Positively responsive collective choice rules and majority rule: a generalization of May's theorem to many alternatives

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This version corrects an error in the statement of May's theorem and a typo in the published paper.

Abstract

May's theorem (1952) shows that if the set of alternatives contains two members, an anonymous and neutral collective choice rule is positively responsive if and only if it is majority rule. We show that if the set of alternatives contains three or more alternatives only the rule that assigns to every problem its strict Condorcet winner satisfies the three conditions plus Nash's version of "independence of irrelevant alternatives" for the domain of problems that have strict Condorcet winners. We show also that no rule satisfies the four conditions for domains that are more than slightly larger.

Keywords: Collective choice, majority rule, May's theorem, positive responsiveness, Condorcet winner.

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1. Introduction

May's theorem (1952, p. 682) says that for collective choice problems with two alternatives, majority rule is the only anonymous, neutral, and positively responsive collective choice rule.[1] These three conditions are attractive. Anonymity says that the alternative chosen does not depend on the names of the individuals and neutrality says that it does not depend on the names of the alternatives. Positive responsiveness says that the rule responds sensibly when any individual's favorite alternative changes: if for some profile of preference relations the rule selects alternative a (possibly together with b, in a tie) and some individual switches from preferring b to preferring a, then the rule selects a alone.

What happens if there are more than two alternatives? A standard answer points to Arrow's "general possibility theorem" (1963, Theorem 2, p. 97), which says that no preference aggregation rule[2] is Pareto efficient,[3] independent of irrelevant alternatives,[4] and nondictatorial.[5] This answer is unsatisfying because Arrow's conditions are disjoint from May's, which leaves open some basic questions. When there are three or more alternatives, which collective choice rules satisfy May's conditions? For such an environment, does May's theorem have a natural generalization?

Arrow himself writes that a complete characterization of the collective choice rules satisfying May's conditions when there are three or more alternatives "does not appear to be easy to achieve" (1963, footnote 26, p. 101). We agree. Further, Arrow's theorem and the subsequent vast literature suggest that when preferences are unrestricted, no collective choice rule for problems with many alternatives satisfies conditions like May's plus a condition requiring consistency across problems with different numbers of alternatives.

[1]A collective choice rule is a function that associates with every profile of preference relations a set of alternatives.

[2]A preference aggregation rule is a function that associates with every profile of preference relations a ("social") preference relation. Preference aggregation rules are sometimes called social welfare functions.

[3]If all individuals prefer a to b, then a is socially preferred to b.

[4]If every individual's preference between a and b is the same in two profiles of preference relations, the social preference between a and b is the same for both profiles.

[5]For no individual is it the case that the social preference between any pair of alternatives is the same as the individual's preferences regardless of the other individuals' preferences.

But what if we restrict to a limited set of preference profiles? Is a natural generalization of majority rule characterized by May's conditions plus a consistency condition across problems with different numbers of alternatives?

We show that the answer is affirmative. The consistency condition we add is a set-valued version of the independence condition used by Nash (1950, condition 7 on p. 159) in the context of his bargaining model, which we call Nash independence.[6] This condition says that removing unchosen alternatives does not affect the set of alternatives selected. We show that for the domain of collective choice problems that have a strict Condorcet winner,[7] an adaptation of May's conditions plus Nash independence characterize the collective choice rule that selects the strict Condorcet winner (Theorem 1). We show also that when the preference profile is one step or more away from having a strict Condorcet winner, no collective choice rule satisfies a slight variant of these conditions if there are at least three individuals and three alternatives (Theorem 2). When preferences are strict, a similar result holds if there are either at least three individuals and four alternatives or at least four individuals and three alternatives (Theorem 3).

A strict Condorcet winner is an appealing outcome if it exists, and the conditions of anonymity, neutrality, positive responsiveness, and Nash independence also are appealing. We interpret Theorem 1 to increase the appeal of both the collective choice rule that selects the strict Condorcet winner and the four conditions. It shows that the combination of the conditions is "just right" for collective choice problems with a strict Condorcet winner: it implies the collective choice rule that selects the strict Condorcet winner. If no collective choice rule were to satisfy the conditions on this domain, the combination of conditions would be too strong, and our subsequent results that no collective choice rule satisfies the conditions on any domain that is more than slightly larger would be less significant. If collective choice rules other than the one that selects the strict Condorcet winner were to satisfy the combination of conditions on the domain of collective choice problems with a strict Condorcet winner, then the combination of conditions would be too weak.

[6]The condition is known by several other names, including the strong superset property. See Brandt and Harrenstein (2011) (who call the condition $\widehat{\alpha}$) for an analysis of the condition and an account of its previous use. It neither implies nor is implied by the related Chernoff condition (see Section 4.2).

[7]An alternative a such that for every other alternative b a strict majority of individuals prefer a to b.

The key condition in our results is an adaptation of May's positive responsiveness to an environment with many alternatives. Suppose that the alternatives a and b are both selected, in a tie, for some problem, and some individual ranks b above a. Now suppose that the individual's preferences change to rank a above b. Our condition requires that a remains one of the selected alternatives, b is no longer selected, and no alternative that was not selected originally is now selected.[8] This condition captures the spirit of May's condition: a change in the relative ranking of two alternatives by a single individual breaks a tie between the alternatives. More loosely, the condition ensures that every individual's preferences matter.

2. Model

Throughout we fix a finite set N of individuals and a finite set A of all possible alternatives, and assume that both sets contain at least two elements. In any given instance, the set of individuals has to choose an alternative from the set of available alternatives, which is a subset of A.

Definition 1 (Collective choice problem)**.** A *collective choice problem* is a pair $(X, \succcurlyeq)$, where X is a subset of A with at least two members and $\succcurlyeq$ is a profile $(\succcurlyeq_i)_{i \in N}$ of complete and transitive binary relations (*preference relations*) on A.

We require the preferences specified by a collective choice problem to rank alternatives outside the set of available alternatives because doing so allows us to write the problem derived from $(X, \succcurlyeq)$ by shrinking the set of alternatives to $Z \subset X$ simply as $(Z, \succcurlyeq)$.

For every collective choice problem we would like to identify an alternative that the individuals collectively like best. However, for some problems, we cannot select a single alternative without discriminating among individuals or alternatives (i.e., without violating the anonymity and neutrality conditions we subsequently describe).[9] For this

[8]A stronger condition is sometimes given the same name. See the discussion after Definition 8.

[9]Consider, for example, the problem with two alternatives, x and y, and two individuals, one of whom prefers x to y and the other of whom prefers y to x. To avoid discrimination, it is necessary to declare ties whenever the number of alternatives is the sum of the divisors of the number of individuals different from 1 (Moulin 1988, Exercise 9.9(b), p. 253).

reason, we allow for ties among alternatives, defining a collective choice rule to be a function that associates with every collective choice problem $(X, \succcurlyeq)$ a *subset* of X.

Definition 2 (Collective choice rule)**.** A *collective choice rule* is a function F that associates with every collective choice problem $(X, \succcurlyeq)$ a nonempty subset $F(X, \succcurlyeq)$ of X with the property that $F(X, \succcurlyeq) = F(X, \succcurlyeq')$ whenever $\succcurlyeq$ and $\succcurlyeq'$ agree on X.

While a collective choice problem requires the individuals to rank all *possible* alternatives, Definition 2 requires that the set of alternatives assigned by a collective choice rule depends only on the individuals' preferences over the *available* alternatives.[10]

For the collective choice rules we discuss, Condorcet winners play a prominent role. As usual, we use $\succ_i$ to denote strict preference: $x \succ_i y$ if and only if not $y \succcurlyeq_i x$.

Definition 3 (Condorcet winner)**.** For the collective choice problem $(X, \succcurlyeq)$, an alternative $x \in X$ is

- a *Condorcet winner* if for each alternative $y \in X \setminus \{x\}$, the number of individuals $i \in N$ for whom $x \succ_i y$ is at least the number for whom $y \succ_i x$
- a *strict Condorcet winner* if for each alternative $y \in X \setminus \{x\}$ the number of individuals $i \in N$ for whom $x \succ_i y$ exceeds the number for whom $y \succ_i x$.

A collective choice problem may have more than one Condorcet winner, but can have at most one strict Condorcet winner. Every problem with two alternatives has a Condorcet winner, but some problems with more than two alternatives do not: suppose that $N = \{1, 2, 3\}$ and consider the problem $(X, \succcurlyeq)$ for which $X = \{x, y, z\}$ and $x \succ_1 y \succ_1 z$, $y \succ_2 z \succ_2 x$, and $z \succ_3 x \succ_3 y$ (a Condorcet cycle).

We take the standard axiomatic approach to finding collective choice rules that select alternatives that reflect the individuals' preferences: we look for rules that satisfy a list of apparently desirable properties for all collective choice problems in certain sets, which we refer to as domains.

[10] In some circumstances it might be reasonable to allow the choice from X to depend on the individuals' preferences over A. For instance, individuals who strongly dislike some alternative may differ from other individuals in a way that is socially relevant. However, these circumstances raise questions about other conditions we impose (notably neutrality), so we do not pursue this approach.

