AN ANALYSIS OF POWER IN EXCHANGE ECONOMIES

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By

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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The aim of this study is to understand how the allocation of goods in an economy depends on the set of possible actions available to each group of individuals. Onto a standard exchange economy we add the structure of a strategic game, and study the outcomes generated by the set of nontransferable utility Harsanyi-Shapley values. We make several different sets of assumptions about the strategies available to each group of agents, and also analyze how the outcome varies as these assumptions are varied within a certain class.

We first study the consequences of three sets of assumptions which involve the political structure of majority rule. For example, under one set of assumptions any coalition containing a majority of the population has the power to expropriate all the goods which the members of the complementary minority attempt to trade in the market, while the best any member of the minority can do is to consume his initial endowment. The solution concept gives us a set of allocations of the available goods, which could be achieved simply be redistributing the goods themselves. Since the allocations are efficient, however, we know from a result of welfare economics that they can also be realized as the outcome of the individuals in the economy trading at fixed prices, after their initial wealth has been modified by a system of taxation. Aumann and Kurz, using a model similar to ours, have shown that under their assumptions the tax an individual pays depends solely on his wealth; in this sense their

model provides an explanation for the existence of wealth taxes. Under the assumptions which we mentioned above we show that the taxes which result are an idealized form of income taxes. Under some other assumptions which draw a distinction between the classical categories of "labor" and "land" on the basis that labor-time can be destroyed (an individual can choose not to work, can go on strike) while land cannot be, we find that the tax rate on the wealth derived from the ownership of land is very high, while that derived from labor is much lower.

We also study the question of how sensitive the results of Aumann and Kurz are to the precise assumptions they make. We find that wealth taxation is the outcome under a wide range of assumptions which includes those of Aumann and Kurz. In particular the dichotomized power distribution entailed in the assumption of majority rule is quite inessential; what are important, rather, are the factors upon which power depends. One consequence of this result is that since the set of outcomes under majority rule coincides with that under a large variety of other "power distributions" (for example, the one where power is proportional to size), one can view the political structure of majority rule as a way of realizing the outcome implied by the power structure, rather than as an exogenous feature of the economy.

AN ANALYSIS OF POWER IN EXCHANGE ECONOMIES*

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CHAPTER 1: Introduction

Classical economic theory attempts to explain the surface phenomena of market economies--relative prices, profits, and the distribution of output--without looking far below the surface for an explanation. In particular, the only actions available to agents in that theory are the purchase and sale of goods in the marketplace. As far as the explanation of the distribution of output is concerned, this abstraction seems to ignore many of the most important factors, especially those involving the relative "power" of groups of agents. Thus, though individuals certainly do acquire income by selling some of the goods with which they are endowed, and use that income to buy other goods, the prices of those goods are not always outside their control (wages are certainly an object of bargaining), and neither are the taxes they pay and subsidies they receive (they can form and vote for political parties to effect taxation schemes to their liking). Here, in an attempt to take into account these factors in a study of the determinants of the distribution of output, we model the economy as a strategic game in which individual players are powerless, but the power which groups

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possess is of central importance. Thus we simply assume that there is available to each group of individuals in the economy a set of strategies, and corresponding to each selection of strategies there is a payoff to each individual. One possible strategy of a group might be to go on strike, another might be to form a government and impose certain taxes. The outcome in this setting is determined by the strategies which the individuals choose to employ. These choices may depend on the whole sets of possible actions which are available to the other individuals, and not simply on those which they actually choose to carry out. (It is in this sense that in the approach here we look below the "surface".) For example, the wage a capitalist decides to pay may well depend on the fact that his workforce could go on strike. Game theory provides a number of models of the way individuals choose their strategies--i.e. it provides a number of "solution concepts" for strategic games. Here we use one which is associated with a specific model of a bargaining procedure. The question we study is how the distribution of output predicted by this solution concept depends on the strategies which are available to the groups in the economy. In particular, we ask what factors confer upon the groups the power to obtain a large share of the output for themselves. We do so by investigating the nature of the set of outcomes predicted by the solution concept for a number of different sets of assumptions on what are the strategic possibilities of the groups.

Before explaining our model in more detail (in Chapter 2) we shall discuss a general issue. In classical economic theory the actions of agents which are significant to the solution concept--or "equilibrium notion"--used are all ones which the individuals might actually carry out; they are important insofar as they benefit the agents taking them. For this reason we can base our assumptions about them on the actions we observe that individuals take. Thus it is possible to make the judgement that the actions which are possible for agents in classical economic theory constitute a reasonable abstraction of the opportunities actually available to individuals in an economy, insofar as these opportunities relate to aspects of the operation of an economy which the theory attempts to capture. In any solution concept which looks a little deeper, and gives a central role to the actions which individuals can threaten to make (but which, at least in equilibrium, would never actually be carried out) we cannot base our assumptions about the strategy sets of the groups of agents in the economy on the choices which we observe individuals actually making. To some extent we can glean information about what are possible strategies for groups of individuals in market economies by observing "non-equilibrium" actions (like strikes), but this still leaves us a good deal of lattitude when it comes to making specific assumptions. What we do here is study the outcome of a number of different assumptions on the strategies available to the groups of agents in the economy. In doing so we get some idea of the range of outcomes our model can generate, as well as obtaining characterizations of the outcomes under specific assumptions.

In the next chapter we outline our model in more detail, describe some of the previous work within the framework we are using, and summarize our results. In Chapter 3 we introduce the game theory we shall subsequently use; in Chapter 4 we describe the economic model, cite a number of results we shall use, and prove a slight generalization of a well-known result. The next four chapters contain our results. In Chapters 5, 6, and 7 we study in detail the outcomes predicted under three different groups of assumptions on the strategy sets. In Chapter 8 we study how the outcome changes as we vary our assumptions within a certain class.

CHAPTER 2: Outline of the Model and Summary of the Results

The setting for the games we shall analyze is an exchange economy with a continuum of agents (i.e. the economy is specified by a collection of utility functions and endowment densities, one for each agent). The games are defined by assigning to each set of agents (each "coalition") a strategy set, and by specifying a payoff function which associates with every collection of strategies a payoff density to each agent. To such a strategic game there are a number of solution concepts which we might apply. We should like to use one which is derived directly from a coherent model of bargaining in which the final outcome is a compromise that depends on the availability to all individuals of strategies which can be used as "threats". Harsanyi's [1963] Bargaining Solution meets this criterion (though one might argue that it is not without flaws), but is very difficult to work with. The solution concept we choose to employ is the set of Harsanyi-Shapley values, which has some features in common with Harsanyi's Bargaining Solution, and may approximate it well under some circumstances. We shall provide precise definitions, and discuss some criticisms of the set of Harsanyi-Shapley values in the next chapter (see Sections 3.2 and 3.3).

Aumann and Kurz [1977] initiated the use of the set of Harsanyi-Shapley values as a solution concept for the sort of games we study. They explored the consequences of one set of assumptions about the strategy sets of the groups of agents in the economy. A central element in the procedure for calculating the set of Harsanyi-Shapley values is the bilateral bargaining between a group of players and its complement in the population, so when we specify the strategic possibilities for the groups of agents we need to do so with this game in mind. Aumann and Kurz assume that when a group contains a majority of the population (so that its complement is a minority) it can expropriate the endowment of its complement, while the best its complement can do in response is to destroy its own endowment. They show under these assumptions that (given their conditions on the utility functions and endowment density) in every economy there is at least one allocation which generates a Harsanyi-Shapley value, and they provide a characterization of such allocations. Under their assumptions each such allocation is efficient, and hence can certainly be "supported" as a competitive allocation after lump-sum wealth taxation--in the sense that there is a price, and a tax for each agent which depends on his utility function and endowment, such that if each agent maximizes his utility given his after-tax income he will choose the quantities of goods assigned him by the Harsanyi-Shapley value allocation. One of the major consequences of the characterization which Aumann and Kurz establish is that the tax of each individual depends only on his utility function and the <u>value</u> of his initial endowment at the supporting prices, and not independently on the endowment itself, so that the tax which results is really a tax on "income"--or, more properly, "wealth". They also establish that the marginal tax rate is always at least 50%.

One might argue that for a private ownership economy the assumption that a majority coalition can expropriate the entire endowment of its complement is a little extreme. Rather one might claim that a majority has the power to tax away only that part of the endowment of the minority which is traded in the market. For example, if we think of agents as being endowed with leisure, it seems reasonable to assume that a majority can tax only that part of this endowment which is offered in the market as labor-time, and cannot directly expropriate the leisure-time of the agents. If, under these conditions, the minority can redistribute its endowment among itself in any way it pleases without entering into any trade, we are back to a situation where the minority and the majority have the same possibilities open to them. To give the majority some power we can assume that the only way the minority can redistribute its endowment is via trade, and that the majority can tax away any goods which minority members attempt to trade--i.e. the majority can effectively prohibit the members of the minority from trading among themselves. We examine the consequences of these assumptions in Chapter 5 below. Our main result there (Theorem A) states that each economy possesses at least one Harsanyi-Shapley value allocation, and provides a characterization of such allocations. Naturally, when we interpret the outcome as the result of a tax, it is not a wealth tax as in Aumann and Kurz [1977]; it is a tax which is related to the extent to which agents benefit from trading, and under some circumstances can be interpreted as an idealized form of income tax (see Section 5.6).

Within our basic model, restricting the power of the majority to the prohibition of trade among the members of the minority is among those assumptions which give the majority the least power, without reverting to a situation where the majority and minority are symmetric (as in a market game). In Chapter 6 we consider a modification of the assumptions of Aumann and Kurz in the opposite direction: we allow a majority to expropriate the endowment of its complement, and assume that the latter can do nothing to prevent this expropriation. The consequences of this set of assumptions are interesting not only because a majority is given as much power as it can possibly expect to have, so that we have a "boundary" case, but also because our analysis allows us to investigate (in Chapter 7) the case where the members of a minority can destroy some goods, but not others. Furthermore, under these assumptions we might expect the outcome to be very "egalitarian" since the power of a group of agents depends solely on its size (in addition to the utility functions of its members), and not at all on its endowment. In one class of cases where it is possible to compute the set of Harsanyi-Shapley values, this is indeed the case (see Theorem B): an allocation is a Harsanyi-Shapley value allocation in this case if and only if it is a competitive allocation in an economy in which every individual has the same wealth.

In Chapter 7 we study an economy in which some goods (like land) cannot be destroyed, while others (like labor-time) can be destroyed. Proposition 7.5 provides a characterization of the set of Harsanyi-Shapley value allocations under these assumptions for a class of economies. A consequence which is of interest concerns the way such allocations can be supported as competitive allocations after taxation of the wealth derived from possession of the different sorts of goods at different rates. In the classical theory of public finance, where the question asked is how taxes should be set so as to minimize the amount of "distortion", the prescription is to tax goods in inelastic supply at high rates, and those in elastic supply at low rates. In our case, where the question is how the outcome of bargaining can be supported as a competitive equilibrium after taxation, we find that the taxes on those goods which cannot be destroyed are high relative to the taxes on the goods which can be destroyed. Given that the property of being in inelastic supply has features in common with the property of being not destroyable ("land" is a typical good in both cases), there is some connection between our result and that of the classical theory (though the underlying model is quite different).

The results of Chapters 5, 6, and 7 give us some idea of the range of outcomes which is possible within our framework, but we should like to be more precise. In particular, we should like to be able to answer questions like "what is it about the assumptions on the strategic game which leads to an outcome which can be supported by a wealth tax, or by a tax on trade?", and "what characteristics of the assumptions make the outcome more or less 'egalitarian'?". In Chapter 8 we establish some results which provide some answers to these questions. We show that the Harsanyi-Shapley value allocations can be supported as competitive allocations after wealth taxation (as is the case under the assumptions of Aumann and Kurz [1977]) under a wide range of conditions (see Proposition 8.10). What are important are the factors upon which the strategic possibilities of the groups of individuals depend, rather than the specific form of that dependence. For example, the dichotomized power distribution involved in the assumption of majority rule made by Aumann and Kurz (which we also maintain in Chapters 5, 6, and 7) is quite inessential: quite general power distributions lead to the same result. This result allows us to view the political system of majority rule as a method of implementing the outcome implied by the actual distribution of power (in which, perhaps, power is proportional to size), rather than simply as a given institution.

CHAPTER 3: The Solution Concept

In this chapter we define a strategic game, and the solution we shall subsequently apply to such games (the set of Harsanyi-Shapley values); we then relate this solution to Harsanyi's [1963] Bargaining Solution. It turns out that the procedure for calculating the set of Harsanyi-Shapley values of a strategic game can be decomposed into two stages. First, a class of coalitional form games is derived from the strategic game, and then the (Shapley) value of each of these coalitional forms is calculated. Thus in order to describe the set of Harsanyi-Shapley values of a strategic game we need to define the value of a game in coalitional form. In the following chapters we shall need a number of results concerning this value for games with a continuum of players; in order not to interrupt the argument at a later stage, we collect together all the results concerning values of coalitional games in Section 3.1 below.

3.1 The Shapley Value of a Game in Coalitional Form

A game in coalitional form (or simply a game) consists of a measurable space $(T,\mathcal{C})$ which is isomorphic to $([0,1],\mathcal{B})$, where $\mathcal{B}$ is the σ-field of Borel subsets of $[0,1]$, and a function $v: \mathcal{C} \to \mathbb{R}$ such that $v(\emptyset) = 0$. T is the set of players, $\mathcal{C}$ the collection of coalitions, and $v(S)$ for $S \in \mathcal{C}$ is the worth of S. We denote such a game by $((T,\mathcal{C}),v)$, or simply by v. If T is finite and $\mathcal{C} = 2^T$ then the game is finite.

A non-decreasing sequence of sets of the form

$$\emptyset = S_0 \subset S_1 \subset \ldots \subset S_m = T$$

is a <u>chain</u>. If v is a game the <u>variation norm</u> $\|v\|$ of v is defined by

$$\|v\| = \sup \sum |v(S_i) - v(S_{i-1})| \quad ,$$

where the supremum is taken over all chains. The space of all games v for which $\|v\|$ is finite (i.e. which are of bounded variation) is denoted BV. BV is in fact a Banach space with the norm $\|\cdot\|$ (Proposition 4.3 of Aumann and Shapley [1974]). The subspace of BV consisting of all bounded finitely-additive games is denoted FA; a member of FA is called a <u>payoff</u>. If v is finite, we can think of this payoff as a vector.

A (Shapley) <u>value on finite games</u> is a function ϕ which assigns to every finite game v a payoff vector ϕv such that

(3.1) $(\phi v)(T) = v(T)$ (efficiency) ,

(3.2) $(\phi v)(\{i\}) = (\phi v)(\{j\})$ whenever $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \in \mathcal{C}$ with $S \not\ni i$ and $S \not\ni j$ (symmetry) ,

(3.3) $(\phi v)(\{i\}) = 0$ whenever $v(S \cup \{i\}) = v(S)$ for all $S \in \mathcal{C}$ with $S \not\ni i$ (ineffective players get nothing) ,

and

(3.4) $\phi(v + w) = \phi v + \phi w$ for every pair of finite games v and w (additivity) .

Shapley [1953] proved the following.

Proposition 3.5 (Shapley): There is one and only one value ϕ on finite games; it is defined by

(3.6) $(\phi v)(\{i\}) = E[v(S_i \cup \{i\}) - v(S_i)]$ for each $i \in T$,

where S_i is the set of players preceding i in a random order on T, and E is the expectation operator when all $|T|!$ such orders are assigned equal probability.

(3.6) allows us to calculate the value of every finite game; Aumann and Shapley [1974] studied a number of ways of extending the value concept to games with a continuum of players. One way is to view such a game as the limit of a sequence of finite games. This leads to the asymptotic value, which has been studied extensively (see for example Aumann and Shapley [1974], Chapter III, and Neyman [1978]), and which we shall use in this study. We shall frequently cite the results of Aumann and Shapley [1974]; in order to avoid excessive repetition we shall refer to this source simply as "Aumann and Shapley" throughout. We can now define the asymptotic value of a game.

A partition Π of (T,C) is a collection of disjoint members of C the union of which is T. The partition Π_2 is a refinement of the partition Π_1 if each member of Π_1 is the union of members of Π_2. A sequence of partitions $\{\Pi_m\}_{m=1}^{\infty}$ is admissible if Π_{m+1} is a refinement of Π_m for all m and for each $s, t \in T$ there exists m such that s and t are in different members of Π_m. Let $((T,C),v)$

be a game, and Π a partition of (T,C). Then we can define a finite game v with player set $\{a: a \in \Pi\}$ by

$$v_{\Pi}(A) = v(\bigcup_{a \in A} a) \quad \text{for each} \quad A \subset \{a: a \in \Pi\} \quad .$$

A payoff ϕv is then the <u>asymptotic value of</u> v if for every $S \in C$ and every admissible sequence of partitions $\{\Pi_m\}_{m=1}^{\infty}$ in which Π_1 is a refinement of $(S, T \backslash S)$, $\lim_{k \to \infty} (\phi v_{\Pi_k})(S_k)$ exists and equals $(\phi v)(S)$, where $S_k = \{a: a \in \Pi_k$ and $a \subset S\}$. The set of all games in BV which possess an asymptotic value is denoted ASYMP. From Theorem F of Aumann and Shapley, ASYMP is a closed linear subspace of BV.

The conditions under which a game v belongs to ASYMP have been a major object of study. We shall now state a result in this area which we shall use in the sequel to establish that the games which arise do in fact have asymptotic values.

The subspace of BV consisting of all nonatomic measures (i.e. countably additive set functions) is denoted NA; the subset of NA consisting of nonnegative measures is denoted NA^+, and that consisting of probability measures is denoted NA^1. bv'NA is the subspace of BV spanned by games of the form $f \circ \mu$ where $f: [0,1] \to \mathbb{R}$ is of bounded variation, $f(0) = 0$, f is continuous at 0 and 1, and $\mu \in NA^1$. pNA is the closed subspace of bv'NA spanned by powers of members of NA^1. If Q_1 and Q_2 are subsets of BV, we write $Q_1 * Q_2$ to denote the closed linear symmetric subspace spanned by all games of the form

v_1v_2 where $v_1 \in Q_1$ and $v_2 \in Q_2$ (and $(v_1v_2)(S) = v_1(S)v_2(S)$ for all $S \in \mathcal{C}$). We can now state the following, which is a special case of Theorem 4.1 of Neyman [1978].

Theorem 3.7 (Neyman): pNA*bv'NA ⊂ ASYMP.

Now, let DIAG be the set of all $v \in$ BV such that there is a positive integer k, a k-dimensional vector η of measures in NA^1, and a neighborhood M in $\mathbb{R}^k$ of the diagonal $D = \{(x,x,\ldots,x) \in \mathbb{R}^k : x \in [0,1]\}$ such that if $\eta(S) \in M$ then $v(S) = 0$. Thus DIAG consists of those games which vanish "close" to the diagonal D. Define pNAD to be the variation closure of pNA + DIAG (so that pNAD consists of games which behave like those of pNA in a neighborhood of the diagonal). Then we have the following corollary of Theorem 3.7.

Corollary 3.8: pNAD*bv'NA ⊂ ASYMP.

Proof: By the definition of DIAG we have DIAG*bv'NA ⊂ DIAG, and by Proposition 43.11 of Aumann and Shapley, DIAG ⊂ ASYMP. So from Theorem 3.7 and the fact that ASYMP is a linear space, (pNA + DIAG)*bv'NA ⊂ ASYMP. But ASYMP is closed, and pNAD*bv'NA is contained in the closure of (pNA + DIAG)*bv'NA, so pNAD*bv'NA ⊂ ASYMP.

Now, define the supremum norm $\|v\|'$ of a bounded game $v: \mathcal{C} \to \mathbb{R}$ by

$$\|v\|' = \sup \{|v(S)| : S \in \mathcal{C}\} .$$

Let BS be the Banach space of bounded games with the supremum norm, and let pNA' be the subspace of BS spanned by powers of members of NA^1. If $v \in pNAD \cap pNA'$ then not only does v have an asymptotic value, but there is an elegant formula for it (due to Aumann and Shapley) which is analogous to (3.6). We shall now explain this formula (see also Chapter IV of Aumann and Shapley). First we need to generalize the notion of a coalition. An <u>ideal subset of</u> $(T,\mathcal{C})$ (or simply an <u>ideal coalition</u>) is a measurable function from $(T,\mathcal{C})$ to $([0,1],\mathcal{B})$. The family of all ideal subsets of $(T,\mathcal{C})$ is denoted $\mathcal{I}$. The coalition $S \in \mathcal{C}$ is not itself an ideal coalition; however, we can associate it with the ideal coalition χ_S, and under this identification regard $\mathcal{C}$ as a subset of $\mathcal{I}$. Members $t \in T$ either belong or fail to belong to any coalition $S \in \mathcal{C}$; we can think of them as belonging to an ideal coalition with some "density" between zero and one.

Define the <u>supremum norm</u> $\|v^*\|'$ of a bounded ideal game $v^*: \mathcal{I} \to \mathbb{R}$ by

$$\|v^*\|' = \sup \{|v^*(\underset{\sim}{f})| : \underset{\sim}{f} \in \mathcal{I}\} .$$

Aumann and Shapley establish (see Proposition 22.16 and Remark 22.20) that there is a unique mapping that associates each game $v: \mathcal{C} \to \mathbb{R}$ which is a member of pNA' with an ideal game $v^*: \mathcal{I} \to \mathbb{R}$ in such a way that

(3.9) $(\alpha v + \beta w)^* = \alpha v^* + \beta w^*$,

(3.10) if f is a continuous real-valued function then $(f \circ v)^* = f \circ v^*$,

(3.11) $\mu^*(\underset{\sim}{f}) = \int \underset{\sim}{f} d\mu$,

and

(3.12) the mapping $v \mapsto v^*$ is continuous in the supremum norm

whenever $v, w \in pNA'$, $\alpha, \beta \in \mathbb{R}$, $\mu \in NA$, and $\underset{\sim}{f} \in \mathcal{I}$. Moreover, $v^*(\chi_S) = v(S)$ for each $S \in \mathcal{C}$, so we can regard v^* as an extension of v to the collection of ideal coalitions.

For any game v, coalition $S \in \mathcal{C}$, and $\theta \in [0,1]$, let

$$(3.13) \qquad \partial v^*(\theta,S) = \lim_{\tau \to 0} \frac{v^*(\theta\chi_T + \tau\chi_S) - v^*(\theta\chi_T)}{\tau}$$

(which may or may not exist). The following is a consequence of Propositions 44.22 and 43.13 of Aumann and Shapley.

<u>Theorem 3.14</u> (Aumann and Shapley): For each $v \in pNAD \cap pNA'$ and each $S \in \mathcal{C}$, $\partial v^*(\theta,S)$ exists for almost all $\theta \in [0,1]$ and is integrable over $[0,1]$ as a function of θ; moreover, the asymptotic value ϕv of v is defined by

$$(3.15) \qquad (\phi v)(S) = \int_0^1 \partial v^*(\theta,S)d\theta \quad \text{for each } S \in \mathcal{C} \quad .$$

The similarity between (3.15) and (3.6) is apparent when we take into account the fact that the characteristics of the members of a subset drawn "randomly" from $T = [0,1]$ will almost certainly be the same as those of T: if we choose an ideal coalition "at random" from $\mathcal{I}$ then it will almost certainly be of the form $\theta\chi_T$ with $\theta \in [0,1]$.

In the sequel we shall need to calculate the asymptotic values of truncations of games in pNAD $\cap$ pNA'; we shall use the following result.

Proposition 3.16: Let $\mu \in NA^1$, let $v \in pNAD \cap pNA'$, and let $\alpha \in (0,1)$. Define the game $q: C \to \mathbb{R}$ by

$$q(S) = \begin{cases} v(S) & \text{if } \mu(S) \geqq \alpha \\ 0 & \text{if } \mu(S) < \alpha \end{cases} .$$

Then $q \in$ ASYMP and the asymptotic value of q is defined by

$$(\phi q)(S) = v^*(\alpha\chi_T)\mu(S) + \int_0^1 \partial v^*(\theta,S)d\theta \quad \text{for all} \quad S \in C \quad .$$

Proof: Define $f: [0,1] \to [0,1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \geqq \alpha \\ 0 & \text{if } x < \alpha \end{cases} .$$

Then $f \circ \mu \in bv'NA$, so $q = (f \circ \mu)*v \in bv'NA*pNAD$. So by Corollary 3.8, $q \in$ ASYMP. But then the formula follows from Proposition 13.1 and Remark 12.1 of Aumann and Kurz [1977].

Now, Theorem 3.14 and Proposition 3.16 provide us with formulas for the asymptotic value ϕv of games which we shall subsequently study. These formulas involve the derivative $\partial v^*(\theta,S)$, and in general we should expect to have to know the form of the extension v^* before we could calculate $\partial v^*(\theta,S)$. However, for a certain class of games it is possible to give a formula for this derivative which does not directly involve the extension v^* (see Lemmas 3.17 and 3.18); for another class we can establish that the derivative behaves in a very regular manner (see Lemma 3.19).

<u>Lemma 3.17</u>: Let $v \in pNA$ be of the form $v = g \circ \nu$, where ν is an n-vector of members of NA. Then for each $S \in \mathcal{C}$, $\partial v^*(\theta,S)$ exists for almost all $\theta \in [0,1]$, and when it exists we have

$$\partial v^*(\theta,S) = g_{\nu(S)}(\theta\nu(T)) \quad ,$$

where $g_{\nu(S)}$ is the derivative of g in the direction $\nu(S)$.

<u>Proof</u>: The first claim follows immediately from Theorem 3.14. To establish the second claim, note that since $v \in pNA$, certainly g is continuous on the range of ν, so by (3.10) we have $(g \circ \nu)^* = g \circ \nu^*$. Now suppose $\partial v^*(\theta,S)$ exists for $\theta = \theta_o$. Then

$$\begin{aligned}\partial v^*(\theta_o,S) &= \lim_{\tau\to 0} \frac{v^*(\theta_o\chi_T + \tau\chi_S) - v^*(\theta_o\chi_T)}{\tau} \\ &= \lim_{\tau\to 0} \frac{(g\circ\nu^*)(\theta_o\chi_T + \tau\chi_S) - (g\circ\nu^*)(\theta_o\chi_T)}{\tau} \\ &= \lim_{\tau\to 0} \frac{g(\theta_o\nu(T) + \tau\nu(S)) - g(\theta_o\nu(T))}{\tau}\end{aligned}$$

(using (3.11)). Thus $g_{\nu(S)}(\theta_o\nu(T))$ exists (since $\partial v^*(\theta_o,S)$ exists by assumption), and we have $\partial v^*(\theta_o,S) = g_{\nu(S)}(\theta_o\nu(T))$, completing the proof.

If, under the assumptions made in the lemma, the components of ν are linearly independent, then note that $g_{\nu(S)}(\theta_o\nu(T)) = \sum_i \nu_i(S)g_i(\theta_o\nu(T))$ (where g_i is the partial derivative of g with respect to the i-th component).

If the game v is a function of a vector of games, rather than measures, then we have the following (which we shall use in Chapter 8).

Lemma 3.18: Let $v \in pNAD \cap pNA'$ be of the form $v = g \circ w$, where $w = (w_1,\ldots,w_n)$ is an n-vector of members of $pNAD \cap pNA'$, and $g: \mathbb{R}^n \to \mathbb{R}$ is differentiable. Then for each $S \in \mathcal{C}$, $\partial v^*(\theta,S)$ and $\partial w_i^*(\theta,S)$ for $i = 1,\ldots,n$ exist for almost all $\theta \in [0,1]$, and when they exist we have

$$\partial v^*(\theta,S) = \sum_{i=1}^{n} g_i(w^*(\theta\chi_T))\partial w_i^*(\theta,S) \quad ,$$

where g_i is the partial derivative of g with respect to the i-th component.

Proof: Once again the first claim follows immediately from Theorem 3.14. To establish the second claim, note that by (3.10) we have $(g \circ w)^* = g \circ w^*$. Now fix $S \in C$ and suppose that each $\partial w_i^*(\theta,S)$ for $i = 1,\ldots,n$ exists for $\theta = \theta_o$. Then

$$\partial v^*(\theta_o,S) = \lim_{\tau\to 0} \frac{v^*(\theta_o\chi_T + \tau\chi_S) - v^*(\theta_o\chi_T)}{\tau}$$

$$= \lim_{\tau\to 0} \frac{g(w^*(\theta_o\chi_T + \tau\chi_S)) - g(w^*(\theta_o\chi_T))}{\tau} \quad ,$$

or

$$\partial v^*(\theta_o, S) = \left(\frac{d}{d\tau} g(w^*(\theta_o \chi_T + \tau \chi_S))\right)_{\tau=0} \quad .$$

Now, by assumption

$$\left(\frac{d}{d\tau} w_i^*(\theta_o \chi_T + \tau \chi_S))\right)_{\tau=0}$$

exists for each $i = 1,\ldots,n$ and equals $\partial w_i^*(\theta_o, S)$, and g is differentiable, so by the chain rule we know that $(dg(w^*(\theta_o \chi_T + \tau \chi_S))/d\tau)_{\tau=0}$ exists, and we have

$$\left(\frac{d}{d\tau} g(w^*(\theta_o \chi_T + \tau \chi_S))\right)_{\tau=0} = \sum_{i=1}^{n} g_i(w^*(\theta_o \chi_T))\partial w_i^*(\theta_o, S) \quad ,$$

so that

$$\partial v^*(\theta_o, S) = \sum_{i=1}^{n} g_i(w^*(\theta_o \chi_T))\partial w_i^*(\theta_o, S) \quad .$$

This completes the proof.

Finally, we say that a game $v: \mathcal{C} \to \mathbb{R}$ is <u>homogeneous of degree</u> α if $v^*(k\chi_S) = k^\alpha v(S)$ for all $k \in [0,1]$ and all $S \in \mathcal{C}$. The proof of the following result is modelled closely on that of Lemma 27.2 of Aumann and Shapley.

Lemma 3.19: Let $v \in pNAD \cap pNA'$ be homogeneous of degree $\alpha \in [0,1]$. Then $\partial v^*(\theta,S)$ exists for all $\theta \in (0,1)$ and is homogeneous of degree $\alpha - 1$ in θ for each $S \in \mathcal{C}$.

Proof: By Theorem 3.14, for each $S \in \mathcal{C}$, $\partial v^*(\theta,S)$ exists for almost all $\theta \in (0,1)$. Suppose it exists for $\theta = \theta_o \in (0,1)$. Let $\theta_1 = k\theta_o$, for some $k \in (0,1)$. Then for each $S \in \mathcal{C}$

$$\frac{v^*(\theta_1\chi_T + \tau\chi_S) - v^*(\theta_1\chi_T)}{\tau} = \frac{k^\alpha v^*(\theta_o\chi_T + (\tau/k)\chi_S) - k^\alpha v^*(\theta_o\chi_T)}{\tau}$$

$$= k^{\alpha-1}\frac{v^*(\theta_o\chi_T + \tau'\chi_S) - v^*(\theta_o\chi_T)}{\tau'}$$

where $\tau' = \tau/k$. So since $\tau' \to 0$ as $\tau \to 0$,

$$\frac{v^*(\theta_o\chi_T + \tau'\chi_S) - v^*(\theta_o\chi_T)}{\tau'} \to \partial v^*(\theta_o,S)$$

as $\tau \to 0$, so that $\partial v^*(\theta_1,S)$ exists and

$$\partial v^*(\theta_1,S) = \partial v^*(k\theta_o,S) = k^{\alpha-1}\partial v^*(\theta_o,S) \quad .$$

Since we can take θ_o arbitrarily close to 1, this establishes that $\partial v^*(\theta,S)$ exists for all $\theta \in (0,1)$ and is homogeneous of degree $\alpha - 1$ for each fixed $S \in \mathcal{C}$.

3.2 The Harsanyi-Shapley Values of a Strategic Game

The primitive notion in the sequel is a strategic game, which we now define. A strategic game Γ consists of

(a) a measure space $(T,\mathcal{C},\mu)$, where T is the set of players, $\mathcal{C}$ is the collection of coalitions, and μ is the population measure;

(b) a set X^S for each $S \in \mathcal{C}$, called the strategy set of S; and

(c) for each $S \in \mathcal{C}$, $\sigma \in X^S$, and $\tau \in X^{T\setminus S}$ a function $\mathbf{h}^S_{\sigma\tau}: T \to \mathbb{R}$, with $\mathbf{h}^S_{\sigma\tau} = \mathbf{h}^{T\setminus S}_{\tau\sigma}$; $\mathbf{h}^S_{\sigma\tau}(t)$ is the payoff density of t when S uses the strategy σ and $T\setminus S$ the strategy τ.

We assume that

(3.20) $\mu(T) = 1$,

(3.21) $X^{\emptyset}$ is a singleton ,

and

(3.22) $\mathbf{h}^S_{\sigma\tau}$ is measurable in t for all $S \in \mathcal{C}$, $\sigma \in X^S$, and $\tau \in X^{T\setminus S}$.

Throughout, we shall write functions on T in boldface; if $\mathbf{f}$ and $\mathbf{g}$ are such functions we shall sometimes write $\int_S \mathbf{f}$ instead of $\int \mathbf{f}(t)\mu(dt)$, $\int \mathbf{f}$ instead of $\int_T \mathbf{f}$, and $\mathbf{fg}$ for the function on T whose value at t is $\mathbf{f}(t)\mathbf{g}(t)$. If $\mathbf{f}(t)$ is the payoff density of t, we say that $\mathbf{f}(t)\mu(dt)$ is his payoff.

In view of (3.21) we write $\mathbf{h}^{\emptyset}_{\sigma\tau} = \mathbf{h}^T_{\tau\sigma}$ as $\mathbf{h}^T_{\tau}$ for every $\tau \in X^T$. We sometimes use $\mathbf{h}^S(t)$ to denote the real-valued function on $X^S \times X^{T\setminus S}$ the value of which at (σ,τ) is $\mathbf{h}^S_{\sigma\tau}(t)$. A strategic game Γ is finite if T is finite, $\mathcal{C} = 2^T$, and $\mu(\{t\}) = 1/|T|$ for each $t \in T$; if $|T| = 2$ then Γ is a two-person strategic game. If Γ is a finite strategic game, we allow $\mathbf{h}^S_{\sigma\tau}(t)$ to be defined as an extended real number, the set of which we denote $\mathbb{R}^*$. If Γ is a two-person strategic game with $T = \{1,2\}$ then we call $h^{\{i\}}(i): X^{\{1\}} \times X^{\{2\}} \to \mathbb{R}^*$ the <u>payoff function of</u> i, for $i = 1,2$; if $h_{\sigma\tau}^{\{1\}}(1) + h_{\sigma\tau}^{\{2\}}(2) = 0$ for every $(\sigma,\tau) \in X^{\{1\}} \times X^{\{2\}}$, then Γ is a <u>two-person zero-sum</u> strategic game.

We can now explain the procedure for calculating the set of Harsanyi-Shapley values of a strategic game Γ; we shall discuss its motivation in the next subsection. We call a real-valued measurable function on $(T,\mathcal{C})$ which is a.e. (with respect to μ) positive a <u>comparison function</u>. The following is a brief outline of the procedure. Fix a comparison function λ. First we shall look at the two-person strategic game between S and $T\setminus S$, for each $S \in \mathcal{C}$, in which the payoff function of S is $\int_S \lambda h^S: X^S \times X^{T\setminus S} \to \mathbb{R}^*$. The Nash variable threat bargaining solution gives at least one pair of optimal threats in this game. Let the payoffs to S and $T\setminus S$ when they carry out these optimal threats be $q_\lambda(S)$ and $q_\lambda(T\setminus S)$. This process defines a game in coalitional form $q_\lambda: \mathcal{C} \to \mathbb{R}$ of which we can calculate the value. It may be that the payoffs assigned by this value cannot be attained in the game without transfers; if λ is such that they can, then the resulting payoff is a Harsanyi-Shapley value of Γ.

Formally, the procedure is as follows. Let λ be a comparison function. Let

$$H_\lambda^S(\sigma,\tau) = \int_S \lambda h_{\sigma\tau}^S - \int_{T\setminus S} \lambda h_{\sigma\tau}^S$$

for each $S \in \mathcal{C}$, $\sigma \in X^S$, and $\tau \in X^{T\setminus S}$. If, for a given $S \in \mathcal{C}$, $H^S_{\underset{\sim}{\lambda}}(\sigma,\tau)$ is defined (as an extended real number) for all $\sigma \in X^S$ and $\tau \in X^{T\setminus S}$, we can regard $H^S_{\underset{\sim}{\lambda}}: X^S \times X^{T\setminus S} \to \mathbb{R}^*$ as the payoff function of a two-person zero-sum game with player set $\{S, T\setminus S\}$; in this case we shall say that $H^S_{\underset{\sim}{\lambda}}$ is <u>defined</u>. Suppose that $(\sigma_o,\tau_o) \in X^S \times X^{T\setminus S}$ is such that

$$(3.23)\quad H^S_{\underset{\sim}{\lambda}}(\sigma,\tau_o) \leqq H^S_{\underset{\sim}{\lambda}}(\sigma_o,\tau_o) \leqq H^S_{\underset{\sim}{\lambda}}(\sigma_o,\tau) \quad \text{for all } \sigma \in X^S \text{ and all } \tau \in X^{T\setminus S}$$

and $H^S_{\underset{\sim}{\lambda}}(\sigma_o,\tau_o)$ is finite, for every $S \in \mathcal{C}$. (If $H^S_{\underset{\sim}{\lambda}}$ is defined, $H^S_{\underset{\sim}{\lambda}}(\sigma_o,\tau_o)$ is then the (unique) minmax value of the two-person zero-sum game.) Define the game $q_{\underset{\sim}{\lambda}}: \mathcal{C} \to \mathbb{R}$ by

$$(3.24)\quad q_{\underset{\sim}{\lambda}}(S) = \int_S \underset{\sim}{\lambda}\underset{\sim}{h}^S_{\sigma_o\tau_o} \quad \text{for each } S \in \mathcal{C} .$$

If $q_{\underset{\sim}{\lambda}}$ possesses an asymptotic value, and there exists $\sigma \in X^T$ such that

$$(3.25)\quad \int_S \underset{\sim}{\lambda}\underset{\sim}{h}^T_\sigma = (\phi q_{\underset{\sim}{\lambda}})(S) \quad \text{for each } S \in \mathcal{C} ,$$

then $\underset{\sim}{h}^T_\sigma: T \to \mathbb{R}$ is a <u>Harsanyi-Shapley value</u>, or simply a <u>value</u>, of Γ.

If there exists $(\sigma_o,\tau_o) \in X^S \times X^{T\setminus S}$ satisfying (3.23) with $H^S_{\underset{\sim}{\lambda}}(\sigma_o,\tau_o)$ finite, we shall say that (σ_o,τ_o) is a <u>finite saddle point</u> of $H^S_{\underset{\sim}{\lambda}}$. For each $S \in \mathcal{C}$ such a pair (σ_o,τ_o) is a pair of <u>optimal threats</u> in the two-person strategic game between S and $T\setminus S$ in which the payoff function of S is $\int_S \underset{\sim}{\lambda}\underset{\sim}{h}^S$ (see the discussion above). Each such game may possess a number of pairs of optimal threats. However, the following result shows that if there are two collections of optimal threats satisfying our conditions then the Harsanyi-Shapley values they generate (if indeed they do so) must be the same.

