoff of firm i (= 1, 2) is

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[5]Note that if consumers' preferences differ, this may not be the case.

$v_i(p_i, p_j) = h_i(p_i, p_j) + [Z(k) - h_i(p_i, p_j) - h_j(p_j, p_i)]/2$. Thus the excess of the monopoly profit over the sum of the threat payoffs is split equally between the firms; a large firm is powerful only because it can issue more damaging threats. Using (2.3) we have

$$(3.1)\quad v_i(p_i, p_j) = \begin{cases} [Z(k) + L_i(p_i) - M_j(p_j)]/2 & \text{if } p_i < p_j \\ [Z(k) + \phi_i(p) - \phi_j(p)]/2 & \text{if } p_i = p_j = p \\ [Z(k) + M_i(p_i) - L_j(p_j)]/2 & \text{if } p_i > p_j. \end{cases}$$

Let $V(k_1, k_2)$ be the (constant-sum) game in which the (pure) strategy set of each firm is S, and the payoff to firm i if it chooses p_i and firm j chooses p_j is $v_i(p_i, p_j)$. A pair of equilibrium strategies in (the mixed extension of) $V(k_1, k_2)$ is a pair of <u>optimal threats</u>, and the equilibrium payoffs are the <u>negotiated payoffs</u> in our model. We denote the negotiated payoff of firm i by $v_i^*(k_1, k_2)$. (Note that since $V(k_1, k_2)$ is constant-sum, the optimal threats guarantee the negotiated payoffs.)

4. <u>The Outcome of Negotiation for Fixed Capacities</u>

Here we characterize the optimal threats and negotiated payoffs for each fixed pair of capacities, and study their properties. For each $x \in [0, d(0)]$, suppose that

$$(4.1)\quad b(x) \text{ maximizes } p(d(p) - x) \text{ over } p \in S,$$

and let $B(x) = b(x)(d(b(x)) - x)$. If firm i sets the price p_i and $b(k_i) > P(k)$, then the best price for firm j to charge out of all those in excess of p_i is $b(k_i)$, independently of p_i (i.e. $b(k_i)$ maximizes M_j in this case). The price $b(k_2)$ figures prominently in the description of the optimal threats and negotiated payoffs; in particular, no firm threatens to charge a price higher than $b(k_2)$. It is easy to show that b is decreasing, with $b(0) = 1$, $b(d(0)) = 0$, and $b(x) < P(x)$ for each $x \in [0, d(0)]$.

The qualitative features of the optimal threats depend on the relation between $b(k_2)$, $P(k)$, and 0. The relevant "regions" are shown in Figure 2. (For each (k_1, k_2), the value of $a(k_1, k_2)$ is defined in Proposition 4.6. The boundaries between regions IIIa and IIIb, and between IIIb and IIIc, are upward-sloping, and if (k_1, k_2) is in region IIIb then $k_1 < d(0)$.) If $b(k_2) < P(k)$ (region I) or $b(k_2) < 0$ (region II), then the optimal threats are pure strategies; otherwise (in region III) they are mixed. The appearance of mixed strategies for some values of (k_1, k_2) is neither unreasonable nor unnatural. The negotiated outcome is still a pure outcome--only the optimal threats involve randomization. As a threat, there is no reason why a firm should not choose its price according to some probability distribution. It is also possible to interpret a mixed strategy in this context as a strategy of threatening to hold "sales" at various "reduced" prices over a period of time (e.g. Varian [1980] has given this interpretation). If the game is repeated, each firm can base its actions at any point on all the previous actions of its opponent, so that the strategy set of each player contains much more than simply repetitions of one-period actions. However, the new strategies do not affect the maximum joint payoff available to the firms. Moreover, independent repetitions of the one-period equilibrium strategies constitute an equilibrium of the repeated game. Thus, if one firm threatens to carry out such a repetition, a best response of its rival is to do the same. Since the game $V(k_1, k_2)$ is constant-sum, the fact that there may exist other optimal threats in the repeated game (involving strategies which depend on the previous actions of the other player) does not concern us: the payoffs to them all must be the same. Thus, we can consistently interpret our model in a setting where actions are repeated over time.

The regions of pure strategy equilibrium are covered in the following result.

Proposition 4.2: If (k_1, k_2) is in region I, then $(P(k), P(k))$ is a pair of optimal threats; if it is in region II, then $(0, 0)$ is a pair of optimal threats. The negotiated payoff of firm i $(= 1, 2)$ is as follows.

$$(4.3) \quad v_i^*(k_1, k_2) = \begin{cases} k_i P(k) & \text{in region Ia} \\ [1 + (k_i - k_j)P(k)]/2 & \text{in region Ib} \\ 1/2 & \text{in region II.} \end{cases}$$

Proof: From (2.3), note that L_i is increasing up to $P(k_i)$, which exceeds $P(k)$, and M_i is decreasing after $b(k_j)$, $i = 1, 2$. Since $b(k_1) \leq b(k_2)$, $b(k_2) \leq P(k)$ in region I, and $L_i(P(k)) = \phi_i(P(k)) = M_i(P(k))$ $(i = 1, 2)$, this means that $(P(k), P(k))$ is an equilibrium in this case; the payoffs follow from substitution. A similar argument can be made for region II. This completes the proof.

For each pair (k_1, k_2) in region III the unique optimal threats are mixed. Fix (k_1, k_2), and let $\mathbf{S}$ be the set of mixed strategies (i.e. cumulative probability distribution functions on S) in $V(k_1, k_2)$. Let $(F_1, F_2) \in \mathbf{S} \times \mathbf{S}$ be a pair of optimal threats. We show that the support of F_i, which we denote supp F_i, is equal to $[a, b(k_2)]$ for each $i = 1, 2$, for some $\max(0, P(k)) < a < b(k_2)$. The strategy F_i has the form

$$(4.4) \quad F_i(x) = \begin{cases} 0 & \text{if } x \leq a \\ G_i(x) & \text{if } a \leq x < b(k_2) \\ 1 & \text{if } b(k_2) \leq x, \end{cases}$$

where G_i: $[a, b(k_2)] \to [0, 1]$ is continuous, $G_i(a) = 0$ $(i = 1, 2)$, $G_1(b(k_2)) < 1$, and $G_2(b(k_2)) = 1$ (so that F_2 is in fact continuous).