Definition 4 (Domain)**.** A *domain* is a function $\mathscr{D}$ that associates with every set $X \subseteq A$ of available alternatives with two or more members a set $\mathscr{D}(X)$ of preference profiles. For any domains $\mathscr{D}$ and $\mathscr{D}'$ we denote by $\mathscr{D} \cap \mathscr{D}'$ the domain defined by $(\mathscr{D} \cap \mathscr{D}')(X) = \mathscr{D}(X) \cap \mathscr{D}'(X)$ for all $X \subseteq A$, and if $\mathscr{D}(X) \subseteq \mathscr{D}'(X)$ for all $X \subseteq A$ then we write $\mathscr{D} \subseteq \mathscr{D}'$.

The first two properties we impose on a collective choice rule stipulate that it does not discriminate among individuals or alternatives.

For any profile $\succsim$ of preference relations and any one-to-one function $\pi : N \to N$ (an N-*permutation*), let $\succsim^{\pi}$ denote the profile of preference relations such that $\succsim_i^{\pi} = \succsim_{\pi(i)}$ for each $i \in N$. Similarly, for any one-to-one function $\sigma : A \to A$ (an A-*permutation*), let $\succsim^{\sigma}$ denote the profile of preference relations such that, for each $i \in N$, $x \succsim_i y$ if and only if $\sigma(x) \succsim_i^{\sigma} \sigma(y)$.

Definition 5 (Anonymity)**.** For any domain $\mathscr{D}$, a collective choice rule F is *anonymous on* $\mathscr{D}$ if for every collective choice problem $(X, \succsim)$ with $\succsim \in \mathscr{D}(X)$ and every N-permutation $\pi : N \to N$ with $\succsim^{\pi} \in \mathscr{D}(X)$, we have $F(X, \succsim) = F(X, \succsim^{\pi})$.

Definition 6 (Neutrality)**.** For any domain $\mathscr{D}$, a collective choice rule F is *neutral on* $\mathscr{D}$ if, for every collective choice problem $(X, \succsim)$ with $\succsim \in \mathscr{D}(X)$ and every A-permutation $\sigma : A \to A$ with $\succsim^{\sigma} \in \mathscr{D}(X)$, we have $F(X, \succsim) = \sigma^{-1}(F(\sigma(X), \succsim^{\sigma}))$.

May's (1952) positive responsiveness condition is a key element in his characterization of majority rule. It applies to collective choice problems in which the two alternatives are tied according to the collective choice rule. It stipulates that the tie is broken when one alternative improves relative to the other in some individuals' rankings.

Definition 7 (Improvement)**.** For a collective choice problem $(X, \succsim)$, the preference profile $\succsim'$ is an *improvement of alternative* $x \in X$ *relative to alternative* $y \in X$ *in the preference profile* $\succsim$ if there exists a set of individuals $J \subseteq N$ such that for every individual $i \in N \setminus J$, $\succsim_i' = \succsim_i$, and for every individual $j \in J$,

- either $y \succsim_j x$ and $x \succ_j' y$, or $y \succ_j x$ and $x \succsim_j' y$
- $w \succsim_j' z$ if and only if $w \succsim_j z$ for all $w, z \in X \setminus \{x\}$.

If $|J| = 1$, then $\succsim'$ is an *improvement of* x (relative to y in $\succsim$) *for one individual.*

May's condition may be generalized to problems with more than two alternatives in many ways. Our approach requires that if an alternative x selected by a collective choice rule improves relative to some other alternative y, then (*i*) x is still selected for the new problem, (*ii*) y is not selected for the new problem, and (*iii*) no alternative is selected for the new problem unless it was selected for the original problem.[11]

Definition 8 (Positive responsiveness)**.** For any collective choice problem $(X, \succsim)$, the collective choice rule F is *positively responsive to an improvement* $\succsim'$ *of* $x \in F(X, \succsim)$ *relative to* $y \in X$ *in* $\succsim$ if (*i*) $x \in F(X, \succsim')$, (*ii*) $y \notin F(X, \succsim')$, and (*iii*) $F(X, \succsim') \subseteq F(X, \succsim)$.

For any domain $\mathscr{D}$, the collective choice rule F is *positively responsive on* $\mathscr{D}$ if, for every collective choice problem $(X, \succsim)$ such that $\succsim \in \mathscr{D}(X)$ and every alternative $x \in F(X, \succsim)$, F is positively responsive to every improvement $\succsim'$ of x for one individual such that $\succsim' \in \mathscr{D}(X)$. The rule is *fully positively responsive* if the same condition holds without the proviso "for one individual".

The difference between positive responsiveness and full positive responsiveness is slight: the only reason an improvement may not be achievable by a sequence of improvements for single individuals is that every sequence that replicates the improvement contains a problem outside the domain.

Positive responsiveness is weaker than another condition (sometimes given the same name) that requires a selected alternative to become the unique selected alternative after any improvement. An example of a collective choice rule that satisfies our condition but not the stronger condition is a variant of *Black's rule* that selects the set of Condorcet winners if this set is nonempty and otherwise selects the set of alternatives with the lowest *Borda count.*[12]

[11] In a different context, Núñez and Valletta (2015, p. 284) formulate a similar condition.

[12] The rule Black (1958, p. 66) suggests differs in that it selects the strict Condorcet winner if one exists and otherwise selects the set of alternatives with the lowest Borda count. (The Borda count of an alternative is the sum of its ranks in the individuals' preferences.) To see that the rule we define violates the stronger version of positive responsiveness, consider a problem with two individuals whose preferences are $x \succ z \succ y$ and $y \succ x \succ z$. The set of Condorcet winners for this problem is $\{x, y\}$, which does not change when the first individual's preferences change to $x \succ y \succ z$.

May (1952) shows that if the set A of all possible alternatives has two members, the only anonymous, neutral, and positively responsive collective choice rule is majority rule, which selects the alternative favored by a majority of individuals, or both alternatives in the case of a tie.

Definition 9 (Majority rule)**.** If the set A of all possible alternatives has two members, *majority rule* is the collective choice rule that associates with every collective choice problem its set of Condorcet winners.

May's result, as we state it, applies to the domain of all preference profiles, the domain of profiles for which no individual is indifferent between any alternatives (so that each individual's preference relation is a linear order), and the domain of profiles that have a strict Condorcet winner.

Definition 10 (Domains $\mathscr{U}$ and $\mathscr{L}$)**.** For the domain $\mathscr{U}$, for every set $X \subseteq A$, the set $\mathscr{U}(X)$ consists of all preference profiles over X. For the domain $\mathscr{L} \subseteq \mathscr{U}$, for each set $X \subseteq A$ the set $\mathscr{L}(X)$ consists of the preference profiles in $\mathscr{U}(X)$ for which no individual is indifferent between any two members of X.

Definition 11 (Domain $\mathscr{C}$)**.** For the domain $\mathscr{C}$, for each set $X \subseteq A$ the set $\mathscr{C}(X)$ consists of all preference profiles $\succsim$ over X for which $(X, \succsim)$ has a strict Condorcet winner.

May's Theorem. *Suppose that the set A of all possible alternatives has two members, and let $\mathscr{D}$ be a domain with the property that for any set $X \subset A$ and preference profile $\succsim \in \mathscr{D}(X)$, any profile obtained from $\succsim$ by an N-permutation or an A-permutation is in $\mathscr{D}(X)$.*

a. If $\mathscr{D} \supseteq \mathscr{L}$ then the only collective choice rule that is anonymous, neutral, and positively responsive on $\mathscr{D}$ is majority rule.

b. If $\mathscr{D} \supseteq \mathscr{C} \cap \mathscr{L}$ and the number of individuals is even, then the only collective choice rule that is anonymous, neutral, and fully positively responsive on $\mathscr{D}$ is majority rule.

May's original result (1952) is for the domain $\mathscr{U}$; Moulin (1988, Exercise 11.2, p. 313) shows that it holds also for $\mathscr{L}$. If the number of individuals is odd, $\mathscr{C} \cap \mathscr{L} = \mathscr{L}$, but if the number of individuals is even, $\mathscr{C} \cap \mathscr{L} \subset \mathscr{L}$ and an improvement for one individual may lead to a profile outside the domain;[13] in this case full positive responsiveness is needed to obtain the result.[14]

For problems with more than two alternatives, many collective choice rules are anonymous, neutral, and positively responsive. Strict scoring rules, for instance, satisfy these three conditions on the domain $\mathscr{L}$.[15] However, every strict scoring rule has an undesirable feature: for some set of individuals and some preference profile $\succcurlyeq$ for these individuals, the rule selects x for $(\{x,y,z\},\succcurlyeq)$ and y for $(\{x,y\},\succcurlyeq)$.[16] That is, the alternative selected depends on the presence of an alternative that is not selected.

To rule out this possibility, we add the condition that removing unchosen alternatives from the set of available alternatives does not affect the set of alternatives selected. This condition is an adaptation to collective choice rules of a condition that Nash (1950, condition 7 on p. 159) originally proposed for point-valued bargaining solutions.[17] We call it Nash independence.[18]

Definition 12 (Nash independence)**.** For any domain $\mathscr{D}$, the collective choice rule F is *Nash independent on* $\mathscr{D}$ if for every collective choice problem $(X,\succcurlyeq)$ with $\succcurlyeq\, \in \mathscr{D}(X)$ and every set $X' \subset X$ for which $\succcurlyeq\, \in \mathscr{D}(X')$, $F(X,\succcurlyeq) \subseteq X'$ implies $F(X',\succcurlyeq) = F(X,\succcurlyeq)$.