<u>Lemma 3.26</u>: Suppose there are pairs (σ_o^1,τ_o^1) and (σ_o^2,τ_o^2) which satisfy (3.23) for each $S \in \mathcal{C}$, and that $H_{\tilde{\lambda}}^S(\sigma_o^1,\tau_o^1)$ is finite for each $S \in \mathcal{C}$. Define the game $q_{\tilde{\lambda}}^i\colon \mathcal{C} \to \mathbb{R}$ by

$$\int_S \tilde{\lambda}\tilde{h}^S_{\sigma_o^i\tau_o^i} = q_{\tilde{\lambda}}^i(S) \quad \text{for each} \quad S \in \mathcal{C} \quad ,$$

for $i = 1,2$. Then if $q_{\tilde{\lambda}}^i \in \text{ASYMP}$ for $i = 1,2$, we have $\phi q_{\tilde{\lambda}}^1 = \phi q_{\tilde{\lambda}}^2$.

<u>Proof</u>: First, the minmax value of a two-person zero-sum game is unique, so $H_{\tilde{\lambda}}^S(\sigma_o^2,\tau_o^2) = H_{\tilde{\lambda}}^S(\sigma_o^1,\tau_o^1)$, and in particular is finite. Now, let the games $v_{\tilde{\lambda}}^i\colon \mathcal{C} \to \mathbb{R}$ for $i = 1,2$ be defined by

$$v_{\tilde{\lambda}}^i(S) = \tfrac{1}{2}(q_{\tilde{\lambda}}^i(S) + q_{\tilde{\lambda}}^i(T) - q_{\tilde{\lambda}}^i(T\backslash S)) = \tfrac{1}{2}(q_{\tilde{\lambda}}^i(S) + q_{\tilde{\lambda}}^{i\#}(S))$$
$$\text{for each} \quad S \in \mathcal{C} \quad ,$$

where the dual $v^{\#}\colon \mathcal{C} \to \mathbb{R}$ of a game $v\colon \mathcal{C} \to \mathbb{R}$ is defined by $v^{\#}(S) = v(T) - v(T\backslash S)$ for each $S \in \mathcal{C}$ ($v^{\#}(\emptyset) = 0$, so $v^{\#}$ is a game). For any game v, if ϕv exists then $\phi v^{\#}$ exists, and $\phi v = \phi v^{\#}$, so $v_{\tilde{\lambda}}^i$ possesses an asymptotic value for $i = 1,2$, and

$$\phi v_{\tilde{\lambda}}^i = (\phi q_{\tilde{\lambda}}^i + \phi q_{\tilde{\lambda}}^{i\#})/2 = \phi q_{\tilde{\lambda}}^i \quad \text{for} \quad i = 1,2 \quad .$$

But $q_{\lambda}^{1}(S) - q_{\lambda}^{1}(T\backslash S) = H_{\lambda}^{S}(\sigma_o^1,\tau_o^1) = H_{\lambda}^{S}(\sigma_o^2,\tau_o^2) = q_{\lambda}^{2}(S) - q_{\lambda}^{2}(T\backslash S)$ for each $S \in \mathcal{C}$, so that in particular $q_{\lambda}^{1}(T) = q_{\lambda}^{2}(T)$, and

$$v_{\lambda}^{1}(S) = v_{\lambda}^{2}(S) \quad \text{for all} \quad S \in \mathcal{C} \ .$$

So $\phi v_{\lambda}^{1} = \phi v_{\lambda}^{2}$. Hence $\phi q_{\lambda}^{1} = \phi q_{\lambda}^{2}$. This completes the proof.

Thus corresponding to any given comparison function the game Γ has at most one Harsanyi-Shapley value, and in order to determine if in fact it does possess one it suffices to locate one collection of pairs of optimal threats for which the game q_{λ} defined in (3.24) has an asymptotic value. Note however that in general Γ possesses a number of Harsanyi-Shapley values, corresponding to different comparison functions.

3.3 Harsanyi's Bargaining Solution and its Relation to the Set of Harsanyi-Shapley Values

As we saw in the previous section, the procedure for calculating the set of Harsanyi-Shapley values consists of two parts: first a coalitional form is derived from the strategic game for each choice of weights for the players (see (3.24)), and then the weights are chosen so that the value of the coalitional game is feasible (see (3.25)). If we are given a game which is in coalitional form to begin with, then we can apply the second part of the procedure directly; if we do so, the result is the set of "non-transferable utility (NTU) values" of the game in coalitional form. It has been argued by Shapley [1969], Aumann [1975] (Section 6), and others that this solution concept for a game in coalitional form has merits of its own. On the other hand, Roth [1978] produces a class of examples where the unique NTU value gives quite unsatisfactory results: players who are completely powerless, and whom we should expect to get zero payoffs if they were bargaining over the outcome, are assigned positive payoffs. Of course, the existence of one such class of examples does not mean that the solution concept is worthless (indeed, no solution concept can be expected to give good results for all games); but we believe that the examples do highlight the fact that the argument which has been made to support the solution is unsatisfactory. Equally clearly, this is no reason to abandon the solution altogether: there might be another interpretation which makes it very attractive. Moreover, in a number of applications (e.g. Aumann [1975], Aumann and Kurz [1977]) it has given intuitively appealing results which contribute to our understanding of the way an economy operates.

Aumann and Kurz, who use the full Harsanyi-Shapley procedure, argue that this also has merits of its own (see, for example, Aumann and Kurz [1978], pp. 144-145). They view the two steps of the procedure (see the previous paragraph) as having separate justifications: the solution applied to the transferable game for each set of weights is "reasonable", as is the method of finding "equilibrium" weights. We do not believe that such an argument provides a justification for the whole procedure, and prefer to think merely of applying a single solution concept to the original strategic game.

Harsanyi [1963] constructs a model of bargaining which applies to every finite strategic game. He generates a solution in two stages: first the basic bargaining process leads to a number of possible outcomes; these provide the basis for a further round of bargaining in which a unique outcome is selected. We regard the second stage as rather *ad hoc*, and shall be concerned only with the possible outcomes of the first stage; we shall call this set *Harsanyi's Bargaining Solution*, and define it precisely below. For games with transferable utility it coincides with the set of Harsanyi-Shapley values, which is in fact a singleton. For games without transferable utility this is not in general so, though Shapley was attempting to approximate Harsanyi's Bargaining Solution when he constructed his set of non-transferable utility values (see Shapley [1969], p. 260), on which the set of Harsanyi-Shapley values is based; it is not clear how good an approximation it is (we shall comment further on this matter below).

Given that Harsanyi's Bargaining Solution is based on a well-motivated model of bargaining, we regard it as a suitable solution concept for our purposes. The reason we do not use it is that it is very difficult to calculate. Instead, we use the set of Harsanyi-Shapley values, viewing it as an approximation to Harsanyi's Bargaining Solution. For this reason, we wish to understand the relation between the two. As we remarked earlier, when utility is transferable they coincide and generate a unique outcome; not only that, but Selten [1964] has provided an axiomatization of the solution in this case. We shall now define Harsanyi's Bargaining Solution in the general case, and remark on the circumstances under which the set of Harsanyi-Shapley values might approximate it well.

Let Γ be a finite strategic game, with $|T| = n$. For clarity, we shall write $h_t(\sigma^S,\tau^{T\setminus S})$ instead of $h^S_{\sigma\tau}(t)$ in the remainder of this section. We shall also make a distinction between the symbols $\subset$ and $\subseteq$: if $A \subset B$ then $A \neq B$. We shall assume that for every $S = \{t_1,\ldots,t_k\} \subset T$ and each strategy $\tau^{T\setminus S}$ of $T\setminus S$,

$$\{(h_{t_1}(\sigma^S,\tau^{T\setminus S}),\ldots,h_{t_k}(\sigma^S,\tau^{T\setminus S})) : \sigma^S \in X^S\}$$

is convex. Let λ be a comparison function (i.e. a real-valued function on T with $\lambda(t) > 0$ for all $t \in T$). As in the previous section, for each $\sigma^S \in X^S$ and $\tau^{T\setminus S} \in X^{T\setminus S}$ let

$$H_\lambda(\sigma^S,\tau^{T\setminus S}) = \sum_{t\in S}\lambda(t)h_t(\sigma^S,\tau^{T\setminus S}) - \sum_{t\in T\setminus S}\lambda(t)h_t(\sigma^S,\tau^{T\setminus S}) \quad .$$

For each $S \in C$ suppose there exists $(\sigma^S_o,\tau^{T\setminus S}_o) \in X^S \times X^{T\setminus S}$ such that

$$(3.27)\quad H_\lambda(\sigma^S_o,\tau^{T\setminus S}) = \max_{\sigma^S\in X^S}\ \min_{\tau^{T\setminus S}\in X^{T\setminus S}} \{ H_\lambda(\sigma^S,\tau^{T\setminus S}):$$

$$\lambda(t)(x^S(t)-y^S(t)) = \lambda(i)(x^S(i)-y^S(i)) \text{ if } t\in S \text{ and } i\in S, \text{ and}$$

$$\lambda(t)(x^{T\setminus S}(t)-y^{T\setminus S}(t)) = \lambda(i)(x^{T\setminus S}(i)-y^{T\setminus S}(i)) \text{ if } t\in T\setminus S \text{ and } i\in T\setminus S\}$$

where $x^S(t) = h_t(\sigma^S_o,\tau^{T\setminus S}_o)$ for each $t \in S$, $x^{T\setminus S}(t) = h_t(\sigma^S_o,\tau^{T\setminus S}_o)$ for each $t \in T\setminus S$, and

(3.28) $\underset{\sim}{y}^S(t) = \sum_{\substack{R \ni t \\ R \subseteq S}} (-1)^{s-r+1} \underset{\sim}{x}^R(t)$ (where $s = |S|$ and $r = |R|$) .

Then a payoff vector $\underset{\sim}{x}$ is a member of Harsanyi's Bargaining Solution if there exists $\sigma^T \in X^T$ such that

$$\underset{\sim}{x}(t) = \underset{\sim}{h}_t^T(\sigma^T, \tau^{\emptyset}) \text{ for all } t \in T ,$$

where $\tau^{\emptyset}$ is the single member of $X^{\emptyset}$. (This is equivalent to the definition on pp. 214-215 of Harsanyi [1963]: (3.27) above implies that Harsanyi's equations (10.1), (10.2), and (10.5) are satisfied, and conversely.)

Now, for each comparison function $\underset{\sim}{\lambda}$ and each $S \subseteq T$ define the coalitional form game $v_{\underset{\sim}{\lambda}}^S$ on S by

(3.29) $v_{\underset{\sim}{\lambda}}^S(R) = \sum_{t \in R} \lambda(t) \underset{\sim}{x}^R(t)$ for all $R \subseteq S$,

where the collection of payoff vectors $\{\underset{\sim}{x}^R\}$ for $R \subseteq T$ is defined by the simultaneous solution of the optimization problems in (3.27). We demonstrate in Appendix 1 that we can deduce from the constraints in (3.27) that for each $S \subseteq T$

(3.30) $\underset{\sim}{\lambda}(t) \underset{\sim}{x}^S(t) = (\phi v_{\underset{\sim}{\lambda}}^S)(\{t\})$ for each $t \in S$.

The fact that this is true for $S = T$ reveals the similarity between the set of Harsanyi-Shapley values and Harsanyi's Bargaining Solution. The only difference between the two lies in the way the "optimal threat" pair $(\sigma_o^S, \tau_o^{T \setminus S})$ for each $S \subseteq T$ is defined. Thus, (3.23) and (3.27) differ only in that the latter optimization problem is subject to some constraints, (3.24) and (3.29) for $S = T$ are the same, and (3.25) and (3.30) for $S = T$ are the same. So if it turns out when we calculate the threat pairs without restrictions that they in fact satisfy the constraints in (3.27), the set of Harsanyi-Shapley values coincides with Harsanyi's Bargaining Solution. This is the case, for example, when utility is transferable. For then by (3.27) for $S = T$ we have $\underset{\sim}{\lambda}(t) = 1$ for all $t \in T$, and from (3.29) and (3.30) for each $S \subseteq T$ only $\sum_{t \in S} \underset{\sim}{x}^S(t)$, and not $\underset{\sim}{x}^S(t)$ for each $t \in S$, is of relevance; but then having solved (3.27) ignoring the constraints, we can always choose values $\underset{\sim}{x}^S(t)$ so that the constraints are satisfied and $\sum_{t \in S} \underset{\sim}{x}^S(t)$ is preserved. When utility is not transferable, it is possible to make the following argument. In a large game, there is a sense in which "most" of the coalitions have almost the same composition as T, and the value depends only on the worths of these coalitions. So in such a case we need to verify that the constraints in (3.27) are satisfied only for coalitions whose composition is almost the same as that of T. But then if the structure of the game has a certain homogeneity, and the constraints are satisfied for some such coalitions, they will be satisfied for all; on the other hand, for feasibility, they must be satisfied for the coalition T. This argument suggests that in a game with a continuum of players which is homogeneous in some sense the set of Harsanyi-Shapley values coincides with Harsanyi's Bargaining Solution. To make it precise we should first have to generalize the definition of Harsanyi's Bargaining

Solution so that it can be applied to games with a continuum of players. Within a limited framework this can in fact be done, using the fact that not only do the constraints in (3.27) imply that (3.30) is satisfied, but also the converse of this statement is true. A limited result on equivalence in the presense of homogeneity can also certainly be established[1/]; the limited nature of the result does not justify the lengthy explanation which it requires, so we shall not go into the details here. It is an open question whether a stronger result, which covers some of the games which we shall subsequently study, can be established.

We have yet to explain how the constraints in (3.27) arise and why they take the form they do. To do so in detail would necessitate a lengthy argument, so we shall merely outline Harsanyi's [1963] model. The final payoff $\underset{\sim}{x}^T(t)$ of each player is made up of "dividends" $\underset{\sim}{w}^S(t)$ which he receives from each coalition S of which he is a member--i.e. $\underset{\sim}{x}^T(t) = \sum_{S \ni t} \underset{\sim}{w}^S(t)$. The dividends which any coalition pays must be backed by threats, in the sense that each coalition must possess a strategy which, given that its complement chooses its strategy in an optimal fashion, yields payoffs which allow it to pay the dividends it proposes. Thus the payoff $\underset{\sim}{x}^S(t)$ which the threat strategy of S gives $t \in S$ must be such that $\underset{\sim}{x}^S(t) = \sum_{\substack{R \ni t \\ R \subseteq S}} \underset{\sim}{w}^R(t)$. Now, the threat strategy of a coalition has to be agreed upon by all its members, and is thus constrained by any agreements which might be reached between its members as to the distribution of dividends. Harsanyi in fact assumes that each pair of players bargain over the distribution of the dividends which are available to them, given the dividends to all other players, in all coalitions to which they both belong. In this bargaining, the dividends received in small coalitions will serve as the threat points for the bargaining in the larger coalitions, and knowing this each player will only accept dividends in small coalitions which "protect" his position in the larger coalitions. The agreements thus reached between each pair of players generate the constraints in (3.27). (Using the fact that $\underset{\sim}{x}^S(t) = \sum_{\substack{R \ni t \\ R \subseteq S}} \underset{\sim}{w}^R(t)$ for each $S \subseteq T$, and (3.28), it can be seen that $\underset{\sim}{x}^S(t) - \underset{\sim}{y}^S(t)$ is just $\underset{\sim}{w}^S(t)$, the dividend which S pays t.) Note that the presence of the constraints in (3.27) makes the process of calculating the optimal threats much more complicated: what are allowed as threats in the game between S and $T \setminus S$ depend on the optimal threats in the games between all other pairs R and $T \setminus R$ --the whole collection of optimal threats has to be determined simultaneously.

We close this section by remarking that the fact that in a large game the value depends only on the worths of those coalitions whose composition is close to that of T derives from the implicit assumption that every player is equally willing to cooperate with any coalition. In some situations this may be a bad assumption: for example in a game in which there are two types of players whose characteristics are very different, it is quite conceivable that it would be to the advantage of each group for its members to refuse to cooperate with the members of the other group. Recently some attempts have been made to build game theoretic solution concepts which predict which coalitions will form "cartels" in this way;

it seems that the application of such solution concepts could well enrich our understanding of the phenomena which we study in the subsequent chapters. For the present, we have restricted ourselves to the use of the set of Harsanyi-Shapley values; the fact that it is based on assumptions which may not be applicable must be borne in mind.

CHAPTER 4: The Economic Framework

In this chapter we describe the economic model which will be used in the sequel, discuss some properties of efficient allocations, study a class of games derived from the economic model, and finally describe the solution concept we shall subsequently apply to economies. Throughout, when $x, y \in \mathbb{R}^n$ we write $x = (x^1,\ldots,x^n)$, $x \geqq y$ if and only if $x^i \geqq y^i$ for all i, $x > y$ if and only if $x \geqq y$ and $x \neq y$, and $x >> y$ if and only if $x^i > y^i$ for all i; we also write $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n: x \geqq 0\}$, and $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n: x >> 0\}$.

4.1 Markets and Efficient Allocations

A market $\mathcal{M}$ consists of

(a) a measure space $(T,\mathcal{C},\mu)$ where T is the set of agents, $\mathcal{C}$ the collection of coalitions, and $\mu \in NA^+$ is the population measure;

(b) a positive integer ℓ, the number of goods;

(c) a function $u: T \times \Omega \to \mathbb{R}_+$, where $\Omega = \mathbb{R}^\ell_+$; $u(t,\cdot): \Omega \to \mathbb{R}_+$ is the utility function of t, and we often write $u_t(x)$ instead of $u(t,x)$ when $x \in \Omega$; and

(d) an integrable function $\underset{\sim}{e}: T \to \Omega$, the initial endowment density.

We shall denote the partial derivative $\partial u_t/\partial x^i$ by u^i_t for $i = 1,\ldots,\ell$, and write $u'_t = (u^1_t,\ldots,u^\ell_t)$. We assume that $(T,\mathcal{C})$ is isomorphic to $([0,1],\mathcal{B})$, where (as before) $\mathcal{B}$ is the σ-field of Borel subsets of $[0,1]$. If we make no further assumptions about $\underset{\sim}{e}$ and u we shall refer to the market as a general market; whenever we refer to M simply as a <u>market</u> we mean that $\mu(T) = 1$ and $\underset{\sim}{e}$ and u satisfy the following five assumptions (such markets will be the main object of our studies).

(4.1) $\int \underset{\sim}{e} >> 0$.

(4.2) For each $t \in T$, u_t is increasing (i.e. $x > y \Rightarrow u_t(x) > u_t(y)$), concave, and continuous on Ω .

(4.3) $u_t(0) = 0$ for all $t \in T$.

(4.4) u is measurable in the product field $\mathcal{B}_\ell \times \mathcal{C}$, where $\mathcal{B}_\ell$ is the σ-field of Borel subsets of Ω .

(4.5) For each $t \in T$ and $i = 1,\ldots,\ell$, the partial derivative u_t^i of u_t exists and is continuous at each $x \in \Omega$ with $x^i > 0$.

In the following chapters we shall have to further restrict the characteristics of the utility functions u_t. We shall say that a market M is <u>bounded</u> if

(4.6) u is uniformly bounded (i.e. $\sup\{u(t,x)\colon t \in T,\ x \in \Omega\} < \infty$) ,

and

(4.7) $u_t(1,1,\ldots,1)$ is uniformly positive

(i.e. $\inf\{u_t(1,1,\ldots,1)\colon t \in T\} > 0$) .

We shall say that M is <u>homogeneous of degree</u> $\beta \in (0,1)$ if

(4.8) every utility function u_t is homogeneous of degree β (i.e. $u_t(kx) = k^\beta u_t(x)$ for every $k > 0$ and $x \in \Omega$) , and we shall say that M is <u>homogeneous</u> if it is homogeneous of degree β for some $\beta \in (0,1)$. Note that if u_t is homogeneous of degree $\beta \in (0,1)$ and $x \in \Omega$ is such that $x^i > 0$ for all $i = 1,\ldots,\ell$ then by Euler's Theorem we have

$$\sum_{i=1}^{\ell} x^i u_t^i(x) = \beta u_t(x) \quad .$$

In the general case in which some $x^i = 0$, the restriction of u_t to those components which are positive is homogeneous of degree β in those components (fixing the others at zero), so that we have

$$(4.9) \qquad \sum_{\{i:x^i>0\}} x^i u_t^i(x) = \beta u_t(x) \quad .$$

If g_t is a real-valued function on $\mathbb{R}^\ell$ for each $t \in T$ and $\underset{\sim}{x}: T \to \mathbb{R}^\ell$ then we write $g(\underset{\sim}{x})$ for the real-valued function on T the value of which at $t \in T$ is $g_t(\underset{\sim}{x}(t))$. If $\underset{\sim}{f}: T \to \mathbb{R}$ is measurable, we often write $f(S)$ rather than $\int_S \underset{\sim}{f}$ in the sequel. If $a \in \mathbb{R}^\ell_+$, an <u>S-allocation of</u> a is a measurable function $\underset{\sim}{x}: S \to \Omega$ with $\int_S \underset{\sim}{x} = a$; we call an S-allocation of $e(S)$ simply an <u>S-allocation</u>, and a T-allocation simply an <u>allocation</u>. When we say that a function $\underset{\sim}{f}$ on T is the unique such function satisfying a certain property, we mean that a function $\underset{\sim}{g}$ on T satisfies the property if and only if $\underset{\sim}{f} = \underset{\sim}{g}$ a.e.. (Here and subsequently "a.e." refers to the measure μ.) A <u>price vector</u>, or simply a <u>price</u>, is an element p of $\mathbb{R}^\ell_{++}$. Recall that we call a real-valued measurable function on (T,C) which is a.e. positive a

<u>comparison function</u>. If $\underset{\sim}{\lambda}$ is a comparison function, the market obtained from M by multiplying u_t by $\underset{\sim}{\lambda}(t)$ for each $t \in T$ is denoted $\underset{\sim}{\lambda}M$.

Now, we say an allocation $\underset{\sim}{x}$ is <u>efficient</u> if there is no allocation $\underset{\sim}{y}$ with $u_t(\underset{\sim}{y}(t)) > u_t(\underset{\sim}{x}(t))$ a.e.. We shall now state some properties of efficient allocations which are used in the sequel (see Section 9 of Aumann and Kurz [1977]). With each efficient allocation $\underset{\sim}{x}$ in M we can associate a price vector p, unique up to multiplication by a positive constant, such that

(4.10) the maximum of $u_t(x)$ over $\{x \in \Omega: px \leq p\underset{\sim}{x}(t)\}$ is a.e. achieved at $x = \underset{\sim}{x}(t)$.

Such a price is called an <u>efficiency price for</u> $\underset{\sim}{x}$. (We are assured that $p >> 0$ since we have assumed u_t to be increasing for each $t \in T$.) From (4.10) we can deduce the existence of a function $\underset{\sim}{\lambda}: T \to \mathbb{R}_+$ such that

(4.11) the maximum of $\underset{\sim}{\lambda}(t)u_t(x) - px$ over $x \in \Omega$ is a.e. achieved at $x = \underset{\sim}{x}(t)$.

Given p, $\underset{\sim}{\lambda}$ is unique if $\underset{\sim}{x}(t) >> 0$ a.e.. We call $(\underset{\sim}{\lambda},p)$ an <u>efficiency pair</u> for $\underset{\sim}{x}$; if $\underset{\sim}{\lambda}$ is a comparison function, we say that $\underset{\sim}{\lambda}$ is an <u>efficiency comparison function</u> for $\underset{\sim}{x}$. From (4.11) it follows that

(4.12) the maximum of $\int\underset{\sim}{\lambda}u(\underset{\sim}{y})$ over all allocations $\underset{\sim}{y}$ is achieved at $\underset{\sim}{x} = \underset{\sim}{y}$.

(Note that this maximum may be infinite, however.) Conversely, any one of the statements (4.10), (4.11), or (4.12) implies that $\underset{\sim}{x}$ is efficient. Also, from (4.11) we have a.e.

$$(4.13) \qquad \underset{\sim}{\lambda}(t)u_t^i(\underset{\sim}{x}(t)) = p^i \quad \text{if} \quad \underset{\sim}{x}^i(t) > 0 \quad .$$

Note that from (4.13) we have a.e. $\underset{\sim}{\lambda}(t) > 0$ if $\underset{\sim}{x}(t) \neq 0$, so if $\underset{\sim}{x}(t) \neq 0$ a.e. and $(\underset{\sim}{\lambda},p)$ is an efficiency pair for $\underset{\sim}{x}$, $\underset{\sim}{\lambda}$ is in fact a comparison function.

A transferable utility competitive equilibrium (t.u.c.e.) in M is a pair $(\underset{\sim}{x},p)$ where $\underset{\sim}{x}$ is an allocation and p is a price vector such that a.e.

(4.14) the maximum of $u_t(x) - p(x - \underset{\sim}{e}(t))$ over $x \in \Omega$ is attained at $x = \underset{\sim}{x}(t)$.

Though M may possess many t.u.c.e.'s, the competitive payoff density $u_t(\underset{\sim}{x}(t)) - p(\underset{\sim}{x}(t) - \underset{\sim}{e}(t))$ is unique (see Proposition 32.3 of Aumann and Shapley). From (4.11) we have that $(\underset{\sim}{x},p)$ is a t.u.c.e. in $\underset{\sim}{\lambda}M$ if and only if $(\underset{\sim}{\lambda},p)$ is an efficiency pair for $\underset{\sim}{x}$.

Finally, a Walrasian equilibrium in M is a pair $(\underset{\sim}{x},p)$ where $\underset{\sim}{x}$ is an allocation and p is a price vector such that p is an efficiency price for x and $p\underset{\sim}{x}(t) = p\underset{\sim}{e}(t)$ a.e.. If $(\underset{\sim}{x},p)$ is a Walrasian equilibrium in M then $\underset{\sim}{x}$ is a Walrasian allocation in M.

4.2 Market Games

Let M be a market. Define a game $r: C \to \mathbb{R}$ by

(4.15) $$r(S) = \sup \{\int_S u(\underset{\sim}{x}): \int_S \underset{\sim}{x} = e(S)\} \text{ for all } S \in C .$$

The game r is the market game derived from M. Under the assumptions we are making ((4.1) through (4.5)) it may be that r(S) is not attained, or is infinite, or both. In order to apply the results of Aumann and Shapley (and, in some cases, to make economic sense) we need to make an assumption which ensures that this does not happen. One such assumption was provided by Aumann and Perles [1965]; we shall now state their result. The market M is integrably sublinear if for each $\varepsilon > 0$ there exists an integrable function $\underset{\sim}{\eta}: T \to \mathbb{R}$ such that if $\|x\| \geqq \underset{\sim}{\eta}(t)$ then $u_t(x) \leqq \varepsilon\|x\|$, where we can take $\|\cdot\|$ to be the norm on $\mathbb{R}^\ell$ defined by $\|x\| = \sum_{i=1}^{\ell} |x^i|$ for each $x \in \mathbb{R}^\ell$. We also say that the function $u: T \times \Omega \to \mathbb{R}_+$ is integrably sublinear in this case. For each $S \in C$ define the function $u_S: \mathbb{R}_+^\ell \to \mathbb{R}_+$ by

$$u_S(a) = \sup \{\int_S u(\underset{\sim}{x}): \int_S \underset{\sim}{x} = a\} \text{ for each } a \in \mathbb{R}_+^\ell .$$

$u_S(a)$ is attained if it is finite and there is an S-allocation $\underset{\sim}{x}$ of a such that $u_S(a) = \int_S u(\underset{\sim}{x})$.

Proposition 4.16 (Aumann and Perles): If M is integrably sublinear then for each $S \in C$, $u_S(a)$ is attained for each $a \in \mathbb{R}_+^\ell$.

Since $r(S) = u_S(e(S))$, this gives us a condition under which $r(S)$ is attained for each $S \in \mathcal{C}$. Now, in the sequel we shall be concerned with the market game derived from the market λM for any comparison function λ; we shall denote this game by r_λ. Unfortunately, if M is integrably sublinear it may not be the case that λM is integrably sublinear for all comparison functions λ. In order to ensure that this is so, we need to make additional assumptions about M. In Chapter 5 we shall assume that M is bounded. The following result (Proposition 14.9 of Aumann and Kurz [1977]) is sufficient for our needs.

Proposition 4.17 (Aumann and Kurz): Let λ be a comparison function. Then if M is bounded and $r_\lambda(T)$ is finite, λM is integrably sublinear.

In Chapter 6 we shall assume that M is homogeneous. In this case we have the following.

Proposition 4.18: If M is homogeneous and $r(T)$ is finite then M is integrably sublinear.

Proof: From Corollary 3.7 of Hart [1979] we know that if $r(T)$ is finite then there exist $A > 0$ and an integrable function $B: T \to \mathbb{R}$ such that for each $t \in T$

$$u_t(y) \leqq B(t) + A\|y\| \quad \text{for all } y \in \Omega \quad .$$

(A is positive here, rather than simply nonnegative, because u_t is increasing for all $t \in T$.) From this we have

(4.19) $\quad u_t(y) - A\|y\| \leq \underset{\sim}{B}(t)$ for all $y \in \Omega$.

Since u_t is homogeneous of degree β for some $\beta \in (0,1)$, we know that $\max\{u_t(y) - A\|y\|: y \in \Omega\}$ is finite and attained; let it be attained at $\underset{\sim}{y}^*(t)$. From the first-order conditions for a maximum we can deduce that

$$u_t(\underset{\sim}{y}^*(t)) = A\|\underset{\sim}{y}^*(t)\| \quad .$$

Now let $\underset{\sim}{z}^*(t) = \underset{\sim}{y}^*(t)/\|\underset{\sim}{y}^*(t)\|$. Then $\|\underset{\sim}{z}^*(t)\| = 1$ and $u_t(\underset{\sim}{y}^*(t)) = \|\underset{\sim}{y}^*(t)\|^\beta u_t(\underset{\sim}{z}^*(t))$, so

$$\underset{\sim}{y}^*(t) = (\beta u_t(\underset{\sim}{z}^*(t))/A)^{1/(1-\beta)}$$

(since $A > 0$). Hence

$$\begin{aligned} & u_t(\underset{\sim}{y}^*(t)) - A\|\underset{\sim}{y}^*(t)\| \\ &= (\beta u_t(\underset{\sim}{z}^*(t))/A)^{\beta/(1-\beta)} u_t(\underset{\sim}{z}^*(t)) - A(\beta u_t(\underset{\sim}{z}^*(t))/A)^{1/(1-\beta)} \\ &= (\beta/A)^{\beta/(1-\beta)}(1-\beta)(u_t(\underset{\sim}{z}^*(t))^{1/(1-\beta)} \quad . \end{aligned}$$

So from (4.19)

(4.20) $\quad (u_t(z^*(t))^{1/(1-\beta)}$ is integrable .

Now, let $z \in \Omega$ be such that $\|z\| = 1$, and let $y = \|\underset{\sim}{y}^*(t)\|z$. Then $\|y\| = \|y^*(t)\|$, and

$$u_t(y) - A\|y\| \leq u_t(\underset{\sim}{y}^*(t)) - A\|\underset{\sim}{y}^*(t)\| \quad ,$$

so $u_t(y) \leq u_t(\underset{\sim}{y}^*(t))$. Hence $\|\underset{\sim}{y}^*(t)\|^\beta u_t(z) \leq \|\underset{\sim}{y}^*(t)\|^\beta u_t(\underset{\sim}{z}^*(t))$, so that $u_t(z) \leq u_t(\underset{\sim}{z}^*(t))$. Thus for all $t \in T$,

$$(4.21) \qquad u_t(z) \leq u_t(\underset{\sim}{z}^*(t)) \text{ for all } z \in \Omega \text{ with } \|z\| = 1 \quad .$$

Now fix $\varepsilon > 0$ and let $\underset{\sim}{\eta}: T \to \mathbb{R}$ be defined by

$$\underset{\sim}{\eta}(t) = (u_t(\underset{\sim}{z}^*(t))/\varepsilon)^{1/(1-\beta)} \text{ for each } t \in T \quad ;$$

$\underset{\sim}{\eta}$ is integrable by (4.20). Suppose $x \in \Omega$ is such that $\|x\| \geq \underset{\sim}{\eta}(t)$. Then using (4.21),

$$\|x\| \geq (u_t(\underset{\sim}{z}^*(t))/\beta)^{1/(1-\beta)} \geq (u_t(x/\|x\|)/\varepsilon)^{1/(1-\beta)}$$
$$= (u_t(x)/\varepsilon\|x\|^\beta)^{1/(1-\beta)} \quad ,$$

so that $\|x\|^{1/(1-\beta)} \geq (u_t(x)/\varepsilon)^{1/(1-\beta)}$, or $\|x\| \geq u_t(x)/\varepsilon$. Hence $u_t(x) \leq \varepsilon\|x\|$. Thus we have shown that

$$\|x\| \geq \underset{\sim}{\eta}(t) \Rightarrow u_t(x) \leq \varepsilon\|x\| \quad ,$$

or in other words M is integrably sublinear. This completes the proof.

<u>Corollary 4.22</u>: Let $\underset{\sim}{\lambda}$ be a comparison function. Then if M is homogeneous and $r_{\underset{\sim}{\lambda}}(T)$ is finite, $\underset{\sim}{\lambda}M$ is integrably sublinear.

<u>Proof</u>: If M is homogeneous then $\lambda \underset{\sim}{M}$ is homogeneous; the result then follows from Proposition 4.18.

Now, in the sequel we shall need results which give conditions under which the game r possesses an asymptotic value. Aumann and Shapley studied this question extensively; we shall now state some of their results which we shall subsequently use. The market M is of <u>finite type</u> if there is a finite set of functions $\{f_1,\ldots,f_n\}$ with $f_i\colon \Omega \to \mathbb{R}_+$ for all $i = 1,\ldots,n$, such that for every $t \in T$, $u_t = f_i$ for some $i = 1,\ldots,n$. Recall that unless we explicitly say otherwise, a market M is assumed to satisfy (4.1) through (4.5). The following is a consequence of Proposition 31.5 of Aumann and Shapley.

<u>Proposition 4.23</u> (Aumann and Shapley): If M is integrably sublinear and of finite type then $r \in pNA$.

From Theorem 3.7 we can conclude that r possesses an asymptotic value. If M is not necessarily of finite type then under some circumstances we can approximate r by a market game derived from a finite type market, as in the following result, which is a consequence of Propositions 40.24, 35.6, and 36.3 of Aumann and Shapley (Proposition 36.3 is needed since we are assuming that u_t is concave for all $t \in T$).

<u>Proposition 4.24</u> (Aumann and Shapley): If M is integrably sublinear and

(4.25) for all $t \in T$ either $\underset{\sim}{e}(t) \gg 0$ or $\underset{\sim}{e}(t) = 0$, then for each $\varepsilon > 0$ there is an integrably sublinear finite type market $\hat{M}$ which differs from M only in the utility functions of the agents, such that

$$\|r - \hat{r}\| < \varepsilon \quad ,$$

where $\hat{r}$ is the market game derived from $\hat{M}$.

From these last two results we have the following (which is Proposition 40.26 of Aumann and Shapley).

Corollary 4.26 (Aumann and Shapley): If M is integrably sublinear and (4.25) is satisfied, then $r \in pNA$.

If we drop assumption (4.25) then we have the following (which is a consequence of Corollary 45.8 and Proposition 45.10 of Aumann and Shapley).

Proposition 4.27 (Aumann and Shapley): If M is integrably sublinear then $r \in pNAD \cap pNA'$.

Given Corollary 3.8 these results establish that r possesses an asymptotic value if M is integrably sublinear. Not only does it possess a value in this case, but we can give an expression for that value in terms of the components of M. The following is a consequence of Propositions 45.10, 31.7, and 32.3 of Aumann and Shapley, and Lemma 3.18.

Proposition 4.28 (Aumann and Shapley): If M is integrably sublinear then r is homogeneous of degree one, so that for all $S \in \mathcal{C}$, $\partial r^*(\theta,S)$ is a constant independent of θ and for any $\theta \in [0,1]$ we have

$$(\phi r)(S) = \partial r^*(\theta, S) \quad .$$

In fact, ϕr coincides with the (unique) competitive payoff distribution, so that we have

$$(\phi r)(S) = \int_S (u(\underset{\sim}{x}) - p(\underset{\sim}{x} - \underset{\sim}{e}))$$

for all $S \in \mathcal{C}$, where $(\underset{\sim}{x}, p)$ is a t.u.c.e. in M.

In Chapter 6 we shall have occasion to consider (in the proof of Proposition 6.14) a general market in which u_t is merely non-decreasing, rather than increasing, for some $t \in T$. Theorem D of Hart [1977] establishes that any such integrably sublinear general market possesses an asymptotic value, but this result is not strong enough for our purposes--we shall need a result which generalizes Proposition 4.24. We shall lead up to this result--Proposition 4.50--via a series of lemmas.

We shall use the following two assumptions:

(4.29) for each $t \in T$ either u_t is increasing, concave, and continuous on Ω, or $u_t(x) = 0$ for all $x \in \Omega$,

and

(4.30) $Z \in \mathcal{C}$ where $Z = \{t \in T: u_t(x) = 0 \text{ for all } x \in \Omega\}$;

we shall call a general market which satisfies (4.1), (4.3), (4.4), (4.29), and (4.30) a <u>quasi-market with zero-utility agents</u>. The main object of our attention is now an integrably sublinear quasi-market with zero-utility agents which we shall denote M_o. We shall use the symbols (T,C,μ), ℓ, $\underset{\sim}{e}$, and u to denote the components of M_o. <u>From now through the end of this section these symbols will be reserved for the general market M_o, rather than for a market satisfying (4.1) through (4.5)</u>. We shall also write $R = T\backslash Z$. Note that we do not assume that $\mu(T) = 1$. If $\mu(Z) = 0$ and (4.25) is satisfied then we are back to the case covered by Proposition 4.24, since the behavior of M_o on R conforms with (4.1) through (4.5). However, if $\mu(Z) > 0$ then since the utility functions of the agents in Z are not increasing, we need to generalize Proposition 4.24. To do so, we shall first approximate the market M_o by a related market M_α in which the utility functions of all agents are increasing. Then we shall use Proposition 4.24 to conclude that there is a finite type approximation to M_α in which all the agents in Z have utility functions which are identically zero. This will give us a finite type approximation to M_o in which all the agents in Z have utility functions which are identically zero.

Thus, for any $\alpha > 0$ let M_α be the market with player space (T,C,μ), ℓ goods, and endowment density $\underset{\sim}{e}$ (i.e. M_α is the same as M_o in these respects), in which the utility function of each agent $t \in T$ is u_t^α, where

(4.31) if $t \in R$ then $u_t^\alpha = u_t$

and

(4.32) if $t \in Z$ then $u_t^\alpha = u_o^\alpha$, where u_o^α is increasing, concave, and for each $i = 1,\ldots,\ell$, $u_o^{\alpha i}$ exists and is continuous at each $x \in \Omega$ for which $x^i > 0$ and

$$u_o^\alpha(x) < \alpha \quad \text{for all} \quad x \in \Omega, \text{ and}$$

$$u_o^{\alpha i}(x) < \alpha \quad \text{for all} \quad x \in \Omega, \text{ and all} \quad i = 1,\ldots,\ell \quad ,$$

and u^α satisfies (4.4). Under these assumptions u^α satisfies all the conditions of Proposition 4.24. In M_o, each agent $t \in Z$ has a utility function which is identically zero on Ω; in M_α he has a function which is bounded by α and the derivatives of which are also bounded by α.

