ENTRY-DETERRING POLICY DIFFERENTIATION BY ELECTORAL CANDIDATES

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Abstract

This paper studies the equilibria of a one-dimensional spatial model in which three candidates seek to maximize their probabilities of winning, are uncertain about the voters' preferences, and may move whenever they wish. In the presence of enough uncertainty there is an equilibrium in which two candidates enter simultaneously at distinct positions in the first period and either the third candidate does not enter or enters between the first two in the second period.

1. Introduction

In most elections the candidates' positions are dispersed. One explanation for the dispersion appeals to the existence of potential entrants: the candidates distance themselves from each other in order to deter the entry of additional candidates, or at least to minimize the impact of such entry.[1] This explanation is not supported by the one-dimensional three-candidate (simultaneous-move) Hotelling–Downs spatial model under the assumption that each candidate is perfectly informed about the voters' preferences and has preferences that satisfy

$$\text{win} \ \succ \ \text{tie for first place} \ \succ \ \text{stay out} \ \succ \ \text{lose}.$$

If, in this model, two candidates adopt distinct positions on each side of the median voter's favorite position then a single entrant at any position loses, no matter how close the candidates' positions. Hence either candidate can move closer to the median without inducing entry, and thereby win. But if both candidates adopt the median voter's favorite position then an entrant at a point sufficiently close to the median wins. Thus the incentive for entry does not increase smoothly as the candidates' positions converge, but rather rises discontinuously from zero, when the candidates' positions are distinct, to one, when the positions are the same. The model fails to capture the idea that entry is deterred by a dispersal in the candidates' positions not because the incentives upon which the idea hinges are absent, but because they appear

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[1]Two other explanations build on the ideas that the candidates care about the winning position, not simply about winning (Wittman (1983), Roemer (1994)), and that citizens are uncertain about the candidates' positions (Hug (1995)). These explanations are others are surveyed in Osborne (1995, Section III).

with too much vigor: an arbitrarily small amount of dispersion is enough to discourage entry.

I show that if two ingredients are added to the model then its equilibria capture the idea. The ingredients are uncertainty by the candidates about the voters' preferences, and the possibility that the candidates can move asynchronously. The addition of one of these ingredients without the other in a three-candidate model is not sufficient to generate pure strategy equilibria in which two candidates enter, at different positions. If the three candidates move simultaneously then there is no pure strategy equilibrium, whether or not the candidates are uncertain about the distribution of voters' preferences. If the candidates are certain about the distribution of voters' preferences and move in a fixed order, then there is a unique pure subgame perfect equilibrium outcome, in which the first candidate to move enters at the median voter's favorite position, the second candidate stays out, and the third candidate enters at the median.[2] If the candidates are certain about the distribution of voters' preferences and each candidate may move whenever she wishes, then in every subgame perfect equilibrium only one candidate enters, and for every distribution of favorite positions there is an equilibrium in which the position of the single entrant is the median (Osborne (1993, Proposition 6)).

I study a model in which there are three players, each of whom is uncertain about the distribution of the voters' favorite positions and wishes to maximize her probability of winning. For each player, the minimal acceptable probability

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[2]Weber's (1992) conclusion is different because he assumes that each player's objective is vote maximization, an objective inconsistent with a preference for winning over losing (see the discussion of Palfrey's model below).

of winning is p_0: each player prefers to stay out of the competition than to enter and win with probability less than p_0. (The expected payoff to running may exceed the expected cost only if the probability of winning exceeds p_0.) Each player may choose a position whenever she wishes: time is discrete, and in each period any player who has not yet chosen a position may do so; once chosen, a position is immutable.

I show that if there is sufficient uncertainty, the game has an equilibrium (essentially a subgame perfect equilibrium) in which two players enter at distinct positions simultaneously in the first period and the third player either stays out, or, if there is enough uncertainty, enters in the second period at a position between those of the other candidates. This equilibrium supports the intuition that candidates may differentiate their positions in order to deter the entry of an additional candidate, or at least to minimize the impact of such entry.

The following logic lies behind the equilibrium. If there is relatively little uncertainty, then each of a pair of candidates has an incentive to choose a position close to the other, in order to obtain the support of a large number of moderate voters. But a candidate, say i, who moves too close to the center invites the entry of the third player, who, by taking a position slightly more extreme than i's, can win with sufficiently high probability to justify her entry, reducing the probability with which candidate i wins. Thus each of the two entrants is motivated to maintain some distance from the other. As the degree of uncertainty increases, the extent of the separation necessary to prevent such entry increases, until entry either on the wings or in the center becomes

inevitable. At this point, the first two entrants choose positions far enough apart that the best position for the third candidate is in the middle (where she does least damage to each of the other candidates).

The model I study is closely related to that of Palfrey (1984). The most important respect in which the models differ is in the candidates' preferences. Palfrey assumes that each candidate prefers the profile x of vote totals to the profile y if and only if she obtains more votes in x than in y. When there are three or more candidates this assumption is inconsistent with the candidates' preferring to win than to lose: in some circumstances, a candidate who wins outright may increase the number of votes she receives by moving her position closer to that of a neighbor, but in doing so may also sufficiently increase the number of votes received by another neighbor that she causes herself to lose rather than to win. I take a basic feature of electoral competition to be that candidates prefer to win than to lose, and thus argue that the criterion of vote maximization is inappropriate for electoral candidates. I do not claim that winning is all that matters—the margin of victory, for example, may be significant—but I do suggest that a candidate should prefer any outcome in which she wins to any in which she loses.

A less significant respect in which Palfrey's model differs from mine is in the temporal structure of choices: Palfrey assumes that two candidates choose positions simultaneously, then a third candidate has the opportunity to do so, while I allow each candidate to move whenever she wishes. The remaining difference between the models is that I allow the candidates to be uncertain

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about the voters' preferences.[3]

Palfrey shows that his game has a subgame perfect equilibrium in which the first two candidates choose distinct positions, and the third candidate subsequently enters between them. As I have argued, such an equilibrium owes its existence to the unappealing assumption that candidates are vote-maximizers. However, my results show that if each candidate aims to maximize the probability of winning, then, in the presence of uncertainty by candidates about the voters' preferences, Palfrey's intuition is strongly rehabilitated: not only is there an equilibrium in which the pattern of choices is the same as it is in the equilibrium of his model, but also the timing of actions that he posits arises endogenously.

My results illustrate two broader points. First, two-candidate models can be misleading. My results support the intuition that the presence of more than two potential candidates (as in almost every election) significantly affects the behavior of any players that choose to run. Second, models in which all the parties are certain of the parameters can be misleading. In most elections both candidates and voters are significantly uncertain about each others' preferences. In some prominent elections (e.g. US presidential races), candidates' uncertainty about voters is tempered by extensive polling. But in many elections the residual uncertainty is significant. My results suggest that the presence of uncertainty can dramatically affect the outcome: under perfect information, all the equilibria of the model I study entail the entry of a single candidate (Osborne (1993, Proposition 6)), while my main result here shows

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[3]Palfrey (1984, Remark 2 on pp. 154–155) briefly discusses the role for candidate uncertainty in his model, but does not explore its consequences.

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that when candidates are uncertain about the voters' preferences, the equilibria have an entirely different flavor.

2. The model

2.1   Extensive game

The model is an extensive game that differs from the one studied in Osborne (1993, Section 4) only in its payoffs, which are derived from a model in which the candidates are imperfectly, rather than perfectly, informed of the voters' preferences. The players are three potential candidates, each of whom chooses whether to enter an electoral competition, and if so at what position. Each player may enter where she wants, when she wants. Once a player has chosen a position, she has no further choices.

Precisely, the space of positions is the real line, $\mathbb{R}$. An election is to be held at a fixed date. Before this date, there is an infinite sequence of times, starting with "period 1", at each of which each player who has not yet chosen a position either chooses a position (a point in $\mathbb{R}$) or waits (W). In each period the players' decisions are made simultaneously.

There is a continuum of voters, each with single-peaked symmetric preferences over positions. I assume (like Palfrey (1984)) that voting is sincere. Specifically, each voter endorses the candidate whose position is closest to her favorite position; two or three candidates who share the same position share the votes associated with that position.

The potential candidates regard the outcome of a pure strategy profile as uncertain because they are uncertain about the location of the distribution of

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the voters' favorite positions. Let F be a nonatomic probability distribution function on $\mathbb{R}$ whose support is a finite interval; denote its median by m^*. For each real number α, let F_α be the probability distribution function on $\mathbb{R}$ defined by $F_\alpha(x) = F(x - (\alpha - m^*))$. The median of F_α is α; the distributions F_α share the same shape, but differ in their locations. (Note that $F_{m^*} = F$.) Each potential candidate knows that the shape of the distribution of the citizens' favorite positions is F, but is uncertain about its location, given by the value of α. They share the belief that the probability distribution function of α is G, which is nonatomic and has median m^* and support equal to a finite interval. (Thus, m^* is the median of the median distribution F_{m^*}.)

