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EQUILIBRIUM IN HOTELLING'S MODEL OF SPATIAL COMPETITION

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We study Hotelling's two-stage model of spatial competition, in which two firms first simultaneously choose locations in the unit interval, then simultaneously choose prices. Under Hotelling's assumptions (uniform distribution of consumers, travel cost proportional to distance, inelastic demand of one unit by each consumer) the price-setting subgames possess equilibria in pure strategies for only a limited set of location pairs. Because of this problem (pointed out independently by Vickrey (1964) and d'Aspremont et al. (1979)), Hotelling's claim that there is an equilibrium of the two-stage game in which the firms locate close to each other is incorrect.

A result of Dasgupta and Maskin (1986) guarantees that each price-setting subgame has an equilibrium in mixed strategies. We first study these mixed strategy equilibria. We are unable to provide a complete characterization of them, although we show that for a subset of location pairs all equilibria are of a certain type. We reduce the problem of finding an equilibrium of this type to that of solving three or fewer highly nonlinear equations. At each of a large number of location pairs we have computed approximate solutions to the system of equations.

Next, we use our analytical results and computations to study the equilibrium location choices of the firms. There is a unique (up to symmetry) subgame perfect equilibrium in which the location choices of the firms are pure; in it, the firms locate 0.27 from the ends of the market. At this equilibrium, the support of the subgame equilibrium price strategy is the union of two short intervals. Most of the probability weight is in the upper interval, so that this strategy is reminiscent of occasional "sales" by the firms. We also find a subgame perfect equilibrium in which each firm uses a mixed strategy in locations. In fact, in the class of strategy pairs in which the firms use the same mixed strategy over locations, and this strategy is symmetric about 0.5, there is a single equilibrium. In this equilibrium most of the probability weight of the common strategy is between 0.2 and 0.4, and between 0.6 and 0.8. There is a wide range of pure Nash (as opposed to subgame perfect) equilibrium location pairs: the subgame strategies in which each firm threatens to charge a price of zero in response to a deviation support all but those location pairs in which the firms are very close.

KEYWORDS: Spatial competition, product differentiation, Hotelling's location model.

1. INTRODUCTION

HOTELLING (1929) formulated the following model of the choice of location and price in a duopoly. Consumers are uniformly distributed over a line segment. A single good is produced at zero cost by two firms, each of which chooses a location in the line segment and a price. Each consumer pays a travel cost which

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is proportional to distance, and buys one unit of the good from the firm for which price plus travel cost is lowest.

We can think of this as a two-stage game between the two firms. In the first stage each (simultaneously) selects a location at which to operate. Then, having observed the locations selected, each (simultaneously) names a price. The consumers choose according to the criterion above, and the firms receive their profits.

Fixing the location of one firm, the other has an incentive to move closer, so as to capture more customers. But since price is named after locations are set, and since firms that are close can be expected (following the logic of Bertrand) to compete fiercely, there is a countervailing pressure on each firm to keep its distance. Nonetheless, Hotelling argued that the firms will locate fairly closely together; he presented an argument in the form of a supposed equilibrium.

Vickrey (1964, pp. 323–334) and (independently) d'Aspremont, Gabszewicz, and Thisse (1979) show that Hotelling's argument is flawed: for locations that are close the pair of price strategies proposed by Hotelling is not an equilibrium. Moreover, d'Aspremont et al. show that no pure strategy price equilibrium exists for such locations. A number of authors have studied variations of Hotelling's model in which pure strategy equilibria do exist, but (to our knowledge) no one has yet produced an equilibrium for Hotelling's original formulation.

We study equilibria in which the firms use mixed strategies in the second stage.[2] We present a strategy pair for which the locations are manifestly the first-stage actions of a Nash equilibrium, and we argue (although are unable to prove) that the strategy pair is a (subgame) perfect equilibrium. Moreover, we argue (with somewhat less conviction) that it is the unique perfect equilibrium in which the firms use pure strategies in the first stage. In this equilibrium, firms locate (on the unit interval) symmetrically, at the distance 0.27 from the two endpoints; it is worth noting that this is close to the (transportation cost) efficient placement of 0.25 from each endpoint. The subsequent price-setting stage requires the firms to randomize. The subgame equilibrium strategies we find are shown in Figure 1 for a number of symmetric location pairs; their qualitative features at our location equilibrium are reminiscent of occasional "sales."

Our analysis is complicated (and, in the end, less than complete) because of the difficulty of the price-setting subgames. Our first result identifies a region P (in the space of location pairs) such that:

(i) If the firms choose a pair of locations in P, then there is a unique subgame equilibrium, which is in pure strategies.

(ii) If the firms choose a pair of locations in the complement of P then there is no pure strategy subgame equilibrium.

We specify the game so that, by the results of Dasgupta and Maskin (1986), a subgame equilibrium exists for each pair of locations. It remains to characterize these equilibria for locations in the complement of P. To this end, we identify a type of (mixed strategy) subgame equilibrium with the following qualitative

[2] Gal-Or (1982) and Shilony and Zamir (in unpublished work reported to us in private correspondence) have previously obtained preliminary results on the outcome of allowing firms to randomize in Hotelling's model.