[13]Take a profile in which one alternative has a majority of two. If the other alternative improves in one individual's preferences, then the alternatives become tied, so that the resulting profile has no strict Condorcet winner.

[14]In this case, rules other than majority rule are anonymous, neutral, and (simply) positively responsive. An example is the rule that selects x if x has a majority of more than two, y if y has a majority of more than two, and otherwise selects $\{x,y\}$.

[15]For each $j = 1,\ldots,k$, where k is the number of members of A, let $\alpha(j)$ be a real number (called a *weight*) with $\alpha(1) \geq \cdots \geq \alpha(k)$. For each individual $i \in N$ and each alternative $x \in X$, denote by $\rho_i(x,X)$ the rank of x in the set X according to $\succcurlyeq_i$. For a collective choice problem $(X,\succcurlyeq)$, the *scoring rule* with weights $\alpha(1), \ldots, \alpha(k)$ selects the set of alternatives $x \in X$ that maximize $\sum_{i\in N} \alpha(\rho_i(x,X))$. If $\alpha(1) > \cdots > \alpha(k)$ then the rule is *strict*.

[16]Fishburn (1984) shows this result using the preference profile for seven individuals in which $y \succ x \succ z$ for three individuals, $x \succ z \succ y$ for two individuals, $x \succ y \succ z$ for one individual, and $z \succ y \succ x$ for the remaining individual.

[17]While Nash did not use the name, the condition is called "independence of irrelevant alternatives" in the bargaining literature. It differs from the axiom with the same name used by Arrow (1963, p. 27).

[18]Chernoff (1954, p. 430) calls it Postulate 5*; some authors refer to it as "Aizerman", after its appearance in Aizerman and Malishevski (1981, p. 1033) (who call it "Independence of rejecting the outcast variants"). (Another adaptation of Nash's condition to collective choice rules is a different condition proposed by Chernoff, which we discuss in Section 4.2.)

If there are two individuals and any number of alternatives, the collective choice rule that selects the set of Condorcet winners (or, equivalently, the set of Pareto efficient alternatives) satisfies all four of our conditions. If the number of individuals is at least three, collective choice rules that satisfy Nash independence and any two of our other conditions (anonymity, neutrality, and positive responsiveness) are easy to find.[19] We show (Theorem 1) that for any number of individuals a collective choice rule is anonymous, neutral, fully positively responsive, and Nash independent for the domain of collective choice problems that have a strict Condorcet winner and in which no individual is indifferent between any alternatives if and only if the rule selects the strict Condorcet winner. We show also (Theorems 2 and 3) that no rule is anonymous, neutral, positively responsive, and Nash independent on most larger domains.

3. Results

3.1 Domains with strict Condorcet winners

Theorem 1. *A collective choice rule F is anonymous, neutral, fully positively responsive, and Nash independent on the domain $\mathscr{C} \cap \mathscr{L}$ (strict Condorcet winner, strict preferences) if and only if, for every $X \subseteq A$ and preference profile $\succcurlyeq \in \mathscr{C} \cap \mathscr{L}(X)$, $F(X, \succcurlyeq)$ contains only the strict Condorcet winner of $(X, \succcurlyeq)$.*

Proof. Any collective choice rule F that selects the strict Condorcet winner (whenever it exists) satisfies the four properties on $\mathscr{C} \cap \mathscr{L}$.

Now let $X \subseteq A$ and $\succcurlyeq \in \mathscr{C} \cap \mathscr{L}(X)$, and let $c \in X$ be the strict Condorcet winner for the collective choice problem $(X, \succcurlyeq)$. Let F be a collective choice rule that is anonymous, neutral, fully positively responsive, and Nash independent on $\mathscr{C} \cap \mathscr{L}$ and suppose, contrary to our claim, that $F(X, \succcurlyeq)$ contains an alternative different from c.

[19] Anonymity and neutrality: The rule that assigns the set X of all available alternatives to the problem $(X, \succcurlyeq)$. Anonymity and positive responsiveness: For any irreflexive and transitive (i.e., quasi-transitive) binary relation $\triangleright$ on A, the rule that selects the set of undominated elements according to $\triangleright$. Neutrality and positive responsiveness: The serial dictatorship rule that selects the favorite alternatives of individual 1, breaking ties according to the preferences of individual 2, further breaking any ties according to the preferences of individual 3, and so on.

Step 1. *$F(X,\succcurlyeq)$ contains at least two alternatives different from c.*

Proof. If $F(X,\succcurlyeq)=\{x,c\}$ or $F(X,\succcurlyeq)=\{x\}$ for some $x \in X\setminus\{c\}$ then by Nash independence we have $F(\{x,c\},\succcurlyeq)=F(X,\succcurlyeq)$. Since c is the strict Condorcet winner of $(\{x,c\},\succcurlyeq)$ however, $F(\{x,c\},\succcurlyeq)=\{c\}$ by May's theorem, a contradiction. $\lhd$

Step 2. *Let $x_1,\dots,x_k$ be the distinct alternatives in $F(X,\succcurlyeq)\setminus\{c\}$. Then for each individual i there exist alternatives x_j and x_l in $\{x_1,\dots,x_k\}$ such that $x_j \succ_i c \succ_i x_l$.*

Proof. By Step 1, $k\geq 2$. Suppose, contrary to the claim, that for some individual i either $c\succ_i x_l$ for all $l\in\{1,\dots,k\}$ or $x_l \succ_i c$ for all $l\in\{1,\dots,k\}$. Assume, without loss of generality, that $x_l \succ_i x_k$ for all $l\in\{1,\dots,k-1\}$.

Now consider the improvement $\succcurlyeq'$ of $\succcurlyeq$ obtained by raising x_k above every other chosen alternative except c in individual i's preference (so that $x_k \succ'_i x_j$ for $j=1,\dots,k-1$ but $c\succ'_i x_k$ if $c\succ_i x_l$ for all $l\in\{1,\dots,k\}$). Since the ranking of c relative to every other alternative is the same in $\succcurlyeq$ and $\succcurlyeq'$, c is the strict Condorcet winner for $(X,\succcurlyeq')$. Then, $x_j\notin F(X,\succcurlyeq')$ for $j=1,\dots,k-1$ by positive responsiveness. So either $F(X,\succcurlyeq')=\{x_k\}$ or $F(X,\succcurlyeq')=\{x_k,c\}$, which contradicts Step 1 (applied to $\succcurlyeq'$). $\lhd$

Step 3. *$F(X,\succcurlyeq)$ contains at least three alternatives different from c.*

Proof. By Step 1, $F(X,\succcurlyeq)$ contains at least two alternatives different from c. If it contains exactly two such alternatives, say x and y, then by Step 2 either $x\succ_i c\succ_i y$ or $y\succ_i c\succ_i x$ for every individual i. As a result, c is not the strict Condorcet winner for $(X,\succcurlyeq)$, contradicting our assumption. $\lhd$

Now let $x_1,\dots,x_k$ be the alternatives in $F(X,\succcurlyeq)\setminus\{c\}$ and denote by $x\in\{x_1,\dots,x_k\}$ the alternative that individual 1 ranks lowest among the alternatives in $\{x_1,\dots,x_k\}$ that she ranks above c. (By Step 2, such an alternative exists.) Consider the improvement $\succcurlyeq'$ of $\succcurlyeq$ obtained by raising x to the top of individual 1's preference. Since the ranking of c relative to every other alternative remains the same in $\succcurlyeq$ and $\succcurlyeq'$, c is the strict Condorcet winner for $(X,\succcurlyeq')$. So, by positive responsiveness, $x\in F(X,\succcurlyeq')$ and the only other alternatives that can belong to $F(X,\succcurlyeq')$ are c and the alternatives in $\{x_1,\dots,x_k\}$ that individual 1 ranks lower than c.

Denote the alternatives in $\{x_1,\ldots,x_k\}$ that belong to $F(X,\succcurlyeq')$ by $\{y_1,\ldots,y_r\}$. (By Step 3, $r \geq 2$.) Denote the alternative in $F(X,\succcurlyeq')$ that individual 1 ranks lowest by y. Consider the improvement $\succcurlyeq''$ of $\succcurlyeq'$ obtained by raising y in individual 1's preference above every alternative ranked lower than c. Then (as with the last improvement) c remains the strict Condorcet winner for $(X,\succcurlyeq'')$. So, by positive responsiveness, $y \in F(X,\succcurlyeq'')$ and the only other alternatives that can belong to $F(X,\succcurlyeq'')$ are x and c, contradicting Step 3. □

If the number of individuals is odd, then the result is true also when full positive responsiveness is replaced by positive responsiveness, because this condition is used only in Step 1 of the proof, when May's theorem is invoked.

3.2 Larger domains

We now argue that no collective choice rule is anonymous, neutral, positively responsive, and Nash independent on any domain that is more than slightly larger than the strict Condorcet domain, $\mathscr{C} \cap \mathscr{L}$. We show also that this conclusion does not depend on individuals' preferences being strict. We first define the domain of problems that have Condorcet winners, but not necessarily strict ones.

Definition 13 (Domain $\mathscr{W}$)**.** For the domain $\mathscr{W} \supseteq \mathscr{C}$, for every set $X \subseteq A$ the set $\mathscr{W}(X)$ consists of all preference profiles for which $(X,\succcurlyeq)$ has a Condorcet winner (but not necessarily a strict Condorcet winner).