For each $S \in \mathcal{C}$ define $u_S^\alpha\colon \mathbb{R}_+^\ell \to \mathbb{R}_+$ by

$$u_S^\alpha(a) = \sup\, \{\int_S u^\alpha(\underset{\sim}{x}) \colon \int_S \underset{\sim}{x} = a\} \quad \text{for each} \quad a \in \mathbb{R}_+^\ell \quad .$$

The partial derivative of u_S with respect to a^i is denoted u_S^i, and the vector of partials is written u_S'; similarly for u_S. The coalitional form of the market M_o is then $w\colon \mathcal{C} \to \mathbb{R}_+$ defined by $w(S) = u_S(e(S))$ for each $S \in \mathcal{C}$, and that of M_α is $w_\alpha\colon \mathcal{C} \to \mathbb{R}_+$ defined by $w_\alpha(S) = u_S^\alpha(e(S))$ for each $S \in \mathcal{C}$.

We need to introduce one more concept (from Aumann and Shapley) before beginning our formal arguments. If $f\colon \Omega \to \mathbb{R}$, let

$$\|f\| = \sup_{x\in\Omega} [|f(x)|/(1 + \sum_{i=1}^{\ell} x^i)] \quad .$$

Let $u: T \times \Omega \to \mathbb{R}_+$ be integrably sublinear and satisfy (4.2) through (4.5). Then for $\delta > 0$ we say that $\hat{u}: T \times \Omega \to \mathbb{R}_+$ is a δ-<u>approximation</u> to $u: T \times \Omega \to \mathbb{R}_+$ if $\hat{u}$ satisfies (4.2) through (4.5) and there exists $E \in \mathcal{C}$ such that $\mu(E) \leq \delta$ and

$$\|\hat{u}_t - u_t\| \leq \delta \quad \text{for all} \quad t \in T \setminus E$$

and

$$\hat{u}_t(x) = \sqrt{\sum_{i=1}^{\ell} x^i} \quad \text{for all} \quad t \in E \quad .$$

If $K \in \mathcal{C}$ and $K \subset T$, and $u: T \times \Omega \to \mathbb{R}_+$ is integrably sublinear and satisfies (4.2) through (4.5) then we say that $\hat{u}: T \times \Omega \to \mathbb{R}_+$ is a δ-<u>approximation on</u> K <u>to</u> u if the restriction of $\hat{u}$ to $K \times \Omega$ is a δ-approximation to the restriction of u to $K \times \Omega$ and $u(t,x) = \hat{u}(t,x)$ for all $t \in T \setminus K$ and all $x \in \Omega$. If $\hat{u}$ is a δ-approximation on K to u then we shall write $\hat{w}: \mathcal{C} \to \mathbb{R}_+$ for the game defined by

$$\hat{w}(S) = \hat{u}_S(e(S)) \quad \text{for each} \quad S \in \mathcal{C} \quad .$$

If $\hat{u}$ is a δ-approximation on R to u then we define $\hat{u}^\alpha$ to be the δ-approximation on R to u^α given by $\hat{u}^\alpha_t = \hat{u}_t$ if $t \in R$ (and $\hat{u}^\alpha_t = u^\alpha_t$ if $t \in Z$) (see Diagram 1). We write $\hat{w}_\alpha$ for the market game associated with $\hat{u}^\alpha$.

We shall now show that given some positive number β, if α and δ are sufficiently small $\hat{w}_\alpha(S)$ is small whenever $\mu(S \cap R)$ is sufficiently small and $\hat{u}$ is a δ-approximation on R to u (see Corollary 4.35),

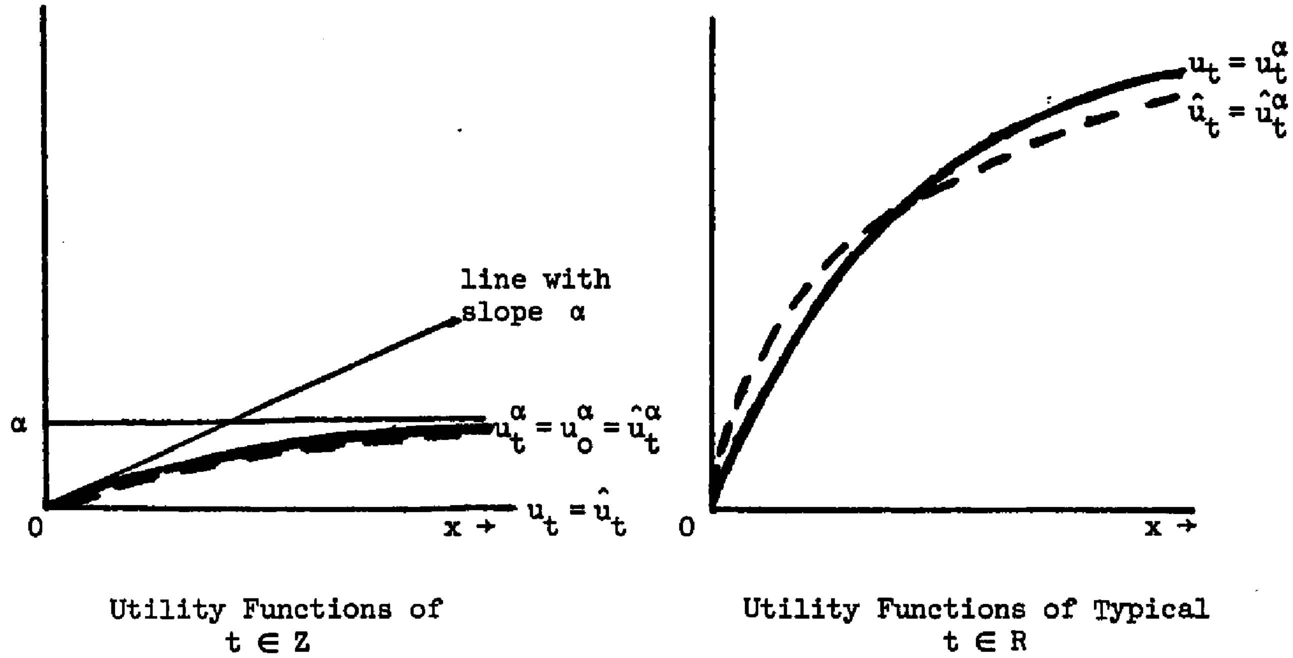


Diagram 1: The One Commodity Case

and $\hat{w}_\alpha(S) = \hat{w}(S)$ whenever $\mu(S \cap R) > \beta$ and $\hat{u}$ is a δ-approximation on R to u (see Corollary 4.39). These results will allow us to show that for sufficiently small α and δ the games $\hat{w}$ and $\hat{w}_\alpha$ are close in variation whenever $\hat{u}$ is a δ-approximation on R to u (see Lemma 4.43). Through the proof of Lemma 4.43 we shall assume that $\mu(Z) > 0$.

Lemma 4.33: For each $\varepsilon > 0$ there exists $\gamma > 0$ such that if $\mu(S) < \gamma$ then $u_S(e(T)) < \varepsilon$.

Proof: From Proposition 4.16, $u_S(e(T))$ is attained, so we have

$$u_S(e(T)) = \int_S u(\underset{\sim}{x})$$

where $\underset{\sim}{x}: T \to \Omega$ is integrable and $\int \underset{\sim}{x} = e(T)$. From Lemma 37.9 of Aumann and Shapley we know that if $\rho > 0$ then there exists an integrable function $\underset{\sim}{\zeta}_\rho : T \to \mathbb{R}_+$ such that for all $t \in T$ and $x \in \Omega$,

$$u_t(x) \leqq \rho(\underset{\sim}{\zeta}_\rho(t) + \|x\|) \quad .$$

Now, fix $\varepsilon > 0$. Then for any $\rho > 0$ we have

$$u_S(e(T)) = \int_S u(\underset{\sim}{x}) \leqq \int_S \rho(\underset{\sim}{\zeta}_\rho(t) + \|\underset{\sim}{x}(t)\|) \leqq \int_S \rho \underset{\sim}{\zeta}_\rho(t) + \rho\|e(T)\| \quad .$$

Set $\rho = \varepsilon/2\|e(T)\|$. We know (see for example Proposition 13 on p. 85 of Royden [1968]) that

there exists $\gamma > 0$ such that if $\mu(S) < \gamma$ then $\int_S \rho \underset{\sim}{\zeta}_\rho < \varepsilon/2$

(since $\rho \underset{\sim}{\zeta}_\rho$ is a fixed nonnegative integrable function on T). Using this γ we conclude that if $\mu(S) < \gamma$ then

$$u_S(e(T)) < \varepsilon/2 + \varepsilon/2 = \varepsilon \quad ,$$

completing the proof of the lemma.

<u>Corollary 4.34</u>: For each $\varepsilon > 0$ there exist $\gamma > 0$ and $\delta > 0$ such that if $\mu(S) < \gamma$ and $\hat{u}$ is a δ-approximation on R to u then $\hat{u}_S(e(T)) < \varepsilon$.

<u>Proof</u>: Fix $\varepsilon > 0$, and take the γ given in Lemma 4.33 corresponding to $\varepsilon/2$. Then we have

$$\mu(S) < \gamma \Rightarrow u_S(e(T)) < \varepsilon/2 \quad .$$

Now take the δ given in Proposition 37.11 of Aumann and Shapley corresponding to $\varepsilon/2(1 + \sum e^i(T))$. Then if $\hat{u}$ is a δ-approximation on R to u (and hence certainly a δ-approximation to u),

$$|\hat{u}_S(e(T)) - u_S(e(T))| < \varepsilon/2 \quad \text{for all} \quad S \in \mathcal{C} \quad .$$

Combining these two conditions we conclude that if $\hat{u}$ is a δ-approximation on R to u, then

$$\mu(S) < \gamma \Rightarrow \hat{u}_S(e(T)) < \varepsilon \quad ,$$

as was to be shown.

Corollary 4.35: For each $\varepsilon > 0$ there exist $\gamma > 0$, $\delta > 0$, and $\alpha > 0$ such that if $\mu(S \cap R) < \gamma$ and $\hat{u}$ is a δ-approximation on R to u then $\hat{w}_\alpha(S) < \varepsilon$.

Proof: From Proposition 4.16, $\hat{w}_\alpha(S)$ is attained for each $\alpha > 0$, each δ-approximation $\hat{u}$ on R to u, and each $S \in \mathcal{C}$, so we have

$$\hat{w}_\alpha(S) = \int_S \hat{u}^\alpha(\tilde{x})$$

where $\tilde{x}\colon T \to \Omega$ is integrable and $\int_S \tilde{x} = e(S)$. So

$$\hat{w}_\alpha(S) = \int_{S\cap Z} \hat{u}^\alpha(\tilde{x}) + \int_{S\cap R} \hat{u}^\alpha(\tilde{x}) = \int_{S\cap Z} u^\alpha(\tilde{x}) + \int_{S\cap R} \hat{u}(\tilde{x})$$

$$\leqq \alpha\mu(S \cap Z) + \hat{u}_{S\cap R}(e(S)) \quad ,$$

and hence $\hat{w}_\alpha(S) \leqq \alpha\mu(Z) + \hat{u}_{S\cap R}(e(T))$. (We have used the fact that $u_t^\alpha(x) < \alpha$ if $t \in Z$ and $x \in \Omega$.)

Now, fix $\varepsilon > 0$, take the γ and δ given in Corollary 4.34 which correspond to $\varepsilon/3$, and set $\alpha = \varepsilon/3\mu(Z)$. Then if $\mu(S \cap R) < \gamma$ and $\hat{u}$ is a δ-approximation on R to u, $\hat{u}_{S\cap R}(e(T)) < \varepsilon/3$, and hence

$$\hat{w}_\alpha(S) \leqq \varepsilon/3 + \varepsilon/3 < \varepsilon \quad ,$$

as was to be shown.

<u>Lemma 4.36</u>: For each $\beta > 0$ there exists $\alpha > 0$ such that if $\mu(S \cap R) \geqq \beta$ and $e^j(S) > 0$ then $u_S^j(e(S)) > \alpha$.

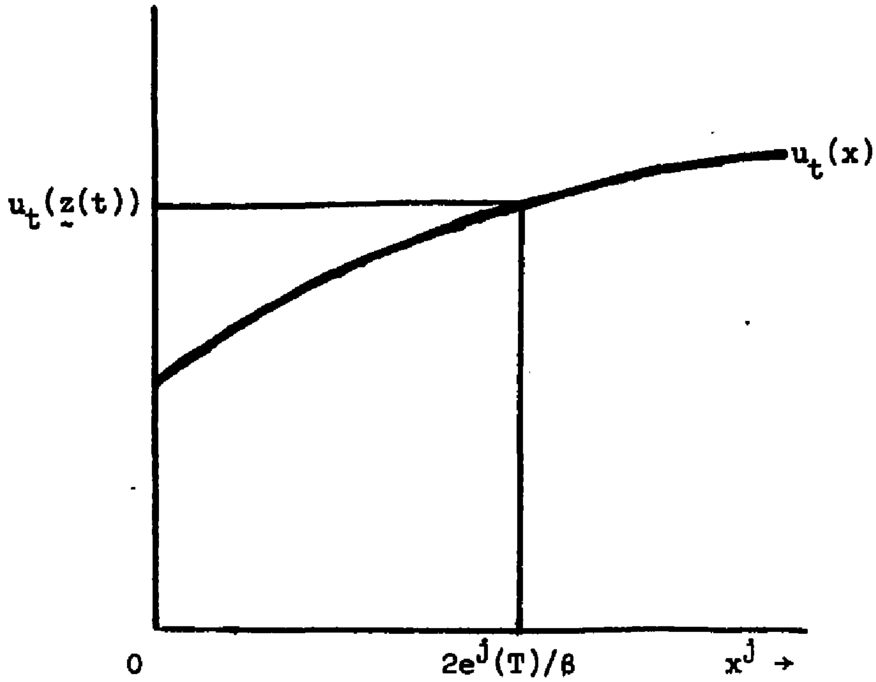
<u>Proof</u>: If $e^i(S) = 0$ for some i, we can just ignore the i-th good in our analysis of $u_S(e(S))$; thus we can assume without loss of generality that $e(S) >> 0$ in this proof. Now, fix $\beta > 0$ and some $j \in \{1,\ldots,\ell\}$. Let $\underset{\sim}{y}$ be an S-allocation at which $u_S(e(S))$ is attained, and let $\underset{\sim}{z}: T \to \Omega$ be defined by

$$\underset{\sim}{z}(t) = (\underset{\sim}{y}^1(t),\ldots,\underset{\sim}{y}^{j-1}(t),\ 2e^j(T)/\beta,\ \underset{\sim}{y}^{j+1}(t),\ldots,\underset{\sim}{y}^\ell(t))$$

for each $t \in T$ (see Diagram 2). Since u is measurable over $\mathcal{B}_\ell \times \mathcal{C}$ (see (4.4)), $u_t^j(\underset{\sim}{z}(t))$ is measurable in t, so we can define the function $g: \mathbb{R}_+ \to \mathbb{R}_+$ by

$$g(b) = \mu\{t \in R: u_t^j(\underset{\sim}{z}(t)) \leqq b\} \quad \text{for each} \quad b \in \mathbb{R}_+ \quad .$$

g is continuous from the right, and $g(0) = 0$ (because u_t is increasing for every $t \in R$), so



<u>Diagram 2</u>: Cross-Section of Typical u_t with $x^i = \tilde{y}^i(t)$ for all $i \neq j$

for all $\varepsilon > 0$ there exists $\alpha_j > 0$ such that
$\mu\{t \in R: u_t^j(\tilde{z}(t)) \leqq \alpha_j\} < \varepsilon$.

Let α_j correspond to $\varepsilon = \beta/3$. Then

(4.37) $\quad \mu\{t \in R: u_t^j(\tilde{z}(t)) \leqq \alpha_j\} < \beta/3$.

Let $S \in \mathcal{C}$ be such that $\mu(S \cap R) \geqq \beta$, and define the set $D \subset S \cap R$ by

$$D = \{t \in S \cap R: u_t^j(\tilde{z}(t)) > \alpha_j\} \quad .$$

Then from (4.37) we have $\mu(D) > 2\beta/3$. We shall now show (by contradiction) that $u^j_S(e(S)) > \alpha_j$. From Proposition 38.5 of Aumann and Shapley, setting $x = \underset{\sim}{z}(t)$ and using concavity, we have

$$u^j_{S\cap R}(e(S)) \geqq \frac{u_t(\underset{\sim}{z}(t)) - u_t(\underset{\sim}{y}(t))}{2e^j(T)/\beta - \underset{\sim}{y}^j(t)} \geqq u^j_t(\underset{\sim}{z}(t))$$

for almost all $t \in S \cap R$ for which $\underset{\sim}{y}^j(t) < 2e^j(T)/\beta$.

But for all $t \in D$, $u^j_t(\underset{\sim}{z}(t)) > \alpha_j$ and $\mu(D) > 0$, so if we are to have $u^j_{S\cap R}(e(S)) \leqq \alpha_j$ then for almost all $t \in D$ it must be the case that $\underset{\sim}{y}^j(t) \geqq 2e^j(T)/\beta > 0$. But then (from Proposition 38.5 of Aumann and Shapley again)

$$u^j_{S\cap R}(e(S)) = u^j_t(\underset{\sim}{y}(t)) \quad \text{for almost all} \quad t \in D \quad .$$

So if $u^j_{S\cap R}(e(S)) \leqq \alpha_j$ then $u^j_t(\underset{\sim}{y}(t)) \leqq \alpha_j$ for almost all $t \in D$; but $u^j_t(\underset{\sim}{z}(t)) > \alpha_j$ for all $t \in D$, so using concavity we must have $\underset{\sim}{y}^j(t) > 2e^j(T)/\beta$ for almost all $t \in D$. But then

$$\int_{S\cap R} \underset{\sim}{y}^j \geqq \int_D \underset{\sim}{y}^j > 2\mu(D)e^j(T)/\beta > 4e^j(T)/3 \quad .$$

But $\underset{\sim}{y}^j(t) = 0$ if $t \in Z$, so $\int_{S\cap R} \underset{\sim}{y}^j = \int_S \underset{\sim}{y}^j = e^j(S)$ and we have a contradiction. Hence we must have $u^j_{S\cap R}(e(S)) > \alpha_j$. But since $\underset{\sim}{y}^j(t) = 0$ if $t \in Z$, $u_{S\cap R}(a) = u_S(a)$ for all $a \in \mathbb{R}^\ell_+$ and all $S \in \mathcal{C}$, so $u^j_{S\cap R}(e(S)) = u^j_S(e(S))$. Hence we have shown that $u^j_S(e(S)) > \alpha_j$ if $\mu(S \cap R) \geqq \beta$.

Now set $\alpha = \min_j \alpha_j$. Then $\alpha > 0$, and if $\mu(S \cap R) \geqq \beta$ then $u'_S(e(S)) > (\alpha,\alpha,\ldots,\alpha)$. This completes the proof.

Corollary 4.38: For each $\beta > 0$ there exist $\alpha > 0$ and $\delta > 0$ such that if $\mu(S \cap R) \geqq \beta$ and $\hat{u}$ is a δ-approximation on R to u then $\hat{u}^j_S(e(S)) > \alpha$ if $e^j(S) > 0$.

Proof: Once again we can restrict attention to the subset of goods for which $e^i(S) > 0$, and so can without loss of generality assume that $e(S) \gg 0$ in this proof. Fix $\beta > 0$, and let α_o be the value of α given in Lemma 4.36. Choose $\alpha > 0$ such that $\alpha < \min \{\alpha_o/2, \beta, \min_i e^i(S)\}$. Then from Lemma 4.36

$$\mu(S \cap R) \geqq \beta \Rightarrow u'_S(e(S)) > (\alpha_o,\alpha_o,\ldots,\alpha_o) > (2\alpha,2\alpha,\ldots,2\alpha) \quad .$$

Now choose the $\delta > 0$ given in Proposition 38.14 of Aumann and Shapley which corresponds to α. Then if $\hat{u}$ is a δ-approximation on R to u and $\mu(S) \geqq \alpha$, we have

$$|\hat{u}^j_S(e(S)) - u^j_S(e(S))| < \alpha \quad \text{for all} \quad j \in \{1,\ldots,\ell\} \ .$$

(We can apply this result of Aumann and Shapley since we chose α so that $\alpha < \min_i e^i(S)$.) But we chose α so that $\beta > \alpha$, and $\mu(S) \geqq \beta$ if $\mu(S \cap R) \geqq \beta$, so we have

$$\mu(S \cap R) \geqq \beta \Rightarrow \hat{u}'_S(e(S)) > (\alpha,\alpha,\ldots,\alpha) \quad ,$$

which is what we needed to establish.

Corollary 4.39: For each $\beta > 0$ there exist $\bar{\alpha} > 0$ and $\delta > 0$ such that if $\mu(S \cap R) \geqq \beta$ and $\hat{u}$ is a δ-approximation on R to u then $\hat{w}_\alpha(S) = \hat{w}(S)$ for all $\alpha \leqq \bar{\alpha}$.

Proof: If $e^i(S) = 0$ for some i, this good can certainly not contribute to any difference between $\hat{w}_\alpha(S)$ and $\hat{w}(S)$, so we can restrict attention to the set of i for which $e^i(S) > 0$, and can thus assume without loss of generality that $e(S) >> 0$. Fix $\beta > 0$ and let $\bar{\alpha} > 0$ and $\delta > 0$ be the values of α and δ given in Corollary 4.38. Then if $\hat{u}$ is a δ-approximation on R to u,

$$\mu(S \cap R) \geqq \beta \Rightarrow \hat{u}_S'(e(S)) > (\bar{\alpha},\bar{\alpha},\ldots,\bar{\alpha}) \quad .$$

Let $\hat{w}(S) = \hat{u}_S(e(S))$ be attained at the S-allocation $\underset{\sim}{x}$. Then from Proposition 38.5 of Aumann and Shapley we have

(4.40) $\quad \hat{u}_t^j(\underset{\sim}{x}(t)) > \bar{\alpha}$ for almost all $t \in S$ for which $\underset{\sim}{x}^j(t) > 0$, for all $j \in \{1,\ldots,\ell\}$.

Let $\alpha \leqq \bar{\alpha}$. Now, $\hat{u}_t^\alpha$ differs from $\hat{u}_t$ only if $t \in Z$, in which case

(4.41) $\quad \hat{u}_t^\alpha(x) \geqq \hat{u}_t(x)$ for all $x \in \Omega$

and

(4.42) $\quad \hat{u}_t^{\alpha j}(x) < \alpha \leqq \bar{\alpha}$ for all $j \in \{1,\ldots,\ell\}$ and all $x \in \Omega$.

Let $\hat{w}_\alpha(S) = \hat{u}_S^\alpha(e(S))$ be attained at the S-allocation $\underset{\sim}{y}$. From (4.41) we have $\hat{w}_\alpha(S) \geqq \hat{w}(S)$, but from (4.40) and (4.42) we have

$$\hat{w}_\alpha(S) = \hat{u}_S^\alpha(e(S)) < \alpha \sum_{i=1}^{\ell} y^i(S \cap Z) + \int_{S\cap R} \hat{u}(\underset{\sim}{x}) - \alpha \sum_{i=1}^{\ell} y^i(S \cap Z)$$

if $\sum_{i=1}^{\ell} y^i(S \cap Z) > 0$, and $\int_{S\cap R} \hat{u}(\underset{\sim}{x}) = \hat{u}_S(e(S)) = \hat{w}(S)$. Hence $\sum_{i=1}^{\ell} y^i(S \cap Z) = 0$, and $\hat{w}_\alpha(S) = \hat{w}(S)$. This completes the proof.

Lemma 4.43: For every $\varepsilon > 0$ there exist $\alpha > 0$ and $\delta > 0$ such that if $\hat{u}$ is a δ-approximation on R to u then $\|\hat{w}_\alpha - \hat{w}\| < \varepsilon$.

Proof: Fix $\varepsilon > 0$. We need to show that there exist $\alpha > 0$ and $\delta > 0$ such that if $\hat{u}$ is a δ-approximation on R to u then

$$\sum_{k=0}^{m} |(\hat{w}_\alpha(S_{k+1}) - \hat{w}(S_{k+1})) - (\hat{w}_\alpha(S_k) - \hat{w}(S_k))| < \varepsilon$$

for all chains $\emptyset = S_0 \subset S_1 \subset \ldots \subset S_m \subset S_{m+1} = T$. Let γ, δ', and α' be the values of γ, δ, and α corresponding to $\varepsilon/2$ which are given in Corollary 4.35, and let α'' and δ'' be the values of $\bar{\alpha}$ and δ corresponding to $\beta = \gamma/2$ which are given in Corollary 4.39. Let $\alpha = \min(\alpha', \alpha'')$ and let $\delta = \min(\delta', \delta'')$. Then from Corollaries 4.35 and 4.39, if $\hat{u}$ is a δ-approximation on R to u we have

(4.44) $\quad \mu(S \cap R) < \gamma \Rightarrow \hat{w}_\alpha(S) < \varepsilon/2$

(since $w_\alpha(S) \leqq w_{\alpha^*}(S)$ for all $S \in \mathcal{C}$ if $\alpha \leqq \alpha^*$) and

$$(4.45)\qquad \mu(S \cap R) \geqq \gamma/2 \Rightarrow \hat{w}_\alpha(S) = \hat{w}(S) \quad .$$

Let $\emptyset = S_o \subset S_1 \subset \ldots \subset S_m \subset S_{m+1} = T$ be a chain. We can insert finitely many additional sets in this chain so that the difference in size between any two adjacent sets is at most $\gamma/2$; this cannot reduce the sum in which we are interested. Relable the chain $\emptyset = U_o \subset U_1 \subset \ldots \subset U_p \subset U_{p+1} = T$. Let q be such that $\mu(U_q \cap R) < \gamma$ and $\mu(U_{q+1} \cap R) \geqq \gamma$. Then $\mu(U_q \cap R) \geqq \gamma/2$, so by (4.45),

$$(4.46)\qquad \sum_{k=q}^{p} |(\hat{w}_\alpha(U_{k+1}) - \hat{w}(U_{k+1})) - (\hat{w}_\alpha(U_k) - \hat{w}(U_k))| = 0 \quad .$$

But

$$(4.47)\qquad \sum_{k=0}^{q-1} |(\hat{w}_\alpha(U_{k+1}) - \hat{w}(U_{k+1})) - (\hat{w}_\alpha(U_k) - \hat{w}(U_k))|$$

$$\leqq \sum_{k=0}^{q-1} (\hat{w}_\alpha(U_{k+1}) - \hat{w}_\alpha(U_k)) + \sum_{k=0}^{q-1} (\hat{w}(U_{k+1}) - \hat{w}(U_k)) = \hat{w}_\alpha(U_q) + \hat{w}(U_q) \quad ,$$

so since $\mu(U_q \cap R) < \gamma$ and $\hat{w}(S) \leqq \hat{w}_\alpha(S)$ for all $S \in \mathcal{C}$ and all $\alpha > 0$, we can conclude from (4.44) that

$$(4.48)\qquad \hat{w}_\alpha(U_q) + \hat{w}(U_q) < \varepsilon/2 + \varepsilon/2 = \varepsilon \quad .$$

From (4.46), (4.47), and (4.48) we have

$$\sum_{k=0}^{p} |(\hat{w}_\alpha(U_{k+1}) - \hat{w}(U_{K+1})) - (\hat{w}_\alpha(U_k) - \hat{w}(U_k))| < \varepsilon \quad ,$$

completing the proof of the lemma.

The following result generalizes Corollary 4.26 and Proposition 4.27.

Corollary 4.49: If M_o is an integrably sublinear quasi-market with zero-utility agents then $w \in pNAD \cap pNA'$, where w is the market game derived from M_o; if in addition M_o satisfies (4.25) then $w \in pNA$.

Proof: If $\mu(Z) = 0$ the result follows from Proposition 4.24. If $\mu(Z) > 0$, then from Lemma 4.43 for each $\varepsilon > 0$ there exists $\alpha > 0$ such that $\|w_\alpha - w\| < \varepsilon$, where w_α is the market game derived from M_α (since u is certainly a δ-approximation on R to itself for all $\delta > 0$). But M_α is an integrably sublinear market (not just a general market), so by Proposition 4.27 we have $w_\alpha \in pNAD \cap pNA'$, and if (4.25) is satisfied then by Corollary 4.27 we in fact have $w_\alpha \in pNA$. Since pNAD, pNA', and pNA are all closed in the variation norm, the result follows.

This result allows us to establish in Chapters 6 and 7 that the games we study there are members of pNA and $pNAD \cap pNA'$ respectively, but in order to calculate the values of these games we need the following stronger result, which generalizes Proposition 4.24.

Proposition 4.50: Let M_o be an integrably sublinear quasi-market with zero-utility agents which satisfies (4.25) and let w be the market game derived from M_o. Then for every $\varepsilon > 0$ there is an integrably sublinear finite type quasi-market with zero-utility agents $\hat{M}_o$ which differs from M_o only in the utility functions of the agents such that

$$\|w - \hat{w}\| < \varepsilon$$

where $\hat{w}$ is the market game derived from $\hat{M}_o$.

The strategy of the proof is to first use Lemma 4.43 to approximate the original market M_o by a market M_α which satisfies all of the assumptions of Aumann and Shapley, then to use their results to find a finite type approximation $\hat{M}_\alpha$ to M_α, and finally to use Lemma 4.43 again to show that there is a market $\hat{M}_o$ which is of finite type and in which the utility function of every $t \in Z$ is identically zero which approximates $\hat{M}_\alpha$. We shall then have approximated M_o by $\hat{M}_o$.[2/]

<u>Proof of Proposition 4.50</u>: If $\mu(Z) = 0$ then the result follows immediately from Proposition 4.24, so assume that $\mu(Z) > 0$. Fix $\varepsilon > 0$ and let α and δ' be the values of α and δ given in Lemma 4.43 which correspond to $\varepsilon/3$. Then if $\hat{u}$ is a δ-approximation on R to u, we have

(4.51) $$\|\hat{w}_\alpha - \hat{w}\| < \varepsilon/3 \quad ,$$

where $\hat{w}$ is the coalitional form of the market where the utility function of t is $\hat{u}_t$ (and $\hat{u}_t = u_t$ if $t \in Z$), and $\hat{w}_\alpha$ is the coalitional form of the market where the utility function of t is $\hat{u}^\alpha_t$ (and $\hat{u}^\alpha_t = \hat{u}_t$ if $t \in R$ and $\hat{u}^\alpha_t = u^\alpha_t$ if $t \in Z$). (See Diagram 1, p. 51). In particular we have

(4.52) $$\|w - w_\alpha\| < \varepsilon/3 \quad ,$$

since u is certainly a δ'-approximation on R to itself. But the market M_α associated with w_α satisfies all the assumptions of Aumann and Shapley (including (4.25))--i.e. all the utility functions are increasing--so from their Proposition 40.24 (on which Proposition 4.24 is based) we can deduce that there exists $\delta'' > 0$ such that if $\hat{u}^\alpha$ is a δ''-approximation to u^α, then $\|\hat{w}_\alpha - w_\alpha\| < \varepsilon/3$. Now set $\delta = \min(\delta', \delta'')$, and note that it can be assumed throughout Section 35 of Aumann and Shapley that every utility function is concave. So among the δ-approximations $\hat{u}^\alpha$ to u^α there is one which is concave and of finite type, and for which $\hat{u}^\alpha_t = u^\alpha_t$ if $t \in Z$ (since u^α_t on Z is already of finite type). Choose such a $\hat{u}^\alpha$. Then since $\delta \leqq \delta''$, certainly

$$(4.53) \qquad \|w_\alpha - \hat{w}_\alpha\| < \varepsilon/3 \quad ,$$

Now let $\hat{u}$ be defined by

$$\hat{u}_t = \begin{cases} u_t & \text{if } t \in Z \\ \hat{u}^\alpha_t & \text{if } t \in R \end{cases} .$$

Then since $\delta \leqq \delta'$, and since $\hat{u}^\alpha$ is a δ-approximation on R to u^α and $\hat{u}^\alpha_t = \hat{u}_t$ if $t \in R$, $\hat{u}$ is a δ'-approximation on R to u. So from (4.51) we have

$$(4.54) \qquad \|\hat{w}_\alpha - \hat{w}\| < \varepsilon/3 \quad .$$

Combining (4.52), (4.53), and (4.54) we have

$$\|w - \hat{w}\| \leqq \|w - w_\alpha\| + \|w_\alpha - \hat{w}_\alpha\| + \|\hat{w}_\alpha - \hat{w}\| < \varepsilon \quad .$$

But the general market underlying $\hat{w}$ is an integrably sublinear finite type quasi-market with zero-utility agents which differs from M_o only in the utility functions of the agents, so the proof is complete.

4.3 Economies

An <u>economy</u> is a pair $E = (M, \Gamma(M))$, where M is a market and $\Gamma(M)$ is a strategic game the players of which are the agents in M. In the following chapters we shall consider a number of such economies, making different assumptions on the nature of the strategic game. An allocation $\underset{\sim}{x}$ in M is a <u>value allocation</u> of $E = (M, \Gamma(M))$ if $u(\underset{\sim}{x})$ is a Harsanyi-Shapley value of $\Gamma(M)$ (see Section 3.2 above). The set of value allocations of E is the solution concept we shall use in what follows.

CHAPTER 5: Economies in which any Majority Can Control the Pattern of Trade

5.1 Introduction

In this chapter we shall characterize the value allocations of economies in which the strategic opportunities of each coalition depend solely on whether it contains a majority of the population or not; our intention is to model an economy in which there is "majority rule". If a coalition contains a majority of the population we shall assume that it can expropriate all the goods which the members of the complementary minority attempt to trade (with each other, or with members of the majority), and redistribute them, in addition to its own endowment, in any way it pleases. If a coalition contains a minority of the population we shall assume that each of its members can assure himself of the utility derived from consuming his initial endowment (simply by not attempting to trade anything), and that the coalition has no strategy which allows its members to receive higher payoffs (the only way it can redistribute its endowment is by trading). These assumptions give a majority coalition the minimal amount of power it might expect to possess in a majority rule private ownership economy. They can be contrasted with the assumptions of Aumann and Kurz [1977], where a majority can expropriate the entire endowment of the complementary minority; as we argued in Chapter 1, this seems to give a majority more power than it could expect to have in a private ownership economy. Roughly speaking we can think of a majority in Aumann and Kurz [1977] exercising its power by imposing a 100% wealth tax--where the "wealth" of an agent includes the total value of his leisure time--while here it does so by imposing a 100% sales tax. Of course the final outcome will not in general involve such a high tax rate in either case; what is being assumed is merely that it is possible for any majority to impose such a tax. (In Chapter 8 we shall investigate the case where the size of the tax a majority coalition can impose depends on the size of the coalition--a coalition containing a bare majority of the population being less powerful than one containing almost all of the population.)

We now formally state the assumptions we shall make about the economy $E = (M, \Gamma(T))$ in this chapter. Let M be a market. Then $\Gamma(M)$ is a strategic game (with player space (T,C,μ)) in which the payoff functions and strategy sets satisfy the following three conditions:

(5.1) if $S \in C$ is such that $\mu(S) > 1/2$, then for each S-allocation $\underset{\sim}{x}$ there is a strategy σ of S (i.e. there exists $\sigma \in X^S$) such that for every strategy τ of $T\setminus S$ (i.e. for all $\tau \in X^{T\setminus S}$)

$$\underset{\sim}{h}^S_{\sigma\tau}(t) \begin{cases} \geqq u_t(\underset{\sim}{x}(t)) & \text{if } t \in S \\ \leqq u_t(\underset{\sim}{e}(t)) & \text{if } t \in T\setminus S \end{cases} ;$$

(5.2) if $S \in C$ is such that $\mu(S) \geqq 1/2$, then there is a strategy τ of $T\setminus S$ such that for each strategy σ of S, there is an S-allocation $\underset{\sim}{x}$ such that

$$\underset{\sim}{h}^S_{\sigma\tau}(t) \begin{cases} \leqq u_t(\underset{\sim}{x}(t)) & \text{if } t \in S \\ \geqq u_t(\underset{\sim}{e}(t)) & \text{if } t \in T\setminus S \end{cases} ;$$

and

(5.3) if $S \in \mathcal{C}$ is such that $\mu(S) = 1/2$, then for each S-allocation $\underset{\sim}{x}$ there is a strategy σ of S such that for each strategy τ of $T \setminus S$ there is a $T \setminus S$-allocation $\underset{\sim}{y}$ such that

$$\underset{\sim}{h}^S_{\sigma\tau}(t) \begin{cases} \geqq u_t(\underset{\sim}{x}(t)) & \text{if } t \in S \\ \leqq u_t(\underset{\sim}{y}(t)) & \text{if } t \in T \setminus S \end{cases} .$$

We do not need to specify the strategy sets or payoff functions in any more detail: every strategic game satisfying (5.1) through (5.3) is equivalent for our purposes. The following is the main result of this chapter.

<u>Theorem A</u>: Let M be a bounded market and assume that $\Gamma(M)$ satisfies (5.1) through (5.3). Then an allocation $\underset{\sim}{x}$ in M is a value allocation of the economy $E = (M, \Gamma(M))$ if and only if it is efficient and a.e.

$$(5.4) \quad \underset{\sim}{\lambda}(t)(u_t(\underset{\sim}{x}(t)) - u_t(\underset{\sim}{e}(t))) - \int \underset{\sim}{\lambda}(u(\underset{\sim}{x}) - u(\underset{\sim}{e})) = p(\underset{\sim}{e}(t) - \underset{\sim}{x}(t))$$

where $(\underset{\sim}{\lambda}, p)$ is an efficiency pair for $\underset{\sim}{x}$. Moreover, such an allocation exists.

Throughout this chapter we shall use $\Gamma(M)$ to denote a strategic game associated with the market M which satisfies (5.1) through (5.3).

5.2 Optimal Threats

The first step in establishing the characterization (5.4) is to determine the nature of a collection of optimal threats in the two-person games between S and $T\setminus S$ for each $S \in \mathcal{C}$. We first study the case where $\underset{\sim}{\lambda}(t) = 1$ for all $t \in T$. Let $r: \mathcal{C} \to \mathbb{R}$ be the market game derived from M (see (4.15)); recall that when we say that $r(S)$ is "attained" we mean that it is finite and there exists an S-allocation $\underset{\sim}{x}$ such that $r(S) = \int_S u(\underset{\sim}{x})$. If $r(T)$ is attained, then since $\int u(\underset{\sim}{e}) \leqq r(T)$ we can define a totally finite measure η on $(T,\mathcal{C})$ by

$$(5.5) \quad \eta(S) = \int_S u(\underset{\sim}{e})d\mu \quad \text{for each } S \in \mathcal{C}$$

($u(\underset{\sim}{e})$ is a measurable function of t by (4.4)). Since $\mu \in NA$, we have $\eta \in NA$. The following result establishes the nature of a collection of pairs of optimal threats in the two-person games between S and $T\setminus S$ for each $S \in \mathcal{C}$.

<u>Lemma 5.6</u>: Assume that $r(S)$ is attained for every $S \in \mathcal{C}$. Then for each $S \in \mathcal{C}$ there exists a pair $(\sigma_o,\tau_o) \in X^S \times X^{T\setminus S}$ such that

$$(5.7) \quad H^S(\sigma,\tau_o) \leqq H^S(\sigma_o,\tau_o) \leqq H^S(\sigma_o,\tau) \quad \text{for all } \sigma \in X^S,\ \tau \in X^{T\setminus S},$$

$H^S(\sigma_o,\tau_o)$ is finite, and

$$\int_{S} h^{S}_{\tilde{}\sigma_o\tau_o} = \begin{cases} r(S) & \text{if } \mu(S) \geq 1/2 \\ \eta(S) & \text{if } \mu(S) < 1/2 \end{cases} .$$

<u>Remark</u>: The strategies (σ_o, τ_o) of course depend on S; we refrain from explicitly incorporating this fact into the notation for the sake of clarity.