First we show that if a pair of strategies (F_1, F_2) of type (4.4) is such that $v_i(p, F_j)$ $(i = 1, 2)$ is constant for $a < p < b(k_2)$, then $v_i(p, F_j) \leq v_i(F_i, F_j)$ for all $p \in S$, and $v_1(b(k_2), F_2) = v_1(F_1, F_2)$, so that (F_1, F_2) is a pair of optimal threats.

Lemma 4.5: *Fix (k_1, k_2) in region III. If (F_1, F_2) is of type (4.4) and $v_i(p, F_j) = v_i(F_i, F_j)$ for all $a < p < b(k_2)$ then (F_1, F_2) is a pair of optimal threats.*

Proof: To save on notation, we use b to denote $b(k_2)$ in this proof. First suppose that $p \leq a$. Then for $i = 1, 2$ we have $v_i(p, F_j) = \int_a^b [Z(k) + L_i(p) - M_j(r)]dF_j(r)/2$ (using (3.1)). But L_i is increasing up to $\max(1, P(k_i)) \geq b$, so $v_i(p, F_j)$ is increasing up to $p = a$. A similar argument, using the fact that M_i is decreasing after $b(k_j) \leq b$, establishes that $v_i(p, F_j)$ is decreasing after $p = b$. Now, since F_2 is continuous, $v_1(p, F_2)$ is continuous in p, so that the above implies that $v_1(p, F_2) = v_1(F_1, F_2)$ if $a \leq p \leq b$ and $v_1(p, F_2) \leq v_1(F_1, F_2)$ for all $p \in S$. The strategy F_1 may not be continuous at b, so we need to check that $v_2(p, F_1) \leq v_2(F_1, F_2)$ if $p \geq b$. But this follows from the fact that, since $b \geq 0$ in region III, we have $M_2(b) \leq \phi_2(b) \leq L_2(b)$, and hence $v_2(F_1, p) \leq v_2(F_1, b) \leq v_2(F_1, F_2)$ if $b < p$, completing the proof.

We can now characterize the optimal threats and negotiated payoffs in region III. For each $p \in S$, let $K_i(p) = L_i(p) - M_i(p)$ for $i = 1, 2$. It follows from (2.3) that $K_i(p) \geq 0$ if $p \geq 0$, and $K_1(p) = K_2(p)$ for all $p \in S$. In view of the latter, we write $K(p)$ rather than $K_i(p)$.

<u>Proposition 4.6</u>: <u>For each</u> (k_1, k_2) <u>in region III there is a pair of optimal threats of type (4.4), with</u>

$$(4.7) \quad G_i(p) = \frac{1}{2(K(p))^{1/2}} \int_a^p \frac{L_j'(x)}{(K(x))^{1/2}}\, dx, \quad \text{for } i = 1, 2,$$

<u>where</u> a <u>is such that</u> $G_2(b(k_2)) = 1$ (<u>so that both</u> G_i <u>and</u> a <u>may depend on both</u> k_1 <u>and</u> k_2). <u>The negotiated payoff of firm</u> 1 <u>is</u>

$$(4.8) \quad v_1^*(k_1, k_2) = \begin{cases} [1 + (k_1 - k_2)B(k_2)/k_1]/2 & \text{in region IIIa} \\ [1 + P(k_1)(Y(k_1, k_2))^2/k_1]/2 & \text{in region IIIb} \\ [1 + \pi(a)]/2 & \text{in region IIIc} \end{cases}$$

<u>where</u> $Y(k_1, k_2) = k_1 - k_2G_2(P(k_1))$, <u>and the negotiated payoff of firm</u> 2 <u>is</u> $v_2^*(k_1, k_2) = 1 - v_1^*(k_1, k_2)$.

<u>Proof</u>: Fix (k_1, k_2) in region III. Note that $Z(k) = 1$, and $b > \max(0, P(k))$. (We again use b to denote $b(k_2)$.) Suppose $F_j \in \mathbf{S}$ is of type (4.4) and is differentiable on $(P(k), b)\backslash P(k_1)$; assign $G_j'(P(k_1))$ an arbitrary value. If $P(k) < p < b$ we have

$$(4.9) \quad 2v_i(p, F_j) = 1 + \int_a^p [M_i(p)-L_j(r)]G_j'(r)dr + \int_p^b [L_i(p)-M_j(r)]G_j'(r)dr$$
$$+ [L_i(p)-M_j(b)](1 - G_j(b)).$$

If we set the derivative of this equal to zero we obtain

$$(4.10) \quad 2K(p)G_j'(p) + K'(p)G_j(p) - L_i'(p) = 0.$$

It is easy to check that for each $t > P(k)$, the function G_j^t defined by

$$G_j^t(p) = \frac{1}{2(K(p))^{1/2}} \int_t^p \frac{L_i'(x)}{(K(x))^{1/2}}\, dx$$

solves (4.10) on (t, b). (If $P(k_1) \in (t, b)$, we mean that G_j^t solves (4.10) on $(t, P(k_1)) \cup (P(k_1), b)$ and is continuous at $P(k_1)$.) Now, $G_2^b(b) = 0$ and as t decreases, $G_2^t(b)$ increases. In fact, since $K(x) < kx$ for all $x \in S$, and $K(x) = 0$ if $x \leq \max(0, P(k))$, we have $G_2^t(b) \uparrow \infty$ as $t \downarrow \max(0, P(k))$. Hence there exists $a > \max(0, P(k))$ such that $G_2^a(b) = 1$. Let $G_2 = G_2^a$, and let $G_1 = G_1^a$, so that G_i $(i = 1, 2)$ is the function defined in (4.7).