We now define a domain consisting of all problems that are close to problems with a strict Condorcet winner. For any preference profile in the domain, improving one alternative in one individual's preferences generates a profile with a strict Condorcet winner.

Definition 14 (Domain $\mathscr{C}^+$)**.** For the domain $\mathscr{C}^+$, for every set $X \subseteq A$, the set $\mathscr{C}^+(X)$ consists of all preference profiles $\succcurlyeq$ over X for which either the collective choice problem $(X,\succcurlyeq)$ has a strict Condorcet winner or there is an improvement $\succcurlyeq'$ for one individual of the preference profile $\succcurlyeq$ such that $(X,\succcurlyeq')$ has a strict Condorcet winner.

The domain $\mathscr{C}^+$ includes profiles outside $\mathscr{W}$ (for example, a Condorcet cycle), and $\mathscr{W}$ includes profiles outside $\mathscr{C}^+$ (for example, Fishburn 1977, (E8) on p. 481). Our results are for the domain $\mathscr{C}^+ \cap \mathscr{W}$, which includes profiles outside the strict Condorcet domain $\mathscr{C}$. When the individuals' preferences are strict, the distinction is more nuanced: $\mathscr{C}^+ \cap \mathscr{W} \cap \mathscr{L}$ differs from $\mathscr{C} \cap \mathscr{L}$ only when the number of individuals is even. (When the individuals' preferences are strict and the number of individuals is odd, a Condorcet winner is a strict Condorcet winner.)

The following result is a key step in our arguments: no rule that satisfies our four conditions assigns a singleton to any collective choice problem that lacks a strict Condorcet winner.

Lemma 1. *Let $\mathscr{D}$ be any domain with $\mathscr{D} \supseteq \mathscr{C}^+ \cap \mathscr{W} \cap \mathscr{L}$, and suppose that $(X, \succsim)$ is a collective choice problem without a strict Condorcet winner. If the collective choice rule F is anonymous, neutral, positively responsive, and Nash independent on $\mathscr{D}$ and $\succsim \in \mathscr{D}(X)$, then $F(X, \succsim)$ contains more than one alternative.*

Proof. Suppose $F(X, \succsim) = \{x\}$ for some alternative $x \in X$. Since x is not the strict Condorcet winner, there is an alternative $y \in X \setminus \{x\}$ such that the number of individuals i for whom $y \succ_i x$ is at least the number for whom $x \succ_i y$. So y is a Condorcet winner of $(\{x, y\}, \succsim)$. Now, $\mathscr{C}^+(\{x, y\}) = \mathscr{W}(\{x, y\})$ $(= \mathscr{U}(\{x, y\}))$ and $\mathscr{W} \cap \mathscr{L}(\{x, y\}) = \mathscr{L}(\{x, y\})$, so by (a) of May's theorem, $y \in F(\{x, y\}, \succsim)$. Since $\succsim \in \mathscr{D}(\{x, y\})$, Nash independence implies $F(\{x, y\}, \succsim) = F(X, \succsim) = \{x\}$, a contradiction. □

3.2.1 Unrestricted individual preferences

When there are at least three individuals and three alternatives, no collective choice rule satisfies our four conditions on any domain that contains all preference profiles that have Condorcet winners that are either strict or close to strict.

Theorem 2. *If both the set A of all possible alternatives and the set N of individuals have at least three members, then no collective choice rule is anonymous, neutral, positively responsive, and Nash independent on any domain $\mathscr{D}$ with $\mathscr{D} \supseteq \mathscr{C}^+ \cap \mathscr{W}$ (Condorcet winner that is strict or close to strict).*

Proof. Let F be a collective choice rule that is anonymous, neutral, positively responsive, and Nash independent on $\mathscr{C}^+ \cap \mathscr{W}$. Let n denote the number of individuals (members of N). Choose a subset X of A with three members, x, y, and z. Let $\succsim = (\succsim_1, \ldots, \succsim_n)$ be the preference profile in which the preferences of individuals 1, 2, and 3 over X are

$$x \succ_1 y \succ_1 z$$
$$y \succ_2 z \sim_2 x$$
$$z \succ_3 x \succ_3 y$$

(notice that individual 2 is indifferent between x and z), and every other individual $i \in \{4, \ldots, n\}$ is indifferent among the three alternatives:

$$x \sim_i y \sim_i z.$$

The problem $(X, \succsim)$ lacks a strict Condorcet winner: only individual 1 prefers x to z, only individual 3 prefers z to x and only individual 2 prefers y to x. However, x is a Condorcet winner. Also, x becomes the strict Condorcet winner by any improvement of x in individual 2 or 3's preference. So $\succsim \in \mathscr{D}(X)$ and similarly $\succsim \in \mathscr{D}(\{x, y\})$.

We first argue that $z \notin F(X, \succsim)$. Let $(X, \succsim')$ be an improvement of $\succsim$ for individual 2 where z is made indifferent to the top-ranked alternative y:

$$z \sim_2' y \succ_2' x.$$

Since z is a Condorcet winner for $(X, \succsim')$ and becomes the strict Condorcet winner when it improves in individual 1 or 2's preferences, $\succsim' \in \mathscr{D}(X)$; similarly $\succsim' \in \mathscr{D}(\{y, z\})$. So if $z \in F(X, \succsim)$ then $F(X, \succsim') = \{z\}$ by positive responsiveness and hence $F(\{y, z\}, \succsim') = \{z\}$ by Nash independence. Since y is a Condorcet winner for $(\{y, z\}, \succsim')$, this conclusion contradicts May's theorem.

Since $z \notin F(X, \succsim)$ and $(X, \succsim)$ lacks a strict Condorcet winner, $F(X, \succsim) = \{x, y\}$ by Lemma 1. Let $\succsim^*$ be the improvement of $\succsim$ for individual 1 where y is made indifferent to the top-ranked alternative x:

$$y \sim_1^* x \succ_1^* z.$$

Since y is a Condorcet winner of $(X, \succsim^*)$ and becomes the strict Condorcet winner when it improves in individual 3's preference, $\succsim^* \in \mathscr{D}(X)$; similarly $\succsim^* \in \mathscr{D}(\{x, y\})$. So $F(X, \succsim^*) = \{y\}$ by positive responsiveness and hence $F(\{x, y\}, \succsim^*) = \{y\}$ by Nash independence. Since x is a Condorcet winner for $(\{x, y\}, \succsim^*)$, this conclusion contradicts May's theorem. □

3.2.2 Strict individual preferences

A result similar to Theorem 2 holds when individuals' preferences are strict. If there are at least three individuals and four alternatives, or four individuals and three alternatives, then no collective choice rule satisfies our four conditions on $\mathscr{C}^+ \cap \mathscr{L}$. When the number of individuals is even, the same is true on $\mathscr{C}^+ \cap \mathscr{W} \cap \mathscr{L}$.

Theorem 3. *If both the set A of all possible alternatives and the set N of individuals have at least three members and at least one of these two sets has at least four members, then no collective choice rule is anonymous, neutral, positively responsive, and Nash independent on any domain $\mathscr{D}$ with $\mathscr{D} \supseteq \mathscr{C}^+ \cap \mathscr{L}$ (strict or close to strict Condorcet winner, strict preferences). If N has an even number of members, then this statement is true also on any domain $\mathscr{D}$ with $\mathscr{D} \supseteq \mathscr{C}^+ \cap \mathscr{W} \cap \mathscr{L}$ (weak Condorcet winner that is strict or close to strict, strict preferences).*

Proof. Denote by n the number of individuals (members of N). In the rest of the proof, parenthetical statements are reserved for the case of n even. Let F be a collective choice rule that is anonymous, neutral, positively responsive, and Nash independent on $\mathscr{D}$.

First suppose that $n \geq 4$. Choose a subset X of A with three members, x, y, and z. Define the preference relations $\succsim^a$, $\succsim^b$, $\succsim^c$, and $\succsim^d$ over X by

$$x \succ^a y \succ^a z$$
$$y \succ^b z \succ^b x$$
$$z \succ^c x \succ^c y$$
$$x \succ^d z \succ^d y.$$

Denote by $\succsim = (\succsim_1, \ldots, \succsim_n)$ the preference profile consisting of $(\succsim^a, \succsim^b, \succsim^c, \succsim^a)$ followed by the first $n-4$ preference relations in the alternating sequence $(\succsim^b, \succsim^d, \succsim^b, \succsim^d, \ldots)$. Expressing $n-4$ as $2s+t$ for integers $s \geq 0$ and $t \in \{0,1\}$, the profile $\succsim$ has 2 copies of $\succsim^a$, $s+t+1$ copies of $\succsim^b$, 1 copy of $\succsim^c$, and s copies of $\succsim^d$. Then $(X, \succsim)$ has

- $2-t$ more individuals who prefer x to y than individuals who prefer y to x
- $2+t$ more individuals who prefer y to z than individuals who prefer z to y
- t more individuals who prefer z to x than individuals who prefer x to z.

