Draft of

MODELS IN POLITICAL ECONOMY

Collective choice, voting, elections, bargaining, and rebellion

Version 2025.4.22

Latest version available at
https://www.economics.utoronto.ca/osborne/mpe

Comments are welcome. If you notice errors or have suggestions for improvements, including the addition of new material, please send them to `martin.j.osborne@gmail.com`.

Martin J. Osborne
University of Toronto

Version: 2025.4.22

Contents

Preface vii

Part I Collective choice 1

1 Collective choice with known preferences 3
- 1.1 Collective choice rules 5
- 1.2 Anonymity and neutrality 11
- 1.3 Two alternatives: majority rule 13
- 1.4 Three or more alternatives: Condorcet winners 19
- 1.5 Two domains with Condorcet winners 29
- 1.6 Condorcet winners for two-dimensional sets of alternatives 39
- 1.7 Preference aggregation: Arrow's theorem 49
- 1.8 Preference intensities and interpersonal comparisons 58
- Notes 67
- Solutions to exercises 69

2 Collective choice with privately-known preferences 77
- 2.1 Strategy-proofness of strict Condorcet winner 79
- 2.2 Strategy-proofness for domain of all preference profiles 86
- 2.3 General mechanisms 90
- Notes 93
- Solutions to exercises 93

Part II Voting 97

3 Voting with two alternatives 99
- 3.1 Voting as a strategic game 101
- 3.2 Costly voting with uncertainty about cost 104
- 3.3 The incentive to vote in a large population 118
- 3.4 Preferences with regret 121
- Notes 123
- Solutions to exercises 124

4 Voting with many alternatives: plurality rule 129
4.1 Plurality rule voting games 130
4.2 Spatial model with concave payoff functions 138
4.3 Strategic and sincere voting: divided majority 139
Notes 149
Solutions to exercises 150

5 Sequential pairwise voting 155
5.1 Binary agendas 157
5.2 Successive agendas 168
5.3 Amendment agendas 171
5.4 Agenda formation 179
5.5 Optimistically sincere strategies, single-peaked preferences, convex agendas 180
Notes 186
Solutions to exercises 187

6 Ethical voting and expressive voting 191
6.1 Ethical voting 192
6.2 Expressive voting 197
Notes 199

7 Voting with asymmetric information 201
7.1 Strategic abstention 203
7.2 Unanimity rule 226
7.3 Majority rule with diverse preferences: an example 233
Notes 234
Solutions to exercises 235

Part III Electoral competition 241

8 Electoral competition: two office-motivated candidates 243
8.1 General model 247
8.2 Spatial model 253
8.3 Candidates uncertain of citizens' preferences 258
8.4 Costly voting 266
8.5 Citizens with preferences over candidates 274
8.6 Electing a legislature 282
8.7 Interpreting Nash equilibrium 287
Notes 288
Solutions to exercises 288

9 Electoral competition: two policy-motivated candidates 295
9.1 Basic model 297
9.2 Uncertain median 308
9.3 Candidates privately informed about policies 314
9.4 Repeated elections 319
Notes 323
Solutions to exercises 323

10 Electoral competition: endogenous candidates 331
10.1 Simultaneous entry with sincere voting 333
10.2 Simultaneous entry with strategic voting 336
10.3 Sequential entry 344
10.4 Citizen-candidates 350
Notes 365
Solutions to exercises 366

11 Distributive politics 375
11.1 Two-candidate competition with exogenous incomes under uncertainty 377
11.2 Voting over transfer systems when income is endogenous 384
11.3 Voting over linear transfer systems 392
11.4 Coalitional bargaining over redistribution 400
Notes 410
Solutions to exercises 411

12 Money in electoral competition 413
12.1 Mobilizing citizens to vote 415
12.2 Informing citizens of candidates' qualities 428
Notes 437
Solutions to exercises 437

13 Two-period electoral competition with imperfect information 439
13.1 Selecting and controlling incumbents with unknown preferences 441
13.2 Inducing an incumbent to exert effort 447
13.3 Candidates who pander to voters 449
Notes 463

Part IV Exercising power 465

14 Bargaining 467

14.1 Bargaining game with voting 469
14.2 Recurrent distributive bargaining game with voting 496
Notes 501
Solutions to exercises 501

15 Rulers threatened by rebellion 503

15.1 Threat of revolt 504
15.2 Coordinating rebellion 517
Notes 533
Solutions to exercises 533

Part V Appendix 537

16 Preferences, profiles, games, and optimization 539

16.1 Preferences and payoffs 539
16.2 Sets of individuals and profiles 544
16.3 Brouwer's fixed-point theorem 545
16.4 Strategic games 545
16.5 Bayesian games 549
16.6 Extensive games 552
16.7 Coalitional games 562
16.8 Optimization 566

References 569

Index 585

Preface

Collective choice, voting, electoral competition, and rebellion. I present some of the main formal models in these areas, which are central to the field of political economy. I focus on a few models, making no attempt to survey related work. I include full proofs of almost all the results I state (because I fully appreciate a result only when I understand its proof).

Political economy concerns many topics beside these four, and uses various methodologies other than the formal modeling. I cover only some models in a few domains of the field.

The models I present are ones that I find appealing—they elegantly express original ideas and help me organize my thoughts about aspects of the world. One way in which they do so is by drawing out common threads in disparate situations. For example, the model of collective decision-making in Chapter 1 highlights the elements common to the problems of the residents of a country choosing a national health policy and a group of friends choosing a restaurant for dinner. In linking situations in this way, the model makes me think that I better understand some aspect of the world, although the precise nature of that improvement is often hard to pinpoint.

In models, as in many other spheres, tastes differ. I prefer relatively general models over ones that assume specific functional forms, and am wary of refinements of the basic solution concepts of game theory, Nash equilibrium and subgame perfect equilibrium. I find models that assume specific functional forms unsatisfying because they leave open the possibility that their properties depend on the forms, and models that consider only a subset of equilibria unsatisfying because they leave open the possibility that other equilibria do not have the same properties. But in both cases, the dividing line is fuzzy. After all, every model is an example of a more general model. In some of the models I discuss, the decision-makers' payoff functions are linear, which is certainly a specific functional form, and the players are assumed not to use weakly dominated actions, a refinement of the standard notions of equilibrium.

One perspective is that the models I present, in concentrating on the surface phenomena of elections and votes, miss the underlying forces that determine political outcomes: power, class, and wealth. I am sympathetic with this view, perhaps more so than many other practitioners of the theories I discuss. Money and power make brief appearances in Chapters 12, 14, and 15, but for the most part the models assume that individuals interact as equals, in isolation from each other. When relating the models to political reality, it is wise to bear in mind that the factors they capture are only part of the story, and not necessarily the most significant part. For example, while formulating the problem of the residents of a country choosing a national health policy as a collective choice problem brings out features shared with the problem of a group of friends choosing a restaurant for dinner, other features of the problem, relating to power, class, and wealth, may be key to understanding it.

Overview

The subject matter of the book is outlined in the panels on the next few pages. These panels list the main results in each chapter. With a few exceptions, only results stated formally as propositions in the body of the book are listed. Several chapters include, in addition, sections that analyze models without stating formal results.

Part I presents the general theory of collective choice. A group of individuals whose members disagree about the desirability of the options it faces has to select one of the alternatives. Do methods for doing so exist for which the selected alternative depends sensibly on the individuals' preferences?

Part II explores methods of collective choice based on voting. The models include ones in which the voters differ in their preferences, and ones in which they differ in their information; ones in which all the alternatives are considered at once, and ones in which pairs of alternative are considered sequentially; and ones in which individuals are motivated by the possibility that their vote will bring about an outcome from which they personally gain, and ones in which they have other motivations.

Part III considers models in which citizens elect representatives who select alternatives on their behalf. The representatives may be motivated by the desire to hold office or by the opportunity to select an alternative, and may be modeled as distinct from the set of citizens or as drawn from that set. The models may be used to study redistributive policies, may be adapted to study the influence of an interest group, and may be extended to analyze sequences of elections.

Part IV studies models of bargaining and regime change induced by the threat of rebellion.

Format and conventions

The formal content of the book is contained in definitions and propositions. The text is intended to make this content digestible, but the definitions and proposi-

I. COLLECTIVE CHOICE

1. Collective choice with known preferences

- Collective choice rule maps preference profile to set of alternatives
- Any attractive collective choice rules?
  - 2 alternatives: anonymous, neutral, positively responsive ⟺ majority rule
  - ≥ 3 alternatives, problems with strict Condorcet winner: anonymous, neutral, positively responsive, Nash independent ⟺ strict Condorcet winner
  - ≥ 3 alternatives, general problems: no anonymous, neutral, positively responsive, Nash independent rule exists
- Single-peaked preferences ⇒ strict Condorcet winner is median favorite alternative
- Single-crossing preferences ⇒ strict Condorcet winner is favorite alternative of median individual
- Alternatives are points in the plane: city-block preferences ⇒ Condorcet winners are component-wise medians of favorite alternatives, Euclidean preferences ⇒ generally no Condorcet winner
- Preference aggregation function maps preference profile to preference relation over set of alternatives
- Any attractive preference aggregation functions?
  If ≥ 3 alternatives, ranking of any x and y depends only on individuals' preferences between x and y, and x is ranked above y whenever all individuals prefer x to y,
  - for domain of all collective choice problems, function must be dictatorial
  - for domain of collective choice problems for which every subset of alternatives has a strict Condorcet winner, majority relation satisfies conditions
- If we can compare individuals' welfares,
  - equity axiom ⇒ leximin ordering (which ranks profiles according to welfare of worse-off individual)
  - welfare differences/ratios but not levels can be compared across individuals ⇒ utilitarian/Nash ordering

2. Collective choice with privately-known preferences

- Collective choice function maps reported preference profile to alternative
- Any reasonable collective choice function that induces truthful reporting?
  - Every preference profile has strict Condorcet winner ⇒ yes
  - Generally ⇒ no: any such function that selects x when x is every individual's favorite alternative is dictatorial
  - Cannot do better with collective choice function that induces non-truthful reporting
- If individuals' preferences single-peaked, in game in which each individual reports an alternative and collective choice function assigns median reported alternative, reporting favorite alternative weakly dominant

---

Dominant = best regardless of others' actions
Condorcet winner = beats every other alternative one-on-one

II. Voting

3. Voting with two alternatives

- Dominant strategy to vote for favorite alternative
- Uncertain cost of voting ⇒ equilibrium in which each individual votes only if cost at most some threshold
- Large population ⇒ few individuals vote (vote unlikely to affect outcome)
- Individuals minimize worst regret ⇒ high turnout possible in large population

4. Voting with many alternatives: plurality rule

- Voting for least-favored alternative is dominated, but not voting for others
- Equilibrium exists in which no individual's vote is weakly dominated
- Set of alternatives is interval of numbers, payoff functions strictly concave ⇒ at most two alternatives tie for winner, every alternative possible in equilibrium
- Divided majority, large population: coordination in equilibrium to defeat minority-favored alternative depends on nature of uncertainty

5. Sequential pairwise voting

- Possible outcomes of undominated voting = top cycle set (alternatives that beat or indirectly beat every other alternative)
- If Condorcet winner exists, it is sole member of top cycle set
- If procedure is series of votes that successively eliminate alternatives, possible outcomes of undominated voting = Banks set ⊆ top cycle set (x in Banks set ⇔ x beats all members of some sequence of alternatives, each member of sequence beats all that follow, and no alternative beats x and all members of sequence)

6. Ethical voting and expressive voting

- Individual may choose voting rule that would generate best outcome for society if everyone like her used it
- Or they may vote to express their preferences rather than to affect outcome
- In both cases, turnout does not depend on size of electorate

7. Voting with shared values and asymmetric information

- Individuals agree on best outcome given state, but some do not know state
- Plurality rule:
  - In equilibrium, partisans and informed individuals vote, and uninformed individuals vote in numbers sufficient to minimize impact of partisans' votes
  - If signals vary in quality, individual votes if her signal quality is high enough
- Unanimity rule:
  - Outcome a unless everyone votes for b and every individual either perfectly informed or uninformed ⇒ all uninformed individuals vote for b
  - Signals vary in quality and number of individuals is large ⇒ each individual's voting for alternative likely to be best according to her signal is not equilibrium; game has mixed strategy equilibrium

III. Electoral competition

8. Electoral competition: two office-motivated candidates

- Nash equilibrium with simultaneous moves: each candidate's position is Condorcet winner (and hence median favorite position/favorite position of median individual for single-peaked/single-crossing preferences)
- Subgame perfect equilibrium with sequential moves: position of each candidate is Condorcet winner if one exists, else second-mover wins
- Uncertain median: Nash equilibrium ⇒ each candidate chooses median of distribution of median
- Candidates privately-informed about median: Nash equilibrium position of candidate with signal t is median of distribution when both signals are t
- Costly voting: if citizen's payoff function is concave, position of each candidate maximizes payoff of citizens with voting cost zero
- Citizens have preferences over candidates about which candidates are uncertain: if payoffs concave, distributions of citizens' net biases same, and Nash equilibrium exists, position of each candidate maximizes sum of citizens' base payoffs

9. Electoral competition: two policy-motivated candidates

- If strict Condorcet winner exists, Nash equilibrium in which both candidates choose it
- If there exists candidate who prefers x to y whenever majority prefers x to y, in Nash equilibrium with positions same, common position is Condorcet winner.
- Uncertain median: candidates' Nash equilibrium positions differ; equilibrium exists if candidates' payoff functions concave, distribution of median log concave
- Repeated elections: outcomes different from median of citizens' favorite positions possible if payoffs convex or incumbents constrained by previous policies

10. Electoral competition: endogenous candidates

- Many politicians choose whether to run and if so positions to take:
  - citizens vote sincerely ⇒ no equilibrium for most distributions of citizens' favorite positions
  - citizens vote strategically, politicians' payoff functions strictly concave, and citizens' actions undominated ⇒ every candidate's equilibrium position = median of citizens' favorite positions
  - politicians act sequentially: 2 politicians ⇒ both enter at median; 3 politicians ⇒ first and last enter at median, second stays out
- Politicians = citizens (citizen-candidates):
  - one-candidate equilibria with position median or close to median
  - sincere voting, $< \frac{1}{3}$ of citizens with favorite position m ⇒ no equilibrium with more than 1 candidate at median
  - strategic voting, large win payoff ⇒ equilibrium with many candidates at median
  - two-candidate equilibria with positions that differ ⇒ positions symmetric about median; such equilibria exist under some conditions

III. Electoral competition, continued

11. Distributive politics

- Electoral competition in which position is distribution of wealth, citizens have preferences over candidates about which candidates are uncertain, payoffs concave, distributions of citizens' net biases same, and Nash equilibrium exists ⇒ candidates propose same distribution; in examples, swing voters and citizens' whose votes more likely to be pivotal are assigned more wealth
- Individuals differ in earning power and choose hours of work
  - Collective choice problem in which alternatives are individuals' favorite tax-subsidy schemes
    - individuals care only about consumption ⇒ no Condorcet winner
    - individuals care about consumption and hours of work ⇒ for some parameters, favorite scheme of individual with median earning power is Condorcet winner
  - Collective choice problem in which alternatives are finitely many linear tax-subsidy schemes and individuals are ordered by pre-tax income independently of tax-subsidy scheme ⇒ single-crossing preferences, so favorite alternative of median individual is strict Condorcet winner
- Tax-subsidy scheme outcome of society-wide bargaining when any majority can expropriate and any minority can withhold ⇒ 50% tax rate, revenue shared equally for Shapley value and for dissatisfaction-minimizing distribution

12. Money in electoral competition

- Interest group that can mobilize voters can move outcome towards its favored alternative, more so if it can target mobilization according to individuals' preferences
- Interest group with preferences contrary to voter's that can inform voter about candidate quality in exchange for candidate taking position group favors ⇒ voter is worse off if uninformed voter believes candidate quality is likely to be high

13. Two-period electoral competition with imperfect information

- Incumbent of unknown quality chooses policy, citizen with limited information observes outcome and either reelects incumbent or elects challenger
  - Citizen does not know incumbent's preferences and incumbent's payoff to holding office high ⇒ equilibrium in which all incumbents choose policy citizen likes
  - All incumbents' preferences conflict with citizen's, effort is required by incumbent to produce outcome citizen likes, and incumbent's payoff to holding office high ⇒ equilibria in which incumbent exerts low effort and in which she exerts high effort and is reelected only if outcome is good for citizen
  - Citizen uncertain about policy best for her, which depends on state incumbent knows and incumbent's payoff to holding office high ⇒ some incumbents choose policy citizen believes is best for her, even if incumbents know it is not; policy chosen by random citizen may be better for citizen

IV. Exercising power

14. Bargaining

- Individual selected randomly to propose alternative, all individuals vote for or against; if majority in favor, proposal is implemented and game ends, else procedure is repeated until a proposal is accepted by a majority
  - Every alternative is outcome of subgame perfect equilibrium
  - If individuals' votes are observable, most alternatives are outcomes of subgame perfect equilibrium with undominated voting
  - If only outcomes, not individuals' votes, are observable and individuals are sufficiently patient
    - ∗ if alternatives are distributions of fixed amount of a good, every alternative is outcome of subgame perfect equilibrium with undominated voting
    - ∗ if set of alternatives is interval of numbers and individuals' preferences are single-peaked, outcome of subgame perfect equilibrium with undominated voting is close to median of individuals' favorite alternatives
  - If alternatives are distributions of fixed amount of a good, unique stationary equilibrium outcome
- Individual selected randomly to propose distribution of fixed amount of good, all individuals vote for or against; if majority in favor, proposal is implemented in period, else status quo is implemented. In both cases, procedure is repeated, with status quo in each period equal to previous period's outcome.
  - Examples in which almost any distribution is outcome of stationary subgame perfect equilibrium, including distributions in which some of the good is wasted

15. Rulers threatened by rebellion

- Rich initially controls distribution of wealth; in some periods environment is favorable to successful revolt by Poor, which destroys some wealth and makes remainder permanently available to Poor: favorable environment rare ⇒ subgame perfect equilibrium in which Rich hands control of distribution permanently to Poor ("democratizes") when favorable environment occurs
- Members of Poor need to coordinate for revolt to succeed: there exists equilibrium in which they coordinate perfectly if they observe dictator's action, but not if they do not observe that action

tions are designed to be entirely self-contained: if you read nothing else you will miss only discussion and motivation, not any formal content.

In the electronic version of the book, every term in the boxes containing the formal content that has a technical meaning, other than basic mathematical terms, is hyperlinked to its definition. If you click on the hyperlink you are taken to the definition; your pdf viewer probably allows you to return to where you were by pressing `Backspace` or `Alt+left arrow`. My intention is that these hyperlinks allow you to read any definition or result independently of the other material.

The structure of many of the definitions resembles that of the first one (1.1), "A *society* $\langle N, X\rangle$ consists of a set N (of *individuals*) and a set X (of *alternatives*)." This sentence is a short form of the logically superior statement "A *society* consists of two sets, one whose members are interpreted to be individuals and one whose members are interpreted to be alternatives. I denote by $\langle N, X\rangle$ a society in which the set of individuals is N and the set of alternatives is X."

With a few exceptions, the names I attach to concepts, models, and results relate to their content rather than the people who originated them, even in cases in which the names of the originators are commonly used by researchers in the field. Names that relate to the originators are convenient shorthands for the cognoscenti, but are unhelpful for the uninitiated or for those of us who are memory-challenged. Further, many models have mixed and unclear parentage, and in those cases assigning one name implies a misplaced certainty regarding their origin.

Acknowledgments

For comments and discussions I am grateful to many people, listed here in a random order: Nageeb Ali, Francesco De Sinopoli, John Duggan, Richard van Weelden, Ariel Rubinstein, Jeffrey S. Rosenthal, Dominik Peters, Helios Herrera, Jean-Pierre Benoît, Colin Stewart, Avidit Acharya, Peter Buisseret, Haruo Imai, Yoram Halevy, Carolyn Pitchik, Daniel Bernhardt, Sean Horan, Hervé Moulin, John Weymark, Jean Guillaume Forand, Craig Brett, Alejandro Saporiti, Navin Kartik, Matthew Turner, Jon Eguia, Ruolong Xiao, Remzi Sanver, and Huseyin Yildirim.

I am particularly grateful to Sean Horan, Carolyn Pitchik, and Ariel Rubinstein, each of whom made detailed comments on several chapters of a draft of the book, which led to big improvements.

Jon Eguia read the entire book and provided countless comments. He pointed out errors, made suggestions for removing and adding material, and showed me how to improve the exposition. His advice led me make major improvements in every chapter. I am enormously grateful for the many hours he spent with my drafts.

My discussions with Ariel Rubinstein over the many years of our collaboration have been enormously influential in my thinking about economic modeling and the presentation of economic theory.

I am grateful to Hsin-Po Wang (StackExchange user `Symbol1`) for technical help with the ternary plots in Chapters 4 and 14.

During the long gestation period for this book, I had the pleasure of working on it at various locations. I am grateful for the hospitality I was generously

afforded during my visits to the Kyoto Institute of Economic Research, the Department of Economics at the National University of Singapore, the School of Economics at the University of New South Wales, the College of Administrative Sciences and Economics at Koç University, and the Centro de Economía Aplicada at the University of Chile.

I cite the sources of the models that I discuss in the "Notes" section at the end of each chapter. In addition, I found Moulin (1988), Austen-Smith and Banks (1999, 2005), and Mueller (2003), as well as unpublished notes generously made available by John Duggan and by Stephen Coate, particularly helpful in understanding and appreciating work in the field.

MODELS IN POLITICAL ECONOMY

I Collective choice

1 Collective choice with known preferences 3
2 Collective choice with privately-known preferences 77

1 Collective choice with known preferences

1.1 Collective choice rules 5
1.2 Anonymity and neutrality 11
1.3 Two alternatives: majority rule 13
1.4 Three or more alternatives: Condorcet winners 19
1.5 Two domains with Condorcet winners 29
1.6 Condorcet winners for two-dimensional sets of alternatives 39
1.7 Preference aggregation: Arrow's theorem 49
1.8 Preference intensities and interpersonal comparisons 58

The residents of a country have to choose a national health policy. The members of an organization have to choose a board of directors. A group of friends has to choose a restaurant for dinner. These problems all involve collective choice: in each case, a group of people whose members may disagree about the desirability of the alternatives has to select a common action.

The study of collective choice lies at the heart of political economy. One line of inquiry analyzes the properties of specific mechanisms for choosing an action. This chapter discusses a more ambitious avenue that involves formulating a list of properties that appear to be desirable and determining the mechanisms that satisfy these properties.

What do we mean by a mechanism? In this chapter I assume that the individuals' preferences are known; a mechanism takes these preferences and the set of alternatives as inputs and generates a subset of the alternatives as output. (Ideally, this subset consists of a single alternative.) In the next chapter, I consider models in which the individuals' preferences are not known.

What can we know about the individuals' preferences? In Sections 1.1 through 1.7 I present models that include information only about the individuals' rankings of alternatives. In Section 1.8 I present models that include information about the intensity of each individual's preference for one alternative rather than another, about the well-being each individual derives from each alternative, and about the differences in these well-beings among individuals.

Synopsis

A collective choice problem consists of a set of individuals, a set of alternatives, and a specification of the individuals' preferences over the alternatives. Proposition 1.1, known as May's theorem, shows that for a collective choice problem with two alternatives, the mechanism that selects the alternative favored by a majority of individuals is the only one for which the outcome does not depend on the names of the individuals (anonymity) or the names of the alternatives (neutrality) and responds sensibly when the individuals' preferences change (positive responsiveness).

For collective choice problems with three or more alternatives, the existence of mechanisms for selecting alternatives with desirable properties depends on the nature of the individuals' preferences. Suppose that for a given problem there exists an alternative x such that, for every other alternative y, a majority of individuals prefer x to y. Such an alternative is called the strict Condorcet winner of the problem. Proposition 1.2 shows that if every member of a set of problems has a strict Condorcet winner then the mechanism that selects that alternative for each problem in the set is the only one that satisfies properties similar to, though less obviously compelling than, the properties in May's theorem.

Under what conditions does a collective choice problem have a strict Condorcet winner? One such condition is that the alternatives may be ordered so that every individual's preferences are single-peaked: as we move through the ordering, every individual initially becomes better off, and then, after we pass her favorite alternative, worse off. Proposition 1.4 shows that for a problem that satisfies this condition, the median of the individuals' favorite alternatives is the strict Condorcet winner if the number of individuals is odd. Another condition under which a collective choice problem has a strict Condorcet winner is that the individuals' preferences satisfy the single-crossing condition: the individuals may be ordered so that for every individual i and any alternatives x and y, if i likes x at least as much as y then either (*a*) all individuals who precede i in the ordering or (*b*) all individuals who follow i in the ordering prefer x to y. Proposition 1.5 shows that for a problem that satisfies this condition, if each median individual with respect to the ordering of individuals has a unique favorite alternative then each such alternative is a Condorcet winner of the problem. Further, if the number of individuals is odd then for any alternatives x and y, the (unique) median individual prefers x to y if and only if a majority of individuals do so, and hence in particular the favorite alternative of the median individual is the strict

Condorcet winner.

For sets of problems that are more than slightly larger than the set of all problems with a strict Condorcet winner, the conclusion is negative. Proposition 1.3 shows that for any set of problems that includes all problems with a strict Condorcet winner plus all problems that would have a strict Condorcet winner if the preferences of a single individual were changed in a certain way, no mechanism for selecting an alternative satisfies the properties in Proposition 1.2.

Section 1.7 presents a different approach. Instead of considering mechanisms for selecting alternatives in collective choice problems, it studies the problem of aggregating the individuals' preferences. The objective is to find a single (societal) preference relation that reasonably reflects the individuals' preferences. One motivation for finding such a preference relation is that we do not know the collective choice problem the society will face, and we want to be prepared for whatever problem arises. Proposition 1.9 (Arrow's impossibility theorem) shows that no mechanism for constructing a societal preference relation from the individuals' preference relations satisfies three appealing properties.

The model in Section 1.8 includes information about each individual's welfare for each alternative, not merely her preferences. A social welfare ordering ranks welfare profiles. Three examples are the utilitarian ordering, which ranks profiles according to their sum, the Nash ordering, which ranks positive profiles according to their product, and the leximin ordering, which ranks profiles according to the welfare of the worst-off individual. All three of these orderings are anonymous and increase the ranking of a welfare profile when all individuals' welfares increase. Propositions 1.11, 1.12, and 1.13 show the implications of adding one more requirement. Proposition 1.11 shows that the leximin ordering results when the additional requirement entails a particular form of equity. Proposition 1.12 shows that the utilitarian ordering results when the additional requirement is that the welfare index is invariant to transformations of the individuals' welfares that preserve the rankings of welfare differences but not necessarily those of welfare levels. Proposition 1.13 shows that the Nash ordering results when the additional requirement is that the welfare index is invariant to transformations of the individuals' welfares that preserve the rankings of welfare ratios.

1.1 Collective choice rules

A society consists of a set of individuals and a set of alternatives.

Definition 1.1: Society

A *society* $\langle N, X\rangle$ consists of a set N (of *individuals*) and a set X (of *alternatives*). The society $\langle N, X\rangle$ is *finite* if N and X are finite.

A collective choice problem consists of a society and, for each individual, a preference relation over the set of alternatives. The preference relation $\succsim_i$ of individual i models her preferences: for all alternatives x and y, we interpret $x \succsim_i y$ to mean that i likes x at least as much as y. For any preference relation $\succsim_i$ we define the binary relation $\succ_i$ by

$$x \succ_i y \quad \Leftrightarrow \quad x \succsim_i y \text{ and not } y \succsim_i x \tag{1.1}$$

and interpret $x \succ_i y$ to mean that i prefers x to y, and we define the binary relation $\sim_i$ by

$$x \sim_i y \quad \Leftrightarrow \quad x \succsim_i y \text{ and } y \succsim_i x \tag{1.2}$$

and interpret $x \sim_i y$ to mean that i is indifferent between x and y (that is, she likes them equally well). (See Section 16.1 for more discussion of preference relations.) I refer to an assignment of preference relations to the individuals as a preference profile.

Definition 1.2: Preference profile

A *preference profile* for a society $\langle N, X\rangle$ is a function that associates with each individual (member of N) a preference relation on X. A preference profile is *strict* if every individual's preference relation is strict (i.e. no individual is indifferent between any two alternatives).

I denote by $(\succsim_i)_{i\in N}$ the preference profile in which the preference relation of each individual $i \in N$ is $\succsim_i$. I denote this profile also simply by $\succsim$; the absence of a subscript indicates that the symbol denotes a profile rather than a preference relation.

Definition 1.3: Collective choice problem

A *collective choice problem* $\langle N, X, \succsim\rangle$ consists of a society $\langle N, X\rangle$ for which both N and X have at least two members and a preference profile $\succsim$ for $\langle N, X\rangle$. The problem is *finite* if the society is finite.

Example 1.1: Collective choice problem

An example of a collective choice problem is $\langle\{1,2,3\},\{a,b,c\},\succsim\rangle$ where

$$a \succ_1 c \succ_1 b$$
$$b \succ_2 a \sim_2 c$$
$$a \sim_3 b \sim_3 c$$

(with $\succ_i$ and $\sim_i$ derived from $\succsim_i$ in (1.1) and (1.2)). Here is an attractive representation of this problem:

1	2	3
a	b	abc
c	ac	
b		

Each column shows the preference relation of the individual whose name heads the column. In each column, the alternatives are listed in order of preference, with the best at the top. Multiple alternatives in a cell indicate indifferences. For example, the middle column indicates that individual 2 likes b best and is indifferent between a and c.

A target of the analysis in this chapter is to specify, for each collective choice problem, alternatives that are reasonable compromises given the individuals' (possibly divergent) preferences. A function that specifies a set of alternatives for each collective choice problem is called a collective choice rule. I formulate properties for collective choice rules that seem to be desirable and look for rules that satisfy these properties.

To require that the properties hold for all collective choice problems is demanding. In some environments, some preference profiles are not plausible, and we may be content if the properties are satisfied for only a limited set of profiles. For example, if we are studying the choice of a political position from the set {*left*, *center*, *right*}, we might assume that *center* is not the worst alternative for any individual: an individual whose favorite position is *left* prefers *center* to *right*, and an individual whose favorite position is *right* prefers *center* to *left*. So in this environment it may be enough that a collective choice rule specifies outcomes for preference profiles in which *center* is not the worst alternative for any individual. To accommodate such cases, I allow a collective choice rule to apply to only a subset of the set of all collective choice problems. Following conventional terminology, I call such a set of collective choice problems a domain.

A domain that fits the example in the previous paragraph is the set of collective choice problems $\langle N, \{left, center, right\}, \succsim\rangle$ for which the preference relation $\succsim_i$ of each individual $i \in N$ satisfies *left* $\succsim_i$ *center* $\succsim_i$ *right*, *center* $\succsim_i$ *left* $\succsim_i$ *right*, *center* $\succsim_i$ *right* $\succsim_i$ *left*, or *right* $\succsim_i$ *center* $\succsim_i$ *left*. Another example of a domain consists of all problems $\langle N, X, \succsim\rangle$ for a given set N where X has three members. In this case the preference relation $\succsim_i$ of each individual $i \in N$ is one of the thirteen possible preference relations over a three-member set (the one for which all three alternatives are indifferent, the six for which exactly two alternatives are indifferent, and the six for which no two alternatives are indifferent). In some models, an appropriate domain is the set of collective choice problems for a given set of individuals in which every individual's preference relation is strict.

A collective choice rule intended to recommend the alternative to be chosen ideally specifies a single alternative for each collective choice problem. However, this requirement is incompatible with the requirement that the alternatives be treated symmetrically. Suppose, for example, that the number of individuals is even, the set of alternatives is $\{a, b\}$, and half of the individuals prefer a to b while the other half prefer b to a. Then if we treat the alternatives symmetrically we have to declare a tie between a and b. For this reason I define a collective choice rule for any given domain to specify a *set* of alternatives for each collective choice problem in the domain.

Definition 1.4: Collective choice rule

For any set D of collective choice problems, a *collective choice rule for D* is a function that associates with every collective choice problem $\langle N, X, \succsim\rangle$ in D a nonempty subset of X (the alternatives selected by the rule).

Perhaps the most well-known collective choice rule is plurality rule, which selects the alternative (or alternatives, in the case of a tie) that is ranked first by the largest number of individuals. To define this rule precisely, I first define an individual's favorite alternatives: the alternatives she likes at least as much as every other alternative.

Definition 1.5: Favorite alternatives

For any set X (of alternatives) and any preference relation $\succsim_i$ on X, the set of *favorite alternatives in X for $\succsim_i$* is

$$\{x \in X : x \succsim_i y \text{ for all } y \in X\}.$$

If, for example, $X = \{a, b, c\}$ and $a \sim_i b \succ_i c$, then the set of favorite alternatives in X for $\succsim_i$ is $\{a, b\}$. Note that if X has infinitely many members then the set of favorite alternatives for a preference relation on X may be empty. Let $X = [0, 1)$

and consider, for example, the preference relation $\succsim_i$ is defined by $x \succsim_i y$ if and only if $x \geq y$.

Plurality rule assigns to a collective choice problem the alternatives that are favorites of the largest number of individuals.[1]

Definition 1.6: Plurality rule

Let D be a set of collective choice problems $\langle N, X, \succsim \rangle$ for which N is finite and for each $i \in N$ the set X_i^* of favorite alternatives in X for $\succsim_i$ is nonempty. *Plurality rule* is the collective choice rule for D that assigns to each collective choice problem $\langle N, X, \succsim \rangle \in D$ the set

$$\{x \in X : |\{i \in N : x \in X_i^*\}| \geq |\{i \in N : y \in X_i^*\}| \text{ for all } y \in X\}.$$

For the collective choice problem in Example 1.1, plurality rule selects $\{a, b\}$, because a and b are both favorite alternatives of two individuals and c is a favorite alternative of only one individual.

The alternatives selected by the plurality rule collective choice rule depend only on the individuals' favorite alternatives. A rule that takes into account the individuals' preferences among the alternatives they rank below their favorite alternatives was proposed by Jean-Charles de Borda (1733–1799). This rule is defined only for collective choice problems in which all preference relations are strict (no individual is indifferent between any two alternatives). For each individual, it assigns to each alternative x a number of points equal to the number of alternatives the individual ranks lower than x. Then it chooses the alternative (or alternatives, in the case of a tie) for which the sum of the number of points over all individuals is largest.

Definition 1.7: Borda rule

Let D be a set of collective choice problems $\langle N, X, \succsim \rangle$ for which N is finite and the preference relation $\succsim_i$ of each individual $i \in N$ is strict. The *Borda rule* is the collective choice rule for D that assigns to each collective choice problem $\langle N, X, \succsim \rangle \in D$ the set of alternatives $x \in X$ that maximize

$$\sum_{i \in N} p_i(x),$$

where $p_i(x) = |\{z \in X : x \succ_i z\}|$ for each $i \in N$ and $x \in X$, the number of

[1] The name plurality rule is used also in a different context, for a voting mechanism in which each individual selects (votes for) one alternative (not necessarily her favorite), and the alternative selected by the most individuals wins. I analyze this voting mechanism in Chapters 3 and 4.

alternatives that $\succsim_i$ ranks below x.