<u>Proof of Lemma 5.6</u>: First consider the case where $\mu(S) > 1/2$. Let $\underset{\sim}{x}$ be an S-allocation which attains $r(S)$; let σ_o be the strategy of S corresponding to $\underset{\sim}{x}$ given in (5.1), and let τ_o be the strategy of $T\setminus S$ given in (5.2). Then from (5.1),

$$\int_{S} h^{S}_{\tilde{}\sigma_o\tau} \geq \int_{S} u(\underset{\sim}{x}) \quad \text{for all} \quad \tau \in X^{T\setminus S}$$

and from (5.2), setting $\sigma = \sigma_o$, there exists an S-allocation $\underset{\sim}{z}$ such that

$$\int_{S} h^{S}_{\tilde{}\sigma_o\tau_o} \leq \int_{S} u(\underset{\sim}{z}) \quad .$$

But $\int_{S} u(\underset{\sim}{z}) \leq \int_{S} u(\underset{\sim}{x})$ for all S-allocations $\underset{\sim}{z}$ by definition, so

$$(5.8) \qquad \int_{S} h^{S}_{\tilde{}\sigma_o\tau_o} \leq \int_{S} u(\underset{\sim}{x}) \leq \int_{S} h^{S}_{\tilde{}\sigma_o\tau} \quad \text{for all} \quad \tau \in X^{T\setminus S} \quad .$$

Similarly from (5.1) we have

$$\int_{T\setminus S} h^{S}_{\tilde{}\sigma_o\tau} \leq \int_{T\setminus S} u(\underset{\sim}{e}) \quad \text{for all} \quad \tau \in X^{T\setminus S}$$

and, setting $\sigma = \sigma_o$ in (5.2) we obtain

$$\int_{T\setminus S} h^S_{\tilde{}\sigma_o\tau_o} \geqq \int_{T\setminus S} u(\underset{\sim}{e}) \quad ,$$

so that

$$(5.9) \quad \int_{T\setminus S} h^S_{\tilde{}\sigma_o\tau_o} \geqq \int_{T\setminus S} u(\underset{\sim}{e}) \geqq \int_{T\setminus S} h^S_{\tilde{}\sigma_o\tau} \quad \text{for all} \quad \tau \in X^{T\setminus S} \quad .$$

Hence

$$(5.10) \quad H^S(\sigma_o,\tau_o) = \int_S h^S_{\tilde{}\sigma_o\tau_o} - \int_{T\setminus S} h^S_{\tilde{}\sigma_o\tau_o} \leqq \int_S h^S_{\tilde{}\sigma_o\tau} - \int_{T\setminus S} h^S_{\tilde{}\sigma_o\tau} = H^S(\sigma_o,\tau)$$

$$\text{for all} \quad \tau \in X^{T\setminus S} \quad .$$

In a similar way we can set $\tau = \tau_o$ in (5.1) and use (5.2) to conclude that

$$(5.11) \quad \int_S h^S_{\tilde{}\sigma\tau_o} \leqq \int_S u(\underset{\sim}{x}) \leqq \int_S h^S_{\tilde{}\sigma_o\tau_o} \quad \text{for all} \quad \sigma \in X^S$$

and

$$(5.12) \quad \int_{T\setminus S} h^S_{\tilde{}\sigma\tau_o} \geqq \int_{T\setminus S} u(\underset{\sim}{e}) \geqq \int_{T\setminus S} h^S_{\tilde{}\sigma_o\tau_o} \quad \text{for all} \quad \sigma \in X^S \quad ,$$

so that

$$(5.13) \quad H^S(\sigma,\tau_o) = \int_S h^S_{\tilde{}\sigma\tau_o} - \int_{T\setminus S} h^S_{\tilde{}\sigma\tau_o} \leqq \int_S h^S_{\tilde{}\sigma_o\tau_o} - \int_{T\setminus S} h^S_{\tilde{}\sigma_o\tau_o} = H^S(\sigma_o,\tau_o)$$

$$\text{for all} \quad \sigma \in X^S \quad .$$

But (5.10) and (5.13) imply that (5.7) is satisfied by the pair (σ_o, τ_o) we have chosen. Moreover, from (5.8) and (5.11)

$$\int_S h^S_{\sigma_o \tau_o} = \int_S u(x) \quad ,$$

and $\int_S u(x) = r(S)$ by the way we chose σ_o; from (5.9) and (5.12)

$$\int_{T\setminus S} h^S_{\sigma_o \tau_o} = \int_{T\setminus S} u(e) = \eta(S) \quad .$$

So for any S with $\mu(S) > 1/2$, and, by reversing the roles of S and $T\setminus S$, for any S with $\mu(S) < 1/2$, we have proved what we need.

Now consider the case where $\mu(S) = 1/2$. Let x attain $r(S)$ and let y attain $r(T\setminus S)$; let σ_o be the strategy of S corresponding to the S-allocation x given by (5.3), and let τ_o be the strategy of $T\setminus S$ corresponding to the $T\setminus S$-allocation y when the roles of S and $T\setminus S$ are reversed in (5.3). Then we have

$$(5.14) \quad \int_S h^S_{\sigma\tau_o} \leqq \int_S u(x) \leqq \int_S h^S_{\sigma_o\tau_o} \leqq \int_S u(x) \leqq \int_S h^S_{\sigma_o\tau} \quad \text{for all } \sigma \in X^S \text{ and all } \tau \in X^{T\setminus S} \quad ,$$

and

$$(5.15) \quad \int_{T\setminus S} h^S_{\sigma\tau_o} \geqq \int_{T\setminus S} u(y) \geqq \int_{T\setminus S} h^S_{\sigma_o\tau_o} \geqq \int_{T\setminus S} u(y) \geqq \int_{T\setminus S} h^S_{\sigma_o\tau} \quad \text{for all } \sigma \in X^S \text{ and all } \tau \in X^{T\setminus S} \quad .$$

so that

$$H^S(\sigma,\tau_o) \leqq H^S(\sigma_o,\tau_o) \leqq H^S(\sigma_o,\tau) \quad \text{for all} \quad \sigma \in X^S \quad \text{and all}$$
$$\text{and all} \quad \tau \in X^{T\setminus S} \quad .$$

Moreover, from (5.14) and (5.15),

$$\int_S h^S_{\tilde{}\sigma_o\tau_o} = \int_S u(\underset{\sim}{x}) = r(S) \quad \text{and} \quad \int_{T\setminus S} h^S_{\tilde{}\sigma_o\tau_o} = \int_{T\setminus S} u(\underset{\sim}{y}) = r(T\setminus S) \quad .$$

This completes the proof of the lemma.

What this result shows is that the strategy pair where the members of the minority simply consume their endowment while the majority threatens to expropriate any goods which minority members attempt to trade and redistributes its endowment among its members in an optimal fashion constitutes a pair of optimal threats. Define the game $q: \mathcal{C} \to \mathbb{R}_+$ by

$$(5.16) \qquad q(S) = \int_S h^S_{\tilde{}\sigma_o\tau_o} = \begin{cases} r(S) & \text{if} \quad \mu(S) \geqq 1/2 \\ \eta(S) & \text{if} \quad \mu(S) < 1/2 \end{cases} \quad .$$

We have not established that (σ_o,τ_o) is the only pair of optimal threats in the game between S and $T\setminus S$. However, since it will turn out that the game q always possesses an asymptotic value we shall, by virtue of Lemma 3.26, locate all the Harsanyi-Shapley values of the economy E under consideration by restricting attention to the family of pairs (σ_o,τ_o) given in Lemma 5.6. Before proceeding to the calculation of the value of the game q, we shall establish two results which will be needed later. The following is used in the proof of Proposition 5.25.

<u>Lemma 5.17</u>: Suppose that H^T possesses a finite saddle point. Then $r(T)$ is attained.

<u>Proof</u>: From the fact that H^T possesses a finite saddle point we know there exists $\sigma_o \in X^T$ such that

$$\int h_{\sigma}^T = H^T(\sigma,\tau_o) \leqq H^T(\sigma_o,\tau_o) = \int h_{\sigma_o}^T \quad \text{for all} \quad \sigma \in X^T$$

(where τ_o is the single strategy of $\emptyset$). Now, from (5.2) we know there exists an allocation x such that $h_{\sigma_o}^T(t) \leqq u_t(x(t))$ for all $t \in T$; applying (5.1) to the allocation x we obtain a strategy σ_1 of T such that $h_{\sigma_1}^S(t) \geqq u_t(x(t))$ for all $t \in T$. Combining these facts we have

$$(5.18) \qquad \int h_{\sigma}^T \leqq \int h_{\sigma_o}^T \leqq \int u(x) \leqq \int h_{\sigma_1}^T \quad \text{for all} \quad \sigma \in X^T \quad ,$$

so that, taking the case $\sigma = \sigma_1$, we can conclude that

$$(5.19) \qquad H^T(\sigma_o,\tau_o) = \int h_{\sigma_o}^T = \int u(x) \quad .$$

By hypothesis $H^T(\sigma_o,\tau_o)$ is finite. Suppose $r(T)$ is not attained at x. Then there exists an allocation y such that $\int u(y) > \int u(x)$. Let σ_2 be the strategy of T corresponding to y given by (5.1). Then

$$\int u(\underset{\sim}{x}) = \int \underset{\sim}{h}^T_{\sigma_o} \geqq \int \underset{\sim}{h}^T_{\sigma_2} \geqq \int u(\underset{\sim}{y}) > \int u(\underset{\sim}{x})$$

(using (5.18) for $\sigma = \sigma_2$, and (5.19)), which is not possible. Hence $r(T) = \int u(\underset{\sim}{x}) < \infty$, where $\underset{\sim}{x}$ is an allocation, establishing the lemma.

The definition of a value allocation of E involves the payoff function $\underset{\sim}{h}^T_\sigma$ of $\Gamma(M)$ (see Section 4.3). The following lemma allows us to work exclusively with the elements of the market M. Aumann and Kurz [1977] proved the same result under their assumptions (see their Proposition 11.12). The proof under our assumptions is very similar; it is included for completeness. Recall that given a comparison function $\underset{\sim}{\lambda}$, $q_{\underset{\sim}{\lambda}}$ is the game defined by $q_{\underset{\sim}{\lambda}}(S) = \int_S \underset{\sim}{\lambda}\underset{\sim}{h}^S_{\sigma_o\tau_o}$ for each $S \in \mathcal{C}$, where (σ_o, τ_o) is a pair of optimal threats in the two-person strategic game between S and $T\setminus S$ in which the payoff function of S is $\int_S \underset{\sim}{\lambda}\underset{\sim}{h}^S$ (see Section 3.2, and in particular (3.24)).

Lemma 5.20: An allocation $\underset{\sim}{x}$ is a value allocation of $E = (M, \Gamma(M))$ if and only if there exists a comparison function $\underset{\sim}{\lambda}$ such that $H^S_{\underset{\sim}{\lambda}}$ has a finite saddle point for every $S \in \mathcal{C}$ and

$$(\phi q_{\underset{\sim}{\lambda}})(S) = \int_S \underset{\sim}{\lambda} u(\underset{\sim}{x}) \quad \text{for each} \quad S \in \mathcal{C} \ ,$$

where $q_{\underset{\sim}{\lambda}}$ is the game defined by (3.24).

Proof: If $\underset{\sim}{x}$ is a value allocation of E then by definition $u(\underset{\sim}{x}) = \underset{\sim}{h}^T_\sigma$ for some $\sigma \in X^T$ and $\int_S \underset{\sim}{\lambda}\underset{\sim}{h}^T_\sigma = \int_S \underset{\sim}{\lambda} u(\underset{\sim}{x}) = (\phi q_{\underset{\sim}{\lambda}})(S)$ for each $S \in \mathcal{C}$ for some comparison function $\underset{\sim}{\lambda}$.

On the other hand, suppose $\underset{\sim}{x}$ is an allocation such that $\int_S \underset{\sim}{\lambda} u(\underset{\sim}{x}) = (\phi q_{\underset{\sim}{\lambda}})(S)$ and $H^S_{\underset{\sim}{\lambda}}$ has a finite saddle point for each $S \in \mathcal{C}$. By (5.1) there exists $\sigma \in X^T$ such that

$$\text{(5.21)} \qquad \underset{\sim}{h}^T_\sigma(t) \geqq u_t(\underset{\sim}{x}(t)) \quad \text{for all} \quad t \in T \ .$$

So

$$\text{(5.22)} \qquad \int \underset{\sim}{\lambda} \underset{\sim}{h}^T_\sigma \geqq \int \underset{\sim}{\lambda} u(\underset{\sim}{x}) = (\phi q_{\underset{\sim}{\lambda}})(T) = q_{\underset{\sim}{\lambda}}(T)$$

(by (3.1)). But by the definition of $q_{\underset{\sim}{\lambda}}(T)$,

$$H^T_{\underset{\sim}{\lambda}}(\sigma', \tau_o) \leqq H^T_{\underset{\sim}{\lambda}}(\sigma_o, \tau_o) = q_{\underset{\sim}{\lambda}}(T) \quad \text{for all} \quad \sigma' \in X^T \ ,$$

for some $\sigma_o \in X^T$, where τ_o is the single strategy of $\emptyset$. So

$$q_{\underset{\sim}{\lambda}}(T) \geqq \int \underset{\sim}{\lambda} \underset{\sim}{h}^T_{\sigma'}, \quad \text{for all} \quad \sigma' \in X^T \ ,$$

and in particular $q_{\underset{\sim}{\lambda}}(T) \geqq \int \underset{\sim}{\lambda} \underset{\sim}{h}^T_\sigma$. Hence we have equality in (5.22), and so from (5.21)

$$\text{(5.23)} \qquad \underset{\sim}{h}^T_\sigma(t) = u_t(\underset{\sim}{x}(t)) \quad \text{for almost all} \quad t \in T \ .$$

So $(\phi q_{\underset{\sim}{\lambda}})(S) = \int_S \underset{\sim}{\lambda} \underset{\sim}{h}^T_\sigma$ for each $S \in \mathcal{C}$. Hence $\underset{\sim}{h}^T_\sigma$ is a value of $\Gamma(M)$; so by (5.23), $u(\underset{\sim}{x})$ is a value of $\Gamma(M)$. Thus $\underset{\sim}{x}$ is a value allocation of E, as was to be proved.

5.3 The Calculation of the Value of the Game q

From Lemma 5.6 we know that in order to characterize the Harsanyi-Shapley values of E (see the definition in Section 3.2, in particular (3.24) and (3.25)) we have to calculate the value of the game q defined in (5.16); this we shall now do. Our argument depends heavily on the results of Aumann and Shapley, and of Aumann and Kurz [1977].

Proposition 5.24: If M is integrably sublinear then $q \in ASYMP$, and ϕq is given by

$$(\phi q)(S) = (r(T) - \eta(T))\mu(S)/2 + \int_S [u(\tilde{x}) - p(\tilde{x} - \tilde{e})]/2 + \int_S u(\tilde{e})/2$$

for each $S \in \mathcal{C}$ where $(\tilde{x},p)$ is a t.u.c.e. in M.

Proof: Define the function $g: [0,1] \to [0,1]$ by

$$g(x) = \begin{cases} 1 & \text{if} \quad x \in [1/2,1] \\ 0 & \text{if} \quad x \in [0,1/2) \end{cases} .$$

Define the games $q^1: \mathcal{C} \to \mathbb{R}_+$ and $q^2: \mathcal{C} \to \mathbb{R}_+$ by

$$q^1(S) = g(\mu(S))r(S) \quad \text{and} \quad q^2(S) = g(\mu(S))\eta(S) \quad \text{for each} \quad S \in \mathcal{C} .$$

We have $g \circ \mu \in bv'NA$, $r \in pNAD$ by Proposition 4.27, and $\eta \in NA$, so $q^1 \in bv'NA*pNAD$ and $q^2 \in bv'NA$. Hence by Corollary 3.8, $q^1 \in ASYMP$ and $q^2 \in ASYMP$. But

$$q(S) = q^1(S) + (\eta(S) - q^2(S)) \quad \text{for each} \quad S \in \mathcal{C} ,$$

so that $q \in$ ASYMP, and by the linearity of the value operator ϕ (see (3.4)) we have

$$\phi q = \phi q^1 + \phi \eta - \phi q^2 \quad .$$

Now, in fact $r \in pNAD \cap pNA'$ by Proposition 4.27; hence by Propositions 3.16 and 4.28,

$$(\phi q^1)(S) = r(T)\mu(S)/2 + \int_S [u(\underset{\sim}{x}) - p(\underset{\sim}{x} - \underset{\sim}{e})]/2 \quad \text{for each} \quad S \in \mathcal{C} \quad ,$$

where $(\underset{\sim}{x},p)$ is a t.u.c.e. in M. Also, by Theorem 3.14 and (3.11), we have $(\phi\eta)(S) = \eta(S)$ for each $S \in \mathcal{C}$. Finally, from Propositions 3.16 and 4.28 once again we have

$$(\phi q^2)(S) = [\eta(T)\mu(S) + \eta(S)]/2 \quad \text{for each} \quad S \in \mathcal{C} \quad .$$

Hence we have

$$\begin{aligned}(\phi q)(S) &= r(T)\mu(S)/2 + \int_S [u(\underset{\sim}{x}) - p(\underset{\sim}{x} - \underset{\sim}{e})]/2 + \eta(S) \\ &\quad - [\eta(T)\mu(S) + \eta(S)]/2 \\ &= (r(T) - \eta(T))\mu(S)/2 + \int_S [u(\underset{\sim}{x}) - p(\underset{\sim}{x} - \underset{\sim}{e})]/2 + \int_S u(\underset{\sim}{e})/2\end{aligned}$$

for each $S \in \mathcal{C}$, where $(\underset{\sim}{x},p)$ is a t.u.c.e. in M, completing the proof of the proposition.

5.4 The Characterization of the Value Allocations of the Economy $(M, \Gamma(M))$ when M Is Bounded

First we have the following, which is analogous to Proposition 14.10 of Aumann and Kurz [1977].

Proposition 5.25: Let M be a bounded market, $\underset{\sim}{x}$ an allocation in M, and $\underset{\sim}{\lambda}$ a comparison function. Then we can choose a finite saddle point (σ_o,τ_o) for $H^S_{\underset{\sim}{\lambda}}$ for each $S \in C$ so that the game $q_{\underset{\sim}{\lambda}}$ defined in (3.24) has a value $\phi q_{\underset{\sim}{\lambda}}$ which satisfies $(\phi q_{\underset{\sim}{\lambda}})(S) = \int_S \underset{\sim}{\lambda} u(\underset{\sim}{x})$ for each $S \in C$ if and only if $\underset{\sim}{x}$ is efficient with efficiency pair $(\underset{\sim}{\lambda},p)$ and a.e. (5.4) is satisfied.

Proof: First suppose that we can choose a finite saddle point (σ_o,τ_o) for $H^S_{\underset{\sim}{\lambda}}$ for each $S \in C$ in such a way that $(\phi q_{\underset{\sim}{\lambda}})(S) = \int_S \underset{\sim}{\lambda} u(\underset{\sim}{x})$ for each $S \in C$. Then by Lemma 5.17, $r_{\underset{\sim}{\lambda}}(T)$ is finite, so that by Proposition 4.17, $\underset{\sim}{\lambda}M$ is integrably sublinear. Also we have

$$\int \underset{\sim}{\lambda} u(\underset{\sim}{x}) = (\phi q_{\underset{\sim}{\lambda}})(T) = q_{\underset{\sim}{\lambda}}(T) = r_{\underset{\sim}{\lambda}}(T)$$

(using the fact that $q_{\underset{\sim}{\lambda}}(T)$ is independent of the collection of optimal threats we choose, and Lemma 5.6). But then $\underset{\sim}{x}$ is efficient and so there exists a price p such that $(\underset{\sim}{\lambda},p)$ is an efficiency pair for $\underset{\sim}{x}$, so that $(\underset{\sim}{x},p)$ is a t.u.c.e. in $\underset{\sim}{\lambda}M$ (see Section 4.1). So by Proposition 5.24 we have

$$(\phi q_{\underset{\sim}{\lambda}})(S) = (r_{\underset{\sim}{\lambda}}(T) - \eta_{\underset{\sim}{\lambda}}(T))\mu(S)/2 + \int_S [\underset{\sim}{\lambda} u(\underset{\sim}{x}) - p(\underset{\sim}{x} - \underset{\sim}{e})]/2 + \int_S \underset{\sim}{\lambda} u(\underset{\sim}{e})/2$$

for each $S \in \mathcal{C}$, where $\eta_{\lambda}(T) = \int \lambda u(e)$. But then from our hypothesis that $(\phi q_{\lambda})(S) = \int_S \lambda u(x)$ for each $S \in \mathcal{C}$ we have

$$\int_S \lambda u(x) = \mu(S) \int \lambda (u(x) - u(e)) - p \int_S (x - e) + \int_S \lambda u(e)$$

or

$$\int_S \lambda (u(x) - u(e)) - \mu(S) \int \lambda (u(x) - u(e)) = p \int_S (e - x)$$

for each $S \in \mathcal{C}$, so a.e. (5.4) is satisfied. This completes the proof of necessity.

Now suppose that x is efficient with efficiency pair (λ, p) and a.e. (5.4) is satisfied. Then $\int \lambda u(x)$ is finite and $\int \lambda u(x) = r_{\lambda}(T)$ (see (4.12)). Hence λM is integrably sublinear, so that by Proposition 4.16, $r_{\lambda}(S)$ is attained for each $S \in \mathcal{C}$. Thus from Lemma 5.6 we can choose a finite saddle point (σ_o, τ_o) for H_{λ}^S for each $S \in \mathcal{C}$ such that the game q_{λ} defined in (3.24) is given by

$$q_{\lambda}(S) = \begin{cases} r_{\lambda}(S) & \text{if} \quad \mu(S) \geqq 1/2 \\ \eta_{\lambda}(S) & \text{if} \quad \mu(S) < 1/2 \end{cases} .$$

Then from Proposition 5.24 we have $q_{\lambda} \in \text{ASYMP}$ and

(5.26) $$(\phi q_{\lambda})(S) = (r_{\lambda}(T) - \eta_{\lambda}(T))\mu(S)/2 + \int_S [\lambda u(x) - p(x - e)]/2 + \int_S \lambda u(e)/2$$

for each $S \in \mathcal{C}$. But integrating (5.4) over $S \in \mathcal{C}$ gives

$$\int_S \lambda[u(\underset{\sim}{x}) - u(\underset{\sim}{e})] - \mu(S)(r_{\underset{\sim}{\lambda}}(T) - \eta_{\underset{\sim}{\lambda}}(T)) = \int_S p(\underset{\sim}{e} - \underset{\sim}{x})$$

for each $S \in \mathcal{C}$.

Combining this with (5.26) gives $(\phi q_{\underset{\sim}{\lambda}})(S) = \int_S \lambda u(\underset{\sim}{x})$ for each $S \in \mathcal{C}$, as was to be shown. This completes the proof.

This gives us the characterization part of Theorem A.

Theorem 5.27: Let M be a bounded market. Then $\underset{\sim}{x}$ is a value allocation of the economy $E = (M, \Gamma(M))$ if and only if it is efficient with efficiency pair $(\underset{\sim}{\lambda}, p)$ and a.e. (5.4) is satisfied.

Proof: The result follows immediately from Lemma 5.20 and Proposition 5.25.

5.5 The Existence of a Value Allocation of the Economy $(M, \Gamma(M))$ when M Is Bounded

Our argument will follow that of Aumann and Kurz [1977] quite closely. Let p be a price vector; for each $t \in T$ define the indirect utility function $u_t^p\colon \mathbb{R}_+ \to \mathbb{R}_+$ of t at the price vector p by

(5.28) $u_t^p(y) = \max\{u_t(y)\colon x \in \Omega \text{ and } px \leqq y\}$ for each $y \in \mathbb{R}_+$.

Define the function $u^p\colon T \times \mathbb{R}_+ \to \mathbb{R}_+$ by $u^p(t,y) = u_t^p(y)$ for each $t \in T$ and $y \in \mathbb{R}_+$. The following, a consequence of Lemmas 16.1, 17.4, and 17.5 of Aumann and Kurz [1977], states some of the properties of the function u^p.

Lemma 5.29: For each price vector p, u^p satisfies (4.2) through (4.5). Also, for each $t \in T$ the inverse function $(u_t^p)^{-1}(y)$ is continuous in (p,y) for $y > 0$.

By virtue of this result, for each price vector p we can define a market M^p with agent space (T,C,μ) and $\ell = 1$, in which the utility function of $t \in T$ is $u_t^p \colon \mathbb{R}_+ \to \mathbb{R}_+$ and the initial endowment density is $p\underset{\sim}{e}$. The following is a consequence of Lemmas 16.1 and 9.10 of Aumann and Kurz [1977].

Lemma 5.30: If M is bounded then so is M^p for each price vector p. Also, if $\underset{\sim}{x}$ is an efficient allocation in M with efficiency pair $(\underset{\sim}{\lambda},p)$ then $p\underset{\sim}{x}$ is an efficient allocation in M^p with efficiency pair $(\underset{\sim}{\lambda},1)$, $u(\underset{\sim}{x}) = u^p(p\underset{\sim}{x})$, and $\underset{\sim}{\lambda}(t) = 1/u_t^{p\prime}(p\underset{\sim}{x}(t))$ if $\underset{\sim}{x}(t) \neq 0$.

We can now state the following alternative characterization of the value allocations of M, which we shall use to prove existence.

Proposition 5.31: Let M be a bounded market. Then an allocation $\underset{\sim}{x}$ is a value allocation of the economy $E = (M, \Gamma(M))$ if and only if there is a price vector p and an efficient allocation $\underset{\sim}{y}^p$ in M^p such that a.e.

$$(5.32)\quad \lambda(t)[u_t^p(y^p(t)) - u_t(e(t))] - \int\lambda[u^p(y^p) - u(e)] = pe(t) - y^p(t)$$

where $(\lambda, 1)$ is an efficiency pair for y^p in M^p, and a.e.

$$(5.33)\quad x(t) \text{ maximizes } u_t(x) \text{ over } \{x \in \Omega: px \leqq y^p(t)\} .$$

<u>Proof</u>: First assume x is a value allocation of E. Then by Theorem 5.27, x is efficient with efficiency pair (λ, p) and a.e. (5.4) is satisfied. Set $y^p = px$. Then by Lemma 5.30, y^p is an efficient allocation in M^p, with efficiency pair $(\lambda, 1)$. Since p is an efficiency price for x, we know that a.e. (5.33) is satisfied (see (4.10)), so that $u_t^p(y^p(t)) = u_t(x(t))$. But then (5.32) is a.e. satisfied, completing the proof of necessity.

Now assume that x is an allocation, p is a price vector, and y^p is an efficient allocation in M^p such that a.e. (5.32) and (5.33) are satisfied. From (5.33) we have $px(t) = y^p(t)$ a.e., so x is efficient and there is a comparison function λ such that (λ, p) is an efficiency pair for $y^p = px$ in M^p (using Lemma 5.30 again), so from (5.32) we have that a.e. (5.4) is satisfied, completing the proof of sufficiency.

Given this result, we shall locate a value allocation of E in the following way. First, for each price vector p we shall find an efficient allocation y^p of M^p which a.e. satisfies (5.32). Then we shall argue that for some price vector p the function x defined by (5.33) is an allocation; by Proposition 5.31 this allocation x is a value allocation of E.

<u>Lemma 5.34</u>: Let M be a bounded market and assume $\underset{\sim}{e}$ is not an efficient allocation in M. Then for each price vector p there exists a unique efficient allocation $\underset{\sim}{y}^p$ in M^p such that (5.32) is a.e. satisfied; moreover, $\underset{\sim}{y}^p(t)$ is a.e. continuous as a function of p.

<u>Proof</u>: Fix a price vector p, and suppose that the efficient allocation $\underset{\sim}{y}^p$ a.e. satisfies (5.32). Since $\underset{\sim}{e}$ is not an efficient allocation in M there exists an allocation $\underset{\sim}{x}$ with $u_t(\underset{\sim}{x}(t)) > u_t(\underset{\sim}{e}(t))$ a.e.. Let $\underset{\sim}{z} = p\underset{\sim}{x}$. Then $\underset{\sim}{z}$ is an allocation of $p\int\underset{\sim}{e}$ in M^p and $u_t^p(\underset{\sim}{z}(t)) \geqq u_t(\underset{\sim}{x}(t))$. So applying (4.12) to M^p we obtain

$$\int\underset{\sim}{\lambda}\underset{\sim}{u}^p(\underset{\sim}{y}^p) \geqq \int\underset{\sim}{\lambda}\underset{\sim}{u}^p(\underset{\sim}{z}) \geqq \int\underset{\sim}{\lambda}\underset{\sim}{u}(\underset{\sim}{x}) > \int\underset{\sim}{\lambda}\underset{\sim}{u}(\underset{\sim}{e}) \quad ,$$

so $\int\underset{\sim}{\lambda}[u^p(\underset{\sim}{y}^p) - u(\underset{\sim}{e})] > 0$. But then from (5.32) it cannot be the case that $\underset{\sim}{y}^p(t) = 0$. Hence $\underset{\sim}{y}^p(t) > 0$ a.e..

Given this fact, from Lemma 5.30 we know that $\underset{\sim}{\lambda}(t) = 1/u_t^{p'}(\underset{\sim}{y}^p(t))$ a.e., so an efficient allocation $\underset{\sim}{y}^p$ in M^p a.e. satisfies (5.32) if and only if it a.e. satisfies

$$(5.35) \qquad \frac{u_t^p(\underset{\sim}{y}^p(t)) - u_t(\underset{\sim}{e}(t))}{u_t^{p\prime}(\underset{\sim}{y}^p(t))} + \underset{\sim}{y}^p(t) = p\underset{\sim}{e}(t) + \int\frac{u^p(\underset{\sim}{y}^p) - u(\underset{\sim}{e})}{u^{p\prime}(\underset{\sim}{y}^p)} \quad .$$

Define $\underset{\sim}{y}_o^p\colon T \to \mathbb{R}_+$ by

$$(5.36) \qquad \underset{\sim}{y}_o^p(t) = (u_t^p)^{-1}(u_t(\underset{\sim}{e}(t))) \quad \text{for all} \quad t \in T \quad .$$

Then $\underset{\sim}{y}_o^p$ is measurable, and since $\underset{\sim}{y}_o^p(t) \leqq p\underset{\sim}{e}(t)$ for all $t \in T$ it is integrable. Also, from Lemma 5.29, $\underset{\sim}{y}_o^p(t)$ is continuous as a function of p for each $t \in T$. Now, as we argued above we have $\int[(u^p(\underset{\sim}{y}^p) - u(\underset{\sim}{e}))/u^{p\prime}(\underset{\sim}{y}^p)] > 0$, so from (5.35) and the fact that $\underset{\sim}{y}_o^p(t) \leqq p\underset{\sim}{e}(t)$ for all $t \in T$ we must have a.e.

$$(5.37) \qquad \underset{\sim}{y}^p(t) > \underset{\sim}{y}_o^p(t) \quad .$$

Consider a general market $\bar{M}^p$ with agent space (T,C,μ) and one good in which the utility function of $t \in T$ is $v_t\colon \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$v_t(y) = u_t^p(y + \underset{\sim}{y}_o^p(t)) - u_t(\underset{\sim}{e}(t)) \quad \text{for each} \quad y \in \mathbb{R}_+$$

and the initial endowment density is $p\underset{\sim}{e} - \underset{\sim}{y}_o^p$. Since $\underset{\sim}{e}$ is not efficient we have $\int(p\underset{\sim}{e} - \underset{\sim}{y}_o^p) > 0$; $v_t(0) = 0$, and (4.2) and (4.4) through (4.7) are satisfied, so $\bar{M}^p$ is in fact a bounded market. Hence by Theorem C of Aumann and Kurz [1977] there is a unique efficient allocation $\underset{\sim}{z}^p$ in $\bar{M}^p$ satisfying $\underset{\sim}{z}^p(t) > 0$ a.e. and

$$(5.38) \qquad \underset{\sim}{z}^p(t) + \frac{v_t(\underset{\sim}{z}^p(t))}{v_t'(\underset{\sim}{z}^p(t))} = p\underset{\sim}{e}(t) - \underset{\sim}{y}_o^p(t) + \int\frac{v(\underset{\sim}{z}^p)}{v'(\underset{\sim}{z}^p)} \quad ;$$

and by Lemma 17.6 of Aumann and Kurz [1977], $\underset{\sim}{z}^p(t)$ is a.e. continuous in p. Now, rewriting (5.38) we can conclude that there is a unique efficient allocation $\underset{\sim}{z}^p$ in $\bar{M}^p$ satisfying $\underset{\sim}{z}^p(t) > 0$ a.e. and

$$z^p(t) + y_o^p(t) + \frac{u_t^p(z^p(t) + y_o^p(t)) - u_t(e(t))}{u_t^{p\prime}(z^p(t) + y_o^p(t))}$$

$$= pe(t) + \int \frac{u^p(z^p + y_o^p) - u(e)}{u^{p\prime}(z^p + y_o^p)} \quad .$$

But from (5.37) any efficient allocation y^p a.e. satisfying (5.35) must be of the form $y_1^p + y_o^p$ where $y_1^p(t) > 0$ a.e. and $\int(y_1^p + y_o^p) = \int pe$. Since we have $\int(z^p + y_o^p) = \int pe$, this establishes that $y^p = z^p + y_o^p$ is the unique efficient allocation a.e. satisfying (5.35), completing the proof of the first part of the result.

Finally, as stated above both $y_o^p(t)$ and $z^p(t)$ are a.e. continuous in p, so $y^p(t) = z^p(t) + y_o^p(t)$ is also, and the proof is complete.

In the remainder of this section we shall let y^p be the unique efficient allocation in M^p which a.e. satisfies (5.32) which is provided by Lemma 5.34 when e is not an efficient allocation in M. For each $t \in T$ and each price vector p let

$$B^p(t) = \{x \in \Omega\colon px \leqq y^p(t)\} \quad ,$$

let

$$D^p(t) = \{x \in B^p(t)\colon u_t(x) \geqq u_t(z) \quad \text{for all} \quad z \in B^p(t)\} \quad ,$$

and let

$$(5.39) \qquad Z(p) = \int \underset{\sim}{D}^p - \int \underset{\sim}{e} \quad .$$

If we can find a price vector p such that $0 \in Z(p)$, then there is an allocation $\underset{\sim}{x}$ which a.e. satisfies (5.33), and so by Proposition 5.31 the economy E possesses a value allocation. To show that there is a price vector p such that $0 \in Z(p)$, we shall use the following version of Debreu's lemma (see Lemma 1 on p. 150 of Hildenbrand [1974][3/]). Let $\Delta = \{p \in \Omega: \sum_{i=1}^{\ell} p^i = 1\}$, let $\text{int}\,\Delta$ be the relative interior of Δ, and let $\partial\Delta = \Delta \setminus \text{int}\,\Delta$.

<u>Lemma 5.40</u> (Debreu): Let Z be a compact- and nonempty-valued correspondence from $\text{int}\,\Delta$ to $\mathbb{R}^{\ell}$ that is bounded from below and has a graph which is closed in $\text{int}\,\Delta \times \mathbb{R}^{\ell}$ for which $pz = 0$ for all $z \in Z(p)$ and for which the following is satisfied:

(5.41) if $\{p_n\}$ is a sequence with $p_n \in \text{int}\,\Delta$ for all n, and $p_n \to p_o \in \partial\Delta$ then $\inf \{\sum_{i=1}^{\ell} z^i: z \in Z(p)\} > 0$ for large enough n .

Then there exists $p \in \text{int}\,\Delta$ such that 0 is a member of the convex hull of $Z(p)$.

In order to establish that the correspondence Z defined in (5.39) satisfies the hypotheses of this lemma, we shall need the following two results.

Lemma 5.42: If M is a bounded market and $\underset{\sim}{e}$ is not an efficient allocation in M then for each compact subset C of $\text{int}\,\Delta$ there is a constant c such that $\underset{\sim}{y}^p(t) \leqq c + p\underset{\sim}{e}(t)$ for all $t \in T$ and all $p \in C$.

Proof: From the proof of Lemma 5.34 we have $\underset{\sim}{y}^p(t) = \underset{\sim}{y}_o^p(t) + \underset{\sim}{z}^p(t)$ for all $t \in T$, where $\underset{\sim}{z}^p$ is an allocation in the market $\bar{M}^p$. By applying Corollary 17.10 of Aumann and Kurz [1977] to the market $\bar{M}^p$ we know that for each compact subset C of $\text{int}\,\Delta$ there is a constant c such that $\underset{\sim}{z}^p(t) \leqq c + p\underset{\sim}{e}(t) - \underset{\sim}{y}_o^p(t)$. But then $\underset{\sim}{z}^p(t) + \underset{\sim}{y}_o^p(t) = \underset{\sim}{y}^p(t) \leqq c + p\underset{\sim}{e}(t)$, completing the proof.

Lemma 5.43: If M is a bounded market and $\underset{\sim}{e}$ is not an efficient allocation in M then there exists $\delta > 0$ such that a.e.

$$\underset{\sim}{y}^p(t) + \frac{u_t^p(\underset{\sim}{y}^p(t))}{u_t^{p\prime}(\underset{\sim}{y}^p(t))} \geqq \delta \quad \text{for all} \quad p \in \text{int}\,\Delta \quad .$$

Proof: Let $\delta = \inf\{\int(p\underset{\sim}{e} - \underset{\sim}{y}_o^p) : p \in \Delta\}$, where $\underset{\sim}{y}_o^p$ is defined in (5.36). $\underset{\sim}{y}_o^p$ is continuous by Lemma 5.29, and Δ is compact, so the infimum is attained. But if $\int(p\underset{\sim}{e} - \underset{\sim}{y}_o^p) = 0$ for some $p \in \Delta$ then $\underset{\sim}{e}$ is efficient, contrary to our assumption. So $\delta > 0$. But by the concavity of u_t^p, we have

$$\frac{u_t^p(\underset{\sim}{y}^p(t)) - u_t(\underset{\sim}{e}(t))}{u_t^{p\prime}(\underset{\sim}{y}^p(t))} \geqq \underset{\sim}{y}^p(t) - \underset{\sim}{y}_o^p(t) \quad \text{for all} \quad t \in T \quad ,$$

so

$$\int \frac{u^p(\underset{\sim}{y}^p) - u(\underset{\sim}{e})}{u^{p\prime}(\underset{\sim}{y}^p)} \geqq \int (\underset{\sim}{y}^p - \underset{\sim}{y}_o^p) = \int (p\underset{\sim}{e} - \underset{\sim}{y}_o^p) \geqq \delta \quad .$$

But then from (5.35) we have a.e.

$$\underset{\sim}{y}^p(t) + \frac{u_t^p(\underset{\sim}{y}^p(t))}{u_t^{p\prime}(\underset{\sim}{y}^p(t))} = p\underset{\sim}{e}(t) + \frac{u_t(\underset{\sim}{e}(t))}{u_t^{p\prime}(\underset{\sim}{y}^p(t))} + \int \frac{u^p(\underset{\sim}{y}^p) - u(\underset{\sim}{e})}{u^{p\prime}(\underset{\sim}{y}^p)}$$
$$\geqq \delta \quad ,$$

as was to be shown.

We now have the following.

Proposition 5.44: If M is a bounded market and $\underset{\sim}{e}$ is not an efficient allocation in M then there exists a price vector p such that $0 \in Z(p)$.