Thus $(X, \succsim)$ has no strict Condorcet winner. But any improvement of x in individual 2's or 3's preference makes x into the strict Condorcet winner. (If n is even, then $t = 0$ and x is a Condorcet winner of $(X, \succsim)$.) So $\succsim \in \mathscr{D}(X)$; x is a strict Condorcet winner of $(\{x, y\}, \succsim)$, so $\succsim \in \mathscr{D}(\{x, y\})$.

We first argue that $z \notin F(X, \succsim)$. Suppose to the contrary that $z \in F(X, \succsim)$. Let $\succsim^*$ be the improvement of $\succsim$ where z becomes the top-ranked alternative for individual 1:

$$z \succ_1^* x \succ_1^* y.$$

Since t more individuals in $\succsim^*$ prefer y to z than prefer z to y, any improvement of z in individual 2's preferences makes z into the strict Condorcet winner. (Also, if $t = 0$, then z is a Condorcet winner for $(X, \succsim^*)$.) So $\succsim^* \in \mathscr{D}(X)$ and similarly $\succsim^* \in \mathscr{D}(\{y, z\})$. Then $F(X, \succsim^*) = \{z\}$ by positive responsiveness and $F(\{y, z\}, \succsim^*) = \{z\}$ by Nash independence. Since y is a Condorcet winner for $(\{y, z\}, \succsim^*)$, this conclusion contradicts May's theorem.

Since $z \notin F(X, \succsim)$ and $(X, \succsim)$ lacks a strict Condorcet winner, $F(X, \succsim) = \{x, y\}$ by Lemma 1. So $F(\{x, y\}, \succsim) = \{x, y\}$ by Nash independence. Since x is the strict Condorcet winner for $(\{x, y\}, \succsim)$, this conclusion contradicts May's theorem, completing the proof[20] for $n \geq 4$.

Now suppose that $n = 3$ and A has at least four members. Choose a subset X of A with four members, w, x, y, and z. Define the preference profile $\succsim = (\succsim_1, \succsim_2, \succsim_3)$ over X

[20]This proof is closely related to the proof of Theorem 2. The profile $\succsim$ approximates the profile used in the proof of Theorem 2, in which individual 2 is indifferent between x and z, by adding a copy of individual 1.

by

$$z \succ_1 w \succ_1 x \succ_1 y$$
$$x \succ_2 y \succ_2 z \succ_2 w$$
$$y \succ_3 w \succ_3 x \succ_3 z.$$

Any improvement of x in individual 1's or 3's preferences makes x the strict Condorcet winner. So $\succcurlyeq \in \mathscr{D}(X)$; w is the strict Condorcet winner of $(\{w,x\},\succcurlyeq)$, so $\succcurlyeq \in \mathscr{D}(\{w,x\})$.

First we argue that $z \notin F(X,\succcurlyeq)$. Suppose to the contrary that $z \in F(X,\succcurlyeq)$. Let $\succcurlyeq^*$ be the improvement of $\succcurlyeq$ in which z becomes the second-ranked alternative for individual 3:

$$y \succ_3^* z \succ_3^* w \succ_3^* x.$$

Since y becomes the strict Condorcet winner when it improves in individual 2's preferences, $\succcurlyeq^* \in \mathscr{D}(X)$; y is the strict Condorcet winner of $(\{y,z\},\succcurlyeq)$, so $\succcurlyeq^* \in \mathscr{D}(\{w,x\})$. We have $z \in F(X,\succcurlyeq^*)$ by positive responsiveness and $z \in F(\{y,z\},\succcurlyeq^*)$ by Nash independence. Since y is the strict Condorcet winner for $(\{y,z\},\succcurlyeq^*)$, this conclusion contradicts May's theorem. Thus $F(X,\succcurlyeq) \subseteq \{w,x,y\}$.

Next we argue that $y \in F(X,\succcurlyeq)$. Suppose to the contrary that $y \notin F(X,\succcurlyeq)$. Then $F(X,\succcurlyeq) = \{w,x\}$ by Lemma 1 and $F(\{w,x\},\succcurlyeq) = \{w,x\}$ by Nash independence. Since w is the strict Condorcet winner for $(\{w,x\},\succcurlyeq)$, this conclusion contradicts May's theorem. Since the same argument applies to the alternatives w and x, $F(X,\succcurlyeq) = \{w,x,y\}$.

Finally let $\succcurlyeq'$ be the improvement of $\succcurlyeq$ where w is top-ranked for individual 3:

$$w \succ_3' y \succ_3' x \succ_3' z.$$

The problem $(X,\succcurlyeq')$ lacks a strict Condorcet winner, but w becomes the strict Condorcet winner when it improves in individual 1's or 2's preferences. So $\succcurlyeq' \in \mathscr{D}(X)$; w is the strict Condorcet winner of $(\{w,x\},\succcurlyeq')$, so $\succcurlyeq' \in \mathscr{D}(\{w,x\})$. Since $F(X,\succcurlyeq) = \{w,x,y\}$, positive responsiveness and Lemma 1 imply $F(X,\succcurlyeq') = \{w,x\}$. So $F(\{w,x\},\succcurlyeq') = \{w,x\}$ by Nash independence. Since w is the strict Condorcet winner for $(\{w,x\},\succcurlyeq')$, this conclusion contradicts May's theorem. $\square$

For a society with three individuals and three alternatives, a collective choice rule that satisfies our four conditions on $\mathscr{C}^+ \cap \mathscr{L}$ (which in this case coincides with $\mathscr{L}$) *does* exist: the rule that assigns the top cycle[21] to each problem. This rule is anonymous, neutral, and Nash independent on any domain.[22] By the following argument, it is also positively responsive on the specified domain. If the top cycle contains a single alternative then this alternative is the strict Condorcet winner, and remains the strict Condorcet winner when it improves. Otherwise, the top cycle contains all three alternatives. Though none of these alternatives is a Condorcet winner, each becomes the strict Condorcet winner when it improves.

Theorems 2 and 3 show that no collective choice rule satisfies our four conditions on domains more than slightly larger than the domain $\mathscr{C} \cap \mathscr{L}$ of problems with strict Condorcet winners and strict preferences. For some domains even closer to $\mathscr{C} \cap \mathscr{L}$, rules satisfying our conditions do exist. Consider the domain $\mathscr{C}^\varepsilon$ where, for each set $X \subseteq A$, $\mathscr{C}^\varepsilon(X)$ consists of $\mathscr{C}(X)$ plus *one* additional preference profile $\succcurlyeq$. The rule F^ε that assigns the strict Condorcet winner to any problem that has such a winner and otherwise assigns the set X of all available alternatives is clearly anonymous, neutral, and Nash independent on any domain. It is also positively responsive on $\mathscr{C}^\varepsilon$ by the following argument. If x is a strict Condorcet winner, then it remains such a winner when it improves. If it is not a strict Condorcet winner, then $F^\varepsilon(X, \succcurlyeq) = X$, and improving x leads to another profile in $\mathscr{C}^\varepsilon$ only when it makes x the strict Condorcet winner, and hence the unique alternative chosen by F^ε. (Improving x cannot make any *other* alternative the strict Condorcet winner.)

3.2.3 A generalization

Our negative results for minor extensions of the strict Condorcet domain persist under weaker conditions than those we have specified. In fact, for Theorem 2, the same proof goes through when we replace anonymity and neutrality with the requirement that the

[21]The smallest (nonempty) set with respect to set inclusion such that for every x in the set and every y outside the set, x is the strict Condorcet winner for the problem $(\{x, y\}, \succcurlyeq)$.

[22]Nash independence follows from the fact that the rule maximizes the transitive closure of a binary relation (the majority relation).

collective choice rule coincides with majority rule for sets of two alternatives and we weaken positive responsiveness and Nash independence to the following conditions.[23]

Definition 15 (Majoritarianism)**.** For any domain $\mathscr{D}$, the collective choice rule F is *majoritarian* on $\mathscr{D}$ if, for every collective choice problem $(\{x,y\},\succcurlyeq)$, $F(\{x,y\},\succcurlyeq)$ is given by majority rule (i.e. is the set of Condorcet winners of $(\{x,y\},\succcurlyeq)$).

Definition 16 (Weak positive responsiveness)**.** For any domain $\mathscr{D}$, the collective choice rule F is *weakly positively responsive on* $\mathscr{D}$ if it is positively responsive to all improvements that raise a chosen alternative to the top of one individual's preferences.

Definition 17 (Weak Nash independence[24])**.** For any domain $\mathscr{D}$, the collective choice rule F is *weakly Nash independent on* $\mathscr{D}$ if for every collective choice problem $(X,\succcurlyeq)$ with $\succcurlyeq\,\in\mathscr{D}(X)$ and every set $X'\subset X$ for which $\succcurlyeq\,\in\mathscr{D}(X')$, $F(X,\succcurlyeq)\subseteq X'$ implies $F(X',\succcurlyeq)\subseteq F(X,\succcurlyeq)$.

Provided that there are at least four alternatives *and* four individuals, Theorem 3 also holds for this set of conditions. Details are contained in the supplementary file.

4. Alternative approaches

Our results show that the existence of collective choice rules that satisfy our four conditions (or natural weakenings of these conditions) on domains larger than the strict Condorcet domain $\mathscr{C}$ is limited. To extend May's theorem to a wider range of domains, a more radical departure from our conditions is necessary. In this section, we discuss potential alternatives to positive responsiveness and Nash independence.