Denote by $\pi_i(x_1, x_2, x_3)$ the probability that player i wins when all three players enter and the position of each player j is x_j. If x_1, x_2, and x_3 are all different, for example, we have

$$\pi_i(x_1, x_2, x_3) = \mu_G(\{\alpha \colon v_i(\alpha|x_1, x_2, x_3) > v_j(\alpha|x_1, x_2, x_3) \text{ for all } j \neq i\})$$

where $v_k(\alpha|x_1, x_2, x_3)$ is the fraction of the votes received by candidate k when the distribution of favorite positions is F_α and the candidates' positions are x_1, x_2, and x_3, and μ_G is the measure associated with G.

Each potential candidate is indifferent between entering and staying out of the competition if and only if her probability of winning is p_0, where $p_0 \in (0, \frac{1}{3})$. For any $p > p_0$ each candidate prefers to win with probability p than to win with any probability less than p, and prefers to stay out than to win with probability less than p_0. (Such preferences are consistent with there being a benefit to winning and a cost to running. Under perfect information, when a candidate's probability of winning is either 0 (lose), $\frac{1}{3}$ (tie with two other

candidates), $\frac{1}{2}$ (tie with one other candidate), or 1 (win), they imply that each candidate prefers to win than to tie for first place than to stay out than to lose, as I assume in Proposition 6 of my 1993 paper, mentioned in the introduction.)

2.2 Equilibrium

The notion of equilibrium I use is a slight variant of pure strategy subgame perfect equilibrium. A strategy profile in an extensive game with perfect information and simultaneous moves (Osborne and Rubinstein (1994, Section 6.3.2)) is a subgame perfect equilibrium if no player can increase her payoff by *unilaterally* changing her strategy at any history after which it is her turn to move. Thus the behavior the players' strategies specifies after histories in which two or more players have simultaneously deviated has no bearing on whether a strategy profile is a subgame perfect equilibrium, as long as this behavior constitutes an equilibrium in the resulting subgame. Precisely, let σ be a subgame perfect equilibrium, and let σ' be another strategy profile. Suppose that every history h at which σ and σ' specify different actions for some player has a subhistory at which σ and σ' specify different actions *for at least two players*. Suppose also that the strategy profile induced by σ' after every such history h is a subgame perfect equilibrium of the subgame following h. Then σ' is a subgame perfect equilibrium with the same outcome as σ.

Thus in a game in which every subgame has a subgame perfect equilibrium, we can check that a strategy profile is a subgame perfect equilibrium without knowing the players' complete strategies: we need to know only the action every player takes after every history in which, in every period, at most *one*

player has deviated from her strategy.

Precisely, define a *substrategy* σ_i of player i to be a function that assigns an action of player i to every member of a *subset* of the set of histories at which it is player i's turn to move (i.e., in the game being studied, player i has not already chosen a position). Denote the set of histories after which σ_i specifies an action by $H(\sigma_i)$ and define an *equilibrium** to be a profile σ of substrategies for which (1) for every player i, $H(\sigma_i)$ includes all histories that result when at most one player deviates from σ in any given period, and (2) after any such history, no player can increase her payoff by changing her strategy, given that the other players continue to adhere to σ.

Condition (1) requires that the substrategies contain enough information to determine whether condition (2), the optimality condition for strategies in a subgame perfect equilibrium, is satisfied.

To illustrate the notion of an equilibrium*, consider the two-fold play of a two-player strategic game in which each player has two actions, A and B. If the strategic game has two Nash equilibria, (A, A) and (B, B), then the pair of substrategies of the repeated game in which each player chooses A in the first period and A in the second period after any history in which at least one player chose A in the first period is an equilibrium*; there is no need to specify the players' actions when, contrary to both of their strategies, they both choose B in the first period. This equilibrium* corresponds to two subgame perfect equilibria—one in which each player's strategy calls for her to choose A after the history (B, B) and one in which each player's strategy calls for her to choose B after the history (B, B).

In the game I study here, the difference between a strategy profile and a substrategy profile is greater than it is in this example. For instance, while a strategy for player 1 must specify an action in period 2 for *every* first-period action profile (W, s_2, s_3), where s_2 and s_3 are members of $\mathbb{R} \cup \{W\}$, a substrategy profile in which the strategies of players 1 and 2 call for them to enter in period 1, say at x_1 and x_2 respectively, and player 3's strategy calls for her to wait in period 1, requires player 1's substrategy to specify an action for period 2 only after a single history, namely (W, x_2, W). (After any history in which player 1 chooses a position she has no further choice; any history in which player 1 chooses W and either player 2 takes an action different from x_2 or player 3 takes an action different from W involves deviations by two or more players.)

The notions of equilibrium* and subgame perfect equilibrium are very closely related: a subgame perfect equilibrium is an equilibrium*, and in a game in which every subgame has a subgame perfect equilibrium, any equilibrium* is a substrategy profile of a subgame perfect equilibrium. In particular, the notions can differ only in a game in which some subgame does not possess a subgame perfect equilibrium. In such a game, there may be no subgame perfect equilibrium corresponding to some equilibrium* σ because a subgame reached when two or more players simultaneously deviate from σ fails to possess a subgame perfect equilibrium.

An equilibrium* models a steady state in which no player can increase her payoff by deviating, knowing how the other players will behave both if she adheres to her strategy and if she deviates from it. (A player may accumulate

such knowledge by observing the other players' behavior when she adheres to her strategy and when she occasionally deviates from it.) The requirement imposed by the notion of subgame perfect equilibrium that an equilibrium exist in every subgame reached after two or more players simultaneously deviate is not relevant to the question of whether a pattern of behavior is a steady state; as a model of such a steady state, the notion of equilibrium* fits better than that of subgame perfect equilibrium.

I use the notion of equilibrium* in my 1993 paper to study the version of the game studied here in which candidates know the voters' preferences. When the candidates are uncertain of these preferences, an equilibrium* appears not to exist; to get insights from the model we need to relax the equilibrium conditions. The problem is that a player's payoff may increase as she moves her position closer to another player's position, but then fall precipitously when the locations coincide; she wants to be close to the other player, but not at the same point. The relaxation I make in the equilibrium conditions is to consider "ϵ-equilibrium", allowing responses that are *almost* optimal.[4]

Precisely, an **equilibrium** is a sequence $\{\sigma^n\}$ of substrategy profiles for which

- for each n, the profile σ^n satisfies condition (1) of an equilibrium*

- the induced sequence of profiles of positions converges

- for every $\epsilon > 0$ there exists an integer N such that whenever $n > N$

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[4]An alternative way around the difficulty is to take limits of subgame perfect equilibria for increasingly fine discretizations of the action space. The results of this approach appear to be the same as the ones I obtain. Palfrey's (1984) "limit equilibrium" is closely related to the notion I use.

the payoff of every player i under σ^n is within ϵ of the supremum of her payoffs over all her substrategies, given σ^n_{-i}.

For example, a subgame in which two candidates are located symmetrically around the median m^* at $m^* - \delta$ and $m^* + \delta$, for $\delta > 0$ sufficiently small, has an equilibrium in which the third candidate enters at positions approaching $m^* - \delta$ from below. If F is symmetric about its median then the limit of the third candidate's winning probabilities in this equilibrium is $G(m^* - \delta/2)$, the probability that the median of the distribution of favorite positions is at most $m^* - \delta/2$. (There is another equilibrium in which the third candidate's position approaches $m^* + \delta$ from above.)

I refer to an equilibrium in which a player's positions converge from below (respectively, above) to the position x as one in which the player "chooses the position $x-$ (respectively, $x+$)". Similarly, I write $\pi_i(x_1, x_2, x_1-)$ for the limit of the probability that player i wins when players 1 and 2 are at x_1 and x_2 respectively and player 3's position approaches x_1 from below.

In summary, an equilibrium differs from a subgame perfect equilibrium in two ways. First, an equilibrium does not necessarily specify the players' actions after all histories, but only after histories that result when, in every period, at most one player deviates from her strategy. Second, rather than requiring the players' strategies to be optimal, an equilibrium requires that no player gain more than an arbitrarily small amount by deviating from her strategy.

3. Symmetric single-peaked distributions

I begin by considering the case in which the densities of the distributions F and G are single-peaked and symmetric about their medians. (The equilibria I find may not survive large departures from the assumptions that F and G are symmetric, but in Appendix 2 I argue that they do survive small departures from these assumptions.)

3.1   An equilibrium with two entrants

In one of the equilibria I study, denoted $\{\sigma^{*n}\}$, two players enter in the first period, one at a position $x_1^* < m^*$ and the other at a position $x_2^* > m^*$, and the third player does not enter. The positions x_1^* and x_2^* are such that

(a) the third player's probability of winning if she enters at $x_1^* -$ or at $x_2^* +$ is p_0

(b) at no position can the third player win with probability greater than p_0.