We now argue that G_1 and G_2 are cumulative probability distribution functions. First, since $M_1'(p) \geq 0$ if $p \leq b$ we have, from (4.10), $G_2'(p) > 0$ whenever $G_2(p) < 1$. Now $G_2(b) = 1$, $G_2'(b) = 0$ (from (4.10)) and $G_2''(b) < 0$ from the derivative of (4.10) and the fact that $M_1''(b) < 0$, since π is strictly concave), so $G_2'(p) > 0$ for all $a \leq p < b$. As for G_1, if $a < p < P(k_1)$ then $L_i'(p) = k_i$, so $G_1(p) = k_2G_2(p)/k_1$, and hence $0 \leq G_1(p) \leq 1$ and $G_1'(p) > 0$. If $P(k_1) < p < b$ then $G_1(p) = G_1(P(k_1)) + G_1^t(p)$ for $t = P(k_1)$, and since $K(p) = k_2p$, $G_1^t(p)$ can be solved explicitly; it lies between 0 and 1 and is increasing. Hence G_1 is a cumulative probability distribution function. By Lemma 4.5, (F_1, F_2) is thus a pair of optimal threats.

To find the negotiated payoffs, let $i = 1$ and $p \to b$ in (4.9). Since $v_1(p, F_2)$ is constant on (a, b), equal to $v_1^*(k_1, k_2)$, and F_2 is continuous on $[a, b]$, we obtain

$$(4.11)\quad 2v_1^*(k_1, k_2) = 1 + \int_a^b [M_1(b)-L_2(r)]G_2'(r)dr = 1 + b(d(b)-k_2) - k_2\int_a^b rG_2'(r)dr.$$

Now, let $y = \min(b(k_2), \max(a, P(k_1)))$. Then

$$(4.12)\quad \int_a^b rG_2'(r)dr = \int_a^y rG_2'(r)dr + \int_y^b rG_2'(r)dr.$$

By using the explicit form of $G_2(p)$ for $a < p < P(k_1)$ we can integrate the first term on the right-hand side by parts; by using (4.10) and the explicit forms of $K(p)$ and $L_1(p)$ for $p > P(k_1)$, we can calculate the second integral. Substituting the results into (4.11), we obtain (4.8). (For the details, see Appendix 1.) Since $Z(k) = 1$, we have $v_1^*(k_1, k_2) + v_2^*(k_1, k_2) = 1$, completing the proof.

We have not shown that the optimal threats defined in Propositions 4.2 and 4.6 are unique, since the main line of our argument requires only the uniqueness of the negotiated payoffs. However, a straightforward adaptation of the argument of Karlin [1959] (solutions to Problems 16-18, pp. 293-295) shows that this is indeed the case. The threats can be given clear intuitive interpretations. We have chosen the units in which quantity is measured so that the output of a monopolist without any capacity constraint (the "unconstrained monopoly output") is 1. In region Ia the total capacity of the firms is less than this output, and the price at which output is sold in the agreement--the monopoly price--is $P(k)$. Thus, under these conditions of undercapacity in the industry, the optimal threats are not really "threats" at all: a firm does not improve its bargaining position by threatening to undercut the price charged by its rival. In region Ib the total capacity of the firms exceeds 1, so that the monopoly price is 1, which is larger than $P(k)$. Thus in this case, the optimal threats do involve some undercutting. In region III, the optimal threats of the firms differ. Substituting the explicit forms of K and L_j into (4.7) it is easy to show that in region IIIa, the mean price threatened by firm 1 exceeds that threatened by firm 2, and that the probability that the threatened price of firm 1 exceeds that of firm 2 is greater than one-half. Also, if the capacities increase relative to demand, while the relative size of the firms is constant, the mean threatenend prices decrease. In regions IIIb and IIIc the optimal threats are more complex, and such precise conclusions cannot be drawn. However, we can show that in region IIIc the optimal threats are independent of the value of k_1. In region II, where each firm has more than enough capacity to serve the total demand when price is equal to unit cost ($p = 0$ under our normalization), the best threat of each firm is to set precisely this breakeven price. Thus in this case each firm's threat is one of dramatic price-cutting.

In a negotiated outcome, no threat is ever carried out. However, Nash [1953] argued that his solution is the limit of equilibrium outcomes in games with some incomplete information, as this incompleteness goes to zero. In each of the games in the sequence, there is some probability that the threats will be carried out[6]. Thus, in a situation with a small imperfection of information, the model predicts threats close to those described in Propositions 4.2 and 4.6.

5. Properties of the Negotiated Payoffs for Fixed Capacities

We first show that the negotiated profit of each firm is a nondecreasing function of its capacity. This must, of course, be true in any reasonable model; it almost[7] follows directly from the fact that the Nash variable-threat solution satisfies Selten's condition of "payoff monotonicity" (see p. 263 of Selten [1960], or result E5 on p. 583 of Selten [1964]). We also show that the negotiated payoffs satisfy a concavity property which we shall use later.

Proposition 5.1: $v_i^*(k_1, k_2)$ is continuous in k_1 and k_2, for $i =$

[6]Binmore [1981] examines this argument formally under a number of assumptions.

[7]"Almost", because in region Ia a change in capacities affects the maximal joint payoff (monopoly profit), so that the assumption in Selten's condition (that the latter is constant) is violated.