4.1 Monotonicity

One possibility is to weaken positive responsiveness to monotonicity by dropping condition (*ii*) in Definition 8, requiring only that when a chosen alternative x improves, it re-

[23] In particular, Lemma 1 (which we use in the proof of Theorem 2) holds under these conditions. Theorem 2 continues to hold if weak Nash independence is further weakened to the requirement that if $\{x,y\}\subset X$ and $F(X,\succcurlyeq)=\{x\}$ then $F(\{x,y\},\succcurlyeq)\neq\{x,y\}$, a condition Richelson (1978, p. 344) calls δ^*.

[24] The condition is called ε^+ by Bordes (1983, p. 125), Aizerman by Moulin (1985, p. 154), and δ^* by Deb (2011, p. 340) (although it differs from the condition given that name by Richelson 1978).

mains chosen and no alternative that was not previously chosen becomes chosen. This condition is weak. For any number of alternatives, a rule that is anonymous, neutral, monotonic, and Nash independent on any domain assigns to each problem $(X, \succsim)$ the set of alternatives that at least one individual prefers to every other alternative in X (the set of *top alternatives*).[25] Another rule that satisfies these conditions on any domain assigns to each problem its top cycle set (see footnote 21). In addition, the rule that assigns to each problem $(X, \succsim)$ the set X of all available alternatives satisfies the conditions.[26]

One option for narrowing down the set of rules that satisfy the conditions is to look for rules that are most selective with respect to set inclusion. For two alternatives, the most selective rule that is anonymous, neutral, and monotonic is majority rule.[27] This characterization provides an appealing alternative to May's theorem (and, accordingly, we are surprised not to have found a reference to it in prior work). But for more alternatives, selectivity does not isolate a single rule. (For example, for some problems the set of top alternatives and the top cycle set are disjoint.)

Another obvious way to narrow down the set of rules is to impose additional conditions. One possible condition is Pareto efficiency, which is satisfied in its weak form by the set of top alternatives but not by the top cycle set. Another possible condition requires the set of alternatives specified by the rule to depend only on the majority relation. This condition is satisfied by the top cycle set, but, when there are at least four alternatives and three individuals, is incompatible with every rule that is monotonic, Nash independent, Pareto efficient, and majoritarian on[28] $\mathscr{C}^+$.

4.2 The Chernoff condition

The Chernoff condition,[29] like Nash independence, is an adaptation to collective choice problems of a condition used by Nash in his analysis of bargaining problems. It re-

[25]Nash independence follows from the fact that the set of top alternatives is the union of sets, each of which maximizes a preference (of some individual), and hence is *path independent* (see Moulin 1985).

[26]The set of all rules that satisfy these conditions when there are two alternatives is characterized by Moulin (1988, Exercise 11.2 (b) and (c), p. 313).

[27]Theorem 5 in the supplementary file.

[28]Theorem 6 in the supplementary file.

[29] Due to Chernoff (1954, Postulate 4, p. 429); called property α by Sen (1969, p. 384).

quires that an alternative y selected from X is selected also from any subset of X that contains y.

Definition 18 (Chernoff condition)**.** For any domain $\mathscr{D}$, the collective choice rule F *satisfies the Chernoff condition on* $\mathscr{D}$ if for every collective choice problem $(X, \succsim)$ with $\succsim \in \mathscr{D}(X)$ and every set $X' \subset X$ for which $\succsim \in \mathscr{D}(X')$, $F(X, \succsim) \cap X' \subseteq F(X', \succsim)$.

Unlike Nash independence, which treats the selected set as a unit, the Chernoff condition treats each selected alternative individually. Given our interest in set-valued collective choice rules, Nash independence is a more appealing condition: if the *set* of alternatives selected from X is contained in $Y \subset X$, then it should be selected from Y; the fact that a *member* of the set of alternatives selected from X is a member of Y is not convincing evidence that this alternative should be a member of the set selected from Y. (The conditions are independent: Nash independence does not imply the Chernoff condition[30] and the Chernoff condition does not imply Nash independence.[31])

Nevertheless, the effect on our results of replacing Nash independence with the Chernoff condition is of interest. Theorem 1 continues to hold,[32] an analog of Theorem 2 holds for any domain $\mathscr{D}$ that contains a problem without a Condorcet winner, and

[30]Nash independence does not imply the Chernoff condition because the latter never prevents an alternative that is *not* selected in X from being selected in $X' \subset X$. For example, the rule that assigns the top cycle (footnote 21) to each problem is Nash independent but does not satisfy the Chernoff condition. To see this, take a Condorcet cycle with three alternatives. The top cycle consists of all three alternatives, but when one alternative is removed, the top cycle selects only one of the remaining two alternatives (the majority winner). (Lemma 7 of Moulin 1986 shows that the violation of the Chernoff condition is not specific to the top cycle set.)

[31]The Chernoff condition does not imply Nash independence because the latter requires an alternative selected from X to be selected from $X' \subset X$ only when *no* alternative in $X \setminus X'$ is selected from X. For example, let k be the number of possible alternatives (members of A), let n be the number of individuals (members of N), and let K be the smallest integer greater than k/n. The rule that selects every alternative that appears among the top K alternatives of at least two individuals satisfies the Chernoff condition, but is not Nash independent. (For any values of k and n, at least one alternative is selected by this rule.) The rule satisfies the Chernoff condition because an alternative remains chosen when another alternative is removed. (Note that K does not change when alternatives are removed: it depends on the number of possible alternatives, not the number of available alternatives.) But the rule is not Nash independent. If, for example, $k = 3$, $n = 2$, individual 1 prefers a to b to c, and individual 2 prefers a to c to b, then $K = 2$ and $\{a\}$ is selected from $\{a, b, c\}$ whereas $\{a, b\}$ is selected from $\{a, b\}$.

[32]Nash independence is used only in Step 1 where it can be replaced by the Chernoff condition.

an analog of Theorem 3 holds for any corresponding domain[33] $\mathscr{D} \cap \mathscr{L}$. However, the rule that assigns the set of Condorcet winners is anonymous, neutral, positively responsive, and satisfies the Chernoff condition on $\mathscr{W}$.

5. Related work

Two main lines of work are related to our characterization of the collective choice rule that selects the strict Condorcet winner (Theorem 1).[34]

Much like us, Alemante et al. (2016) extend May's result to three or more alternatives. A key difference is that they require the choice rule to be single-valued.[35] In our view, this restriction is not appealing since it may force the rule to discriminate among individuals and alternatives (when the choice problem lacks a strict Condorcet winner).

Dasgupta and Maskin (2008) (who extend earlier work by Maskin 1995) take a different approach to highlight the appeal of "simple majority rule". They show that this rule satisfies a set of appealing conditions on the largest possible domain. Two significant differences between our conditions and theirs are that they require a collective choice rule to be almost always single-valued and they impose the Chernoff condition rather than Nash independence.[36] As we have discussed, we have concerns about the appeal

[33] Let y be an alternative assigned by the collective choice rule to a problem $(X, \succcurlyeq)$ without a Condorcet winner. Then for some alternative x, a majority of individuals prefer some x to y, and the Chernoff condition requires that y is chosen for the problem $(\{x, y\}, \succcurlyeq)$, which violates May's theorem. The comment of Bordes (1979, p. 178) that "Since it is known that [... the Chernoff condition] leads to impossibility results of the Arrow kind" suggests that he may have been aware of this result, although we cannot find a formal statement of it. (Note that our Theorems 2 and 3 do not follow from it.)

[34] Less closely related to our work is Goodin and List (2006), who restrict attention to rules that depend only on the individuals' top alternatives. With this restriction, they show that only "plurality rule" (which selects the alternatives that are most preferred by the largest number) satisfies natural adaptations of May's conditions. In our view, a significant limitation of their approach is that information about favorite alternatives is not always rich enough to capture "collective preference". Indeed, this information may not be sufficient to identify the strict Condorcet winner of a choice problem that has one.

[35] Campbell and Kelly (2003, 2015, 2016) also consider single-valued rules, but the axioms they use to characterize the rule that selects the strict Condorcet winner are unrelated to May's conditions.

[36] Dasgupta and Maskin call this condition "independence of irrelevant alternatives" (though it differs from Arrow's condition of the same name). We discuss the differences between the Chernoff condition and Nash independence in Section 4.2.

of single-valuedness and we have argued that Nash independence is more compelling than the Chernoff condition for collective choice rules.

Our nonexistence results (Theorems 2 and 3) are part of the huge literature initiated by Arrow (1951). One of our key conditions, positive responsiveness, is a variant of a condition called "positive association of social and individual values" that Arrow (1963, pp. 25–26) discusses but does not require for his classic result about aggregating individual preferences into a transitive collective preference (Theorem 2, p. 97).

The first authors to impose positive responsiveness were Mas-Colell and Sonnenschein (1972). Their Theorem 2 (p. 186) shows that if the aggregation rule satisfies this condition for sets with two alternatives, Arrow's result holds under the weaker requirement that the collective preference is quasi-transitive (i.e., its strict component is transitive). Since a quasi-transitive preference induces a Nash independent choice rule, their result has implications in our setting. However, it is difficult to compare with our results. On the one hand, Arrow's conditions are more permissive than May's for sets with two alternatives. On the other, quasi-transitive preferences impose rationality requirements beyond Nash independence, like the Chernoff condition and Sen's property γ.[37]

Blair et al. (1976) shift the focus from preference aggregation to collective choice. Broadly, they establish incompatibility between certain "normative binary" conditions (for choice from sets of two alternatives) and certain "rationality" conditions (linking choices from larger sets to binary sets). In terms of binary conditions, their Theorem 6 imposes Arrow's conditions[38] and the strong positive responsiveness condition that we discuss after Definition 8, showing that these requirements are incompatible with the rationality imposed by the Chernoff condition.