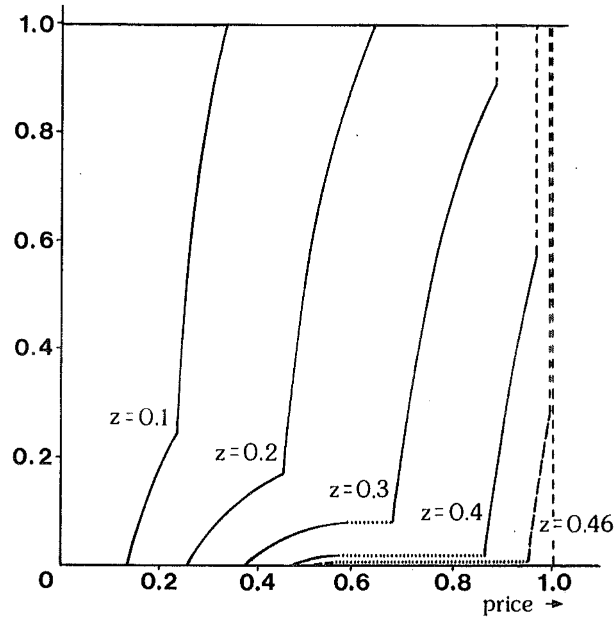

FIGURE 1—The equilibrium price strategies at some symmetric location pairs.

Each curve is the cumulative distribution function of prices used by the firms at a symmetric location pair; the value of z is the distance between the firms. Horizontal sections of the distribution functions are indicated by dotted lines. When $z=0$ the equilibrium strategy of each firm is to set a price of 0 with probability one; when $z=0.5$, it is to set a price of 1 with probability one. The value of z of 0.46 corresponds to the (pure) location equilibrium in which each firm is at the distance 0.27 from an endpoint.

features. The support of each firm's strategy is either a single interval, or the union of two intervals. If the support of each strategy is the union of two intervals, then each price in each firm's lower interval is just low enough to attract all the consumers when the other firm charges some price in its upper interval. (In this case the lower prices can be thought of as "sale" prices.) We are able (in Proposition 3) to give a fairly tight characterization of any subgame equilibrium that is of this type, and we are able to identify a subset S (see Figure 2) of the complement of P in which every subgame equilibrium is of this type. This leaves the complement of $P \cup S$ to worry about. We are unable to show that a subgame equilibrium of our type exists for location pairs in this set, although we show (in Proposition 2) that as the locations get closer together, all equilibria approach the Bertrand equilibrium, in which prices and profits are zero.

Our characterization of the (mixed strategy) subgame equilibria comes in the form of the solution of three or fewer highly nonlinear equations, together with some side inequalities. We have used computational methods to obtain approximate solutions to these equations for a large number of location pairs in the complement of P (both inside and outside S). The strategy pairs associated with these approximate solutions are ε-equilibrium for $\varepsilon < 10^{-7}$. For location pairs in a subset $T2$ of the complement of P (see Figure 2) we can show that the approximate equilibria are very close to actual equilibria.[3] For location pairs in the complement of $P \cup T2$ we cannot show that our approximate equilibria are close to actual equilibria. A result of Kuhn and MacKinnon (1975) (see also Anderson (1986)) ensures that any approximate solution of our equations is close to an exact solution when the degree of approximation is sufficiently small. While this has no formal implication for our calculations (since it is not possible to know what is "sufficiently small"), the result is suggestive. In fact, since the approximate solutions we found, given a wide variety of initial conditions, vary systematically with the location pair, we believe that at each location pair in the complement of P, not only is our approximate subgame equilibrium close to an exact equilibrium, but also this exact equilibrium is the unique subgame equilibrium of our type. We cannot prove this, but we believe the evidence is persuasive.

With these results, we return to the choice of location. We identify a pair of locations (falling in S, but not in $T2$) that gives a Nash equilibrium. We know that this is a Nash equilibrium, because we can find (imperfect) out-of-equilibrium subgame strategies that support it. But this is not quite satisfactory; a wide range of location pairs give imperfect equilibria (see the discussion in Section 4).

We believe that the equilibrium we identify is perfect. The subgame strategies which make it perfect are those we found by computation. We are prevented from being certain that the equilibrium is perfect because we cannot be sure that the payoffs to the approximate subgame equilibrium we found in the complement of $P \cup T2$ are close to those of exact equilibria.

Moreover, if we have indeed identified the unique subgame equilibrium for each location pair, then our computations show that the equilibrium we find is the only perfect equilibrium in which the location choices are pure. (In Section 4 we discuss the existence of perfect equilibria in which the location choices are mixed.)

We are sorry to be reporting such incomplete results. We hope that our work will provide another researcher with enough leads (finally) to nail down the perfect equilibrium (or equilibria) of Hotelling's game.

In Section 2 we specify the game precisely, in Section 3 we discuss our results on the subgame equilibria, and in Section 4 we consider location choice. In Appendix 1 we give outlines of the proofs of the results in Section 3; in Appendix 2 we discuss our computational methods.

2. THE MODEL

Consumers are uniformly distributed on the line segment $[0, 1]$. We normalize the cost of travel to 1 per unit distance. Each of two firms chooses a location in $[0, 1]$. Let x_1 be the distance of firm 1 from 0, and let x_2 be the distance of firm 2 from 1. For each pair of locations (x_1, x_2), let $\Gamma(x_1, x_2)$ be the game in which the firms simultaneously choose prices. Consider the case in which firm 1 locates

[3] It follows that, at each of the finite number of location pairs in $T2$ which we examined, an equilibrium of our type exists.

