Correction to "A model of political competition with citizen-candidates"

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Proposition 8 (p. 81) of Osborne and Slivinski (1996) asserts that if $b \neq 4c$ then in all three-candidate equilibria under a runoff system in which the positions of all three candidates are not the same, each candidate obtains one-third of the votes on the first round. Benjamin Solow has pointed out to us that this result is incorrect. The configurations described in the proposition are equilibria, but for some distributions F of the citizens' favorite positions the model has also three-candidate equilibria in which the first-round vote shares of two candidates, say i and j, are equal, and the vote share of the remaining candidate, say k, is larger. In such equilibria, the second round is between i and k with probability $\frac{1}{2}$ and between j and k with probability $\frac{1}{2}$. Solow provides an explicit example: if F is uniform on $[0,1]$ then the game has an equilibrium in which the candidates' positions are 0.2, 0.4, and 0.8 if $b \geq 4c + 0.2$ and $c \geq 0.1$. The vote shares on the first round are 0.3, 0.3, and 0.4, so the second round is between the candidates at 0.2 and 0.8 with probability $\frac{1}{2}$ and between the ones at 0.4 and 0.8 with probability $\frac{1}{2}$. In the first case the second round is a tie and in the second case the candidate at 0.4 wins.

Part of the message of our original paper is that multicandidate equilibria are more likely under a runoff system than they are under plurality rule; the fact that additional multicandidate equilibria exist under a runoff system reinforces this message.

The error in our original argument occurs on page 81. Consider a three-candidate equilibrium. Suppose that the probability that candidate i wins the first round is zero, and consider the implications of candidate i's withdrawal. If, when i is present, a candidate wins the first round with more than half of the votes, then she does so also after i withdraws. Otherwise, the candidates on the second round are the two candidates other than i, regardless of whether i participates in the first round. Thus i's withdrawal does not affect the ultimate winner. Consequently, in any three-candidate equilibrium each candidate's probability of winning the first round must be positive. But this requirement does *not* imply, contrary to our claim on page 81 of the paper, that all three candidates tie on the first round. Rather, *either* they all tie *or* two of them tie and the third obtains a larger vote share. The latter possibility admits equilibria not considered by Proposition 8 in our paper. Here is a corrected version of the result.

Proposition 8 (three candidate equilibria under a runoff system)**.** *In all three-candidate equilibria in which not all the candidates' positions are the same, these positions are equal either to*

$$a_1 = m + t_1 - t_2,\ a_2 = t_1 + t_2 - m,\ \textit{and}\ a_3 = t_2 + m - t_1, \tag{1}$$

where $t_j = F^{-1}(j/3)$ *for* $j = 1, 2$*, or to*

$$a_1 = m - d,\ a_2 = m + e,\ \textit{and}\ a_3 = m + d \tag{2}$$

for some $d > 0$ *and* $|e| \le d$*. In the second case, either*

$$F(\tfrac{1}{2}(a_1 + a_2)) = F(\tfrac{1}{2}(a_2 + a_3)) - F(\tfrac{1}{2}(a_1 + a_2)) < 1 - F(\tfrac{1}{2}(a_2 + a_3)) \le \tfrac{1}{2} \tag{3}$$

or

$$1 - F(\tfrac{1}{2}(a_2 + a_3)) = F(\tfrac{1}{2}(a_2 + a_3)) - F(\tfrac{1}{2}(a_1 + a_2)) < F(\tfrac{1}{2}(a_1 + a_2)) \le \tfrac{1}{2} \tag{4}$$

if $|e| < d$*.*

In the first type of equilibrium each candidate obtains one third of the votes on the first ballot. In the second type of equilibrium, on the first ballot if (3) then candidate 3 obtains the most votes and candidates 1 and 2 tie for second place, and if (4) then candidate 1 obtains the most votes and candidates 2 and 3 tie for second place. A necessary condition for the first type of equilibrium is $b \ge 6c$ *and a necessary condition for the second type of equilibrium is* $b \ge 4c$ *if* $|e| < d$ *and* $b = 4c$ *if* $|e| = d$*.*

Proof. For each candidate to have a positive probability of winning the first round, either the vote shares on the first round are all $\frac{1}{3}$, or two of them are equal and the third is larger, but not more than $\frac{1}{2}$ (otherwise there is no second round, and both of the other players are better off withdrawing). The argument for the first case is given in the paper.

Consider the second case. Number the candidates so that candidate 1 has the most votes on the first round. Then with probability $\frac{1}{2}$ the second round is between candidates 1 and 2 and with probability $\frac{1}{2}$ it is between candidates 1 and 3. We cannot have $a_2 < a_1 < a_3$ or $a_3 < a_1 < a_2$, because then candidate 1 wins the second round against each of the other candidates. So $a_1 < a_2 < a_3$ or $a_3 < a_2 < a_1$. In each case, candidate 2 wins the second round against candidate 1, so for both candidate 1 and candidate 3 to have a positive probability of winning the second round, they must tie on that round, so that $\frac{1}{2}(a_1 + a_3) = m$. For the first-round vote shares of candidates 2 and 3 to be the same and for candidate 1's vote share to be at most $\frac{1}{2}$, the candidates' positions have to satisfy the conditions in (2).

The cases $a_1 < a_2 = a_3$ and $a_2 = a_3 < a_1$ ($|e| = d$) are covered by the argument in the first paragraph of the proof in the paper.

Suppose that $a_1 < a_2 < a_3$ (the case $a_3 < a_2 < a_1$ is symmetric). With probability $\frac{1}{2}$ the second round is between candidates 1 and 2, and candidate 2 wins, and with probability $\frac{1}{2}$ the second round is between candidates 1 and 3, who tie. Thus the ultimate winner is candidate 1 with probability $\frac{1}{4}$, candidate 2 with probability $\frac{1}{2}$, and candidate 3 with probability $\frac{1}{4}$. Hence candidate 3's payoff in such an equilibrium is

$$-\tfrac{1}{4}(a_3 - a_1) - \tfrac{1}{2}(a_3 - a_2) + \tfrac{1}{4}b - c.$$

If she withdraws, then candidate 2 wins outright, so that candidate 3's payoff is

$$-(a_3 - a_2).$$

Thus for candidate 3 to not want to withdraw, we need the first payoff to be at least the second, which implies that

$$b \geq 4c - a_1 + 2a_2 - a_3.$$

A similar calculation for candidate 1 leads to the condition

$$b \geq 4c + a_1 - 2a_2 + a_3.$$

Thus a necessary condition for an equilibrium of this type is $b \geq 4c$. □

References

Osborne, Martin J. and Al Slivinski (1996), "A model of political competition with citizen-candidates." *Quarterly Journal of Economics*, 111, 65–96.