Much of the subsequent literature maintains some version of the Chernoff condition (see e.g., Deb 2011). One notable exception is Duggan (2019), who imposes weak Nash independence and Sen's property γ. His Theorem 12 shows that these rationality conditions are incompatible with normative binary conditions that are even weaker than those imposed by Blair et al. (with the main difference being a less demanding re-

[37]Property γ requires that an alternative chosen from the collective choice problems $(X, \succeq)$ and $(Y, \succeq)$ must be chosen from $(X \cup Y, \succeq)$ (Sen 1971, p. 314).

[38]They impose a strong version of Arrow's non-dictatorship that also rules out "weak" dictatorship.

sponsiveness condition). Like Mas-Collel and Sonnenschein's result, Duggan's result is difficult to compare with ours: he imposes stronger rationality restrictions on choice from larger sets[39] but much weaker restrictions on choices from binary sets.[40]

The results that appear to be closest to ours relate to majoritarian collective choice rules. For rules within this class, Theorem 5 of Ferejohn and Grether (1977) shows that Pareto efficiency is incompatible with a condition (called weak path independence) that strengthens weak Nash independence. In turn, Theorem 2 of Richelson (1978) relies on a strong neutrality assumption to establish the same kind of incompatibility with a condition (which he calls δ^* (see footnote 23)) that is weaker than weak Nash independence. The key difference is that our results do not impose an efficiency requirement, showing instead that Richelson's condition δ^* is *directly* incompatible with natural extensions of May's original conditions.[41]

These negative results motivated Bordes to identify majoritarian rules that *do* satisfy some appealing properties. In a sequence of papers (1976, 1979, 1983) culminating in joint work with Banks (1988), he shows that versions of the top cycle, uncovered set, and Banks set satisfy weak Nash independence and a weak version of monotonicity (which imposes only the first condition of positive responsiveness) as well as a range of other appealing (but relatively weak) properties, including weak Pareto efficiency and a weak version of Sen's property γ. This work, like ours, considers majoritarian rules; it differs in imposing conditions other than positive responsiveness.

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[39]As we have discussed, Nash independence can be weakened to weak Nash independence in our results.

[40]A similar observation applies to recent work by Brandt, Brill, and Harrenstein, who show that, on the domain of strict preferences $\mathscr{L}$, Pareto efficiency and non-dictatorship are incompatible with the strong version of positive responsiveness discussed after Definition 8. (We are grateful to the authors for making available to us a draft of their paper.)

[41]As we have discussed (see footnote 23), Nash independence can be weakened to δ^* in our Theorem 2.

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Supplement for

Positively responsive collective choice rules and majority rule: a generalization of May's theorem to many alternatives

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1. Generalization of (Theorem 2 and) Theorem 3

The proof of Lemma 1 uses anonymity (Definition 5), neutrality (Definition 6), and positively responsiveness (Definition 8) only through May's theorem, so that these conditions can be replaced by majoritarianism (Definition 15). Also, since the proof uses Nash independence (Definition 12) only when the collective choice is a singleton, it continues to hold with weak Nash independence (Definition 17).

Like Lemma 1, the proof of Theorem 2 uses anonymity and neutrality only through May's theorem and uses Nash independence only when the collective choice is a singleton. The proof uses positive responsiveness directly, but does so only for improvements that move an alternative to the top of one individual's preferences. For such improvements, weak positive responsiveness (Definition 16) is sufficient.

The next result generalizes Theorem 3.

Theorem 4. *If both the set A of all possible alternatives and the set N of individuals have at least four members, then no collective choice rule is majoritarian, weakly positively responsive, and weakly Nash independent on any domain $\mathscr{D}$ with $\mathscr{D} \supseteq \mathscr{C}^+ \cap \mathscr{L}$. If N has an even number of members, then this statement is true also on any domain $\mathscr{D}$ with $\mathscr{D} \supseteq \mathscr{C}^+ \cap \mathscr{W} \cap \mathscr{L}$ (even when A has three members).*

Proof. Let k denote the number of alternatives in A and n the number of individuals in N. For n even, the argument given in the proof of Theorem 3 for $n \geq 4$ and $k \geq 3$ goes through with weak positive responsiveness in place of positive responsiveness and weak Nash independence in place of Nash independence until the sentence "Since $z \notin$

$F(X,\succsim)$ and $(X,\succsim)$ lacks a strict Condorcet winner, $F(X,\succsim)=\{x,y\}$ by Lemma 1." After this sentence, it should be modified as follows.

Let $\succsim^*$ be the improvement of $\succsim$ where y becomes top-ranked for individual 1:

$$y \succ_1^* x \succ_1^* z.$$

Since $t = 0$, x and y are Condorcet winners for $(X,\succsim^*)$, and any improvement in individual 3's preference makes y into the strict Condorcet winner. So, $\succsim^* \in \mathscr{D}(X)$ and similarly $\succsim^* \in \mathscr{D}(\{x,y\})$. Then, $F(X,\succsim^*) = \{y\}$ by weak positive responsiveness and $F(\{x,y\},\succsim^*) = \{y\}$ by weak Nash independence. Since x is a Condorcet winner for $(\{x,y\},\succsim^*)$, this contradicts majoritarianism.

A different argument is required to establish the result for n odd. For $n \geq 5$ and $k \geq 4$, define the preference relations $\succsim^a$ to $\succsim^e$ on $X=\{x,y,z,w\}$ by

$$\begin{aligned}
&x \succ^a y \succ^a z \succ^a w\\
&y \succ^b z \succ^b w \succ^b x\\
&z \succ^c w \succ^c x \succ^c y\\
&w \succ^d x \succ^d y \succ^d z\\
&w \succ^e z \succ^e y \succ^e x.
\end{aligned}$$

Denote by $\succsim = (\succsim_1,\dots,\succsim_n)$ the preference profile consisting of $(\succsim^a,\succsim^b,\succsim^c,\succsim^d,\succsim^a)$ followed by the first $n-5$ preference relations in the sequence $(\succsim^e,\succsim^a,\succsim^e,\succsim^a,\dots)$. Expressing $n-5$ as $2s$ for an integer $s \geq 0$, the profile $\succsim^*$ has $s+2$ copies of $\succsim^a$, 1 copy each of $\succsim^b$, $\succsim^c$, and $\succsim^d$, and s copies of $\succsim^e$. In $(X,\succsim)$, it follows that there are majorities of

- 3 votes for: x over y, y over z and z over w
- 1 vote for: x over z, y over w and w over x.

These observations imply that any improvement of x in individual 4's preferences makes x into the strict Condorcet winner. So $\succsim \in \mathscr{D}(X)$; x is the strict Condorcet winner of $(\{x,y\},\succsim)$, so $\succsim \in \mathscr{D}(\{x,y\})$.

First we argue that $y \notin F(X, \succsim)$. Suppose to the contrary that $y \in F(X, \succsim)$. Let $\succsim^*$ be the improvement of $\succsim$ in which y becomes top-ranked for individual 3:

$$y \succ_3^* z \succ_3^* w \succ_3^* x.$$

Since any improvement in individual 1's or 4's preferences makes y into the strict Condorcet winner, $\succsim^* \in \mathscr{D}(X)$; x is the strict Condorcet winner of $(\{x, y\}, \succsim^*)$, so $\succsim^* \in \mathscr{D}(\{x, y\})$. So $F(X, \succsim^*) = \{y\}$ by weak positive responsiveness and $F(\{x, y\}, \succsim^*) = \{y\}$ by weak Nash independence. Since x is the strict Condorcet winner for $(\{x, y\}, \succsim^*)$, this conclusion contradicts majoritarianism. Similar arguments rule out $z \in F(X, \succsim)$ and $w \in F(X, \succsim)$. So, $F(X, \succsim) = \{x\}$. Since $(X, \succsim)$ lacks a strict Condorcet winner, this conclusion violates Lemma 1. $\square$

2. Monotonicity

Monotonicity is defined precisely as follows.

Definition 19 (Monotonicity)**.** For any domain $\mathscr{D}$, the collective choice rule F is *monotonic on $\mathscr{D}$* if, for every collective choice problem $(X, \succsim)$ such that $\succsim \in \mathscr{D}(X)$ and every alternative $x \in F(X, \succsim)$, all improvements $\succsim'$ of x such that $\succsim' \in \mathscr{D}(X)$ satisfy $x \in F(X, \succsim')$ and $F(X, \succsim') \subseteq F(X, \succsim)$.

This criterion is very weak. To narrow down the set of rules, we can restrict to rules that are most selective (with respect to set inclusion). For any domain $\mathscr{D}$, a collective choice rule F is *more selective on $\mathscr{D}$* than another rule F' if

$$F(X, \succsim) \subseteq F'(X, \succsim)$$

for every collective choice problem $(X, \succsim)$ such that $\succsim \in \mathscr{D}(X)$. Within a set of collective choice rules $\mathfrak{F}$, $F \in \mathfrak{F}$ is *most selective on $\mathscr{D}$* if no $F' \in \mathfrak{F} \setminus \{F\}$ is more selective on $\mathscr{D}$.