to the left of firm 2 (i.e., $x_1+x_2\leq 1$). Let p_i be the price charged by firm i, and let $z=1-x_1-x_2$, the distance between the firms. If $p_i<p_j-z$, all consumers buy from firm i, while if $p_j-z\leq p_i\leq p_j+z$, the fraction $x_i+(p_j-p_i+z)/2=(p_j-p_i+1+x_i-x_j)/2$ does so. (Whenever the indices i and j appear in the same expression, $i\neq j$; the division of consumers when $p_i=p_j-z$ is unimportant.) The cost of production is zero. Thus the payoff of firm i in $\Gamma(x_1,x_2)$ when $x_1+x_2\leq 1$ is

$$K_i(p_i,p_j)=\begin{cases} p_i & \text{if} \quad p_i<p_j-z, \\ p_i(p_j-p_i+m_i)/2 & \text{if} \quad p_j-z\leq p_i<p_j+z, \\ 0 & \text{if} \quad p_j+z\leq p_i, \end{cases}$$

where $m_i=1+x_i-x_j$. Given the symmetry of the problem, we can use these payoffs to define the payoffs in $\Gamma(x_1,x_2)$ for every location pair (x_1,x_2).

Let Γ be the two-stage game in which the firms first simultaneously choose locations, and then, for each location pair (x_1,x_2), play the price-setting (sub)game $\Gamma(x_1,x_2)$. We are interested in the equilibria of Γ. In particular, we seek a subgame perfect equilibrium of Γ. First we study the Nash equilibria of the subgame $\Gamma(x_1,x_2)$ for each location pair (x_1,x_2).

3. EQUILIBRIUM IN THE PRICE-SETTING SUBGAMES

By the following result, each of the price-setting subgames has a Nash equilibrium. (Here and subsequently we allow the firms to use mixed strategies.)

LEMMA: *For each pair (x_1,x_2) of locations, the subgame $\Gamma(x_1,x_2)$ has a Nash equilibrium.*

PROOF: Consider the restricted subgame in which the pure strategy set of each firm is $[0,m]$, for some $m>1$. By Theorem 3 of Dasgupta and Maskin (1986) this game has a Nash equilibrium. But if m is large enough (greater than 3, for example), the payoff function K_i is nonincreasing in p_i when $p_i\geq m$, for each $0\leq p_j\leq m$, so that any equilibrium of the restricted game is an equilibrium of the unrestricted game.

For a subset of location pairs (x_1,x_2), the only equilibrium of $\Gamma(x_1,x_2)$ is in pure strategies, as described in the following result. (Since the problem is symmetric, we restrict attention here and subsequently to the case $x_1+x_2\leq 1$.)

PROPOSITION 1: *If $(1+(x_i-x_j)/3)^2\geq 4(x_i+2x_j)/3$ for $i=1$, 2 (region P1 of Figure 2) then $\Gamma(x_1,x_2)$ has a unique equilibrium, which is pure, in which i sets the price $p_i=1+(x_i-x_j)/3$ and obtains a profit of $(1+(x_i-x_j)/3)^2/2$, for $i=1$, 2. If $x_1+x_2=1$ (region P2) and every consumer has a finite reservation price, then $\Gamma(x_1,x_2)$ has a unique equilibrium, which is pure, with $p_1=p_2=0$ and profits of zero. For no other location pair is there a pure equilibrium.*

The equilibrium in region P $(=P1\cup P2)$ is the one found by Hotelling. The extent of P is established by d'Aspremont et al. (1979), who also show that both equilibria are unique within the class of pure equilibria. We prove in

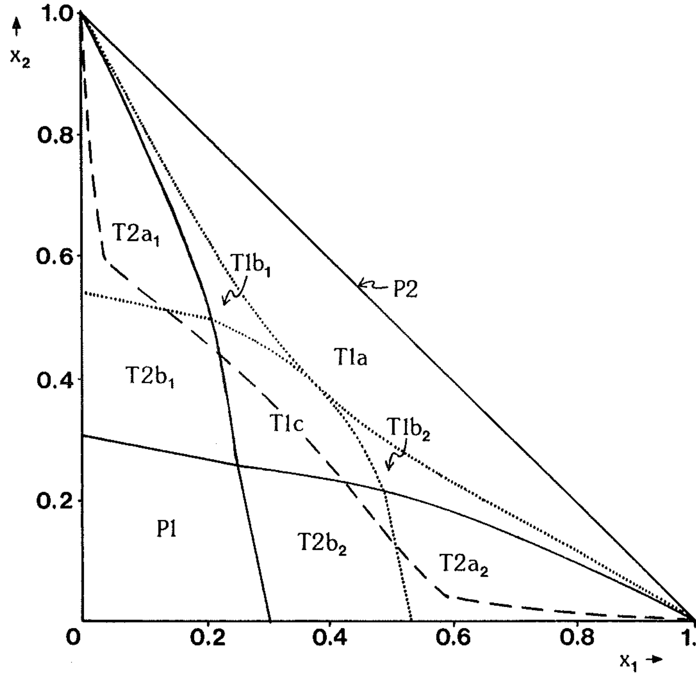

FIGURE 2—Types of equilibrium in the price-setting subgames $\Gamma(x_1, x_2)$.

The solid lines separate the regions $T1$, $T2$, and $P1$; the dotted lines subdivide $T1$ and $T2$. Region $P2$ is the line segment joining (1, 0) and (0, 1); region S is the area below the dashed line. If (x_1, x_2) is in $P = P1 \cup P2$ then the unique equilibrium of $\Gamma(x_1, x_2)$ is pure. In region S an equilibrium must be of type T. In regions $T1$ and $T2$ we find approximate equilibria of type T satisfying the following conditions:

$T1a$: $b_i - a_i = 2z$ for $i = 1, 2$,
$T1b_i$: $b_i - a_i < 2z$ and $b_j - a_j = 2z$,
$T1c$: $b_i - a_i < 2z$ for $i = 1, 2$,
$T2a_i$: $b_i = b_j - z$,
$T2b_i$: $b_i > b_j - z$.