Example 1.2: Borda rule

Consider the collective choice problem $\langle\{1,2,3\},\{a,b,c,d\},\succsim\rangle$ in which the individuals' preferences are given in the following table.

1	2	3
a	a	b
b	b	c
c	c	d
d	d	a

We have $p_1(a) = p_2(a) = 3$ and $p_3(a) = 0$, $p_1(b) = p_2(b) = 2$ and $p_3(b) = 3$, and $p_1(c) = p_2(c) = 1$ and $p_3(c) = 2$, and $p_1(d) = p_2(d) = 0$ and $p_3(d) = 1$, so the Borda rule selects $\{b\}$. By contrast, plurality rule selects $\{a\}$. The Borda rule takes into account that even though b is ranked first by only one individual, it is ranked second by the other two, whereas a is ranked last by the third individual.

The model of a collective choice problem includes only information about the individuals' preference rankings. One interpretation of the Borda rule is that it imbues these rankings with interpersonally-comparable cardinal significance: it treats each rung in each individual's ranking as equivalent to each rung in every other individual's ranking. If, for example, an alternative goes up one rung in one individual's preferences and down one rung in another's, the total number of points it receives remains the same. In some circumstances, using the individuals' preference rankings in this way seems inappropriate. For example, if, in the situation that the collective choice problem in Example 1.2 models, there is a meaningful scale on which individuals 1 and 2 regard b as much worse than a whereas individual 3 regards c, d, and a as only slightly worse than b, then $\{a\}$ may be a more reasonable choice for the group than $\{b\}$.

For a collective choice problem with k alternatives, the Borda rule assigns $k-1$ points to the top alternative in an individual's preferences, $k-2$ points to the next alternative, and so on. A *scoring rule* is a generalization of the Borda rule in which for some numbers $r^1 \geq r^2 \geq \cdots \geq r^k$, r^1 points are assigned to the top alternative in an individual's preference, r^2 points are assigned to the next alternative, as so forth. Like the Borda rule, such a rule may be interpreted as imbuing the individuals' preference rankings with interpersonally-comparable cardinal significance. It differs from the Borda rule in the weights it assigns to the rungs in the individuals' rankings. Note that a scoring rule for which r^1 is a positive number and $r^j = 0$ for $j = 2, \ldots, k$ is plurality rule.

Exercise 1.1: Scoring rules

Consider the collective choice problem $\langle N, \{a, b, c\}, \succsim \rangle$ for ten individuals in which one individual prefers a to b to c, four prefer b to a to c, three prefer c to a to b, and two prefer c to b to a. For each number $p \in (0, 1)$ find the alternatives selected by the scoring rule that assigns 1 point to the top alternative in each individual's preferences, p points to the middle alternative, and 0 points to the bottom alternative.

1.2 Anonymity and neutrality

Plurality rule and the Borda rule both treat the individuals symmetrically—every individual's preferences have the same influence on the outcome. They also treat the alternatives symmetrically—no alternative has any special significance. I now define these properties precisely.

First define a permutation on a finite set to be a one-to-one function from the set to itself, and a permutation of a profile to be a one-to-one reassignment of the elements in the profile.

Definition 1.8: Permutation

For any finite set Y, a *permutation on* Y is a one-to-one function from Y to Y. For any finite set N and profile $(x_i)_{i \in N}$, the profile $(y_i)_{i \in N}$ is a *permutation of* $(x_i)_{i \in N}$ if for some permutation π on N we have $y_i = x_{\pi(i)}$ for all $i \in N$.

One permutation on the set $N = \{1, 2, 3\}$, for example, is the function π for which $\pi(1) = 2$, $\pi(2) = 1$, and $\pi(3) = 3$, and the corresponding permutation of the profile (x_1, x_2, x_3) is the profile (x_2, x_1, x_3). There are five other permutations on the set $\{1, 2, 3\}$, including one that maps each member of the set to itself.

Now let N be a set of individuals, $\succsim$ a preference profile for N, and $\succsim'$ a permutation of $\succsim$. For example, for the collective choice problem $\langle \{1, 2, 3\}, \{a, b, c\}, \succsim \rangle$ in Example 1.1 and the permutation π on $\{1, 2, 3\}$ for which $\pi(1) = 2$, $\pi(2) = 1$, and $\pi(3) = 3$, the preference profile $\succsim'$ is

1	2	3
b	a	abc
ac	c	
	b	

A collective choice rule is anonymous if it assigns the same set of alternatives to the collective choice problems $\langle N, X, \succsim \rangle$ and $\langle N, X, \succsim' \rangle$ whenever $\succsim'$ is a permutation of $\succsim$. That is, the set of alternatives assigned by an anonymous collective choice rule depends only on the collection of preference relations, not on which individual has which preference relation.

Definition 1.9: Anonymous collective choice rule

Let D be a set of finite collective choice problems. A collective choice rule F for D is *anonymous* if for every collective choice problem $\langle N, X, \succsim \rangle \in D$ and every permutation $\succsim'$ of $\succsim$ for which $\langle N, X, \succsim' \rangle \in D$ we have $F(N, X, \succsim') = F(N, X, \succsim)$.

Plurality rule and the Borda rule are both anonymous. Dictatorship is decidedly not.

Example 1.3: Dictatorship

Let N be a set (of individuals). For any individual $i \in N$, *dictatorship by individual* i is the collective choice rule (for any set of collective choice problems) that selects for each collective choice problem $\langle N, X, \succsim \rangle$ the favorite alternatives in X for $\succsim_i$.

Although dictatorship does not treat the individuals equally, it treats the alternatives equally, as do plurality rule and the Borda rule. To be precise, let $\langle N, X, \succsim \rangle$ be a finite collective choice problem, let σ be a permutation on X, and consider the collective choice problem $\langle N, X, (\succsim_i^\sigma)_{i \in N} \rangle$ in which, for each $i \in N$, $\succsim_i^\sigma$ is the preference relation defined by

$$x \succsim_i y \text{ if and only if } \sigma(x) \succsim_i^\sigma \sigma(y).$$

That is, i's preference between x and y according to $\succsim_i$ is her preference between $\sigma(x)$ and $\sigma(y)$ according to $\succsim_i^\sigma$.

Consider, for example, the collective choice problem $\langle \{1, 2, 3\}, \{a, b, c\}, \succsim \rangle$ in Example 1.1. Define the permutation σ on X by $\sigma(a) = c$, $\sigma(b) = a$, and $\sigma(c) = b$. Then $(\succsim_i^\sigma)_{i \in N}$ is given by

1	2	3
c	a	abc
b	bc	
a		

A collective choice rule is neutral if for any permutation on the set of alternatives, the alternatives the rule selects for the permuted problem are the permutations of the alternatives it selects for the original problem. If, for example, $\{a\}$ is selected for the problem in Example 1.1, then $\sigma(a)$, namely c, is selected for the problem $\langle\{1,2,3\},\{a,b,c\},(\succsim_i^\sigma)_{i\in N}\rangle$ just defined.

Definition 1.10: Neutral collective choice rule

Let D be a set of finite collective choice problems. For every collective choice problem $\langle N,X,\succsim\rangle \in D$, every permutation σ on X, and every individual $i \in N$, define the preference relation $\succsim_i^\sigma$ by

$$x \succsim_i y \text{ if and only if } \sigma(x) \succsim_i^\sigma \sigma(y).$$

A collective choice rule F for D is *neutral* if for every collective choice problem $\langle N,X,\succsim\rangle \in D$ and every permutation σ on X for which $\langle N,X,\succsim^\sigma\rangle \in D$, we have

$$F(N,X,\succsim^\sigma) = \{x \in X : x = \sigma(z) \text{ for some } z \in F(N,X,\succsim)\}.$$

The set of alternatives selected by a neutral collective choice rule depends only on the individuals' preferences over the alternatives, not on the names of the alternatives. For example, if a neutral rule selects alternative a but not alternative b for some collective choice problem, then for the problem in which a and b are interchanged in the individuals' rankings, the rule selects b but not a. In particular, a neutral collective choice rule gives no significance to any given alternative, like the status quo, if one exists. The idea is that the priority of any given alternative is reflected in the individuals' preferences; no alternative has any special status independent of its rankings by the individuals.

Many collective choice rules, including ones that do not seem sensible, are anonymous and neutral. For example, the rule that selects the alternative ranked *lowest* by the largest number of individuals is anonymous and neutral. I now discuss additional properties that appear to be desirable. I first analyze the case of two alternatives, which turns out to differ significantly from that of three or more alternatives.

1.3 Two alternatives: majority rule

For collective choice problems in which the set of alternatives contains two members, many collective choice rules, including both plurality rule and the Borda rule, are equivalent to majority rule. That is, if one alternative is favored by more than 50% of the individuals, they select that alternative, and if both alternatives are favored by exactly 50% of the individuals, they select both alternatives, in a tie.

Definition 1.11: Majority rule

Majority rule is the collective choice rule for any set of finite collective choice problems with two alternatives that selects both alternatives (i.e. declares a tie) when each alternative is a favorite alternative of the same number of individuals, and otherwise selects the alternative that is a favorite alternative of a strict majority of individuals. An alternative with the latter property is a *strict majority winner*.

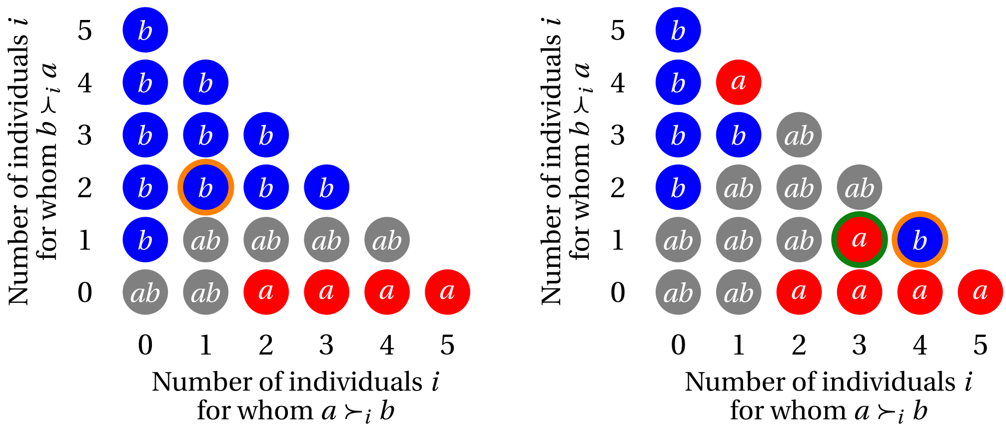
Note that if majority rule selects a single alternative, that alternative is not necessarily preferred to the other alternative by a majority of individuals. For example, if one individual prefers a to b and all the remaining individuals are indifferent between a and b then a is the alternative selected by majority rule.

One anonymous and neutral rule that is not equivalent to majority rule is minority rule, which selects the alternative favored by *fewer than* 50% of the individuals. What property distinguishes majority rule from minority rule?

We can represent an anonymous collective choice rule for the set of problems with two alternatives and no restrictions on the individuals' preferences in a diagram like those in Figure 1.1. These diagrams represent rules for problems with five individuals in which the alternatives are a and b. In each diagram, each disk represents a set of collective choice problems. (Ignore for the moment the letters inside the disks.) The disk at the point (x,y) represents the problems in which x individuals prefer a to b, y prefer b to a, and the remainder are indifferent between a and b. For example, the disk surrounded by an orange circle in Figure 1.1a represents problems in which one individual prefers a to b, two prefer b to a, and two are indifferent between a and b.

In each diagram, the letters inside the disks define an anonymous collective choice rule: they indicate the outcome selected by the rule for the problems the disk represents. For example, in Figure 1.1a the label ab on the disk at position $(1,0)$ indicates that the rule depicted assigns a tie to every problem in which one individual prefers a to b and the remainder are indifferent between a and b. For easy identification, disks labeled a are red, those labeled b are blue, and those labeled ab are gray.

The rule shown in Figure 1.1a is not neutral. Denote by $G(x,y)$ the set of alternatives assigned by the rule when x individuals prefer a to b, y prefer b to a, and the remainder are indifferent between a and b. For the rule to be neutral, we need $G(x,y)=\{a\}$ if and only if $G(y,x)=\{b\}$, and $G(x,y)=\{a,b\}$ if and only if $G(y,x)=\{a,b\}$. In particular, we need $G(x,x)=\{a,b\}$ for all values of x. Thus a rule is neutral if and only if the pattern of outcomes in a diagram like those



(a) A non-neutral rule. (b) A neutral rule.

Figure 1.1 Two anonymous collective choice rules for collective choice problems with five individuals and two alternatives, a and b. (The rules indicated are not intended to be sensible.)

in Figure 1.1 is symmetric about the main diagonal, so that in particular every outcome on the diagonal is $\{a, b\}$. The rule in Figure 1.1a is not neutral because, for example, it assigns b to problems in which two individuals prefer a to b, two prefer b to a, and the remaining individual is indifferent between a and b.

The rule given in Figure 1.1b is neutral, but it has an unattractive feature: as the disk circled in green indicates, when three individuals prefer a to b, one prefers b to a, and the remaining individual is indifferent between a and b, the outcome is a, but, as the disk circled in orange indicates, if the individual who is indifferent switches to preferring a then the outcome switches to b. That is, when *more* individuals favor a, the outcome moves *away* from a. (Symmetrically, the outcome is b when three individuals prefer b to a, one prefers a to b, and the remaining individual is indifferent between a and b, and is a when the individual who is indifferent switches to preferring b.)

To eliminate collective choice rules that behave in this way, we can require that if an alternative moves up in some individuals' preferences then the outcome specified by the collective choice rule moves in the direction of that alternative. To define this requirement precisely, I first define a preference profile $\succsim'$ to be an improvement of a profile $\succsim$ for an alternative x relative to an alternative y if the profiles differ only in the preferences of the individuals in some set J and for every individual $j \in J$, either x is ranked equal to or below y by $\succsim_j$ and above y by $\succsim'_j$, or x is ranked below y by $\succsim_j$ and equal to or above y by $\succsim'_j$, while all other alternatives are ranked the same by $\succsim_j$ and $\succsim'_j$. I give a definition of the property for collective choice problems with any number of alternatives, because I use it later for problems with more than two alternatives.

1	2	3
a	b	abc
c	ac	
b		

Original problem

1	2	3
a	b	abc
bc	ac	

Improvement of b relative to c

1	2	3
a	b	b
c	ac	ac
b		

Improvement of b relative to c

Figure 1.2 The collective choice problem in Example 1.1, at the left, and two problems in which the preference profiles are improvements for b relative to c. For the problem on the right, the preference profile is also an improvement for b relative to a.

Definition 1.12: Improvement of preference profile for alternative

Let $\langle N, X, \succsim \rangle$ be a finite collective choice problem, let $\succsim'$ be a preference profile for $\langle N, X \rangle$ that differs from $\succsim$, and let $J \subseteq N$ be the set of individuals j for whom $\succsim_j$ differs from $\succsim'_j$. The profile $\succsim'$ is an *improvement of* $\succsim$ *for* $x \in X$ *relative to* $y \in X$ if for each $j \in J$ we have

$$\text{either } y \succsim_j x \text{ and } x \succ'_j y, \text{ or } y \succ_j x \text{ and } x \succsim'_j y$$

and

$$w \succsim'_j z \text{ if and only if } w \succsim_j z \text{ for all } w \in X \setminus \{x\} \text{ and all } z \in X \setminus \{x\}.$$

Consider the collective choice problem in Example 1.1, which is shown at the left in Figure 1.2. The preference profiles for the other two problems in the figure are improvements for b relative to c. In the problem in the middle, b moves up in individual 1's preferences, to become indifferent with c. (If b moves further up, to lie between a and c, to be indifferent with a, or to be above a, the resulting profile is also an improvement for b relative to c.) In the problem on the right, b moves up in individual 3's preferences to become preferred to a and c rather than indifferent with them. (This profile is also an improvement for b relative to a.)

For a collective choice problem $\langle N, X, \succsim \rangle$ with two alternatives, a and b, the preference profile $\succsim'$ is an improvement of $\succsim$ for a relative to b if for every individual j in some set J we have (*i*) $b \succ_j a$ and $a \succ'_j b$, (*ii*) $a \sim_j b$ and $a \succ'_j b$, or (*iii*) $b \succ_j a$ and $a \sim'_j b$, and all other individuals' preferences remain the same. Thus the number of individuals for whom $a \succ_i b$ either remains the same or increases and the number for whom $b \succ_i a$ either remains the same or decreases. In terms of Figures 1.1a and 1.1b, for any position (x, y), the problems at a position (x', y') to the east, southeast, or south of (x, y) are the improvements for a

relative to b of the problems at (x, y).

I now define the property of positive responsiveness, which rules out the unattractive feature of the rule shown in Figure 1.1b. For now I define the property only for problems with two alternatives. In this case, a collective choice rule is positively responsive if, when an alternative x that is selected (possibly in a tie with other alternatives) goes up in some individuals' rankings, the rule selects it uniquely. (In Definition 1.15 I extend the definition to problems with three or more alternatives.)

Definition 1.13: Positively responsive collective choice rule for two alternatives

Let $\langle N, X\rangle$ be a finite society for which X consists of two alternatives, and let D be a set of finite collective choice problems $\langle N, X, \succsim\rangle$. A collective choice rule F for D is *positively responsive* if for every preference profile $\succsim$ for $\langle N, X\rangle$ for which $\langle N, X, \succsim\rangle \in D$ and every improvement $\succsim'$ of $\succsim$ for alternative x relative to the other alternative for which $\langle N, X, \succsim'\rangle \in D$ we have

$$x \in F(N, X, \succsim) \quad \Rightarrow \quad F(N, X, \succsim') = \{x\}.$$

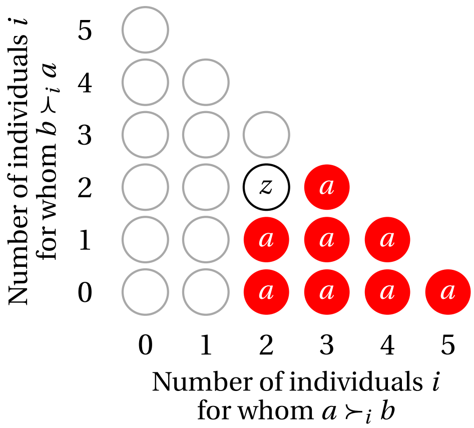
In terms of the diagrams, a rule is positively responsive if for every position assigned to $\{a\}$ or $\{a, b\}$, like the one labeled z in Figure 1.3a, the positions to the east, southeast, and south are assigned to $\{a\}$, and for every problem assigned to $\{b\}$ or $\{a, b\}$, the problems to the west, northwest, and north are assigned to $\{b\}$.

Majority rule, shown in Figure 1.3b, is positively responsive: if a goes up in some individuals' preferences, then either the number of individuals for whom a is a favorite increases or the number of individuals for whom b is a favorite decreases, or both, so that if the rule originally selected $\{a\}$ then it still does so, and if it originally selected $\{a, b\}$ then it switches to selecting $\{a\}$.

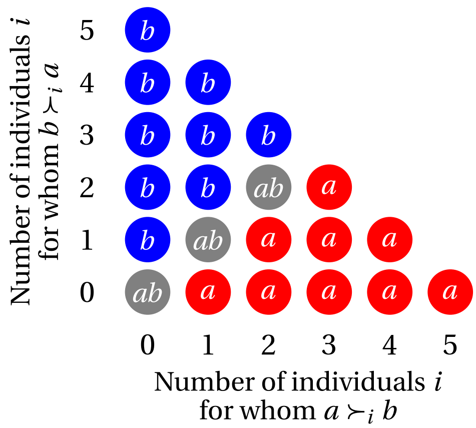
In fact, majority rule is the only anonymous, neutral, and positively responsive collective choice rule for collective choice problems with two alternatives: the requirements of neutrality (Figure 1.1b) and positive responsiveness (Figure 1.3a) generate Figure 1.3b. This result is known as May's theorem, after its originator, Kenneth O. May (1915–1977).

Proposition 1.1: May's theorem

Let $\langle N, X\rangle$ be a finite society for which X has two members, and let D be (*a*) the set of all collective choice problems $\langle N, X, \succsim\rangle$, (*b*) the set of all such problems in which each individual's preference relation is strict, or (*c*) the set of all such problems with a strict majority winner. A collective choice



(a) If z is a or ab then for a positively responsive collective choice rule, the outcomes indicated are a.



(b) Majority rule, the only anonymous, neutral, and positively responsive collective choice rule.

Figure 1.3 Anonymous collective choice rules for collective choice problems with five individuals and two alternatives, a and b. (The diagram on the left only partially specifies a rule.) See the text discussing Figure 1.1 for an explanation of the way in which the diagrams represent collective choice rules.

rule for D is anonymous, neutral, and positively responsive if and only if it is majority rule.

Proof

I have argued that majority rule is anonymous, neutral, and positively responsive (for any domain).

Now consider a collective choice rule that is anonymous, neutral, and positively responsive on D. I argue that it is majority rule. Given that the rule is anonymous, it may be represented in a diagram like Figure 1.1. (For the domains in cases (*b*) and (*c*), the relevant diagrams contain only a subset of the positions in the triangle.) By neutrality, every position on the main diagonal (if any exist for the domain D) is labeled ab. Suppose that some position (x, y) with $x < y$ (so that (x, y) is above the main diagonal) is labeled a. Then by neutrality, (y, x) is labeled b. But each problem associated with (x, y) is an improvement for b relative to a for a problem associated with (y, x), so by positive responsiveness (x, y) is labeled b, a contradiction. Similarly, every position (x, y) with $x > y$ is labeled a. Thus the collective choice rule is majority rule (Figure 1.3b).

Exercise 1.2: Collective choice rules consistent with two of the three conditions

As I note, the rule shown in Figure 1.1b is anonymous and neutral but not positively responsive. Give examples of collective choice rules for problems with two alternatives that are (a) anonymous and positively responsive but not neutral and (b) neutral and positively responsive but not anonymous.

The condition of positive responsiveness requires that if the alternatives are tied and one of them improves relative to the other, it becomes the outcome. Suppose that we relax the condition to nonnegative responsiveness: if a preference profile is an improvement of another profile for x relative to the other alternative, then if the outcome was originally $\{x\}$, it remains $\{x\}$, and if it was originally a tie, it either changes to $\{x\}$ (as for positive responsiveness) *or remains a tie*. More precisely, the collective choice rule F is *nonnegatively responsive* if for any improvement $\succsim'$ of $\succsim$ for a relative to b we have

$$F(N,X,\succsim)=\{a\} \Rightarrow F(N,X,\succsim')=\{a\} \quad \text{and} \quad F(N,X,\succsim)=\{a,b\} \Rightarrow a \in F(N,X,\succsim')$$

and symmetrically for an improvement for b relative to a. Rules other than majority rule are consistent with anonymity, neutrality, and nonnegative responsiveness.

Exercise 1.3: Nonnegatively responsive collective choice rules

In a diagram like Figure 1.1, characterize the collective choice rules for sets of alternatives with two members that are anonymous, neutral, and nonnegatively responsive (but not necessarily positively responsive).

1.4 Three or more alternatives: Condorcet winners

For a collective choice problem with two alternatives, an alternative is selected by majority rule if it beats or ties the other alternative. For collective choice problems with three or more alternatives, a natural extension of majority rule selects the alternatives that beat or tie every other alternative in pairwise contests. These alternatives are called Condorcet winners, after Marie Jean Antoine Nicolas de Caritat, Marquis of Condorcet (1743–1794).

Definition 1.14: Condorcet winner

Let $\langle N, X, \succsim \rangle$ be a collective choice problem for which N is finite and let $x \in X$ and $y \in X$ be alternatives. Then x *beats* y if the number of individuals $i \in N$ for whom $x \succ_i y$ exceeds the number for whom $y \succ_i x$, and x *ties with* y if these two numbers are equal. The alternative x is

- a *Condorcet winner* of $\langle N, X, \succsim \rangle$ if it beats or ties every other alternative
- a *strict Condorcet winner* of $\langle N, X, \succsim \rangle$ if it beats every other alternative.

A problem can have at most one strict Condorcet winner; if it has no strict Condorcet winner then it may have more than one Condorcet winner.

Exercise 1.4: Collective choice problem with no strict Condorcet winner but unique Condorcet winner

Given an example of a collective choice problem with no strict Condorcet winner but a unique Condorcet winner.

Exercise 1.5: Alternative that ties with Condorcet winner

Is an alternative that ties with a Condorcet winner necessarily a Condorcet winner?

For a collective choice problem with two alternatives, the set of Condorcet winners is the set of alternatives selected by majority rule, and if one alternative is preferred to the other by a strict majority of individuals, then that alternative is the strict Condorcet winner.

The collective choice problem in Example 1.1, in which there are three alternatives, has two Condorcet winners, a and b, and no strict Condorcet winner. The next example gives a problem with a strict Condorcet winner in which that alternative differs from the ones selected by plurality rule and the Borda rule.

Example 1.4: Condorcet winner, plurality winner, and Borda winner

For the following collective choice problem, c is the strict Condorcet winner: four individuals prefer it to a, four individuals prefer it to b, and four individuals prefer it to d. By contrast, a is selected by plurality rule and b is selected by the Borda rule.

1	2	3	4	5	6	7
a	a	a	c	c	b	d
c	b	b	b	b	d	c
b	c	d	d	d	c	b
d	d	c	a	a	a	a

Some collective choice problems with three or more alternatives have no Condorcet winner.

Example 1.5: Condorcet cycle

For the following collective choice problem, known as a Condorcet cycle, c beats a, a beats b, and b beats c, so that no alternative is a Condorcet winner.

1	2	3
a	c	b
b	a	c
c	b	a

Among all logically possible collective choice problems with three alternatives in which all individuals' preferences are strict, the percentage with a Condorcet winner declines as the number n of individuals increases; for all values of n it is at least 91%. For problems with more alternatives, this bound on the percentage is smaller. For example, it is about 58% for problems with eight alternatives (Gehrlein and Fishburn 1976). In the next section (1.5) I discuss conditions on preference profiles that ensure the existence of a Condorcet winner.

For the domain of all collective choice problems that have Condorcet winners, we can generalize May's theorem. To do so, we need to first generalize the property of positive responsiveness to many alternatives. One generalization requires that if an alternative x selected for a collective choice problem improves relative to some other alternative y, then (*i*) x is still selected for the new problem, (*ii*) y is not selected for the new problem, and (*iii*) an alternative is selected for the new problem only if it was selected for the original problem.

Definition 1.15: Positively responsive collective choice rule

For any set D of collective choice problems, a collective choice rule F for D is *positively responsive* if, for every collective choice problem $\langle N, X, \succsim\rangle \in D$, every alternative $x \in F(N, X, \succsim)$, every alternative $y \in X$, and every improvement $\succsim'$ of $\succsim$ for x relative to y for which $\langle N, X, \succsim'\rangle \in D$, *i.* $x \in F(N, X, \succsim')$

ii. $y \notin F(N, X, \succsim')$

iii. $F(N, X, \succsim') \subseteq F(N, X, \succsim)$.

Note that if a collective choice rule selects $\{x\}$ for some collective choice problem and an alternative y improves relative to x, then the property of positive responsiveness has no implications for the alternatives selected by the rule for the new problem.

For the domain of collective choice problems with a strict Condorcet winner, the collective choice rule that assigns the strict Condorcet winner to each collective choice problem is positively responsive: if a strict Condorcet winner improves in some individuals' preferences then it remains a strict Condorcet winner.

Plurality rule satisfies condition (*i*) of positive responsiveness: if x is one of the alternatives the rule selects (maybe the only one) and goes up in some individuals' preferences, it remains one of the selected alternatives. But plurality rule does not satisfy condition (*ii*): if x and y are both members of the set the rule selects, and y improves relative to x, then x may still be among the selected alternatives. The reason is that the alternatives plurality rule selects are not affected by the ranking of alternatives that are not at the top of an individual's preferences. Consider the following example.

1	2	3
a	b	c
b	a	b
c	c	a

1	2	3
a	b	c
b	a	a
c	c	b

In both of these problems, plurality rule selects $\{a, b, c\}$ (all three alternatives, in a tie). This behavior violates condition (*ii*) of positive responsiveness because the preference profile for the problem on the right is an improvement for a relative to b (via individual 3), so that condition (*ii*) demands that b not be selected for that problem.

Exercise 1.6: Plurality rule with runoff

Plurality rule with runoff is the collective choice rule that selects the alternative that is the favorite of the largest number of individuals if this number is more than half the number of individuals, and otherwise identifies the two alternatives that are the favorites of the largest number of individuals and selects the one that beats the other in a two-alternative contest (or both, if they tie). The idea behind the rule is that the first round determines which two alternatives have the most support, and the second round allows everyone to express a preference between these alternatives. Use the following example, in which each column is a preference ordering, to show that this rule does not satisfy condition (i) of positive responsiveness.

6 individuals	5 individuals	4 individuals	2 individuals
a	c	b	b
b	a	c	a
c	b	a	c

That plurality rule is not positively responsive when there are three or more alternatives is not surprising. When there are two alternatives, knowing an individual's favorite alternative tells us the individual's complete preference relation. When there are three or more alternatives, that is no longer the case, and the identity of the individuals' favorite alternatives, the only information used by plurality rule, seems inadequate to make a collective decision that fully reflects the individuals' preferences.

The Borda rule is designed to address this problem: it gives weight to individuals' rankings of alternatives below their favorites, and is indeed positively responsive. If an alternative x improves in an individual's preferences relative to y, then the number of points assigned to x goes up and the number of points assigned to every other alternative either goes down or remains the same. So if x was selected for the original problem it is the only alternative selected for the new problem.

However, when there are three or more alternatives, the Borda rule has an undesirable property. Consider the following collective choice problem.

1	2	3	4	5
b	b	b	a	a
a	a	a	c	c
c	c	c	b	b

The Borda rule selects a for this problem. (It gets 7 points; b gets 6 points, and c gets 2 points.) Now suppose that c is no longer available, so that we have the following problem.

1	2	3	4	5
b	b	b	a	a
a	a	a	b	b

The Borda rule selects b for this problem: the unavailability of c causes b to be chosen rather than a. This behavior seems odd.

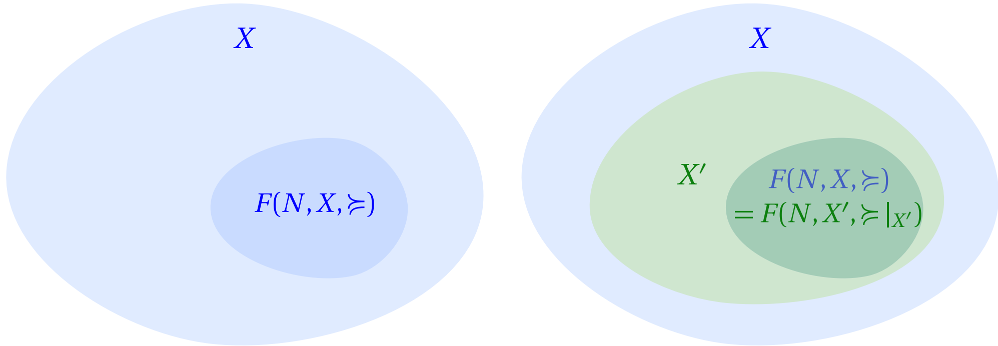


Figure 1.4 An illustration of Nash independence. In the problem on the right, X' is a subset of X that contains $F(N,X,\succsim)$. Nash independence requires that the set $F(N,X',\succsim|_{X'})$ of alternatives selected for $\langle N,X',\succsim|_{X'}\rangle$ is equal to the set $F(N,X,\succsim)$ selected for $\langle N,X,\succsim\rangle$.

To restrict to collective choice rules that do not behave in this way we can require, in addition to anonymity, neutrality, and positive responsiveness, that removing unchosen alternatives does not affect the set of alternatives selected. More precisely, if, when the set of alternatives is X, the set of alternatives selected is Y, then when the set of alternatives is a subset of X that includes Y, the set of alternatives selected remains Y. This condition is illustrated in Figure 1.4. It is a version of one proposed by Nash (1950) and is named after him. In the following definition, $\succsim|_{X'}$ is the restriction of the preference profile $\succsim$ to the set X': for a, $b \in X'$, $a \succsim|_{X'} b$ if and only if $a \succsim b$.

Definition 1.16: Nash independence

For any set D of collective choice problems, the collective choice rule F for D is *Nash independent* if for every collective choice problem $\langle N,X,\succsim\rangle \in D$ and every set $X' \subset X$ for which $\langle N,X',\succsim|_{X'}\rangle \in D$ and $F(N,X,\succsim) \subseteq X'$, we have $F(N,X',\succsim|_{X'}) = F(N,X,\succsim)$.

On the domain of problems that have a strict Condorcet winner, the rule that selects the strict Condorcet winner satisfies this property: if an alternative beats every other in pairwise contests then it continues to do so if some of these beaten alternatives are eliminated.

Thus for the domain of collective choice problems that have a strict Condorcet winner, among the examples of collective choice rules I have described only the one that selects the strict Condorcet winner is anonymous, neutral, positively responsiveness, and Nash independent. In fact, if every individual's preference relation is strict, it is the only rule that satisfies these properties among *all* possible rules.

Definition 1.17: Strict Condorcet domain

For any finite sets N and A, the *strict Condorcet domain for* (N,A), denoted $C(N,A)$, is the set of all collective choice problems $\langle N,X,\succsim\rangle$ for which (*i*) $X \subseteq A$, (*ii*) $\succsim_i$ is strict for all $i \in N$, and (*iii*) $\langle N,X,\succsim\rangle$ has a strict Condorcet winner.

Proposition 1.2: Generalization of May's theorem to many alternatives

For any finite sets N and A, a collective choice rule F for the strict Condorcet domain $C(N,A)$ is anonymous, neutral, positively responsive, and Nash independent if and only if, for every collective choice problem $\langle N,X,\succsim\rangle \in C(N,A)$, $F(N,X,\succsim)$ contains only the strict Condorcet winner of $\langle N,X,\succsim\rangle$.

Proof

I have argued that the collective choice rule for $C(N,A)$ that selects the strict Condorcet winner satisfies the four properties.

Now let F be a collective choice rule for $C(N,A)$ that is anonymous, neutral, positively responsive, and Nash independent and let $\langle N,X,\succsim\rangle \in C(N,A)$. If A contains two alternatives then by May's theorem $F(N,X,\succsim) = \{c\}$, where c is the strict Condorcet winner of $\langle N,X,\succsim\rangle$.

Now suppose that A contains three or more alternatives.

Step 1 *Let* $\langle N,X,\succsim\rangle$ *be a collective choice problem in* $C(N,A)$ *and let* c *be its strict Condorcet winner. If* $F(N,X,\succsim)$ *contains an alternative different from* c *then it contains at least two alternatives different from* c.