The following characterization of majority rule shares the same simplicity as May's theorem.

Theorem 5. *Suppose that the set A of all possible alternatives has two members, and let $\mathscr{D}$ be a domain. If $\mathscr{D} \supseteq \mathscr{L}$ then the most selective collective choice rule that is anonymous, neutral, and monotonic on $\mathscr{D}$ is majority rule.*

The following two rules are anonymous, neutral, monotonic, and Nash independent for any number of alternatives, on any domain.[42]

Top alternatives (TA) is the rule that, for every problem $(X, \succeq)$, assigns the set of alternatives $x \in X$ that at least one individual $i \in N$ prefers to every other alternative in X (i.e., $x \succeq_i y$ for all $y \in X$).

Top cycle (TC) is the most selective rule F that, for every problem $(X, \succeq)$, satisfies *Condorcet transitivity*: if $x \in F(X, \succeq)$ and $y \notin F(X, \succeq)$, then alternative x is the strict Condorcet winner for the problem $(\{x, y\}, \succeq)$.

When there are two alternatives, selectivity provides a sound basis to distinguish between the two rules: the latter (which coincides with majority rule) is more selective than the former (which coincides with unanimity rule). With more alternatives, this is no longer true, even on the domain $\mathscr{C}^+$. With three alternatives (and four individuals), for instance, there are problems $(X, \succeq)$ where $TA(X, \succeq)$ is a strict subset of $TC(X, \succeq)$, and, with as few as four alternatives (and three individuals), problems where $TA(X, \succeq)$ and $TC(X, \succeq)$ are disjoint. The second observation illustrates the more general phenomenon that Nash independence is not closed under intersection: even if rules F and G satisfy Nash independence, their set-wise intersection $F \cap G$ need not.[43] In practice, this means that one cannot use selectivity to identify a unique majoritarian rule that satisfies Nash independence (plus additional conditions).

This discussion highlights a related challenge for using Theorem 5 to extend majority rule. Since there is no common refinement of the top alternatives and the top cycle rules on $\mathscr{C}^+$ (when there are at least four alternatives and three individuals), one must

[42]Nash independence is the only condition that is not immediate. Since it is the union of sets, each of which maximizes a preference (of some individual), the top alternatives rule is *path independent* (see Moulin 1985) and thus Nash independent. For the top cycle rule, Nash independence follows from the fact that the rule maximizes the transitive closure of a binary relation (the majority relation).

[43]Formally, $[F \cap G](X, \succeq) = F(X, \succeq) \cap G(X, \succeq)$ for every collective choice problem $(X, \succeq)$.

ultimately prioritize the features of one rule over the other. We see this as problematic because each rule captures a different normatively appealing feature of majority rule. On the one hand, the top alternatives rule is weakly Pareto efficient.

Definition 20 (Pareto efficiency)**.** For any domain $\mathscr{D}$, the collective choice rule F is *Pareto efficient on* $\mathscr{D}$ if, for every collective choice problem $(X, \succsim)$ such that $\succsim \in \mathscr{D}(X)$ and every alternative $x \in F(X, \succsim)$, no alternative $y \in X$ satisfies

- $y \succsim_i x$ for every individual $i \in N$ and
- $y \succ_j x$ for at least one individual $j \in N$

and is *weakly Pareto efficient on* $\mathscr{D}$ if no alternative $y \in X$ satisfies

- $y \succ_i x$ for every individual $i \in N$.

On the other hand, the top cycle rule is informationally parsimonious (see Fishburn 1977) in the sense that it relies only on the structure of the majority relation.

Definition 21 (Relationality[44])**.** Two collective choice problems $(X, \succsim)$ and $(X, \succsim')$ are *majority equivalent* if, for all $x, y \in X$, x is a Condorcet winner for $(\{x, y\}, \succsim)$ if and only if it is a Condorcet winner for $(\{x, y\}, \succsim')$. For any domain $\mathscr{D}$, the collective choice rule F is *relational on* $\mathscr{D}$ if it is neutral on $\mathscr{D}$ and $F(X, \succsim) = F(X, \succsim')$ for all majority equivalent collective choice problems $(X, \succsim)$ and $(X, \succsim')$ such that $\succsim, \succsim' \in \mathscr{D}(X)$.

More generally, when there are at least four alternatives and three individuals, Pareto efficiency and relationality are incompatible for *every* majoritarian rule that is monotonic and Nash independent on $\mathscr{C}^+$.

Theorem 6. *If the set A of all possible alternatives has at least four members and the set N of individuals has at least three members, then a collective choice rule that is majoritarian, monotonic, Nash independent, and relational on any domain $\mathscr{D}$ with $\mathscr{D} \supseteq \mathscr{C}^+$ is not Pareto efficient.*

[44] In Bordes (1983), this condition is called structural majoritarian independence. In Fishburn (1977), collective choice rules that satisfy this condition and select the strict Condorcet winner whenever it exists are called C1 choice functions.

Proof. Let F be a collective choice rule that is majoritarian, monotonic, Nash independent, and relational on $\mathscr{C}^+$. Let n denote the number of individuals (members of N). Choose a subset X of A with four members x, y, z, and w. Let $\succcurlyeq = (\succcurlyeq_1, \ldots, \succcurlyeq_n)$ be the preference profile in which individuals 1, 2, and 3 have preferences over X given by

$$x \succ_1 y \succ_1 z \succ_1 w$$
$$y \succ_2 z \succ_2 w \succ_2 x$$
$$z \succ_3 w \succ_3 x \succ_3 y$$

and every other individual $i \in \{4, \ldots, n\}$ exhibits complete indifference:

$$x \sim_i y \sim_i z \sim_i w.$$

Since x becomes the strict Condorcet winner when it improves to the top position in individual 2's or 3's preferences, $\succcurlyeq \in \mathscr{C}^+(X)$. Also, notice that z weakly Pareto dominates w in $(X, \succcurlyeq)$ (and, in fact, z strictly Pareto dominates w when $n = 3$).

First note that Lemma 1 holds for a majoritarian and Nash independent collective choice rule (in the proof, replace "May's theorem" with "majoritarianism").

To show that F violates Pareto efficiency, suppose that $w \notin F(X, \succcurlyeq)$. By Lemma 1, $F(X, \succcurlyeq)$ contains at least two alternatives. If $F(X, \succcurlyeq) = \{x, y\}$, then $F(\{x, y\}, \succcurlyeq) = \{x, y\}$ by Nash independence, contradicting May's theorem since x is the strict Condorcet winner for $(\{x, y\}, \succcurlyeq)$. Symmetric arguments hold for $\{x, z\}$ and $\{y, z\}$. So, $F(X, \succcurlyeq) = \{x, y, z\}$.

Let $\succcurlyeq^*$ be the improvement of $\succcurlyeq$ in which w becomes top-ranked for individual 3:

$$w \succ_3^* z \succ_3^* x \succ_3^* y.$$

Since x (still) becomes the strict Condorcet winner when it is raised to the top in individual 2's or 3's preferences, $\succcurlyeq^* \in \mathscr{C}^+(X)$. Since z is (still) preferred by a majority over w, $(X, \succcurlyeq^*)$ is majority equivalent to $(X, \succcurlyeq)$ and so $F(X, \succcurlyeq^*) = \{x, y, z\}$ by relationality.

Let $\succcurlyeq^{**}$ be the improvement of $\succcurlyeq^*$ in which x becomes second-ranked for individual 3:

$$w \succ_3^{**} x \succ_3^{**} z \succ_3^{**} y.$$

Since x (still) becomes the strict Condorcet winner when it is raised to the top in individual 2's or 3's preference, $\succsim^{**} \in \mathscr{C}^+(X)$. So $F(X, \succsim^{**}) \subseteq \{x, y, z\}$ by monotonicity. However, we obtain a problem that is majority equivalent to $(X, \succsim^*)$ by permuting X according to the rotation $(x\,y\,z\,w)$. So $F(X, \succsim^{**}) = \{x, y, w\}$ by majoritarianism and relationality. This conclusion contradicts $F(X, \succsim^{**}) \subseteq \{x, y, z\}$. $\square$

This result can be strengthened and partially extended to the setting with strict preferences. For an odd number of individuals, no rule with the features specified in Theorem 6 is weakly Pareto efficient and relational on the domain $\mathscr{C} \cap \mathscr{L}$. To see this, let $\succsim' = (\succsim_1, \ldots, \succsim_n)$ be the preference profile consisting of the preferences $(\succsim_1, \succsim_2, \succsim_3)$ from Theorem 6 followed by the first $n-3$ preference relations in the alternating sequence $(\succsim^a, \succsim^b, \succsim^a, \succsim^b, \ldots)$ where

$$x \succ^a y \succ^a z \succ^a w$$
$$z \succ^b w \succ^b y \succ^b x.$$

Then the argument in the proof of Theorem 6 applies equally with $(X, \succsim')$ in place of $(X, \succsim)$. The key difference is that z strictly Pareto dominates w in $(X, \succsim')$, which allows us to strengthen the statement of the result to weak Pareto efficiency.

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