If either entrant chooses a position closer to the center then the third player enters beside her at a slightly more extreme position. If either entrant moves away from the center then the third player enters between the two entrants so as to equalize their probabilities of victory (or, if this is not possible, to minimize the difference between these probabilities) if by doing so she wins with probability at least p_0, and otherwise stays out.

Before defining the substrategy profile σ^* precisely, I consider informally the circumstances under which it is an equilibrium. Given player 3's reactions to deviations by players 1 and 2, it is plausible that neither entrant can

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profitably deviate. Further, if either entrant moves closer to the center then plausibly it is optimal for the third player to enter at a slightly more extreme position (where she wins with probability greater than p_0). It remains to consider player 3's optimal reaction to a move *away* from the center by either entrant.

To study this optimal reaction, first consider how the players' vote shares depend on the value of α (the median of the distribution of the citizens' favorite positions) for any positions x_1, x_2, and x_3 with $x_1 < x_2$ and $x_3 = x_1-$. For such positions, player 3's vote share $v_3(\alpha|x_1, x_2, x_3)$ is nonincreasing in α, player 2's vote share is nondecreasing in α, and player 1's vote share first increases then decreases. By the symmetry and single-peakedness of F, candidate 1's vote share is maximal when candidate 2's and 3's vote shares are equal. There are two cases.

(i) If x_1 and x_2 are sufficiently separated, player 1 wins with positive probability (for some range of values of α player 1's vote share exceeds those of players 2 and 3), so that the vote shares have the shapes shown in Figure 1, and player 3 wins if and only if $\alpha < \underline{\alpha}$, the point at which her vote share is equal to that of player 1.

(ii) If x_1 and x_2 are relatively close then player 1 does not win for any value of α (the maximum of player 1's vote share is less than the value of player 2's and player 3's vote shares when they are equal), and player 3 wins if and only if α is less than the point at which her vote share is equal to that of player 2.
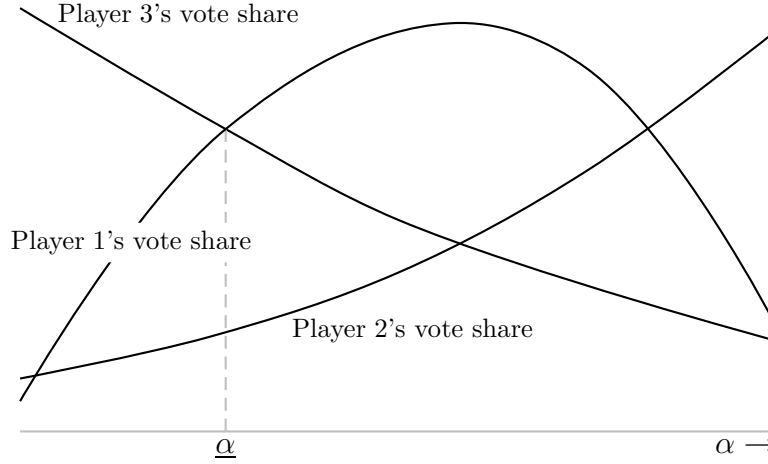
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Figure 1. The candidates' votes shares when their positions are x_1, x_2, and x_3, where $x_3 = x_1-$, $x_1 < x_2$, and $x_2 - x_1$ is large enough.

Now, a move away from the center by player 2 decreases her vote share for all values of α and increases that of player 1. (The curve for player 1 in Figure 1 shifts up for all values of α and that for player 2 shifts down.) The implication for player 3's optimal response differs between cases (i) and (ii). In case (i) player 3 competes with player 1 at the margin, and player 3's probability of winning at x_1^*- *decreases* when player 2 moves away from the center. In case (ii) player 3 competes with player 2 on the margin, and her probability of winning at x_1^*- *increases* when player 2 moves away from the center. A symmetric argument applies to a move by player 1 away from the center.

This informal analysis suggests that if x_1^* and x_2^* are sufficiently separated then player 3's optimal response to a move away from the center by either entrant is to enter between them. But if x_1^* and x_2^* are close together then player 3's optimal response to such a move is to enter at a position just more extreme than that of the other entrant, making the move desirable for its per-

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petrator. In conclusion, it seems that we need the points x_1^* and x_2^* satisfying (a) and (b) above to be sufficiently separated in order for the substrategy profile σ^* defined informally at the start of the section to be an equilibrium.

Subsequently I show that if the voters' preferences are uncertain enough then sufficiently separated points x_1^* and x_2^* satisfying (a) exist. These points may not satisfy (b) because they are too far apart, so that player 3 can win with probability p_0 at some point between them. I show that in this case an alternative equilibrium exists in which the third candidate enters.

I now define x_1^* and x_2^* and the associated equilibrium precisely, and then define the alternative equilibrium. In Section 3.3 I present the main result, giving conditions under which the equilibria exist.

The positions x_1^* and x_2^* are defined by the conditions that player 3 wins with probability p_0 at x_1^*- and at x_2^*+, that player 1 wins with positive probability when player 3 enters at x_1^*-, and that player 2 wins with positive probability when player 3 enters at x_2^*+. Given that the vote shares take the forms shown in Figure 1, these conditions are equivalent to player 3's tying for first place when she locates at x_1^*- and α is such that $G(\alpha) = p_0$, and when she locates at x_2^*+ and α is such that $G(\alpha) = 1 - p_0$:

$$1 - F_{G^{-1}(p_0)}\left(\tfrac{1}{2}(x_1 + x_2)\right) \leq$$

$$F_{G^{-1}(p_0)}(x_1) = F_{G^{-1}(p_0)}\left(\tfrac{1}{2}(x_1 + x_2)\right) - F_{G^{-1}(p_0)}(x_1)$$

$$F_{G^{-1}(1-p_0)}\left(\tfrac{1}{2}(x_1 + x_2)\right) \leq$$

$$1 - F_{G^{-1}(1-p_0)}(x_2) = F_{G^{-1}(1-p_0)}(x_2) - F_{G^{-1}(1-p_0)}\left(\tfrac{1}{2}(x_1 + x_2)\right).$$

These conditions are equivalent to

$$2F_{G^{-1}(p_0)}(x_1) = F_{G^{-1}(p_0)}\left(\tfrac{1}{2}(x_1 + x_2)\right) \geq \tfrac{2}{3} \tag{1}$$

$$2(1 - F_{G^{-1}(1-p_0)}(x_2)) = 1 - F_{G^{-1}(1-p_0)}\left(\tfrac{1}{2}(x_1 + x_2)\right) \geq \tfrac{2}{3}. \tag{2}$$

If there is little uncertainty about the distribution F (i.e. the dispersion in G is small), then the two inequalities are inconsistent, so that the conditions have no solution. If there is enough uncertainty about F—specifically, if $F_{G^{-1}(p_0)}(m^*) \geq \tfrac{2}{3}$—then the conditions have a solution. By the symmetry of F and G we have $F_{G^{-1}(1-p_0)}(x) = 1 - F_{G^{-1}(p_0)}(2m^* - x)$ for any value of x, so that if $x_2 = 2m^* - x_1$ then (1) and (2) are the same condition, namely $2F_{G^{-1}(p_0)}(x_1^*) = F_{G^{-1}(p_0)}(m^*) \geq \tfrac{2}{3}$. The equation in this condition has a unique solution because the support of F is an interval. Thus we have the following result.

Lemma 1 *Suppose that the densities of F and G are symmetric about their medians. If $F_{G^{-1}(p_0)}(m^*) \geq \tfrac{2}{3}$ then (1) and (2) have a solution (x_1^*, x_2^*) defined by $F_{G^{-1}(p_0)}(x_1^*) = \tfrac{1}{2}F_{G^{-1}(p_0)}(m^*)$ and $\tfrac{1}{2}(x_1^* + x_2^*) = m^*$.*

(If, in addition, the density of F is single-peaked, then (x_1^*, x_2^*) is the only solution of (1) and (2).)

The conditions defining x_1^* and x_2^* are illustrated in Figure 2: the second tertile t_2 of $F_{G^{-1}(p_0)}$ is at most m^* and the area under $f_{G^{-1}(p_0)}$ (the density of $F_{G^{-1}(p_0)}$) up to x_1^* is equal to the area between x_1^* and m^*; x_2^* is symmetric with x_1^* about m^*.