Proof: The proof of Lemma 17.22 of Aumann and Kurz [1977] uses only those properties of $\underset{\sim}{y}^p$ (and hence of the correspondence Z defined in (5.39)) which we have established in Lemmas 5.34, 5.42, and 5.43, so their proof demonstrates that Z satisfies all the hypotheses of Lemma 5.40, and is convex. Hence there is a price vector p such that $0 \in Z(p)$.

This allows us to establish the existence part of Theorem A.

Theorem 5.45: If M is a bounded market then there exists a value allocation of the economy $E = (M, \Gamma(M))$.

Proof: First we shall deal with the case in which $\underset{\sim}{e}$ is an efficient allocation in M. Then $\underset{\sim}{x} = \underset{\sim}{e}$ satisfies (5.4), so that from Theorem 5.27, $\underset{\sim}{e}$ is a value allocation of E.

If $\underset{\sim}{e}$ is not an efficient allocation in M then by Proposition 5.44 there exists a price vector p such that $\int \underset{\sim}{x}^p = \int \underset{\sim}{e}$ where $\underset{\sim}{x}^p(t)$ a.e. maximizes u_t over $\{x \in \Omega: px \leqq \underset{\sim}{y}^p(t)\}$, and $\underset{\sim}{y}^p$ is the unique efficient allocation in M^p which a.e. satisfies (5.32) (the existence of which is assured by Lemma 5.34). But then by Proposition 5.31, $\underset{\sim}{x}^p$ is a value allocation of E. This completes the proof.

Proof of Theorem A: Theorems 5.27 and 5.45 immediately yield Theorem A.

5.6 Discussion and Examples

We shall now investigate the properties of the value allocations characterized in Theorem A. First, we have the following:

Lemma 5.46: If M is a bounded market and $\underset{\sim}{x}$ is a value allocation of $E = (M, \Gamma(M))$ then $u_t(\underset{\sim}{x}(t)) \geqq u_t(\underset{\sim}{e}(t))$ a.e..

Proof: If $\underset{\sim}{x}$ is a value allocation of E then by Theorem A we know that (5.4) is a.e. satisfied; and by (4.12) we have $\int \underset{\sim}{\lambda}(u(\underset{\sim}{x}) - u(\underset{\sim}{e})) \geqq 0$. Now suppose that $u_t(\underset{\sim}{x}(t)) < u_t(\underset{\sim}{e}(t))$ for all $t \in S$ where $S \in \mathcal{C}$ and $\mu(S) > 0$. Then since p is an efficiency price for $\underset{\sim}{x}$, $p\underset{\sim}{x}(t) < p\underset{\sim}{e}(t)$ a.e. in S. But then $p(\underset{\sim}{e}(t) - \underset{\sim}{x}(t)) + \int \underset{\sim}{\lambda}(u(\underset{\sim}{x}) - u(\underset{\sim}{e})) > 0$ a.e. in S, so that (5.4) is not satisfied by $\underset{\sim}{x}$. Hence $u_t(\underset{\sim}{x}(t)) \geqq u_t(\underset{\sim}{e}(t))$ a.e..

In the proof of Theorem 5.45 we aruged that if $\underset{\sim}{e}$ is an efficient allocation in M then it is a value allocation of $E = (M, \Gamma(M))$. Using Lemma 5.46, we can make the following stronger claim.

Corollary 5.47: If M is a bounded market and $\underset{\sim}{e}$ is an efficient allocation in M then $\underset{\sim}{x}$ is a value allocation of $E = (M, \Gamma(M))$ if and only if $u_t(\underset{\sim}{x}(t)) = u_t(\underset{\sim}{e}(t))$ and $p\underset{\sim}{x}(t) = p\underset{\sim}{e}(t)$ a.e., where p is an efficiency price for $\underset{\sim}{x}$.

Proof: If $\underset{\sim}{x}$ is a value allocation of E then from Lemma 5.46 and the fact that $\underset{\sim}{e}$ is efficient we have $u_t(\underset{\sim}{x}(t)) = u_t(\underset{\sim}{e}(t))$ a.e.. But then from (5.4) we have $p\underset{\sim}{e}(t) = p\underset{\sim}{x}(t)$ a.e.. Conversely, if $\underset{\sim}{x}$ is an allocation for which $u_t(\underset{\sim}{x}(t)) = u_t(\underset{\sim}{e}(t))$ a.e. it is efficient, and if in addition $p\underset{\sim}{x}(t) = p\underset{\sim}{e}(t)$ a.e. then (5.4) is satisfied, so that by Theorem A, $\underset{\sim}{x}$ is a value allocation of E.

This allows us to deal very quickly with the one good case.

Corollary 5.48: If M is a bounded market and $\ell = 1$ then $\underset{\sim}{x} = \underset{\sim}{e}$ is the unique value allocation of $E = (M, \Gamma(M))$.

Proof: If $\ell = 1$ then $\underset{\sim}{e}$ is efficient, and we can take $p = 1$. The result then follows from Corollary 5.47.

This result is of course entirely to be expected: if there is only one good then any redistribution of it will make someone worse off, and every coalition $S \in C$ has a strategy which ensures it a payoff of $\int_S u(\underset{\sim}{e})$. If $\underset{\sim}{e}$ is efficient in the multi-good case then we can make the same argument, except that there may be redistributions of $\underset{\sim}{e}$ in this case which preserve everyone's utility, so that we can only make the statement of Corollary 5.47. If every utility function u_t is strictly concave, however, any redistribution will make someone worse off, and we can conclude that $\underset{\sim}{x}$ is a value allocation if and only if $\underset{\sim}{x}(t) = \underset{\sim}{e}(t)$ a.e..

The more interesting case arises if there are many goods and the initial endowment is not efficient (this is also the realistic case). We shall now study a class of examples which illuminates what happens in general.

Assume that $u_t = u$ for all $t \in T$, and u is homogeneous of degree $\alpha \in (0,1]$. A market with these utility functions does not actually satisfy our boundedness assumption (4.6). However, our calculations will apply to an economy in which each u_t is bounded but $u_t(x)$ coincides with $u(x)$ so long as x is less than some real number B. Let $\underset{\sim}{x}$ be a value allocation in such an economy. Since u is homogeneous all efficient allocations consist of bundles lying on the ray from the origin through the aggregate initial endowment $\int\underset{\sim}{e}$. So we can set $p = u'(\int\underset{\sim}{e})$, and

$$\underset{\sim}{x}(t) = \underset{\sim}{k}(t)\int\underset{\sim}{e} \quad \text{where} \quad \underset{\sim}{k}: T \to \mathbb{R}_+ \quad \text{is measurable and} \quad \int\underset{\sim}{k} = 1 \quad .$$

Assume that $\underset{\sim}{x}(t) \gg 0$ a.e.. Then from (4.13) we have $\underset{\sim}{\lambda}(t) = p^i/u^i(\underset{\sim}{x}(t))$. So from (5.4) we obtain a.e.

$$\frac{p^i u(\underset{\sim}{x}(t))}{u^i(\underset{\sim}{x}(t))} - \frac{p^i u(\underset{\sim}{e}(t))}{u^i(\underset{\sim}{x}(t))} - \int\underset{\sim}{\lambda}[u(\underset{\sim}{x}) - u(\underset{\sim}{e})] = p(\underset{\sim}{e}(t) - \underset{\sim}{x}(t)) \quad .$$

Let $\int\lambda[u(\underset{\sim}{x}) - u(\underset{\sim}{e})] = c$. Then we have a.e.

$$p^i x^i(t) - p^i x^i(t)\frac{u(\underset{\sim}{e}(t))}{u(\underset{\sim}{x}(t))} = [c + p(\underset{\sim}{e}(t) - \underset{\sim}{x}(t))]\frac{u^i(\underset{\sim}{x}(t))x^i(t)}{u(\underset{\sim}{x}(t))} \quad ;$$

or, summing over i, and using the fact that the homogeneity of degree α of u means that $\sum_{i=1}^{\ell} u^i(\underset{\sim}{x}(t))x^i(t)/u(\underset{\sim}{x}(t)) = \alpha$,

$$p\underset{\sim}{x}(t)\left[1 - \frac{u(\underset{\sim}{e}(t))}{u(\underset{\sim}{x}(t))}\right] - \alpha c = \alpha p(\underset{\sim}{e}(t) - \underset{\sim}{x}(t)) \quad \text{a.e.,}$$

or

$$(5.49) \qquad p\underset{\sim}{x}(t)\left[1 + \alpha - \frac{u(\underset{\sim}{e}(t))}{u(\underset{\sim}{x}(t))}\right] = \alpha(c + p\underset{\sim}{e}(t)) \quad \text{a.e.} \quad .$$

Now recalling that $\underset{\sim}{x}(t) = \underset{\sim}{k}(t)\int\underset{\sim}{e}$, we have

$$(1 + \alpha)\underset{\sim}{k}(t) - \frac{u(\underset{\sim}{e}(t))}{u(\int\underset{\sim}{e})}(\underset{\sim}{k}(t))^{1-\alpha} = \frac{\alpha(c + p\underset{\sim}{e}(t))}{p\int\underset{\sim}{e}} \quad \text{a.e.} \quad .$$

It is easy to solve this equation for $\underset{\sim}{k}(t)$ if $\alpha = 1$; if not, the solution is rather complex. In the case $\alpha = 1$ we have a.e.

$$\underset{\sim}{k}(t) = \frac{c + p\underset{\sim}{e}(t)}{2p\int\underset{\sim}{e}} + \frac{u(\underset{\sim}{e}(t))}{2u(\int\underset{\sim}{e})} \quad ,$$

so that a.e.

$$\underset{\sim}{x}(t) = \frac{1}{2}\int\underset{\sim}{e}\left[\frac{c + p\underset{\sim}{e}(t)}{p\int\underset{\sim}{e}} + \frac{u(\underset{\sim}{e}(t))}{u(\int\underset{\sim}{e})}\right] \quad .$$

Integrating, and using the fact that $\int \underset{\sim}{x} = \int \underset{\sim}{e}$, we obtain

$$c = p\int \underset{\sim}{e}\left[1 - \frac{\int u(\underset{\sim}{e})}{u(\int \underset{\sim}{e})}\right] .$$

Hence a.e.

$$\underset{\sim}{x}(t) = \frac{1}{2}\left[1 + \frac{p\underset{\sim}{e}(t)}{p\int \underset{\sim}{e}} + \frac{u(\underset{\sim}{e}(t)) - \int u(\underset{\sim}{e})}{u(\int \underset{\sim}{e})}\right]\int \underset{\sim}{e} ,$$

where $p = u'(\int \underset{\sim}{e})$. So a.e.

$$p\underset{\sim}{e}(t) - p\underset{\sim}{x}(t) = \frac{1}{2}\left\{p\underset{\sim}{e}(t) - p\int \underset{\sim}{e}\left[1 + \frac{u(\underset{\sim}{e}(t)) - \int u(\underset{\sim}{e})}{u(\int \underset{\sim}{e})}\right]\right\}$$

$$= \frac{1}{2}\left(p\underset{\sim}{e}(t) - \frac{u(\underset{\sim}{e}(t))}{u(\int \underset{\sim}{e})}p\int \underset{\sim}{e}\right) - c .$$

So a.e.

(5.50) $$p\underset{\sim}{e}(t) - p\underset{\sim}{x}(t) = \frac{1}{2}(p\underset{\sim}{e}(t) - u(\underset{\sim}{e}(t))) - c$$

since $p = u'(\int \underset{\sim}{e})$ and $u'(\int \underset{\sim}{e})\int \underset{\sim}{e}/u(\int \underset{\sim}{e}) = 1$ since u is homogeneous of degree one.

Now, for each price vector p, let $u^p\colon \mathbb{R}_+ \to \mathbb{R}_+$ be the indirect utility function at p (see (5.28)) and define $\underset{\sim}{y}_o^p\colon T \to \mathbb{R}_+$ by

$$\underset{\sim}{y}_o^p(t) = (u^p)^{-1}(u(\underset{\sim}{e}(t)) \quad \text{for each} \quad t \in T$$

(see Diagram 3 for the two good case). Since u is homogeneous of degree one, u^p is homogeneous of degree one in y -- i.e. $u^p(y) = ay$

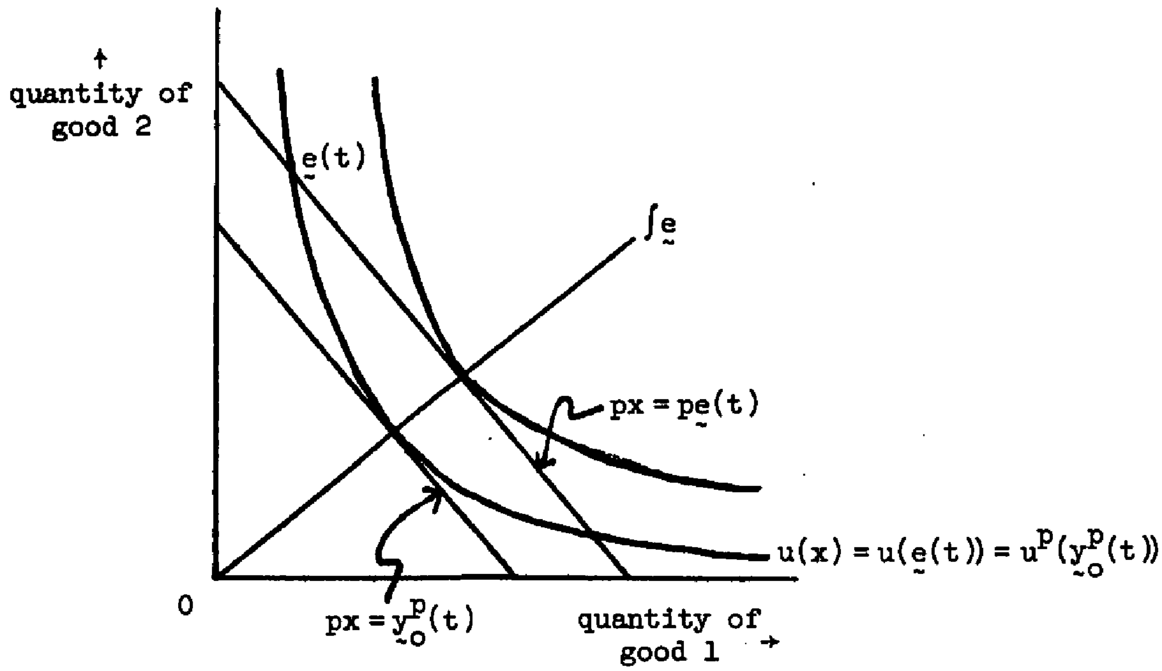


Diagram 3

for some $a > 0$, for all $y \in \mathbb{R}_+$. But we have $u^p(p\int e) = u(\int e)$ since $\int e$ lies on the efficient ray, so because of the way we normalized the price vector, we have

$$a = \frac{u^p(p\int e)}{p\int e} = \frac{u(\int e)}{u'(\int e)\int e} = 1 \quad .$$

So we have $y_o^p(t) = u(e(t))$. Hence from (5.50), x is a value allocation in the economy if a.e.

$$(5.51) \qquad pe(t) - px(t) = \tfrac{1}{2}(pe(t) - y_o^p(t)) - c \quad .$$

Since p $(= u'(\int e))$ is an efficiency price for x, this allocation can be realized in a competitive equilibrium at prices p in which each agent $t \in T$ starts with the wealth $p\underset{\sim}{x}(t)$ rather than $p\underset{\sim}{e}(t)$ -- i.e. he pays the "tax" $p\underset{\sim}{e}(t) - p\underset{\sim}{x}(t)$. Now, the quantity $\underset{\sim}{y}_o^p(t)$ measures how badly off t is when he is prevented from trading (at prices p)--it is his implicit "wealth" in this circumstance. Thus (5.51) shows that at the value allocation $\underset{\sim}{x}$ each agent is taxed at the marginal rate of 50% on the increase in his wealth due to the possibilities for trade. When utility is transferable we have from Proposition 5.24 that for all $t \in T$,

$$(\phi q)(S) = \mu(S)\frac{1}{2}[r(T) - \eta(T)] + \int_S u(\underset{\sim}{e}) + \frac{1}{2}[\int_S [u(\underset{\sim}{x}) - p(\underset{\sim}{x} - \underset{\sim}{e})] - \int_S u(\underset{\sim}{e})]$$

for each $S \in \mathcal{C}$, so that we can see that the value is the result of a 50% "tax" on $\int_S [u(\underset{\sim}{x}) - p(\underset{\sim}{x} - \underset{\sim}{e})] - \int_S u(\underset{\sim}{e})$, which can be interpreted as the "gains from trade" since $\int_S [u(\underset{\sim}{x}) - p(\underset{\sim}{x} - \underset{\sim}{e})]$ is the t.u.c.e. payoff of S. Given this, we obtain the result above when all agents have the same utility function which is homogeneous of degree one because the price is then independent of the efficient allocation chosen, and income is proportional to utility: $p\underset{\sim}{e}(t) - \underset{\sim}{y}_o^p(t)$ measures precisely the "gains from trade".

Aumann and Kurz [1977] established that in general the tax density of an agent implicit in the allocations their model generates depends solely on his wealth density $p\underset{\sim}{e}(t)$; we might hope that <u>in general</u> the tax in our cases is levied on $p\underset{\sim}{e}(t) - \underset{\sim}{y}_o^p(t)$. But this is not so. When all agents have the same homogeneous utility function then the tax does depend solely on $p\underset{\sim}{e}(t)$ and $\underset{\sim}{y}_o^p(t)$, but it is not true that agents for which $p\underset{\sim}{e}(t) - \underset{\sim}{y}^P(t)$ is the same pay the same tax (a calculation for the case $\alpha = 1/2$ confirms that this is so). The tax here depends on the utility loss from being prohibited from trading and this utility loss can be different for agents with identical preferences, so that the tax is not simply related to the quantity of trade in which an agent wants to engage at the equilibrium prices. Thus the value allocations in the game which we have studied in this chapter involve an "ideal" form of tax on the "value" of trades which depends on the cardinal characteristics of the agents' utility functions.

CHAPTER 6: Economies in which the Entire Endowment of Society Is Available to any Majority

6.1 Introduction

Here we shall study the consequences of giving majority coalitions as much power as they can possibly expect to have; we shall retain the idealization of "majority rule" employed in the previous chapter. What we assume is that any coalition containing a majority of the population has a strategy which allows it to expropriate the entire endowment of the complementary minority, while the latter can do nothing to retaliate. We choose to investigate the consequences of this extreme assumption about the power of majorities for a number of reasons. Firstly, we are interested in the range of outcomes which can be generated by different assumptions about the strategic game, and in studying an extreme case we shall obtain a "bound" on these possible outcomes. Secondly, under our assumptions the worth of each coalition is independent of its endowment, so that the final allocation of each agent is independent of his endowment, and depends solely on his utility function. Thus the outcome is "egalitarian" in a certain sense; it is interesting to see what form of "egalitarianism" the Harsanyi-Shapley values give rise to. Finally, it appears to be realistic to assume that it is not feasible to threaten to destroy some goods (like land), though this is clearly not true of all goods (consider, for example, labor-time). Our study of the extreme case where no good can be destroyed provides the basis for the study of an economy in which some goods can and some cannot be destroyed (see Chapter 7). Note that with respect to majority coalitions our assumption here is the same as that of Aumann and Kurz [1977], but that they assume that minorities can threaten to destroy their endowments in response to any threat of a majority to expropriate them, thereby preventing the majority from obtaining use of the entire endowment of society.

Let M be a market. The strategy sets and payoff functions of the stategic game $\Gamma(M)$ of the economy $E = (M, \Gamma(M))$ which we shall study here satisfy the following conditions:

(6.1) if $S \in \mathcal{C}$ is such that $\mu(S) > 1/2$, then for every S-allocation $\underset{\sim}{x}$ of the total endowment $e(T)$ there is a strategy σ of S such that for every strategy τ of $T \setminus S$,

$$\underset{\sim}{h}^S_{\sigma\tau}(t) \begin{cases} = u_t(\underset{\sim}{x}(t)) & \text{if } t \in S \\ = 0 & \text{if } t \in T \setminus S \end{cases} ;$$

(6.2) if $S \in \mathcal{C}$ is such that $\mu(S) \geqq 1/2$, then for all strategies τ of $T \setminus S$ and all strategies σ of S, there exists an S-allocation $\underset{\sim}{x}$ of the total endowment $e(T)$ such that

$$\underset{\sim}{h}^S_{\sigma\tau}(t) \begin{cases} \leqq u_t(\underset{\sim}{x}(t)) & \text{if } t \in S \\ \geqq 0 & \text{if } t \in T \setminus S \end{cases} ;$$

and

(6.3) if $S \in \mathcal{C}$ is such that $\mu(S) = 1/2$, then for each S-allocation $\underset{\sim}{x}$ there is a strategy σ of S such that for each strategy τ of $T \setminus S$ there is a $T \setminus S$-allocation $\underset{\sim}{y}$ such that

$$h^S_{\sigma\tau}(t)\begin{cases} \geq u_t(x(t)) & \text{if } t \in S \\ \leq u_t(y(t)) & \text{if } t \in T\setminus S \end{cases} .$$

(Note that (6.3) is the same as (5.3).)

The main result in this chapter is the following.

Theorem B: Let M be a homogeneous market and assume that $\Gamma(M)$ satisfies (6.1) through (6.3). Then an allocation x is a value allocation of the economy $E = (M, \Gamma(M))$ if and only if it is efficient and a.e.

$$px(t) = pe(T) \quad ,$$

where p is an efficiency price for x. Moreover, such an allocation exists.

Throughout this chapter $\Gamma(M)$ will denote a strategic game associated with the market M which satisfies (6.1) through (6.3).

6.2 Optimal Threats

As in the previous chapter, the first step in characterizing the set of value allocations of E is to establish the nature of a collection of optimal threats in the two-person games between S and $T\setminus S$ for each $S \in \mathcal{C}$. We first study the case where $\lambda(t) = 1$ for all $t \in T$. Recall that for each $S \in C$ the function $u_S\colon \mathbb{R}^\ell_+ \to \mathbb{R}_+$ is defined by

(6.4) $$u_S(a) = \sup \{\int_S u(x) : \int_S x = a\} \quad \text{for each } a \in \mathbb{R}^\ell_+$$

and that we say that $u_S(a)$ is "attained" if it is finite and there is an S-allocation $\underset{\sim}{x}$ of a such that $u_S(a) = \int_S u(\underset{\sim}{x})$ (see Section 4.2). Now, define the game $v\colon \mathcal{C} \to \mathbb{R}_+$ by

$$(6.5) \qquad v(S) = u_S(e(T)) \quad \text{for all} \quad S \in \mathcal{C} \quad ,$$

and let $r\colon \mathcal{C} \to \mathbb{R}_+$ be the market game derived from M (i.e. $r(S) = u_S(e(S))$ for each $S \in \mathcal{C}$). Then we have the following.

Lemma 6.6: Assume that both $v(S)$ and $r(S)$ are attained for every $S \in \mathcal{C}$. Then for each $S \in \mathcal{C}$ there exists a pair $(\sigma_o, \tau_o) \in X^S \times X^{T\setminus S}$ such that

$$(6.7) \qquad H^S(\sigma, \tau_o) \leqq H^S(\sigma_o, \tau_o) \leqq H^S(\sigma_o, \tau) \quad \text{for all} \quad \sigma \in X^S,\ \tau \in X^{T\setminus S} \quad ,$$

$H^S(\sigma_o, \tau_o)$ is finite, and

$$\int_S \underset{\sim}{h}^S_{\sigma_o \tau_o} = \begin{cases} v(S) & \text{if} \quad \mu(S) > 1/2 \\ r(S) & \text{if} \quad \mu(S) = 1/2 \\ 0 & \text{if} \quad \mu(S) < 1/2 \ . \end{cases}$$

Proof: Consider a coalition $S \in \mathcal{C}$ with $\mu(S) > 1/2$. Let $\underset{\sim}{x}$ be an S-allocation of $e(T)$ which attains $v(S)$. Let σ_o be the strategy of S corresponding to $\underset{\sim}{x}$ given in (6.1). Then

$$\int_S \underset{\sim}{h}^S_{\sigma_o \tau} = \int_S u(\underset{\sim}{x}) \quad \text{for all} \quad \tau \in X^{T\setminus S} \quad .$$

But by (6.2), setting $\tau = \tau_o$ (an arbitrary strategy of $T\setminus S$), we know that there exists an S-allocation $\underset{\sim}{z}$ of $e(T)$ such that

$$\int_S h^S_{\underset{\sim}{\sigma}\tau_o} \leqq \int_S u(\underset{\sim}{z}) \quad \text{for all} \quad \sigma \in X^S \quad .$$

So by the definition of $\underset{\sim}{x}$ we have

$$\int_S h^S_{\underset{\sim}{\sigma}\tau_o} \leqq \int_S h^S_{\underset{\sim}{\sigma}_o\tau_o} = \int_S u(\underset{\sim}{x}) = \int_S h^S_{\underset{\sim}{\sigma}_o\tau} \quad \text{for all} \quad \sigma \in X^S \quad , \quad \tau \in X^{T\setminus S} \quad .$$

Also, from (6.1) we have

$$\int_{T\setminus S} h^S_{\underset{\sim}{\sigma}_o\tau} = 0 \quad \text{for all} \quad \tau \in X^{T\setminus S} \quad ,$$

and from (6.2)

$$\int_{T\setminus S} h^S_{\underset{\sim}{\sigma}\tau_o} \geqq 0 \quad \text{for all} \quad \sigma \in X^S \quad .$$

So

$$\int_{T\setminus S} h^S_{\underset{\sim}{\sigma}\tau_o} \geqq \int_{T\setminus S} h^S_{\underset{\sim}{\sigma}_o\tau_o} = 0 = \int_{T\setminus S} h^S_{\underset{\sim}{\sigma}_o\tau} \quad \text{for all} \quad \sigma \in X^S \quad , \quad \tau \in X^{T\setminus S} \quad .$$

Hence

$$H^S(\sigma,\tau_o) = \int_S h^S_{\underset{\sim}{\sigma}\tau_o} - \int_{T\setminus S} h^S_{\underset{\sim}{\sigma}\tau_o} \leqq H^S(\sigma_o,\tau_o) = \int_S h^S_{\underset{\sim}{\sigma}_o\tau_o} - \int_{T\setminus S} h^S_{\underset{\sim}{\sigma}_o\tau_o}$$

$$= \int_S h^S_{\underset{\sim}{\sigma}_o\tau} - \int_{T\setminus S} h^S_{\underset{\sim}{\sigma}_o\tau} = H^S(\sigma_o,\tau) \quad \text{for all} \quad \sigma \in X^S \quad , \quad \tau \in X^{T\setminus S} \quad .$$

So (6.7) is satisfied by the pair (σ_o,τ_o) when $\mu(S) > 1/2$. Moreover,

$$\int_S h^S_{\sigma_o\tau_o} = v(S) \quad \text{and} \quad \int_{T\setminus S} h^S_{\sigma_o\tau_o} = 0 \quad ,$$

so $H^S(\sigma_o,\tau_o) = v(S)$, and we have proved what is claimed in the lemma in the case when $\mu(S) > 1/2$. By reversing the roles of S and $T\setminus S$ this completes the proof for those $S \in C$ with $\mu(S) < 1/2$ also. Finally, if $\mu(S) = 1/2$ the last part of the proof of Lemma 5.6 establishes what is claimed, since (5.3) and (6.3) are identical.

Thus, as we should expect, an optimal threat of a majority is to expropriate the resources of its complement and distribute them optimally among itself, while a minority can do nothing to prevent this expropriation. Define the game $q: C \to \mathbb{R}_+$ by

$$(6.8) \qquad q(S) = \int_S h^S_{\sigma_o\tau_o} = \begin{cases} v(S) & \text{if} \quad \mu(S) > 1/2 \\ r(S) & \text{if} \quad \mu(S) = 1/2 \\ 0 & \text{if} \quad \mu(S) < 1/2 \end{cases} .$$

As in the previous chapter we have not established that for each $S \in C$, (σ_o,τ_o) is the only pair of optimal threats. However, once again since it will turn out that the game defined in (6.8) always possesses an asymptotic value we shall, by virtue of Lemma 3.25, locate all the value allocations of E by restricting attention to the family of pairs (σ_o,τ_o) given in Lemma 6.6. Before establishing that q does indeed possess an asymptotic value, we shall state two results which we shall use later. They are the counterparts of Lemmas 5.17 and 5.20, and since the proofs under the assumptions on $\Gamma(M)$ which we are making here are the same as those given in Chapter 5, we shall omit them.

Lemma 6.9: Suppose that H^T possesses a finite saddle point. Then $r(T) = v(T)$ is attained.

Lemma 6.10: An allocation $\underset{\sim}{x}$ is a value allocation of $E = (M, \Gamma(M))$ if and only if there exists a comparison function $\underset{\sim}{\lambda}$ such that $H^S_{\underset{\sim}{\lambda}}$ has a finite saddle point for every $S \in C$ and

$$(\phi q_{\underset{\sim}{\lambda}})(S) = \int_S \underset{\sim}{\lambda} u(\underset{\sim}{x}) \quad \text{for each} \quad S \in C \ ,$$

where $q_{\underset{\sim}{\lambda}}$ is the game defined by (3.24).

Lemma 6.9 is used in the proof of Lemma 6.37. Lemma 6.10 has a role parallel to that of Lemma 5.20. Now, from Lemma 6.6 we know that in order to characterize the value allocations of E we need to study the properties of the game q defined in (6.8). With that goal in mind, we shall now study the game v defined in (6.5).

6.3 The Game v

We shall show here that the game v defined in (6.5) is a member of pNA, and can in fact be approximated in the bounded variation norm by a similar game derived from a finite type market (we shall need the latter result in the calculation of the value of the game q).

To do so we shall construct a market game which is related to v, approximate this game by a finite type market game and then show that this generates for us a finite type approximation to v. The finite type market we construct contains agents whose utility functions are identically zero, and hence in particular not strictly increasing; it is at this point that we use Proposition 4.50 (the generalization of Proposition 4.24).

Proposition 6.11: Let M be an integrably sublinear market. Let v be the coalitional form derived from M defined by (6.5). Then $v \in pNA$, and for each $\varepsilon > 0$ there is an integrably sublinear finite type market $\hat{M}$, which differs from M only in the utility functions of the agents, such that

$$\|v - \hat{v}\| < \varepsilon$$

where $\hat{v}$ is the coalitional form derived from $\hat{M}$ defined by (6.5).

Proof: Let (Z,C',μ') be a copy[4/] of (T,C,μ), and let $\bar{T} = Z \cup T$; let $\bar{C}$ be the σ-algebra of subsets of $\bar{T}$ generated by C and C', and let $\bar{\mu}$ be the measure on $(\bar{T}, \bar{C})$ generated by μ and μ'. Define a general market $\bar{M}$ in the following way. Let $(\bar{T},\bar{C},\bar{\mu})$ be the space of agents, $\bar{u}_t$ the utility function of $t \in \bar{T}$, and $\bar{e}$ the initial endowment density, with

$$\bar{u}_t(x) = \begin{cases} 0 & \text{for all } x \in \Omega \quad \text{if } t \in Z \\ u_t(x) & \text{for all } x \in \Omega \quad \text{if } t \in T \end{cases}$$

and

$$\bar{\underset{\sim}{e}}(t) = \begin{cases} \underset{\sim}{e}(t) & \text{if} \quad t \in Z \\ 0 & \text{if} \quad t \in T \end{cases} .$$

Let the coalitional form of $\bar{M}$ be denoted by $w: \bar{\mathcal{C}} \to \mathbb{R}_+$ --i.e.

$$w(\bar{S}) = \sup \{\int_{\bar{S}} \bar{u}(\underset{\sim}{x}) : \int_{\bar{S}} \underset{\sim}{x} = \bar{\underset{\sim}{e}}(\bar{S})\} \quad \text{for all} \quad \bar{S} \in \bar{\mathcal{C}} \quad .$$

Let $S \in \mathcal{C}$. Then

$$(6.12) \qquad w(Z \cup S) = \sup \{ \int_{Z \cup S} \bar{u}(\underset{\sim}{x}) : \int_{Z \cup S} \underset{\sim}{x} = \bar{\underset{\sim}{e}}(Z \cup S)\}$$
$$= \sup \{\int_S u(\underset{\sim}{x}) : \int_S \underset{\sim}{x} = e(T)\} = v(S) \quad .$$

Now, $\bar{M}$ is an integrably sublinear quasi-market with zero-utility agents which satisfies (4.25), so by Corollary 4.49 we have $w \in pNA$. It is easy to verify that from (6.12) we then have $v \in pNA$.

To demonstrate the second part of the result, fix $\varepsilon > 0$. By Proposition 4.50 there is an integrably sublinear finite type market $\hat{\bar{M}}$ which differs from $\bar{M}$ only in the utility functions of the agents and which has coalitional form $\hat{w}$ such that

$$(6.13) \qquad \|w - \hat{w}\| < \varepsilon/2 \quad .$$

Define the market $\hat{M}$ in the following way. $(T, \mathcal{C}, \mu)$ is the space of agents, $\underset{\sim}{e}$ the initial endowment density, and $\hat{u}_t$ the utility function of $t \in T$, where $\hat{u}_t$ is the utility function of $t \in T$ in $\hat{\bar{M}}$.

Let $\hat{v}: \mathcal{C} \to \mathbb{R}$ be the game derived from $\hat{M}$ defined by (6.5). Then for $S \in \mathcal{C}$,

$$(6.14) \qquad \hat{v}(S) = \sup \{\int_S \hat{u}(\underset{\sim}{x}) : \int_{S} \underset{\sim}{x} = e(T)\} = \sup \{ \int_{Z\cup S} \hat{u}(\underset{\sim}{x}) : \int_{Z\cup S} \underset{\sim}{x} = \bar{e}(Z \cup S)\}$$

$$= \hat{w}(Z \cup S) \quad .$$

We shall now show that $\|v - \hat{v}\| < \varepsilon$. Let

$$\emptyset = S_o \subset S_1 \subset \ldots \subset S_m \subset S_{m+1} = T$$

be a chain (of subsets of T). Then

$$\emptyset \subset Z \cup S_o = Z \subset Z \cup S_1 \subset \ldots \subset Z \cup S_m \subset Z \cup S_{m+1} = Z \cup T = \bar{T}$$

is a chain of subsets of $\bar{T}$. So from (6.13) we have

$$|w(Z) - \hat{w}(Z)| + \sum_{k=0}^{m} |(w(Z \cup S_{k+1}) - \hat{w}(Z \cup S_{k+1})) - (w(Z \cup S_k) - \hat{w}(Z \cup S_k))|$$

$$< \varepsilon/2 \quad .$$

But $w(Z) = \hat{w}(Z) = 0$, so

$$\sum_{k=0}^{m} |(v(S_{k+1}) - \hat{v}(S_{k+1})) - (v(S_k) - \hat{v}(S_k))| < \varepsilon/2 \quad ,$$

using (6.12) and (6.14). Hence $\|v - \hat{v}\| \leq \varepsilon/2 < \varepsilon$, and the proof is complete.

In the calculation of the value of the game q (in Section 6.5) we shall also need a result concerning the nature of the game v in a finite type market. Suppose M is of finite type, with $u_t \in \{f_1,\ldots,f_n\}$ for each $t \in T$. Let $S_i = \{t \in T: u_t = f_i\}$ for each $i = 1,\ldots,n$, and define the function $g: \mathbb{R}^n_+ \times \Omega \to \mathbb{R}_+$ by

$$(6.15)\quad g(y,z) = \max \{ \sum_{i=1}^{n} y^i f_i(x_i) : x_i \in \Omega \text{ for all } i = 1,\ldots,n \text{ and } \sum_{i=1}^{n} y^i x_i \leqq z \}$$

for each $(y,z) \in \mathbb{R}^n_+ \times \Omega$. This maximum is always attained, g is concave, nondecreasing, and continuous, and $\partial g/\partial a^i$ exists and is continuous at each $a = (y,z)$ for which $a^i > 0$ $(i = 1,\ldots,n + \ell)$ (see Lemmas 39.9 and 39.13 of Aumann and Shapley). The result which we shall need is the following.

Lemma 6.16: If M is of finite type with $u_t \in \{f_1,\ldots,f_n\}$ for each $t \in T$, and v is the coalitional form derived from M defined in (6.5), then $v = \hat{g} \circ \eta$, where $\eta = (\eta_1,\ldots,\eta_n)$, η_i is the member of NA defined by $\eta_i(S) = \mu(S \cap S_i)$ for each $S \in \mathcal{C}$, for each $i = 1,\ldots,n$, and $\hat{g}: \mathbb{R}^n_+ \to \mathbb{R}_+$ is defined by $\hat{g}(y) = g(y,e(T))$ for each $y \in \mathbb{R}^n_+$.

Proof: From Lemma 39.8 of Aumann and Shapley we have $u_S(z) = g(\eta(S),z)$ for each $S \in \mathcal{C}$ and each $z \in \Omega$. Hence $v(S) = u_S(e(T))$ $= g(\eta(S),e(T)) = \hat{g}(\eta(S))$ for each $S \in \mathcal{C}$, completing the proof.

(Note that this gives us an alternative way to show that $v \in pNA$ (cf. Lemma 39.16 of Aumann and Shapley).) For our purposes in this chapter we do not need to study the value of the game v, but rather that of the game q. However, it turns out that the value of the game v is of some interest, and we discuss it in Appendix 2, using the results of this section.

6.4 The Value of the Game q

From Lemma 6.6 the game in which we are interested is $q: \mathcal{C} \to \mathbb{R}$ defined in (6.8). Define the game $k: \mathcal{C} \to \mathbb{R}$ by

$$k(S) = \begin{cases} v(S) & \text{if } \mu(S) \geqq 1/2 \\ 0 & \text{if } \mu(S) < 1/2 \end{cases} . \tag{6.17}$$

Define the function $g: [0,1] \to [0,1]$ by

$$g(x) = \begin{cases} 1 & \text{if } x \geqq 1/2 \\ 0 & \text{if } x < 1/2 \end{cases} .$$

Then $g \circ \mu \in bv'NA$, and $k = (g \circ \mu)*v$. In this section we shall argue that k and q both possess asymptotic values, and that they coincide. We shall then use a result of Aumann and Kurz [1977] (Proposition 3.16) to derive an expression for ϕk, and hence ϕq.

Let $b: \mathcal{C} \to \mathbb{R}$ be the game given by $b = k - q$. Define the function $f: [0,1] \to [0,1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x = 1/2 \\ 0 & \text{otherwise} \end{cases} .$$

Then $f \circ \mu \in bv'NA$ and $b = (f \circ \mu)*(v - r)$. Recall that if $v: \mathcal{C} \to \mathbb{R}$ is a game, v^* denotes the extension of v to the collection of ideal coalitions (see Section 3.1).

Lemma 6.18: If M, is integrably sublinear then $q \in ASYMP$, and ϕq is given by

$$(6.19) \qquad (\phi q)(S) = v^*(\chi_T/2)\mu(S) + \int_{1/2}^{1} \partial v^*(\theta,S)d\theta \quad \text{for each} \quad S \in \mathcal{C} \quad .$$

Proof: If M is integrably sublinear then by Proposition 6.11 we have $v \in pNA$, so by Proposition 3.16 we have $k \in ASYMP$, with ϕk defined by

$$(\phi k)(S) = v^*(\chi_T/2)\mu(S) + \int_{1/2}^{1} \partial v^*(\theta,S)d\theta \quad \text{for each} \quad S \in \mathcal{C} \quad .$$

Now, by Proposition 4.27 we have $r \in pNAD$. Hence $v - r \in pNAD$, and $b = (f \circ \mu)*(v - r) \in bv'NA*pNAD$. So by Corollary 3.8 we have $b \in ASYMP$; it is easy to see that $(\phi b)(S) = 0$ for all $S \in \mathcal{C}$. But $q = k - b$. So $q \in ASYMP$; since $(\phi b)(S) = 0$ for all $S \in \mathcal{C}$, we have $\phi q = \phi k$, completing the proof.