Appendix 1 (see (c), and the discussion after (i)) that there is no mixed equilibrium[4] in region P.

In Appendix 1 we establish a number of properties of the equilibria for location pairs in the complement of P. In particular, we show the following.

PROPOSITION 2: *Every equilibrium of* $\Gamma(x_1, x_2)$ *converges to the pure equilibrium* $(p_1, p_2) = (0, 0)$ *as* $x_1 + x_2 \to 1$.

To describe our results further, let (F_1, F_2) be an equilibrium (each F_i is a cumulative probability distribution function over prices), and let a_i and b_i be respectively the smallest and largest prices in the support of F_i, for $i = 1, 2$. We

[4] The restriction of finite reservation prices is very weak. Without it, there are mixed equilibria when $x_1 + x_2 = 1$ in which each firm charges arbitrarily high prices with positive probability. (Shmuel Zamir pointed this out to us in private correspondence.) An equilibrium of this type does not exist for any other location pair (see (i) of Appendix 1).

show that for a range of location pairs, (F_1, F_2) must take a specific form. Define an equilibrium to be of *type T* if $b_i - a_i \leq 2z$, each F_i is atomless except possibly at b_i, and either (i) the support of each F_i is $[a_i, b_j - z] \cup [a_j + z, b_i]$, and each F_i has an atom at b_i if and only if $b_i - a_i < 2z$ (*type* $T1$), or (ii) the support of F_j is $[a_j, b_j]$, that of F_i is $[a_j - z, b_j - z] \cup \{b_i\}$ with $b_i \geq b_j - z (i = 1 \text{ or } 2)$, F_i has an atom at b_i, and F_j has an atom at b_j if and only if $b_i > b_j - z$ (*type* $T2$). The nature of the supports of F_1 and F_2 in a type $T1$ equilibrium with $b_i - a_i < 2z$ for $i = 1, 2$ is shown in Figure 3. Our result (the proof of which is outlined in Appendix 1) is as follows.

PROPOSITION 3: *Every equilibrium of $\Gamma(x_1, x_2)$ in which $b_i - a_i \leq 2z$ for $i = 1, 2$ is of type T. If (x_1, x_2) is in region S (see Figure 2) then $b_i - a_i \leq 2z$ for $i = 1, 2$ in every equilibrium of $\Gamma(x_1, x_2)$, so that every equilibrium of $\Gamma(x_1, x_2)$ is of type T.*

For (F_1, F_2) to be an equilibrium of type T it is necessary and sufficient that for $i = 1, 2$, (1) F_j is such that the profit $K_i(p, F_j)$ of i is constant (say equal to E_i) on the interior of the support of F_i, and on the union of this with b_i if F_i has an atom at b_i (roughly, each firm is indifferent between actions taken with positive probability), and (2) $K_i(p, F_j) \leqq E_i$ for all p outside the support of F_i.

The condition that $K_i(p, F_j)$ be constant on the interior of the support of F_i is equivalent, upon differentiation with respect to p, to the condition that F_j satisfy an integral-differential equation. (A standard argument[5] shows that each

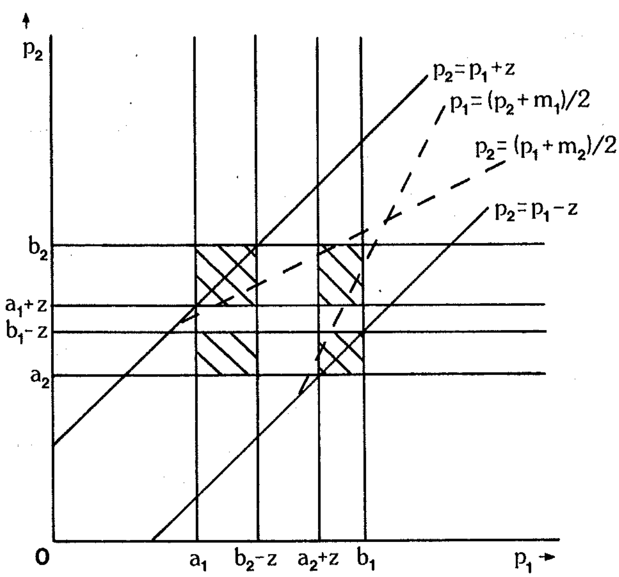

FIGURE 3—The supports of the equilibrium strategies in a type $T1c$ equilibrium of $\Gamma(x_1, x_2)$.

In region $T1c$, the supports of the equilibrium strategies in $\Gamma(x_1, x_2)$ take the form shown. (For each value of p_j, $p_i = (p_j + m_i)/2$ maximizes the payoff of firm i in $(p_j - z, p_j + z)$.) In the other regions, the forms of the supports are indicated in Figure 2.

[5] See, for example, Solution to Problem 17 on p. 294 of Karlin (1959).