I now specify the substrategy profile σ^{*n} precisely. The players' actions after any history depend only on the positions currently occupied, not on the

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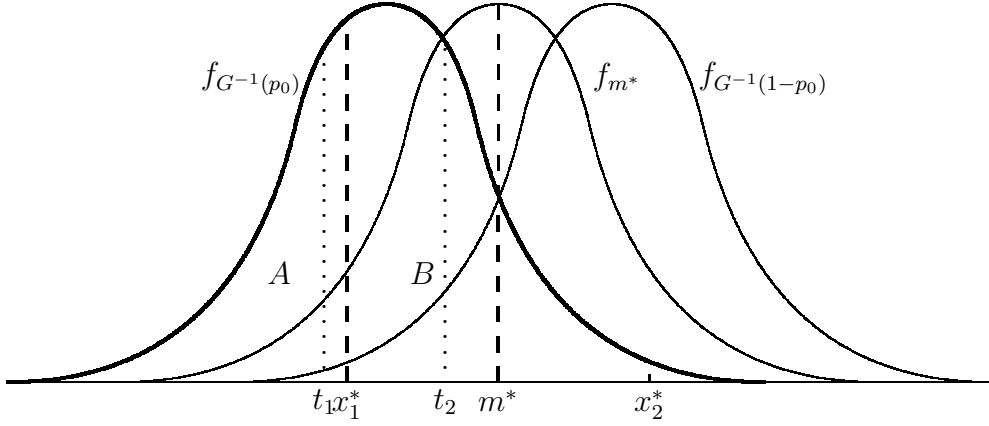
Figure 2. The positions x_1^* and x_2^*. The points t_1 and t_2 are the tertiles of $F_{G^{-1}(p_0)}$: $t_i = F_{G^{-1}(p_0)}(i/3)$. The areas A and B under $f_{G^{-1}(p_0)}$ delimited by the dashed lines are equal.

period in which they were occupied, and are defined as follows:

- Player 1 enters at x_1^* at the start of the game and after any history in which the only position occupied is x_2^*.

- Player 2 enters at x_2^* at the start of the game and after any history in which the only position occupied is x_1^*.

- Player 3

  · stays out at the start of the game and after any history in which the set of occupied positions is $\{x_1^*\}$, $\{x_2^*\}$, or $\{x_1^*, x_2^*\}$

  · enters at $\min\{x - 1/n, x_2^* - 1/n\}$ after any history in which the set of occupied positions is $\{x, x_2^*\}$, where $x > x_1^*$

  · enters at

$$\min\{x_2^* - 1/n, 2m^* - \tfrac{1}{2}(x + x_2^*)\}$$

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if her probability of winning there is at least p_0, and otherwise does not enter, after any history in which the set of occupied positions is $\{x, x_2^*\}$, where $x < x_1^*$

· behaves analogously after histories in which the set of occupied positions is $\{x_1^*, x\}$ for $x \neq x_2^*$.

(It is straightforward to check that this description characterizes a triple of substrategies.)

In the proposition below I give conditions under which $\{\sigma^{*n}\}$ is an equilibrium. Players 1 and 2 are deterred from moving closer to the center by the "threat" of player 3 to enter beside them, at a slightly more extreme position, a move that is rational for her. Note that if player 1 stays out in period 1 then player 3 expects her to enter in the next period, so player 3 optimally does not enter.[5] Note also that although all the action in every σ^{*n} takes place in a single period, the simultaneous move game does not have an equilibrium in which player 1 chooses x_1^*, player 2 chooses x_2^*, and player 3 stays out: player 3's threat to enter after a deviation by one of the other players is essential in maintaining the equilibrium.

3.2 An equilibrium with three entrants

In the presence of great uncertainty about the location of F, the positions x_1^* and x_2^* are far enough apart that player 3's probability of winning at a point between them exceeds p_0, so that $\{\sigma^{*n}\}$ is not an equilibrium. Instead, there is

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[5] Alternatively, there is an equilibrium in which player 3 enters at x_1^* in the second period if player 1 does not enter in the first period, and player 1 stays out.

an equilibrium $\{\hat{\sigma}^n\}$ in which players 1 and 2 enter at points $\hat{x}_1$ and $\hat{x}_2 > \hat{x}_1$ in the first period, and player 3 enters at m^* in the second period, where $\hat{x}_1$ and $\hat{x}_2$ are such that player 3's probabilities of winning at m^*, at $\hat{x}_1-$, and at $\hat{x}_2+$ are equal, and $\frac{1}{2}(\hat{x}_1 + \hat{x}_2) = m^*$. Player 3's reactions to deviations are the same as they are for the equilibrium $\{\sigma^{*n}\}$: if either entrant chooses a position closer to the center then the third player enters beside her at a slightly more extreme position, while if either entrant moves away from the center then the third player enters between the two entrants so as to equalize their probabilities of victory (or, if this is not possible, to minimize the difference between these probabilities).

If players 1 and 2 follow their strategies then player 3 is indifferent between staying out, entering at m^*, entering at $\hat{x}_1-$, and entering at $\hat{x}_2+$, and is worse off entering at any other point. If either player 1 or player 2 deviates then player 3's reactions are also optimal, for the same reasons as they are in the equilibrium $\{\sigma^{*n}\}$. Finally, the reactions of player 3 deter deviations by players 1 and 2, again as in $\{\sigma^{*n}\}$. Note that it is *not* an equilibrium for all three candidates to enter in the first period, since in this case player 3 has no means by which to "punish" a deviation by player 1 or player 2, since she has already committed herself to a position.

The following result gives conditions under which such positions $\hat{x}_1$ and $\hat{x}_2$ exist. (A proof is in Appendix 1.)

Lemma 2 *Suppose that the densities of F and G are symmetric about their medians and the density of F is single-peaked. For any position x_1, let $h(x_1) =$*

$2m^* - x_1$. *Then there exists $\hat{x}_1$ such that*

$$\pi_3(\hat{x}_1, h(\hat{x}_1), m^*) = \pi_3(\hat{x}_1, h(\hat{x}_1), \hat{x}_1-) = \pi_3(\hat{x}_1, h(\hat{x}_1), h(\hat{x}_1)+).$$

If there exists such a value of $\hat{x}_1$ for which the common value $\hat{p}$ of these probabilities is positive then there is no other value of $\hat{x}_1$ that satisfies the equations. We have $\hat{p} < \frac{1}{3}$ and $F_{G^{-1}(\hat{p})}(m^*) > \frac{2}{3}$ if $\hat{p} > 0$.

The substrategy profile $\hat{\sigma}^n$ is defined precisely as follows. As in the profile σ^{*n}, the players' actions after any history depend only on the positions currently occupied, not on the period in which they were occupied. They are defined as follows:

- Player 1 enters at $\hat{x}_1$ at the start of the game and after any history in which the only position occupied is $\hat{x}_2$.

- Player 2 enters at $\hat{x}_2$ at the start of the game and after any history in which the only position occupied is $\hat{x}_1$.

- Player 3

  · stays out at the start of the game and after a history in which the set of occupied positions is $\{\hat{x}_1\}$ or $\{\hat{x}_2\}$

  · enters at m^* after a history in which the set of occupied positions is $\{\hat{x}_1, \hat{x}_2\}$

  · enters at $\min\{x - 1/n, \hat{x}_2 - 1/n\}$ after any history in which the set of occupied positions is $\{x, \hat{x}_2\}$, where $x > \hat{x}_1$

· enters at

$$\min\{\hat{x}_2 - 1/n, 2m^* - \tfrac{1}{2}(x + \hat{x}_2)\}$$

after any history in which the set of occupied positions is $\{x, \hat{x}_2\}$, where $x < \hat{x}_1$

· behaves analogously after histories in which the set of occupied positions is $\{\hat{x}_1, x\}$ for $x \neq \hat{x}_2$.

3.3   Main result

My main result (proved in Appendix 1) shows that if there is sufficient uncertainty then either $\{\sigma^{*n}\}$ or $\{\hat{\sigma}^n\}$ is an equilibrium.

Proposition *Suppose that the densities of F and G are single-peaked and symmetric about their medians, and $F_{G^{-1}(p_0)}(m^*) \geq \tfrac{2}{3}$ (i.e. the location of F is sufficiently uncertain).*

(a) If

$$F_{G^{-1}(\frac{1}{2}(1-p_0))}(\tfrac{1}{2}(m^* + x_1^*)) \geq \tfrac{1}{2}F_{G^{-1}(\frac{1}{2}(1-p_0))}(\tfrac{1}{2}(m^* + x_2^*)) \qquad (3)$$

(so that player 3's probability of winning at m^* is at most p_0 when players 1 and 2 enter at x_1^* and x_2^*) then $\{\sigma^{*n}\}$ is an equilibrium. In this equilibrium players 1 and 2 enter in period 1 at x_1^* and x_2^* respectively and player 3 does not enter.

(b) If the inequality in (3) is reversed then $\{\hat{\sigma}^n\}$ is an equilibrium. In this equilibrium players 1 and 2 enter in period 1 at $\hat{x}_1$ and $\hat{x}_2$ respectively and player 3 enters at m^* in period 2.

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3.4   Comparative statics

When the minimal acceptable probability p_0 of winning is small, (3) is violated, and hence the three-candidate equilibrium $\{\hat{\sigma}^n\}$ exists. As p_0 increases, the candidates' positions remain the same as long as (3) is still violated. At some point (3) is satisfied, in which case the two-candidate equilibrium $\{\sigma^{*n}\}$ replaces $\{\hat{\sigma}^n\}$. As p_0 increases further, x_1^* and x_2^* converge until $F_{G^{-1}(p_0)}(m^*) < \frac{2}{3}$, at which point neither equilibrium exists.