Proof. Suppose, to the contrary, that $F(N,X,\succsim) = \{x,c\}$ or $F(N,X,\succsim) = \{x\}$ for some $x \in X \setminus \{c\}$. The alternative c is a strict Condorcet winner of $\langle N,\{x,c\},\succsim|_{\{x,c\}}\rangle$, so this problem is in $C(N,A)$, and $F(N,X,\succsim) \subseteq \{x,c\}$, so by Nash independence $F(N,\{x,c\},\succsim|_{\{x,c\}}) = F(N,X,\succsim)$ and hence $F(N,\{x,c\},\succsim|_{\{x,c\}}) = \{x,c\}$ or $F(N,\{x,c\},\succsim|_{\{x,c\}}) = \{x\}$. Now, given that F is anonymous, neutral, and positively responsive, it satisfies these properties in particular for two-alternative problems in $C(N,A)$. Thus by May's theorem $F(N,\{x,c\},\succsim|_{\{x,c\}}) = \{c\}$, a contradiction. $\lhd$

Step 2 *If for some problem* $\langle N,X,\succsim\rangle \in C(N,A)$ *the set* $F(N,X,\succsim)$ *has* k *members with* $k \geq 2$, *there exists a problem* $\langle N,X,\succsim'\rangle \in C(N,A)$ *for which*

$F(N,X,\succsim')$ has fewer than k members, including at least one different from its strict Condorcet winner.

Proof. Let $\langle N,X,\succsim\rangle \in C(N,A)$, let c be its strict Condorcet winner, and let $F(N,X,\succsim) = \{x_1,x_2,\ldots,x_k\}$ with $k \geq 2$. By Step 1, at least two members of $F(N,X,\succsim)$ differ from c. Suppose that $x_1 \neq c$ and $x_2 \neq c$. A strict majority of individuals prefer c to x_1 and a strict majority prefer c to x_2, so at least one individual prefers c to both x_1 and x_2. Suppose that $c \succ_i x_1 \succ_i x_2$. Modify i's preference relation by raising x_2 to come between x_1 and c; keep the preference relation of every other individual the same. Denote the new preference profile by $\succsim'$. The alternative c is the strict Condorcet winner for $\langle N,X,\succsim'\rangle$, so in particular this problem has a strict Condorcet winner and hence is in $C(N,A)$. Thus by positive responsiveness $F(N,X,\succsim')$ is a subset of $\{x_1,x_2,\ldots,x_k\}$ that contains x_2 but not x_1. Hence it contains fewer alternatives than does $F(N,X,\succsim)$, among them an alternative different from c. ◁

An implication of Step 2 is that for some problem $\langle N,X,\succsim\rangle$ the set $F(N,X,\succsim)$ contains a single alternative different from c, contradicting Step 1.

Is any collective choice rule anonymous, neutral, positively responsive, and Nash independent for a domain larger than the strict Condorcet domain? For domains that are more than slightly larger than the strict Condorcet domain, the answer is no. Define the Condorcet-plus domain to consist of all problems with a strict Condorcet winner *plus* all problems for which some improvement for a single individual generates a problem that has a strict Condorcet winner. (This domain, unlike the strict Condorcet domain, contains problems for which the preference relations are not strict.)

Definition 1.18: Condorcet-plus domain

For any finite sets N and A, the *Condorcet-plus domain for* (N,A), denoted $C^+(N,A)$, is the set of collective choice problems $\langle N,X,\succsim\rangle$ such that $X \subseteq A$ and either (*i*) $\langle N,X,\succsim\rangle$ has a strict Condorcet winner or (*ii*) there is a preference profile $\succsim'$ for $\langle N,X\rangle$ that differs from $\succsim$ only in the preference relation of one individual and is an improvement of $\succsim$ such that the problem $\langle N,X,\succsim'\rangle$ has a strict Condorcet winner.

If A contains two alternatives then for any finite set N the Condorcet-plus domain for (N,A) contains all collective choice problems with two alternatives:

for any two-alternative problem that does not have a strict Condorcet winner, the alternatives tie, and an improvement for any individual breaks this tie.

If A is a finite set containing three or more alternatives then for any finite set N the Condorcet-plus domain for (N,A) contains the Condorcet cycle in Example 1.5: if a is raised above c in individual 2's preferences then it becomes a strict Condorcet winner. But some three-alternative problems are not in $C^+(N,A)$. An example is the following problem, which does not have a strict Condorcet winner and for which no improvement for a single individual produces a strict Condorcet winner.

1	2	3	4	5	6
a	a	c	c	b	b
b	b	a	a	c	c
c	c	b	b	a	a

Each alternative beats one of the other alternatives four to two and loses to the remaining alternative two to four. If an alternative improves in any one individual's preferences, it improves its performance against the alternative that previously beat it only to a tie, so the new problem, like the old one, has no strict Condorcet winner.

I now show that if N and A are finite sets each containing at least three elements, then no collective choice rule is anonymous, neutral, positively responsive, and Nash independent on $C^+(N,A)$ or any domain that contains $C^+(N,A)$.

Proposition 1.3: No rule is anonymous, neutral, positively responsive, and Nash independent on Condorcet-plus domain

Let N and A be finite sets, each containing at least three elements. No collective choice rule for any set of finite collective choice problems that contains the Condorcet-plus domain $C^+(N,A)$ is anonymous, neutral, positively responsive, and Nash independent.

Proof

Let F be an anonymous, neutral, positively responsive, and Nash independent collective choice rule for a set of collective choice problems that includes $C^+(N,A)$. Let $N = \{1,2,\ldots,n\}$ with $n \geq 3$, and $X = \{a,b,c\}$, and suppose that the preferences of individuals 1, 2, and 3 form the Condorcet cycle in Example 1.5, while the other individuals are indifferent between a, b, and c:

1	2	3	4	$\cdots$	n
a	c	b	abc	$\cdots$	abc
b	a	c			
c	b	a			

This problem, $\langle N, X, \succsim \rangle$, is in $C^+(N,A)$ because raising b above a in individual 1's preferences makes it a strict Condorcet winner.

I first argue that the neutrality and anonymity of F imply that $F(N, X \succsim) = \{a, b, c\}$. Consider the permutation σ on X given by $\sigma(a) = c$, $\sigma(b) = a$, and $\sigma(c) = b$. The problem $\langle N, X, \succsim^\sigma \rangle$, where $\succsim^\sigma$ is the preference profile derived from $\succsim$ as given in Definition 1.10, is

1	2	3	4	$\cdots$	n
c	b	a	abc	$\cdots$	abc
a	c	b			
b	a	c			

This problem is in $C^+(N,A)$ because raising a above c in individual 1's preferences makes it a strict Condorcet winner. So by neutrality, (*i*) $F(N, X, \succsim^\sigma)$ consists of the alternatives $\sigma(z)$ for each $z \in F(N, X, \succsim)$. But $\succsim^\sigma$ is the permutation of $\succsim$ obtained by mapping individual 1 into 3, 2 into 1, and 3 into 2. Thus by anonymity (*ii*) $F(N, X, \succsim^\sigma) = F(N, X, \succsim)$. Conditions (*i*) and (*ii*) are satisfied if and only if $F(N, X, \succsim) = \{a, b, c\}$.

Now, starting with $\langle N, X, \succsim \rangle$, raise b to be indifferent with a in individual 1's preferences, generating the problem

1	2	3	4	$\cdots$	n
ab	c	b	abc	$\cdots$	abc
c	a	c			
	b	a			

Denote this problem $\langle N, X, \succsim' \rangle$. It is in $C^+(N,A)$ because raising b above a in individual 1's preferences makes b a strict Condorcet winner, so that by positive responsiveness, $b \in F(N, X, \succsim')$ and $a \notin F(N, X, \succsim')$. Thus $F(N, X, \succsim')$ is either $\{b\}$ or $\{b, c\}$.

If $F(N, X, \succsim') = \{b\}$, remove c from the set of alternatives, to generate the problem

1	2	3	4	$\cdots$	n
ab	a	b	ab	$\cdots$	ab
	b	a			

This problem is in $C^+(N,A)$ because raising b above a in individual 1's preferences makes b a strict Condorcet winner, so by Nash independence F assigns $\{b\}$ to it, contradicting May's theorem, which assigns $\{a,b\}$ to it.

If $F(N,X,\succsim') = \{b,c\}$, remove a from the set of alternatives, to generate the problem

1	2	3	4	$\cdots$	n
b	c	b	bc	$\cdots$	bc
c	b	c			

This problem is in $C^+(N,A)$ because b is a strict Condorcet winner, so by Nash independence F assigns $\{b,c\}$ to it, contradicting May's theorem, which assigns $\{b\}$ to it.

Exercise 1.7: Nash independent rules consistent with two of the three other conditions

For each pair of the conditions anonymity, neutrality, and positive responsiveness, find a collective choice rule that satisfies that pair of conditions and is Nash independent for the domain of all collective choice problems.

Proposition 1.2 shows that the collective choice rule that selects the strict Condorcet winner is the only rule for the domain of problems that have a strict Condorcet winner that is anonymous, neutral, positively responsive, and Nash independent. Proposition 1.3 shows that this result cannot be extended much beyond this domain. For any domain that includes problems for which no improvement in a single individual's preferences creates a strict Condorcet winner, no collective choice rule satisfies the four properties.

1.5 Two domains with Condorcet winners

1.5.1 Single-peaked preferences

Suppose that for some ordering $\trianglerighteq$ of the alternatives, individual i has a favorite alternative, say a_i^*, and prefers a_i^* to b to c whenever $c \lhd b \lhd a_i^*$ or $a_i^* \lhd b \lhd c$. We say that her preference relation is single-peaked with respect to the ordering.

Definition 1.19: Single-peaked preference relation

Let X be a set (of alternatives) and let $\trianglerighteq$ be a linear order on X. A preference relation $\succsim_i$ over X is *single-peaked with respect to* $\trianglerighteq$ if it has a single favorite alternative, say a^*, and

$$c \lhd b \lhd a^* \quad \text{or} \quad a^* \lhd b \lhd c \quad \Rightarrow \quad a^* \succ_i b \succ_i c. \tag{1.3}$$

For any strict preference relation, there are linear orders of the alternatives with respect to which the preference relation is single-peaked. (One such order, for example, arranges the alternatives from most preferred to least preferred.) If a preference profile has the property that every individual's preference relation is single-peaked with respect to the *same* linear order of the alternatives, we say that the profile is single-peaked (with respect to the order). Assuming that a preference profile has this property may be reasonable, for example, if the alternatives are the amounts of money society spends on a certain endeavor: some people like low spending better than moderate spending better than high spending, some people have the reverse preference, and some people like moderate spending best, but no one likes both low spending and high spending better than moderate spending.

Definition 1.20: Single-peaked preference profile

Let $\langle N, X\rangle$ be a society and let $\trianglerighteq$ be a linear order on X. A preference profile $\succcurlyeq$ for $\langle N, X\rangle$ is *single-peaked with respect to* $\trianglerighteq$ if every preference relation $\succcurlyeq_i$ for $i \in N$ is single-peaked with respect to $\trianglerighteq$. A collective choice problem $\langle N, X, \succcurlyeq\rangle$ has *single-peaked preferences* if for some linear order $\trianglerighteq$ on X the preference profile $\succcurlyeq$ is single-peaked with respect to $\trianglerighteq$.

The name single-peaked comes from the shape of the payoff functions that represent the preferences. An example is shown in Figure 1.5. In this diagram, there is a small colored disk for each alternative and each individual, and the fact that the y-coordinate of the disk for some alternative x of a certain color is larger than the y-coordinate of the disk for another alternative x' of the same color means that the individual associated with the color prefers x to x'. (The magnitude of the difference between the y-coordinates has no significance.) For example, individual 3's preference relation $\succcurlyeq_3$ is given by $d \succ_3 c \succ_3 e \succ_3 b \succ_3 a$. Note that each individual's preference relation in a single-peaked preference profile is strict on each side of the individual's favorite alternative, but the individual may be indifferent between alternatives on different sides of her favorite alternative. In the preference profile in Figure 1.5, for example, individual 2 is indifferent between a and c.

The next result involves the median of the individuals' favorite alternatives. An odd number of ordered alternatives has a single median, namely the middle alternative in the order; an even number of ordered alternatives has two

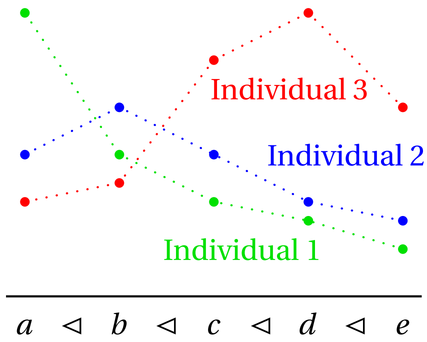


Figure 1.5 A single-peaked preference profile.

medians, namely the two middle alternatives.

Definition 1.21: Median of finite set with respect to ordering

Let $X = \{x_1, \ldots, x_n\}$ be a finite set and $\trianglerighteq$ a linear order on X, with $x_1 \trianglerighteq x_2 \trianglerighteq \cdots \trianglerighteq x_n$. If n is odd then the *median of X with respect to* $\trianglerighteq$ is x_k for $k = \frac{1}{2}(n+1)$, and if n is even then the *medians of X with respect to* $\trianglerighteq$ are x_l and x_{l+1} for $l = \frac{1}{2}n$.

For example, for the preferences illustrated in Figure 1.5, for which the individuals' favorite alternatives are a, b, and d, we have $a \triangleleft b \triangleleft d$, so that b is the median of the favorite alternatives with respect to $\trianglerighteq$.

I now show that for a collective choice problem with single-peaked preferences, an alternative is a median of the individuals' favorite alternatives if and only if it is a Condorcet winner, and if the number of individuals is odd then the unique median favorite alternative is the strict Condorcet winner. I follow convention in calling the result a "median voter theorem", although the median is defined with respect to an ordering of the alternatives, not an ordering of the individuals (voters).

Proposition 1.4: Median voter theorem for single-peaked preferences

Consider a collective choice problem in which the number of individuals is finite. If the problem has single-peaked preferences with respect to a linear order $\trianglerighteq$, an alternative is a Condorcet winner if and only if it is a median of the individuals' favorite alternatives with respect to $\trianglerighteq$. If the number of individuals is odd, the unique median is the strict Condorcet winner.

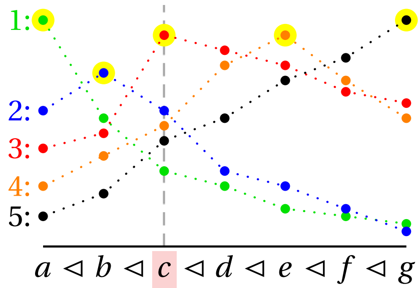


Figure 1.6 An illustration of Proposition 1.4 for a collective choice problem with five individuals. The median of the individuals' favorite positions is c.

Proof

Denote by n the number of individuals and let m be a median of the individuals' favorite alternatives with respect to the ordering $\trianglerighteq$. (In Figure 1.6, for example, the individuals' favorite alternatives are a, b, c, e, and g (indicated in yellow), so that $m = c$.)

For every individual i, denote i's favorite alternative by x_i^*. If $x \triangleright m$ then for all individuals i for whom $m \trianglerighteq x_i^*$ we have $m \succ_i x$ by the single-peakedness of preferences. Given that m is a median of the favorite alternatives, the number of such individuals is $\frac{1}{2}(n+1)$ if n is odd, so that m beats x, and at least $\frac{1}{2}n$ if n is even, so that m beats or ties with x. Similarly, if $m \triangleright x$ then for all i with $x_i^* \trianglerighteq m$ we have $m \succ_i x$, and the number of such individuals is $\frac{1}{2}(n+1)$ if n is odd and at least $\frac{1}{2}n$ if n is even. Thus m is a Condorcet winner of the collective choice problem, and is a strict Condorcet winner if n is odd.

If an alternative x is not a median of the individuals' favorite alternatives, it is beaten by any median, and hence is not a Condorcet winner.

Exercise 1.8: Median rule and Borda winner

Give an example to show that for a single-peaked preference profile the median favorite alternative may differ from the alternative selected by the Borda rule.

In some models, the assumption that the individuals' preference profile is single-peaked is unreasonable, but a weaker version of this condition is acceptable. One such version allows each individual's preferences to have a single plateau, rather than a single peak, with strict preferences on each side of the plateau. Precisely, let $\langle N, X, \succsim \rangle$ be a collective choice problem for which for some linear order $\trianglerighteq$ on X and, for each individual $i \in N$, alternatives $\underline{a}^i$ and $\overline{a}^i$ (not necessarily distinct), we have $a \prec_i b$ if $a \lhd b \unlhd \underline{a}^i$, $a \sim_i b$ if $\underline{a}^i \unlhd a \lhd b \unlhd \bar{a}^i$, and $a \succ_i b$ if $\bar{a}^i \lhd a \lhd b$. Denote the number of individuals by n and let $z_1, z_2, \ldots, z_{2n}$ be an ordering of $\{\underline{a}^1, \underline{a}^2, \ldots, \underline{a}^n, \bar{a}^1, \bar{a}^2, \ldots, \bar{a}^n\}$ such that $z_1 \unlhd z_2 \unlhd \cdots \unlhd z_{2n}$. Fishburn (1973, Theorem 9.3) shows that an alternative x is a Condorcet winner of $\langle N, X, \succsim \rangle$ if and only if $z_n \unlhd x \unlhd z_{n+1}$. The next exercise asks you to find an example in which the preference profile has a single plateau and the median of some collection of the individuals' favorite alternatives does not satisfy this condition, and hence is not a Condorcet winner.

Exercise 1.9: Median of collection of favorite alternatives and Condorcet winner for single-plateau preferences

Give an example of a collective choice problem $\langle N, X, \succsim \rangle$ with an odd number of individuals in which the preference profile $\succsim$ has a single plateau with respect to a linear order $\unrhd$ on X, as defined in the previous paragraph (but is not single peaked), and the median with respect to $\unrhd$ of a set $\{a_i^*\}_{i \in N}$ of alternatives for which each a_i^* is a favorite alternative of individual i differs from the unique Condorcet winner.

Now suppose that each individual has a single favorite alternative but may be indifferent among alternatives on the same side of her favorite. Then the (unique) median of the individuals' favorite alternatives may differ from the Condorcet winner, as you are asked to demonstrate in the next exercise.

Exercise 1.10: Median of favorite alternatives and Condorcet winner for variant of single-peaked preferences with indifference

Consider a collective choice problem $\langle N, X, \succsim \rangle$ with an odd number of individuals in which each individual has a single favorite alternative and, for some linear order $\unrhd$ of the alternatives, each individual's preferences satisfy the variant of (1.3) in which the right-hand side is $a^* \succsim_i b \succsim_i c$. Give an example of such a problem that does not have a Condorcet winner and also an example with a unique Condorcet winner that differs from the median of the individuals' favorite alternatives.

The next exercise leads you through an inductive proof that a finite collective choice problem with an odd number of individuals has a strict Condorcet winner (part of the median voter theorem, Proposition 1.4). A variant of this proof establishes an elegant generalization of that result, discussed after the exercise.

1	2	3
a	a	a
b	d	c
c	b	d
d	c	b

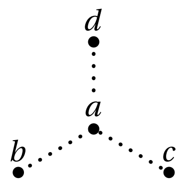


Figure 1.7 A collective choice problem (left) with preferences that are single-peaked on the tree on the right, but are not single-peaked on a line.

Exercise 1.11: Another proof of existence of Condorcet winner when preferences are single-peaked

Let $\langle N, X, \succsim\rangle$ be a finite collective choice problem with an odd number of individuals that has single-peaked preferences with respect to a linear order $\trianglerighteq$. Denote the alternatives arranged according to this order by $(x_1, x_2, \ldots, x_k)$, let $X_1 = \varnothing$, $Z_k = \{x_1, \ldots, x_k\}$, and for any $t \in \{2, \ldots, k\}$ let $X_t = \{x_1, \ldots, x_{t-1}\}$ and $Z_t = \{x_t, \ldots, x_k\}$. (*a*) Show that if for some $t \in \{1, \ldots, k\}$ (*i*) x_t is the favorite alternative in Z_t of a majority of individuals and (*ii*) no alternative $x_j \in X_t$ is the favorite alternative in Z_j of a majority of individuals, then x_t is the strict Condorcet winner of $\langle N, X, \succsim\rangle$. (*b*) Use this result to give an inductive proof of the existence of a strict Condorcet winner of $\langle N, X, \succsim\rangle$ in which at each step the first remaining alternative according to $\trianglerighteq$ is selected if it is the favorite among the remaining alternatives of a majority of individuals and otherwise is removed from the set of alternatives.

A tree is a connected graph with no cycles. Define a preference profile to be single-peaked on a tree if for some tree each individual's preferences are single-peaked on every path through the tree. The proof in Exercise 1.11 may be adapted to show that every finite collective choice problem in which the preference profile is single-peaked on a tree has a strict Condorcet winner. At each step in the inductive argument, one of the terminal nodes of the tree is either selected or removed. An example of a collective choice problem with preferences that are single-peaked on a tree but not on a line is given in Figure 1.7. These preferences are not single-peaked in the sense of Definition 1.20 because the individuals' worst alternatives (d, c, and b) number more than two. But they are single-peaked on all the paths in the tree on the right of Figure 1.7, and a is the strict Condorcet winner of the problem.

1.5.2 Single-crossing preferences

The single-crossing property is another condition on preference profiles that is sufficient for the existence of a Condorcet winner.

Suppose that the individuals can be ordered in such a way that for any alternatives x and y, if a median individual in the ordering likes x at least as much as y then either every individual earlier in the ordering or every individual later in the ordering prefers x to y. Then if a median individual prefers x to y, a majority of individuals prefer x to y, so that if a median individual has a unique favorite alternative then that alternative is a Condorcet winner. Further, if the number of individuals is odd and the (unique) median individual has a unique favorite alternative, then that alternative is the strict Condorcet winner. In addition, under these conditions the preferences of the median individual coincide with those of the majority, in the sense that for any alternatives x and y, the median individual prefers x to y if and only if a majority of individuals do so. That is, the median individual's preference relation coincides with the majority relation, defined as follows.

Definition 1.22: Majority relation

Let $\langle N, X, \succsim \rangle$ be a finite collective choice problem. The *majority relation for* $\langle N, X, \succsim \rangle$ is the binary relation $\trianglerighteq$ on X defined by

> $x \trianglerighteq y$ if and only if the number of individuals i for whom $x \succ_i y$ is at least the number for whom $y \succ_i x$.

A preference profile has the single-crossing property if the individuals can be ordered in such a way that the preferences of every individual, not only those of the median individual, satisfy the condition in the previous paragraph. That is, there is a linear order $\geq$ of the individuals such that for every individual i and all alternatives x and y, if $x \succsim_i y$ then either (*a*) $x \succ_j y$ for every individual $j < i$ or (*b*) $x \succ_j y$ for every individual $j > i$.

Definition 1.23: Single-crossing preferences

Let $\langle N, X, \succsim \rangle$ be a collective choice problem for which N is finite and let $\geq$ be a linear order on N. The problem $\langle N, X, \succsim \rangle$ has *single-crossing preferences with respect to* $\geq$ if for every $i \in N$, $x \in X$, and $y \in X \setminus \{x\}$,

$$x \succsim_i y \quad \Rightarrow \quad x \succ_j y \text{ either } (a) \text{ for all } j < i \text{ or } (b) \text{ for all } j > i.$$

The problem has *single-crossing preferences* if this condition is satisfied for some linear order $\geq$ on N.

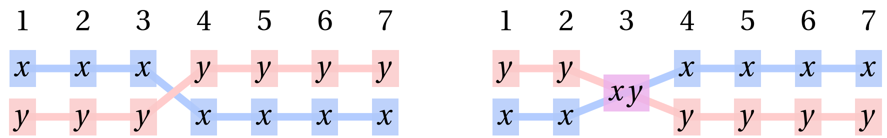


Figure 1.8 An illustration of the individuals' preferences between the alternatives x and y for two collective choice problems with single-crossing preferences and seven individuals. In each case, the individuals are ordered by their names. Each column in each diagram indicates the preference between x and y for one individual, with the individual's preferred alternative at the top. In the diagram on the right, individual 3 is indifferent between x and y.

If a collective choice problem has single-crossing preferences and $x \succsim_i y$ for some individual i then either $x \succ_j y$ for every individual j or there is a unique individual i^* such that $x \succsim_{i^*} y$ and either (a) $x \succ_i y$ for all $i < i^*$ and $x \prec_i y$ for all $i > i^*$, or (b) $x \prec_i y$ for all $i < i^*$ and $x \succ_i y$ for all $i > i^*$. Figure 1.8 shows two examples and in doing so motivates the term "single-crossing". This implication of Definition 1.23 leads to an equivalent definition of single-crossing that involves a linear order of the alternatives in addition to a linear order of the individuals.

Exercise 1.12: Alternative definition of single-crossing preferences

Show that a collective choice problem $\langle N, X, \succsim \rangle$ has single-crossing preferences if and only if there is a linear order $\geq$ of the individuals and a linear order $\trianglerighteq$ of the alternatives such that whenever $x \succsim_i y$, (a) $x \lhd y \Rightarrow x \succ_j y$ for all $j < i$ and (b) $y \lhd x \Rightarrow x \succ_j y$ for all $j > i$.

Individuals' preferences plausibly satisfy the single-crossing property in some economic models in which individuals differ in a characteristic like their earning-power or their degree of risk aversion or altruism, by which they can be ordered. Section 11.3.2 contains an example.

The next exercise concerns the relation between single-peaked and single-crossing preferences: (a) single-peaked preferences may not be single-crossing, (b) if every individual's preferences are strict and every alternative is some individual's favorite, single-crossing preferences are single-peaked, and (c) otherwise single-crossing preferences may not be single-peaked.

Exercise 1.13: Single-crossing and single-peakedness

a. Show that the collective choice problem $\langle\{1,2,3\},\{a,b,c,d\},\succsim\rangle$ for which $a \succ_1 b \succ_1 c \succ_1 d$, $b \succ_2 c \succ_2 d \succ_2 a$, and $c \succ_3 b \succ_3 a \succ_3 d$ has single-peaked but not single-crossing preferences.

b. Suppose that a collective choice problem with a finite number of individuals in which every individual's preference relation is strict has single-crossing preferences with respect to a linear order $\geq$, and every alternative is the favorite of some individual. Show that the problem has single-peaked preferences with respect to the ordering of the alternatives given by the preference relation of the first individual according to $\geq$.

c. Specify a collective choice problem that has single-crossing preferences but does not satisfy the conditions in (*b*) and does not have single-peaked preferences. (Such a problem exists with three individuals and three alternatives.)

Here is a precise version of the claims at the start of this section.

Proposition 1.5: Median voter theorem for single-crossing preferences

Consider a collective choice problem with a finite number of individuals that has single-crossing preferences with respect to a linear order $\geq$. Suppose that each median individual with respect to $\geq$ has a unique favorite alternative.

a. The favorite alternative of each median individual is a Condorcet winner of the problem, and if the number of individuals is odd, the favorite alternative of the (unique) median individual is the strict Condorcet winner of the problem.

b. If the number of individuals is odd, the preference relation of the (unique) median individual coincides with the majority relation.

Proof

Denote the collective choice problem by $\langle N, X, \succsim \rangle$ and the number of individuals (members of N) by n. Let i be a median individual according to $\geq$ and let x^* be her favorite alternative.

a. For every alternative $x \in X \setminus \{x^*\}$ we have $x^* \succ_i x$, and hence by single-crossing either $x^* \succ_j x$ for all $j < i$ or $x^* \succ_j x$ for all $j > i$. Thus $x^* \succ_j x$ for at least $\frac{1}{2}(n+1)$ individuals j if n is odd and for at least $\frac{1}{2}n$ individuals j if n is even. Hence x^* is a Condorcet winner, and a strict one if n is odd.

b. If $x \succ_i y$ then by single-crossing $x \succ_j y$ either for all individuals $j <$

i or for all individuals $j > i$, so that a majority of individuals prefer x to y. If $x \sim_i y$ then by single-crossing either $x \succ_j y$ for all individuals $j < i$ and $y \succ_j x$ for all individuals $j > i$, or vice versa, so that the number of individuals who prefer x to y is equal to the number who prefer y to x.

For a problem for which the number of individuals is even, note that, unlike the companion result Proposition 1.4 for single-peaked preferences, this result claims only that the favorite alternative of a median individual is a Condorcet winner, not the converse. Here is an example that shows that the converse is false. There are three alternatives, a, b, and c, and four individuals. Two individuals prefer a to c to b and the other two prefer b to c to a. This problem has single-crossing preferences with respect to any ordering of the individuals in which the two individuals who prefer a to c to b are either first and second or third and fourth. The alternative c is a Condorcet winner but for no ordering of the individuals is it the favorite alternative of either median individual. (Alternatives a and b are also Condorcet winners.)

The proof of part a of the result applies the single-crossing condition only to the case in which i is the median individual and x is her favorite alternative. Thus, as the argument at the start of this section suggests, the assumption in the result that the problem has single-crossing preferences may be replaced by a weaker assumption. Let $\geq$ be a linear order on N, and denote by i^* the median individual according to $\geq$ and by x^* her favorite alternative. Then the following assumption suffices: for all $y \in X \setminus \{x^*\}$,

$$x^* \succsim_{i^*} y \quad \Rightarrow \quad \text{either } (a)\ x^* \succ_j y \text{ for all } j < i^* \text{ or } (b)\ x^* \succ_j y \text{ for all } j > i^*.$$

The property in part b of the result does not hold for problems with single-peaked preferences: for some such problems, no individual's preference relation coincides with the majority relation.

Exercise 1.14: Single-peaked preferences and no individual with preferences of majority

Give an example of a collective choice problem with single-peaked preferences in which for no individual i is it the case that for all alternatives x and y, $x \succsim_i y$ if and only if $x \succsim_j y$ for a majority of individuals j. (An example exists with three individuals and four alternatives.)

If the assumption in Proposition 1.5 that each median individual has a unique favorite alternative is removed, each favorite alternative of the median individual is a Condorcet winner (but not necessarily a strict one) if the number of in-

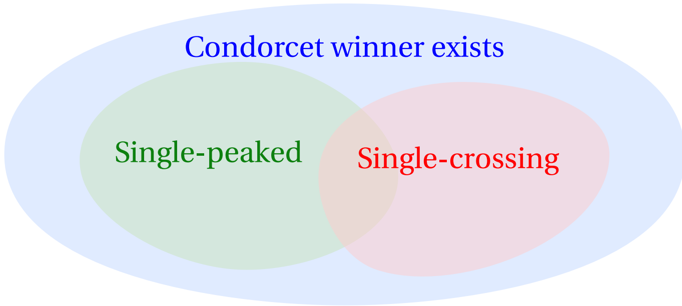


Figure 1.9 The structure of the set of collective choice problems that have a Condorcet winner.

dividuals is odd, but a favorite alternative of a median individual may not be a Condorcet winner if the number of individuals is even, as you are asked to show in the next exercise.

Exercise 1.15: Single-crossing preferences and individual with multiple favorite alternatives

Give an example of a collective choice problem with single-crossing preferences and an even number of individuals in which a favorite alternative of a median individual with multiple favorite alternatives is not a Condorcet winner.

Exercise 1.16: Condorcet winner exists but preferences not single-peaked or single-crossing

Give an example of a collective choice problem that has a strict Condorcet winner but for which the preference profile is not single-peaked and does not satisfy the single-crossing property.

Propositions 1.4 and 1.5 and the results in Exercises 1.13 and 1.16 imply that the sets of collective choice problems that have a Condorcet winner, have single-peaked preferences, and have single-crossing preferences are related in the way indicated in Figure 1.9.

1.6 Condorcet winners for two-dimensional sets of alternatives

For a collective choice problem in which the set of alternatives is a subset of a a one-dimensional space, the assumptions of single-peaked or single-crossing preferences, which require the alternatives to be linearly ordered, may be appropriate. If they are, then the collective choice problem has a Condorcet winner by Proposition 1.4 or 1.5. For a problem in which the set of alternatives is a subset of a space with two or more dimensions, these assumptions seem unlikely to be appropriate. In this section I consider conditions unrelated to the conditions of single-peaked or single-crossing preferences under which a collective choice problem for which the set of alternatives is $\mathbb{R}^2$, the set of pairs of real numbers, has a Condorcet winner. As in the previous section, I assume that the set of individuals is finite.

1.6.1 City block preferences

In a city with streets that form a grid with square blocks, you have a favorite location and you evaluate every other location according to its walking distance from that location. Suppose that roads that run north–south are called avenues and ones that run east–west are called streets. Then if, for example, your favorite location is (5th Avenue, 10th Street), you are indifferent between (3rd Avenue, 9th Street) and (2nd Avenue, 10th Street): both of these locations are three blocks away. We say you have city block preferences. The following definition does not restrict the alternatives to be located on a grid: the set of alternatives is $\mathbb{R}^2$.

Definition 1.24: City block preferences

A preference relation $\succsim_i$ on $\mathbb{R}^2$ reflects *city block preferences* if for some alternative $(x_{i1}^*, x_{i2}^*) \in \mathbb{R}^2$ it is represented by the payoff function u_i defined by

$$u_i(x_1, x_2) = -|x_1 - x_{i1}^*| - |x_2 - x_{i2}^*| \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2.$$

The alternative x_i^* in this definition is the individual's favorite alternative, and each of her indifference sets has the form

$$\{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - x_{i1}^*| + |x_2 - x_{i2}^*| = c\}$$

for some number c, a square with center x_i^* whose diagonals are vertical and horizontal. Figure 1.10 shows an example. For such a preference relation, two alternatives are indifferent if they are equidistant from x^*, where the distance between two alternatives is the length of the shortest path between them that consists only of horizontal and vertical line segments.

An individual with city block preferences cares independently about the two dimensions. The fact that an alternative diverges from an individual's favorite in one dimension does not affect the impact on her of a given divergence in the other dimension.

I show that every collective choice problem in which each individual has city block preferences has a Condorcet winner. Each component of this winner is a

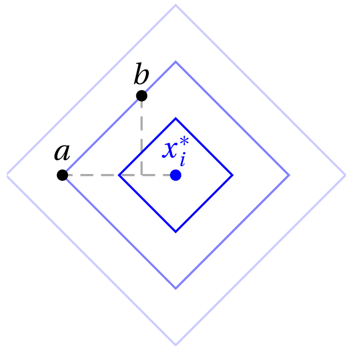
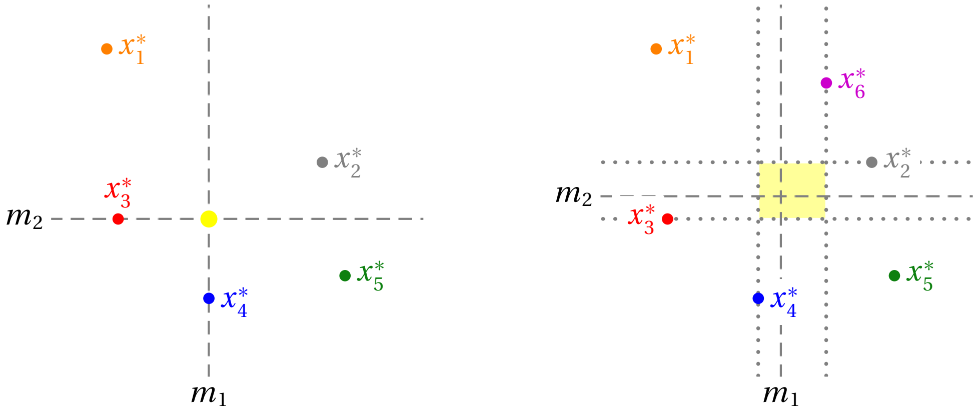


Figure 1.10 Some indifference sets for a preference relation reflecting city block preferences. The individual's favorite alternative is x_i^*, and darker indifference sets correspond to more preferred alternatives. The city-block distance from x_i^* to a, for example, is the same as the city-block distance from x_i^* to b.

median of the individuals' favorite values of that component. More precisely, for a problem in which the number of individuals is odd, denote by m_j, for $j = 1$, 2, the (unique) median of $(x_{ij}^*)_{i \in N}$, the collection of the jth components of the individuals' favorite alternatives. The alternative (m_1, m_2) is the strict Condorcet winner of such a problem. (Figure 1.11a shows an example.) For a problem in which the number of individuals is even, every pair (m_1, m_2) for which, for $j = 1$, 2, $m_j \leq x_{ij}^*$ for exactly half of the individuals (and hence $m_j \geq x_{ij}^*$ for the other half) is a Condorcet winner. (Figure 1.11b shows an example.)