F_i is differentiable on the interior of its support). This equation may be differentiated again and, in the case of a type $T1$ equilibrium, solved, subject to the condition $F_j(a_j)=0$, to give

$$(1)\quad F_j(p)=\begin{cases} 1-\exp\left(\dfrac{p-a_j}{2x_i}\right)+A_j\exp\left(\dfrac{p+z}{2x_i}\right)\displaystyle\int_{-p}^{-a_j} h(s,x_i,z)\,ds & \text{if } a_j\leqslant p\leqslant b_i-z,\\ (1-\delta_j)\exp\left(\dfrac{b_j-p}{2x_j}\right)-B_j\exp\left(-\dfrac{p-z}{2x_j}\right)\displaystyle\int_{p}^{b_j} h(s,x_j,z)\,ds & \text{if } a_i+z\leqslant p<b_j,\end{cases}$$

for some A_j and B_j, where δ_j is the size of the atom in F_j at b_j, and $h(s,x,z)=(s-z)^{-2}\exp((s-z)/2x)$. (The integrals can be expressed as infinite series by making the substitution $t=(s-z)/2x$, integrating by parts, and using the fact that $\int(e^t/t)\,dt=\ln|t|+\sum_{n=1}^{\infty}t^n/nn!$.) The case of a type $T2$ equilibrium can be dealt with in a similar fashion.

If F_j is defined by (1), the derivative of $K_i(p,F_j)$ is constant on the interior of the support of F_i. By substituting F_j into the expression for $K_i(p,F_j)$ for $i=1,2$, we obtain conditions on $(a_i,b_i,\delta_i,A_i,B_i)$ for $i=1,2$ which ensure that this derivative is zero. A number of other conditions have to be satisfied for (F_1,F_2) to be an equilibrium: if $K_i(p,F_j)=E_i$ for $a_i\leqslant p\leqslant b_j-z$, then we need $K_i(p,F_j)$ to be equal to the same constant E_i for $a_j+z\leqslant p<b_i$; we need $F_j'(p)\geqslant 0$ for all p in the support of F_i, and $F_j(b_i-z)=F_j(a_i+z)$, so that F_j is a distribution function; and we need $K_i(p,F_j)\leqq E_i$ for all p outside the support of F_i. We obtain from these conditions ten equations and eight inequalities which the ten variables $(a_i,b_i,\delta_i,A_i,B_i)$ $(i=1,2)$ must satisfy. Simple algebraic manipulations reduce this system to three or fewer equations in as many variables (depending on the type of equilibrium), together with some inequalities.

These arguments establish that a solution of this system defines an equilibrium; Proposition 3 guarantees that if (x_1,x_2) is in S then every equilibrium of $\Gamma(x_1,x_2)$ is associated with a solution of the system.

As discussed in the Introduction and in Appendix 2, we computed approximate solutions to the equations, and checked that they satisfied the inequalities, at a large number of location pairs (x_1,x_2). In region $T2$ (see Figure 2) the system consists of a single equation in one unknown, together with some inequalities. At each of the location pairs in $T2$ for which we made computations, we checked that on each side of our approximate solution the function involved has opposite signs, so that (by the Intermediate Value Theorem) an exact solution exists close to our approximate solution. Thus at each of these location pairs, an equilibrium of type T exists close to our ε-equilibrium. In region $T2$ we have to solve two or more equations, so that there is no straightforward computation which demonstrates that there are exact equilibria close to our approximate ones.

Contours of the profit of firm 1 for our collection of approximate subgame equilibria are shown in Figure 4.

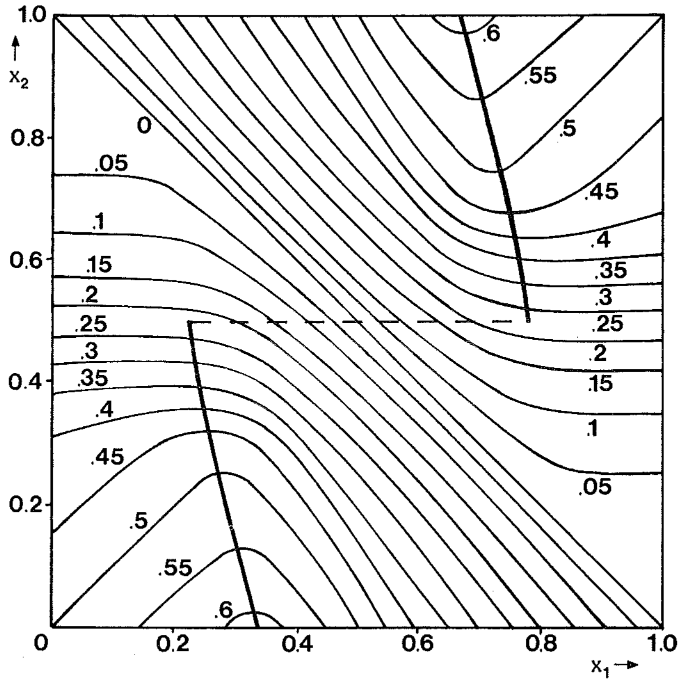

FIGURE 4—Contours of the profit of firm 1 in the approximate subgame equilibria.

The variable x_i is the distance of firm i from 0 ($i = 1$) or 1 ($i = 2$). The number beside each contour is the profit to which it corresponds. The heavy (discontinuous) line is the best response function of firm 1; for each value of x_2 it selects the value of x_1 which maximizes the profit of firm 1.

4. EQUILIBRIUM IN LOCATIONS

To study the pure perfect equilibrium location pairs, we rely on the computations of the approximate subgame equilibria described in the previous section. The best response function of firm 1 in the location game is shown in Figure 4. There is a unique (up to symmetry) pure equilibrium (x, x) with $0.266 < x < 0.274$. The subgame equilibrium price strategy when $x = 0.27$ is shown in Figure 1.