6.5 The Calculation of ϕq in Homogeneous Markets

In order to calculate the value of the game q we know from (6.19) that we have to understand the behavior of $\partial v^*(\theta,S)$ for $\theta \in [1/2,1]$. Unlike any market game, which possesses a strong homogeneity property independent of the utility functions of the agents comprising the market, the game v can behave quite irregularly unless we restrict the characteristics of the utility functions, making a direct calculation of ϕq based on (6.19) impossible. For this reason we shall restrict our attention here to homogeneous markets (for the definition of which see Chapter 4); in this section we shall calculate ϕq for such markets. The proof of the following result closely follows that of Lemma 39.16 of Aumann and Shapley.

<u>Lemma 6.20</u>: If M is homogeneous of degree $\beta \in (0,1)$ and $v(T)$ if finite then v is homogeneous of degree $1 - \beta$.

<u>Proof</u>: We have to show that $v^*(\alpha\chi_S) = \alpha^{1-\beta}v(S)$ for all $\alpha \in [0,1]$ and all $S \in \mathcal{C}$. Fix $\alpha \in [0,1]$. First suppose that M is of finite type, with $u_t \in \{f_1,\ldots,f_n\}$ for each $t \in T$. Then from Lemma 6.16 we have

$$v(S) = \hat{g}(\eta(S)) \quad \text{for each} \quad S \in \mathcal{C} \quad ,$$

where η is an n-vector of nonnegative members of NA, and $\hat{g}\colon \mathbb{R}_+^n \to \mathbb{R}_+$ is continuous. Since each component of η is in NA, the domain of $\hat{g}$ can be taken to be compact; denote it by K. Then by the Weierstrass approximation theorem (see, for example, Royden [1968], Corollary 29, p. 174), for each $j = 1,2,\ldots$ we can find a polynomial $h_j\colon K \to \mathbb{R}_+$ such that

$$|h_j(y) - g(y)| < 1/j \quad \text{for all} \quad y \in K \quad .$$

But by (3.9), (3.10), and (3.11) we have

$$(h_j \circ \eta)^*(\alpha\chi_S) = (h_j \circ \eta^*)(\alpha\chi_S) = h_j(\alpha\eta(S))$$

for each $\alpha \geqq 0$, $S \in \mathcal{C}$ and $j = 1,2,\ldots$. Now let $j \to \infty$; using (3.12) we have

$$v^*(\alpha\chi_S) = (\hat{g} \circ \eta)^*(\alpha\chi_S) = \hat{g}(\alpha\eta(S)) \quad .$$

Now, we are assuming that M is homogeneous of degree β, so f_i is homogeneous of degree β for all $i = 1,\ldots,n$, and we have

$$\begin{aligned}
\hat{g}(\alpha\eta(S)) &= \max \{ \sum_{i=1}^{n} \alpha\eta_i(S) f_i(x_i) : x_i \in \Omega \text{ for all } i = 1,\ldots,n \\
&\qquad \text{and } \sum_{i=1}^{n} \alpha\eta_i(S) x_i \leqq e(T) \} \\
&= \max \{ \sum_{i=1}^{n} \alpha^{1-\beta}\eta_i(S) f_i(\alpha x_i) : \alpha x_i \in \Omega \text{ for all } i = 1,\ldots,n \\
&\qquad \text{and } \sum_{i=1}^{n} \eta_i(S)\alpha x_i \leqq e(T) \} \\
&= \alpha^{1-\beta} \max \{ \sum_{i=1}^{n} \eta_i(S) f_i(z_i) : z_i \in \Omega \text{ for all } i = 1,\ldots,n \\
&\qquad \text{and } \sum_{i=1}^{n} \eta_i(S) z_i \leqq e(T) \} \\
&= \alpha^{1-\beta}\hat{g}(\eta(S))
\end{aligned}$$

for all $S \in \mathcal{C}$. Hence

$$v^*(\alpha\chi_S) = \hat{g}(\alpha\eta(S)) = \alpha^{1-\beta}\hat{g}(\eta(S)) = \alpha^{1-\beta}v(S)$$

for all $S \in C$. So if M is of finite type we have proved the lemma.

Now consider a market M which satisfies the conditions of the lemma, and is not necessarily of finite type. Then by Proposition 4.18, M is integrably sublinear (since $v(T) = r(T)$), so by Proposition 6.11, for each $\varepsilon > 0$ there is a finite type market $\hat{M}$ such that $\|v - \hat{v}\| < \varepsilon$, where $\hat{v}$ is the coalitional form derived from $\hat{M}$ defined by (6.5). By our argument above, $\hat{v}$ is homogeneous of degree $1 - \beta$; but the space of games in pNA which are homogeneous of degree γ for any $\gamma \in [0,1]$ is closed in BV (see the proof of Proposition 27.12 of Aumann and Shapley), so v is also homogeneous of degree $1 - \beta$. This completes the proof.

This result allows us to say something about the behavior of $\partial v^*(\theta,S)$ as θ varies, for a fixed $S \in C$.

<u>Corollary 6.21</u>: If M is homogeneous of degree $\beta \in (0,1)$ and $v(T)$ is finite then $\partial v^*(\theta,S)$ exists for all $\theta \in (0,1)$ and is homogeneous of degree $-\beta$ in θ for each $S \in C$.

<u>Proof</u>: This follows immediately from Lemma 6.20, Proposition 4.18, Proposition 6.11, and Lemma 3.18.

Given this homogeneity of $\partial v^*(\theta,S)$ we shall be able to calculate $(\phi q)(S)$ for any $S \in C$ using (6.19) if we can evaluate it at one point. Now, for any game w, $\partial w^*(\theta,S)$ is the rate of change of w^* at the point $\theta\chi_T$ in the direction of χ_S. Fix $S \in C$ with $\mu(S) < 1$ and consider a market M_S in which the space of agents, number of goods, and utility function of each agent are the same as in M, but in which the endowment density $\underset{\sim}{e}_S$ satisfies $e_S(S) = 0$, $e_S(T) = e(T)$, and for each $t \in T$ either $\underset{\sim}{e}_S(t) = 0$ or $\underset{\sim}{e}_S(t) >> 0$. Let $r_S: C \to \mathbb{R}$ be the market game derived from M_S --i.e.

$$(6.22) \qquad r_S(R) = \sup \{\int_R u(\underset{\sim}{x}): \int_R \underset{\sim}{x} = e_S(R)\} \quad \text{for each} \quad R \in C \quad .$$

In this game the ideal coalition $\theta\chi_T$ has endowment $\theta e(T)$, and adding a small replica of χ_S to $\theta\chi_T$ does not increase this endowment (since $e_S(S) = 0$), but only affects the worth through the addition of the utility functions of the members of S. But it is also the case for the game v that adding a small replica of χ_S to $\theta\chi_T$ does not affect the quantity of goods available to the coalition ($e(T)$ is available to all coalitions), so that the worth is only affected through the additions of the utility functions of the members of S. The only difference is that the quantity of goods available to $\theta\chi_T$ in the game v is $e(T)$, rather than $\theta e(T)$ in the case of the game r_S. But for θ close to 1 these are close, so that under such circumstances this argument suggests that $\partial r_S^*(\theta,S)$ and $\partial v^*(\theta,S)$ are close. Since r_S is a market game, $\partial r_S^*(\theta,S)$ is actually independent of θ. Thus the above reasoning suggests that $\partial v^*(\theta,S)$ converges to $\partial r_S^*(\theta_o,S)$ as $\theta \to 1$, for any given $\theta_o \in (0,1)$. We shall now make this precise.

<u>Lemma 6.23</u>: Let M be homogeneous, and suppose $v(T)$ is finite. Fix $S \in C$ with $\mu(S) < 1$ and let r_S be the market game derived from M_S (see (6.22)). Then $\partial r_S^*(\theta,S)$ exists and is the same for all $\theta \in (0,1)$ and

$$(6.24) \qquad \lim_{\theta \to 1} \partial v^*(\theta,S) = \partial r_S^*(\theta_o,S)$$

for any $\theta_o \in (0,1)$.

<u>Proof</u>: That $\partial r_S^*(\theta,S)$ exists and is the same for all $\theta \in (0,1)$ follows from Proposition 4.27 and Lemma 3.19.

To establish the second claim of the lemma, first consider the case where M is of finite type, with $u_t \in \{f_1,\ldots f_n\}$ for all $t \in T$. Let $S_i = \{t \in T: u_t = f_i\}$ for $i = 1,\ldots,n$. Then from the proof of Lemma 39.16 of Aumann and Shapley we have

$$r_S(R) = g(\eta(R),e_S(R)) \text{ for each } R \in \mathcal{C} \quad ,$$

where $\eta_i(S) = \mu(S \cap S_i)$ for each $S \in \mathcal{C}$ and each $i = 1,\ldots,n$, so that $\eta = (\eta_1,\ldots,\eta_n)$ is an n-dimensional vector of nonnegative members of NA and $g: \mathbb{R}_+^n \times \Omega \to \mathbb{R}_+$ is defined in (6.15). Thus we have $r_S = g \circ (\eta,e_S)$, where (η,e_S) is an $(n + \ell)$-dimensional vector of nonnegative members of NA. Also, from Lemma 6.16 we have

$$v(R) = g(\eta(R),e(T)) \text{ for each } R \in \mathcal{C} \quad ,$$

or $v = \hat{g} \circ \eta$ (where $\hat{g}: \mathbb{R}_+^n \to \mathbb{R}_+$ is defined by $g(y) = g(y,e(T))$ for each $y \in \mathbb{R}_+^n$).

Now, the range of η is n-dimensional (since $\eta(S_i)$ is the i-th unit vector in $\mathbb{R}^n$). Suppose the range of e_S is r-dimensional, and let $z_1,\ldots,z_r$ (with $z_i \in \mathbb{R}_+^\ell$ for $i = 1,\ldots,r$) be a basis for the smallest linear subspace containing the range of e_S. Then there are members $\nu_1,\ldots,\nu_r$ of NA^+ such that

$$e_S(R) = \sum_{j=1}^{r} \nu_j(R) z_j \quad \text{for all} \quad R \in \mathcal{C} \quad .$$

Then since $r_S = g \circ (\eta, e_S) \in pNA$ (by Proposition 4.23), from Lemma 3.18 we have

$$\partial r_S^*(\theta,S) = \sum_{i=1}^{n} \eta_i(S) g_i(\theta\eta(T), \theta e_S(T)) + \sum_{j=1}^{r} \nu_j(S) g_{z_j}(\theta\eta(T), \theta e_S(T))$$

(where g_{z_j} is the derivative of g in the direction z_j) for all $\theta \in (0,1)$. But $\nu_j(S) = 0$ for all $j = 1,\ldots,r$ because $e_S(S) = 0$, so

$$\partial r_S^*(\theta,S) = \sum_{i=1}^{n} \eta_i(S) g_i(\theta\eta(T), \theta e(T)) \quad .$$

Similarly, from the fact that $v = \hat{g} \circ \eta \in pNA$ (by Lemma 6.16) we have

$$\partial v^*(\theta,S) = \sum_{i=1}^{n} \eta_i(S) g_i(\theta\eta(T), e(T))$$

for all $\theta \in (0,1)$. But g_i is continuous at positive values of its argument for all $i = 1,\ldots,n$, so

$$\lim_{\theta \to 1} |g_i(\theta\eta(T), \theta e(T)) - g_i(\theta\eta(T), e(T))| = 0$$

for all $i = 1,\ldots,n$. Hence

$$(6.25) \qquad \lim_{\theta \to 1} |\partial r_S^*(\theta,S) - \partial v^*(\theta,S)| = 0 \ .$$

But $\partial r_S^*(\theta,S)$ is independent of θ, so we have established (6.24) in the case where M is of finite type.

Now consider the case where M is not necessarily of finite type. We know from Proposition 4.24 that for each $\varepsilon > 0$ there is a finite type market $\hat{M}_S$ such that $\|r_S - \hat{r}_S\| < \varepsilon$, where $\hat{r}_S\colon C \to \mathbb{R}_+$ is the market game derived from $\hat{M}_S$. Both r_S and $\hat{r}_S$ are members of pNA which are homogeneous of degree 1 (see Corollary 4.26) so that by Proposition 4.28 we have

$$(\phi r_S)(R) = \partial r_S^*(\theta,R) \quad \text{and} \quad (\phi \hat{r}_S)(R) = \partial \hat{r}_S^*(\theta,R)$$

for all $R \in C$ and $\theta \in (0,1)$. But

$$\|r_S - \hat{r}_S\| < \varepsilon \Rightarrow \|\phi r_S - \phi \hat{r}_S\| < \varepsilon \Rightarrow |(\phi r_S)(R) - (\phi \hat{r}_S)(R)| < \varepsilon$$

for all $R \in C$ (using Proposition 18.1 of Aumann and Shapley). Hence for all $\varepsilon > 0$ there exists a finite type market $\hat{M}_S$ such that

$$(6.26) \qquad |\partial r_S^*(\theta,R) - \partial \hat{r}_S^*(\theta,R)| < \varepsilon \quad \text{for all } \theta \in (0,1) \text{ and all } R \in C \ .$$

We shall now use the same sort of approximation argument for the game v. By Propositions 4.18 and 6.14, for each $\varepsilon > 0$ there is a finite type market $\hat{M}$ such that $\|v - \hat{v}\| < \varepsilon$, where $\hat{v}\colon C \to \mathbb{R}_+$ is the coalitional form derived from $\hat{M}$ defined by (6.5). By Proposition 6.11 both v and $\hat{v}$ are members of pNA, so by Theorem 3.14

$$(\phi v)(R) = \int_0^1 \partial v^*(\theta,R)d\theta \quad \text{and} \quad (\phi\hat{v})(R) = \int_0^1 \partial\hat{v}^*(\theta,R)d\theta$$

for all $R \in C$. Using Corollary 6.21 we then have

$$(\phi v)(R) = \lim_{\theta\to 1} \partial v^*(\theta,R) \int_0^1 \theta^{-\beta}d\theta = \partial v^*(\theta,R)/(1-\beta)$$

for each $R \in C$, where $\beta \in (0,1)$ is the degree of homogeneity of M. Similarly

$$(\phi\hat{v})(R) = \partial\hat{v}^*(\theta,R)/(1-\beta) \quad \text{for each} \quad R \in C \quad .$$

But

$$\|v - \hat{v}\| < \varepsilon \Rightarrow \|\phi v - \phi\hat{v}\| < \varepsilon \Rightarrow |(\phi v)(R) - (\phi\hat{v})(R)| < \varepsilon$$

for all $R \in C$. Hence for all $\varepsilon > 0$ there exists a finite type market $\hat{M}$ such that

$$(6.27) \quad |\partial v^*(\theta,R) - \partial\hat{v}^*(\theta,R)| < \varepsilon \quad \text{for all} \quad \theta \in (0,1) \quad \text{and all} \quad R \in C \quad .$$

Finally, from (6.25) we have

$$\lim_{\theta\to 1} |\partial\hat{r}_S^*(\theta,S) - \partial\hat{v}^*(\theta,S)| = 0 \quad .$$

Combining this with (6.26) and (6.27) we have

$$\lim_{\theta\to 1} |\partial r_S^*(\theta,S) - \partial v^*(\theta,S)| = 0 \quad .$$

Combined with the first claim of the lemma, this completes the proof.

We are now in a position to calculate ϕq for a homogeneous market.

Proposition 6.28: Let M be homogeneous of degree $\beta \in (0,1)$ and suppose $v(T)$ is finite. Then $q \in \text{ASYMP}$ and for each $S \in C$

$$(\phi q)(S) = 2^{\beta-1}v(T)\mu(S) + (1 - 2^{\beta-1})\int_S u(\underset{\sim}{x})$$

where $\underset{\sim}{x}$ is an allocation at which $v(T)$ is achieved.

Proof: From Proposition 4.18 and Lemma 6.18 we have $q \in \text{ASYMP}$, and

$$(\phi q)(S) = v^*(\chi_T/2)\mu(S) + \int_{1/2}^{1} \partial v^*(\theta,S)d\theta \quad \text{for all} \quad S \in C \quad .$$

From Lemma 6.20 we have $v^*(\chi_T/2) = (1/2)^{1-\beta}v^*(\chi_T) = 2^{\beta-1}v(T)$. From Lemma 6.23 and Corollary 6.21 we have

$$\int_{1/2}^{1} \partial v^*(\theta,S)d\theta = \partial r_S^*(\theta_o,S) \int_{1/2}^{1} \theta^{-\beta}d\theta$$

$$= \partial r_S^*(\theta_o,S)(1 - 2^{\beta-1})/(1 - \beta)$$

for all $S \in C$ with $\mu(S) < 1$, for any $\theta_o \in (0,1)$. But from Proposition 4.28.

$$\partial r_S^*(\theta_o,S) = (\phi r_S)(S) = \int_S (u(\underset{\sim}{x}) - p\underset{\sim}{x})$$

for all $S \in C$, since $e_S(S) = 0$, where $(\underset{\sim}{x},p)$ is any t.u.c.e. of M_S. Now, M_S differs from M only in endowment density, so the t.u.c.e.'s of M_S are the same as the t.u.c.e.'s of M. Hence

$$(\phi q)(S) = 2^{\beta-1}v(T)\mu(S) + (1 - 2^{\beta-1})\int_S (u(\underset{\sim}{x}) - p\underset{\sim}{x})/(1 - \beta)$$

for any $S \in C$ with $\mu(S) < 1$, where $(\underset{\sim}{x},p)$ is a t.u.c.e. in M. But if $(\underset{\sim}{x},p)$ is a t.u.c.e. in M, a.e. $u_t^i(\underset{\sim}{x}(t)) = p^i$ if $\underset{\sim}{x}^i(t) > 0$ (see (4.13)). So a.e.

$$p\underset{\sim}{x} = \sum_{i=1}^{\ell} p^i \underset{\sim}{x}^i(t) = \sum_{\{i:\ \underset{\sim}{x}^i(t)>0\}} p^i \underset{\sim}{x}^i(t) = \sum_{\{i:\ \underset{\sim}{x}^i(t)>0\}} \beta u_t^i(\underset{\sim}{x}(t))\underset{\sim}{x}^i(t)$$

$$= \beta u_t(\underset{\sim}{x}(t)) \quad ,$$

using (4.9). Hence

$$\int_S (u(\underset{\sim}{x}) - p\underset{\sim}{x}) = (1 - \beta)\int_S u(\underset{\sim}{x}) \quad ,$$

and so for any $S \in C$ with $\mu(S) < 1$,

$$(\phi q)(S) = 2^{\beta-1}v(T)\mu(S) + (1 - 2^{\beta-1})\int_S u(\underset{\sim}{x}) \quad ,$$

where $\underset{\sim}{x}$ is an allocation at which $r_S(T) = v(T)$ is achieved. If $\mu(S) = 1$, then $\mu(T\setminus S) = 0$, so $(\phi q)(S) = (\phi q)(T) = q(T) = v(T)$, in accordance with the formula for $(\phi q)(S)$ in the lemma. Hence the proof is complete.

6.6 An Alternative Method of Calculating ϕq

In the previous section we exploited the similarity between the game v and the members of a collection of market games to study the characteristics of v for a homogeneous market, and consequently calculate ϕq. Here we shall outline, in an unrigorous fashion, a direct method of calculating the "derivatives" $\partial v^*(\theta,S)$. Though this method does not involve any results about market games, it is closely related to a method of calculating the value of a market game (see Aumann [1975], Section 8) and essentially merely rephrases the analysis of the previous section. We include it here because of the intuitive appeal of its simple line of reasoning; we feel that it illuminates rather clearly what is happening in the game v. (However, to make our arguments precise would be a major task; this is why we have chosen to rely on the relatively straightforward, if not transparent, arguments of the previous section to establish our results formally.)

Our line of reasoning here involves infinitesimal subsets dt of T; t is a representative member of dt. We shall denote the ideal coalition $\theta\chi_T$ (for $\theta \in [0,1]$) by θT, and throughout treat it as though it were an actual coalition. In particular, if $v: \mathcal{C} \to \mathbb{R}$ is a game, we shall write the extension v^* of v to the collection of ideal coalitions simply as v (no confusion will arise). Using this notation, if v possesses a value it seems reasonable to suppose that

$$(\phi v)(dt) = \int_0^1 (v(\theta T) - v(\theta T \setminus dt))d\theta \quad .$$

We shall not justify this, but shall merely rely on its intuitive reasonableness (cf. (3.15)). By phrasing Lemma 6.18 in these terms, we obtain the following expression for the value of the game q: for any infinitesimal $dt \subset T$,

$$(6.29)\quad (\phi q)(dt) = v(T/2)\mu(dt) + \int_{1/2}^{1} (v(\theta T) - v(\theta T \setminus dt))d\theta \quad .$$

We shall now go about characterizing the allocations at which $v(\theta T)$ is attained, and calculating $v(\theta T) - v(\theta T \setminus dt)$ for all $\theta \in [0,1]$, in order to evaluate $(\phi q)(dt)$ for any $dt \subset T$.

In accordance with our viewing θT as a coalition (to which each $t \in T$ "belongs" with "density" θ), we have

$$v(\theta T) = \max_{\underset{\sim}{x}} \{ \int_{\theta T} u(\underset{\sim}{x}) : \int_{\theta T} \underset{\sim}{x} = e(T) \} \quad \text{for all} \quad \theta \in [0,1] \quad .$$

To avoid complications, we shall assume throughout this informal demonstration that any maximizer $\underset{\sim}{x}^{\theta T}: T \to \mathbb{R}_+^{\ell}$ satisfies $\underset{\sim}{x}^{\theta T} >> 0$ a.e.. Then under our assumptions on u a necessary and sufficient condition for $\underset{\sim}{x}^{\theta T}: T \to \mathbb{R}_+^{\ell}$ to be a maximizer is that there exist a constant $\xi^{\theta T} \in \mathbb{R}^{\ell}$ such that

$$(6.30)\quad u_t'(\underset{\sim}{x}^{\theta T}(t)) = \xi^{\theta T} \quad \text{for almost all} \quad t \in T$$

and

$$(6.31)\quad \int_{\theta T} \underset{\sim}{x}^{\theta T} = e(T) \quad .$$

Henceforth $\underset{\sim}{x}^{\theta T}: T \to \mathbb{R}^{\ell}_{+}$ will denote such a maximizer.

Claim 6.32: For $dt \subset T$ and $\theta \in [0,1]$ we have

$$(6.33)\quad v(\theta T) - v(\theta T \setminus dt) = [u_t(\underset{\sim}{x}^{\theta T}(t)) - u_t'(\underset{\sim}{x}^{\theta T}(t))\underset{\sim}{x}^{\theta T}(t)]\mu(dt) \quad .$$

Demonstration: We have $v(\theta T) = \int_{\theta T} u(\underset{\sim}{x}^{\theta T})$. Consider the effect of dt leaving θT. $\theta T \setminus dt$ gains the resources which dt was consuming, namely $\underset{\sim}{x}^{\theta T}(t)\mu(dt)$, and loses the utility dt received from them, namely $u_t(\underset{\sim}{x}^{\theta T}(t))\mu(dt)$. Given (6.30), the best way for $\theta T \setminus dt$ to use the resources $\underset{\sim}{x}^{\theta T}(t)\mu(dt)$ is to distribute them evenly, leading to a gain in utility for each $ds \in \theta T \setminus dt$ of $(\xi^{\theta T}\underset{\sim}{x}^{\theta T}(t)\mu(dt)/\mu(\theta T \setminus dt))\mu(ds)$ (using (6.30) again), so that the gain in utility to $\theta T \setminus dt$ is just $\xi^{\theta T}\underset{\sim}{x}^{\theta T}(t)\mu(dt)$. Hence

$$v(\theta T \setminus dt) = v(\theta T) + \xi^{\theta T}\underset{\sim}{x}^{\theta T}(t)\mu(dt) - u_t(\underset{\sim}{x}^{\theta T}(t))\mu(dt) \quad ,$$

or

$$v(\theta T) - v(\theta T \setminus dt) = [u_t(\underset{\sim}{x}^{\theta T}(t)) - u_t'(\underset{\sim}{x}^{\theta T}(t))\underset{\sim}{x}^{\theta T}(t))]\mu(dt)$$

(using (6.30)), as we claimed.

From Claim 6.32 and (6.29) we have

$$(\phi q)(dt)$$
$$= v(T/2)\mu(dt) + \int_{1/2}^{1} \{[u_t(\underset{\sim}{x}^{\theta T}(t)) - u_t'(\underset{\sim}{x}^{\theta T}(t))\underset{\sim}{x}^{\theta T}(t)]\mu(dt)\}d\theta \quad .$$

As before, it is the possibly irregular behavior of the integrand which prevents us from calculating $(\phi q)(dt)$ for all markets. As in the previous section, we shall restrict attention to homogeneous markets.

<u>Claim 6.34</u>: If M is homogeneous then $\underset{\sim}{x}^{\theta T} = \underset{\sim}{x}^{T}/\theta$ for all $\theta \in (0,1]$.

<u>Demonstration</u>: We have $u_t'(\underset{\sim}{x}^T(t)/\theta) = \theta^{1-\beta}u_t'(\underset{\sim}{x}^T(t)) = \theta^{1-\beta}\xi^T$ by (6.30) and $\int_{\theta T} \underset{\sim}{x}^T/\theta = \theta\int_T \underset{\sim}{x}^T/\theta = \int_T \underset{\sim}{x}^T = e(T)$ by (6.31). So $\underset{\sim}{x}^T/\theta$ satisfies (6.30) and (6.31) with $\xi^{\theta T} = \theta^{1-\beta}\xi^T$. Hence $\underset{\sim}{x}^{\theta T} = \underset{\sim}{x}^T/\theta$.

<u>Claim 6.35</u>: If M is homogeneous of degree $\beta \in (0,1)$ then

$$(\phi q)(dt) = 2^{\beta-1}v(T)\mu(dt) + (1 - 2^{\beta-1})u_t(\underset{\sim}{x}^T(t))\mu(dt)$$

for $dt \subset T$, where $\underset{\sim}{x}^T$ is an allocation at which $v(T)$ is achieved.

<u>Demonstration</u>: From Claim 6.34 we have

$$v(T/2) = \int_{T/2} u(\underset{\sim}{x}^{T/2}) = \int 2^{\beta}u(\underset{\sim}{x}^T)/2 = 2^{\beta-1}\int u(\underset{\sim}{x}^T) = 2^{\beta-1}v(T) \quad ;$$

from Claims 6.32 and 6.34 we have

$$\begin{aligned} v(\theta T) - v(\theta T\backslash dt) &= (1 - \beta)u_t(\underset{\sim}{x}^{\theta T}(t))\mu(dt) \\ &= (1 - \beta)u_t(\underset{\sim}{x}^{T}(t))\mu(dt)/\theta^{\beta} \end{aligned}$$

(using the homogeneity of u_t once again). Hence by (6.29)

$$(\phi q)(dt) = 2^{\beta-1}v(T)\mu(dt) + (1-\beta)u_t(\underset{\sim}{x}^T(t))\mu(dt)\int_{1/2}^{1}\theta^{-\beta}d\theta$$

$$= 2^{\beta-1}v(T)\mu(dt) + (1-\beta)u_t(\underset{\sim}{x}^T(t))\mu(dt)(1-2^{\beta-1})/(1-\beta)$$

$$= 2^{\beta-1}v(T)\mu(dt) + (1-2^{\beta-1})u_t(\underset{\sim}{x}^T(t))\mu(dt) \quad ,$$

as we claimed.

Claim 6.35 can be seen to be a translation of Proposition 6.28 into the language we are using here. This completes what we have to say about this approach to the calculation of ϕq.

6.7 The Proof of Theorem B: The Existence and Characterization of the Value Allocations of the Economy $(M, \Gamma(M))$ when M Is Homogeneous

First we shall prove the following, which constitutes the major part of Theorem B, the main result of this chapter.

Theorem 6.36: Let M be a homogenous market. Then an allocation $\underset{\sim}{x}$ in M is a value allocation of the economy $E = (M, \Gamma(M))$ if and only if it is efficient and a.e.

$$p\underset{\sim}{x}(t) = pe(T) \quad ,$$

where p is an efficiency price for $\underset{\sim}{x}$.

To prove this result we shall use the following, which is modelled on Proposition 14.10 of Aumann and Kurz [1977].

Lemma 6.37: Let M be a homogeneous market, $\underset{\sim}{x}$ an allocation, and $\underset{\sim}{\lambda}$ a comparison function. Then we can choose a finite saddle point (σ_o, τ_o) for H^S_{λ} for each $S \in C$ so that the game q_{λ} defined in (3.24) has a value ϕq_{λ} which satisfies $(\phi q_{\lambda})(S) = \int_S \lambda u(x)$ for each $S \in C$ if and only if x is efficient with efficiency comparison function λ and

(6.38) $\lambda(t)u_t(x(t)) = \int \lambda u(x)$ a.e. .

Proof: First suppose that we can choose a finite saddle point for H^S_{λ} for each $S \in C$ in such a way that $(\phi q_{\lambda})(S) = \int_S \lambda u(x)$ for each $S \in C$. Then from Lemma 6.9 we know that $v_{\lambda}(T) = r_{\lambda}(T)$ is finite, and we have $\int \lambda u(x) = (\phi q_{\lambda})(T) = q_{\lambda}(T) = v_{\lambda}(T)$ (using Lemma 6.6 and the fact that $q_{\lambda}(T)$ is independent of the collection of optimal threats we choose (see Section 3.2)). Thus x is an allocation at which $v_{\lambda}(T)$ is achieved. Now since $v_{\lambda}(T)$ is finite and M is homogeneous, so that λM is homogeneous, from Proposition 4.18 we have that λM is integrably sublinear. Hence $r_{\lambda}(S)$ and $v_{\lambda}(S)$ are attained for all $S \in C$, so that from Lemma 6.6, Proposition 6.28, and the fact that ϕq_{λ} is independent of the collection of optimal threats which we choose to define q_{λ} (see Lemma 3.26),

$$(\phi q_{\lambda})(S) = 2^{\beta-1} v_{\lambda}(T)\mu(S) + (1 - 2^{\beta-1}) \int_S \lambda u(x) \quad \text{for all} \quad S \in C \quad ,$$

where β is the degree of homogeneity of M, and hence of λM. But we are assuming that $(\phi q_{\lambda})(S) = \int_S \lambda u(x)$ for each $S \in C$, so

$$\int_S \lambda u(x) = 2^{\beta-1} v_{\lambda}(T)\mu(S) + (1 - 2^{\beta-1}) \int_S \lambda u(x)$$

or

$$\int_S \underset{\sim}{\lambda} u(\underset{\sim}{x}) = \mu(S) \int \underset{\sim}{\lambda} u(\underset{\sim}{x}) \quad \text{for all} \quad S \in \mathcal{C} \; .$$

Hence $\underset{\sim}{\lambda}(t)u_t(\underset{\sim}{x}(t)) = \int \underset{\sim}{\lambda} u(\underset{\sim}{x})$ a.e., where $\underset{\sim}{x}$ is an allocation at which $v_{\underset{\sim}{\lambda}}(T)$ is achieved, and hence is efficient. This completes the proof of necessity.

Now suppose that $\underset{\sim}{x}$ is efficient with efficiency comparison function $\underset{\sim}{\lambda}$ and a.e. $\underset{\sim}{\lambda}(t)u_t(\underset{\sim}{x}(t)) = \int \underset{\sim}{\lambda} u(\underset{\sim}{x})$. Then $v_{\underset{\sim}{\lambda}}(T) = \int \underset{\sim}{\lambda} u(\underset{\sim}{x})$ is finite, so by Proposition 4.18, $\underset{\sim}{\lambda}M$ is integrably sublinear. Hence $v_{\underset{\sim}{\lambda}}(S)$ and $r_{\underset{\sim}{\lambda}}(S)$ are attained for all $S \in \mathcal{C}$ (Proposition 4.16) and so by Lemma 6.6 we can choose a finite saddle point (σ_o, τ_o) for $H^S_{\underset{\sim}{\lambda}}$ for each $S \in \mathcal{C}$ such that the game $q_{\underset{\sim}{\lambda}}$ defined in (3.24) is given by

$$q_{\underset{\sim}{\lambda}}(S) = \begin{cases} v_{\underset{\sim}{\lambda}}(S) & \text{if} \quad \mu(S) > 1/2 \\ r_{\underset{\sim}{\lambda}}(S) & \text{if} \quad \mu(S) = 1/2 \\ 0 & \text{if} \quad \mu(S) < 1/2 \; . \end{cases}$$

Then from Proposition 6.28 we have

$$(\phi q_{\underset{\sim}{\lambda}})(S) = 2^{\beta-1} v_{\underset{\sim}{\lambda}}(T)\mu(S) + (1 - 2^{\beta-1}) \int_S \underset{\sim}{\lambda} u(\underset{\sim}{x}) \quad \text{for all} \quad S \in \mathcal{C} \; .$$

But we are assuming that $\underset{\sim}{\lambda}(t)u_t(\underset{\sim}{x}(t)) = \int \underset{\sim}{\lambda} u(\underset{\sim}{x})$ a.e., so

$$\begin{aligned} (\phi q_{\underset{\sim}{\lambda}})(S) &= 2^{\beta-1}\mu(S)\int \underset{\sim}{\lambda} u(\underset{\sim}{x}) + (1 - 2^{\beta-1}) \int_S \underset{\sim}{\lambda} u(\underset{\sim}{x}) \\ &= \int_S \underset{\sim}{\lambda} u(\underset{\sim}{x}) \quad \text{for every} \quad S \in \mathcal{C} \; . \end{aligned}$$

This completes the proof.

We also need the following.

Lemma 6.39: Let M be a homogeneous market, $\underset{\sim}{x}$ an allocation and $\underset{\sim}{\lambda}$ a comparison function. Then $\underset{\sim}{x}$ is efficient with efficiency comparison function $\underset{\sim}{\lambda}$ and (6.38) is satisfied if and only if $\underset{\sim}{x}$ is efficient and $p\underset{\sim}{x}(t) = pe(T)$ a.e., where p is an efficiency price for $\underset{\sim}{x}$.

Proof: Suppose that $\underset{\sim}{x}$ is efficient with efficiency comparison function $\underset{\sim}{\lambda}$ and a.e. $\underset{\sim}{\lambda}(t)u_t(\underset{\sim}{x}(t)) = \int\underset{\sim}{\lambda}u(\underset{\sim}{x})$. Then there exists $p \in \mathbb{R}^{\ell}_{++}$ such that $(\underset{\sim}{\lambda},p)$ is an efficiency pair for $\underset{\sim}{x}$, so that a.e.

$$\underset{\sim}{\lambda}(t)u^i_t(\underset{\sim}{x}(t))\underset{\sim}{x}^i(t) = p^i\underset{\sim}{x}^i(t) \quad \text{if} \quad \underset{\sim}{x}^i(t) > 0$$

(see (4.13)). Hence a.e.

$$\begin{aligned} p\underset{\sim}{x}(t)u_t(\underset{\sim}{x}(t)) &= \sum_{i=1}^{\ell} p^i\underset{\sim}{x}^i(t)u_t(\underset{\sim}{x}(t)) = \sum_{\{i:\,\underset{\sim}{x}^i(t)>0\}} p^i\underset{\sim}{x}^i(t)u_t(\underset{\sim}{x}(t)) \\ &= \sum_{\{i:\,\underset{\sim}{x}^i(t)>0\}} \underset{\sim}{\lambda}(t)u^i_t(\underset{\sim}{x}(t))\underset{\sim}{x}^i(t)u_t(\underset{\sim}{x}(t)) \\ &= \beta u_t(\underset{\sim}{x}(t))\int\underset{\sim}{\lambda}u(\underset{\sim}{x}) \quad , \end{aligned}$$

using (4.9) and our assumption that $\underset{\sim}{\lambda}(t)u_t(\underset{\sim}{x}(t)) = \int\underset{\sim}{\lambda}u(\underset{\sim}{x})$ a.e., where β is the degree of homogeneity of M. But since $\underset{\sim}{\lambda}(t) > 0$ a.e., u_t is increasing for all $t \in T$, and $\int\underset{\sim}{e} >> 0$, we have $\int\underset{\sim}{\lambda}u(\underset{\sim}{x}) > 0$, so from (6.38) we have $u_t(\underset{\sim}{x}(t)) > 0$ a.e. Hence

$$p\underset{\sim}{x}(t) = \beta\int\underset{\sim}{\lambda}u(\underset{\sim}{x}) \quad \text{a.e.} \quad .$$

So $p\int \underset{\sim}{x} = pe(T) = \beta\int \underset{\sim}{\lambda} u(\underset{\sim}{x})$ a.e., and hence $p\underset{\sim}{x}(t) = pe(T)$ a.e., as was to be shown.

Now assume that $p\underset{\sim}{x}(t) = pe(T)$ a.e., where p is an efficiency price for $\underset{\sim}{x}$. Let $(\underset{\sim}{\lambda}, p)$ be an efficiency pair for $\underset{\sim}{x}$. Then since $e(T) \gg 0$, $\underset{\sim}{x}(t) \neq 0$ a.e., so that $\underset{\sim}{\lambda}(t) > 0$ a.e.--i.e. $\underset{\sim}{\lambda}$ is a comparison function. Then using (4.13) and (4.9) we have a.e.

$$pe(T) = \sum_{i=1} p^i \underset{\sim}{x}^i(t) = \sum_{\{i:\, \underset{\sim}{x}^i(t)>0\}} p^i \underset{\sim}{x}^i(t) = \sum_{\{i:\, \underset{\sim}{x}^i(t)>0\}} \underset{\sim}{\lambda}(t) u^i(\underset{\sim}{x}(t)) \underset{\sim}{x}^i(t)$$
$$= \beta \underset{\sim}{\lambda}(t) u_t(\underset{\sim}{x}(t))$$

(where β is the degree of homogeneity of M), so that $\int \underset{\sim}{\lambda} u(\underset{\sim}{x}) = pe(T)/\beta$, and hence $\underset{\sim}{\lambda}(t) u_t(\underset{\sim}{x}(t)) = \int \underset{\sim}{\lambda} u(\underset{\sim}{x})$ a.e., completing the proof.

Proof of Theorem 6.36: The result follows immediately from Lemmas 6.10, 6.37, and 6.39.

Theorem 6.36 can be rephrased with the aid of the following definition. An allocation $\underset{\sim}{x}$ is an equal income competitive allocation of M if it is a Walrasian allocation of the market M' which differs from M only as regards the initial endowment density, which is given by $\underset{\sim}{e}'(t) = e(T)$ for all $t \in T$.

Proposition 6.40: Let M be a homogeneous market. Then an allocation $\underset{\sim}{x}$ is a value allocation of the economy $E = (M, \Gamma(M))$ if and only if it is an equal income competitive allocation of M.