Now consider the effect of an increase in the degree of uncertainty. Consider a family of distributions G_β, indexed by the parameter β, for which $G_\beta(\alpha)$ is increasing in β for all $\alpha < m^*$ and decreasing in β for all $\alpha > m^*$. (As β increases, G_β becomes more uncertain.) If β takes the smallest value under which either equilibrium exists (i.e. $F_{G_\beta^{-1}(p_0)}(m^*) = \frac{2}{3}$) then (3) is satisfied, so that the equilibrium $\{\sigma^{*n}\}$ exists: when the uncertainty about the distribution of voters' preferences is relatively small, there are two entrants. As the uncertainty increases (β increases), the positions x_1^* and x_2^* separate. However, given the change in G, the probability of player 3's winning at some position between x_1^* and x_2^* does not necessarily increase, with the consequence that the nature of the equilibrium may go back and forth between $\{\sigma^{*n}\}$ (two entrants) and $\{\hat{\sigma}^n\}$ (three entrants) as the degree of uncertainty increases.

3.5   Discussion

The proposition is silent about the existence of an equilibrium when the uncertainty about the location of F is small. My investigation of this issue suggests, but does not prove, that no (pure) equilibrium exists in this case.

The equilibrium $\{\sigma^{*n}\}$ does not survive for an interesting reason: each entrant can increase her chances of winning by making her position a little more *extreme*. When there is little uncertainty, the positions x_1^* and x_2^* are close to each other, so that if the third player enters at x_1^*- then she competes with player 2, and player 1 has no chance of winning (the graph of player 1's vote share in the analogue of Figure 1 lies entirely below that of player 2's and player 3's shares). In this case, a move by player 2 to a slightly more extreme position *increases* player 3's probability of winning at x_1^*- above p_0, thereby making player 3's entry worthwhile. Her entry makes player 2 better off, since it splits the vote on the left between players 1 and 3. Thus the deviation by player 2 is profitable. Informally, candidate 2 takes a more extreme position in order to make it worthwhile for her opposition to splinter.

The proposition does not show that $\{\sigma^{*n}\}$ and $\{\hat{\sigma}^n\}$ are the *only* equilibria. It is easy to argue that there is no equilibrium in which all three players enter simultaneously in the first period (whether or not the distributions F and G are single-peaked and symmetric), but it is more complicated to rule out other configurations. In particular, to study the possibility of equilibria in which one player enters in period 1, we need to study the best responses of a player to all possible pairs of positions for the other two players, a task that appears to be complex. (In the simpler model in which two players are constrained to enter in the first period and the third is restricted to the second period (as in Palfrey's (1984) model), $\{\sigma^{*n}\}$ and $\{\hat{\sigma}^n\}$ are the only equilibria.)

In Appendix 2 I argue that the equilibria $\{\sigma^{*n}\}$ and $\{\hat{\sigma}^n\}$ do not depend sensitively on the assumption that the distributions F and G are symmetric:

if these distributions are slightly asymmetric then similar equilibria continue to exist. If F and G are far from symmetric, however, it appears that there may be no equilibrium outcomes like those generated by $\{\sigma^{*n}\}$ and $\{\hat{\sigma}^n\}$. Certainly other equilibria may exist when F and G are far from symmetric. For instance, for a case in which F is very asymmetric I give an example in Osborne (1993, p. 146) of an equilibrium in the case of certainty in which one candidate enters in the first period (at a position more extreme than any citizen's favorite position), and no further candidates enter; this equilibrium survives the introduction of a little uncertainty about F.

4. Concluding remarks

My result provides a set of circumstances under which the intuition that the protagonists in a two-candidate competition are under pressure to differentiate their positions in order to deter entry is logically consistent. The result reveals that the intuition relies on both the candidates' uncertainty about the voters' preferences and the possibility of their acting asynchronously.

The case of three potential candidates is the simplest one in which the intuitive argument can possibly make sense. The extent to which the equilibria depend on there being exactly three potential candidates is unclear. At a minimum, the precise forms of the equilibria appear to depend on this assumption. If the policy differentiation in the equilibria depends on the assumption—if, when there are four or more potential candidates, the equilibria involve little or no policy differences—then the model shows another dimension in which the intuition is not robust. (A full analysis of the model with more than three

potential candidates appears to be difficult.)

An important question in the study of elections concerns the implications of different electoral systems for the number of candidates, the dispersion in their positions, and ultimately the representativeness of the winner's position. The fact that tractable multicandidate equilibria exist for almost no distribution of the voters' preferences in the standard Hotelling–Downs model has inhibited approaches to this question. My model, building upon Palfrey's, provides a vehicle with which to address the question.

As an example, consider briefly the different implications of plurality rule and a two-ballot runoff system. In the equilibrium $\{\sigma^{*n}\}$, the third player's entry beside one of the other players leads her to win with probability close to p_0. At this position she also has a positive probability of coming second, so that her probability of getting into the second round exceeds p_0. In the second round she wins with probability $\frac{1}{2}$, so that if her probability of coming second in the first round exceeds p_0, she will choose to enter. Consequently the existing candidates will have to adopt somewhat more extreme positions under a runoff system then under plurality rule in order to keep the third candidate out of the race. This tendency contrasts with that in the citizen-candidate model (Osborne and Slivinski (1996), Besley and Coate (1997)), where in two-candidate equilibria the dispersion in the entrants' positions is designed to prevent entry *between* the two candidates' positions, so that under a runoff system *less* dispersion is possible.

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Appendix 1: Proofs

Proof of Lemma 2. The function $\pi_3(x_1, h(x_1), m^*)$ is decreasing in x_1 when it is positive, because for all values of α the vote shares of players 1 and 2 increase with x_1, while that of player 3 decreases. (For any given x_1, player 1's vote share as a function of α decreases, that of player 3 increases, and that of player 2 increases then decreases.) The function $\pi_3(x_1, h(x_1), x_1-)$, on the other hand, is increasing in x_1 where it is positive, because for all values of α the vote share of player 1 decreases with x_1, that of player 2 remains the same, and that of player 3 increases. Thus there exists $\hat{x}_1$ satisfying the first equation; the symmetry of F and G implies that the same value of x_1 satisfies the second equation. Further, if there is a solution of the first equation for which $\pi_3(\hat{x}_1, h(\hat{x}_1), m^*) > 0$ then there is no other solution.

If $F_{G^{-1}(\hat{p})}(m^*) \leq \frac{2}{3}$ then for the profile of positions $(\hat{x}_1, \hat{x}_2, \hat{x}_1-)$, player 3 ties for first place with player 2 when $\alpha = G^{-1}(\hat{p})$ (because player 3 wins with probability $\hat{p}$). Thus $G^{-1}(\hat{p})$ is the midpoint of $[\hat{x}_1, m^*]$. It follows that the maximal vote share, as α varies, of a candidate whose constituency has length $\frac{1}{2}[\hat{x}_2 - \hat{x}_1]$ is less than $\frac{1}{3}$ (using the fact that $F_{G^{-1}(\hat{p})}(m^*) \leq \frac{2}{3}$). Because player 3 is such a candidate when the profile of positions is $(\hat{x}_1, \hat{x}_2, m^*)$, we have $\pi_3(\hat{x}_1, \hat{x}_2, m^*) = \hat{p} = 0$.

It follows that if $\hat{p} > 0$ then $F_{G^{-1}(\hat{p})}(m^*) > \frac{2}{3}$, so that for the profile of positions $(\hat{x}_1, \hat{x}_2, \hat{x}_1-)$, players 1 and 3 tie for first place when $\alpha = G^{-1}(\hat{p})$ (player 2 receives less than $\frac{1}{3}$ of the vote): $v_1(G^{-1}(\hat{p})|\hat{x}_1, \hat{x}_2, \hat{x}_1-) = v_3(G^{-1}(\hat{p})|\hat{x}_1, \hat{x}_2, \hat{x}_1-)$. Now, the constituency of player 3 at the profile $(\hat{x}_1, \hat{x}_2, m^*)$ has the same length as the constituency of player 1 at the profile $(\hat{x}_1, \hat{x}_2, \hat{x}_1-)$, so by the single-

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peakedness of F, $v_3(G^{-1}(\hat{p})|\hat{x}_1, \hat{x}_2, m^*) < v_1(G^{-1}(\hat{p})|\hat{x}_1, \hat{x}_2, \hat{x}_1-)$. Further, $v_3(G^{-1}(\hat{p})|\hat{x}_1, \hat{x}_2, \hat{x}_1-) < v_1(G^{-1}(\hat{p})|\hat{x}_1, \hat{x}_2, m^*)$. Thus $v_3(G^{-1}(\hat{p})|\hat{x}_1, \hat{x}_2, m^*) < v_1(G^{-1}(\hat{p})|\hat{x}_1, \hat{x}_2, m^*)$, so that player 1 loses for the profile $(\hat{x}_1, \hat{x}_2, m^*)$ when $\alpha = G^{-1}(\hat{p})$. It follows that $\pi_3(\hat{x}_1, \hat{x}_2, m^*) < 1 - 2\hat{p}$. By definition we have $\pi_3(\hat{x}_1, \hat{x}_2, m^*) = \hat{p}$, so that $\hat{p} < \frac{1}{3}$. $\qquad\square$

In the proof of the proposition I use the following result.