1, 2. <u>$v_1^*(k_1, k_2)$ is increasing in k_1 on regions I, IIIa, and IIIb, and is constant in k_1 otherwise. $v_2^*(k_1, k_2)$ is increasing in k_2 in regions I and III, and constant in k_2 otherwise. $v_1^*(k_1, k_2)$ is differentiable in k_1, and concave in k_1 on the union of regions Ib, II, and III. $v_2^*(k_1, k_2)$ is differentiable in k_2 on region I, and concave and differentiable in k_2 on the union of regions II and III.</u>

<u>Proof</u>: Continuity follows from substituting the equations for the boundaries between regions into (4.3) and (4.8). Also, it is immediate that $v_1^*(k_1, k_2)$ is differentiable in k_1 on the interior of each region. The remainder of the result follows from a tedious explicit differentiation of v_i^* in each region; the details are given in Appendix 2. This completes the proof.

Next we show that the profit per unit of capacity of the small firm is at least as high as that of the large one, and if the industry capacity exceeds the unconstrained monopoly output, the inequality is strict.

<u>Proposition 5.2</u>: <u>If</u> $k > 1$ <u>and</u> $k_1 > k_2$ <u>then</u> $v_1^*(k_1, k_2)/v_2^*(k_1, k_2) < k_1/k_2$; <u>otherwise</u> $v_1^*(k_1, k_2)/v_2^*(k_1, k_2) = k_1/k_2$.

<u>Proof</u>: If $k \leq 1$ or $k_1 = k_2$ the result follows immediately from (4.3) and (4.8). If $k > 1$ then $Z(k)$ (the monopoly profit) is 1 (see (2.4)) so that $v_1^*(k_1, k_2) + v_2^*(k_1, k_2) = 1$. Hence it is enough to show that $v_1^*(k_1, k_2) < k_1/k$. First consider region Ib. Since $k > 1$ we have $kP(k) < 1$, so that, using (4.3),

$$v_1^*(k_1, k_2) = [1 + (k_1 - k_2)P(k)]/2 < [1 + (k_1 - k_2)/k]/2 = k_1/k.$$

In region II the result is immediate from (4.3).

Now consider region III. Define $f(k_1, k_2) = [1 + (k_1 - k_2)\times B(k_2)/k_1]/2$. We shall first argue that $v_1^*(k_1, k_2) \leq f(k_1, k_2)$ for all (k_1, k_2) in this region. In region IIIa, we have equality (see (4.8)). If (k_1, k_2) is in region IIIb, we need to show that $g(k_1, k_2) \equiv (k_1 - k_2)B(k_2) - P(k_1)(Y(k_1, k_2))^2 \geq 0$ (using (4.8)). Differentiating, using (A.3) of Appendix 2, we find that $\partial g(k_1, k_2)/\partial k_1 = B(k_2) - P(k_1)Y(k_1, k_2)$. On the boundary between regions IIIa and IIIb we have $P(k_1) = b(k_2)$, so that $B(k_2) = (k_1 - k_2)P(k_1)$ and $Y(k_1, k_2) = k_1 - k_2$, and hence $\partial g(k_1, k_2)/\partial k_1 = 0$. Differentiating again, using the fact that G_2 satisfies (4.10) for $i = 1$, we find that $\partial^2 g(k_1, k_2)/\partial k_1^2 = -k_2 P(k_1)P'(k)G_2'(P(k_1)) > 0$, where G_2' here denotes the right-hand derivative. Thus $\partial g(k_1, k_2)/\partial k_1 > 0$ in region IIIb, and hence $g(k_1, k_2) \geq 0$ (since $g(k_1, k_2) = 0$ on the boundary). Finally, note that in region IIIc, a is independent of k_1 (see the discussion of region IIIc in Appendix 2). Hence $v_1^*(k_1, k_2)$ is constant in k_1 in region IIIc and so, since v_1^* is continuous in k_1, we have $v_1^*(k_1, k_2) < f(k_1, k_2)$.

It remains to show that $f(k_1, k_2) < k_1/k$ if (k_1, k_2) is in region III. Rearranging this inequality, we need to show that $k_1(1 - B(k_2)) - k_2B(k_2) > 0$. Since $B(k_2) < 1$ and $k_1 > d(b(k_2)) - k_2 = B(k_2)/b(k_2)$ in region III, we have $k_1(1 - B(k_2)) - k_2B(k_2) > B(k_2)[1 - B(k_2) - k_2b(k_2)]/b(k_2) = B(k_2)[1 - b(k_2)d(b(k_2))]/b(k_2) > 0$ (the last inequality since $pd(p) < 1$ if $p < 1$, and $b(k_2) < 1$). This completes the proof.

The following result allows us to interpret our model as one of negotiation over output quotas. We show that there is a feasible output level $x_i \leq k_i$ for firm i (= 1, 2) such that the negotiated payoff of i is precisely the profit earned when the output x_i is sold at the monopoly price. For each value of k, let $m(k)$ be the monopoly price (so that

$m(k) = P(k)$ if $k \leq 1$, and $m(k) = 1$ otherwise).

Proposition 5.3: $v_i^*(k_1, k_2) \leq k_i m(k)$ for $i = 1, 2$, for all (k_1, k_2).

Proof: In region Ia we have $v_i^*(k_1, k_2) = k_i P(k)$ (see (4.3)) and $m(k) = P(k)$, so the result follows. In the remaining regions we have $k > 1$, so that $m(k) = 1$. Given this, and Proposition 5.2, it is enough to show that $v_2^*(k_1, k_2) \leq k_2$. Now, since $v_1^*(k_1, k_2)$ is increasing in k_1 (by Proposition 5.1) and $v_1^*(k_1, k_2) + v_2^*(k_1, k_2) = 1$ in regions Ib, II, and III, $v_2^*(k_1, k_2)$ is decreasing in k_1 there. Hence if $k_2 \leq 1/2$ then $v_2^*(k_1, k_2) \leq v_2^*(1-k_2, k_2) = k_2$, while if $k_2 > 1/2$ then $v_2^*(k_1, k_2) \leq v_2^*(k_2, k_2) = 1/2 < k_2$, completing the proof.