Since we have not fully characterized the equilibrium payoffs in the subgames, we cannot show that there is perfect equilibrium of Γ in which the firms use mixed strategies in the first stage (as well as in the second). However, given the symmetry of the game, it is reasonable that such an equilibrium exists. To make a specific calculation, we used our approximate subgame equilibrium payoffs to construct an approximation of the first stage of Γ, in which each firm has 21 strategies (the locations 0, 0.05, 0.1, . . . , 0.95, 1). Among the class of mixed strategy location pairs (x_1, x_2) in which $x_1 = x_2$ and each x_i is symmetric about 0.5, there is a unique equilibrium.[6] In this equilibrium, the support of each

[6] For each of the 2^{11} possible supports X for a symmetric strategy, we calculated the strategy of i with support X which makes j indifferent between all pure strategies in X, and checked if j's payoff inside X exceeds that outside X.

location strategy extends from 0.2 to 0.8; each strategy is bimodal, most of the probability mass being concentrated between 0.2 and 0.4, and between 0.6 and 0.8.

As mentioned in the Introduction, there is a wide range of Nash equilibrium locations. The second-stage action of i which minimizes j's profit is a price of zero (assuming that negative prices are not allowed[7]), in which case j's best action is to locate at $(1-x_i)/2$ and charge a price of (slightly less than) $(1-x_i)/2$, earning a profit of $(1-x_i)^2/4$. Thus the strategy pair in which i locates at x_i^*, follows the subgame equilibrium strategy if j locates at x_j^*, and otherwise sets a price of zero $(i=1,2)$, is a Nash equilibrium if i's profit is at least $(1-x_j^*)^2/4$ $(i=1,2)$. Since we have no analytical expression for the subgame equilibrium payoffs, we cannot determine precisely the extent of the Nash equilibrium location pairs. However, our arguments in Appendix 1 put a lower bound on the subgame equilibrium payoffs, since they put a lower bound on a_i (see the discussion of (i)). This lower bound implies, for example, that any symmetric location pair (x, x) is a Nash equilibrium if $0 \leqq x < 0.46$ (i.e., only those location pairs in which the firms are very close are not Nash equilibria); most asymmetric location pairs are also Nash equilibria.

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APPENDIX 1: OUTLINE OF PROOFS

Here we outline proofs of Propositions 1, 2, and 3. (Full details are available upon request.)

If (F_1, F_2) is an equilibrium of $\Gamma(x_1, x_2)$ then $a_i \geqslant 0$ for $i=1,2$, since each firm can guarantee a profit of zero by setting a price of zero. We also have the following.

(a) $a_j - z \leqslant a_i \leqslant a_j + z$ *and* $b_j - z \leqslant b_i \leqslant b_j + z$ *for* $i=1,2$: This follows from an examination of i's profit when $p < a_j - z$ and when $p > b_j + z$.

(b) *If* $p>0$ *is an atom of* F_i *and* $x_i > 0$, *then* $p-z$ *is not an atom of* F_j: Under these conditions, the profit of j jumps down at $p-z$, so that this cannot be an atom of F_j.

(c) *If* $z=0$ *and* b_i *exists for some* i, *or if every consumer has a finite reservation price, then* $a_i = b_i = 0$ *for* $i=1,2$ (i.e., *the only equilibrium is pure, each firm charging the price zero*): If b_i exists then b_j exists and $b_j = b_i$ (by (a)). Let $b_i = b_j = b$. If $b>0$ then it is not an atom of both F_i and F_j (by (b), since $z=0$ means that $x_i > 0$ for some i). Suppose b is not an atom of F_j. Then $K_i(b, F_j)$ is equal to i's equilibrium profit (see (d) of Fact (B) in Osborne and Pitchik (1986)). But $K_i(b, F_j) = 0$, while $K_i(p, F_j) > 0$ for $0<p<b$. Hence we must have $b=0$. If every consumer has a finite reservation price, then $K_i(p, q) = 0$ for all q if p is large enough, say if $p \geqslant \bar{p}$. Hence $b_i \leqslant \bar{p}$; the argument above establishes that $b_i = b_j = 0$.

This proves the second sentence of Proposition 1. *From now on, we assume that* $z>0$.

(d) $a_i > 0$ *for* $i=1,2$, *and the equilibrium profit of each firm is positive*: This follows from the fact that firm i can guarantee a positive profit by setting the price $z/2$.

(e) *If* $x_j > 0$ *and* $b_i = b_j - z$, *then* b_i *is an atom of* F_i *and* b_j *is not an atom of* F_j: If b_i is not an atom of F_i, then $K_j(b_j, F_i) = 0$ is the equilibrium profit of j (see (d) of Fact (B) in Osborne and Pitchik (1986)). This contradicts (d), so that b_i is an atom of F_i, and so b_j is not an atom of F_j (by (b)).

[7] Note that if the cost of production is positive (rather than zero), then p_i can be interpreted as the excess of price over unit cost, so that negative values of p_i are meaningful.

Subsequently we assume that $x_i > 0$ for $i = 1, 2$. (All our results hold when $x_i = 0$ for some i, but messy arguments are then needed.) The next result implies Proposition 2.

(f) *$b_i \to 0$ for $i = 1, 2$ as $z \to 0$*: Domination arguments show that $K_j(b_i - z, b_i) \geqq K_j(b_i - 3z, b_i)$, which is equivalent to $b_i \leqq (2 + x_i)z/x_i$.