Lemma 3 *Suppose that the density of F is single-peaked. If the candidates' positions are $x_1 < x_2 < x_3$ and candidate 2 wins with positive probability, then the set of values of α for which she wins is an interval of the form $\left(\frac{1}{2}(x_1 + x_2) + \delta_1, \frac{1}{2}(x_2 + x_3) - \delta_2\right)$, where $\delta_1 > 0$ and $\delta_2 > 0$. The values of δ_1 and δ_2, and hence the length of the interval, are independent of x_2. If the density of F is symmetric about its median then $\delta_1 = \delta_2$.*

Proof. As the value of α increases, the vote share of candidate 1 decreases, that of candidate 2 increases then decreases, and that of candidate 3 increases. Thus if candidate 2 wins with positive probability then she wins for all values of α such that

$$F_\alpha\left(\tfrac{1}{2}(x_2 + x_3)\right) - F_\alpha\left(\tfrac{1}{2}(x_1 + x_2)\right) >$$
$$\max\{F_\alpha\left(\tfrac{1}{2}(x_1 + x_2)\right), 1 - F_\alpha\left(\tfrac{1}{2}(x_2 + x_3)\right)\}. \quad (4)$$

By the single-peakedness of F, this set is an interval (refer to Figure 3). If $\alpha \le \frac{1}{2}(x_1 + x_2)$ then player 1 wins, because she obtains more than one half of the votes. Thus the smallest point in the interval exceeds $\frac{1}{2}(x_1+x_2)$. Similarly, the largest point in the interval is less than $\frac{1}{2}(x_2 + x_3)$. Because the length of

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the interval $\left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(x_2 + x_3)\right)$ of positions that are closer to x_2 than to either of the other candidates is independent of x_2, so too is the length of the interval of values of α that satisfy (4). Finally, if the density of F is symmetric about its median, the interval takes the form $\left(\frac{1}{2}(x_1 + x_2) + \delta, \frac{1}{2}(x_2 + x_3) - \delta\right)$ for some $\delta > 0$. $\qquad\square$
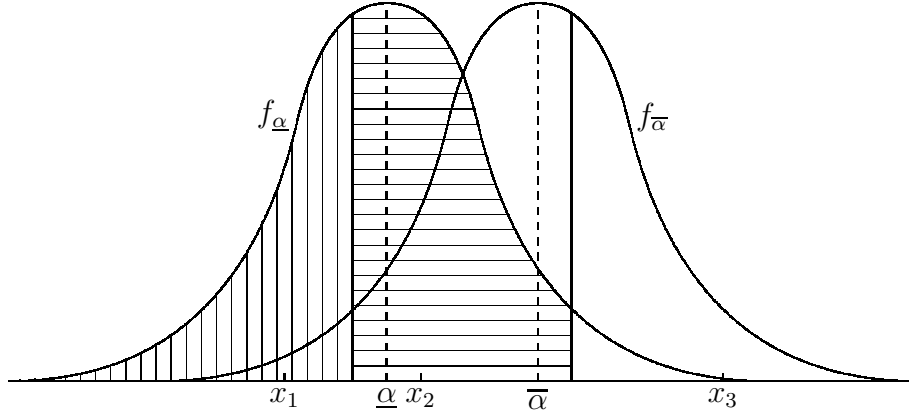
Figure 3. Candidate 2, at x_2, wins for all values of α between $\underline{\alpha}$ and $\overline{\alpha}$, where $\underline{\alpha}$ is determined by the condition that the shaded areas are equal and $\overline{\alpha}$ is determined by a symmetric condition. The thick vertical lines indicate the midpoints of $[x_1, x_2]$ and $[x_2, x_3]$.

Proof of Proposition. (*a*) *The equilibrium* $\{\sigma^{*n}\}$: *optimality of player 3's strategy* I begin by arguing that player 3's action is optimal after each history in which in each period at most one player deviated from her equilibrium strategy. I consider the possible sets of occupied positions in turn.

$\varnothing$ and $\{x_1^*, x_2^*\}$:   In both cases every action of player 3 results in the same set of occupied positions, because σ^{*n} calls for players 1 and 2 to enter in period 1 at x_1^* and x_2^* respectively. If player 3 enters at $x_1^* -$ then she wins with probability p_0; if she enters at $x < x_1^*$ then she wins with smaller probability. Symmetrically, her probability of winning at any

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point greater than x_2^* is at most p_0. I now show that entry in (x_1^*, x_2^*), or at x_1^* or x_2^*, leads also to a win with probability at most p_0.

$x_3 \in (x_1^*, x_2^*)$: From Lemma 3 and the single-peakedness and symmetry of G, the position that maximizes 3's probability of winning is m^* $(= \frac{1}{2}(x_1^* + x_2^*))$. The inequality (3) implies that the probability of player 3's winning at m^* is at most p_0.

$x_3 = x_1^*$ or $x_3 = x_2^*$: First suppose that $x_3 = x_1^*$. Then players 1 and 3 split the votes associated with the position x_1^*. Thus $\pi_1(x_1^*, x_2^*, x_1^*) = \pi_3(x_1^*, x_2^*, x_1^*)$. Now, for all α, the sum of player 1's and player 3's vote shares for the profile of positions (x_1^*, x_2^*, x_1^*) is the same as it is for the profile (x_1^*, x_2^*, x_1^*-). For the profile (x_1^*, x_2^*, x_1^*), however, both player 1 and player 3 need to beat player 2 in order for one of them to win, while for the profile (x_1^*, x_2^*, x_1^*-) only one of them needs to beat player 2 in order for one of them to win. Thus

$$\pi_1(x_1^*, x_2^*, x_1^*) + \pi_3(x_1^*, x_2^*, x_1^*) \le \pi_1(x_1^*, x_2^*, x_1^*-) + \pi_3(x_1^*, x_2^*, x_1^*-).$$

Because $\pi_1(x_1^*, x_2^*, x_1^*-) = \pi_3(x_1^*, x_2^*, x_1^*+)$, it follows that

$$\pi_3(x_1^*, x_2^*, x_1^*) \le \tfrac{1}{2}[\pi_3(x_1^*, x_2^*, x_1^*+) + \pi_3(x_1, x_2, x_1^*-)].$$

Each term on the right-hand side is at most p_0 by the previous arguments, so the probability of player 3's winning at x_1^* is at most p_0. A symmetric argument applies to the case $x_3 = x_2^*$.

I conclude that after any history in which either no position is occupied or the positions x_1^* and x_2^* are occupied, there is no position at which

player 3's probability of winning exceeds p_0, so that an optimal action for her is to stay out.

$\{x_1, x_2^*\}$ where $x_1 \in (x_1^*, x_2^*)$: Of the points outside $[x_1, x_2^*]$, clearly none except possibly x_1- and x_2^*+ are optimal for player 3. If $x_3 = x_1-$ then player 3 wins for all values of α up to some critical value greater than x_1, say $x_1 + \delta$. If $x_3 = x_2^*+$ then player 3 wins for all values of α in excess of some critical value; by the symmetry of F, this critical value is $x_2^* - \delta$. But now because x_1 is closer to m^* than is x_2^*, and the density of G is symmetric about m^*, it follows that player 3's probability of winning at x_2^*+ is less than her probability of winning at x_1-.

I now argue that $\pi_3(x_1, x_2^*, x_1-) > p_0$. Relative to the case in which the profile of positions is (x_1^*, x_2^*, x_1^*-), for all values of α player 3's share of the vote is higher when the profile of positions is (x_1, x_2^*, x_1-). Further, for $\alpha \leq G^{-1}(p_0)$, player 1's share is lower, because her constituency is shorter, $F_{G^{-1}(p_0)}(x_1^*) = \frac{1}{2}F_{G^{-1}(p_0)}(m^*) \geq \frac{1}{3}$, and F is symmetric and single-peaked. Thus $\pi_3(x_1, x_2^*, x_1-) > \pi_3(x_1^*, x_2^*, x_1^*-) = p_0$.

Next I argue that at all points in $[x_1, x_2^*]$ player 3's probability of winning is less than p_0. For $x_3 \in (x_1, x_2^*)$, $\pi_3(x_1, x_2^*, x_3)$ is decreasing in x_1 (for each value of α, player 1's share of the vote increases and player 3's share decreases as x_1 increases). Because $\pi_3(x_1^*, x_2^*, x_3) \leq p_0$ by the argument for a history in which the set of occupied positions is $\{x_1^*, x_2^*\}$, we have $\pi_3(x_1, x_2^*, x_3) < p_0$ for $x_1 > x_1^*$ and all $x_3 \in (x_1, x_2^*)$.