Suppose that the number of individuals is odd and (x_1, x_2) is a point for which $|m_1 - x_1| \neq |m_2 - x_2|$. Then the reason that the number of individuals who prefer (m_1, m_2) to (x_1, x_2) exceeds the number with the opposite preference may be seen geometrically. Figure 1.12a shows a case in which (x_1, x_2) lies northeast of (m_1, m_2). An individual whose favorite alternative is in the area shaded green prefers (m_1, m_2) to (x_1, x_2). This area includes all alternatives (z_1, z_2) with $z_2 \leq m_2$, so that by the definition of m_2 it includes the favorite positions of a majority of individuals. The arguments for the cases in which (x_1, x_2) lies in one of the three other quadrants are symmetric.

If (x_1, x_2) is a point for which $|m_1 - x_1| = |m_2 - x_2|$ then the favorite positions of the individuals who prefer (m_1, m_2) to (x_1, x_2) lie in the area shaded green in Figure 1.12b. The argument that the favorite positions of a majority of individuals lie in this area is given in the proof of the result.



(a) A problem with five individuals. The strict Condorcet winner is (m_1, m_2), indicated by the yellow disk.

(b) A problem with six individuals. Each alternative in the yellow rectangle (for example (m_1, m_2)) is a Condorcet winner.

Figure 1.11 Collective choice problems in which the set of alternatives is $\mathbb{R}^2$ and each individual has city block preferences.

Proposition 1.6: Condorcet winner when alternatives are two-dimensional , with city block preferences

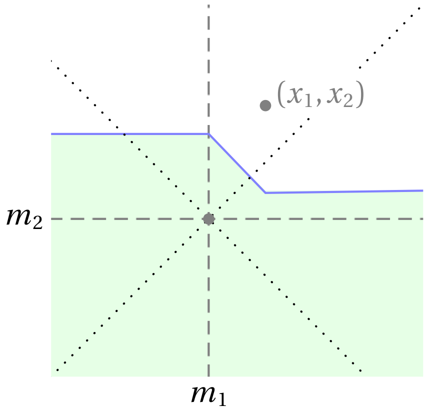
Let $\langle N, X, \succsim \rangle$ be a collective choice problem for which the set N of individuals is finite, the set X of alternatives is $\mathbb{R}^2$, and each individual has city block preferences.

If the number of individuals is odd, for $j \in \{1, 2\}$ let m_j be the median of the jth components of the individuals' favorite alternatives with respect to the linear order $\geq$. Then the alternative (m_1, m_2) is the strict Condorcet winner of $\langle N, X, \succsim \rangle$.

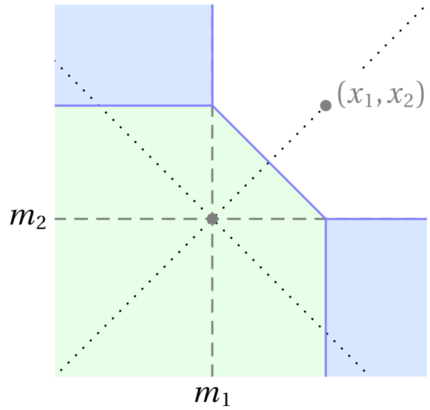
If the number of individuals is even, for $j \in \{1, 2\}$ let $\underline{m}_j$ and $\overline{m}_j$, with $\underline{m}_j \leq \overline{m}_j$, be the medians of the jth components of the individuals' favorite alternatives with respect to $\geq$. Then the Condorcet winners of $\langle N, X, \succsim \rangle$ are the alternatives (m_1, m_2) with $m_j \in [\underline{m}_j, \overline{m}_j]$ for $j = 1, 2$.

Proof

Denote the number of individuals by n. First suppose that n is odd. Let $(x_1, x_2) \in \mathbb{R}^2$ be an alternative different from (m_1, m_2). I argue that a strict majority of the individuals prefer (m_1, m_2) to (x_1, x_2), so that (m_1, m_2) is a strict Condorcet winner. Denote the favorite position of each individual i by x_i^*.



(a) $|m_1 - x_1| \neq |m_2 - x_2|$.



(b) $|m_1 - x_1| = |m_2 - x_2|$.

Figure 1.12 Collective choice problems in which the set of alternatives is $\mathbb{R}^2$ and each individual has city block preferences. In each case, an individual whose favorite alternative is on the blue line or in the area shaded blue is indifferent between (m_1, m_2) and (x_1, x_2), and one whose favorite alternative is in the area shaded green prefers (m_1, m_2) to (x_1, x_2).

First suppose that $x_2 > m_2 + |x_1 - m_1|$ (as in the example in Figure 1.12a). Then every individual i whose favorite position is at most m_2 (individuals 3, 4, and 5 in Figure 1.11a) prefers (m_1, m_2) to (x_1, x_2):

$$\begin{aligned} |x_1 - x^*_{i1}| + |x_2 - x^*_{i2}| &> |x_1 - x^*_{i1}| + |m_2 - x^*_{i2} + |x_1 - m_1|| \\ &= |x_1 - x^*_{i1}| + |m_1 - x_1| + |m_2 - x^*_{i2}| \text{ (given } m_2 \geq x^*_{i2}) \\ &\geq |m_1 - x^*_{i1}| + |m_2 - x^*_{i2}|. \end{aligned}$$

By the definition of m_2, the number of such individuals is $\frac{1}{2}(n+1)$, a strict majority.

Now suppose that $x_2 = m_2 + |x_1 - m_1|$, with $x_1 > m_1$, as in the example in Figure 1.12b, so that (x_1, x_2) lies on the black line in Figure 1.13 (which excludes the point (m_1, m_2)). Divide the space into four regions, as in the figure; each region consists of an area shaded in a light color plus the boundaries with the corresponding dark color, if any. Denote the numbers of the individuals' favorite alternatives in the regions by n_1, n_2, n_3, and n_4, as shown in the figure. (In the case shown in Figure 1.11a, for example, $n_1 = 2$ and $n_2 = n_3 = n_4 = 1$.) Given the definitions of m_1 and m_2, we have $n_1 + n_2 > n_3 + n_4$ and $n_1 + n_3 > n_2 + n_4$, so that $n_1 > n_4$. By variants of the argument for the previous case, the n_1 individuals with favorite alternatives in the orange (southwest) region prefer (m_1, m_2) to (x_1, x_2) and the $n_2 + n_3$ individuals with favorite alternatives in the green and violet regions like

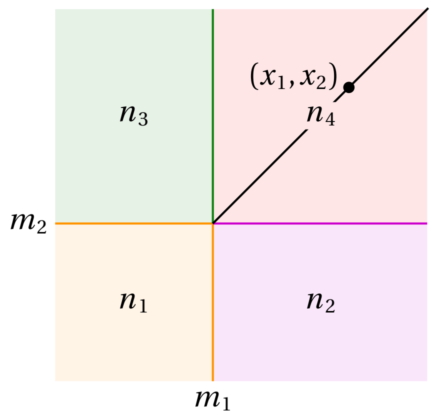


Figure 1.13 The regions in the second part of the proof of Proposition 1.6.

(m_1, m_2) at least as much as (x_1, x_2). Thus given $n_1 > n_4$, more individuals prefer (m_1, m_2) to (x_1, x_2) than prefer (x_1, x_2) to (m_1, m_2).

Similar arguments apply to every alternative (x_1, x_2) with $|m_1 - x_1| = |m_2 - x_2|$ and $x_1 < m_1$, so that (m_1, m_2) is a strict Condorcet winner.

If n is even, for any (m_1, m_2) satisfying the condition in the result, similar arguments establish that for any other alternative (x_1, x_2), at least as many individuals prefer (m_1, m_2) to (x_1, x_2) as prefer (x_1, x_2) to (m_1, m_2), so that (m_1, m_2) is a Condorcet winner.

This result does not generalize to problems in higher dimensional spaces: not all problems with city block preferences in spaces of three or more dimensions have Condorcet winners (Wendell and Thorson 1974, Example 3.1).

1.6.2 Max preferences

Suppose that each individual focuses exclusively on the dimension for which an alternative differs most from her favorite alternative.

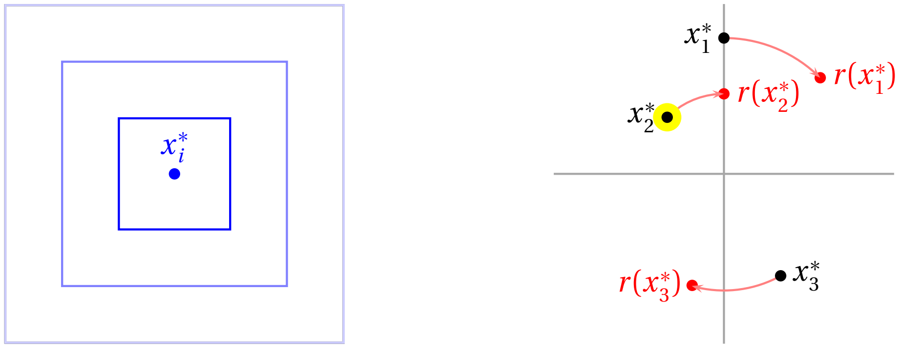
Definition 1.25: Max preferences

A preference relation $\succsim_i$ on $\mathbb{R}^2$ reflects *max preferences* if for some alternative $(x_{i1}^*, x_{i2}^*) \in \mathbb{R}^2$ it is represented by the payoff function u_i defined by

$$u_i(x_1, x_2) = -\max\{|x_1 - x_{i1}^*|, |x_2 - x_{i2}^*|\} \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2.$$

The indifference sets for such preferences are squares centered at x_i^* with sides parallel to the axes, as in Figure 1.14a.

Given Proposition 1.6 and the fact that these indifference sets are rotations



(a) Indifference sets of an individual with max preferences and favorite alternative x_i^*. Darker indifference sets correspond to higher payoffs.

(b) A collective choice problem with three individuals in which each individual has max preferences. The Condorcet winner is x_2^*.

Figure 1.14

by 45 degrees of the indifference sets for city block preferences, we can find the Condorcet winners of a collective choice problem in which each individual has max preferences as follows. Rotate the set of the individuals' favorite positions by 45 degrees and for $j \in \{1,2\}$ let m_j be a median of the jth components of the rotated alternatives. A Condorcet winner of the problem is the alternative obtained by applying the inverse of the rotation to (m_1, m_2). (The outcome is independent of the center and direction of the rotation.)

Proposition 1.7: Condorcet winner when alternatives are two-dimensional , with max preferences

Let $\langle N, X, \succsim \rangle$ be a collective choice problem for which the set N of individuals is finite, the set X of alternatives is $\mathbb{R}^2$, and each individual has max preferences. Denote the favorite alternative of each individual $i \in N$ by x_i^* and let $r : \mathbb{R}^2 \to \mathbb{R}^2$ be a 45° rotation of the plane. An alternative x^* is a Condorcet winner of $\langle N, X, \succsim \rangle$ if and only if for each $j \in \{1,2\}$ the number $r_j(x^*)$ is a median of the set of points $r_j(x_i^*)$ for $i \in N$.

Figure 1.14b shows an example with three individuals in which I use a clockwise rotation about $(0,0)$. For an arbitrary problem, a Condorcet winner is not necessarily the favorite alternative of any individual, but in this example the strict Condorcet winner is the favorite alternative of individual 2. Note that the value of its first dimension, x_{21}^*, is the smallest value of the first dimension over all the individuals' favorite alternatives. It is the strict Condorcet winner of the problem because for each of the other individuals, the second dimension, not the first, is salient: $|x_{j2}^* - x_{22}^*| > |x_{j1}^* - x_{21}^*|$ for $j \in \{1,3\}$. To verify directly that it is the strict Condorcet winner, let x be an alternative different from x_2^*. If $x_2 < x_{22}^*$ then individuals 1 and 2 prefer x_2^* to x; if $x_2 > x_{22}^*$ then individuals 2 and 3 prefer x_2^* to x; and if $x_2 = x_{22}^*$ then individual 2 prefers x_2^* to x and individuals 1 and 3 are indifferent between x_2^* and x.

1.6.3 Euclidean preferences

Finally, suppose that each individual evaluates an alternative x according to the length of the line segment between her favorite alternative and x, the Euclidean distance between these alternatives.

Definition 1.26: Euclidean preferences

A preference relation $\succsim_i$ on $\mathbb{R}^2$ reflects *Euclidean preferences* if for some alternative $x_i^* \in \mathbb{R}^2$ it is represented by the payoff function u_i defined by

$$u_i(x) = -\|x - x_i^*\| \quad \text{for all } x \in \mathbb{R}^2$$

where $\|x - x_i^*\|$ is the Euclidean distance between x and x_i^* (the length of the line segment joining x and x_i^*).

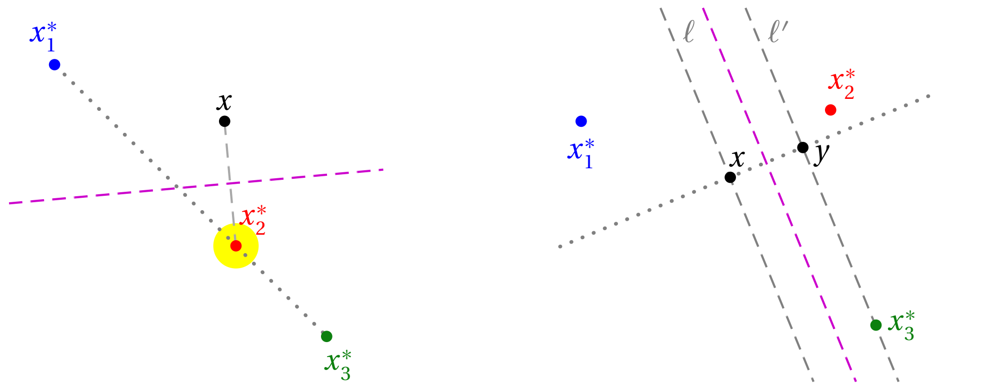
Each indifference set of an individual with Euclidean preferences has the form

$$\{x \in \mathbb{R}^2 : \|x - x_i^*\| = c\}$$

for some number c, a circle with center x_i^*.

Some collective choice problems in which every individual has Euclidean preferences have Condorcet winners. Suppose, for example, that there are three individuals and their favorite alternatives are the ones given in Figure 1.15a. The favorite alternative of individual 2, x_2^*, is a Condorcet winner of this collective choice problem. Take any other alternative, x. An individual prefers x_2^* to x if and only if x_2^* is closer to her favorite alternative than is x. The alternatives equidistant from x_2^* and x lie on the perpendicular bisector of the line segment joining x_2^* and x, indicated by the violet dashed line in Figure 1.15a. All individuals with favorite alternatives on the x_2^*-side of this perpendicular bisector prefer x_2^* to x, and all individuals with favorite alternatives on the x-side of it prefer x to x_2^*. Thus x_2^* beats the alternative x in the figure two to one, and you should be able to convince yourself that it beats every other alternative by at least two to one. Thus x_2^* is a Condorcet winner—in fact, the strict Condorcet winner.

The fact that x_3^* is a Condorcet winner in this example depends on the fact that it lies on the line through x_1^* and x_3^*. Consider the collective choice problem in Figure 1.15b, in which x_2^* does not have this property. This problem has no



(a) A collective choice problem with a Condorcet winner (x_2^*).

(b) A collective choice problem with no Condorcet winner.

Figure 1.15 Collective choice problems in which the set of alternatives is $\mathbb{R}^2$, the set of all pairs of real numbers.

Condorcet winner. For every alternative x, there is a line ℓ through x with the property that the favorite alternatives of more individuals lie on one side of ℓ than on the other side, where an alternative on ℓ is counted as being on both sides. For example, in the figure, the favorite alternatives of individuals 2 and 3 lie on the right of ℓ and the favorite alternative of individual 1 lies on the left of it. Now select the favorite alternative on the side of ℓ where a majority of favorite alternatives lie that is closest to ℓ (x_3^* in the figure) and draw a line ℓ' through it parallel to ℓ. Finally, choose the point y that is on both ℓ' and a line through x perpendicular to ℓ. Then more individuals prefer y to x than x to y, showing that x is not a Condorcet winner.

The next result generalizes these arguments to collective choice problems with any number of individuals.

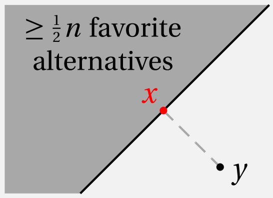
Proposition 1.8: Condorcet winner when alternatives are two-dimensional, with Euclidean preferences

Consider a collective choice problem $\langle N, X, \succsim \rangle$ in which the set N of individuals is finite, the set X of alternatives is $\mathbb{R}^2$, and each individual has Euclidean preferences. An alternative x is a Condorcet winner of $\langle N, X, \succsim \rangle$ if and only if the favorite alternatives of at least half of the individuals lie on each side of every line through x (with a point on the line counted as being on both sides).

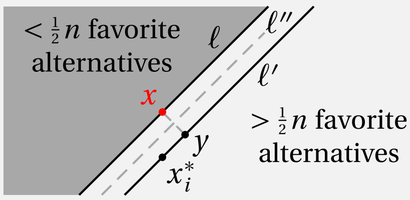
Proof

Denote the number of individuals by n.

I first argue that if x is an alternative with the property that the favorite alternatives of at least half of the individuals lie on each side of every line through x then x is a Condorcet winner. Let y be another alternative. Draw the line through x perpendicular to the line segment joining x and y. By assumption, the favorite alternatives of at least $\frac{1}{2}n$ individuals lie on the side of this perpendicular line opposite to y, so that those individuals prefer x to y. Thus x is a Condorcet winner.



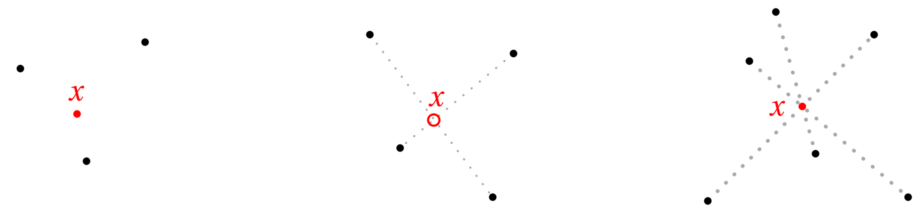
I now argue that if x does not satisfy the property then it is not a Condorcet winner. Given that fewer than $\frac{1}{2}n$ individuals' favorite alternatives lie on one side of some line ℓ through x, more than $\frac{1}{2}n$ individuals' favorite alternatives lie *strictly* on the other side of ℓ. Let x_i^* be a favorite alternative closest to ℓ among those on the other side of ℓ, and draw a line ℓ' parallel to ℓ through x_i^*.



Choose the alternative y at the intersection of ℓ' and the line perpendicular to ℓ (and to ℓ') through x. Let ℓ'' be the line midway between ℓ and ℓ' (the dashed line in the diagram). The alternative y is preferred to x by all individuals with favorite alternatives on the opposite side (relative to x) of ℓ'', who number more than $\frac{1}{2}n$, so x is not a Condorcet winner.

The condition in the result for an alternative to be a Condorcet winner is aptly characterized as requiring the alternative to be a "median in every direction".

For every collective choice problem in which the set of alternatives is two-dimensional and there are four individuals, an alternative exists that satisfies the condition. If one of the individual's favorite alternatives lies within the triangle formed by the other individuals' favorite alternatives, as in Figure 1.16a, then this



(a) Four individuals, Condorcet winner x.

(b) Four individuals, Condorcet winner x.

(c) Seven individuals, (strict) Condorcet winner x.

Figure 1.16 Configurations of favorite alternatives satisfying the condition in Proposition 1.8 for x to be a Condorcet winner. Each small disk, black or red, is an individual's favorite alternative. (In the middle case, x, indicated by a circle, is not a favorite alternative of any individual.)

alternative satisfies the condition and hence is a Condorcet winner. Otherwise, as in Figure 1.16b, the favorite alternatives can be divided into two pairs with the property that the lines connecting the members of each pair intersect inside the quadrilateral formed by the alternatives. In this case, the alternative at the intersection of the line (and only this alternative) is a Condorcet winner.

For most collective choice problems with a two-dimensional set of alternatives and either three individuals or at least five, no alternative is a median in every direction. Suppose, for example, that the number n of individuals is odd. Then $\frac{1}{2}n$ is not an integer, and the condition requires the favorite alternatives of at least $\frac{1}{2}(n+1)$ individuals to lie on each side of every line through x. Thus x must be the favorite alternative of some individual. Further, as a line through x rotates, whenever it passes through an alternative different from x it must at the same time pass through another such alternative, to keep $\frac{1}{2}(n+1)$ alternatives on each side of it. If $n = 3$ only configurations of the individuals' favorite alternatives that lie on a line satisfy this condition; the middle alternative is the Condorcet winner. An example of a configuration satisfying the condition for $n = 7$ is shown in Figure 1.16c.

1.7 Preference aggregation: Arrow's theorem

The object of study of the previous sections is a collective choice rule, which associates with each collective choice problem an alternative or set of alternatives, based on the individuals' preferences. This section considers the possibility of associating with each collective choice problem a preference relation over the set of alternatives; that is, the possibility of aggregating the individuals' preference relations into a single preference relation. Rather than looking for an alternative that best reflects the individuals' possibly diverse preference relations, we look for an entire ranking of the alternatives that best reflects these preference relations. One motivation for this line of inquiry is that the members of the society do not currently know the collective choice problem they will face, and they wish to be prepared for whatever problem arises. Armed with a preference relation for society, for any set of alternatives that arises they can select the alternative that is best according to that preference relation. The main result is due to Kenneth J. Arrow (1921–2017), whose work is the foundation of social choice theory.

I assume that every individual has a strict preference relation over the set of alternatives. Our job is to associate with every profile of such preference relations a single (not necessarily strict) preference relation over the set of alternatives, which I call the social preference relation. That is, we seek a preference aggregation function, defined as follows.

Definition 1.27: Preference aggregation function

Let $\langle N, X\rangle$ be a society and D a set of strict preference profiles for $\langle N, X\rangle$. A *preference aggregation function* for $(\langle N, X\rangle, D)$ is a function that assigns a ("social") preference relation over X to every collective choice problem $\langle N, X, \succsim\rangle$ with $\succsim \in D$.

Example 1.6: Borda aggregation

Assign to each collective choice problem the preference relation $\trianglerighteq$ over the set of alternatives that ranks x at least as highly as y if and only if the number of points $\sum_{i\in N} p_i(x)$ that the Borda rule assigns to x is at least the number it assigns to y. For instance, for the problem in Example 1.2, this preference relation is given by $b \triangleright a \triangleright c \triangleright d$ (b is assigned 7 points, a is assigned 6, c is assigned 4, and d is assigned 1).

Example 1.7: Condorcet aggregation

Assign to each alternative one point for every alternative it beats and half a point for every alternative with which it ties. Rank the alternatives by the number of points received. For example, for the problem in Example 1.2, the preference relation $\trianglerighteq$ thus defined is given by $a \triangleright b \triangleright c \triangleright d$ (a beats b, c, and d, b beats c and d, c beats d, and d beats no alternative).

The following property of a preference aggregation function plays a central role in the analysis: the ranking of two alternatives according to the social preference relation depends only on the individuals' preferences between these two alternatives. More precisely, if for two preference profiles $\succsim$ and $\succsim'$ and two alternatives x and y, the preference of every individual i between x and y is the same according to $\succsim_i$ as it is according to $\succsim'_i$, then the social preference between x and y generated by $\succsim$ is the same as the social preference between these alternatives generated by $\succsim'$. This property is known as *independence of irrelevant alternatives* (a name that seems calculated to convince the reader that the property is reasonable).

Definition 1.28: Independence of irrelevant alternatives (IIA)

Let $\langle N, X\rangle$ be a society and D a set of strict preference profiles for $\langle N, X\rangle$. A preference aggregation function F for $(\langle N, X\rangle, D)$ is *independent of irrelevant alternatives* (IIA) if for all preference profiles $\succsim\, \in D$ and $\succsim' \in D$ for $\langle N, X\rangle$ and any alternatives $x \in X$ and $y \in X$ with

$$x \succ_i y \text{ if and only if } x \succ'_i y \quad \text{for every } i \in N$$

we have

$$x \trianglerighteq y \text{ if and only if } x \trianglerighteq' y,$$

where $\trianglerighteq\, = F(N, X, \succsim)$ and $\trianglerighteq' = F(N, X, \succsim')$, the social preference relations assigned by F to the collective choice problems $(N, X, \succsim)$ and $(N, X, \succsim')$.

The Borda preference aggregation function defined in Example 1.6 does not satisfy this property for a domain that includes the following two collective choice problems.

1	2	3
a	a	a
b	b	b
c	c	d
d	d	c

1	2	3
c	a	d
d	b	a
a	c	b
b	d	c

Each individual ranks c relative to d in the same way for both problems, but for the problem on the left c gets more points than d (2 versus 1) whereas for the problem on the right the opposite is true (c gets 4 points and d gets 5). Thus according to the Borda rule c is socially preferred to d for the left problem but d is socially preferred to c for the right problem.

The Condorcet aggregation function defined in Example 1.7 also does not satisfy independence of irrelevant alternatives for a domain that includes these two problems. For the problem on the left, c beats only d, and d does not beat any alternative, so that c is socially preferred to d, but for the problem on the right, c beats only d, and d beats a and b, so that d is socially preferred to c.

One preference aggregation function that does satisfy independence of irrelevant alternatives assigns to every preference profile the social preference relation in which all alternatives are indifferent. In addition to failing to meaningfully aggregate the individuals' preference relations, this function violates the natural requirement that the social preference relation rank one alternative above another one whenever all the individuals do so. This requirement is called the Pareto property, after Vilfredo Pareto (1848–1923), who introduced the idea that one social state is better than another if all individuals prefer it.

Definition 1.29: Pareto property of preference aggregation function

Let $\langle N, X\rangle$ be a society and let D be a set of strict preference profiles for $\langle N, X\rangle$. A preference aggregation function F for $(\langle N, X\rangle, D)$ has the *Pareto property* if for every preference profile $\succsim \in D$ for which there are alternatives $x \in X$ and $y \in X$ with $x \succ_i y$ for all $i \in N$, we have $x \rhd y$, where $\unrhd = F(N, X, \succsim)$ (and $\rhd$ is the strict relation associated with $\unrhd$).

A simple argument shows that for a society with at least three alternatives, an even number of individuals, and a domain that includes three basic preference profiles, no preference aggregation function satisfies independence of irrelevant alternatives, the Pareto property, and adaptations of anonymity and neutrality to preference aggregation. Consider a domain that includes preference profiles for which (*i*) individuals 1 and 2 have the following preferences over alternatives a, b, and c and are indifferent between any other alternatives, which they rank below a, b, and c, (*ii*) all other odd-numbered individuals have the same preferences as individual 1, and (*iii*) all other even-numbered individuals have the same preferences as individual 2.

1	2
b	c
a	a
c	b

1	2
c	c
b	a
a	b

1	2
b	c
c	a
a	b

Alternatives b and c are symmetric in the first problem, so anonymity and neutrality imply they are socially indifferent. Similarly, a and b must be socially indifferent in the second problem. Now, each individual's rankings of b and c in the first and third problems are the same, so by independence of irrelevant alternatives, the fact that these alternatives are socially indifferent in the first problem means that they are socially indifferent in the third problem. Similarly, each individual's rankings of a and b in the second and third problems are the same, so these alternatives are socially indifferent in the third problem. By the transitivity of social preferences, a and c are thus socially indifferent in the third problem. This conclusion conflicts with the implication of the Pareto property that, because all individuals prefer c to a in this problem, c is socially preferred to a.

The next result, Arrow's impossibility theorem, does not impose the assumptions of anonymity and neutrality on a preference aggregation function. Without these assumptions, preference aggregation functions that satisfy independence of irrelevant alternatives and the Pareto property exist: select any individual $i^* \in N$ and assign i^*'s preference relation, $\succsim_{i^*}$, to every preference profile. Such a preference aggregation function is, naturally, called dictatorial.

Definition 1.30: Dictatorial preference aggregation function

Let $\langle N, X\rangle$ be a society and let D be a set of strict preference profiles for $\langle N, X\rangle$. A preference aggregation function F for $(\langle N, X\rangle, D)$ is *dictatorial* if for some individual $i^* \in N$, for every preference profile $\succsim \in D$ and any alternatives x and y, we have $x \trianglerighteq y$ if and only if $x \succsim_{i^*} y$, where $\trianglerighteq = F(N, X, \succsim)$.

Arrow's theorem shows that for a society with at least three alternatives and the domain of all strict preference profiles, a preference aggregation function satisfies independence of irrelevant alternatives and the Pareto property *only* if it is dictatorial.

Proposition 1.9: Arrow's impossibility theorem

Let $\langle N, X\rangle$ be a finite society for which X contains at least three alternatives and let D be the set of all strict preference profiles for $\langle N, X\rangle$. A preference aggregation function for $(\langle N, X\rangle, D)$ satisfies the Pareto property and independence of irrelevant alternatives if and only if it is dictatorial.

Proof

If a preference aggregation function is dictatorial then it satisfies the Pareto property and independence of irrelevant alternatives (IIA). I now show the converse.

Let F be a preference aggregation function for $(\langle N, X\rangle, D)$ that satisfies the Pareto property and IIA. Let $N = \{1, \ldots, n\}$ and fix $b \in X$.

Step 1 *Let $\succsim$ be a preference profile for $\langle N, X\rangle$ for which b is either at the top or the bottom of each individual's ranking, and let $\trianglerighteq = F(N, X, \succsim)$. Then b is either the unique maximal alternative for $\trianglerighteq$ or the unique minimal alternative for $\trianglerighteq$ (that is, either $b \triangleright x$ for all $x \in X \setminus \{b\}$ or $x \triangleright b$ for all $x \in X \setminus \{b\}$).*

Proof. Assume to the contrary that for two other alternatives a and c we have $a \trianglerighteq b \trianglerighteq c$. Let $\succsim'$ be a preference profile that is obtained from $\succsim$ by moving c just above a for every individual i for whom $a \succ_i c$ (so that it remains below b for all individuals for whom b is best), as illustrated in the following example.

1	2	3	$\cdots$	$n-1$	n
b	b		$\cdots$	a	b
	a	a	$\cdots$		
c	c		$\cdots$	c	a
		c	$\cdots$		c
a		b	$\cdots$	b	
			$\succsim$		

$\rightarrow$

1	2	3	$\cdots$	$n-1$	n
b	b		$\cdots$	c	b
	c	c	$\cdots$	a	
c	a	a	$\cdots$		c
			$\cdots$		a
a		b	$\cdots$	b	
			$\succsim'$		

Let $\trianglerighteq' = F(N, X, \succsim')$. By the Pareto property, $c \triangleright' a$. We have $a \succ_i b$ if and only if $a \succ'_i b$ for every $i \in N$ and $a \trianglerighteq b$, so that $a \trianglerighteq' b$ by IIA. Also, $b \succ_i c$ if and only if $b \succ'_i c$ for every $i \in N$ and $b \trianglerighteq c$, so that $b \trianglerighteq' c$ by IIA. Thus $a \trianglerighteq' c$ by the transitivity of $\trianglerighteq'$, contradicting $c \triangleright' a$. ◁

Step 2 *Consider two preference profiles $\succsim$ and $\succsim'$ for $\langle N, X\rangle$ for which b is either at the top or the bottom of each individual's ranking and the set of individuals who rank b at the top is the same. Let $\trianglerighteq = F(N, X, \succsim)$ and $\trianglerighteq' = F(N, X, \succsim')$. Then b is either the unique maximal alternative for both $\trianglerighteq$ and $\trianglerighteq'$ or the unique minimal alternative for both $\trianglerighteq$ and $\trianglerighteq'$.*

Proof. By Step 1, b is either the unique maximal alternative for $\trianglerighteq$ or the unique minimal alternative for $\trianglerighteq$. Suppose the former. The ranking of b relative to any other alternative x is the same in $\succsim_i$ and $\succsim'_i$ for every individual i and hence by IIA b and x are ranked in the same way by both $\trianglerighteq$ and $\trianglerighteq'$. Thus b is the unique maximal alternative for $\trianglerighteq'$. If b is the unique minimal alternative for $\trianglerighteq$ the argument is analogous. ◁

Step 3 *For some individual $i^* \in N$,*

i. for every preference profile $\succsim$ for $\langle N, X\rangle$ for which $1, \ldots, i^*-1$ rank b at the top and $i^*, \ldots, n$ rank it at the bottom, the preference relation $F(N, X, \succsim)$ ranks b uniquely at the bottom

ii. for every preference profile $\succsim$ for $\langle N, X\rangle$ for which $1, \ldots, i^*$ rank b at the top and $i^*+1, \ldots, n$ rank it at the bottom, the preference relation $F(N, X, \succsim)$ ranks b uniquely at the top.

Proof. Let $\succsim$ be a preference profile in which b is at the bottom of all individuals' preferences. By the Pareto property, b is the unique minimal alternative for $F(N, X, \succsim)$. Now, for each individual i in turn, starting with individual 1, move b from the bottom to the top of i's preferences. By Step 1, b is always either the unique maximal or unique minimal alternative for the resulting social preferences. By the Pareto property, it is the unique maximal alternative after it moves to the top of all individuals' preferences. Let i^* be the first individual for whom the change in her preferences moves b from the bottom to the top of the social preferences. By Step 2, the identity of i^* does not depend on the individuals' rankings of the other alternatives. ◁

Step 4 *For any preference profile $\succsim$ for $\langle N, X\rangle$ and all alternatives a and c different from b we have $a \rhd c$ if and only if $a \succ_{i^*} c$, where $\unrhd = F(N, X, \succsim)$ and i^* is the individual identified in Step 3.*

Proof. Assume to the contrary that for some preference profile $\succsim$ for $\langle N, X\rangle$ we have $a \succ_{i^*} c$ and $c \unrhd a$. Let $\succsim'$ be the profile obtained from $\succsim$ by raising b to the top of the preferences of individuals $1, \ldots, i^*-1$, lowering it to the bottom of the preferences of individuals $i^*+1, \ldots, n$, and moving it between a and c for individual i^*, as in the following example.