Proof: To say that p is an efficiency price for an allocation $\underset{\sim}{x}$ means that the maximum of $u_t(x)$ over $\{x \in \Omega: px \leqq p\underset{\sim}{x}(t)\}$ is a.e. achieved at $x = \underset{\sim}{x}(t)$. So Theorem 6.70 implies that $\underset{\sim}{x}$ is a value allocation if and only if $\max \{u_t(x): x \in \Omega$ and $px \leqq pe(T)\}$ is a.e. achieved at $x = \underset{\sim}{x}(t)$. But this just says that $\underset{\sim}{x}$ is an equal income competitive allocation of M. Conversely, if $\underset{\sim}{x}$ is an equal income competitive allocation of M then it is efficient, and the maximum of $u_t(x)$ over $\{x \in \Omega: px \leqq pe(T)\}$ is a.e. achieved at $x = \underset{\sim}{x}(t)$. But then given our assumptions on u_t, $p\underset{\sim}{x}(t) = pe(T)$ a.e. and p is an efficiency price for $\underset{\sim}{x}$.

Proof of Theorem B: The result follows immediately from Theorem 6.36, Proposition 6.40, and Theorem 2 on p. 151 of Hildenbrand [1974] which implies that M possesses an equal income competitive allocation.

6.8 Discussion

As we explained in Section 6.1 our assumptions here give majority coalitions as much power as they can possibly hope for. The worth of a coalition S depends solely on its size $(\mu(S))$ (in addition to the utility functions of its members) and not at all on the resources at its disposal $(e(S))$, so that naturally the set of value allocations has the same property. For the class of utility functions which we are considering, we have concluded that what actually happens is that an agent's "after-tax income" is also independent of the precise utility function which he possesses: the outcome is, in this sense, the most "equal" possible. It is interesting that it is this form of equality to which the set of value allocations gives rise--even though this solution concept rests on the cardinal properties of the utility functions of the individuals in the market, the outcomes predicted here have more to do with the equal allocation of goods. As under the assumptions of Aumann and Kurz [1977], the outcomes here can obviously be supported by a system of wealth taxes in which the tax each agent $t \in T$ pays is $p\underset{\sim}{e}(t) - pe(T)$ (i.e. the marginal tax rate is 100% and there is a lump-sum subsidy of $pe(T)$).

We cannot in general make the calculations of the previous sections for arbitrary nonhomogeneous markets. However, there is one case for which we can quite easily give a partial characterization of the set of value allocations. If every individual has the same utility function then we can actually calculate $v(S)$ for each $S \in \mathcal{C}$ (where v is the game defined in (6.5)); this allows us to isolate some of the value allocations of the economy. In fact, we have the following.

Lemma 6.41: Let M be a market, and suppose that there is a comparison function $\underset{\sim}{\lambda}^*$ and a utility function u (i.e. a function satisfying (4.2) through (4.5)) such that $\underset{\sim}{\lambda}^*(t)u_t = u$ for all $t \in T$. Then if $\underset{\sim}{x}$ is an efficient allocation in M for which a.e.

$$p\underset{\sim}{x}(t) = pe(T) \quad , \tag{6.42}$$

where p is an efficiency price for $\underset{\sim}{x}$, then it is a value allocation of the economy $E = (M, \Gamma(M))$.

Note that if u is strictly quasi-concave then if $\underset{\sim}{x}$ is efficient and a.e. satisfies (6.42), so that $p\underset{\sim}{x}(t)$ is a.e. constant, we must have $\underset{\sim}{x}(t) = e(T)$ a.e..

Proof of Lemma 6.41: First note that $v_{\underset{\sim}{\lambda}*}(S)$ and $r_{\underset{\sim}{\lambda}*}(S)$ are attained for all $S \in \mathcal{C}$ and in fact (since u is concave by assumption),

$$v_{\underset{\sim}{\lambda}*}(S) = \mu(S)u(e(T)/\mu(S)) \quad \text{for each} \quad S \in \mathcal{C}$$

(where $v\colon \mathcal{C} \to \mathbb{R}_+$ is the game defined in (6.5)), and

$$r_{\underset{\sim}{\lambda}*}(S) = \mu(S)u(e(S)/\mu(S)) \quad \text{for each} \quad S \in \mathcal{C} \quad .$$

So by Theorem B of Aumann and Shapley we have $v_{\underset{\sim}{\lambda}*} \in pNA$ and $r_{\underset{\sim}{\lambda}*} \in pNA$.

Now assume that $\underset{\sim}{x}$ is an efficient allocation in M which a.e. satisfies (6.42). Since both $v_{\underset{\sim}{\lambda}*}(S)$ and $r_{\underset{\sim}{\lambda}*}(S)$ are attained for every $S \in \mathcal{C}$, by Lemma 6.6 we know that $H^S_{\underset{\sim}{\lambda}*}$ has a finite saddle point for every $S \in \mathcal{C}$, and the game $q_{\underset{\sim}{\lambda}*}$ defined by (3.24) is given by

$$q_{\underset{\sim}{\lambda}*}(S) = \begin{cases} v_{\underset{\sim}{\lambda}*}(S) & \text{if} \quad \mu(S) > 1/2 \\ r_{\underset{\sim}{\lambda}*}(S) & \text{if} \quad \mu(S) = 1/2 \\ 0 & \text{if} \quad \mu(S) < 1/2 \quad . \end{cases}$$

Since $v_{\underset{\sim}{\lambda}*} \in pNA$ and $r_{\underset{\sim}{\lambda}*} \in pNA$ we can argue as in the second part of the proof of Lemma 6.18 that the value $\phi q_{\underset{\sim}{\lambda}*}$ of the game $q_{\underset{\sim}{\lambda}*}$ is the same as the value $\phi k_{\underset{\sim}{\lambda}*}$ of the game $k_{\underset{\sim}{\lambda}*}\colon \mathcal{C} \to \mathbb{R}_+$ defined by

$$k_{\underset{\sim}{\lambda}*}(S) = \begin{cases} v_{\underset{\sim}{\lambda}*}(S) & \text{if} \quad \mu(S) \geqq 1/2 \\ 0 & \text{if} \quad \mu(S) < 1/2 \quad . \end{cases}$$

But $k_{\lambda^*} = u(e(T))(f \circ \mu)$ where $f: [0,1] \to [0,1]$ is defined by

$$f(x) = \begin{cases} xu(e(T)/x)/u(e(T)) & \text{if} \quad x \in [1/2, 1] \\ 0 & \text{if} \quad x \in [0, 1/2) \end{cases} .$$

The function f is of bounded variation and is continuous at 0 and 1. Also $f(0) = 0$ and $f(1) = 1$, so by Theorem A of Aumann and Shapley and Theorem 3.7 we have

$$(6.43) \qquad (\phi k_{\lambda^*})(S) = (\phi q_{\lambda^*})(S) = u(e(T))\mu(S) \quad \text{for each} \quad S \in C \quad .$$

But from (6.42) and the fact that x is efficient we have $u(x(t)) = \lambda^*(t)u_t(x(t)) = u(e(T))$ for all $t \in T$, so from (6.43) we have

$$(\phi q_{\lambda^*})(S) = \int_S \lambda^* u(x) \quad \text{for each} \quad S \in C \quad .$$

So by Lemma 6.10, x is a value allocation of the economy $E = (M, \Gamma(M))$, completing the proof.

We cannot give a full characterization of the set of value allocations under the assumptions of Lemma 6.41 because in order to do so we should need to calculate the value of the game q_λ for comparison functions $\lambda \neq \lambda^*$; the restriction that for one comparison function all the weighted utility functions are identical gives us no help in this calculation. It may be that there is a comparison function $\lambda \neq \lambda^*$ for which $(\phi q_\lambda)(S) = \int_S \lambda u(x)$ for each $S \in C$, for some allocation x.

A question which arises is whether we can give an example of a (non-homogeneous) economy for which we can argue that none of the equal income competitive allocations is a value allocation. This we shall now do.

Let M be a market in which the agent space is $(T,\mathcal{C},\mu)$ and there are ℓ goods, with $\ell = n + m$. Let f_t be a utility function on the consumption set $\mathbb{R}^n_+$ which is homogeneous of degree $\alpha \in (0,1)$ for every $t \in T$, and let g_t be a utility function on the consumption set $\mathbb{R}^m_+$ which is homogeneous of degree $\beta \in (0,1)$ for every $t \in T$. Let the utility function of $t \in T$ in the market M be $u_t\colon \mathcal{C} \to \mathbb{R}_+$ defined by $u_t(x) = f_t(x_1) + g_t(x_2)$, where $x = (x_1,x_2)$ with $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^m$; denote the initial endowment density of $t \in T$ by $\underset{\sim}{e}(t) = (\underset{\sim}{e}_1(t),\underset{\sim}{e}_2(t))$, with $\underset{\sim}{e}_1(t) \in \mathbb{R}^n$ and $\underset{\sim}{e}_2(t) \in \mathbb{R}^m$. It is convenient to give names to the two "sub-markets" involved here: let M_1 be the market with agent space $(T,\mathcal{C},\mu)$ and n goods in which the utility function and initial endowment density of $t \in T$ are f_t and $\underset{\sim}{e}_1(t)$ respectively, and let M_2 be the market with agent space $(T,\mathcal{C},\mu)$ and m goods in which the utility function and initial endowment density of $t \in T$ are g_t and $\underset{\sim}{e}_2(t)$ respectively. The fact that the utility outcomes in the market M are the sums of outcomes in the two homogeneous markets M_1 and M_2 means that we can immediately deduce from Proposition 6.28 that the game q defined in (6.8) is a member of ASYMP, and for each $S \in \mathcal{C}$

$$(\phi q)(S) = 2^{\alpha-1}v_1(T)\mu(S) + 2^{\beta-1}v_2(T)\mu(S) + (1 - 2^{\alpha-1})\int_S f(\underset{\sim}{x}_1) + (1 - 2^{\beta-1})\int_S g(\underset{\sim}{x}_2)$$

where v_i is the game defined in (6.5) which is derived from the market M_i for $i = 1,2$, and $\underset{\sim}{x}_i$ is an allocation at which $v_i(T)$ is achieved for $i = 1,2$. Given this, we can argue as in Section 6.7, exploiting the homogeneity of each f_t and g_t once again to deduce that $\underset{\sim}{x} = (\underset{\sim}{x}_1,\underset{\sim}{x}_2)$ is a value allocation of the economy $E = (M, \Gamma(M))$ if and only if it is efficient and a.e.

$$(6.44) \qquad \beta 2^{\alpha-1} p_1(\underset{\sim}{x}_1(t) - e_1(T)) + \alpha 2^{\beta-1} p_2(\underset{\sim}{x}_2(t) - e_2(T)) = 0 \quad ,$$

where $p = (p_1,p_2)$ is an efficiency price for $\underset{\sim}{x} = (\underset{\sim}{x}_1,\underset{\sim}{x}_2)$. Now, to say that p is an efficiency price for $\underset{\sim}{x}$ means that there exists a comparison function $\underset{\sim}{\lambda}$ such that $(\underset{\sim}{x},p)$ is a t.u.c.e. in $\underset{\sim}{\lambda}M$ (see Section 4.1). And if $(\underset{\sim}{x},p) = ((\underset{\sim}{x}_1,\underset{\sim}{x}_2),(p_1,p_2))$ is a t.u.c.e. in $\underset{\sim}{\lambda}M$ then from the definition of a t.u.c.e. (see (4.14)) it is immediate that $(\underset{\sim}{x}_1,p_1)$ is a t.u.c.e. in $\underset{\sim}{\lambda}M_1$, and $(\underset{\sim}{x}_2,p_2)$ is a t.u.c.e. in $\underset{\sim}{\lambda}M_2$. We shall now argue that in general it is not the case that a solution $\underset{\sim}{x}$ to (6.44) involves $p\underset{\sim}{x}(t) = pe(T)$; then we shall provide a specific example. If $p\underset{\sim}{x}(t) = p_1\underset{\sim}{x}_1(t) + p_2\underset{\sim}{x}_2(t) = pe(T) = p_1e_1(T) + p_2e_2(T)$ a.e. then from (6.44) we have a.e.

$$(\beta 2^{\alpha-1} - \alpha 2^{\beta-1})p_1(\underset{\sim}{x}_1(t) - e_1(T)) = 0 \quad ,$$

so that if $\beta 2^{\alpha-1} \neq \alpha 2^{\beta-1}$ we have $p_1\underset{\sim}{x}_1(t) = p_1e_1(T)$ a.e., and hence $p_2\underset{\sim}{x}_2(t) = p_2e_2(T)$ a.e.. But in general there is no comparison function $\underset{\sim}{\lambda}$ such that there is a t.u.c.e. $(\underset{\sim}{x}_1,p_1)$ in $\underset{\sim}{\lambda}M_1$ such that $p_1\underset{\sim}{x}_1(t) = p_1e_1(T)$ a.e. and a t.u.c.e. $(\underset{\sim}{x}_2,p_2)$ in $\underset{\sim}{\lambda}M_2$ such that $p_2\underset{\sim}{x}_2(t) = p_2e_2(T)$

a.e., and hence in general there is no solution $\underset{\sim}{x}$ of (6.44) for which $p\underset{\sim}{x}(t) = pe(T)$ a.e.. To see this in a specific case, let $n = m = 1$ and suppose that for each $t \in T$,

$$f_t(x) = x^{1/2} \quad \text{and} \quad g_t(x) = tx^{1/3} \quad .$$

If we are to have $p_1\underset{\sim}{x}_1(t) = p_1e_1(T)$ a.e. then, given that $n = 1$, we must have $\underset{\sim}{x}_1(t) = e_1(T)$ a.e.. But f_t is the same for all $t \in T$, so for $(\underset{\sim}{x}_1,p_1)$ to be a t.u.c.e. in $\underset{\sim}{\lambda}M_1$ we must have $\underset{\sim}{\lambda}(t) = 1$ for all $t \in T$. Now it is easy to verify that there is a unique t.u.c.e. $(\underset{\sim}{x}_2,p_2)$ in M_2, given by $(\underset{\sim}{x}_2,p_2) = (2^{2/3}/3(5e_2(T))^{2/3}, 2t^{2/3}/5e_2(T))$; this gives

$$p_2(\underset{\sim}{x}_2(t) - e_2(T)) = c(2t^{2/3} - (e_2(T))^2)$$

where $c \neq 0$ is a constant, so that $p_2(\underset{\sim}{x}_2(t) - e_2(T)) \neq 0$ a.e. and hence $p\underset{\sim}{x}(t) \neq pe(T)$ a.e..

If agents have utility functions which are not homogeneous of the same degree, or of a special form like those of the previous paragraph, we cannot calculate the set of value allocations directly. In the transferable case the value of an agent depends on the contribution he makes to the worth of "diagonal" coalitions, and it is clear that agents whose utility functions have a relatively high marginal utility throughout will have a high value, while those with a relatively low marginal utility will do poorly. However, what is crucial in the nontransferable case is the "equilibrium" value which the comparison function takes on; the fact that the way that this depends on the structure of the economy is quite complex makes the nonhomogeneous case very difficult to analyze.

What we have concluded from our study in this chapter is that if, in an economy with majority rule, every majority coalition has as much power as it can possibly expect to have (see assumptions (6.1) through (6.3)), in a certain class of economies the resulting outcome is "egalitarian" in the sense that the tax system gives every agent the same wealth (see Theorem B). Our study of this extreme case gives us a "bound" on the possible outcomes we can generate from the basic model. It also provides the basis for the study of the more realistic case in which the members of a minority can prevent their complement from making use of some of the goods (like labor-time) with which they (the members of the minority) are endowed, but not from making use of others (like land, which cannot be "destroyed"); it is this case which we study in the next chapter.

CHAPTER 7: An Application: An Economy Containing Labor and Land

7.1 Introduction

In this chapter we use the results of Chapter 6 to study an economy in which there are two sorts of goods. On one sort we make the assumption of the previous chapter that if an agent has an endowment of them; he cannot prevent a majority coalition of which he is not a member from expropriating this endowment. We might think of the flow of land services as being a typical good of this type: since land itself cannot be destroyed, a majority coalition can expropriate it and hence obtain the flow of services from it. (In Chapter 5 we assumed that it is only necessary to withdraw a good from the market to avoid taxation; in the case of land we are assuming here that whether or not it is offered in the market it can be taxed.) On the other sort of goods we make the assumption that an agent can destroy his endowment in order to avoid expropriation by a majority coalition. A typical member of this group of goods might be labor-time: a worker can always "destroy" his labor-time by going on strike. On this set of goods for simplicity we actually make the assumption of Aumann and Kurz [1977], rather than that of Chapter 5 above--i.e. if an agent chooses not to destroy his endowment then a majority can expropriate it in its entirety, and not just that part of it which he offers in trade. We could say that the difference between the two sorts of goods lies in their "elasticity of supply"--though no prices are involved here. Thinking of the difference in such terms, we can compare our results with those of classical public finance, where the question asked is at what level <u>should</u> tax rates be set (to meet a certain objective). The conclusion there is that for an efficient outcome tax rates on goods with inelastic supply (like land) should be high. Here, where we are asking what the tax rates <u>will</u> be, given the power endowed upon individuals and groups by their possession of certain goods, the conclusion we reach is similar: the goods which cannot be destroyed (the supply of which is "inelastic") will be taxed at high rates (see the discussion in Section 7.4). The reasons for the high taxes in the two cases are different, however. In the classical theory the prescription is to tax goods in inelastic supply at high rates because to do so will make no difference to the quantity of the goods which is available in the market, so that any inefficiency will be minimized. Here, the theory predicts high tax rates on such goods because their owners can do nothing to reduce the quantity of them which is available for expropriation by any majority coalition, so that their ownership conveys no special benefits.

We shall now formally state the assumptions we shall make about the strategies available to the groups in the economy. Let M be a market. The endowment density of $t \in T$ is $\underset{\sim}{e}(t) = (\underset{\sim}{a}(t), \underset{\sim}{\omega}(t))$, where $\underset{\sim}{a}(t) \in \mathbb{R}^n_+$, $\underset{\sim}{\omega}(t) \in \mathbb{R}^m_+$, and $n + m = \ell$. If $x \in \Omega$, we shall consistently write $x = (y,z)$, where it is to be understood that $y \in \mathbb{R}^n_+$ and $z \in \mathbb{R}^m_+$. We make the following assumptions about the strategic game $\Gamma(M)$, which defines the economy $E = (M, \Gamma(M))$:

(7.1) if $S \in C$ is such that $\mu(S) > 1/2$, then for every S-allocation $\underset{\sim}{x} = (\underset{\sim}{y},\underset{\sim}{z})$ of $(a(S),\omega(T))$ there is a strategy σ of S such that for all strategies τ of $T \setminus S$

$$\underset{\sim}{h}^S_{\sigma\tau}(t) \begin{cases} \geqq u_t(\underset{\sim}{y}(t),\underset{\sim}{z}(t)) & \text{if} \quad t \in S \\ = 0 & \text{if} \quad t \in T\backslash S \quad ; \end{cases}$$

(7.2) if $S \in \mathcal{C}$ is such that $\mu(S) \geqq 1/2$, there is a strategy τ of $T\backslash S$ such that for every strategy σ of S there is an S-allocation $\underset{\sim}{x} = (\underset{\sim}{y},\underset{\sim}{z})$ of $(a(S),\omega(T))$ such that

$$\underset{\sim}{h}^S_{\sigma\tau}(t) \begin{cases} \leqq u_t(\underset{\sim}{y}(t),\underset{\sim}{z}(t)) & \text{if} \quad t \in S \\ \geqq 0 & \text{if} \quad t \in T\backslash S \quad ; \end{cases}$$

and

(7.3) if $S \in \mathcal{C}$ is such that $\mu(S) = 1/2$, then for each S-allocation $\underset{\sim}{x}$ there is a strategy σ of S such that for each strategy τ of $T\backslash S$ there is a $T\backslash S$-allocation $\bar{\underset{\sim}{x}}$ such that

$$\underset{\sim}{h}^S_{\sigma\tau}(t) \begin{cases} \geqq u_t(\underset{\sim}{x}(t)) & \text{if} \quad t \in S \\ \leqq u_t(\bar{\underset{\sim}{x}}(t)) & \text{if} \quad t \in T\backslash S \quad . \end{cases}$$

(Note that (7.3) is the same as (5.3) and (6.3).) We shall now state the result which will be demonstrated in this chapter. We shall need to assume that each utility function u_t for $t \in T$ is of a rather special form: we shall make the following assumption.[5/]

(7.4) for all $t \in T$, $u_t(y,z) = f_t(y) + g_t(z)$ for each $(y,z) \in \Omega$, where f_t satisfies (4.6) and (4.7) and g_t satisfies (4.8) .

(We are also of course maintaining assumptions (4.2) through (4.5).)

Proposition 7.5: Let M be a market satisfying (7.4). Then if $\Gamma(M)$ satisfies (7.1) through (7.3), an allocation $\underset{\sim}{x} = (\underset{\sim}{y},\underset{\sim}{z})$ is a value allocation of the economy $E = (M, \Gamma(M))$ if and only if it is efficient and a.e.

(7.6) $$\underset{\sim}{\lambda}(t)u_t(\underset{\sim}{x}(t)) - \int\underset{\sim}{\lambda}u(\underset{\sim}{x}) = p_1(\underset{\sim}{a}(t) - \underset{\sim}{y}(t)) + ((2^\beta - 1)/\beta)p_2(\omega(T) - \underset{\sim}{z}(t))$$

where $(\underset{\sim}{\lambda},p) = (\underset{\sim}{\lambda}, (p_1,p_2))$ is an efficiency pair for $\underset{\sim}{x} = (\underset{\sim}{y},\underset{\sim}{z})$.

Throughout this chapter $\Gamma(M)$ will denote a strategic game associated with the market M which satisfies (7.1) through (7.3).

7.2 Optimal Threats and the Value of the Game q

To find a collection of pairs of optimal threats in the games between S and $T\setminus S$ for each $S \in \mathcal{C}$ we can use an argument identical to that in Section 6.2. Doing so (we shall not repeat the argument here), we find that the game defined by this collection of optimal threats is the game $q\colon \mathcal{C} \to \mathbb{R}_+$ defined by

(7.7) $$q(S) = \begin{cases} w(S) & \text{if } \mu(S) > 1/2 \\ r(S) & \text{if } \mu(S) = 1/2 \\ 0 & \text{if } \mu(S) < 1/2 \end{cases},$$

where $w\colon \mathcal{C} \to \mathbb{R}_+$ is the game defined by

$$w(S) = u_S(a(S),\omega(T)) \quad \text{for each} \quad S \in \mathcal{C}$$

and $r\colon \mathcal{C} \to \mathbb{R}_+$ is the market game derived from M. Now, for each $S \in \mathcal{C}$,

$$w(S) = \sup \{\int_S f(\underset{\sim}{y}) + \int_S g(\underset{\sim}{z}) : (\int_S \underset{\sim}{y}, \int_S \underset{\sim}{z}) = (a(S), \omega(T))\}$$

$$= \sup \{\int_S f(\underset{\sim}{y}) : \int_S \underset{\sim}{y} = a(S)\} + \sup \{\int_S g(\underset{\sim}{z}) : \int_S \underset{\sim}{z} = \omega(T)\}$$

$$= r_f(S) + v_g(S) \quad ,$$

where $r_f: C \to \mathbb{R}$ is the market game derived from the market M_f with agent space (T,C,μ), n goods, utility functions $f_t: \mathbb{R}^n_+ \to \mathbb{R}$, and initial endowment density $\underset{\sim}{a}: T \to \mathbb{R}^n_+$, and $v_g: C \to \mathbb{R}$ is the game defined as in (6.5) which is derived from the market M_g with agent space (T,C,μ), m goods, utility functions $g_t: \mathbb{R}^m_+ \to \mathbb{R}_+$, and initial endowment density $\underset{\sim}{\omega}: T \to \mathbb{R}^m_+$. If M is integrably sublinear, $r_f(S)$ and $v_g(S)$ are attained for every $S \in C$, and we have $r_f \in pNAD \cap pNA'$ by Proposition 4.27, and $v_g \in pNA$ by Proposition 6.11. Hence $w \in pNAD \cap pNA'$, and we can conclude as in Section 6.4 that $q \in ASYMP$ and that the value ϕq of q is identical to the value of the game $k: C \to \mathbb{R}_+$ defined by

$$k(S) = \begin{cases} w(S) & \text{if } \mu(S) \geqq 1/2 \\ 0 & \text{if } \mu(S) < 1/2 \end{cases} \quad .$$

Now define games $k_f: C \to \mathbb{R}_+$ and $k_g: C \to \mathbb{R}_+$ by

$$k_f(S) = \begin{cases} r_f(S) & \text{if } \mu(S) \geqq 1/2 \\ 0 & \text{if } \mu(S) < 1/2 \end{cases} \quad ,$$

and

$$k_g(S) = \begin{cases} v_g(S) & \text{if } \mu(S) \geqq 1/2 \\ 0 & \text{if } \mu(S) < 1/2 \end{cases} \quad .$$

Then $k = k_f + k_g$, so we have $\phi q = \phi k_f + \phi k_g$. From Propositions 3.16 and 4.28 we have an expression for ϕk_f when M is integrably sublinear, and from Proposition 6.28, we have an expression for ϕk_g in this case. Combining these we obtain

$$(\phi q)(S) = (r_f(T)/2 + 2^{\beta-1}v_g(T))\mu(S) + \int_S (f(\underset{\sim}{y}) - p_1(\underset{\sim}{y} - \underset{\sim}{a}))/2$$
$$+ (1 - 2^{\beta-1}) \int_S g(\underset{\sim}{z}) \quad \text{for each} \quad S \in C \quad ,$$

where $(\underset{\sim}{y}, p_1)$ is a t.u.c.e. in M_f and $\underset{\sim}{z}$ is an allocation in M_g at which $v_g(T)$ is attained. From this last fact there exists $p_2 \in \mathbb{R}^m_{++}$ such that $(\underset{\sim}{z}, p_2)$ is a t.u.c.e. in M_g. But then $((\underset{\sim}{y}, \underset{\sim}{z}), (p_1, p_2))$ is a t.u.c.e. in M, so we have

$$(7.8) \qquad (\phi q)(S) = [\int f(\underset{\sim}{y})/2 + 2^{\beta-1} g(\underset{\sim}{z})]\mu(S) + \int_S [f(\underset{\sim}{y}) - p_1(\underset{\sim}{y} - \underset{\sim}{a})]/2$$
$$+ (1 - 2^{\beta-1}) \int_S g(\underset{\sim}{z}) \quad \text{for each} \quad S \in C \quad ,$$

where $(\underset{\sim}{x}, p) = ((\underset{\sim}{y}, \underset{\sim}{z}), (p_1, p_2))$ is a t.u.c.e. in M.

7.3 The Characterization of the Value Allocations of the Economy $(M, \Gamma(M))$

Once again we can reason exactly as we did in Chapter 6; we shall only sketch the argument here. Let $\underset{\sim}{\lambda}$ be a comparison function. Then from (7.8) we have

$$(7.9) \qquad (\phi q_{\lambda})(S) = [\int \lambda f(y)/2 + 2^{\beta-1}\int \lambda g(z)]\mu(S) + \int_S [\lambda f(y) - p_1(y - a)]/2$$
$$+ (1 - 2^{\beta-1})\int_S \lambda g(z) \quad \text{for each} \quad S \in C \quad ,$$

where $(x,p) = ((y,z),(p_1,p_2))$ is a t.u.c.e. in λM. Now assume that $(\phi q_{\lambda})(S) = \int_S \lambda u(x)$ for each $S \in C$. Then $\int \lambda u(x) = (\phi q_{\lambda})(T) = q_{\lambda}(T)$ $= w_{\lambda}(T) = r_{\lambda}(T)$, so from (7.9), we have

$$(7.10) \qquad \int_S \lambda(f(y) + g(z)) =$$
$$= [\int \lambda f(y)/2 + 2^{\beta-1}\int \lambda g(z)]\mu(S) + \int_S [\lambda f(y) - p_1(y - a)]/2$$
$$+ (1 - 2^{\beta-1})\int_S \lambda g(z) \quad \text{for each} \quad S \in C \quad ,$$

so that a.e.

$$(7.11) \qquad \lambda(t)(f_t(y(t)) + 2^{\beta}g_t(z(t))) = \int \lambda f(y) + 2^{\beta}\int \lambda g(z) + p_1(a(t) - y(t)) \quad ,$$

where $(x,p) = ((y,z),(p_1,p_2))$ is a t.u.c.e. in λM. Conversely, if (7.11) is a.e. satisfied we can deduce (7.10) by integrating over S, and use (7.9) to deduce that $(\phi q_{\lambda})(S) = \int \lambda u(x)$ for all $S \in C$. We can prove a result analagous to Lemma 6.10 under assumptions (7.1) through (7.3), so we have established the following (cf. Lemma 6.37 and Lemma 6.10).

Proposition 7.12: An allocation x is a value allocation of $E = (M, \Gamma(M))$ where M satisfies (7.4) if and only if $x = (y,z)$ is efficient with efficiency pair $(\lambda, (p_1,p_2))$ and (7.11) is a.e. satisfied.

We shall now further exploit the homogeneity properties of g_t for each $t \in T$.

<u>Lemma 7.13</u>: Let M be a market satisfying (7.4) and let $\underset{\sim}{x} = (\underset{\sim}{y},\underset{\sim}{z})$ be an efficient allocation in M with efficiency pair $(\underset{\sim}{\lambda},p) = (\underset{\sim}{\lambda},(p_1,p_2))$. Then $\underset{\sim}{x}$ a.e. satisfies (7.11) if and only if it a.e. satisfies (7.6).

<u>Proof</u>: Suppose $\underset{\sim}{x}$ a.e. satisfies (7.11). Then since $(\underset{\sim}{\lambda},p_2)$ is an efficiency pair for $\underset{\sim}{z}$ in M_g we have

$$\underset{\sim}{\lambda}(t)g_t^i(\underset{\sim}{z}(t))\underset{\sim}{z}^i(t) = p^i\underset{\sim}{z}^i(t) \quad \text{for} \quad i = 1,\ldots,m, \quad \text{for each} \quad t \in T$$

(using the homogeneity of degree β of each g_t). So from (7.11) we have a.e.

$$\underset{\sim}{\lambda}(t)(f_t(\underset{\sim}{y}(t)) + g_t(\underset{\sim}{z}(t))) - (1 - 2^\beta)\underset{\sim}{\lambda}(t)g_t(\underset{\sim}{z}(t)) =$$

$$= \underset{\sim}{\lambda}(t)u_t(\underset{\sim}{x}(t)) - (1 - 2^\beta)p_2\underset{\sim}{z}(t)$$

$$= \int\underset{\sim}{\lambda}f(\underset{\sim}{y}) + \int\underset{\sim}{\lambda}g(\underset{\sim}{z}) - ((1 - 2^\beta)/\beta)\int\underset{\sim}{\lambda}g(\underset{\sim}{z}) + p_1(\underset{\sim}{a}(t) - \underset{\sim}{y}(t))$$

$$= \int\underset{\sim}{\lambda}u(\underset{\sim}{x}) - ((1 - 2^\beta)/\beta)p_2\omega(T) + p_1(\underset{\sim}{a}(t) - \underset{\sim}{y}(t)) \quad ,$$

which gives us (7.6). Conversely, if $\underset{\sim}{x}$ a.e. satisfies (7.6) then we can reverse this argument to show that $\underset{\sim}{x}$ a.e. satisfies (7.11). This completes the proof.

To take $\omega(T) = 0$ is formally illegitimate since it violates (4.1), but if we do so we can assume that $\underset{\sim}{z}(t) = 0$ a.e., and (7.6) reduces to the characterization provided by Aumann and Kurz [1977]--which is as it should be since in this case every good which exists in the economy can be destroyed. Similarly if we take $a(T) = 0$, (7.6) reduces to the characterization given in Chapter 6, namely $p_2\underset{\sim}{z}(t) = p_2\omega(T)$ a.e. (since $\beta < 2^\beta - 1$ for all $\beta \in (0,1)$).

Now, since p is an efficiency price for $\underset{\sim}{x}$ this allocation can be achieved as a competitive allocation after wealth has been redistributed so that $t \in T$ has $p\underset{\sim}{x}(t)$ rather than the amount $p\underset{\sim}{e}(t)$ with which he began; thus $\underset{\sim}{x}$ can be achieved as a competitive allocation in which $t \in T$ pays the "tax" $p\underset{\sim}{e}(t) - p\underset{\sim}{x}(t)$. From (7.6) we can see that, given the characteristics of the other agents, the vector of goods which $t \in T$ receives in the game we are considering depends only on $p_1\underset{\sim}{a}(t)$ and u_t (everything else in the equation is a constant). Hence $p\underset{\sim}{e}(t) - p\underset{\sim}{x}(t)$ equals $p_2\underset{\sim}{\omega}(t)$ plus a function of u_t and $p_1\underset{\sim}{a}(t)$ (and the characteristics of the other agents)--i.e. the tax rate on the value of the second group of goods is 100% (all of $p_2\underset{\sim}{\omega}(t)$ is paid in tax), while that on the first group is in general less than 100%. In order to study the size of the taxes on this first group we shall now consider a class of examples.

7.4 A Class of Examples

We shall consider here the economy derived from a market in which $u_t = u$ for all $t \in T$, and $f_t = f$ is homogeneous of the same degree (namely β) as each $g_t = g$. (As in the examples considered in Section 5.6, f does not actually satisfy our boundedness assumption; however, we can reason as we did there that our arguments will still locate the value allocations of the economy we are studying.) Now u is homogeneous of degree β $(\in(0,1))$, so all efficient allocations in M consist of bundles lying on the ray from the origin through the aggregate initial endowment $\int e = (\int a, \int \omega)$. So we can set $p = u'(\int e)$ and

(7.14) $\quad x(t) = k(t)\int e \quad , \quad$ where $k: T \to \mathbb{R}_+$ is measurable and $\int k = 1$.

Assume that $x(t) >> 0$ a.e.. Then from (4.13) we have $\lambda(t) = p^i/u^i(x(t))$, so from (7.6) we have a.e.

$$p^i u(x(t))/u^i(x(t)) - c = p_1(a(t) - y(t)) + ((2^\beta - 1)/\beta)p_2(\omega(T) - z(T))$$

(where $c = \int \lambda u(x)$), so a.e.

$$p^i x^i(t) = [p_1(a(t) - y(t)) + ((2^\beta - 1)/\beta)p_2(\omega(T) - z(t)) + c] \times \frac{u^i(x(t))x^i(t)}{u(x(t))}$$

(since $x(t) >> 0$ a.e., $u(x(t)) > 0$ a.e.). Hence, summing and using the homogeneity of u, a.e.

$$\begin{aligned} px(t) &= p_1 y(t) + p_2 z(t) \\ &= \beta[p_1(a(t) - y(t)) + ((2^\beta - 1)/\beta)p_2(\omega(T) - z(t)) + c] \end{aligned}$$

so a.e.

$$(1 + \beta)p_1 y(t) + 2^\beta p_2 z(t) = \beta p_1 a(t) + c' \quad ,$$

where $c' = c + (2^\beta - 1)p_2\omega(T)$. Now using (7.14) we have a.e.

$$(1 + \beta)\underset{\sim}{k}(t)p_1\int\underset{\sim}{a} + 2^\beta\underset{\sim}{k}(t)p_2\int\underset{\sim}{\omega} = \beta p_1\underset{\sim}{a}(t) + c' \quad ,$$

so a.e.

$$\underset{\sim}{x}(t) = \underset{\sim}{k}(t)\int\underset{\sim}{e} = \left(\frac{\beta p_1\underset{\sim}{a}(t) + c'}{(1 + \beta)p_1\int\underset{\sim}{a} + 2^\beta p_2\int\underset{\sim}{\omega}}\right)\int\underset{\sim}{e} \quad .$$

Integrating we find that $c' = p_1\int\underset{\sim}{a} + 2^\beta p_2\int\underset{\sim}{\omega}$, so we have a.e.

$$(7.15) \qquad (\underset{\sim}{y}(t),\underset{\sim}{z}(t)) = \left(1 + \frac{\beta p_1(\underset{\sim}{a}(t) - \int\underset{\sim}{a})}{(1 + \beta)p_1\int\underset{\sim}{a} + 2^\beta p_2\int\underset{\sim}{\omega}}\right)\left(\int\underset{\sim}{a},\int\underset{\sim}{\omega}\right) \quad ,$$

where $p = (p_1,p_2) = u'(\int\underset{\sim}{e}) = u'(\int\underset{\sim}{a},\int\underset{\sim}{\omega})$ is an efficiency price for $\underset{\sim}{x} = (\underset{\sim}{y},\underset{\sim}{z})$. From this last fact the allocation $\underset{\sim}{x}$ can be achieved in a competitive equilibrium in which $t \in T$ pays the "tax" $\underset{\sim}{\tau}(t) = p\underset{\sim}{e}(t) - p\underset{\sim}{x}(t)$. We shall now study these taxes in more detail. We have a.e.

$$\begin{aligned}
\underset{\sim}{\tau}(t) &= p_1\underset{\sim}{a}(t) + p_2\underset{\sim}{\omega}(t) - p_1\underset{\sim}{y}(t) - p_2\underset{\sim}{z}(t) \\
&= p_1\underset{\sim}{a}(t) + p_2\underset{\sim}{\omega}(t) - \left(1 + \frac{p_1(\underset{\sim}{a}(t) - a(T))}{(1 + \beta)p_1 a(T) + 2^\beta p_2\omega(T)}\right) \\
&\quad \times (p_1 a(T) + p_2\omega(T)) \\
&= \left(1 - \frac{\beta(p_1 a(T) + p_2\omega(T))}{(1 + \beta)p_1 a(T) + 2^\beta p_2\omega(T)}\right)p_1(\underset{\sim}{a}(t) - a(T)) \\
&\quad + p_2(\underset{\sim}{\omega}(t) - \omega(T)) \quad .
\end{aligned}$$

So a.e.

$$(7.16) \qquad \underset{\sim}{\tau}(t) = \left(\frac{p_1 a(T) + (2^\beta - \beta) p_2 \omega(T)}{(1+\beta) p_1 a(T) + 2^\beta p_2 \omega(T)}\right) p_1(\underset{\sim}{a}(t) - a(T)) + p_2(\underset{\sim}{\omega}(t) - \omega(T)) \quad .$$

Thus, as we argued above in general, the tax rate on the wealth $p_2 \underset{\sim}{\omega}(t)$ associated with the second group of goods is 100%, while the tax rate on the wealth $p_1 \underset{\sim}{a}(t)$ associated with the first group of goods is

$$\underset{\sim}{\tau}_1(t) = \frac{p_1 a(T) + (2^\beta - \beta) p_2 \omega(T)}{(1+\beta) p_1 a(T) + 2^\beta p_2 \omega(T)} \quad .$$

Now, if we assume that all goods can be destroyed--i.e. the economy satisfies the assumptions of Aumann and Kurz [1977]--then from Section 10 of that paper we know that the tax rate on wealth is $1/(1+\beta)$. We shall now compare this with $\underset{\sim}{\tau}_1(t)$. We have

$$\frac{1/(1+\beta)}{\underset{\sim}{\tau}_1(t)} = \frac{p_1 a(T) + 2^\beta p_2 \omega(T)/(1+\beta)}{p_1 a(T) + (2^\beta - \beta) p_2 \omega(T)} \quad ,$$

and $2^\beta/(1+\beta) = 2^\beta - \beta 2^\beta/(1+\beta) > 2^\beta - \beta$ since $2^\beta < 1+\beta$ for all $\beta \in (0,1)$, so

$$\frac{1/(1+\beta)}{\underset{\sim}{\tau}_1(t)} > 1 \quad .$$

Thus in this class of examples the tax rate on wealth derived from goods which can be destroyed is lower if there are also goods in the economy which cannot be destroyed than if all goods can be destroyed. In Aumann and Kurz [1977], where an economy of the latter sort is studied, the tax rate is at least 50%. Here the tax rate on the wealth derived from goods which can be "destroyed" (e.g. labor-time) can be less than 50%, but from the formula above can be shown to be at least $1 - 1/(e \log_e 2) \simeq 0.465$ (where e is the base of hyperbolic logarithms) and comes close to this only if $p_1 a(T)$ is very small in relation to $p_2 \omega(T)$.