Finally, we show that the larger is the joint capacity of the firms relative to demand, the higher the unit profit of the small firm relative to that of the large one.

Proposition 5.4: Fix $z \in [1/2, 1]$. Then $v_1^*(zk, (1-z)k)/v_2^*(zk, (1-z)k)$ is constant in k (equal to $z/(1-z)$) if $k < 1$, decreasing in k if $1 < k < d(0)/(1-z)$, and constant (equal to 1) if $d(0)/(1-z) < k$.

Proof: For regions Ia and II the result is immediate from (4.3). In the remaining cases, let $J(k_1, k_2)$ be such that $v_1^*(k_1, k_2) = (1 + J(k_1, k_2))/2$, so that $v_2^*(k_1, k_2) = (1 - J(k_1, k_2))/2$. Then it is enough to show that $J(zk, (1-z)k)$ is decreasing in k. In region Ib we have $J(zk, (1-z)k) = (2z - 1)kP(k)$, which is decreasing in k by (2.2). In region IIIa, we have $J(zk, (1-z)k) = (2z-1)B((1-z)k)/z$, which is decreasing in k, since B is decreasing. In region IIIb, $J(zk, (1-z)k) = P(zk)\times$

$(Y(zk, (1-z)k))^2/zk$; a calculation using (A.4) and (A.5) shows that the derivative of this with respect to k is $(1-z)(P(zk))^{1/2} \times Y(zk, (1-z)k)[P(zk)^{1/2}G_2(P(zk)) - 2(b((1-z)k)^{1/2}]/zk$, which is negative since $G_2(P(zk)) < 1$ and $P(zk) < b((1-z)k)$ in region IIIb. Finally, in region IIIc we have $J(zk, (1-z)k) = \pi(a((1-z)k))$ (a is independent of k_1 in this region, as remarked in the proof of Proposition 5.2). Since a is decreasing (see the discussion of region IIIc in Appendix 2), $J(zk, (1-z)k)$ is decreasing. This completes the proof.

6. The Capacity Choices of the Firms

We now allow each firm to choose its capacity before entering negotiations. The choices are made simultaneously, and for each pair (k_1, k_2) the (negotiated) payoffs are those described above. We assume that the cost of a unit of capacity is the same for both firms, equal to u. We are interested in the Nash equilibrium of the game $W(u)$ in which the strategic variable of each firm is its capacity, and the payoff $w_i(k_i, k_j)$ of firm i when (k_i, k_j) is chosen is the negotiated payoff corresponding to (k_i, k_j) minus the cost of k_i. Since we have only defined the negotiated payoff $v_i^*(k_1, k_2)$ when $k_1 \geq k_2$, the appropriate definition of w_i is

$$w_i(k_i, k_j) \equiv \begin{cases} v_2^*(k_j, k_i) - uk_i & \text{if } k_i < k_j \\ v_1^*(k_i, k_j) - uk_i & \text{if } k_i \geq k_j. \end{cases}$$

In this section we strengthen the concavity assumption (2.2) as follows.

(6.1) π is strictly concave on $[0, p_0]$.

We show that for each value of $u > 0$ the game $W(u)$ has a unique pure-strategy equilibrium; at this equilibrium there is excess capacity if and only if $u > 1/2$ (i.e. half of the monopoly price); and the more costly is capacity, the smaller is the excess capacity. First we show that (6.1) allows us to strengthen Proposition 5.1 as follows.

Proposition 6.2: Under (6.1), $v_1^*(k_1, k_2)$ is concave in k_1 throughout, and $v_2^*(k_1, k_2)$ is concave in k_2 on the union of regions Ia, II, and III.

Proof: Given Proposition 5.1 we need to show only that v_i^* is concave in k_i on region Ia, $i = 1, 2$. Using (4.3) we have $\partial^2 v_i^*(k_1, k_2)/\partial k_i^2 = 2P'(k) + k_i P''(k) = \pi''(P(k))(P'(k))^2 + (\pi'(P(k)) - k_j)P''(k)$. But if $P''(k) < 0$ this is negative by the first expression, and if $P''(k) > 0$ it is negative by the second one (using (6.1) and the fact that if $k > 1$ then $\pi'(P(k)) < 0$). This completes the proof.

We can now characterize the equilibrium capacity choices of the firms. It is immediate from Proposition 5.1 that if $u = 0$ then (k_1, k_2) is an equilibrium pure-strategy pair of $W(u)$ if and only if $k_i \geq d(0)$ for $i = 1, 2$. The following deals with the case $u > 0$.

Proposition 6.3: For each $u > 0$ the game $W(u)$ has a unique pure-strategy equilibrium. Let the equilibrium strategy pair be $(k_1^*(u), k_2^*(u))$. For each $0 < x < 2d(0)$ let

$$(6.4) \quad f(x) = \begin{cases} 2P(x) + xP'(x) & \text{if } 0 < x \leq 1 \\ P(x) & \text{if } 1 < x < d(b(x/2)) \\ 2B(x/2)/x & \text{if } d(b(x/2)) < x < 2d(0). \end{cases}$$

Then for each $0 < u < P(0)$ there is a unique point $k^*(u) < 2d(0)$ such that $f(k^*(u)) = 2u$. We have $k_i^*(u) = k^*(u)/2$ for $i = 1, 2$, k^* is decreasing in u, and $k^*(u) > 1$ if and only if $u < 1/2$. If $u \geq P(0)$ then $k_i^*(u) = 0$ for $i = 1, 2$.