1	⋯	i^*-1	i^*	i^*+1	⋯	n		1	⋯	i^*-1	i^*	i^*+1	⋯	n
	⋯	c		a	⋯	b		b	⋯	b		a	⋯	
b	⋯		a		⋯				⋯	c	a		⋯	c
a	⋯			c	⋯	c	$\rightarrow$	a	⋯		b	c	⋯	a
	⋯	a	c	b	⋯	a			⋯				⋯	
c	⋯	b	b		⋯			c	⋯	a	c	b	⋯	b
			$\succsim$								$\succsim'$			

Let $\unrhd' = F(N, X, \succsim')$. The relative positions of a and c are the same in $\succsim$ and $\succsim'$, so $c \unrhd' a$ by IIA. In $\succsim'$, the individuals' rankings of a relative to b are the same as they are in any profile in which b is ranked at the top by $1, \ldots, i^*-1$ and at the bottom by the remaining individuals, so that by Step 3*i* and IIA we have $a \rhd' b$. Similarly, using Step 3*ii* and IIA, $b \rhd' c$. Thus by transitivity $a \rhd' c$, contradicting $c \unrhd a$. ◁

Step 4 says that i^* is the dictator regarding any two alternatives other than b. It remains to show that i^* is also the dictator regarding the comparison of b with any other alternative.

Step 5 *For any preference profile* $\succsim$ *for* $\langle N, X\rangle$ *and every alternative* a *we have* $a \rhd b$ *if and only if* $a \succ_{i^*} b$, *where* $\trianglerighteq = F(N, X, \succsim)$ *and* i^* *is the individual identified in Step 3.*

Proof. Let $\succsim$ be a preference profile for which $a \succ_{i^*} b$. Let c be an arbitrary third alternative. Let $\succsim'$ be a preference profile obtained from $\succsim$ by moving c in i^*'s ranking to between b and a (if it is not already there) and raising c to the top of all the other individuals' rankings, as in the following example.

1	$\cdots$	i^*-1	i^*	i^*+1	$\cdots$	n		1	$\cdots$	i^*-1	i^*	i^*+1	$\cdots$	n
	$\cdots$	c		a	$\cdots$	b		c	$\cdots$	c		c	$\cdots$	c
c	$\cdots$		a		$\cdots$				$\cdots$		a	a	$\cdots$	b
a	$\cdots$		b	c	$\cdots$	c	$\rightarrow$	a	$\cdots$		c		$\cdots$	
	$\cdots$	b		b	$\cdots$				$\cdots$	b	b	b	$\cdots$	
b	$\cdots$	a	c		$\cdots$	a		b	$\cdots$	a			$\cdots$	a
			$\succsim$								$\succsim'$			

Let $\trianglerighteq' = F(N, X, \succsim')$. By the Pareto property, $c \rhd' b$. By Step 4, given $a \succ'_{i^*} c$ we have $a \rhd' c$ and hence $a \rhd' b$ by transitivity. Since $a \succ_i b$ if and only if $a \succ'_i b$ for all $i \in N$, we thus have $a \rhd b$ by IIA. $\lhd$

This result has the same flavor as Proposition 1.3: no aggregation method satisfies a list of attractive properties for the domain of all possible preference profiles. The results differ both in the type of aggregation considered—collective choice rule or preference aggregation function—and in the nature of the properties imposed, but the message is similar.

Proposition 1.2 shows that for the domain of collective choice problems that have a strict Condorcet winner, a collective choice rule that satisfies a list of attractive properties *does* exist: the one that assigns to each problem its strict Condorcet winner. An analogous result holds for preference aggregation functions.

Recall that the majority relation for a collective choice problem is the binary relation for which any alternative a is preferred to another alternative b if and only if a majority of individuals prefer a to b. For a problem $\langle N, X, \succsim\rangle$ with the property that for every subset X' of X (including X itself) the problem $\langle N, X', \succsim|_{X'}\rangle$ has a strict Condorcet winner, the majority relation is transitive, so that the function that assigns it to each problem in the domain of problems with this property is a preference aggregation function. This preference aggregation function also satisfies the Pareto property and independence of irrelevant alternatives.

Proposition 1.10: Preference aggregation with a strict Condorcet winner

Let $\langle N, X\rangle$ be a finite society and let D be the set of strict preference profiles $\succcurlyeq$ for $\langle N, X\rangle$ for which for every subset X' of X the collective choice problem $\langle N, X', \succcurlyeq|_{X'}\rangle$ has a strict Condorcet winner. For any preference profile $\succcurlyeq \in D$, the majority relation for $\langle N, X, \succcurlyeq\rangle$ is a preference relation, and the preference aggregation function for $(\langle N, X\rangle, D)$ that for each $\succcurlyeq \in D$ assigns to $\langle N, X, \succcurlyeq\rangle$ the majority relation for $\langle N, X, \succcurlyeq\rangle$ satisfies the Pareto property and independence of irrelevant alternatives.

Proof

By definition the majority relation for $\langle N, X, \succcurlyeq\rangle$ is complete. I now argue that it is transitive. Let $x \in X$, $y \in X$, and $z \in X$ and suppose that a majority of individuals prefer x to y and a majority prefer y to z. Let $X' = \{x, y, z\}$. By assumption, the collective choice problem $\langle N, X', \succcurlyeq|_{X'}\rangle$ has a strict Condorcet winner. This alternative is not y, because a majority of individuals prefer x to y, and it is not z, because a majority of individuals prefer y to z, so it must be x. Thus a majority of individuals prefer x to z and hence the majority relation for $\langle N, X, \succcurlyeq\rangle$ is transitive. We conclude that the majority relation for $\langle N, X, \succcurlyeq\rangle$ is a preference relation.

Denote the majority relation by $\trianglerighteq$. If for any alternatives x and y all individuals prefer x to y then $x \vartriangleright y$, so the Pareto property is satisfied. Whether $x \trianglerighteq y$ or $y \trianglerighteq x$ depends only on the individuals' preferences between x and y, so also independence of irrelevant alternatives is satisfied.

If a preference profile on a set X is single-peaked with respect to a linear order $\trianglerighteq$, then the restriction of the profile to a subset of X is single-peaked with respect to the restriction of $\trianglerighteq$ to the subset, so by Proposition 1.4 the domain of single-peaked preference profiles for a society $\langle N, X\rangle$ for which the number of individuals is odd satisfies the property in this result. Hence for this domain the function that assigns to each collective choice problem its majority relation is a preference aggregation function that satisfies the Pareto property and independence of irrelevant alternatives. By Proposition 1.5, the same is true for the domain of single-crossing preference profiles for a society with an odd number of individuals, and moreover for this domain the majority relation is the preference relation of the median individual.

1.8 Preference intensities and interpersonal comparisons

The information in a collective choice problem about the individuals' preferences concerns only the individuals' rankings of the alternatives. For some collective choice problems, this information appears to be an inadequate basis for the selection of a collective action.

Consider, for example, the simplest of all collective choice problems: two individuals have to decide between two alternatives, a and b. One individual prefers a to b and the other prefers b to a. To select one of these alternatives, we need more information about the individuals' evaluations of the alternatives.

Now suppose that two individuals have to decide among three alternatives, a, b, and c. One individual prefers a to b to c and the other prefers c to b to a. The collective choices consistent with anonymity and neutrality are $\{a, c\}$, $\{b\}$, and $\{a, b, c\}$. The last is a non-choice, and to choose between $\{a, c\}$ and $\{b\}$ again we need more more information about the individuals' evaluations of the alternatives.

One piece of information that we can use to choose an alternative in both of these examples is a comparison of the individuals' welfares for the alternatives. Such information not only allows us to select alternatives in problems like these two, but may also overturn the resolutions discussed in the previous sections for other problems. For example, suppose that three individuals have to decide between the alternatives a and b. Two individuals prefer a and the third prefers b. Then a is the strict Condorcet winner, so that it is selected by any anonymous, neutral, positively responsive, and Nash independent collective choice rule (Proposition 1.2), and is ranked first in the social preference relation generated by a preference aggregation function that satisfies the Pareto property and independence of irrelevant alternatives (Proposition 1.10). But if we have information on the intensity of the individuals' preferences, and the first two individuals' preferences for a over b are slight compared with the last individual's preference for b over a, we may decide to select b rather than a, especially if the first two individuals' welfares for b are higher than the last individual's.

One reason that the models in the previous sections do not include interpersonally comparable measures of welfare is the difficulty of quantifying welfare and comparing it across individuals. We can in principle obtain information about individuals' ordinal preferences by observing their choices, but these choices do not directly reveal the individuals' welfares, and especially do not allow us to compare one individual's welfare with another's. However, individuals' welfares can be assessed and compared in other ways. We can ask individuals to report their well-beings on a common scale; we can assess the extent to which each individual experiences certain states, like hunger and ill-health; and we can observe individuals' incomes and wealths.

I assume for the remainder of this section that associated with each alternative is a profile of numbers, interpreted as the individuals' welfares. How should such profiles be ranked? My approach, as in the previous sections, is axiomatic: I state some properties for an ordering of welfare profiles that embody certain principles and investigate the orderings that satisfy these properties and hence are consistent with the principles.

A social welfare ordering is a complete transitive binary relation—that is, a preference relation—over welfare profiles.

Definition 1.31: Social welfare ordering

Let $N = \{1, \ldots, n\}$ be a set of individuals. A *social welfare ordering* over a set $Z \subseteq \mathbb{R}^n$ of welfare profiles for N is a preference relation over Z.

The interpretation of a social welfare ordering $\succsim$ is that for any profiles u and v of the individuals' welfares, u is at least as socially desirable as v if and only if $u \succsim v$. (In this section I use u and v to denote profiles of numbers, rather than functions, as elsewhere.)

One example of a social welfare ordering is the utilitarian ordering, which ranks welfare profiles according to the sum of the welfares.

Definition 1.32: Utilitarian social welfare ordering

Let $N = \{1, \ldots, n\}$ be a set of individuals. A social welfare ordering $\succsim$ over $\mathbb{R}^n$ for N is the *utilitarian ordering* if $u \succsim v$ if and only if $\sum_{i \in N} u_i \geq \sum_{i \in N} v_i$.

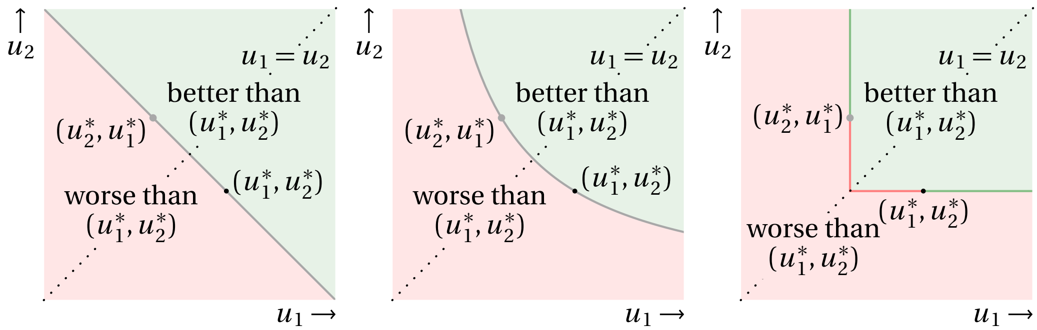
This ordering pays no attention to inequality in welfare. For a society of two individuals, for example, it makes the welfare profile $(1,1)$ indifferent to the profiles $(2,0)$ and $(5,-3)$, and ranks the profile $(101,0)$ above the profile $(50,50)$.

A related social welfare ordering ranks profiles of positive welfares according to the product of the welfares.

Definition 1.33: Nash social welfare ordering

Let $N = \{1, \ldots, n\}$ be a set of individuals. A social welfare ordering $\succsim$ over $\mathbb{R}^n_{++}$ for N is the *Nash social welfare ordering* if $u \succsim v$ if and only if $\prod_{i \in N} u_i \geq \prod_{i \in N} v_i$.

This ordering puts some weight on the equality of welfare. For a society of two individuals, for example, it ranks the welfare profile $(3,3)$ above the profile $(8,1)$.



(a) Utilitarian ordering (b) Nash ordering (c) Leximin ordering

Figure 1.17 The pairs (u_1, u_2) ranked better than (u_1^*, u_2^*) (green), worse than it (red), and equal to it (gray) by three welfare orderings. For the leximin ordering, each region includes its boundaries with a dark shade of the color of the region, and the only pair indifferent to (u_1^*, u_2^*) is (u_2^*, u_1^*).

An example of a social welfare ordering that is even more sensitive to inequality is the leximin ordering, which gives priority to the smallest welfare. When comparing the profiles u and v according to this ordering, we arrange their components according to size, smallest to largest, and rank the profiles according to the first components in this order that differ. For example, for $u = (1,4,2,1,5)$ and $v = (3,1,8,2,1)$, the orderings by component size are $(1,1,2,4,5)$ and $(1,1,2,3,8)$, so because these vectors are the same up to their third components and the fourth component of the first vector, 4, exceeds the fourth component of the second vector, 3, u is ranked above v. (The utilitarian and Nash orderings rank v above u.)

Definition 1.34: Leximin social welfare ordering

Let $N = \{1, \ldots, n\}$ be a set of individuals and for any $u \in \mathbb{R}^n$ and $k \in N$ let $k(u)$ be the kth smallest component of u (so that $u_{k(u)} \le u_{(k+1)(u)}$ for all $k = 1, \ldots, n-1$). The social welfare ordering $\succsim$ over $\mathbb{R}^n$ for N is the *leximin ordering* if $u \succsim v$ if and only if either u is a permutation of v or there exists $k \in N$ such that $u_{j(u)} = v_{j(v)}$ for $j = 1, \ldots, k-1$ and $u_{k(u)} > v_{k(v)}$.

Note that this ordering is not continuous. For example, for a society of two individuals, $(1,3)$ is better than $(1,2)$, but for all $\varepsilon > 0$, $(1-\varepsilon, 3)$ is worse than $(1,2)$.

For the case of two individuals ($n = 2$), the sets of welfare pairs ranked better than, equal to, and worse than a pair (u_1^*, u_2^*) by the three orderings are shown in Figure 1.17.

I now present axiomatic characterizations of these orderings. The characterizations share two properties, adaptations of the anonymity property for collective choice rules and the Pareto property for a preference aggregation function. The anonymity property says that an ordering does not depend on the individuals' names.

Definition 1.35: Anonymous social welfare ordering

A social welfare ordering $\succsim$ over a set $Z \subseteq \mathbb{R}^n$ of welfare profiles for a set $N = \{1, \ldots, n\}$ of individuals is *anonymous* if for every $u \in Z$ and every permutation v of u with $v \in Z$ we have $u \sim v$.

The Pareto property says that if the welfare of every individual is at least as high in u as it is in v and is higher for at least one individual, then u is ranked above v.

Definition 1.36: Strong Pareto condition for social welfare ordering

A social welfare ordering $\succsim$ over a set $Z \subseteq \mathbb{R}^n$ of welfare profiles for a set $N = \{1, \ldots, n\}$ of individuals satisfies the *strong Pareto condition* if for all $u \in Z$ and $v \in Z$,

$$u_i \geq v_i \text{ for all } i \in N \text{ with at least one strict inequality} \quad \Rightarrow \quad u \succ v.$$

Each characterization involves one additional property. For the leximin ordering this property says that the welfare profile u is ranked above the profile v if these profiles differ only in the welfares of two individuals, say i and j, i's welfare is higher in u than in v, j's welfare is higher in v than in u, and j's welfare in u is higher than i's welfare in u. Under these conditions, moving from v to u makes i better off and j worse off, and both before and after the change j is better off than i. The property is named after its originator, Peter J. Hammond.

Definition 1.37: Hammond-equitable social welfare ordering

A social welfare ordering $\succsim$ over $\mathbb{R}^n$ for a set $N = \{1, \ldots, n\}$ of individuals is *Hammond-equitable* if for all $i \in N$ and $j \in N$ with $i \neq j$ and all $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$ with $u_k = v_k$ for all $k \in N \setminus \{i, j\}$,

$$v_j > u_j > u_i > v_i \quad \Rightarrow \quad u \succ v.$$

The utilitarian, Nash, and leximin social welfare orderings are all anonymous and satisfy the strong Pareto condition, but only the leximin ordering is Hammond equitable: both the utilitarian and Nash orderings rank the profile $(1,7)$ above the profile $(2,3)$, in violation of Hammond equity. In fact, among all possible orderings, the leximin ordering is the only one that satisfies all three conditions.

Proposition 1.11: Characterization of leximin social welfare ordering

A social welfare ordering over $\mathbb{R}^n$ for a set of n individuals is anonymous and Hammond-equitable and satisfies the strong Pareto condition if and only if it is the leximin ordering.

I present a proof of this result only for a society consisting of two individuals.

Proof for two individuals

The leximin ordering satisfies the three conditions. I now show that if a social welfare ordering for a set of two individuals satisfies the three conditions then it is the leximin ordering.

First suppose that $u_1^* > u_2^*$ and consider the ordering of the pair (u_1^*, u_2^*) of welfares relative to any other pair. Refer to Figure 1.18.

- By the strong Pareto condition, pairs in the green region are ranked above (u_1^*, u_2^*) and ones in the violet region are ranked below (u_1^*, u_2^*).
- By Hammond equity, pairs in the yellow region are ranked above (u_1^*, u_2^*) and ones in the red region are ranked below (u_1^*, u_2^*).
- Given these rankings, anonymity implies that pairs in the blue region are ranked above (u_1^*, u_2^*) and ones in the brown region are ranked below (u_1^*, u_2^*).
- The pairs that remain are (u_2^*, u_1^*) and the ones on the dashed black line.

  The pair (u_2^*, u_1^*) is equivalent to (u_1^*, u_2^*) by anonymity.

  Finally, for any pair (u_1, u_2) on the dashed black line there is a pair (u_1', u_2') in the yellow region with $u_1 > u_1'$ and $u_2 > u_2'$, so that by the strong Pareto condition (u_1, u_2) is ranked above (u_1', u_2'). Given that every pair in the yellow region is ranked above (u_1^*, u_2^*), the transitivity of a social welfare ordering implies that (u_1, u_2) is ranked above (u_1^*, u_2^*).

A symmetric argument applies to pairs (u_1^*, u_2^*) with $u_1^* < u_2^*$. The only comparisons that remain are between pairs (u_1^*, u_2^*) and (u_1, u_2) with $u_1^* = u_2^*$ and $u_1 = u_2$. The strong Pareto condition implies that (u_1^*, u_2^*) is ranked above (u_1, u_2) if $u_1^* > u_1$ and below it if $u_1^* < u_1$.

We conclude that the rankings of all pairs of welfares are the ones given by the leximin ordering, shown in Figure 1.17c.

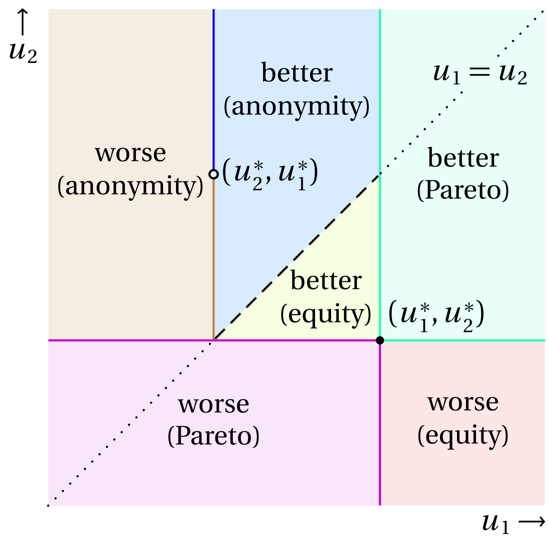


Figure 1.18 The proof of Proposition 1.11 when there are two individuals. (Each region includes any of its boundaries with a dark shade of the color of the region.)

I now present a property that, in addition to anonymity and the Pareto condition, characterizes the utilitarian social welfare ordering. The character of this property is completely different from that of Hammond-equity. It says that if one welfare profile is ranked above another, then certain transformations of the first profile are ranked above the same transformations of the second profile. Specifically, if u is ranked above v, then for any profile $(a_i)_{i \in N}$ of numbers, the profile u' defined by $u'_i = u_i + a_i$ for all $i \in N$ is ranked above the profile v' defined by $v'_i = v_i + a_i$ for all $i \in N$. (For example, if $(2,2)$ is ranked above $(4,-1)$, then $(0,5)$ is ranked above $(2,2)$.)

Definition 1.38: Invariance of social welfare ordering with respect to additive transformations of welfares

A social welfare ordering $\succsim$ over $\mathbb{R}^n$ for a set of n individuals is *invariant with respect to additive transformations of welfares* if whenever $u \succsim v$ we have $u + a \succsim v + a$ for all $a \in \mathbb{R}^n$.

This condition may make sense if we cannot observe the individuals' welfare levels but we can observe whether the difference in one individual's welfare between two welfare profiles is bigger or smaller than the difference in another individual's welfare between two other welfare profiles. The reason is that the transformations of welfare in the definition preserve comparisons of differences in welfare: $u_i - v_i \ge w_j - y_j$ if and only if $u_i + a_i - (v_i + a_i) \ge w_j + a_j - (y_j + a_j)$.

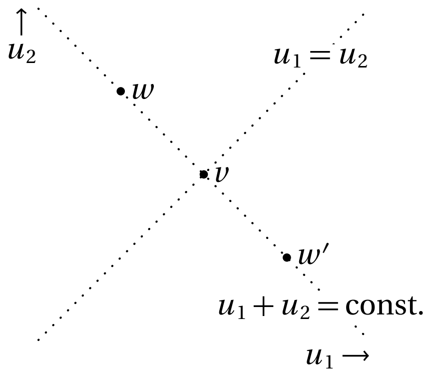


Figure 1.19 An illustration for the proof of Proposition 1.12 for a society consisting of two individuals.

Proposition 1.12: Characterization of utilitarian welfare ordering

A social welfare ordering over $\mathbb{R}^n$ for a set of n individuals is anonymous and invariant with respect to additive transformations of welfares and satisfies the strong Pareto condition if and only if it is the utilitarian ordering.

For a society consisting of two individuals, this result may be given a simple proof. Let v be a welfare pair for which $v_1 = v_2$, let w be a pair for which $w_1 + w_2 = v_1 + v_2$, and let $w' = (w_2, w_1)$, as in Figure 1.19. By anonymity, $w \sim w'$. Now let $a_1 = v_1 - w_1$ and $a_2 = v_2 - w_2$. Then $v_i = w_i + a_i$ and $w'_i = v_i + a_i$ for $i = 1, 2$. Thus if $w \succ v$ then $v \succ w'$ by invariance with respect to additive transformations of welfares, and hence $w \succ w'$, contradicting $w \sim w'$. Similarly, if $w \prec v$ then $v \prec w'$, contradicting $w \sim w'$. Thus $w \sim v$. Finally, the strong Pareto condition implies that $(v_1 + \alpha, v_2 + \alpha) \succ (v_1, v_2)$ for any $\alpha > 0$, so that $v' \succ v$ if and only if $v'_1 + v'_2 > v_1 + v_2$.

I now present a proof of the result for an arbitrary number of individuals. The main part of the argument shows that if a social welfare ordering $\succsim$ satisfies the conditions in the result then for any two welfare profiles u and v for which $\sum_{i=1}^n u_i = \sum_{i=1}^n v_i$ we have $u \sim v$. To reach this conclusion, u and v are repeatedly transformed. Let $u^0 = u$ and $v^0 = v$. First, the components of u^0 and v^0 are put in order, from smallest to largest, to generate $\hat{u}^0$ and $\hat{v}^0$. By the anonymity condition, $\hat{u}^0 \sim u^0$ and $\hat{v}^0 \sim v^0$. Then for each component i, the smaller of $\hat{u}^0_i$ and $\hat{v}^0_i$ is subtracted from both $\hat{u}^0_i$ and $\hat{v}^0_i$, to generate u^1 and v^1. By the invariance condition, $u^1 \sim \hat{u}^0$ and $v^1 \sim \hat{v}^0$, so that $u^1 \sim u^0$ and $v^1 \sim v^0$. For each value of i, either $u^1_i = 0$ or $v^1_i = 0$, and at least one component of u^1 and one component of v^1 is zero (given that the sum of the components of u is equal to the sum of the components of v). The components of u^1 and v^1 are then put in order to generate $\hat{u}^1$ and $\hat{v}^1$, so that the first components of $\hat{u}^1$ and $\hat{v}^1$ are 0, and the process is repeated. At step n, all components of the resulting profiles u^n and v^n are 0, so that $u \sim u^n \sim v^n \sim v$.

Proof of Proposition 1.12

The utilitarian ordering satisfies the conditions in the result.

To show that it is the only social welfare ordering that does so, let $\succsim$ be a social welfare ordering that satisfies the conditions in the result and let $N = \{1, \dots, n\}$. Let $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$ with $\sum_{i=1}^n u_i = \sum_{i=1}^n v_i$. I argue that $u \sim v$.

To do so, I apply a sequence of transformations to u and v and argue that the invariance condition implies that the welfare profiles generated at each step are socially indifferent to the profiles from the previous step. Denote the welfare profiles generated by the transformations at each step $t \geq 1$ by u^t and v^t, and let $u^0 = u$ and $v^0 = v$. For $t \geq 1$, u^t and v^t are generated from u^{t-1} and v^{t-1} as follows.

1. Let $\hat{u}^{t-1}$ be a permutation of u^{t-1} with $\hat{u}_1^{t-1} \leq \hat{u}_2^{t-1} \leq \cdots \leq \hat{u}_n^{t-1}$ and let $\hat{v}^{t-1}$ be a permutation of v^{t-1} with $\hat{v}_1^{t-1} \leq \hat{v}_2^{t-1} \leq \cdots \leq \hat{v}_n^{t-1}$.
2. For each $i \in N$, let $u_i^t = \hat{u}_i^{t-1} - \min\{\hat{u}_i^{t-1}, \hat{v}_i^{t-1}\}$ and $v_i^t = \hat{v}_i^{t-1} - \min\{\hat{u}_i^{t-1}, \hat{v}_i^{t-1}\}$.

Step 1 *For all $t \geq 1$ we have $u^t \sim u^{t-1}$ and $v^t \sim v^{t-1}$.*

Proof. By the anonymity of $\succsim$, $\hat{u}^{t-1} \sim u^{t-1}$ and $\hat{v}^{t-1} \sim v^{t-1}$, and by its invariance with respect to additive transformations of welfares, $u^t \sim \hat{u}^{t-1}$ and $v^t \sim \hat{v}^{t-1}$. ◁

Step 2 *For $t = 1, \dots, n$ we have $\hat{u}_i^t = 0$ and $\hat{v}_i^t = 0$ for $i = 1, \dots, t$.*

Proof. At each step $t \geq 1$, for each $i \in N$ we have

$$\begin{aligned} u_i^t = 0 \text{ and } v_i^t \geq 0 \quad & \text{if } \hat{u}_i^{t-1} \leq \hat{v}_i^{t-1} \\ u_i^t \geq 0 \text{ and } v_i^t = 0 \quad & \text{if } \hat{u}_i^{t-1} \geq \hat{v}_i^{t-1}. \end{aligned}$$

Thus in particular $u_i^t \geq 0$, $v_i^t \geq 0$, and either $u_i^t = 0$ or $v_i^t = 0$ (or both). Further, $\sum_{i=1}^n u_i^t = \sum_{i=1}^n v_i^t$ if $\sum_{i=1}^n u_i^{t-1} = \sum_{i=1}^n v_i^{t-1}$, and thus since $\sum_{i=1}^n u_i^0 = \sum_{i=1}^n v_i^0$ we have $\sum_{i=1}^n u_i^t = \sum_{i=1}^n v_i^t$ for $t = 1, \dots, n$. Hence $u_i^t = 0$ for at least one value of i and $v_i^t = 0$ for at least one value of i.

In particular, $\hat{u}_1^1 = 0$ and $\hat{v}_1^1 = 0$ and hence $\hat{u}_1^t = 0$ and $\hat{v}_1^t = 0$ for all $t \geq 1$.

I now argue by induction. Let $1 \le t \le n-1$ and suppose that $\hat{u}_i^t = 0$ and $\hat{v}_i^t = 0$ for $i = 1, \ldots, t$, so that $\hat{u}_i^{t+1} = 0$ and $\hat{v}_i^{t+1} = 0$ for $i = 1, \ldots, t$. Then $\sum_{i=t+1}^n \hat{u}_i^t = \sum_{i=t+1}^n \hat{v}_i^t$, so that $u_i^{t+1} = 0$ for at least one value of $i \ge t+1$ and $v_i^{t+1} = 0$ for at least one value of $i \ge t+1$. Thus $\hat{u}_{t+1}^{t+1} = 0$ and $\hat{v}_{t+1}^{t+1} = 0$ and hence $\hat{u}_i^{t+1} = 0$ and $\hat{v}_i^{t+1} = 0$ for $i = 1, \ldots, t+1$. ◁

Step 3 $u \sim v$.

Proof. By Step 2, $u_i^n = v_i^n = 0$ for $i = 1, \ldots, n$, so that $u^n \sim v^n$ and hence by Step 1, $u^t \sim v^t$ for $t = 0, \ldots, n$. Thus in particular $u \sim v$. ◁

The strong Pareto condition implies that if $\sum_{i=1}^n u_i > \sum_{i=1}^n v_i$ then $u \succ v$, so that $\succcurlyeq$ is the utilitarian ordering.

If you find the invariance condition appealing, this result may make the utilitarian social welfare ordering more appealing (or less unappealing). I am not in this camp. We can plausibly assess at least imperfectly whether one individual is better off than another, but assessing how one individual's gain in welfare compares with another's seems an order of magnitude more difficult. So it seems backwards to assume that welfares cannot be compared but differences in welfare can. (However, maybe you can think of a situation in which it makes sense.)

The Nash social welfare ordering is characterized by a different invariance property, in conjunction with anonymity and the Pareto condition.

Definition 1.39: Invariance of social welfare ordering with respect to multiplicative transformations of welfares

A social welfare ordering $\succcurlyeq$ over $\mathbb{R}^n_{++}$ for a set of n of individuals is *invariant with respect to multiplicative transformations of welfares* if whenever $u \succcurlyeq v$ we have $b \cdot u \succcurlyeq b \cdot v$ for all $b \in \mathbb{R}^n_{++}$ (where $\cdot$ denotes inner product).

This condition may make sense if we cannot observe the individuals' welfare levels but we can observe whether the ratio u_i/v_i of one individual's welfare between two welfare profiles is bigger or smaller than the ratio w_j/y_j of another individual's welfare between two other welfare profiles, because the transformations of welfare in the definition preserve these ratios. My comment on the assumption that welfare levels cannot be compared but welfare differences can applies also here: it seems backward to assume that welfares cannot be compared but ratios of welfare ratios can.

The next result is a corollary of Proposition 1.12.

Proposition 1.13: Characterization of Nash welfare ordering

A social welfare ordering over $\mathbb{R}^n_{++}$ for a set of n individuals is anonymous and invariant with respect to multiplicative transformations of welfares and satisfies the strong Pareto condition if and only if it is the Nash ordering.

Proof

The Nash ordering satisfies the conditions in the result.

Now let $\succsim$ be a social welfare ordering on $\mathbb{R}^n_{++}$ that satisfies the conditions in the result. Define the social welfare ordering $\succsim^*$ on $\mathbb{R}^n$ by

$$u \succsim^* v \text{ if and only if } (e^{u_1}, e^{u_2}, \ldots, e^{u_n}) \succsim (e^{v_1}, e^{v_2}, \ldots, e^{v_n}).$$

Then $\succsim^*$ is invariant with respect to additive transformations of welfares by the following argument. For any $a \in \mathbb{R}^n$, let $b_i = e^{a_i}$ for all $i \in N$, where N is the set of individuals. Suppose that $u \succsim^* v$, so that $(e^{u_1}, e^{u_2}, \ldots, e^{u_n}) \succsim (e^{v_1}, e^{v_2}, \ldots, e^{v_n})$. Then $(b_1 e^{u_1}, b_2 e^{u_2}, \ldots, b_n e^{u_n}) \succsim (b_1 e^{v_1}, b_2 e^{v_2}, \ldots, b_n e^{v_n})$ by the assumption that $\succsim$ is invariant with respect to multiplicative transformations of welfares, and hence $u + a \succsim^* v + a$.

Now, the fact that $\succsim$ is anonymous and satisfies the strong Pareto condition means that $\succsim^*$ satisfies these conditions, so by Proposition 1.12, $\succsim^*$ is the utilitarian ordering. That is, $u \succsim^* v$ if and only if $\sum_{i \in N} u_i \geq \sum_{i \in N} v_i$. Thus $u \succsim v$ if and only if $\sum_{i \in N} \ln u_i \geq \sum_{i \in N} \ln v_i$, or equivalently $\prod_{i \in N} u_i \geq \prod_{i \in N} v_i$, so that $\succsim$ is the Nash ordering.

In the models of collective choice in the remainder of the book, the outcome is determined either by voting or by the balance of power, as determined by the availability of actions to individuals and groups that can affect other individuals and groups. For the most part, the models assume that all individuals are selfish—no individual's welfare is directly affected by any other individual's well-being—and so considerations of relative welfare, which dominate the analysis of this section, are absent.

Notes

Proposition 1.1 is due to May (1952). The notion of a Condorcet winner is due to Condorcet (1785). Section 1.4 is based on Horan et al. (2019); Proposition 1.2 is Theorem 1 in that paper and Proposition 1.3 is a weak version of Theorem 2.

(The notion I call positive responsiveness is called full positive responsiveness in the paper.) The proof I give for Proposition 1.2 is a simplification due to Ariel Rubinstein of the argument in Horan et al. (2019), and the proof for Proposition 1.3 is due to him. (The stronger Theorem 2 in Horan et al. 2019 requires a different proof.) The results in this section are related to those of Dasgupta and Maskin (2008). Their model has a continuum of individuals and they require a collective choice rule to assign a single alternative to almost every preference profile. They identify a set of conditions that are satisfied by the collective choice rule that assigns to each collective choice problem its set of Condorcet winners and show, roughly, that no other collective choice rule satisfies these conditions on a larger domain of problems. The conditions include anonymity, neutrality, and a relative of Nash independence that treats each alternative, rather than each set of alternatives, as a unit.

Proposition 1.4 is due to Black (1958) (see his theorems on pp. 16 and 18). The generalization to trees discussed after Exercise 1.11 is due to Demange (1982). The use of single-crossing preferences in the context of collective choice has its origin in Roberts (1977); Proposition 1.5 is due to Rothstein (1990) (who uses the term "order-restricted preferences") and Gans and Smart (1996). My presentation of the material in Section 1.5.2 benefitted from discussions with Navin Kartik.

Proposition 1.6 is due to Rae and Taylor (1971, 77) and Wendell and Thorson (1974, Theorem 3.2). Wendell and Thorson (1974, Example 3.1) show that the result does not generalize to three dimensions (contrary to the claim on p. 78 of Rae and Taylor 1971). McKelvey and Wendell (1976) and Humphreys and Laver (2010) further investigate Condorcet winners when alternatives are multidimensional. Proposition 1.8 is due to Davis et al. (1972); see also Enelow and Hinich (1984, Section 3.6).