If $x_3 = x_1$ then player 3 shares with player 1 the votes associated with

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the position x_1, and, as I argued in the previous case, $\pi_3(x_1, x_2^*, x_1) \leq \frac{1}{2}[\pi_3(x_1, x_2^*, x_1+) + \pi_3(x_1, x_2^*, x_1-)]$. Because we have $\pi_3(x_1, x_2^*, x_1+) < p_0$ and $\pi_3(x_1, x_2^*, x_1-) > p_0$ by the previous paragraphs, $\pi_3(x_1, x_2^*, x_1) < \pi_3(x_1, x_2^*, x_1-)$, so that entry by player 3 at x_1- is better than entry at x_1. A symmetric argument leads to the conclusion that entry at x_2^* is not optimal either.

I conclude that player 3's optimal action when the positions $x_1 < x_1^*$ and x_2^* are occupied is to enter at x_1-.

$\{x_1, x_2^*\}$ where $x_1 \geq x_2^*$: In this case x_2^*- is the optimal position for player 3: she wins with probability greater than $\frac{1}{2}$ there, and with probability less than $\frac{1}{2}$ if she enters at any point greater than x_2^* (because then player 2 wins with probability greater than $\frac{1}{2}$).

$\{x_1, x_2^*\}$ where $x_1 < x_1^*$: As before, neither x_1 nor x_2^* is an optimal position for player 3. I now argue that neither are x_1- or x_2^*+. By the symmetry of F and G, player 3's probability of winning at x_2^*+ exceeds her probability of winning at x_1- (as argued previously). Now suppose that player 3 locates at x_2^*+. Compared with the situation in which the positions are (x_1^*, x_2^*, x_2^*+), player 1's share of the votes is lower for all values of α, player 2's is higher, and player 3's is the same; in both cases player 2's probability of winning is positive. (Refer to a diagram analogous to Figure 1.) Thus player 3's probability of winning is smaller than it is for the positions (x_1^*, x_2^*, x_2^*+), when it is p_0.

The remaining possibilities are that player 3 chooses a position in (x_1, x_2^*)

or does not enter. If she chooses a position in (x_1, x_2^*), she wins for an interval of values of α whose form is given in Lemma 3. Given the symmetry and single-peakedness of G, the position that maximizes her probability of winning is thus such that, if possible, this interval is centered at m^*, which implies that $x_3 = 2m^* - \frac{1}{2}(x_1 + x_2^*)$; otherwise the position that maximizes her probability of winning is $x_2^* -$. Thus an optimal action of player 3 is to enter at $\min\{x_2^*-, 2m^* - \frac{1}{2}(x_1 + x_2^*)\}$ if her probability of winning there is at least p_0, and otherwise to stay out.

This completes the argument that the actions of player 3 defined in σ_3^{*n} are ϵ-optimal given σ_1^{*n} and σ_2^{*n}.

Optimality of player 1's and player 2's strategies Now I argue that player 1's action is optimal after every history in which in each period at most one player deviated from σ^{*n}. Such histories come in only two types: the initial history and those in which the set of occupied positions is $\{x_2^*\}$. Because σ_2^{*n} calls for player 2 to enter at x_2^* in period 1, any action by player 1 after any of these histories results in the same set of occupied positions.

If player 1 follows her strategy and enters at x_1^* then player 3 subsequently stays out, and player 1 wins with probability $\frac{1}{2}$. Player 1's other options are to stay out and to enter at a point different from x_1^*. So long as she stays out, player 3 stays out too (in each period she expects player 1 to follow the precepts of σ_1^{*n} and enter in the next period at x_1^*). Thus player 1 is no better off staying out than entering at x_1^*.

If player 1 enters at $x_1 > x_1^*$ then player 3 enters in the next period at $\min\{x_1-, x_2^*-\}$. For $x_1 \in (x_1^*, x_2^*)$ and $x_3 \in (x_1, x_2^*)$, the arguments above

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imply that $\pi_3(x_1, x_2^*, x_3) < p_0$, so that, in particular, $\pi_3(x_1, x_2^*, x_1+) < p_0$. Because $\pi_1(x_1, x_2^*, x_1-) = \pi_3(x_1, x_2^*, x_1+)$, I conclude that player 1's probability of winning when she locates at $x_1 \in (x_1^*, x_2^*)$ and player 3 reacts according to her strategy is less than $p_0 < \frac{1}{3}$. If $x_1 \geq x_2^*$ then player 3 enters in the next period at x_2^*-, so that player 1's probability of winning is less than $\frac{1}{2}$.

If player 1 enters at $x_1 < x_1^*$ then player 3 either stays out or enters at $\min\{x_2^*-, 2m^* - \frac{1}{2}(x + x_2^*)\}$. If she stays out then player 1's probability of winning is less than it is when player 1's position is x_1^*. If she enters then either her position equalizes player 1's and player 2's probabilities of winning, or $x_3 = x_2^*-$, in which case player 2's probability of winning exceeds player 1's. In both cases player 1's probability of winning is less than $\frac{1}{2}(1 - p_0) < \frac{1}{2}$.

I conclude that no position is better for player 1 than x_1^*. Given the symmetry assumptions, the same arguments imply that no position is better for player 2 than x_2^*.

(b) *The equilibrium* $\{\hat\sigma^n\}$: *optimality of player 3's strategy* As for $\{\sigma^{*n}\}$, I begin by arguing that player 3's action is optimal after each history in which in each period at most one player deviated from her equilibrium strategy; I consider the possible sets of occupied positions in turn. First note that given the opposite inequality to (3), we have $\pi_3(x_1^*, x_2^*, m^*) \geq p_0$. Further, $\pi_3(x_1^*, x_2^*, x_1-) = p_0$ by definition. Thus because $\pi_3(x_1, h(x_1), m^*)$ is decreasing in x_1 and $\pi_3(x_1, h(x_1), h(x_1)-)$ is increasing in x_1 (see the proof of Lemma 2), we have $\hat p \geq p_0$.

$\varnothing$ and $\{\hat x_1, \hat x_2\}$: In both cases every action by player 3 results in the same set of occupied positions, because $\hat\sigma^n$ calls for players 1 and 2 to enter

in period 1 at $\hat{x}_1$ and $\hat{x}_2$ respectively. If player 3 enters at $\hat{x}_1-$ then she wins with probability $\hat{p}$; if she enters at $x < \hat{x}_1$ then she wins with smaller probability. Symmetrically, her probability of winning at any point greater than $\hat{x}_2$ is at most $\hat{p}$. I now show that entry in $(\hat{x}_1, \hat{x}_2)$, or at $\hat{x}_1$ or $\hat{x}_2$, leads to a win with probability at most $\hat{p}$.

$x_3 \in (\hat{x}_1, \hat{x}_2)$: From Lemma 3 and the single-peakedness and symmetry of G, m^* maximizes 3's probability of winning, which is equal to $\hat{p}$.

$x_3 = \hat{x}_1$ or $x_3 = \hat{x}_2$: As for the equilibrium $\{\sigma^{*n}\}$, neither of these positions is optimal.

I conclude that after any history in which either no positions are occupied or the positions $\hat{x}_1$ and $\hat{x}_2$ are occupied, there is no position at which player 3's probability of winning exceeds $\hat{p}$. Because $\hat{p} \geq p_0$, an optimal action for player 3 is to enter at m^*.

$\{x_1, \hat{x}_2\}$ where $x_1 \in (\hat{x}_1, \hat{x}_2)$: By the same argument as for the equilibrium $\{\sigma^{*n}\}$, we have $\pi_3(x_1, \hat{x}_2, x_1-) > \pi_3(x_1, \hat{x}_2, \hat{x}_2+)$. I now argue that the probability of 3's winning at x_1- exceeds $\hat{p}$. By the definition of $\hat{x}_1$ and $\hat{x}_2$, player 3 wins when the profile of positions is $(\hat{x}_1, \hat{x}_2, x_1-)$ if and only if $\alpha < G^{-1}(\hat{p})$. Now, $\hat{p} \geq p_0 > 0$, so by Lemma 2 we have $F_{G^{-1}(\hat{p})}(m^*) > \frac{2}{3}$; given this inequality, the argument is the same as the analogous argument for the equilibrium $\{\sigma^{*n}\}$.

Now let $x_3 \in (x_1, \hat{x}_2)$. For all values of α player 1's vote share is higher and player 3's is smaller than when it is $(\hat{x}_1, \hat{x}_2, x_3)$. Thus $\pi_3(x_1, \hat{x}_2, x_3) < \pi_3(\hat{x}_1, \hat{x}_2, x_3) \leq \hat{p}$.

By the same argument as for the equilibrium $\{\sigma^{*n}\}$, entry by player 3 at either x_1 or $\hat{x}_2$ is worse for player 2 than entry at x_1-.

I conclude that player 3's optimal action in this case is to enter at x_1-.