Proof: First we claim that at any pure-strategy equilibrium we have $k_1^*(u) = k_2^*(u)$. To show this, note that if $k_i^*(u) > 0$ for $i = 1, 2$, and w_i is differentiable at $(k_i^*(u), k_j^*(u))$, then we need $\partial w_i(k_i^*(u), k_j^*(u))/\partial k_i = 0$ for $i = 1, 2$. By Proposition 5.1, w_i is differentiable unless $k_i < k_j$ and (k_j, k_i) is on the boundary between regions Ib and IIIa. But it is easy to show that in the exceptional case the right-hand derivative of w_i exceeds the left-hand derivative. Hence an equilibrium cannot lie in this exceptional case, and so we can restrict attention to the remaining values of (k_i, k_j). By calculating the derivatives in each region (see Appendix 2) it is easy to show that $\partial w_i(k_i^*(u), k_j^*(u))/\partial k_i = 0$ for $i = 1, 2$ only if $k_i^*(u) = k_j^*(u) < d(0)$. It is also easy to check that if $k_i^*(u) > 0$ then we must have $k_j^*(u) > 0$, so that the only possible equilibrium in which $k_i^*(u) = 0$ for some i is $(k_i^*(u), k_j^*(u)) = (0, 0)$.

We now argue that for each $u > 0$ there is a unique pure-strategy equilibrium. To show this, we first prove that there is a unique number $k < 2d(0)$ such that the first-order condition $\partial w_i(k/2, k/2)/\partial k_i = 0$ is satisfied for $i = 1, 2$. If we calculate the derivatives (see Appendix 2) we find that $\partial w_i(k/2, k/2)/\partial k_i = 0$ for $i = 1, 2$ if and only if $f(k) = 2u$. To show that for each $0 < u < P(0)$ this equation has a unique solution, note that f is continuous, $f(0) = 2P(0)$, and $f(2d(0)) = 0$. Thus it is enough to show that f is decreasing. On $(1, 2d(0))$ this follows from the fact that P and B are decreasing. On $(0, 1)$ we have $f(k) = P(k) + \Pi'(k)$, where $\Pi(k) \equiv kP(k)$. But it is easy to show that (6.1) implies that Π is concave. Hence $f'(k) < 0$. Thus k^* is decreasing in u for $0 < u < P(0)$. If $u \geq P(0)$ there is no $k > 0$ such that $f(k) = 2u$. In this case it is easy to check that $(k_1, k_2) = (0, 0)$ is an equilibrium.

For notational simplicity, we now fix u and write k instead of $k^*(u)$. It remains to show that for $i = 1, 2$ we have $w_i(x, k/2) \leq w_i(k/2, k/2)$ for all $x \geq 0$ (i.e second-order conditions must be satisfied). If $x \geq k/2$ this follows from the concavity of v_1^* in k_1 (see Proposition 6.2). If $x < k/2$ then since v_2^* is concave in k_2 except possibly on region Ib, we need to consider only what happens if $(k/2, x)$ is in this region.

First suppose that $(k/2, k/2)$ is in region IIIa, so that $2u = 2B(k/2)/k$ (see (6.4)) and $k > d(b(k/2))$. If $x < k/2$ and $(k/2, x)$ is in region Ib, then $x < d(b(x)) - k/2$, and, using (4.3) and the fact that $w_i(k/2, k/2) = 1/2 - uk/2$, we find that the condition $w_i(x, k/2) \leq w_i(k/2, k/2)$ for $x < k/2$ is equivalent to the condition $2u \leq P(x+k/2)$ for $x < k/2$. But $2u = 2B(k/2)/k = 2b(k/2)(d(b(k/2)) - k/2)/k < b(k/2)$ (since $d(b(k/2)) < k$), and $P(x+k/2) > P(d(b(x)) = b(x) > b(k/2)$ (the first inequality since $x < d(b(x)) - k/2$, the second since b is decreasing). Hence $(k/2, k/2)$ is an equilibrium in this case.

Now suppose that $(k/2, k/2)$ is in region Ib, so that $2u = P(k)$ (see (6.4)). If $x < k/2$ and $(k/2, x)$ is in region Ib then from (4.3) we have $w_i(x, k/2) = [1 - (k/2-x)P(x+k/2)]/2 - ux$. Using the fact that $P(x+k/2) > P(k)$ we deduce that $w_i(x, k/2) < [1 - kP(k)/2]/2 = [1 - uk]/2 = w_i(k/2, k/2)$, so that $(k/2, k/2)$ is an equilibrium in this case also.

So far we have established that (k_1, k_2) is a pure-strategy Nash equilibrium of $W(u)$ if and only if $k_1 = k_2 = k^*(u)/2$. Since $f(1) = 1$, it follows that $k^*(u) > 1$ if and only if $u < 1/2$, completing the proof.

Finally, we analyze the effect of changes in demand on the equilibrium capacity choices of the firms. First suppose that before normalizing the price and quantity units (see Section 2), the demand $d(p)$ increases by the same proportion at each price--say $d^2(x) = td^1(x)$ for each price $x \geq 0$, for some $t > 1$. Then at each price $x \geq 0$, the price elasticity of demand is the same for d^1 and d^2. Thus the price which maximizes the monopoly profit $pd^i(p)$ is the same in both cases $(i = 1, 2)$, so that price is normalized in the same way for both demands. Since $d^2 = td^1$, the quantity units are proportionately larger for d^2. Thus, after normalization, d^1 and d^2 coincide, so that the normalized capacity choices are the same in both cases, and hence in the original units, the capacity choices under d^2 are precisely t times as large as those under d^1. Thus we have the following.

Proposition 6.5: If, for each price, demand increases by the same proportion, then the equilibrium capacity choices of the firms increase by the same proportion.

Now fix the point on the demand function where the elasticity is one, and make the function more elastic at every other price. Then the normalization is the same in both cases, and demand increases at every price below the monopoly price. Hence $P(x)$ increases for each $x > 1$ and $B(x)$ increases for each $x > 0$. Thus if $u < 1/2$ then, from Proposition 6.3, the chosen capacities increase (since $k^*(u)$ then exceeds 1). In fact, it is clear that it is enough to assume that the demand increases at every price below 1, while at 1 both demand and the slope of the demand function are fixed; what happens at prices in excess of 1 is irrelevant. Thus we have the following.