Proposition 1.9 is due to Arrow (1951). The proof I give is due to Geanakoplos (2005), and is taken almost verbatim from Osborne and Rubinstein (2023, Proposition 20.1). Other parts of Section 1.7 are based on Chapter 20 of that book; some are quoted verbatim. I am grateful to Ariel Rubinstein for permitting me to include the material here. The preference aggregation function in Example 1.7 (Condorcet aggregation) has a long history, dating back at least to Ramon Llull (c. 1232–1315/16) (see Hägele and Pukelsheim 2001). It is sometimes called the Copeland method, after Arthur H. Copeland (1898–1970), who (re)proposed it in 1951 (see, for example, Goodman 1954, 42; apparently no copy of the mimeographed notes he cites survives).

The study of collective choice based on interpersonal comparisons of welfare was initiated by Sen (1970). The Nash social welfare ordering is named for its relation with the bargaining solution of Nash (1950). Proposition 1.11 is due

points	1×	4×	3×	2×
1	a	b	c	c
p	b	a	a	b
0	c	c	b	a

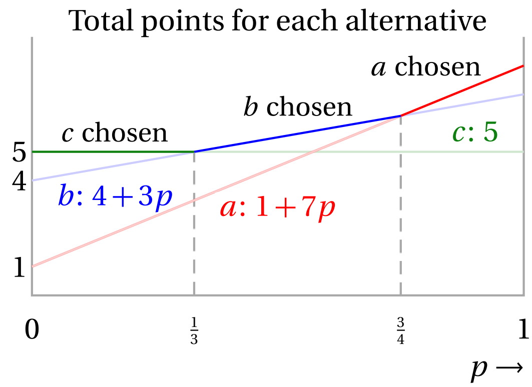


Figure 1.20 The collective choice problem in Exercise 1.1.

to Hammond (1976, Theorem 7.2). The proof for two individuals that I present is taken from Blackorby et al. (1984, Theorem 6.1) and Bossert and Weymark (2004, Theorem 12.2). Proposition 1.12 is due to d'Aspremont and Gevers (1977, Theorem 3). The proof for two individuals is taken from Blackorby et al. (1984, 351–352). My presentation of both results draws upon Bossert and Weymark (2004). Proposition 1.13 is a slight variant of Moulin (1988, Theorem 2.3).

Exercise 1.3 is based on Fishburn (1974, 67) (see also Moulin 1988, Exercise 11.2). The example in Exercise 1.6 is taken from Moulin (1988, 235). The argument in Exercise 1.11 and its extension to trees is taken from Exercise 10.4 in Moulin (1988, 279). The examples in parts *a* and *c* of Exercise 1.13 are taken from Austen-Smith and Banks (1999, Example 4.6). The result in part *b* is Corollary 3 of Puppe (2018); the proof that I give is taken from Elkind et al. (2022, Proposition 3.19). The observation in Exercise 1.14 is taken from Rothstein (1990).

Solutions to exercises

Exercise 1.1

Refer to Figure 1.20. The set of alternatives selected by the rule is

$$\begin{cases} \{c\} & \text{if } 0 < p < \tfrac{1}{3} \\ \{b,c\} & \text{if } p = \tfrac{1}{3} \\ \{b\} & \text{if } \tfrac{1}{3} < p < \tfrac{3}{4} \\ \{a,b\} & \text{if } p = \tfrac{3}{4} \\ \{a\} & \text{if } \tfrac{3}{4} < p < 1. \end{cases}$$

Exercise 1.2

a. For an anonymous and positively responsive collective choice rule, the pattern of outcomes in a diagram like those in Figure 1.1 has to satisfy the

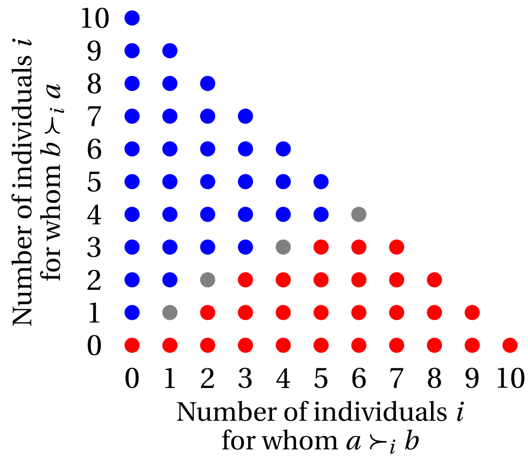


Figure 1.21 A collective choice rule that is anonymous and positively responsive but not neutral. See the discussion of Figure 1.1 for an explanation of the way in which the diagram represents a collective choice rule.

condition in Figure 1.3a and the symmetric condition for b. An example is shown in Figure 1.21. The rule shown in not neutral because the pattern of outcomes is not symmetric about the main diagonal. Another example is the rule that selects $\{a\}$ if more than $\frac{2}{3}$ of the individuals who are not indifferent between a and b prefer a to b, selects $\{b\}$ if fewer do so, and selects $\{a,b\}$ if exactly $\frac{2}{3}$ do so.

b. A collective choice rule that is neutral and positively responsive but not anonymous is the variant of dictatorship for which in the case that the dictator is indifferent between the alternatives, majority rule determines the alternatives chosen.

Exercise 1.3

For an anonymous and neutral collective choice rule, the pattern of outcomes in a diagram like those in Figure 1.1 has to be symmetric about the main diagonal.

If a nonnegatively responsive collective choice rule selects $\{a\}$ for some problem then it selects $\{a\}$ for every problem in the region to the east, southeast, and south; if it selects $\{b\}$ for some problem then it selects $\{b\}$ for every problem in the region to the west, northwest, and north; and if it selects $\{a,b\}$ for some problem then it selects either $\{a\}$ or $\{a,b\}$ for every problem in the region to the east, southeast, and south and either $\{b\}$ or $\{a,b\}$ for every problem to the west, northwest, and north (cf. Figure 1.3a).

A pattern that satisfies these conditions is given in Figure 1.22.

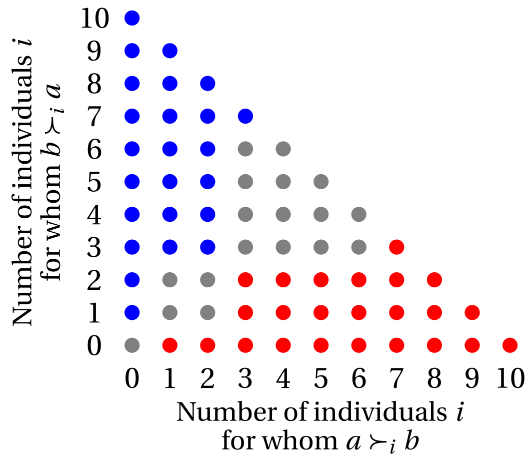


Figure 1.22 A collective choice rule that is anonymous and nonnegatively responsive but not positively responsive. See the discussion of Figure 1.1 for an explanation of the way in which the diagram represents a collective choice rule.

Another example is the rule that selects $\{a\}$ if more than $\frac{2}{3}$ of the individuals who are not indifferent between a and b prefer a to b, selects $\{b\}$ if more than $\frac{2}{3}$ of the individuals who are not indifferent between a and b prefer b to a, and otherwise selects $\{a,b\}$. (The pattern that represents this rule differs from the one in Figure 1.22 only in that the points (3,2), (4,2), (2,3), and (2,4) are assigned to $\{a,b\}$.)

Exercise 1.4

In the following problem, a is the unique Condorcet winner, but is not a strict Condorcet winner. It beats b, but ties with c (which loses to b).

1	2	3
a	c	b
b	a	ac
c	b	

Exercise 1.5

The following collective choice problem shows that the answer is negative: a is a Condorcet winner and b ties with it, but is not a Condorcet winner.

1	2	3	4
a	a	b	c
c	c	a	b
b	b	c	a

Exercise 1.6

On the first round a and b are selected, and on the second round a wins (11 to 6). Now raise a above b in the preferences of the last two individuals, so that their preferences become $a \succ b \succ c$. Then on the first round a and c are selected and on the second round c wins. Thus after a's ranking improves, it is no longer selected.

Exercise 1.7

Neutrality, positive responsiveness, and Nash independence: A variant of dictatorship by individual 1: select 1's favorite alternative, breaking a tie according to the preferences of individual 2, further breaking a tie according to the preferences of individual 3, and so on. (Dictatorship itself is not positively responsive, because if the dictator is indifferent between the alternatives and one of them improves in the preferences of another individual, it does not become the only alternative chosen.)

Anonymity, positive responsiveness, and Nash independence: For an arbitrary alternative, select that alternative for every collective choice problem.

Anonymity, neutrality, and Nash independence: Select the set of all alternatives.

Exercise 1.8

Suppose there are five alternatives, a, b, c, d, and e and five individuals. Two individuals have the preference ordering $b \succ_i a \succ_i c \succ_i d \succ_i e$, one has the ordering $c \succ_i b \succ_i a \succ_i d \succ_i e$, and two have the ordering $d \succ_i c \succ_i b \succ_i e \succ_i a$. This preference profile is single-peaked relative to the ordering $\lhd$ with $a \lhd b \lhd c \lhd d \lhd e$. The median favorite alternative is c and the Borda winner is b. (This alternative gets 15 points; c gets 14 points, d gets 11 points, a gets 8 points, and e gets 1 point.)

Exercise 1.9

Consider the collective choice problem $\langle\{1,2,3\},\{a,b,c\},\succsim\rangle$ in which $a \succ_1 b \succ_1 c$, $a \sim_2 b \succ_2 c$, and $a \sim_3 b \sim_3 c$. The preference profile $\succsim$ satisfies the single-plateau condition for the linear order $\unrhd$ for which $a \lhd b \lhd c$. Individual 1's favorite alternative is a, one of individual 2's favorite alternatives is b, and one of individual 3's favorite alternatives is c. The median of these alternatives with respect to $\unrhd$ is b, but the only Condorcet winner of the collective choice problem is a.

Exercise 1.10

Consider the collective choice problem in which the set of individuals is $\{1,2,3,4,5\}$, the set of alternatives is $\{a,b,c,d,e\}$, and the preference profile is given as follows.

1	2	3	4	5
a	b	c	d	e
$bcde$	a	b	c	d
	cde	a	b	c
		de	a	b
			e	a

Each individual has a single favorite alternative and the profile satisfies the variant of (1.3) for the linear order $\trianglerighteq$ for which $a \triangleleft b \triangleleft \; c$. Alternatives a and b are beaten by c, d and e are beaten by a, and c is beaten by d, so the problem has no Condorcet winner.

Now suppose that with the same set of individuals the set of alternatives is $\{a, b, c\}$ and the preference profile is given as follows.

1	2	3	4	5
a	a	b	c	c
bc	bc	a	ab	ab
		c		

Each individual has a single favorite alternative and the profile satisfies the variant of (1.3) for the linear order $\trianglerighteq$ for which $a \triangleleft b \triangleleft \; c$. The median of the individuals' favorite alternatives with respect to $\trianglerighteq$ is b, but the only Condorcet winner of the collective choice problem is a.

Exercise 1.11

a. By (*i*), x_t beats every other alternative in Z_t. Thus if $t = 1$, x_t is the strict Condorcet winner of $\langle N, X, \succsim \rangle$. Now assume that $t \geq 2$ and suppose, contrary to the claim, that some $x_j \in X_t$ beats x_t. That is, a majority of individuals prefer x_j to x_t. Then by the single-peakedness of preferences, the same individuals prefer x_{t-1} to x_t and hence also prefer x_{t-1} to every other alternative in Z_t, so that x_{t-1} is the favorite alternative in Z_{t-1} of a majority of individuals, contrary to (*ii*) for $j = t - 1$. Thus no $x_j \in X_t$ beats x_t, and hence x_t is the strict Condorcet winner of $\langle N, X, \succsim \rangle$.

b. At step t, if x_t is the favorite alternative in Z_t of a majority of individuals, select it and terminate. Otherwise proceed to step $t + 1$. This procedure terminates at latest at step k. If it terminates at step t then it selects x_t, which by part a is the strict Condorcet winner of $\langle N, X, \succsim \rangle$.

Exercise 1.12

If the condition in the exercise is satisfied then the condition in Definition 1.23 is satisfied because for any alternatives x and y either $x \triangleleft y$ or $y \triangleleft x$.

Now suppose that the condition in Definition 1.23 is satisfied. I construct a linear order on X such that the condition in the exercise is satisfied.

Define the binary relation $\trianglerighteq$ on X as follows. Let x and y be alternatives. If $x \succ_j y$ for all $j \in N$, then $x \triangleleft y$. Otherwise, if $x \not\succ_i y$ for some individual i, let i^* be the unique individual implied by Definition 1.23 for whom $x \not\succ_{i^*} y$ and either (a) $x \succ_j y$ for all $j < i^*$ and $x \prec_j y$ for all $j > i^*$ or (b) $x \prec_j y$ for all $j < i^*$ and $x \succ_j y$ for all $j > i^*$. Define $x \triangleleft y$ in case (a) and $x \triangleright y$ in case (b). If $x \prec_j y$ for all $j \in N$, then $x \triangleright y$.

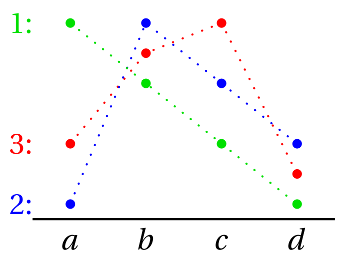
Given that the collective choice problem satisfies the condition in Definition 1.23, the binary relation $\trianglerighteq$ is complete. It is transitive by the following argument. If $x \triangleleft y$ and $y \triangleleft z$ then by the transitivity of each individual's preference relation either $x \succ_j z$ for every individual j or there exists an individual j^* such that $x \succ_j z$ for all $j < j^*$ and $x \prec_j z$ for all $j > j^*$. Thus $x \triangleleft z$, so that $\trianglerighteq$ is transitive. Finally, $x \triangleleft y$ and $y \triangleleft x$ are possible only if $x = y$. Thus $\trianglerighteq$ is a linear order.

If $x \triangleleft y$ then either $x \succ_j y$ for all $j \in N$ or for some individual i^* we have $x \succ_i y$ for all $i < i^*$ and $x \prec_i y$ for all $i > i^*$, so that (a) in the condition in the exercise is satisfied; if $y \triangleleft x$ then (b) is satisfied.

Exercise 1.13

a. The preference profile is given as follows.

1	2	3
a	b	c
b	c	b
c	d	a
d	a	d



The profile is single-peaked for the orderings $a \triangleleft b \triangleleft c \triangleleft d$ and $d \triangleleft c \triangleleft b \triangleleft a$, and only for these orderings.

It is not single-crossing because the condition in Definition 1.23 is not satisfied for any ordering of the individuals:

- $a \succ_1 b$ but $b \succ_2 a$ and $b \succ_3 a$, so the condition is not satisfied by any ordering in which 1 is in the middle
- $d \succ_2 a$ but $a \succ_1 d$ and $a \succ_3 d$, so the condition is not satisfied by any ordering in which 2 is in the middle
- $c \succ_3 b$ but $b \succ_1 c$ and $b \succ_2 c$, so the condition is not satisfied by any ordering in which 3 is in the middle.

b. Denote by $\succsim_1$ the preference ordering of the first individual according to $\geq$. Suppose, contrary to the result, that the preferences of some individual i are not single-peaked with respect $\succsim_1$. That is, for some alternatives x, y, and z with $x \prec_1 y \prec_1 z$ we have $x \succ_i y$ and $z \succ_i y$. Let j be an individual with favorite alternative y, so that $y \succ_j x$ and $y \succ_j z$. Given that $x \prec_1 y$ and $x \succ_i y$, by single-crossing every individual after i prefers x to y. Thus j comes before i. But $y \succ_j z$ and $y \prec_1 z$, so everyone after j prefers y to z, contradicting the fact that i comes after j and prefers z to y.

c. Consider the collective choice problem given as follows.

1	2	3
a	c	c
b	a	b
c	b	a

This problem has single-crossing preferences with respect to the ordering $\geq$ of the individuals for which $1 < 2 < 3$.

The preference profile is not single-peaked with respect to any ordering of the alternatives because for single-peakedness the middle alternative cannot be ranked lowest for any individual, and every alternative is ranked lowest by one individual.

Exercise 1.14

Consider the problem with three individuals and four alternatives in which the individuals' preferences are $a \succ_1 b \succ_1 c \succ_1 d$, $b \succ_2 c \succ_2 d \succ_2 a$, and $c \succ_3 b \succ_3 a \succ_3 d$. This problem has single-peaked preferences with respect to the ordering $a \lhd b \lhd c \lhd d$, but no individual's preference relation coincides with that of the majority, for which b beats c beats a beats d.

Exercise 1.15

Consider the problem with four individuals and four alternatives in which the individuals' preferences are $a \succ_1 b \succ_1 c \succ_1 d$, $a \sim_2 b \succ_2 c \succ_2 d$, $c \succ_3 b \succ_3 d \succ_3 a$, and $d \succ_4 c \succ_4 b \succ_4 a$. This problem has single-crossing preferences with respect to the ordering 1, 2, 3, 4 of individuals, and a, a favorite alternative of individual 2, a median individual, is not a Condorcet winner (it loses to b).

Exercise 1.16

Take the preference profile in the solution of Exercise 1.13*a*, which is single-peaked but not single-crossing, and raise d to between b and c in individual 1's preferences. For the resulting profile, b is a strict Condorcet winner. However, the resulting profile is not single-peaked and does not have the single-crossing property.

It is not single-peaked because the alternatives the individuals rank lowest, c, a, and d, are distinct and thus cannot all be the smallest or largest alternative according to the ordering of alternatives, as single-peakedness requires.

It is not single-crossing because the condition in Definition 1.23 is not satisfied for any ordering of the individuals:

- $a \succ_1 c$ but $c \succ_2 a$ and $c \succ_3 a$, so the condition is not satisfied by any ordering in which 1 is in the middle
- $d \succ_2 a$ but $a \succ_1 d$ and $a \succ_3 d$, so the condition is not satisfied by any ordering in which 2 is in the middle
- $c \succ_3 b$ but $b \succ_1 c$ and $b \succ_2 c$, so the condition is not satisfied by any ordering in which 3 is in the middle.

2 Collective choice with privately-known preferences

2.1 Strategy-proofness of strict Condorcet winner 79
2.2 Strategy-proofness for domain of all preference profiles 86
2.3 General mechanisms 90

A mechanism for selecting an alternative in a collective choice problem should sensibly use information about the individuals' preferences. The models in the previous chapter assume that these preferences are known to the designer of the mechanism. The models in the remainder of the book assume that each individual's preferences are known only to the individual.

One way for a mechanism designer to proceed if she does not know the individuals' preferences is to choose a set P of permitted preference relations, ask each individual to report a preference relation in this set, and select an alternative based on these reports. Denote the alternative chosen by the designer when the profile of reported preference relations is $\succsim$ by $g(\succsim)$, and assume that the individuals know the function g. We can then model the individuals as players in a strategic game in which each individual's set of actions is P and the action profile $\succsim$ (the profile of reported preference relations) results in the alternative $g(\succsim)$ being selected. In this game, each individual i prefers to report a preference relation $\succsim_i^1$ to a preference relation $\succsim_i^2$ if and only if she prefers $g(\succsim^1)$ to $g(\succsim^2)$, where $\succsim^1$ and $\succsim^2$ are the resulting preference profiles, given the other individuals' reports. Assume for expositional convenience that for every true preference profile, the solution concept we apply to the game generates a unique action profile. The mechanism designer's task is then to specify the set P of permitted reports and the function g in such a way that, for each true preference profile $\succsim^*$, the alternative $g(\succsim)$, where $\succsim$ is the preference profile given by the solution concept of the game when the true preference profile is $\succsim^*$, varies reasonably with $\succsim^*$.

Synopsis

Section 2.1 studies environments in which for some set P of preference relations, every collective choice problem in which each individual's preference relation is in P has a strict Condorcet winner. For such an environment, consider the mechanism in which each individual has to report a preference relation in P and the function g selects the strict Condorcet winner of the collective choice problem in which the preference profile is the one reported. Proposition 2.1 shows that for this mechanism, whenever the true preference relation of each individual is in P, no individual can do better than report her true preference relation, regardless of the preference relations reported by the other individuals. In the argot of the field, that is, the mechanism is strategy-proof. If each individual does in fact report her true preference relation, then the mechanism selects the strict Condorcet winner for the true preference profile, which Proposition 1.2 suggests is a reasonable choice.

Suppose, for example, that the number of individuals is odd and for some linear order on the set of alternatives, P is the set of all preference relations that are single-peaked with respect to the order. Then by Proposition 1.4, every collective choice problem in which each individual's preference relation is in P has a strict Condorcet winner. Thus by Proposition 2.1, if every individual's preference relation is in P then for the mechanism that asks each individual to report a preference relation in P and selects the strict Condorcet winner for the reported preference profile, no individual can do better than report her true preference relation. In assessing the significance of this result, a reasonable question to ask is: how could a mechanism designer know that every individual's preference relation is single-peaked?

Suppose that the mechanism designer does not want to pre-judge the character of the individuals' preferences, and thus restricts herself to mechanisms in which the set P of permitted reports is the set of all possible preference relations. Call a collective choice rule *acceptable* if (i) no individual is a dictator—for no individual i is the outcome i's favorite alternative regardless of the other individuals' preferences—and (ii) whenever every individual's favorite alternative is the same, the rule selects that alternative. The main result in Section 2.2, Proposition 2.3 (known as the Gibbard-Satterthwaite theorem) contrasts strongly with Proposition 2.1. It shows that if there are at least three alternatives and every individual is free to report any preference relation, then no acceptable collective choice rule that selects a single alternative for every profile of reported preference relations is strategy-proof.

This result does not by itself imply that a mechanism designer who has no information about the individuals' preferences can implement only collective choice rules that are not acceptable. Suppose that you want to implement a collective choice rule that assigns a single alternative, say $f(N, X, \succsim)$, to each collective choice problem $\langle N, X, \succsim\rangle$. Doing so is possible if, when you announce that you will select the alternative given by the collective choice rule for the reported preference profile, every individual optimally reports her true preference relation. If your criterion for an optimal report is that an individual can do no better regardless of the other individuals' reports, Proposition 2.3 rules out this possibility for any acceptable collective choice rule. Another route you could consider is less direct: you announce that for some mapping g from preference profiles to alternatives that differs from the collective choice rule, you will select the alternative $g(\succsim')$ if the reported preference profile is $\succsim'$, and you design g in such a way that each individual i optimally reports a preference relation $\succsim'_i$, which may differ from her true preference relation $\succsim_i$, such that $g(\succsim') = f(N, X, \succsim)$. That is, you compensate for the distortion in the reported preferences by distorting the mapping from reported preferences to alternatives. The argument in Section 2.3 shows that under certain conditions you cannot gain by such a tactic. Specifically, suppose there is a mechanism that induces the individuals to submit reports that generate the alternative specified by some (single-valued) collective choice rule, where "induces" means that each individual's report is optimal for her regardless of the other individuals' reports. Then Proposition 2.4 (known as a "revelation principle") shows that the collective choice rule is strategy-proof. But then Proposition 2.3 implies that the collective choice rule is not acceptable: it either is dictatorial or does not respect unanimity.

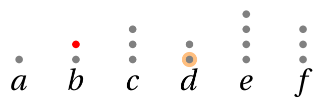
2.1 Strategy-proofness of strict Condorcet winner

Proposition 1.2 in the previous chapter shows that for collective choice problems that have a strict Condorcet winner, the collective choice rule that selects that alternative has singular appeal. Can this rule be implemented if we do not know the individuals' preferences? Consider the mechanism in which each individual reports a preference relation and the alternative chosen is the strict Condorcet winner for the profile of reported preference relations. Can we expect self-interested individuals to report their true preference relations?

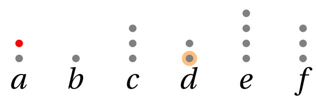
The answer is a qualified yes. Let P be a set of preference relations such that every collective choice problem for which every individual's preference relation is in P has a strict Condorcet winner. (By Proposition 1.4, if the number of individuals is odd, the set of single-peaked preference relations is one such set.) I argue that if each individual is restricted to report a member of P and the alternative selected for any profile of reported preference relations is the strict Condorcet winner for that profile, no individual can do better than report her true preference relation, regardless of the preference relations reported by the other individuals. One way to state this result is to say that the collective choice rule that selects the strict Condorcet winner is strategy-proof over P.

If there are two alternatives, the number of individuals is odd, and P is the set of strict preference relations, this result is clear: an individual's saying that she favors b when in fact she favors a either does not affect the strict Condorcet winner or changes it from $\{a\}$ to $\{b\}$, which is worse for her.

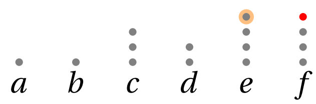
When there are three or more alternatives, the idea behind the result may be conveyed by an example. Suppose that the number of individuals is odd, the alternatives are a, b, c, d, e, and f, and every individual's preferences are single-peaked relative to that ordering. By Proposition 1.4, the strict Condorcet winner for the reported preference profile is the median of the favorite alternatives for the reported profile, so we can think of everyone simply reporting an alternative, rather than her entire preference relation, and the mechanism selecting the median of these reports. Suppose that you favor b and that, given the alternatives reported by everyone else, if you report b then the median reported alternative is d.



What can you do? The crucial point is that you can do nothing to bring the median reported alternative closer to b. If you switch to reporting a or c, the median does not change: the number of reported alternatives less than d remains the same.



If you switch to d the median also does not change. If you switch to e or f, the median might change, but if it does then it changes to e, which is worse for you than c.



Thus by changing your report, you may be able to move the selected alternative away from your favorite, but you cannot move it closer. So whatever the other individuals report, you can do no better than report your favorite alternative.

I now show a general result; subsequently I return to problems with single-peaked preferences. Given a set P of preference relations, call the set of collective choice problems in which every individual's preference relation is in P the domain generated by P.

Definition 2.1: Domain generated by set of preference relations

Let $\langle N, X\rangle$ be a finite society. For any set P of preference relations over X, the *domain* $\mathscr{D}(N, X, P)$ is the set of all collective choice problems $\langle N, X, \succsim\rangle$ for which $\succsim_i \in P$ for each $i \in N$.

The subsequent results concern collective choice rules that assign a single alternative to each collective choice problem. I call such a rule a collective choice *function*.

Definition 2.2: Collective choice function

For any set D of collective choice problems, a *collective choice function for D* is a function that associates with every collective choice problem $\langle N, X, \succsim\rangle$ in D a single member of X (the alternative selected by the rule).

A collective choice function is strategy-proof over a set P of preference relations if, for every collective choice problem in which every individual's preference relation is in P, no individual can induce a better outcome according to her true preference relation by reporting a preference relation in P different from her true preference relation, regardless of the preference relations (in P) submitted by the other individuals.

Definition 2.3: Strategy-proof collective choice function

Let $\langle N, X\rangle$ be a finite society, let P be a set of preference relations over X, and let f be a collective choice function for the domain $\mathscr{D}(N, X, P)$. Then f is *strategy-proof over P* if for every $\langle N, X, \succsim\rangle \in \mathscr{D}(N, X, P)$ and every individual $i \in N$,

$$f(N, X, \succsim) \succsim_i f(N, X, (\succsim'_i, \succsim_{-i})) \quad \text{for every } \succsim'_i \in P,$$

where $(\succsim'_i, \succsim_{-i})$ is the preference profile that differs from $\succsim$ only in that i's preference relation is $\succsim'_i$ rather than $\succsim_i$.

This concept is closely related to that of weak domination for an action in a strategic game. Consider the strategic game in which the players are the individuals, each player's set of actions is a set P of preference relations, and each player i prefers the action profile $\succsim$ to the action profile $\succsim'$ if and only if $f(N, X, \succsim) \succ_i f(N, X, \succsim')$. Then the collective choice function f is strategy-proof over P if and only if, for each individual i and each preference relation $\succsim'_i \in P$ different from i's true preference relation $\succsim_i$, i's action $\succsim_i$ either weakly dominates $\succsim'_i$ or is equivalent to $\succsim_i'$ in the sense that i is indifferent between the two actions regardless of the other individuals' actions. (To see that an individual's true preference relation may be equivalent to another preference relation in this game, suppose that $X = \{a, b, c\}$ and P contains exactly two preference relations: a preferred to b preferred to c, and a preferred to c preferred to b. Then for every preference profile the strict Condorcet winner is a, and in particular the winner is the same whether an individual reports her true preference relation or the other possible preference relation.)

Let P be a set of preference relations and suppose that every collective choice problem in the domain $\mathscr{D}(N, X, P)$ has a strict Condorcet winner. Consider the collective choice function that selects the strict Condorcet winner for the submitted preference profile, let that alternative be a if every individual submits her true preference relation, and let b be an alternative that individual i prefers to a. Can i cause b to become the selected alternative by submitting a preference relation that ranks b higher than it is in her true preferences? As in the example discussed at the start of the section, the answer is no. The fact that a is the strict Condorcet winner for the true preference profile, in which i ranks b above a, means that it beats all other alternatives, including b; i's submitting a preference relation in which b is ranked even higher than it is in her true preferences does not change that fact. Individual i may be able to change the outcome to an alternative, say c, that she ranks *below* a, by submitting a preference relation in which c is ranked above rather than below a, but that makes her worse off, not better off. She cannot change the outcome to one that she prefers to a. Thus the collective function that selects the strict Condorcet winner is strategy-proof.

Proposition 2.1: Collective choice function for strict Condorcet domain is strategy-proof

Let $\langle N, X\rangle$ be a finite society and let P be a set of preference relations over X for which every collective choice problem in the domain $\mathscr{D}(N, X, P)$ has a strict Condorcet winner. The collective choice function for $\mathscr{D}(N, X, P)$ that assigns to each collective choice problem in $\mathscr{D}(N, X, P)$ its strict Condorcet winner is strategy-proof over P.

Proof

Let $\langle N, X, \succsim\rangle \in \mathscr{D}(N, X, P)$ be a collective choice problem and let a be its strict Condorcet winner. Suppose that for $\succsim_i' \in P$, the strict Condorcet winner of $\langle N, X, \succsim'\rangle$, where $\succsim' = (\succsim_i', \succsim_{-i})$, is $b \neq a$. Then the number of individuals j for whom $a \succ_j b$ exceeds the number for whom $b \succ_j a$, and the number for whom $b \succ'_j a$ exceeds the number for whom $a \succ'_j b$. The preference profiles $\succsim$ and $\succsim'$ differ only in i's preference relation, so $a \succ_i b$ (and $b \succ'_i a$), establishing the result.

The next exercise shows that this result cannot be extended to domains with Condorcet winners that are not strict.

Exercise 2.1: Non-strict Condorcet winners and strategy-proofness

Suppose that $N = \{1,2,3\}$, $X = \{a,b,c\}$, and $c \succ_1 b \succ_1 a$, $a \succ_2 c \succ_2 b$, and $a \sim_3 b \succ_3 c$. Consider the collective choice rule that assigns to every collective choice problem its set of (not necessarily strict) Condorcet winners. Suppose that individual 1 prefers one set of alternatives to another if and only if she prefers the alternative she likes best in the first set to the one she likes best in the second set. Show that by reporting a preference relation different from her true relation, individual 1 can induce an outcome that she prefers. Construct an example to show that the same is true if she (pessimistically) evaluates a set of alternatives according to the worst alternative for her in the set.

For a set of collective choice problems for each which problem has a strict Condorcet winner, Proposition 2.1 suggests a way of implementing the rule that selects the strict Condorcet winner even if we do not know the individuals' preferences: ask each individual to submit a preference relation from an appropriate set and select the strict Condorcet winner of the submitted relations. The result establishes that each individual can do no better than submit her true preference relation, regardless of the preference relations submitted by the other individuals.

But if we do not know the individuals' preferences, how can we know whether the collective choice problem they face has a strict Condorcet winner? And how can we select an appropriate set of preference relations from which the individuals are allowed to choose? If the number of individuals is odd, two sufficient conditions for a collective choice problem to have a strict Condorcet winner are that the problem has single-peaked preferences (Proposition 1.4) and that it has single-crossing preferences (Proposition 1.5). If the set of alternatives is naturally one-dimensional (e.g. the amount of money to spend on a public good) we may have reason to believe that one or other of these properties is satisfied. Nevertheless, it is hard to see how we can be sure that is the case, so that if we ask each individual to submit a preference relation, we need to handle submitted relations that are not single-peaked or do not belong to the collection of relations with the single-crossing property that we have in mind.

One option in a one-dimensional environment is to restrict each individual to report a single alternative, rather than a preference relation, and select the median of the submitted alternatives. In a single-peaked or single-crossing domain, the strict Condorcet winner is the median of the individuals' favorite alternatives, so if every individual reports her favorite alternative, this mechanism yields the strict Condorcet winner. I formulate the mechanism as a strategic game.

Definition 2.4: Median-based collective choice game

A *median-based collective choice game* $\langle N, X, \trianglerighteq, (\succsim_i)_{i\in N}\rangle$, where N is a finite set (of individuals) with an odd number of members, X is a set (of alternatives), $\trianglerighteq$ is a linear order on X, and, for each $i \in N$, $\succsim_i$ is a preference relation on X, is a strategic game with the following components.

Players
The set N.

Actions
The set of actions of each player is X.

Preferences
For any action profile z, let $m(z)$ be the median of the individuals' actions with respect to $\trianglerighteq$; this alternative is the outcome of z. Each player i prefers the action profile a to the action profile b if and only if $m(a) \succ_i m(b)$.

An implication of Proposition 2.1 is that in such a game in which each individual's preference relation is single-peaked with respect to the order, no individual can do better than choose her favorite alternative, regardless of the alternatives chosen by the other individuals. In fact, each individual's action of choosing her favorite alternative weakly dominates all her other actions.

Proposition 2.2: Collective choice game with single-peaked preferences

For a median-based collective choice game $\langle N, X, \trianglerighteq, (\succsim_i)_{i\in N}\rangle$ in which the set X of alternatives is finite and the preference relation $\succsim_i$ of each individual i is single-peaked with respect to $\trianglerighteq$, each individual's action of choosing her favorite alternative weakly dominates all her other actions.

Proof

Let P be the set of single-peaked preference relations over X. A collective choice problem $\langle N, X, \succsim\rangle$ is in the domain generated by P if N is finite and

$\succsim_i \in P$ for each $i \in N$. By Proposition 1.4 the median of the individuals' favorite alternatives is the strict Condorcet winner of any such problem. Thus by Proposition 2.1 the collective choice function that assigns the median of the individuals' favorite alternatives to each collective choice problem in the domain generated by P is strategy-proof. That is, for each individual i and each alternative b different from her favorite alternative, a^*, the action a^* either weakly dominates b or is equivalent to b in the collective choice game. But no action is equivalent to a^*: for any action $b \neq a^*$, if the other individuals' actions are equally split between a^* and b, then for i's action a^* the median action is a^* and for her action b the median action is b.

Exercise 2.2: Collective choice game with preferences not single-peaked

Given an example of a median-based collective choice game for which the preference profile is not single-peaked and the action profile in which each individual chooses her favorite alternative is not a Nash equilibrium.