$\{x_1, \hat{x}_2\}$ where $x_1 \geq \hat{x}_2$: By the same argument as for the equilibrium $\{\sigma^{*n}\}$, $\hat{x}_2-$ is the optimal position for player 3.

$\{x_1, \hat{x}_2\}$ where $x_1 < \hat{x}_1$: By the same argument as for the equilibrium $\{\sigma^{*n}\}$, an optimal action of player 3 is to enter at $\min\{\hat{x}_2-, 2m^* - \frac{1}{2}(x_1 + \hat{x}_2)\}$.

This completes the argument that the actions of player 3 defined in $\hat{\sigma}_3^n$ are ϵ-optimal given $\hat{\sigma}_1^n$ and $\hat{\sigma}_2^n$.

Optimality of player 1's strategy and player 2's strategy Now I argue that player 1's action is optimal after every history in which in each period at most one player deviated from $\hat{\sigma}^n$. Such histories come in only two types: the initial history and those in which the set of occupied positions is $\{\hat{x}_2\}$. Because $\hat{\sigma}_2^n$ calls for player 2 to enter at $\hat{x}_2$ in period 1, any action by player 1 after any of these histories results in the same set of occupied positions.

If player 1 follows her strategy and enters at $\hat{x}_1$ then player 3 subsequently enters at m^*. Player 1's other options are to stay out and to enter at a point different from $\hat{x}_1$. So long as she stays out, player 3 stays out too (in each period she expects player 1 to follow the precepts of $\hat{\sigma}_1^n$ and enter in the next period at $\hat{x}_1$). Thus player 1 is no better off staying out than entering at $\hat{x}_1$.

By the same argument as for the equilibrium $\{\sigma^{*n}\}$, if player 1 enters at $x_1 > \hat{x}_1$ then player 3's reaction of entering at $\min\{x_1-, \hat{x}_2-\}$ leads player 1 to win with probability less than $\hat{p}$. Because $\hat{p} < \frac{1}{3}$, player 1 is worse off if she

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makes such a deviation. If $x_1 \geq \hat{x}_2$ then player 3 reacts by entering at $\hat{x}_2-$, so that player 1's probability of winning is less than $\hat{p}$, because it is less than player 3's probability of winning when the profile of positions is $(\hat{x}_1, \hat{x}_2, x_1)$.

If player 1 enters at $x_1 < \hat{x}_1$ then player 3 enters at $\min\{\hat{x}_2-, 2m^* - \frac{1}{2}(x + \hat{x}_2)\}$. In this case either her position equalizes player 1's and player 2's probabilities of winning or $x_3 = \hat{x}_2-$, and player 2's probability of winning exceeds player 1's. In both cases player 1's probability of winning is less than $\frac{1}{2}(1 - \hat{p})$, her probability of winning if she locates at $\hat{x}_1$.

I conclude that no position is better for player 1 than $\hat{x}_1$. Given the symmetry assumptions, the same arguments imply that no position is better for player 2 than $\hat{x}_2$. □

Appendix 2: Slightly Asymmetric Distributions F and G

In this appendix I argue that an analogue of the Proposition continues to hold if F and G are slightly asymmetric.

The conditions (1) and (2) do not depend on the symmetry of F and G. The equation in the first condition associates with each value of x_1 a unique value of x_2, while the equation in the second condition associates with each value of x_2 a unique value of x_1. When F and G are symmetric, the graphs of these two relations strictly intersect. Small enough perturbations in F and G change the relations only slightly, so that if the asymmetries of F and G are sufficiently small, there is a pair (x_1^*, x_2^*) that satisfies the two conditions (as Lemma 1 shows for the symmetric case).

Lemma 2 can similarly be extended to the case in which F and G are

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slightly asymmetric: there are points $\hat{x}_1$, $\hat{x}_2$, and $\hat{x}_3$ such that

$$\pi_3(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \pi_3(\hat{x}_1, \hat{x}_2, \hat{x}_1-) = \pi_3(\hat{x}_1, \hat{x}_2, \hat{x}_2+).$$

When F and G are asymmetric we need to modify the behavior that player 3's equilibrium strategy σ_3^{*n} specifies after a history in which the set of occupied positions is $\{x, x_2^*\}$ with $x < x_1^*$. If, in this case, player 3's optimal action is to enter, the best point at which to do so is in (x, x_2^*) but is no longer necessarily $\min\{x_2^*-, 2m^* - \frac{1}{2}(x + x_2^*)\}$. The third point in the strategy σ_3^{*n} should be replaced by "enters at the point in (x, x_2^*) that maximizes her probability of winning if this maximal probability is at least p_0, and otherwise stays out after any history in which the set of occupied positions is $\{x, x_2^*\}$, where $x < x_1^*$". Similarly, the second point in the strategy $\hat{\sigma}_3^n$ should be replaced by "enters at the point in $(\hat{x}_1, \hat{x}_2)$ that maximizes her probability of winning after a history in which the set of occupied positions is $\{\hat{x}_1, \hat{x}_2\}$" and the fourth point should be replaced by "enters at the point in $(x, \hat{x}_2)$ that maximizes her probability of winning after any history in which the set of occupied positions is $\{x, \hat{x}_2\}$, where $x < \hat{x}_1$".

Whether or not condition (3) in the Proposition is satisfied determines which of the equilibria $\{\sigma^{*n}\}$ and $\{\hat{\sigma}^n\}$ exists. If it is satisfied then x_1^* and x_2^* are close enough together that player 3's probability of winning at every point in (x_1^*, x_2^*) is at most p_0 when players 1 and 2 enter at x_1^* and x_2^*; if it is violated then there is some point in (x_1^*, x_2^*) at which player 3's probability of winning exceeds p_0. If F and G are asymmetric, there appears to be no simple condition like (3) on F and G that determines which case holds; (3) can be replaced simply by the condition that there is no point in (x_1^*, x_2^*) at

which player 3's probability of winning is greater than p_0.

In the case that F and G are asymmetric, the following further requirements need to be added to the Proposition:

$$\pi_3(x_1, x_2^*, x_1-) \text{ is increasing in } x_1$$

$$\pi_3(x_1^*, x_2, x_2+) \text{ is decreasing in } x_2$$

$$\pi_3(x_1, x_2^*, x_1-) > \pi_3(x_1, x_2^*, x_2^*+) \text{ for } x_1^* < x_1 < x_2^*$$ (5)

$$\pi_3(x_1^*, x_2, x_2+) > \pi_3(x_1^*, x_2, x_1^*-) \text{ for } x_1^* < x_2 < x_2^*.$$

When F and G are symmetric, all of these (strict) conditions are satisfied; they continue to be satisfied when the asymmetries in F and G are small enough. (They guarantee, for example, that player 3 enters if player 1's position is less than x_1^*, deterring such a move by player 1.)

The proof of the Proposition can now be modified to cover the case in which F and G are slightly asymmetric. The first two paragraphs need no change. In the third paragraph ("$x_3 \in (x_1^*, x_2^*)$") the modified version of (3), rather than (3) itself, needs to be used. The next two paragraphs need no change. The conclusion of the next two paragraphs (the start of the argument for $\{x_1, x_2^*\}$ where $x_1 \in (x_1^*, x_2^*)$) now follows from (5). The arguments for the remainder of this case and the next case need no change. The argument for the first paragraph of the case $\{x_1, x_2^*\}$ where $x_1 < x_1^*$ also now follows from (5); the next paragraph is no longer needed, given the less specific form of player 3's strategy in the modified result.

The remainder of the proof for the equilibrium $\{\sigma^{*n}\}$ remains the same, except for the last argument, concerning the implications of player 1's entering at $x_1 < x_1^*$. In this case, player 3 either stays out or enters at the point

in (x_1, x_2^*) that maximizes her probability of winning. If she stays out then player 1's probability of winning is less than it is when player 1 enters at x_1^*. If she enters then her probability of winning is at least p_0, and, for sufficiently small asymmetries in F and G, player 1's and player 2's probabilities of winning are approximately equal, so that player 1's probability of winning is less than $\frac{1}{2}$.

The changes in the argument for the equilibrium $\{\hat{\sigma}^n\}$ are similar.

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Figure legends

Figure 1: The candidates' votes shares when their positions are x_1, x_2, and x_3, where $x_3 = x_1-$, $x_1 < x_2$, and $x_2 - x_1$ is large enough.

Figure 2: The positions x_1^* and x_2^*. The points t_1 and t_2 are the tertiles of $F_{G^{-1}(p_0)}$: $t_i = F_{G^{-1}(p_0)}(i/3)$. The areas A and B under $f_{G^{-1}(p_0)}$ delimited by the dashed lines are equal.

Figure 3: Candidate 2, at x_2, wins for all values of α between $\underline{\alpha}$ and $\overline{\alpha}$, where $\underline{\alpha}$ is determined by the condition that the shaded areas are equal and $\overline{\alpha}$ is determined by a symmetric condition. The thick vertical lines indicate the midpoints of $[x_1, x_2]$ and $[x_2, x_3]$.