Proposition 6.6: Suppose the point on the demand function where the price elasticity of demand is unity is fixed, while demand increases at all lower prices. Then if there is originally excess capacity, its size increases.

Appendix 1: The Equilibrium Payoffs in Region III: The Details of the Proof of (4.8)

First, we have

$$\int_a^y rG_2'(r)dr = [rG_2(r)]_a^y - \int_a^y G_2(r)dr,$$

and from (4.7) and (2.3) we have $G_2(r) = k_1 f'(r)f(r)/2$, where $f(r) = \int_a^r [x(k - d(x))]^{-1/2}dx$, so that

$$\int_a^y G_2(r)dr = k_1[(f(r))^2/4]_a^y = k_1(f(y))^2/4$$
$$= k_1(2G_2(y)/k_1 f'(y))^2/4 = (G_2(y))^2 y(k - d(y))/k_1.$$

Hence

$$\text{(A.1)} \quad \int_a^y rG_2'(r)dr = yG_2(y) - y(G_2(y))^2(k - d(y))/k_1$$
$$= yG_2(y)(k_1 - G_2(y)(k - d(y)))/k_1.$$

Next, if we integrate (4.10) for $i = 1$ we obtain

$$2\int_y^b K(p)G_2'(p)dp + \int_y^b K'(p)G_2(p)dp - \int_y^b L_1'(p)dp = 0.$$

But if $y < p < b$ then $K(p) = k_2 p$, since $y > P(k_1)$ if $y < b$, so we have $2k_2\int_y^b pG_2'(p)dp + k_2\int_y^b G_2(p)dp - (L_1(b) - L_1(y)) = 0$, or, performing the second integration by parts and replacing L_1 by π (since $b > y > P(k_1)$),

$$2k_2\int_y^b pG_2'(p)dp + k_2\left([pG_2(p)]_y^b - \int_y^b pG_2'(p)dp\right) - (\pi(b) - \pi(y)) = 0,$$

or

$$k_2\int_y^b pG_2'(p)dp = \pi(b) - \pi(y) - k_2(b - yG_2(y)).$$

Hence

(A.2) $$k_2\int_a^b rG_2'(r)dr = k_2yG_2(y)\left[k_1-G_2(y)(k-d(y))\right]/k_1 + \pi(b) - \pi(y) - k_2(b-yG_2(y)).$$

Now using (4.12), (A.1), (A.2), and (4.11) we find that

$$2v_1^*(k_1, k_2) = 1 + y\left[k_2(G_2(y))^2(k-d(y)) - 2k_2k_1G_2(y) + k_1d(y)\right]/k_1,$$

which gives (4.8).

Appendix 2: The Derivatives of $v_i^*(k_1, k_2)$: The Details of the Proof of Proposition 5.1

Region Ia: For $i = 1, 2$, we have $\partial v_i^*/\partial k_i = P(k) + k_i P'(k) \geq P(k) + kP'(k) = \pi'(P(k))P'(k) > 0$ (since $P(k) > 1$ in this region).

Region Ib: For $i = 1, 2$, we have $\partial v_i^*/\partial k_i = [P(k) + (k_i - k_j) \times P'(k)]/2$. Since $k_2 \leq k_1$ this means that $\partial v_2^*/\partial k_2 \geq P(k)/2 > 0$. Also, in this region we have $b(k_2) < P(k)$, so that $\pi'(P(k)) < \pi'(b(k_2)) = k_2$. Hence $\partial v_1^*/\partial k_1 = [P(k) + kP'(k) - 2k_2P'(k)]/2 = P'(k)[\pi'(P(k)) - 2k_2]/2 > 0$.

Further, $\partial^2 v_1^*/\partial k_1^2 = [2P'(k) + (k_1 - k_2)P''(k)]/2$. But from the concavity of π on $[0, 1]$ we have $P''(k) \leq 2(P'(k))^2/P(k)$, so $\partial^2 v_1^*/\partial k_1^2 \leq P'(k)[P(k) + (k_1 - k_2)P'(k)]/P(k) = P'(k)(\partial v_1^*/\partial k_1)/P(k) \leq 0$.

Region IIIa: $\partial v_1^*/\partial k_1 = k_2 B(k_2)/2k_1^2 > 0$; $\partial^2 v_1^*/\partial k_1^2 = -k_2 B(k_2)/k_1^3 \leq 0$; $\partial v_2^*/\partial k_2 = [B(k_2) + (k_1 - k_2)b(k_2)]/2k_1 > 0$; $\partial^2 v_2^*/\partial k_2^2 = [(k_1 - k_2)b'(k_2) - 2b(k_2)]/2k_1 \leq 0$ (since $b'(k_2) < 0$).

Region IIIb: First, from the definition of a we have

$$\int_a^{P(k_1)} \frac{L_1'(x)}{(K(x))^{1/2}}\,dx = 2(K(b(k_2)))^{1/2} - \int_{P(k_1)}^{b(k_2)} \frac{L_1'(x)}{(K(x))^{1/2}}\,dx.$$

But $K(x) = k_2 x$ and $L_1(x) = \pi(x)$ if $P(k_1) < x < b(k_2)$, so

$$G_2(P(k_1)) = \left(\frac{b(k_2)}{P(k_1)}\right)^{1/2} - \frac{1}{2k_2(P(k_1))^{1/2}} \int_{P(k_1)}^{b(k_2)} \frac{\pi'(x)}{x^{1/2}}\,dx,$$

so, given that $Y(k_1, k_2) = k_1 - k_2 G_2(P(k_1))$, we have

$$Y(k_1, k_2) = k_1 - (P(k_1))^{-1/2}\left\{k_2(b(k_2))^{1/2} - \int_{P(k_1)}^{b(k_2)} \frac{\pi'(x)}{2x^{1/2}}\,dx\right\}.$$