(g) *If p is an atom of F_i, then $p \geqq 2x_i$*: If p is an atom of F_i then $K_j(\cdot, F_i)$ jumps down at $p - z$ (if $p > z$), and at $p + z$, so that supp F_j contains no point in $(p - z, p - z + \varepsilon)$ or in $(p + z, p + z + \varepsilon)$ for some $\varepsilon > 0$. But then $K_i(\cdot, F_j)$ is increasing on $(p, \min(p + \varepsilon, 2x_i))$ if $p < 2x_i$, contradicting the fact that p is an atom of F_i.

(h) *If $p \in \text{supp } F_i$ and p is not an atom of F_i, then either $p - z \in \text{supp } F_j$ or $p + z \in \text{supp } F_j$*: If neither $p - z$ nor $p + z$ is in supp F_j, then $K_i(\cdot, F_j)$ is either increasing, constant equal to zero, or strictly concave on some neighborhood of p. None of these is consistent with $p \in \text{supp } F_i$ and p not an atom of F_i.

(i) *$b_i \leqslant (b_j + m_i)/2$ for $i = 1, 2$, and hence $b_i \leqslant \gamma_i \equiv \min(1 + (x_i - x_j)/3, 2(1 - x_j), 3(1 - x_i) - x_j)$ for $i = 1, 2$*: This follows from domination arguments.

We can now restrict the value of z_i by making further domination arguments. From (i) we have supp $F_i \subset [a_i, \gamma_i]$ for $i = 1, 2$, so that if there exists p_i^* such that $K_i(p_i, p_j) < K_i(p_i^*, p_j)$ for all $a_j \leqslant p_j \leqslant \gamma_j$ and all $p_i < \underline{p}_i$, then we must have $a_i \geqslant \underline{p}_i$. Let $V_i(a_j)$ be the largest such value of $\underline{p}_i$; then $a_i \geqq V_i(a_j)$ for $i = 1, 2$. The precise form of V_i is complex; we omit the details.

Let a_i^* be the minimal value of a_i such that $a_i \geqslant V_i(a_j)$ and $a_j \geqslant V_j(a_i)$ for some a_j. Then $a_i \geqq a_i^*$ in any equilibrium. Thus if $a_i^* = \gamma_i$ for $i = 1, 2$, then the pure equilibrium $(p_1, p_2) = (\gamma_1, \gamma_2)$ is the only possible equilibrium. A very tedious analysis of the functions $V_i (i = 1, 2)$ (the details of which we omit) shows that this is so for every (x_1, x_2) in P; this completes the proof of Proposition 1. Also, if $\gamma_i - a_i^* \leqslant 2z$ then we know that $b_i - a_i \leqslant 2z$. This is useful because our subsequent results use the assumption that $b_i - a_i \leqslant 2z$; we show that the only equilibria satisfying this condition are of type T. A computation shows that $\gamma_i - a_i^* \leqslant 2z$ for $i = 1, 2$ whenever (x_1, x_2) is in S (see Figure 3). Thus the second sentence of Proposition 3 follows from the first, which remains to be proved. *From now on, we assume that $b_i - a_i \leqslant 2z$ for $i = 1, 2$.*

(j) *If p is an atom of F_i then $p = b_i$*: If $\bar{p}$ is an atom of F_i, then supp F_j excludes intervals just above $\bar{p} - z$ and $\bar{p} + z$, so that $K_i(\cdot, F_j)$ is strictly concave, and hence decreasing, on $(\bar{p}, \bar{p} + \varepsilon)$ for some $\varepsilon > 0$. Suppose $\bar{p} < b_i$, and let $\bar{p} + \delta (> \bar{p} + \varepsilon)$ be the smallest price in supp F_i above $\bar{p}$. Then, arguing as in (h), each of the intervals of length δ above $\bar{p} - z$ and $\bar{p} + z$ contains either a single atom of F_j or no point in supp F_j. To complete the proof, we can show that $K_i(\cdot, F_j)$ is dominated by a concave function which coincides with $K_i(\cdot, F_j)$ close to $\bar{p}$ (we omit the details). This shows that $K_i(\bar{p} + \delta, F_j) < K_i(\bar{p}, F_j)$, so that $\bar{p} + \delta \notin \text{supp } F_i$, contrary to assumption.

(k) *If $a_i < b_j - z$ then $[a_i, b_j - z] \subset \text{supp } F_i$*: If $a_i < p < b_j - z$ and $p \notin \text{supp } F_i$ then, given the previous results, $K_j(\cdot, F_i)$ is strictly concave on an interval around $p + z$, so that it is not maximized at the endpoints of the interval, at which it must equal j's equilibrium profit.

(l) *If $a_j + z < b_i$, then $[a_j + z, b_i] \subset \text{supp } F_i$*: This follows from an argument similar to that in (k).

(m) *If $b_j - z < a_j + z$ (i.e. if $b_j - a_j < 2z$), then $\text{supp } F_i \cap (b_j - z, a_j + z) = \varnothing$ or $\{b_i\}$*: If $p \in \text{supp } F_i$ and $b_j - z < p < a_j + z$, then p is an atom of F_i by (h), so that $p = b_i$ by (j).

(n) *If $a_i > a_j - z$ for $i = 1, 2$, then $b_j > a_i + z$*: Since a_i is not an atom of F_i (by (j)) we have $a_i + z \in \text{supp } F_j$ (by (h)). Hence $b_j \geqq a_i + z$. If $b_j = a_i + z$, then $b_j - a_j < 2z$ (since $a_j > a_i - z$), so that a_i is an isolated member of supp F_i (by (m), using $b_j - z = a_i$), contradicting (j).