CHAPTER 8: A Class of Power Distributions

8.1 Introduction

The aim of this study is not simply to find the outcome of specific assumptions about the strategic possibilities of the groups of agents (as we have done in the previous three chapters), but also to understand how the solutions given by our model change as we allow the strategy sets to vary over a wide range: we want to know what it is that makes agents more or less "powerful", and how sensitive the set of solutions is to the precise assumptions we make. Here we shall report the results of an investigation in this area.

Rather than dealing directly with the strategic game, we begin our analysis here with the coalitional game q (see (3.24)) which summarizes the threat possibilities of all the groups of agents. The fact that q is the result of the groups carrying out certain strategies in a game $\Gamma(M)$ associated with the market M puts restrictions on its characteristics. Thus if M is a market we shall say that $q: \mathcal{C} \to \mathbb{R}$ is a bargaining game associated with M if for each $S \in \mathcal{C}$, $q(S) = F_S(u,\underset{\sim}{e},\mu)$ (i.e. depends solely on the data u, $\underset{\sim}{e}$, and μ of M in addition to S) and there is an allocation $\underset{\sim}{x}$ in M such that $q(S) = \int_S u(\underset{\sim}{x})$ and $q(T\setminus S) = \int_{T\setminus S} u(\underset{\sim}{x})$. Let $\underset{\sim}{\lambda}$ be a comparison function. We define the game $q_{\underset{\sim}{\lambda}}: \mathcal{C} \to \mathbb{R}$ by $q_{\underset{\sim}{\lambda}}(S) = F_S(\underset{\sim}{\lambda}u,\underset{\sim}{e},\mu)$ for each $S \in \mathcal{C}$, and say (slightly abusing our terminology) that an allocation $\underset{\sim}{x}$ in M is a value allocation of q if there exists a comparison function $\underset{\sim}{\lambda}$ for which $q_{\underset{\sim}{\lambda}}(T)$ is finite and

(8.1) $$\int_S \lambda u(x) = (\phi q_{\lambda})(S) \quad \text{for each} \quad S \in C \quad .$$

In the remainder of this chapter we first characterize the value allocations of a whole class of bargaining games associated with M, and then discuss the way they reflect the "distribution of power" and study the circumstances under which these allocations can be supported as competitive allocations with a particular form of taxation.

8.2 The Characterization of the Value Allocations of a Class of Bargaining Games Associated with a Bounded Market M.

Throughout this chapter we shall be concerned with a fixed bounded market M. The class of bargaining games q we shall examine here contains those for which

(8.2) $q(S) = f(r(S),e(S),\mu(S))$ for each $S \in C$, where $f: \mathbb{R}^{\ell+2} \to \mathbb{R}$ is increasing and continuously differentiable and $f(r_{\lambda}(T),e(T),\mu(T)) = r_{\lambda}(T)$ for every comparison function λ

(and r is the market game derived from M defined in (4.15)). Note that since $r(\emptyset) = e(\emptyset) = \mu(\emptyset) = 0$, the fact that q is a game implies that $f(0) = 0$. Let $f_{1/2}: [0,1] \to [0,1]$ be defined by

$$f_{1/2}(x) = \begin{cases} 1 & \text{if} \quad x \in [1/2,1] \\ 0 & \text{if} \quad x \in [0,1/2) \quad . \end{cases}$$

Then the bargaining game to which the strategic game of Aumann and Kurz [1977] leads is given by

(8.3) $$q(S) = f_{1/2}(\mu(S))r(S) \quad \text{for each} \quad S \in \mathcal{C} \quad .$$

Since $f_{1/2}$ is not differentiable at $x = 1/2$, q does not fall into the class defined by (8.2). However, it can be approximated by members of that class. The form (8.3) involves a dichotomized distribution of "power", in which the power of any coalition containing more than 50% of the population is radically different from that of any coalition with less than 50%. It seems that for one reason or another in actual economies groups with less than 50% of the population may be quite powerful; included in the class defined by (8.2) are such cases. (We shall discuss these matters further below.)

We shall now characterize the value allocations of a bargaining game satisfying (8.2). First, we have the following.

Proposition 8.4: Assume M is integrably sublinear, and let $q: \mathcal{C} \to \mathbb{R}$ be a bargaining game satisfying (8.2). Then $q \in pNAD \cap pNA'$ and ϕq is given by

(8.5) $$(\phi q)(S) = a_1 \int_S [u(\underset{\sim}{x}) - p(\underset{\sim}{x} - \underset{\sim}{e})] + a_2 e(S) + a_3\mu(S) \quad ,$$

where the constants $a_1 \in \mathbb{R}_+$, $a_2 \in \mathbb{R}^\ell_+$, and $a_3 \in \mathbb{R}_+$ are given by

(8.6) $$b_i = \int_0^1 f_i(\theta r(T), \theta e(T), \theta\mu(T)) \quad \text{for} \quad i = 1,\ldots,\ell + 2$$

where $b = (a_1,a_2,a_3) \in \mathbb{R}^{\ell+2}_+$, and $(\underset{\sim}{x},p)$ is a t.u.c.e. in M.

<u>Proof</u>: We have $\mu \in NA$, e is a vector of members of NA, and, by Proposition 4.27, $r \in pNAD \cap pNA'$. Let w be the $(\ell + 2)$-vector of games (r,e,μ), so that $q = f \circ w$. Since $w_i \in pNA'$ for $i = 1,\ldots,\ell + 2$ we can find a sequence $\{w_i^n\}_{n=1}^{\infty}$ with $w_i^n = \pi_i^n \circ \nu_i^n$ for each n, such that ν_i^n is a vector of members of NA and π_i^n is a polynomial, and w_i^n converges to w_i in the supremum norm. But then $f \circ w^n = f \circ (\pi_1^n \circ \nu_1^n,\ldots,\pi_{\ell+2}^n \circ \nu_{\ell+2}^n) = f \circ \pi^n \circ (\nu_1^n,\ldots,\nu_{\ell+2}^n)$ where $\pi^n\colon \mathbb{R}^{\ell+2} \to \mathbb{R}^{\ell}$ is defined by $\pi^n(x_1,\ldots,x_{\ell+2}) = (\pi_1^n(x_1),\ldots,\pi_{\ell+2}^n(x_{\ell+2}))$, so since $f \circ \pi^n$ is continuously differentiable on the range of $(\nu_1^n,\ldots,\nu_{\ell+2}^n)$ and $(f \circ \pi^n)(0) = 0$ (since $f(0) = 0$), by Theorem B of Aumann and Shapley we have $f \circ w^n \in pNA$. But $f \circ w^n$ converges to $f \circ w$ in the supremum norm, so $f \circ w \in pNA'$.

Similarly, since $w_i \in pNAD$ for $i = 1,\ldots,\ell + 2$, there exists $d_i \in DIAG$ and a sequence $\{\bar{w}_i^n\}_{n=1}^{\infty}$ with $\bar{w}_i^n = \bar{\pi}_i^n \circ \bar{\nu}^n$ for each n, such that $\bar{\nu}^n$ is a vector of members of NA and $\bar{\pi}_i^n$ is a polynomial and $\bar{w}_i^n$ converges to $w_i - d_i$ in the BV norm. But as above we can then argue that $f \circ \bar{w}^n$ is a member of pNA for all n, so that $f \circ (w - d)$ is also. Since $f \circ w = f \circ (w - d) + (f \circ w - f \circ (w - d))$ and $f \circ w - f \circ (w - d) \in DIAG$ (since $d \in DIAG$), we then have $f \circ w \in pNAD$. This, together with the conclusion of the previous paragraph, establishes the first claim of the lemma.

Now from Lemma 3.18 we know that for each $S \in \mathcal{C}$, $\partial q^*(\theta,S)$ and $\partial w_i^*(\theta,S)$ for $i = 1,\ldots,n$ exist for almost all $\theta \in [0,1]$, and when they exist we have

$$\partial q^*(\theta,S) = \sum_{i=1}^{\ell+2} f_i(w^*(\theta\chi_T))\partial w_i^*(\theta,S) \quad ,$$

so that by Theorem 3.14 we have

$$(\phi q)(S) = \int_0^1 \sum_{i=1}^{\ell+2} f_i(w^*(\theta\chi_T))\partial w_i^*(\theta,S)d\theta \quad \text{for each} \quad S \in \mathcal{C} \quad .$$

Now, $w = (r,e,\mu)$ and (e,μ) is a vector of members of NA, so for all $\theta \in [0,1]$ we have $(e^*(\theta\chi_T),\mu^*(\theta\chi_T)) = (\theta e(T),\theta\mu(T))$ and $(\partial e^*(\theta,S),\partial\mu^*(\theta,S)) = (e(S),\mu(S))$ for each $S \in \mathcal{C}$. Also, by Proposition 4.28 we know that for all $\theta \in [0,1]$ we have $r^*(\theta\chi_T) = \theta r(T)$ and $\partial r^*(\theta,S) = (\phi r)(S)$ for each $S \in \mathcal{C}$. Hence for each $S \in \mathcal{C}$

(8.7) $$(\phi q)(S) = a_1(\phi r)(S) + a_2 e(S) + a_3\mu(S)$$

where $a_1 \in \mathbb{R}_+$, $a_2 \in \mathbb{R}_+^\ell$, and $a_3 \in \mathbb{R}_+$ are the constants given by (8.6). But again from Proposition 4.28 we have

$$(\phi r)(S) = \int_S [u(\underset{\sim}{x}) - p(\underset{\sim}{x} - \underset{\sim}{e})] \quad \text{for each} \quad S \in \mathcal{C} \quad ,$$

where $(\underset{\sim}{x},p)$ is a t.u.c.e. in M, so the proof is complete.

This result allows us to characterize the value allocations of q.

<u>Theorem 8.8</u>: Let M be a bounded market and let q be a bargaining game satisfying (8.2). Then $\underset{\sim}{x}$ is a value allocation of q if and only if it is an efficient allocation in M and a.e.

(8.9) $$(1 - a_1)\underset{\sim}{\lambda}(t)u_t(\underset{\sim}{x}(t)) = a_1 p(\underset{\sim}{e}(t) - \underset{\sim}{x}(t)) + a_2\underset{\sim}{e}(t) + a_3$$

where $(\underset{\sim}{x},p)$ is a t.u.c.e. in $\underset{\sim}{\lambda}M$, $a_1 \in \mathbb{R}_+$, $a_2 \in \mathbb{R}_+^{\ell}$, and $a_3 \in \mathbb{R}_+$ are the constants given in Proposition 8.4 corresponding to the game $q_{\underset{\sim}{\lambda}}$.

Proof: First assume that $\underset{\sim}{x}$ is a value allocation of q. Then there exists a comparison function $\underset{\sim}{\lambda}$ such that $q_{\underset{\sim}{\lambda}}(T)$ is finite and (8.1) is satisfied. But then $\int \underset{\sim}{\lambda} u(\underset{\sim}{x}) = (\phi q_{\underset{\sim}{\lambda}})(T) = q_{\underset{\sim}{\lambda}}(T) = r_{\underset{\sim}{\lambda}}(T)$ (using (8.2)), so $\underset{\sim}{x}$ is efficient with efficiency comparison function $\underset{\sim}{\lambda}$, and $r_{\underset{\sim}{\lambda}}(T)$ is finite. But then by Proposition 4.17, $\underset{\sim}{\lambda}M$ is integrably sublinear, and there is a price vector p such that $(\underset{\sim}{\lambda},p)$ is an efficiency price for $\underset{\sim}{x}$, so by Proposition 8.4 we have

$$(\phi q_{\underset{\sim}{\lambda}})(S) = a_1 \int_S [\underset{\sim}{\lambda} u(\underset{\sim}{x}) - p(\underset{\sim}{x} - \underset{\sim}{e})] + a_2 e(S) + a_3 \mu(S)$$
$$\text{for each } S \in C \quad .$$

So, using (8.1), for each $S \in C$

$$\int_S \underset{\sim}{\lambda} u(\underset{\sim}{x}) = a_1 \int_S \underset{\sim}{\lambda} u(\underset{\sim}{x}) - a_1 p \int_S (\underset{\sim}{x} - \underset{\sim}{e}) + a_2 e(S) + a_3 \mu(S)$$

or

$$(1 - a_1) \int_S \underset{\sim}{\lambda} u(\underset{\sim}{x}) = a_1 p \int_S (\underset{\sim}{e} - \underset{\sim}{x}) + a_2 e(S) + a_3 \mu(S) \quad ,$$

giving us (8.9) a.e..

Now assume that $\underset{\sim}{x}$ is efficient and (8.9) is a.e. satisfied. Since $f(0) = 0$ and f is increasing we have $a_1 < 1$, so we can integrate over T to conclude that $r_{\underset{\sim}{\lambda}}(T) = \int \underset{\sim}{\lambda} u(\underset{\sim}{x})$ is finite. Hence by Proposition 4.17 once again, M is integrably sublinear, so that by Proposition 8.4,

$$(\phi q_{\lambda})(S) = a_1 \int_S \lambda u(x) - a_1 p \int_S (x - e) + a_2 e(S) + a_3 \mu(S)$$

for each $S \in C$.

Integrating (8.9) over S then gives us $(\phi q_{\lambda})(S) = \int_S \lambda u(x)$ for all $S \in C$; since $q_{\lambda}(T) = (\phi q_{\lambda})(T) \doteq \int \lambda u(x) = r_{\lambda}(T)$, $q_{\lambda}(T)$ is finite, so that x is a value allocation of q.

This completes the proof.

8.3 Wealth Taxes

Aumann and Kurz [1977] argue that the value allocations of the game they consider can all be supported as competitive allocations after each agent's wealth $pe(t)$ has been modified by a wealth tax. Given Theorem 8.8 it is easy to argue that the same is true for a wide class of bargaining games.

Proposition 8.10: Let M be a bounded market and let q be a bargaining game satisfying (8.2) for which

(8.11) $f(r(S),e(S),\mu(S)) = f(r(S),0,\mu(S))$ for each $S \in C$

(i.e. for which $q(S)$ is independent of $e(S)$). Let x be a value allocation of q. Then there exists a price vector p such that $x(t)$ a.e. maximizes $u_t(x)$ over $\{x \in \Omega: px \leq pe(t) - \tau(t)\}$ where $\tau: T \to \mathbb{R}$ is such that if $u_t = u_s$ and $pe(t) = pe(s)$ then $\tau(t) = \tau(s)$ (i.e. τ is a system of wealth taxes).

Proof: If we define a function $f_1: \mathbb{R}^2 \to \mathbb{R}$ by $f_1(r(S),\mu(S)) = f(r(S),0,\mu(S))$ for all $S \in C$ we can deduce, using an argument as in the proofs of Proposition 8.4 and Theorem 8.8, that $\underset{\sim}{x}$ is a value allocation of q if and only if it is efficient and a.e. (8.9) is satisfied with $a_2 = 0$ --i.e. a.e.

$$(8.12) \qquad (1 - a_1)(\underset{\sim}{\lambda}(t)u_t(\underset{\sim}{x}(t)) - r_{\underset{\sim}{\lambda}}(T)) = a_1 p(\underset{\sim}{e}(t) - \underset{\sim}{x}(t)) \quad ,$$

where $(\underset{\sim}{\lambda},p)$ is an efficiency pair for $\underset{\sim}{x}$. The fact that p is an efficiency price for $\underset{\sim}{x}$ means that $\underset{\sim}{x}(t)$ a.e. maximizes $u_t(x)$ over $\{x \in \Omega: px \leqq p\underset{\sim}{x}(t)\}$. But it $u_t = u_s$ and $p\underset{\sim}{e}(t) = p\underset{\sim}{e}(s)$, any solution of (8.12) for $\underset{\sim}{x}(t)$ must be a solution for $\underset{\sim}{x}(s)$, and vice versa, so if we set $\underset{\sim}{\tau}(t) = p\underset{\sim}{e}(t) - p\underset{\sim}{x}(t)$, $\underset{\sim}{\tau}(t) = \underset{\sim}{\tau}(s)$ under these circumstances, completing the proof.

We can see from this proof that if e(S) is independently an argument of f then in general the value allocations of q cannot be supported as competitive allocations after wealth taxation--for the solution $\underset{\sim}{x}(t)$ of (8.9) then depends independently on $\underset{\sim}{e}(t)$, not just $p\underset{\sim}{e}(t)$ (unless it happens that a_2 is proportional to p).

As we noted above, the bargaining game to which the assumptions of Aumann and Kurz [1977] lead (see (8.3)) is not a member of the class we are considering, but can be approximated by members of that class. Proposition 8.9 shows that their result that each value allocation of their game can be supported as a competitive allocation after wealth taxation is not at all sensitive to the precise characteristics of the game q; all that is important is that for each $S \in \mathcal{C}$, $q(S)$ depends solely on $r(S)$ and $\mu(S)$, and not independently on $e(S)$. We have not shown how a strategic game can be constructed to generate a given bargaining game q, but it is clear that a wide variety of assumptions about the strategic possibilities of groups of agents will lead to a game q which depends solely on $r(S)$ and $\mu(S)$ (we shall return to this matter in the next section).

There is a stronger sense in which the value allocations of Aumann and Kurz [1977] are the result of wealth taxation, which we shall now explain. For each price vector p and $t \in T$ let $u_t^p: \mathbb{R}_+ \to \mathbb{R}_+$ be the indirect utility function of agent t at prices p (see (5.28)). Let M^p be the market with agent space $(T,\mathcal{C},\mu)$ and $\ell = 1$ in which the utility function of $t \in T$ is $u_t^p: \mathbb{R}_+ \to \mathbb{R}_+$ and the initial endowment density is $p\underset{\sim}{e}: T \to \mathbb{R}_+$. By Lemma 5.30, for each price vector p, M^p is bounded if M is. If q is a bargaining game associated with M, with $q(S) = F_S(u,\underset{\sim}{e},\mu)$ for each $S \in \mathcal{C}$, for each price vector p we can construct the bargaining game q^p defined by $q^p(S) = F_S(u^p,p\underset{\sim}{e},\mu)$ (q^p is thus just a one-commodity ("income") version of q). Viewing this as a bargaining game associated with M^p, let $\underset{\sim}{y}^p$ be a value allocation of q^p. (That is, $\underset{\sim}{y}^p$ is a value allocation of the game which is played according to the same rules as q, but in which "money" (or "income") is treated as the sole good.) Finally, let $\underset{\sim}{x}^p(t)$ a.e. maximize $u_t(x)$ over $\{x \in \Omega: px \leq \underset{\sim}{y}^p(t)\}$ (so that in particular $\underset{\sim}{y}^p(t) = p\underset{\sim}{x}^p(t)$). If we have $\int \underset{\sim}{x}^p = \int \underset{\sim}{e}$, we say that $\underset{\sim}{x}^p$ is a <u>value allocation of the wealth redistribution game derived from</u> q.

There is no reason to believe that the value allocations of an arbitrary bargaining game q coincide with the value allocations of the wealth redistribution game derived from q even if the former can be achieved as competitive allocations after wealth taxation. However, for all the bargaining games considered in Proposition 8.10 this coincidence does in fact occur.

<u>Lemma 8.13</u>: Let q be a bargaining game, and suppose that $\underset{\sim}{x}$ is a value allocation of q if and only if it is efficient with efficiency pair $(\underset{\sim}{\lambda},p)$ and a.e. $G(u_t(\underset{\sim}{x}(t)), \underset{\sim}{\lambda}(t), p\underset{\sim}{e}(t), p\underset{\sim}{x}(t)) = 0$ for some function $G: \mathbb{R}^4 \to \mathbb{R}$. Then the set of value allocations of q coincides with the set of value allocations of the wealth redistribution game derived from q.

<u>Proof</u>: First suppose that $\underset{\sim}{x}$ is a value allocation of q, with efficiency pair $(\underset{\sim}{\lambda},p)$. Then from Lemma 5.30 we have $u_t(\underset{\sim}{x}(t)) = u_t^p(p\underset{\sim}{x}(t))$, so that a.e.

$$(8.14) \qquad G(u_t^p(p\underset{\sim}{x}(t)), \underset{\sim}{\lambda}(t), p\underset{\sim}{e}(t), p\underset{\sim}{x}(t)) = 0 \quad .$$

But also from Lemma 5.30, $p\underset{\sim}{x}$ is an efficient allocation in M^p with efficiency pair $(\underset{\sim}{\lambda},1)$, so from (8.14) $p\underset{\sim}{x}$ is a value allocation of q^p. Furthermore, since p is an efficiency price for $\underset{\sim}{x}$, we know that $\underset{\sim}{x}(t)$ a.e. maximizes $u_t(x)$ over $\{x \in \Omega: px \leq px(t)\}$; since $\int\underset{\sim}{x} = \int\underset{\sim}{e}$ by definition, $\underset{\sim}{x}$ is a value allocation of the wealth redistribution game derived from q.

Now suppose that $\underset{\sim}{x}$ is a value allocation of the wealth redistribution game derived from q. Then there is a price vector p such that $\underset{\sim}{x}(t)$ a.e. maximizes $u_t(x)$ over $\{x \in \Omega: px \leq p\underset{\sim}{x}(t)\}$, and hence also a comparison function $\underset{\sim}{\lambda}$ such that $(\underset{\sim}{\lambda},p)$ is an efficiency pair for $\underset{\sim}{x}$. So (8.14) is a.e. satisfied, and so, by Lemma 5.30, $G(u_t(\underset{\sim}{x}(t)), \underset{\sim}{\lambda}(t), p\underset{\sim}{e}(t), p\underset{\sim}{x}(t)) = 0$ a.e.. Hence $\underset{\sim}{x}$ is a value allocation of q.

Corollary 8.15: Let M be a bounded market and let q be a bargaining game satisfying (8.2) and (8.11). Then the set of value allocations of q coincides with the set of value allocations of the wealth redistribution game derived from q.

Proof: The result follows immediately from Lemma 8.13 and the proof of Proposition 8.10, given that for each price vector p, M^p is bounded when M is bounded (see Lemma 5.30).

Thus if, in any bargaining game satisfying (8.2) and (8.11), the agents act as if there is only one good--namely "wealth"--and exercise their power with respect to that good, the outcome is the same as it is in the original economy.

8.4 Discussion: the Distribution of Power

From (8.5) we can see that the value of the game q, and hence its value allocations, depends only of those characteristics of the function f which are reflected in the constants a_1, a_2, and a_3. So from (8.6) we can see that the only pertinent characteristics of f are its derivatives on the "diagonal" ($\{x \in \mathbb{R}^{\ell+2}: x = \theta\nu(T)$ for some $\theta \in [0,1]\}$). This follows from the fact that the value of a game depends only of the behavior of that game in a neighborhood of the diagonal (because almost all "randomly chosen" coalitions are perfect replicas of T), and the market game r is homogeneous of degree one, so that $q(\theta\chi_T) = f(\theta r(T),\theta e(T),\theta\mu(T))$ for each $\theta \in [0,1]$. To understand what this involves, consider a very special case of a game which satisfies (8.2): consider the case where

(8.16) $q(S) = g(\mu(S))r(S)$ for each $S \in \mathcal{C}$

for some increasing continuously differentiable function $g: \mathbb{R} \to \mathbb{R}$ with $g(1) = 1$. We might think of g as reflecting the "distribution of power" in the case where "power" depends only on size. For a game of this form we have

$$a_1 = \int_0^1 g(\theta)d\theta \quad , \quad a_2 = 0 \quad , \text{ and so } \quad a_3 = (1 - \int_0^1 g(\theta)d\theta)r(T)$$

so that the value allocations of q depend only on the area under g. In particular, both the functions g^1 and g^2 shown in Diagram 4 lead to the same set of value allocations: since r is homogeneous of degree one, the "contribution" in r of any coalition is constant along the diagonal, and so the value of q, which is the average of the contributions there, will depend only on the integral over $[0,1]$ of some feature of g.

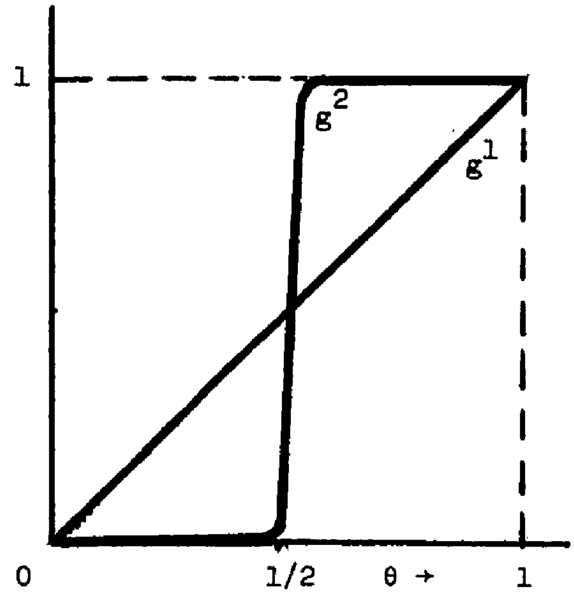


Diagram 4

As we remarked earlier the game q to which the assumptions of Aumann and Kurz [1977] leads--namely the one defined in (8.3)--does not fall into the class studied here because of its lack of differentiability. However, we can use their results to compare the value allocations they obtain with the ones which arise here. Given their formula for the value ϕq of q we can see that the value allocations they obtain are identical to those of any game of the form (8.16) for which $\int_0^1 g(\theta)d\theta = 1/2$. Both g^1 and g^2 shown in Diagram 4 satisfy this property. We might think of the game q associated with g^1 as being one in which "power" is proportional to size. Given that the model of Aumann and Kurz is of an economy in which there is majority rule, if we think that it is reasonable to assume that the underlying power associated with groups in an economy is proportional to their size, then we can view the political system of majority rule as a way of realizing the outcome implied by this power structure, rather than as an exogenous feature of the economy.

Throughout this chapter we have worked with the coalitional game q rather than the "basic" strategic form game, and have said nothing about how the latter might be constructed to generate the former. This does not prevent the games q from standing on their own as models of the economy, though there may be some interest in examining their strategic bases. Now, in constrast to the assumption of Aumann and Kurz [1977] there are <u>some</u> groups in the economy which contain much less than 50% of the population which are very powerful--for example, workers in industries which produce "essential" goods. However, it seems that their power derives from their ability to form a "union" or "syndicate" and refuse to join in coalitions with other groups in the economy--i.e. it derives from considerations which are ruled out in the Harsanyi-Shapley value analysis. In order to capture these phenomena it thus seems necessary to use a different approach; to this extent our study of the role of the distribution of power in this chapter has been somewhat limited.

Appendix 1

Here we shall show that (3.29) can be deduced from the constraints in (3.27). Our argument is merely a rearrangement of that in Harsanyi [1963]. Fix $S \subseteq T$ and let

(A1.1) $\quad \underset{\sim}{w}^S(t) = \underset{\sim}{x}^S(t) - \underset{\sim}{y}^S(t)$ for each $t \in S$.

Then by (3.28),

$$\underset{\sim}{w}^S(t) = \underset{\sim}{x}^S(t) - \sum_{\substack{R \ni t \\ R \subset S}} (-1)^{s-r+1} \underset{\sim}{x}^R(t) \quad ,$$

or

(A1.2) $\quad \underset{\sim}{w}^S(t) = \sum_{\substack{R \ni t \\ R \subseteq S}} (-1)^{s-r} \underset{\sim}{x}^R(t) \quad .$

So we have

$$\sum_{\substack{U \ni i \\ U \subseteq S}} \underset{\sim}{w}^U(t) = \sum_{\substack{U \ni i \\ U \subseteq S}} \sum_{\substack{R \ni i \\ R \subseteq U}} (-1)^{u-r} \underset{\sim}{x}^R(t)$$

$$= \sum_{\substack{Q \ni i \\ Q \subseteq S}} \sum_{k=0}^{s-q} (-1)^k \frac{(s-q)!}{k!(s-q-k)!} \underset{\sim}{x}^Q(t)$$

(collecting the coefficients of $\underset{\sim}{x}^Q(t)$ for each $Q \subseteq S$), so

(A1.3) $\quad \sum_{\substack{U \ni t \\ U \subseteq S}} \underset{\sim}{w}^U(t) = \underset{\sim}{x}^S(t) \quad ,$

since $\sum_{k=0}^{s-q}(-1)^k \frac{(s-q)!}{k!(s-q-k)!} = 0$ unless $s - q = 0$.

Now let $Y^S = \sum_{t\in S} \underset{\sim}{\lambda}(t)\underset{\sim}{w}^S(t)$ and $Z^S = \sum_{t\in S} \underset{\sim}{\lambda}(t)\underset{\sim}{x}^S(t)$. Then by (A1.2),

$$Y^S = \sum_{t\in S} \underset{\sim}{\lambda}(t) \sum_{\substack{R\ni t\\ R\subset S}} (-1)^{s-r}\, \underset{\sim}{x}^R(t)$$

$$= \sum_{R\subseteq S} \Big((-1)^{s-r} \sum_{t\in R} \underset{\sim}{\lambda}(t)\underset{\sim}{x}^R(t)\Big) \quad ,$$

or

$$\text{(A1.4)} \qquad Y^S = \sum_{R\subseteq S} (-1)^{s-r}\, Z^R \quad .$$

But by the constraints in (3.27), and (A1.1),

$$\underset{\sim}{\lambda}(t)\underset{\sim}{w}^S(t) = \underset{\sim}{\lambda}(i)\underset{\sim}{w}^S(i) \quad \text{if} \quad t, i \in S \quad ,$$

so

$$\text{(A1.5)} \qquad \underset{\sim}{\lambda}(t)\underset{\sim}{w}^S(t) = \frac{1}{s} \sum_{i\in S} \underset{\sim}{\lambda}(i)\underset{\sim}{w}^S(i) = \frac{1}{s} Y^S = \frac{1}{s} \sum_{R\subseteq S} (-1)^{s-r}\, Z^R \quad ,$$

using (A1.4). Hence if $t \in S$,

$$\underset{\sim}{\lambda}(t)\underset{\sim}{x}^S(t) = \sum_{\substack{U\ni t\\ U\subseteq S}} \underset{\sim}{\lambda}(t)\underset{\sim}{w}^U(t) \qquad \text{(using (A1.3))}$$

$$= \sum_{\substack{U\ni t\\ U\subseteq S}} \sum_{R\subseteq U} \frac{1}{u}(-1)^{u-r}\, Z^R \qquad \text{(using (A1.5))}$$

$$= \sum_{\substack{R \ni t \\ R \subseteq S}} \frac{(r-1)!(s-r)!}{s!} (Z^R - Z^{T \setminus R}) \quad .$$

$$= \sum_{\substack{R \ni t \\ R \subseteq S}} \frac{(r-1)!(s-r)!}{s!} \left(\sum_{i \in R} \underset{\sim}{\lambda}(i) \underset{\sim}{x}^R(i) - \sum_{i \in T \setminus R} \underset{\sim}{\lambda}(i) \underset{\sim}{x}^{T \setminus R}(i) \right)$$

$$= (\phi v^S_{\underset{\sim}{\lambda}})(\{t\})$$

where $v^S_{\underset{\sim}{\lambda}}$ is the game on S defined in (3.29), as was to be shown.

Appendix 2

Here we study the game v defined in (6.5); we have not included the results in the main body of this study since they are primarily of technical interest.

In Section 8.1 above we define the set of value allocations of a class of coalitional form games which are associated with a market. The game $v: \mathcal{C} \to \mathbb{R}_+$ defined by

$$v(S) = u_S(e(T)) \quad \text{for each} \quad S \in \mathcal{C}$$

(see (6.5)) does not fall into this class since in general there is no allocation $\underset{\sim}{x}$ in M such that $v(S) = \int_S u(\underset{\sim}{x})$ and $v(T \setminus S) = \int_{T \setminus S} u(\underset{\sim}{x})$, but we can still use the definition given in Section 8.1. We can also think of the game v as being derived from a coalitional form game in which utility is not transferable, where the set of utility allocations which the coalition $S \in \mathcal{C}$ can achieve is

$$\{u(\underset{\sim}{x}): \underset{\sim}{x} \text{ is an } S\text{-allocation of } e(T)\} \quad .$$

To this game we can apply the usual "λ-transfer" method of calculating the set of value allocations which was proposed by Shapley [1969]. It is easy to see that, except for the fact that Shapley allows the weighting function $\underset{\sim}{\lambda}$ to vanish on a set of positive measure, the result of doing so is the same as if we use the definition in Section 8.1. The result which we shall prove is that the set of value allocations of v consists of essentially all the efficient allocations. Shapley [1969], p. 259, gives an "equation counting" argument which suggests that generically there is at most a 0-dimensional set of value allocations. Our result is of interest because it shows that the game v is a member of the negligible collection of games for which this set is of higher dimension. The result also means that the "prediction" of the set of value allocations is that anything can happen in this game. This may not be a bad prediction in this case: in the game every coalition has access to the total endowment of society, and one might expect that this would create instability which would make every efficient allocation a candidate for the final outcome. (The fact that $v(S)$ and $v(T \setminus S)$ cannot be simultaneously achieved does not mean that the game makes no economic sense; it is certainly perfectly reasonable simply as a game.)

We shall now establish the claim we have made; we shall rely heavily on the results of Chapter 6. First we have the following.

<u>Lemma A2.1</u>: Let M be a homogeneous market and suppose $v(T)$ is finite. Then $v \in ASYMP$ and for each $S \in \mathcal{C}$

$$(\phi v)(S) = \int_S u(\underset{\sim}{x}) \quad ,$$

where $\underset{\sim}{x}$ is an allocation at which $v(T)$ is achieved.

<u>Proof</u>: By Proposition 4.18 (using the fact that $v(T) = r(T)$) and Proposition 6.11 we have $v \in pNA$. Hence by Theorem 3.14 the asymptotic value ϕv of v exists and is given by

$$(\phi v)(S) = \int_0^1 \partial v^*(\theta, S) d\theta \quad \text{for each} \quad S \in \mathcal{C} \quad .$$

But from Corollary 6.21 and Lemma 6.23 we have

$$\int_0^1 \partial v^*(\theta,S)d\theta = \partial r_S^*(\theta_o,S)\int_0^1 \theta^{-\beta}d\theta = \partial r_S^*(\theta_o,S)/(1-\beta)$$

for all $S \in \mathcal{C}$ with $\mu(S) < 1$ and any $\theta_o \in (0,1)$, where r_S is the market game defined in (6.22) and β is the degree of homogeneity of M. Also, from Proposition 4.28 we have

$$\partial r_S^*(\theta_o,S) = (\phi r_S)(S) = \int_S (u(\underset{\sim}{x}) - p\underset{\sim}{x}) \quad \text{for all} \quad S \in \mathcal{C} \quad ,$$

since $e_S(S) = 0$, where $(\underset{\sim}{x},p)$ is any t.u.c.e. of M_S (the market associated with r_S). Hence

$$(\phi v)(S) = \int_S (u(\underset{\sim}{x}) - p\underset{\sim}{x})/(1-\beta) \quad .$$

But, as in the proof of Proposition 6.28 we can use the homogeneity of each u_t to deduce that

$$\int_S (u(\underset{\sim}{x}) - p\underset{\sim}{x}) = (1-\beta)\int_S u(\underset{\sim}{x}) \quad .$$

So we can conclude that $(\phi v)(S) = \int_S u(\underset{\sim}{x})$ for any $S \in \mathcal{C}$ with $\mu(S) < 1$, where $\underset{\sim}{x}$ is an allocation at which $r_S(T) = v(T)$ is achieved. And if $\mu(S) = 1$ then $(\phi v)(S) = (\phi v)(T) = v(T) = r(T) = \int u(\underset{\sim}{x})$, where $\underset{\sim}{x}$ is an allocation at which $r(T) = v(T)$ is achieved. This completes the proof.

We can now establish the result which is of interest. (Recall that we defined the set of value allocations of a coalitional form game in Section 8.1.)

<u>Proposition A2.2</u>: Let M be a homogeneous market. Then an allocation $\underset{\sim}{x}$ is a value allocation of the game v if and only if it is efficient and there exists an efficiency pair $(\underset{\sim}{\lambda},p)$ for $\underset{\sim}{x}$ for which $\underset{\sim}{\lambda}$ is a comparison function and $v_{\underset{\sim}{\lambda}}(T)$ is finite.

<u>Proof</u>: First let $\underset{\sim}{x}$ be a value allocation of v. Then there exists a comparison function $\underset{\sim}{\lambda}$ for which $v_{\underset{\sim}{\lambda}}(T)$ is finite and (8.1) is satisfied. But then $\int \underset{\sim}{\lambda} u(\underset{\sim}{x}) = (\phi v_{\underset{\sim}{\lambda}})(T) = v_{\underset{\sim}{\lambda}}(T) = r_{\underset{\sim}{\lambda}}(T)$, so there exists an efficiency price p such that $(\underset{\sim}{\lambda},p)$ is an efficiency pair for $\underset{\sim}{x}$, completing the proof of necessity.

Now suppose that $\underset{\sim}{x}$ is efficient with efficiency pair $(\underset{\sim}{\lambda},p)$, where $\underset{\sim}{\lambda}$ is a comparison function, and $v_{\underset{\sim}{\lambda}}(T)$ is finite. Then $v_{\underset{\sim}{\lambda}}(T) = r_{\underset{\sim}{\lambda}}(T)$ is achieved at $\underset{\sim}{x}$, so by Lemma A2.1 we have

$$(\phi v_{\underset{\sim}{\lambda}})(S) = \int_S \underset{\sim}{\lambda} u(\underset{\sim}{x}) \quad \text{for each} \quad S \in \mathcal{C} \quad .$$

But this is just (8.1), so $\underset{\sim}{x}$ is a value allocation of v. This completes the proof.

Note that if $\underset{\sim}{x}$ is efficient with efficiency pair $(\underset{\sim}{\lambda},p)$ and $\underset{\sim}{x}(t) \neq 0$ a.e. then certainly $\underset{\sim}{\lambda}$ is a comparison function (see the remark after (4.13)). Only if $\underset{\sim}{x}(t) = 0$ for some set of positive measure is it possible that $\underset{\sim}{x}$ fail to satisfy the condition of Proposition A2.2.

Footnotes

1/ This observation, together with the precise formulation of the result, is due to Haruo Imai.

2/ It might seem that this is a rather roundabout way to find a finite type approximation to M_o: why not simply approximate the utility functions of M_o directly? The problem is that while we know that among the δ-approximations to u there is one of finite type (see Aumann and Shapley, Proposition 35.6), this may not be true of other sorts of approximations. But to work directly with δ-approximations we should essentially have to duplicate the lengthy and complicated arguments of Aumann and Shapley (see pp. 210ff.); instead we use an approximation procedure which allows us to use their arguments.

3/ Hildenbrand states the result under the assumption that Z is an upper hemi-continuous correspondence, rather than a correspondence with a closed graph (and the two assumptions are not (quite) equivalent in this context). However, using the fixed point theorem for a correspondence with a closed graph, it is easy to prove the result as stated here.

4/ That is, the only difference between the measure spaces (Z,C',μ') and (T,C,μ) is their name.

5/ We cannot use the more general assumption that for each fixed $y \in \mathbb{R}^n_+$, $u_t(y,\cdot)$ is homogeneous of some degree $\beta \in (0,1)$ because then we should have $u_t(y,0) = 0$ for all $y \in \mathbb{R}^n_+$, and so u_t would not be increasing.

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