Differentiating, we find that

$$\text{(A.3)} \quad \frac{\partial Y}{\partial k_1}(k_1, k_2) = \frac{1}{2}\left(1 - \frac{P'(k_1)}{P(k_1)}Y(k_1, k_2)\right).$$

Now we find that

$$\text{(A.4)} \quad \frac{\partial v_1^*}{\partial k_1} = \frac{k_2 P(k_1) G_2(P(k_1)) Y(k_1, k_2)}{2k_1^2} > 0$$

and, using the fact that G_2 satisfies (4.10) for $i = 1$ on $(P(k_1), b(k_2))$,

$$\frac{\partial^2 v_1^*}{\partial k_1^2} = \frac{k_2 P(k_1)\left(-2G_2(P(k_1))Y(k_1, k_2) + k_1^2 P'(k_1) G_2'(P(k_1))\right)}{2k_1^3} < 0,$$

where G_2' here denotes the right-hand derivative. Also,

$$\text{(A.5)} \quad \frac{\partial v_2^*}{\partial k_2} = \frac{(P(k_1)b(k_2))^{1/2} Y(k_1, k_2)}{k_1} > 0$$

and

$$\frac{\partial^2 v_2^*}{\partial k_2^2} = \frac{b'(k_2)(P(k_1))^{1/2} Y(k_1, k_2)/2(b(k_2))^{1/2} - b(k_2)}{k_1} < 0.$$

<u>Region IIIc:</u> Here $a > P(k_1)$, so that $K(p) = k_2 p$ and $L_1(p) = \pi(p)$ if $a < p < b(k_2)$. Hence $G_2(b(k_2)) = 1$ implies that

$$2k_2(b(k_2))^{1/2} = \int_a^{b(k_2)} \frac{\pi'(x)}{x^{1/2}}\, dx$$

(see (4.7)). Thus, a depends only on k_2, so that $\partial v_1^*/\partial k_1 = 0$. Differentiating, we find that $\pi'(a(k_2))a'(k_2) = -2(a(k_2)b(k_2))^{1/2}$, so that $a'(k_2) < 0$ and $\partial(\pi'(a(k_2))a'(k_2))/\partial k_2 = -(a(k_2)b(k_2))^{-1/2}(a'(k_2)b(k_2) + a(k_2)b'(k_2)) > 0$. Hence $\partial v_2^*/\partial k_2 = -\pi'(a(k_2))a'(k_2)/2 > 0$ and $\partial^2 v_2^*/\partial k_2^2 < 0$.

<u>Region II:</u> $\partial v_1^*/\partial k_1 = \partial v_2^*/\partial k_2 = 0$.

Finally, given the above calculations it is easy to check that v_i^*

(i = 1, 2) is differentiable in k_i on all the boundaries with the exception of the one between regions Ib and IIIa for $i = 2$.

References

Binmore, K. [1981], "Nash Bargaining and Incomplete Information", Economic Theory Discussion Paper 45, Department of Applied Economics, Cambridge University, February.

Bittlingmayer, G. [1982], "Price Fixing and the Addyston Pipe Case", mimeo, February.

Dixit, A. [1980], "The Role of Investment in Entry-Deterrence", Economic Journal, 90, pp. 95-106.

Gately, D. [1979], "OPEC Pricing and Output Decisions. A Partition Function Approach to OPEC Stability", in Applied Game Theory, S. J. Brams, A. Schotter, and G. Schwödiauer (eds.), Würzburg: Physica-Verlag, pp. 303-312.

Green, E. J. and R. H. Porter [1981], "Noncooperative Collusion under Imperfect Price Information", Social Science Working Paper 367, California Institute of Technology, January.

Karlin, S. [1959], Mathematical Methods and Theory in Games, Programming, and Economics, Volume II, Reading, MA: Addison-Wesley.

Kreps, D. M. and J. A. Scheinkman [n.d.], "Cournot Pre-Commitment and Bertrand Competition Yield Cournot Outcomes".

Nash, J. F. [1953], "Two Person Cooperative Games", Econometrica, 21, pp. 128-140.

Osborne, M. J. and C. Pitchik [1983a], "Profit-Sharing in a Collusive Industry", Discussion Paper 180, Department of Economics, Columbia University, February.

Osborne, M. J. and C. Pitchik [1983b], "Price Competition in a Capacity-Constrained Duopoly", Discussion Paper 185, Department of Economics, Columbia University, March.

Radner, R. [1980], "Collusive Behavior in Noncooperative Epsilon-Equilibria of Oligopolies with Long but Finite Lives", Journal of Economic Theory, 22, pp. 136-154.

Scherer, F. M. [1980], Industrial Market Structure and Economic Performance, 2nd. Ed., Chicago: Rand McNally.

Selten, R. [1960], "Bewertung Strategischer Spiele", Zeitschrift für die Gesamte Staatswissenschaft, 116, pp. 221-282.

Selten, R. [1964], "Valuation of n-Person Games", in Advances in Game Theory, M. Dresher, L. S. Shapley, and A. W. Tucker (eds.) (Annals of Mathematics Studies, 52), Princeton University Press, pp. 577-626.

Spence, A. M. [1977], "Entry, Capacity, Investment, and Oligopolistic

Pricing", Bell Journal of Economics, 8, pp. 534-544.

Stevens, W. S. [1913], Industrial Combinations and Trusts, New York: Macmillan.

Stigler, G. J. [1964], "A Theory of Oligopoly", Journal of Political Economy, 72, pp. 44-61.

Stocking, G. W. and M. W. Watkins [1946], Cartels in Action, New York: The Twentieth Century Fund.

Varian, H. R. [1980], "A Model of Sales", American Economic Review, 70, pp. 651-659.