(o) *If $a_i > a_j - z$ for $i = 1, 2$ then b_j is an atom of F_j if and only if $b_j - a_j < 2z$*: If $b_j - a_j < 2z$ and b_j is not an atom of F_j, then $b_j - z \in \text{supp } F_i$ by (h) (since $b_j + z > a_i + 2z \geqq b_i$). Since a_j is not an atom of F_j, we also have $a_j + z \in \text{supp } F_i$. But then $K_i(\cdot, F_j)$ is continuous and strictly concave on $[b_j - z, a_j + z]$, which means that i's profit cannot be maximized at both endpoints, where it must attain its equilibrium value. Hence b_j is an atom of F_j. Now assume that $b_j - a_j = 2z$. Then $a_i < a_j + z (= b_j - z) < b_i$ (the second inequality by (n)), so that by (k) and (l) we have supp $F_i = [a_i, b_i]$. If F_j has an atom at b_j, then $K_i(\cdot, F_j)$ jumps down at $a_j + z$, contradicting the (a.e.) constancy of $K_i(\cdot, F_j)$ on supp F_i. So b_j is not an atom of F_j.

We can now show that every equilibrium of $\Gamma(x_1, x_2)$ in which $b_i - a_i \leqslant 2z$ is of type T.

Type T1: If $a_i > a_j - z$ for $i = 1, 2$ then (n), (k), (l), and (m) imply that supp $F_i = [a_i, b_j - z] \cup [a_j + z, b_i]$ for $i = 1, 2$. By (o), b_i is an atom of F_i if and only if $b_i - a_i < 2z$.

Type T2: If $a_i = a_j - z$ then (k), (l), and (m) imply that supp $F_j = [a_j, b_j]$, and supp $F_i = [a_j - z, b_j - z]$ or $[a_j - z, b_j - z] \cup \{b_i\}$. In the first case $b_j - z$ is an atom of F_i and b_j is not an atom of F_j by (e); in the second case b_j is an atom of F_j (otherwise i's payoff in $(b_j - z, b_i)$ exceeds that at $b_j - z$ and at b_i, as in the proof of (o)), and b_i is an atom of F_i.

APPENDIX 2: NOTES ON COMPUTATIONAL TECHNIQUES AND ACCURACY

Techniques

As discussed in Section 3, the problem of finding an equilibrium of type T can be reduced to that of simultaneously solving three or fewer (depending on the value of (x_1, x_2)) equations of the form $g(y_1, \dots, y_k) = 0$ in as many unknowns, and checking that the solution satisfies a number of inequalities. To find an approximate solution of the equations for a particular pair (x_1, x_2) of locations, we evaluated the left-hand sides of the equations at each point in a grid, found the point in the grid which generates the lowest sum of absolute values of these left-hand sides, and then repeated the procedure on a smaller grid. We stopped this iterative procedure when we obtained an absolute value for the sum of the left-hand sides less than 10^{-7}. We then used the resulting parameter values to calculate equilibrium payoffs and equilibrium strategies, and to check that the inequalities are satisfied. We carried out this procedure for about 350 pairs (x_1, x_2). (This involved computations at 175 points, given the symmetry of the problem). All calculations were performed by a DEC 20 computer, programmed in APL, with an internal precision of about 18 decimal digits.

Accuracy

1. The integrals in (1) can be expressed only as infinite series. Let $I(t) = -e^t/t + \ln|t| + \sum_{n=1}^{\infty} t^n/nn!$. Then $\int_a^b h(s, x, z) = (I((b-z)/2x) - I((a-z)/2x))/2x$ (integrating as discussed in the text). We used the first 25 terms to approximate the infinite series in I. This approximation is better, the smaller is the absolute value of the argument of I. For $x_1 = x_2 = 0.27$ (the approximate pure location equilibrium) we have $a = -p$ and $b = -a_i$ for the integral in $F_i(p)$ on $[a_i, b_j - z]$, and $a = p$ and $b = b_i$ for the integral in $F_i(p)$ on $[a_j + z, b_i]$ (see (1)). Given that $a_i \approx 0.5$ and $b_i \approx 1$ for $i = 1, 2$ in this case, $(b-z)/2x \approx -1.78$ and $(a-z)/2x$ ranges between -1.78 and -1.85 in the first integral, and $(b-z)/2x \approx 1$ and $(a-z)/2x$ ranges between 0.93 and 1 in the second integral. This means that the omitted terms in the infinite series are of the form $t^n/nn!$, with $-1.85 < t < 1$ and $n \geq 26$; the absolute value of the sum of all such terms is at most $(1/26)[r^{26}/26! + r^{27}/27! + \cdots]$, where $r = |t|$, which is at most $e^r r^{26}/26 \times 26!$ (using an upper bound for the Lagrange form of the remainder term in the expansion for e^r). Given that $-1.85 < t < 1$, this is less than 10^{-20}, and hence the approximation error is less than the computational error. As x_1 and x_2 vary, this error changes. However, an analysis of the various cases shows that the error does not exceed 10^{-14} at any of the points (x_1, x_2) we studied.

2. The solution we find is also only approximate because we find parameter values which solve the nonlinear equations only to within 10^{-7}. Since the length of the support of every equilibrium strategy is at most 1, and our solution guarantees that the derivative of the payoff is at most 10^{-7}, the payoff of each firm varies by at most 10^{-7} on the support. For prices outside the support, our computations (and in some cases analytical arguments) show that the payoff is less then the equilibrium payoff. Hence the equilibria we find are ε-equilibria for $\varepsilon < 10^{-7}$.

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