Exercise 2.3: Game in which outcome is smallest chosen alternative

Consider a variant of a median-based collective choice game in which the outcome is the smallest chosen alternative (according to the ordering of alternatives) rather than the median. Show that if each individual's preference relation is single-peaked with respect to the ordering of alternatives, each individual's action of choosing her favorite alternative weakly dominates all her other actions, as it does in Proposition 2.2.

The models I have presented so far assume that all individuals participate in the mechanisms. If participation is costly, they may not. To communicate her preferences, an individual may have to attend a meeting or file a report. These activities take time and effort, and some individuals may decide that the expected return does not justify the cost. If so, which information is lost, and how is the chosen alternative affected? The next two exercises ask you to analyze examples.

Exercise 2.4: Collective choice game with costly reporting

Consider an example of a variant of a median-based collective choice game in which reporting an alternative is optional, and an individual who does so incurs a cost. Each individual is restricted to either report her favorite alternative or not submit a report. Alternatives are real numbers, the number of individuals is $2k+1$ for some positive integer k, and each integer from $-k$ to k is the favorite alternative of exactly one individual. The outcome is the mean of the two medians of the submitted reports if the number of individuals who submit reports is positive and even, and the outcome is 0 if no individual submits a report. Finally, the payoff of each individual i is $-|x - x_i^*| - c$ if she submits a report and $-|x - x_i^*|$ if she does not, where x_i^* is her favorite alternative and c, the cost of reporting, is a positive number. Show that for some positive number h, the game has a Nash equilibrium in which an individual submits a report if and only if her favorite alternative is at most $-h$ or at least h.

Exercise 2.5: Collective choice game with shareable reporting costs

Consider a variant of the game in the previous exercise in which each individual may report any alternative (not just her favorite) and the cost she incurs is decreasing in the number of other individuals who report the same alternative (but is always positive). Show that in a Nash equilibrium (*i*) no more than four distinct alternatives are reported, (*ii*) if one alternative is reported then it is reported by exactly one individual, is the individual's favorite alternative, and differs from 0, and (*iii*) if three or four alternatives are reported then exactly one individual reports each middle alternative.

2.2 Strategy-proofness for domain of all preference profiles

If there are two alternatives, the number of individuals is odd, and the individuals' preferences are strict, then the alternative favored by a majority is a strict Condorcet winner, so that the collective choice function that assigns this alternative is strategy-proof over the set of all strict preference relations by Proposition 2.1. Other rules are strategy-proof over this set also. For example, denote the alternatives a and b and let k be a nonnegative integer that is at most the number of individuals. Then the rule that assigns a to a problem if and only if at least k individuals favor a is strategy-proof.

Does a collective choice function exist that is strategy-proof over the domain of *all* preference profiles, regardless of the number of alternatives? A dictatorship has this property: if, for some individual i^*, the alternative chosen is always i^*'s favorite, then no other individual's report affects the outcome. I now show (Proposition 2.3) that if there are three or more alternatives, then among collective choice functions that respect the individuals' unanimous agreement regarding the best alternative, dictatorship is the only collective choice function that is strategy-proof over the set of all strict preference profiles.

Definition 2.5: Unanimous collective choice function

Let D be a set of collective choice problems for which every individual's preference relation is strict. A collective choice function f for D is *unanimous* if for any collective choice problem $\langle N, X, \succsim \rangle \in D$,

$$x \text{ is the favorite alternative in } X \text{ for } \succsim_i \text{ for every } i \in N \;\Rightarrow\; f(N, X, \succsim) = x.$$

The result is named for its originators, Allan Gibbard and Mark A. Satterthwaite. The proof I present uses Arrow's impossibility theorem (Proposition 1.9). For any collective choice function f, it defines a preference aggregation function G and shows that if f is unanimous and strategy-proof then G satisfies the Pareto property and independence of irrelevant alternatives. Thus by Arrow's theorem G is dictatorial, which implies that f is also dictatorial.

Proposition 2.3: Gibbard-Satterthwaite theorem

Let $\langle N, X \rangle$ be a finite society for which X contains at least three alternatives and let P be the set of all strict preference relations over X. Let f be a collective choice function for the domain $\mathscr{D}(N, X, P)$ generated by P. If f is unanimous and strategy-proof over P then it is a dictatorship: for some individual $i^* \in N$, $f(N, X, \succsim)$ is the favorite alternative of individual i^* for every profile $\succsim \in P$.

Proof

Let f be a unanimous and strategy-proof collective choice function for the domain generated by P. Throughout the argument, every preference relation is strict and every preference profile consists of strict preference relations.

Step 1 *Let $f(N, X, \succsim) = x$ and let $\succsim'$ be a preference profile that differs from $\succsim$ only in the preference relation of individual j and, for some alternative $y \neq x$, (i) $a \succ'_j b$ if and only if $a \succ_j b$ for all alternatives a and b different from y and (ii) $y \prec_j z$ and $y \succ'_j z$ for some alternative z. (That is, $\succsim'_i$ differs from $\succsim_i$ in that y is raised relative to at least one other alternative while the ordering of all other alternatives is maintained.) Then $f(N, X, \succsim') \in \{x, y\}$.*

Proof. Suppose to the contrary that $f(N, X, \succsim') = w$ for some $w \notin \{x, y\}$. If

$w \succ_j x$ then f is not strategy-proof because

$$f(N, X, (\succsim'_j, \succsim_{-j})) = f(X, N, \succsim') = w \succ_j f(N, X, \succsim)$$

(if j's preference relation is $\succsim_j$ she is better off reporting $\succsim'_j$ than reporting $\succsim_j$ when every other individual i reports $\succsim_i$). If $x \succ_j w$ then $x \succ'_j w$ (because x and w differ from y), so that f is also not strategy-proof because

$$f(N, X, (\succsim_j, \succsim'_{-j})) = f(X, N, \succsim) \succ'_j w = f(N, X, \succsim')$$

(if j's preference relation is $\succsim'_j$ she is better off reporting $\succsim_j$ than reporting $\succsim'_j$ when every other individual i reports $\succsim'_i$). ◁

Step 2 *For any alternatives x and y and any preference profile $\succsim$ in which x and y are the top two alternatives for every individual, $f(N, X, \succsim) \in \{x, y\}$.*

Proof. Assume to the contrary that for some preference profile $\succsim$, x and y are the top two alternatives for every individual and $f(N, X, \succsim) \notin \{x, y\}$. Let $\succsim$ be such a profile with the maximal number of individuals who prefer x to y, say k, among such profiles. Then $k < |N|$ because by unanimity we have $f(N, X, \succsim) = x$ if x is the favorite alternative of every individual. Let $f(N, X, \succsim) = z$, let j be an individual for whom $y \succ_j x$, and let $\succsim'_j$ be a preference relation for which x is at the top and y is ranked second. Then the number of individuals who prefer x to y according to $(\succsim'_j, \succsim_{-j})$ is $k + 1 > k$, so that $f(N, X, (\succsim'_j, \succsim_{-j})) \in \{x, y\}$, which, given $x \succ_j z$ and $y \succ_j z$, contradicts the strategy-proofness of f. ◁

Step 3 *Let $\succsim$ and $\succsim'$ be preference profiles for which, for every individual, x and y are the top two alternatives and occur in the same order in $\succsim$ and $\succsim'$: $x \succ_i y$ if and only if $x \succ'_i y$ for all $i \in N$. Then $f(N, X, \succsim) = f(N, X, \succsim')$.*

Proof. By Step 2, $f(N, X, \succsim) \in \{x, y\}$. Without loss of generality assume that $f(N, X, \succsim) = x$. We can transform $\succsim$ into $\succsim'$ by a sequence of moves, at each of which we raise one alternative, other than x or y, in one individual's preferences, keeping x and y as the top two alternatives for all individuals. By Step 1 the alternative given by f after each move is either the raised alternative, which is not x or y, or the alternative given by f before the move. By Step 2 the alternative given by f after each move is either x or y. Thus the alternative given by f after each move is x. ◁

Step 4 *For any alternatives a and b and any preference profile $\succsim$, for each individual i define $\succsim_i^{ab}$ to be the preference relation obtained from $\succsim_i$ by moving a and b to the top, keeping them in the same order as they are in $\succsim_i$ and not changing the order of the other alternatives. For any preference profile $\succsim$, define the binary relation $\trianglerighteq$ on X by $x \trianglerighteq y$ if and only if $f(N,X,\succsim^{xy}) = x$, for all $x \in X$ and $y \in X$. The binary relation $\trianglerighteq$ is a strict preference relation.*

Proof. From Step 2, for all alternatives x and y we have $f(N,X,\succsim^{xy}) \in \{x,y\}$, so that either $x \trianglerighteq y$ or $y \trianglerighteq x$, and hence $\trianglerighteq$ is complete.

To verify that $\trianglerighteq$ is transitive, assume to the contrary that there exist alternatives a, b, and c for which $a \trianglerighteq b \trianglerighteq c \trianglerighteq a$. Consider the profile $\succsim''$ obtained from $\succsim$ by moving a, b, and c to the top, preserving their order, in every individual's preference relation. By an argument analogous to the proof of Step 2, $f(N,X,\succsim'') \in \{a,b,c\}$. Without loss of generality, let $f(N,X,\succsim'') = a$. Now let $\gtrsim$ be the preference profile obtained from $\succsim''$ by moving b to the third position in every individual's preference relation. The conclusions of the following two arguments are contradictory. (*i*) By Step 2, $f(N,X,\gtrsim) \in \{a,c\}$ because the top two alternatives in every preference relation $\gtrsim_i$ are a and c. The preference profile $\succsim''$ may be obtained from $\gtrsim$ by a sequence of changes in each of which b is raised in one individual's preferences. Thus by Step 1, $f(N,X,\succsim'') \in \{b,x\}$, where $x = f(N,X,\gtrsim)$. Given that $f(N,X,\succsim'') = a$, we have $x = a$, so that $f(N,X,\gtrsim) = a$. (*ii*) By definition, given $c \trianglerighteq a$ we have $f(N,X,\succsim^{ac}) = c$. For each individual the relative orders of a and c in the profiles $\gtrsim$ and $\succsim$ are the same, so that Step 3 applied to $\succsim^{ac}$ and $\gtrsim$ implies that $f(N,X,\gtrsim) = c$.

Finally, given $\succsim^{xy} = \succsim^{yx}$ for all alternatives x and y, if $x \trianglerighteq y$ then it is not the case that $y \trianglerighteq x$, so that the ordering is strict. $\lhd$

Step 5 *Let G be the preference aggregation function for $(\langle N,X\rangle, P)$ that maps a preference profile $\succsim$ into a preference relation $\trianglerighteq$ as described in Step 4. This preference aggregation function is dictatorial.*

Proof. I argue that G satisfies the conditions of Proposition 1.9 (Arrow's impossibility theorem), so that by this result, G is dictatorial.

Suppose that $x \succ_i y$ for all $i \in N$. Then x is the favorite alternative of every individual in the preference profile $\succsim^{xy}$ defined in Step 4, so that because F is unanimous, $f(N,X,\succsim^{xy}) = x$, and hence $x \trianglerighteq y$. Thus G satisfies the Pareto property.

Let x and y be two alternatives and let $\succsim$ and $\succsim'$ be two preference profiles for which for every $i \in N$ we have $x \succ_i y$ if and only if $x \succ'_i y$. Then by Step 3 we have $f(N, X, \succsim^{xy}) = f(N, X, \succsim'^{xy})$, so that $x \trianglerighteq y$ if and only if $x \trianglerighteq' y$ where $\trianglerighteq = G(N, X, \succsim)$ and $\trianglerighteq' = G(N, X, \succsim')$. Thus G satisfies independence of irrelevant alternatives. ◁

Step 6 *There exists an individual i^* such that $f(N, X, \succsim)$ is the favorite alternative of i^* for all $\succsim \in P$.*

Proof. By Step 5 there is an individual i^* such that $G(N, X, \succsim) = \succsim_{i^*}$ for all $\succsim \in P$. Let $\succsim \in P$, let $f(N, X, \succsim) = x$, and let y be another alternative. By Step 2 we have $f(N, X, \succsim^{xy}) \in \{x, y\}$. The profile $\succsim$ may be obtained from $\succsim^{xy}$ by a sequence of steps in each of which one alternative other than x and y is raised in one individual's preferences. By Step 1, after each step, the alternative selected by f is either the raised alternative or the alternative selected previously. Given that $f(N, X, \succsim) = x$, we conclude that $f(N, X, \succsim^{xy}) = x$. Thus by the definition of G, $x \succ_{i^*} y$. ◁

2.3 General mechanisms

So far I have discussed only mechanisms in which each individual reports a preference relation and the alternative selected is the one given by a collective choice function for the reported preference profile. More generally, a mechanism designer can ask each individual to select a report from a set of the designer's choosing and base her choice of an alternative in an arbitrary fashion on the profile of submitted reports.

Definition 2.6: Mechanism

Let $\langle N, X \rangle$ be a finite society. A *mechanism* $\langle (S_i)_{i \in N}, g \rangle$ *for* $\langle N, X \rangle$ consists of a set S_i (of possible *reports*) for each individual $i \in N$ and a function $g : \times_{i \in N} S_i \to X$ (the *outcome function*).

For the mechanisms considered in the previous sections, each set S_i is a set of preference relations over X and $g(\succsim)$ is the alternative selected by a collective choice function for the collective choice problem $\langle N, X, \succsim \rangle$. The question addressed is: for which collective choice functions does every individual optimally report her true preference relation, regardless of the preference relations reported by the other individuals? If there are at least three alternatives and each set S_i is the set of all strict preference relations, the answer given by the Gibbard-Satterthwaite theorem is: among unanimous rules, only dictatorships. Can we do better with a general mechanism?

To be more precise about the meaning of doing better, define a mechanism to implement a collective choice function in quasi-dominant strategies if, for every preference profile $\succsim$, there is a report profile $\sigma(\succsim)$ such that the outcome $g(\sigma(\succsim))$ of the mechanism is the alternative selected by the collective choice function for $\succsim$ and no individual can induce an outcome that she prefers by choosing a different report, regardless of the other individuals' reports.

Definition 2.7: Mechanism implementing collective choice function in quasi-dominant strategies

Let $\langle N, X\rangle$ be a finite society, let D be the set of all collective choice problems $\langle N, X, \succsim\rangle$, and let f be a collective choice function for D. The mechanism $\langle (S_i)_{i\in N}, g\rangle$ for $\langle N, X\rangle$ *implements f in quasi-dominant strategies* if for every individual $i \in N$ and every preference relation $\succsim_i$ on X for individual i there exists a report $\sigma_i(\succsim_i) \in S_i$ such that

$$g(\sigma(\succsim)) = f(N, X, \succsim), \tag{2.1}$$

where $\sigma(\succsim) = (\sigma_i(\succsim_i))_{i\in N}$, and

$$g(\sigma_i(\succsim_i), s_{-i}) \succsim_i g(s_i', s_{-i}) \text{ for all } s_i' \in S_i, \text{ all } s_{-i} \in S_{-i}, \text{ and all } i \in N. \tag{2.2}$$

This concept is sometimes called implementation in dominant strategies. However, the report $\sigma_i(\succsim_i)$ in the definition does not necessarily weakly dominate individual i's other possible reports in the sense of Definition 16.19 because it may not satisfy the second condition in this definition. For this reason I attach the prefix quasi.

The question now is: are there collective choice functions that are not strategy-proof but can be implemented in quasi-dominant strategies? The answer is negative. Suppose that a collective choice function f can be implemented in quasi-dominant strategies by a mechanism $M = \langle (S_i)_{i\in N}, g\rangle$. I argue that f is strategy-proof. The fact that f can be implemented in quasi-dominant strategies by M means that for every (true) preference profile $\succsim$ there is a report profile $\sigma(\succsim) \in \times_{i\in N} S_i$ for which (*i*) the outcome $g(\sigma(\succsim))$ is the alternative $f(N, X, \succsim)$ and (*ii*) for no individual i does S_i contain a different report that induces an outcome she prefers, according to $\succsim_i$, for any list of the other individuals' reports. Now consider the mechanism $M' = \langle (S_i')_{i\in N}, g'\rangle$ in which the set S_i' of permitted reports for each individual i is the set of preference relations and $g'(\succsim') = g(\sigma(\succsim'))$ for any profile $\succsim'$ of reports. We can think of this mechanism as one in which each individual i reports a preference relation $\succsim'_i$ and then the mechanism operator reports $\sigma_i(\succsim'_i)$ to the mechanism M on her behalf. For any preference profile $\succsim$, the outcome of M when each individual i reports $\sigma_i(\succsim_i)$ is the same as the outcome of M' when each individual i reports $\succsim_i$, namely $g(\sigma(\succsim))$. Further, under M, no individual i is better off changing her report from $\sigma_i(\succsim_i)$ to the report $\sigma_i(\succsim'_i)$ she would submit if her preference relation were $\succsim'_i$, for any $\succsim'_i$, regardless of the other individuals' reports. Thus in particular i is not better off changing her report from $\sigma_i(\succsim_i)$ to $\sigma_i(\succsim'_i)$ if the other individuals' reports are the ones they would choose for any given specification of their preference relations. Hence i is not better off changing her report from $\succsim_i$ to $\succsim'_i$ under M', regardless of the preference relations reported by the other individuals. That is, the collective choice function is strategy-proof.

Proposition 2.4: Revelation principle for implementation in quasi-dominant strategies

Let $\langle N, X\rangle$ be a finite society, let D be the set of all collective choice problems $\langle N, X, \succsim\rangle$, and let f be a collective choice function for D. If some mechanism for $\langle N, X\rangle$ implements f in quasi-dominant strategies then f is strategy-proof over the set of all preference relations over X.

Proof

Suppose that the mechanism $\langle (S_i)_{i\in N}, g\rangle$ for $\langle N, X\rangle$ implements f in quasi-dominant strategies. For each preference profile $\succsim$ for $\langle N, X\rangle$ and each individual $i \in N$, let $\sigma_i(\succsim_i)$ be the member of S_i given in Definition 2.7. Then for any preference profile $\succsim$ and any preference relation $\succsim'_i$ different from $\succsim_i$, substitute $s'_i = \sigma_i(\succsim'_i)$ and $s_{-i} = (\sigma_j(\succsim_j))_{j\in N\setminus\{i\}}$ into (2.2) to obtain

$$g(\sigma(\succsim)) \succsim_i g(\sigma(\succsim'_i, \succsim_{-i})) \text{ for all } \succsim'_i \text{ and all } i \in N.$$

Using (2.1), this condition is equivalent to

$$f(N, X, \succsim) \succsim_i f(N, X, (\succsim'_i, \succsim_{-i})) \text{ for all } \succsim'_i \text{ and all } i \in N,$$

so that f is strategy-proof.

An implication of this result and Proposition 2.3 (the Gibbard-Satterthwaite theorem) is that if a unanimous single-valued collective choice rule for the domain of all strict preference profiles can be implemented in quasi-dominant strategies then it is dictatorial.

Corollary 2.1: Unanimous collective choice rule that can be implemented in dominant strategies is dictatorial

For a finite society in which there are at least three alternatives, any collective choice function for the set of all strict preference profiles that is unanimous and can be implemented in quasi-dominant strategies is dictatorial: for some individual i^* it selects the favorite alternative of i^* for every collective choice problem.

Notes

Proposition 2.1 is due to Black (1948b, 32); it is a special case of Moulin (1980, Proposition 1) (see also Moulin 1988, Lemma 10.3). Proposition 2.3 is due to Gibbard (1973) and Satterthwaite (1975). My presentation of this result and its proof are taken from Osborne and Rubinstein (2023); I am grateful to Ariel Rubinstein for allowing me to use this material. Proposition 2.4 was first established by Gibbard (1973); several versions have been demonstrated in various models subsequently. For discussions of the implementation of a collective choice rule via solution concepts other than equilibrium in dominant strategies, see Osborne and Rubinstein (1994, Sections 10.4 and 10.5) and Austen-Smith and Banks (2005, Section 3.3).

The result in Exercise 2.3 is generalized by Moulin (1980), who characterizes the collective choice rules that are strategy-proof in a single-peaked domain. Saporiti (2009) provides an analogous characterization for a single-crossing domain. Osborne et al. (2000) study a model that generalizes the example in Exercise 2.4 and Osborne and Tourky (2008) study a model that generalizes the example in Exercise 2.5.

Solutions to exercises

Exercise 2.1

If individual 1 reports her true preference relation, then there is a single Condorcet winner, a. (Alternatives a and b tie, a beats c, and c beats b.) If she instead reports the ordering $b \succ_1 c \succ_1 a$ then the Condorcet winners are a and b. (Alternatives a and b tie, and both a and b beat c.) So by reporting this preference relation rather than her true preference relation she can induce the outcome $\{a, b\}$, in which the best outcome for her is b.

Now suppose that $N = \{1, 2, 3\}$, $b \succ_1 a \succ_1 c$, $a \succ_2 b \succ_2 c$, and $c \succ_3 a \sim_3 b$. Then the set of Condorcet winners is $\{a, b\}$. If individual 1 switches to reporting

$b \succ_1 c \succ_1 a$, then the set of Condorcet winners becomes $\{b\}$.

Exercise 2.2

Suppose the game has three individuals and three alternatives, a, b, and c, with the ordering $a \lhd b \lhd c$. Individual 1 prefers a to b to c, individual 2 prefers b to a to c, and individual 3 prefers c to a to b. If every individual chooses her favorite alternative, the outcome is b. If individual 3 deviates and chooses a, the outcome changes to a, which she prefers to b.

Exercise 2.3

Suppose that an individual i chooses her favorite alternative, say a_i^*. Denote the smallest alternative chosen by all individuals by $\underline{a}$. If $a_i^* = \underline{a}$ then the outcome is a_i^*, and i can do no better by choosing another alternative. If $a_i^* > \underline{a}$ then i can affect the outcome only by choosing an alternative smaller than $\underline{a}$, which is worse for her than $\underline{a}$, given that her preferences are single-peaked. Thus the first condition in Definition 16.19 is satisfied. Now let a' be an alternative different from a_i^*. If all the individuals other than i choose alternatives larger than a', the outcome is better for i if she chooses a_i^* than if she chooses a', so that the second condition in Definition 16.19 is satisfied.

Exercise 2.4

I first argue that if $c \geq k$ then the game has a Nash equilibrium in which no individual submits a report (or equivalently an individual submits a report if and only if her favorite alternative is at most $-h$ or at least h for some $h > k$). If no individual submits a report, then the outcome is 0, so that the payoff of each individual i is $-|x_i^*|$. If individual i deviates to submit a report, the outcome changes to x_i^* and i's payoff becomes $-c$. Thus no individual optimally submits a report if $-|x_i^*| \geq -c$ for all i, or equivalently $c \geq k$.

Now suppose that $c < k$. Let $h \leq k$ be a positive integer and consider the action profile in which an individual i submits a report if and only if $|x_i^*| \geq h$. The outcome is 0 (the mean of the two submitted reports that are smallest in absolute value). First suppose that $x_i^* \geq h$. Then i's payoff is $-x_i^* - c$. If she deviates to not submit a report, the outcome changes to $-h$, so that her payoff becomes $-x_i^* - h$. Thus her submission of a report is optimal for her if and only if $h \geq c$. If $x_i^* \leq -h$, a symmetric argument yields the same conclusion. Now suppose that $0 \leq x_i^* < h$. Then i's payoff is $-x_i^*$. If she deviates to submit a report, the outcome changes to her favorite position, x_i^*, and her payoff becomes $-c$. Thus her non-submission of a report is optimal for her if and only if $x_i^* \leq c$. If $-h < x_i^* \leq 0$, a symmetric argument yields the same conclusion.

In summary, the action profile is a Nash equilibrium if and only if $h \geq c$ and

$x_i^* \leq c$ whenever $x_i^* < h$. If c is an integer, these conditions are satisfied if and only if $h = c$ or $h = c + 1$, and if c is not an integer, the conditions are satisfied if and only if h is the smallest integer that is at least c.

Exercise 2.5

Two factors limit the number of distinct alternatives reported. First, by switching from reporting an alternative that is being reported by k individuals (including the individual in question) to one that is being reported by at least k individuals, an individual reduces her reporting cost. Second, some switches in the alternative an individual reports do not affect the outcome. That is the case, for example, if both alternatives are less than an alternative that is in turn less than the outcome. These two factors run through the following arguments.

i. Denote the outcome of the equilibrium x^*. First suppose that three or more distinct alternatives greater than x^* are reported. Denote the two largest alternatives reported by x and y, and suppose that the number of individuals who report x is at least the number who report y. Then if an individual who is reporting y switches to reporting x, she reduces her reporting cost and does not affect the outcome, and hence is better off. Thus in any equilibrium at most two distinct alternatives greater than the outcome are reported. A symmetric argument shows that also at most two distinct alternatives less than the outcome are reported.

We conclude that at most five distinct alternatives are reported, and if five are reported then the outcome is the middle reported alternative. In this last case, let x and y be the two smallest reported alternatives, with the number of individuals reporting x at least the number reporting y. Then, as in the previous paragraph, an individual reporting y who switches to report x reduces her reporting cost and does not change the outcome, and hence is better off. Thus at most four distinct alternatives are reported.

ii. If one alternative is reported, it is reported by only one individual, because if two or more individuals report it, any one of them can switch to not reporting without changing the outcome, and thereby save the cost of reporting. The alternative is the individual's favorite because if it is not, she can switch to reporting her favorite, which changes the outcome to that alternative. Her favorite alternative must differ from 0, because if it is 0 she is better off switching to not reporting.

iii. Suppose that three distinct alternatives are reported, $x < y < z$. Suppose that the outcome is greater than y. If two or more individuals report y

then an individual who switches from reporting x to reporting y, or vice versa, does not affect the outcome. Thus if two or more individuals report y then if the number who report x is at least the number who report y, an individual who reports y can benefit by switching to report x, and if the number who report y is at least the number who report x, an individual who reports x can benefit by switching to report y. So only one individual reports y. A symmetric argument leads to the same conclusion if the outcome is less than y. Now suppose that the outcome is equal to y. If two or more individuals report y then either a deviation by an individual reporting x to not reporting or a deviation by an individual reporting z to not reporting does not affect the outcome, and hence makes the deviating individual better off. Thus in any equilibrium only one individual reports y.

Now suppose that four distinct alternatives are reported, $w < x < y < z$. If the outcome is at most x then an individual benefits from switching from y to z or vice versa, and if the outcome is at least y an individual benefits from switching from w to x or vice versa. So the outcome is the mean of x and y. If two or more individuals report x, then an individual benefits from switching from x to w or vice versa, because doing so does not affect the outcome. Thus exactly one individual reports x. A similar argument shows that exactly one individual reports y.

II Voting

3 Voting with two alternatives 99

4 Voting with many alternatives: plurality rule 129

5 Sequential pairwise voting 155

6 Ethical voting and expressive voting 191

7 Voting with asymmetric information 201

3 Voting with two alternatives

3.1 Voting as a strategic game 101
3.2 Costly voting with uncertainty about cost 104
3.3 The incentive to vote in a large population 118
3.4 Preferences with regret 121

A group of individuals selects one of two alternatives by voting, which may entail a cost. How do the individuals' decisions to vote depend on their preferences and voting costs? How does the fraction of individuals who vote depend on the population size?

Synopsis

Section 3.1 models voting as a strategic game. The players are individuals, each of whom may vote for one of the alternatives or abstain. Each individual prefers an outcome in which her preferred alternative wins (obtains the most votes) to one in which the alternatives tie, and prefers that outcome to one in which her preferred alternative loses. In this game, an individual's voting for her preferred alternative weakly dominates her voting for the other alternative and, if her cost of voting is zero, also weakly dominates abstention (Proposition 3.1). In particular, if every individual's cost of voting is zero, an action profile is a Nash equilibrium in which no individual's action is weakly dominated if and only if each individual who is not indifferent between the alternatives votes for her preferred alternative (Corollary 3.1). If the individuals' voting costs are positive, most voting games in which the costs are not so high that no individual optimally votes have no Nash equilibrium (Section 3.1.2), rendering the model of little use for understanding voting behavior when voting is costly.

A variant of the model that does have equilibria assumes that each individual knows her own voting cost but is uncertain of the other individuals' costs. Sections 3.2 and 3.3 explore a model in which each individual believes that every other individual's voting cost is drawn independently from a given distribution and each individual's preferences are represented by the expected value of a Bernoulli payoff function. As in the model in Section 3.1, each individual is assumed to know the other individuals' preferences between the alternatives. In every equilibrium of the game I present, there is a number c_i^* for each individual i such that i votes for her favorite alternative if her cost is less than c_i^* and abstains if it is greater than c_i^* (Lemma 3.1). Proposition 3.2 characterizes these threshold costs for equilibria in which every individual who favors the same alternative has the same threshold cost (symmetric equilibria) and Proposition 3.3 shows that such an equilibrium exists.

Intuition suggests that if many individuals vote then the probability of any one individual's vote affecting the outcome, which happens only if the vote among the other individuals is tied or nearly tied, is small, so that only individuals with small voting costs vote. Proposition 3.4 formalizes this idea. For a model in which each individual's voting cost is drawn from a given distribution, it gives conditions under which, as the number of individuals increases without bound, the probability that any given individual votes converges to zero. Section 3.3 explores the idea further in a model in which each individual is uncertain of the other individuals' preferences between the alternatives. It asks how, in this model, the probability that an individual's vote affects the outcome of an election changes as the population increases.

The models presented so far assume that each individual's preferences are represented by the expected value of a function that assigns a payoff to each alternative. Section 3.4 studies the implications of a different assumption. Each individual considers, for each of her actions and each list of the other individuals' actions, how much better off she would have been had she chosen a different action, and chooses the action that minimizes the largest value of this gain over all lists of the other individuals' actions. That is, she chooses the action that minimizes her maximal possible regret. If she votes, then in the outcomes that generate the most regret her favored alternative loses or wins by two votes or more, so that the outcome would have been the same had she not voted (and thereby saved the cost of doing so). If she abstains, then in the outcomes that generate the most regret her favored alternative ties or loses by one vote, so that her voting would have increased the probability of her favored alternative's winning by one-half. Thus if her cost of voting is not too high she optimally votes, regardless of the number of individuals, so that this model can generate high turnout even in large populations.

When an individual chooses to abstain in any of these models, she does so because the cost of voting exceeds the benefit from doing so. Another rationalization for abstention is that individuals feel they are insufficiently informed to make a choice, and prefer to delegate the decision to those who are informed. I explore this idea in Chapter 7.

3.1 Voting as a strategic game

I formulate voting as a strategic game. The model includes the option for each individual not to vote and the possibility that voting is costly. The players are individuals, each of whom can vote for one of two alternatives, a and b, or abstain. If one alternative, say z, receives more votes than the other, it wins, and the payoff of each individual i is $u_i(z) - c_i$ if she votes and $u_i(z)$ if she does not, where u_i is a real-valued function on $\{a, b\}$ and c_i is a nonnegative number. If the alternatives receive the same number of votes, the payoff of each individual i is $\frac{1}{2}(u_i(a) + u_i(b)) - c_i$ if she votes and $\frac{1}{2}(u_i(a) + u_i(b))$ if she does not. One rationale for this specification of an individual's payoff in the case of a tie is that in this event each alternative is selected with probability $\frac{1}{2}$ and u_i is a function whose expected value represents the player's preferences over lotteries over outcomes.

Definition 3.1: Two-alternative voting game

The *two-alternative voting game* $\langle N, \{a, b\}, (u_i)_{i \in N}, (c_i)_{i \in N} \rangle$, where N is a finite set (of individuals) with at least two members, a and b are alternatives, each u_i is a real-valued function on $\{a, b\}$, and each c_i is a nonnegative number, is the following strategic game.

Players

The set N.

Actions

For each player i, the set of actions is $\{\textit{vote for } a, \textit{vote for } b, \textit{abstain}\}$.

Payoffs

For any action profile x, denote by $W(x) \subseteq \{a, b\}$ the set of alternatives that receive the most votes: $W(x) = \{z\}$ if more individuals vote for z than for the other alternative and $W(x) = \{a, b\}$ if the same number of individuals vote for each alternative. The payoff of each player i for x is

$$\begin{cases} \sum_{w \in W(x)} u_i(w)/|W(x)| & \text{if } x_i = \textit{abstain} \\ \sum_{w \in W(x)} u_i(w)/|W(x)| - c_i & \text{if } x_i \in \{\textit{vote for } a, \textit{vote for } b\}. \end{cases}$$

3.1.1 Costless voting

A two-alternative voting game with three or more individuals in which every individual's voting cost is zero has many Nash equilibria. For example, every action profile in which every individual chooses (votes for) the same alternative is an equilibrium, because for such an action profile no change in any individual's action affects the outcome. (Remember that in a Nash equilibrium, no change in any player's action makes her better off, but changes may not make her worse off, either.) The next exercise asks you to find all Nash equilibria in the case that (for simplicity) no individual is indifferent between the alternatives.

Exercise 3.1: Nash equilibria of two-alternative voting game with zero costs

Find all the Nash equilibria of a two-alternative voting game $\langle N, \{a,b\}, (u_i)_{i \in N}, (c_i)_{i \in N} \rangle$ in which $c_i = 0$ and $u_i(a) \neq u_i(b)$ for each $i \in N$.

Among the Nash equilibria are ones in which at least one individual votes for the alternative she likes least. Such equilibria seem implausible, because an individual who votes for such an alternative can gain no possible advantage by doing so and, intuitively, risks influencing the outcome in favor of that alternative. The notion of Nash equilibrium assumes that no individual wavers from her equilibrium action, so that a Nash equilibrium is not affected by such risk. The idea is captured, instead, by the notion of a weakly dominated action: an action a_i for which there is another action b_i that yields i at least as high a payoff as does a_i for *all* actions of the other players and a higher payoff than does a_i for *some* actions of the other players. An individual's voting for her favorite alternative weakly dominates her voting for the other alternative: no matter how the other individuals vote, she is not worse off voting for her favorite alternative than voting for the other alternative, and for some configurations of the other individuals' votes, she is better off. If an individual's voting cost is zero, her voting for her favorite alternative also weakly dominates abstention.

Proposition 3.1: Weak domination in two-alternative voting game

Let $\langle N, \{a,b\}, (u_i)_{i \in N}, (c_i)_{i \in N} \rangle$ be a two-alternative voting game and let i be an individual for whom $u_i(a) \neq u_i(b)$. Individual i's action of voting for her favorite alternative weakly dominates her action of voting for the other alternative and, if $c_i = 0$, weakly dominates *abstain*.

This result is closely related to Proposition 2.1, but the following proof employs an argument independent of that result.

Proof

Table 3.1 shows an individual's payoffs as a function of her action and the winning margin in favor of a among the other individuals' votes. For an individual who prefers a to b, for every column the entry in the top cell is

	winning margin for a among other individuals				
	≥ 2	1	0	-1	≤ -2
vote a	$u_i(a) - c_i$	$u_i(a) - c_i$	$u_i(a) - c_i$	$\frac{1}{2}(u_i(a) + u_i(b)) - c_i$	$u_i(b) - c_i$
vote b	$u_i(a) - c_i$	$\frac{1}{2}(u_i(a) + u_i(b)) - c_i$	$u_i(b) - c_i$	$u_i(b) - c_i$	$u_i(b) - c_i$
abstain	$u_i(a)$	$u_i(a)$	$\frac{1}{2}(u_i(a) + u_i(b))$	$u_i(b)$	$u_i(b)$

Table 3.1 The payoffs of individual i in a two-alternative voting game as a function of her action and the winning margin for a among the other individuals.

at least the entry in the middle cell, and for the second and fourth columns the entry in the top cell is larger than the entry in the middle cell. So for such an individual, voting for a weakly dominates voting for b. If $c_i = 0$ then for every column the entry in the top cell is at least the entry in the bottom cell, and for the third and fourth columns the entry in the top cell is larger than the entry in the bottom cell, so that voting for a weakly dominates abstention. Symmetric arguments apply to an individual who prefers b to a.

An immediate corollary of this result is that in any Nash equilibrium of a two-alternative voting game with costless voting in which no individual's action is weakly dominated, every individual votes for her favorite alternative.

Corollary 3.1: Nash equilibrium in weakly undominated actions in two-alternative voting game with zero costs

Let $\langle N, \{a, b\}, (u_i)_{i \in N}, (c_i)_{i \in N} \rangle$ be a two-alternative voting game in which $c_i = 0$ for all $i \in N$. An action profile is a Nash equilibrium in which no individual's action is weakly dominated if and only if every individual who is not indifferent between the alternatives votes for her favorite alternative.

Because this result isolates a single, plausible, action profile among the plethora of Nash equilibria, most models in which individuals vote between two alternatives and voting is costless assume that no individual uses a weakly dominated action.

3.1.2 Costly voting

When voting costs are positive, the nature of the equilibria is completely different: most two-alternative voting games in which every individual's voting cost is positive have no Nash equilibria unless the voting costs are sufficiently high, when they have equilibria in which no one votes or a single individual does so.

When voting is costly, an individual optimally votes only if doing so makes a difference to the outcome—if her vote is *pivotal*. If the numbers of votes for the two alternatives differ, then no vote on the losing side is pivotal, so in any equilibrium with votes for both alternatives, the alternatives tie. Consider such an equilibrium. If individual i votes for a, then by switching to abstention she changes the outcome from a tie to b. Thus her voting for a is optimal only if $\frac{1}{2}(u_i(a)+u_i(b))-c_i \geq u_i(b)$, or equivalently $\frac{1}{2}(u_i(a)-u_i(b)) \geq c_i$. Similarly, her voting for b is optimal only if $\frac{1}{2}(u_i(b)-u_i(a)) \geq c_i$, and her abstaining is optimal only if $\frac{1}{2}(u_i(a)+u_i(b)) \geq \max\{u_i(a), u_i(b)\} - c_i$.

Thus if $c_i \neq \frac{1}{2}|u_i(a)-u_i(b)|$ for all $i \in N$ then in any equilibrium in which both alternatives receive votes, an individual i votes for a if and only if $\frac{1}{2}(u_i(a)-u_i(b)) > c_i$ and votes for b if and only if $\frac{1}{2}(u_i(b)-u_i(a)) > c_i$, so that for the alternatives to tie we need

$$|\{i \in N : \tfrac{1}{2}(u_i(a)-u_i(b)) > c_i\}| = |\{i \in N : \tfrac{1}{2}(u_i(b)-u_i(a)) > c_i\}|.$$

Most two-alternative voting games do not satisfy this condition, and hence have no Nash equilibria in which both alternatives receive votes.

Games in which the voting costs are high enough have equilibria in which no individual votes and may have equilibria in which one individual votes—equilibria that are of little interest.

These arguments lead to the conclusion that the notion of Nash equilibrium for a two-alternative voting game with positive voting costs is not useful for understanding voting behavior. One option is to consider instead the notion of mixed strategy equilibrium. However, interpretations of mixed strategy equilibria do not fit many environments that voting games are intended to model. Instead, I discuss a related approach, in which each individual is uncertain of the other individuals' voting costs.

3.2 Costly voting with uncertainty about cost

I now specify a model in which each individual knows her own voting cost but is uncertain of the other individuals' voting costs. I show that the model has an equilibrium and characterize its equilibria.

A focus of the analysis is the fraction of individuals who vote in a large population. As in the previous model, an individual is motivated to vote by the possibility that doing so affects the outcome, which happens only if the other individuals' votes are split equally or almost equally between the alternatives. Intuition suggests that if the other individuals' characteristics are uncertain and a large number of these individuals vote, then a tie or near tie among their votes is unlikely, so that the remaining individual optimally votes only if her voting cost is

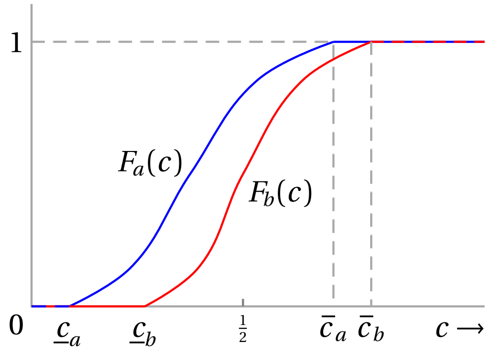


Figure 3.1 An example of distributions of voting costs of the form assumed in a two-alternative voting game with uncertain voting costs.

at most some threshold, which decreases as the number of individuals who vote increases. Thus if the number of individuals is large, an action profile in which a large fraction of them vote is not an equilibrium, because for such an action profile each individual optimally votes only if her voting cost is near the bottom of the distribution of possible costs. The analysis leads to Proposition 3.4, which gives conditions under which this intuition is correct.

I retain the assumption in a two-alternative voting game that the number of individuals who favor each alternative is known. For many elections this assumption is not reasonable, but the model I present captures the fact that each individual is uncertain about the other individuals' voting behavior, and limiting the uncertainty to one source keeps the model relatively simple.

The model of a Bayesian game accommodates these assumptions. It differs from a strategic game in that it includes a specification of the uncertainty that each individual faces (and a payoff function for each individual that applies to the uncertain environment). It models this uncertainty by specifying a set of possible states, the information each individual has about the state, and each individual's belief regarding aspects of the state about which she is not informed.

The uncertainty in the game we wish to analyze concerns the individuals' voting costs, so a state is a profile of such costs. I assume that every individual i knows her own cost, c_i, and believes that the cost of every individual who favors a given alternative is drawn from the same distribution. I assume specifically that the voting cost of every individual who favors $x \in \{a, b\}$ is drawn independently from a nonatomic distribution F_x with support $[\underline{c}_x, \bar{c}_x]$ where $0 < \underline{c}_x < \frac{1}{2} < \bar{c}_x$. The assumption that the distributions are nonatomic means that no single cost has a positive probability, so that the probability distribution functions are continuous, like those in Figure 3.1.

Definition 3.2: Two-alternative voting game with uncertain voting costs

A *two-alternative voting game with uncertain voting costs* $\langle(N_a, N_b), \{a, b\}, (F_a, F_b)\rangle$, where N_a and N_b are finite sets, a and b are alternatives, and F_a and F_b are nonatomic probability distribution functions whose supports are intervals of positive numbers, is the following Bayesian game, where $[\underline{c}_x, \bar{c}_x]$, with $0 < \underline{c}_x < \frac{1}{2} < \bar{c}_x$, denotes the support of F_x for each $x \in \{a, b\}$.

Players

The set $N = N_a \cup N_b$ (N_a consists of individuals who favor a and N_b consists of individuals who favor b).

States

The set of states is the set of profiles $(c_j)_{j\in N}$ of voting costs, with $c_j \in [\underline{c}_a, \bar{c}_a]$ for every player $j \in N_a$ and $c_j \in [\underline{c}_b, \bar{c}_b]$ for every player $j \in N_b$. An individual with a given voting cost is referred to as a *type* of the individual.

Actions

The set of actions of each player is $\{$*vote for a*, *vote for b*, *abstain*$\}$.

Signals

The signal function τ_i of each player i is given by $\tau_i((c_j)_{j\in N}) = c_i$ (every individual knows her own cost, but not the other individuals' costs).

Prior beliefs

Every player believes that for each $x \in \{a, b\}$ the voting cost of each player $i \in N_x$ is drawn from F_x, and that each player's cost is drawn independently of every other player's cost.

Payoffs

The Bernoulli function for each player's preference relation over the set of lotteries over pairs of action profiles and states is defined as follows.

For each player $i \in N_a$ (who favors a), let $u_i(a) = 1$ and $u_i(b) = 0$, and for each $i \in N_b$ (who favors b), let $u_i(b) = 1$ and $u_i(a) = 0$. For any state $(c_j)_{j\in N}$ and any $x \in \{a, b\}$, if more players choose *vote for x* than *vote for y*, the other alternative, then the Bernoulli payoff of player i is $u_i(x) - c_i$ if she votes and $u_i(x)$ if she abstains; if the number of players who vote for each alternative is the same, then $\frac{1}{2}(u_i(a) + u_i(b))$ replaces $u_i(x)$ in each case.

Comments

- The assumption that each individual's preferences are represented by the expected value of a payoff function is not innocuous. Section 3.4 presents an analysis based on a different assumption about the individuals' preferences.
- The assumption that $u_i(x) = 1$ and $u_i(x) = 0$ for an individual i who favors x means that no individual is indifferent between the alternatives. It implies that a voting cost of 1 is equivalent to the difference between an individual's payoff to her favorite alternative and her payoff to the other alternative. Given that the payoffs are the same in every game, they do not appear as parameters in the specification $\langle (N_a, N_b), \{a, b\}, (F_a, F_b)\rangle$.
- The assumption that $\underline{c}_a$ and $\underline{c}_b$ are less than $\frac{1}{2}$ rules out equilibria in which no type of any individual votes, because if no one votes then an individual who deviates to voting for her favorite alternative changes the outcome from a tie to a win for her favorite alternative, changing her payoff from $\frac{1}{2}$ to $1 - c$, which is a increase if $c < \frac{1}{2}$.
- The assumption that $\bar{c}_a$ and $\bar{c}_b$ are greater than $\frac{1}{2}$ means that with positive probability an individual's cost is high enough that she does not optimally vote even if her vote is pivotal. (Perhaps she faces a medical emergency at the time of the vote.) This assumption rules out equilibria in which every type of every individual votes, because if everyone votes then an individual who deviates from voting for x to abstention either does not change the outcome or changes it from a win for x to a tie, or from a tie to a win for the other alternative, and hence increases her payoff by at least $\frac{1}{2} - (1 - c) = 0 - (\frac{1}{2} - c) = c - \frac{1}{2}$, which is positive if $c > \frac{1}{2}$.

An individual's strategy specifies an action for each of her possible types. That is, for each $x \in \{a, b\}$, a strategy for an individual who favors x is a function from $[\underline{c}_x, \bar{c}_x]$ to $\{\textit{vote for } a, \textit{vote for } b, \textit{abstain}\}$. A strategy profile s^* is a Nash equilibrium if the action $s_i^*(c)$ of each type c of each individual i is optimal given i's belief about the other individuals' types and the action $s_j^*(c')$ chosen by each type c' of every other individual j, which together generate a probability distribution in i's mind over the combination of actions of the other individuals.

If you are not a habitué of the world of Bayesian games, you may wonder why we require an individual's equilibrium strategy to specify an action for all of her possible types, given that she knows her type. The reason is that in an equilibrium we want the actions that each individual believes each type of every other individual will take to be the ones that the type would in fact take (given the other individuals' strategies). The way we do that is to have each individual's strategy specify the action chosen by each of her possible types, and require it to be optimal *for each type* given the other individuals' strategies and the individual's beliefs about the other individuals' types. Thus the actions an individual's strategy specifies for types different from her actual type function as the other individuals' beliefs about the actions the individual would take were she to have these types.

Given an individual's belief about the other individuals' behavior, which action should she choose? Intuition suggests that she should vote for her favorite alternative if her voting cost is low and abstain if her voting cost is high. That is, she should employ a threshold strategy.

Definition 3.3: Threshold strategy in two-alternative voting game with uncertain voting costs

Let $\langle (N_a, N_b), \{a, b\}, (F_a, F_b)\rangle$ be a two-alternative voting game with uncertain voting costs and denote the support of F_x by $[\underline{c}_x, \bar{c}_x]$ for each $x \in \{a, b\}$. For each $x \in \{a, b\}$ and each player $i \in N_x$ (who favors x), a strategy $s_i : [\underline{c}_x, \bar{c}_x] \to \{\textit{vote for a}, \textit{vote for b}, \textit{abstain}\}$ of player i is a *threshold strategy* if for some number $c_i^* \in [\underline{c}_x, \bar{c}_x]$ (the *threshold* for individual i) we have

$$s_i(c_i) = \begin{cases} \textit{vote for x} & \text{if } \underline{c}_x \leq c_i < c_i^* \\ \textit{abstain} & \text{if } c_i^* < c_i \leq \bar{c}_x \end{cases}$$

and $s_i(c_i^*) \in \{\textit{vote for x}, \textit{abstain}\}$.

The intuition that each individual's best response to any strategies of the other individuals is a threshold strategy is confirmed by the following analysis.

Consider an individual who favors a. For each of her actions, the outcome generated by each combination of actions of the other individuals, and hence her payoff, depends only on the winning margin for a among the other individuals' actions. Her payoffs when her voting cost is c are given in Table 3.2. For $c > 0$, these payoffs have several relevant features.

- Abstention strictly dominates voting for b.
- If $c > \frac{1}{2}$ then abstention strictly dominates voting for either alternative.
- If $c < \frac{1}{2}$ then voting for a is better than abstaining if and only if the winning margin for a among the other individuals is 0 or -1 (the highlighted cells in Table 3.2), and the gain in payoff to switching from abstention to voting for a for these two winning margins is the same, equal to $\frac{1}{2} - c$.

Specifically, for an individual with voting cost $c > 0$ who favors a, a vote for a is a

	winning margin for a among other individuals				
	≥ 2	1	0	-1	≤ -2
vote for a	$1-c$	$1-c$	$1-c$	$\frac{1}{2}-c$	$-c$
vote for b	$1-c$	$\frac{1}{2}-c$	$-c$	$-c$	$-c$
abstain	1	1	$\frac{1}{2}$	0	0

Table 3.2 The payoffs of an individual of type c who favors a in a two-alternative voting game with uncertain voting costs as a function of her action and the winning margin for a among the other individuals. For the highlighted cells, the individual's voting for a and abstaining generate different outcomes.

best response to the other individuals' strategies if and only if

$$\tfrac{1}{2}\left[\Pr(\text{tie among others' votes}) + \Pr(a \text{ loses by 1 among others' votes})\right] \geq c$$

and abstention is a best response if and only if this inequality is reversed ($\leq$); voting for b is never a best response. In particular, if voting for a is a best response for an individual when her cost is c, it is a best response also when her cost is less than c, and if abstention is a best response when her cost is c, it is a best response also when her cost exceeds c. The same considerations apply to an individual who favors b, so every best response of any individual is a threshold strategy.

Lemma 3.1: Equilibrium of two-alternative voting game with uncertain voting costs

In every Nash equilibrium of a two-alternative voting game with uncertain voting costs, each individual's strategy is a threshold strategy.

Subsequently I restrict attention to Nash equilibria that are symmetric in the sense that every individual who favors a uses the same strategy and every individual who favors b uses the same strategy.

Definition 3.4: Symmetric equilibrium of two-alternative voting game with uncertain voting costs

A Nash equilibrium s^* of a two-alternative voting game with uncertain voting costs $\langle (N_a, N_b), \{a, b\}, (F_a, F_b) \rangle$ is *symmetric* if there are strategies s_a and s_b such that $s_i^* = s_a$ for every individual $i \in N_a$ and $s_i^* = s_b$ for every individual $i \in N_b$.

I now provide a characterization of the symmetric equilibria (Proposition 3.2) and study how these equilibria depend on the parameters of the game. I focus on the behavior of the equilibria as the number of individuals increases without bound, an analysis that culminates in Proposition 3.4. This result gives conditions under which the limit of the cost threshold for voting is the lowest possible cost and the limit of the expected number of individuals who vote is finite.

Consider a symmetric equilibrium in which the thresholds are c_a and c_b. How do these thresholds depend on the individuals' characteristics? Given the other individuals' strategies, an individual who favors x is indifferent between voting for x and abstaining when her cost is equal to the threshold c_x. Her expected payoff in each case depends on the other individuals' strategies and her belief about their types. I now explore these payoffs in detail.

Consider a profile of threshold strategies in which some types of each individual vote and some abstain; let i be an individual who favors a. What probability does i assign to the event that a and b are tied among the other individuals' votes? Only individuals who favor a vote for a, and only those who favor b vote for b, so the maximum number k such that a and b both get k votes among the other individuals is $\min\{n_a-1, n_b\}$, where $n_a = |N_a|$ and $n_b = |N_b|$. The probability i assigns to the event that an individual who favors a votes is the probability that the individual's cost is at most c_a, which is $F_a(c_a)$, and the probability she assigns to the event that an individual who favors b votes is $F_b(c_b)$. Thus the probability i assigns to a tie between a and b among the other individuals' votes is the sum from $k = 0$ to $k = \min\{n_a, -1, n_b\}$ of the probability that the voting cost of k of the other $n_a - 1$ individuals who favor a is at most c_a and the voting cost of k of the individuals who favor b is at most c_b.

To write a compact expression for this probability, I introduce some additional notation. For any positive integer n, integer l with $0 \leq l \leq n$, and number $p \in [0,1]$), denote by $B(n,l,p)$ the probability of exactly l successes in n independent trials when the probability of success on each trial is p:

$$B(n,l,p) = \binom{n}{l} p^l (1-p)^{n-l}.$$

Then the probability that i assigns to a tie among the other individuals' votes when every other individual who favors a votes with probability p_a and every individual who favors b votes with probability p_b is

$$P_a^0(p_a, p_b, n_a, n_b) = \sum_{k=0}^{\min\{n_a-1,n_b\}} B(n_a-1, k, p_a) B(n_b, k, p_b).$$

Similarly, the probability that i assigns to a's losing by one vote among the other individuals is

$$P_a^1(p_a, p_b, n_a, n_b) = \sum_{k=0}^{\min\{n_a-1, n_b-1\}} B(n_a - 1, k, p_a) B(n_b, k+1, p_b).$$

Thus i's expected gain from voting is

$$G_a(p_a, p_b, n_a, n_b) = \tfrac{1}{2}\left[P_a^0(p_a, p_b, n_a, n_b) + P_a^1(p_a, p_b, n_a, n_b)\right]. \tag{3.1}$$

She optimally votes if this gain is greater than her cost, abstains if it is less than her cost, and is indifferent between voting and abstaining if it is equal to her cost. For the threshold strategies we are considering, an individual who favors a votes with probability $F_a(c_a)$, the probability that her voting cost is less than c_a, and an individual who favors b votes with probability $F_b(c_b)$, so the condition in terms of c_a and c_b for a symmetric equilibrium in which some types of each individual who favors a vote and some abstain is $c_a^* = G_a(F_a(c_a^*), F_b(c_b^*), n_a, n_b)$, the condition for an equilibrium in which all types abstain is $c_a^* \geq G_a(F_a(c_a^*), F_b(c_b^*), n_a, n_b)$, and the condition for an equilibrium in which all types vote is $c_a^* \leq G_a(F_a(c_a^*), F_b(c_b^*), n_a, n_b)$.

For an individual who favors b, the expressions differ only in that the roles of a and b are interchanged, so a pair of threshold strategies with thresholds c_a^* and c_b^* is a symmetric equilibrium if and only if

$$\begin{aligned} c_a^* &\left\{\begin{matrix} \geq \\ = \\ \leq \end{matrix}\right\} G_a(F_a(c_a^*), F_b(c_b^*), n_a, n_b) \begin{cases} \text{if } F_a(c_a^*) = 0 \\ \text{if } 0 < F_a(c_a^*) < 1 \\ \text{if } F_a(c_a^*) = 1 \end{cases} \\ c_b^* &\left\{\begin{matrix} \geq \\ = \\ \leq \end{matrix}\right\} G_b(F_a(c_a^*), F_b(c_b^*), n_a, n_b) \begin{cases} \text{if } F_b(c_b^*) = 0 \\ \text{if } 0 < F_b(c_b^*) < 1 \\ \text{if } F_b(c_b^*) = 1. \end{cases} \end{aligned} \tag{3.2}$$

By Lemma 3.1, each individual's strategy in a Nash equilibrium is a threshold strategy, so this argument establishes the following result.

Proposition 3.2: Symmetric equilibrium of two-alternative voting game with uncertain voting costs

A strategy profile s^* of a two-alternative voting game with uncertain voting costs $\langle (N_a, N_b), \{a, b\}, (F_a, F_b)\rangle$ is a symmetric Nash equilibrium if and only if for numbers c_a^* and c_b^* that satisfy (3.2) and threshold strategies s_a and s_b with thresholds c_a^* and c_b^* respectively we have $s_i^* = s_a$ for every individual $i \in N_a$ and $s_i^* = s_b$ for every individual $i \in N_b$.

Exercise 3.2: Two-alternative voting game with uncertain voting costs

Consider a two-alternative voting game with uncertain voting costs $\langle (N_a, N_b), \{a, b\}, (F_a, F_b)\rangle$ in which $|N_a| = |N_b| = 2$.

a. Find a Nash equilibrium of this game when F_a and F_b are uniform on $[0,1]$ (so that $F_a(x) = F_b(x) = x$ for $x \in [0,1]$).

b. If F_a is uniform on $[0,1]$, for which distributions F_b, if any, does the game have a Nash equilibrium in which no type of any individual in N_b votes?

Informal analysis for $n_a = n_b$ and $F_a = F_b$

To get an idea of the nature of a symmetric equilibrium and the way it varies with the number of individuals, suppose first that the number of individuals who favor a is the same as the number who favor b and the probability distribution of every individual's voting cost is the same: $n_a = |N_a| = n_b = |N_b| = m$ $(= \frac{1}{2}|N|)$ and $F_a = F_b = F$, with $\underline{c}_a = \underline{c}_b = \underline{c}$ and $\bar{c}_a = \bar{c}_b = \bar{c}$.

Under these assumptions, it is reasonable to look for an equilibrium in which all individuals have the same cost thresholds for voting, and hence the same probabilities of voting. If this probability is p, then the expected gain of any individual from voting is $G_a(p, p, m, m) = G_b(p, p, m, m)$. Denote the common function by G, and for any value of m define the function g_m by $g_m(p) = G(p, p, m, m)$ for all p. That is, $g_m(p)$ is each individual's expected gain from voting when m individuals favor each alternative and every other individual votes with probability p. By (3.2), the condition for an equilibrium in which $p_a^* = p_b^* = p^*$ with $0 < p^* < 1$ is $F^{-1}(p^*) = g_m(p^*)$.

Consider the function g_m. Let i be an individual who favors a. If the probability that each of the other individuals votes is zero, a vote by i certainly changes the outcome, from a 0–0 tie to a win for a, so that i's gain from voting is $\frac{1}{2}$. If the probability that each of the other individuals votes is positive but close to zero, the probability of a tie between the alternatives among the other individuals' votes is relatively high, and hence i's voting is likely, but not certain, to change the outcome; thus her gain from voting is less than $\frac{1}{2}$, but not much less. As the probability that each of the other individuals votes increases, the probability of a tie among their votes decreases, reducing i's gain from voting. When the probability that each of the other individuals votes is $\frac{1}{2}$, the probability of a tie among their votes reaches a minimum, and hence i's gain from voting is at a minimum. As the probability that each of the other individuals votes increases above $\frac{1}{2}$, the probability of a tie increases, raising i's gain from voting; when the probability reaches 1, this gain is again $\frac{1}{2}$. For any value of p, an increase in the number m of

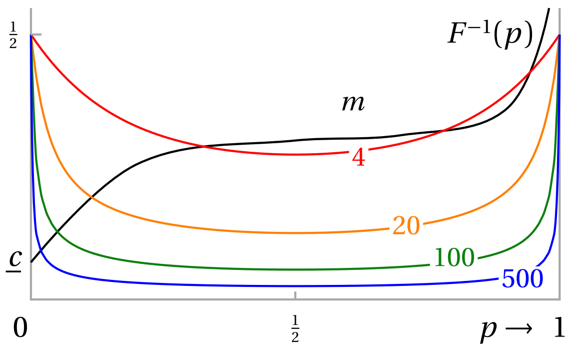


Figure 3.2 The colored curves are graphs of the function g_m defined by $g_m(p) = G(p, p, m, m)$, an individual's expected gain from voting when every other individual votes for her favorite alternative with probability p and abstains with probability $1 - p$, and m individuals favor each alternative, for various values of m.

individuals who favor each alternative reduces the probability of a tie among the other individuals' votes, and hence reduces i's gain from voting.

Figure 3.2 shows examples of the function g_m for a few values of m, as well as an example of F^{-1}. For any given value of m, the equilibrium values of p are those for which $F^{-1}(p) = g_m(p)$.

This analysis suggests the following results.

- If F is continuous, an equilibrium exists: given $\underline{c} \leq \frac{1}{2}$, the graph of F^{-1} crosses each colored line in Figure 3.2 at least once.
- Multiple equilibria may exist: for example, for $m = 4$ in Figure 3.2 the game has three equilibria.
- For every value of p with $0 < p < 1$ the value of $g_m(p)$ decreases to zero as m increases without bound, so that if m is large enough the game has only one equilibrium.
- As m increases without bound the equilibrium probability that an individual votes goes to zero.

The general case

Existence of an equilibrium The first observation for the special case, that the continuity of the probability distribution function of the voting cost ensures that an equilibrium exists, applies also to the general model.

Proposition 3.3: Nash equilibrium of two-alternative voting game with uncertain voting costs

Every two-alternative voting game with uncertain voting costs has a symmetric Nash equilibrium and every such equilibrium is a threshold strategy profile.

Proof

Denote the game $\langle (N_a, N_b), \{a, b\}, (F_a, F_b)\rangle$ and define the function $\widehat{G}_a : [0,1] \times [0,1] \to [0,1] \times [0,1]$ by

$$\widehat{G}_a(c_a, c_b) = G_a(F_a(\min\{\max\{c_a, \underline{c}_a\}, \overline{c}_a\}), F_b(\min\{\max\{c_b, \underline{c}_b\}, \overline{c}_b\}), n_a, n_b)$$

and the function $\widehat{G}_b$ analogously, where $n_a = |N_a|$ and $n_b = |N_b|$. Given that F_a and F_b are continuous and G_a and G_b are continuous in p_a and p_b, $\widehat{G}_a$ and $\widehat{G}_b$ are continuous, so that by Brouwer's fixed point theorem there is a pair $(\hat{c}_a, \hat{c}_b) \in [0,1] \times [0,1]$ such that

$$\begin{aligned} \hat{c}_a &= \widehat{G}_a(\hat{c}_a, \hat{c}_b) \\ \hat{c}_b &= \widehat{G}_b(\hat{c}_a, \hat{c}_b). \end{aligned} \tag{3.3}$$

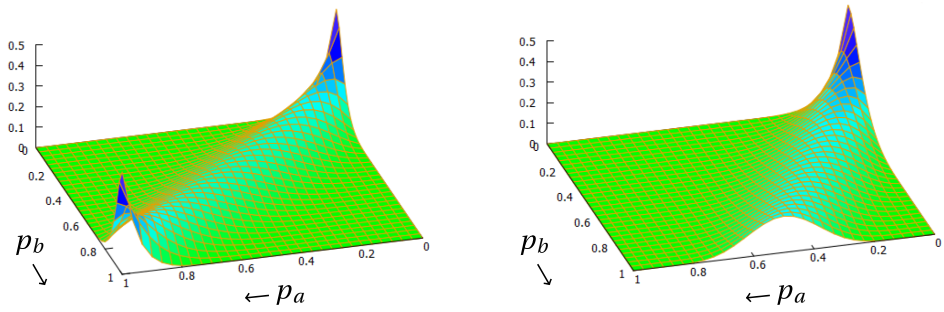
Let $c_a^* = \min\{\max\{\hat{c}_a, \underline{c}_a\}, \overline{c}_a\}$ and $c_b^* = \min\{\max\{\hat{c}_b, \underline{c}_b\}, \overline{c}_b\}$. I argue that (c_a^*, c_b^*) satisfies (3.2), so that by Proposition 3.2 the threshold strategy profile in which the threshold of every individual who favors a is c_a^* and the threshold of every individual who favors b is c_b^* is a Nash equilibrium.

To see that (c_a^*, c_b^*) satisfies (3.2), first suppose that $F_a(c_a^*) = 0$. Then $\hat{c}_a \le \underline{c}_a = c_a^*$ from the definition of c_a^* in terms of $\hat{c}_a$, so that the first condition in (3.3) implies that $c_a^* \ge G_a(F_a(c_a^*), F_b(c_b^*), n_a, n_b)$. Now suppose that $0 < F_a(c_a^*) < 1$. Then $\hat{c}_a = c_a^*$, so that the first condition in (3.3) implies that $c_a^* = G_a(F_a(c_a^*), F_b(c_b^*), n_a, n_b)$. The other cases follow similarly.

Every equilibrium is a threshold strategy profile by Lemma 3.1.

Properties of equilibrium with large number of individuals How does equilibrium turnout vary as the number of individuals increases? My informal analysis suggests that if $n_a = n_b$ and $F_a = F_b$ then when the number of individuals is large the game has only one symmetric equilibrium in which $p_a^* = p_b^*$, and in this equilibrium the common probability of voting goes to 0 as the number of individuals increases.

To study the equilibria more generally, first consider how an individual's ex-



(a) $n_a = n_b = 20$. (b) $n_a = 20$, $n_b = 10$.

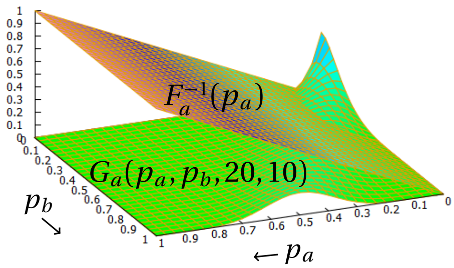
Figure 3.3 The expected gain $G_a(p_a, p_b, n_a, n_b)$ from voting for an individual who favors a as a function of the probability p_a of voting for each of the other $n_a - 1$ individuals who favor a and the probability p_b of voting for each of the n_b individuals who favor b.

pected gain from voting varies with the probabilities with which the other individuals vote. Figure 3.3 shows two examples of this gain for an individual who favors a, as a function of the probability p_a of voting for each of the other $n_a - 1$ individuals who favor a and the probability p_b of voting for each of the n_b individuals who favor b. Figure 3.3a shows an example in which n_a and n_b are equal and Figure 3.3b shows one in which they differ. (The restriction of the surface in Figure 3.3a to the line $p_a = p_b$ is the orange curve in Figure 3.2.) Notice that when n_a and n_b are equal, the expected gain from voting is close to $\frac{1}{2}$ when (p_a, p_b) is close to either $(0,0)$ (no one else votes) or $(1,1)$ (everyone else votes), and when n_a and n_b differ, it is close to $\frac{1}{2}$ only when (p_a, p_b) is close to $(0,0)$.

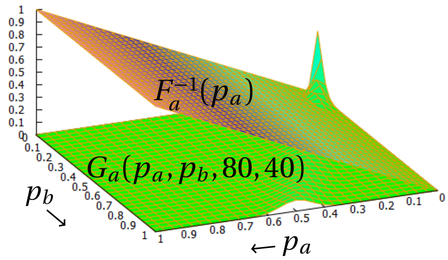
As n_a and n_b increase, the expected gain decreases at all points except $(0,0)$ and, if $n_a = n_b$, $(1,1)$, where it remains $\frac{1}{2}$. Suppose that the numbers of individuals who favor each alternative increase proportionately. That is, consider a sequence of games in which $n_x = r k_x$ for each $x \in \{a, b\}$, where k_x is a given positive integer and $r = 1, 2, \ldots$. For $(k_a, k_b) = (20, 10)$, Figure 3.4 shows two examples of G_a: the left panel is for $r = 1$ and the right panel is for $r = 4$.

Figure 3.4 shows also the function F_a^{-1} in the case that F_a is uniform on $[0,1]$. The first equilibrium condition in (3.2) is $F_a^{-1}(p_a) = G_a(p_a, p_b, n_a, n_b)$ for $0 < p_a < 1$, which means that an equilibrium lies on the intersection of the two surfaces in Figure 3.4. The second equilibrium condition is the analogue for F_b^{-1} and G_b, a function that has the same general form as G_a when n_a and n_b are large. An equilibrium pair (p_a, p_b) lies at the intersection of (*i*) the intersection of the two surfaces in Figure 3.4 and (*ii*) the intersection of the analogous surfaces for F_b^{-1} and G_b. The figures suggest that as r, and hence the number of individuals who favor each alternative, increases, the possible equilibrium values of p_a and p_b converge to 0.

The next result shows that this property of an equilibrium holds generally:



(a) $n_a = 20$, $n_b = 10$.



(b) $n_a = 80$, $n_b = 40$.

Figure 3.4 The expected gain $G_a(p_a, p_b, n_a, n_b)$ from voting for an individual who favors a and the function $F_a^{-1}(p_a)$ in the case that F_a is uniform on $[0,1]$.

if $k_a \neq k_b$ then in the equilibria of the game in which rk_a individuals favor a and rk_b favor b, as r increases without bound the probability that any individual votes goes to zero.

Proposition 3.4: Voting probability converges to zero as population increases

Fix positive integers k_a and k_b with $k_a \neq k_b$. For any positive integer r, let $N_a(r)$ be a set with rk_a members and $N_b(r)$ a set with rk_b members, and let $\Gamma(r) = \langle (N_a(r), N_b(r)), \{a, b\}, (F_a, F_b)\rangle$ be a two-alternative voting game with uncertain voting costs. For each $x \in \{a, b\}$ denote the support of F_x by $[\underline{c}_x, \bar{c}_x]$. Let $(c_a^*(r), c_b^*(r))$ be the thresholds in a symmetric Nash equilibrium threshold strategy profile of $\Gamma(r)$ (which exists by Proposition 3.3). Then the limits of $c_a^*(r)$ and $c_b^*(r)$ as r increases without bound are $\underline{c}_a$ and $\underline{c}_b$, so that the limiting probability that any individual votes is zero.

Proof

The argument uses the following result (suggested by Figure 3.3), which may be derived from the properties of the binomial distribution (see Rosenthal 2020, Corollary 10): given $k_a \neq k_b$, for all values of (p_a, p_b) except those close to $(0,0)$, the values of $G_a(p_a, p_b, rk_a, rk_b)$ and $G_b(p_a, p_b, rk_a, rk_b)$ converge to zero as r increases without bound. Precisely, for every $\varepsilon > 0$ and $x = a, b$,

$$\lim_{r\to\infty} \max_{(p_a,p_b)} \{G_x(p_a, p_b, rk_a, rk_b) : p_a \geq \varepsilon \text{ or } p_b \geq \varepsilon\} = 0. \tag{3.4}$$

Now, let $\varepsilon > 0$ and $\delta = F_a^{-1}(\varepsilon) - \underline{c}_a$, so that $\varepsilon = F_a(\underline{c}_a + \delta)$. Then (3.4)

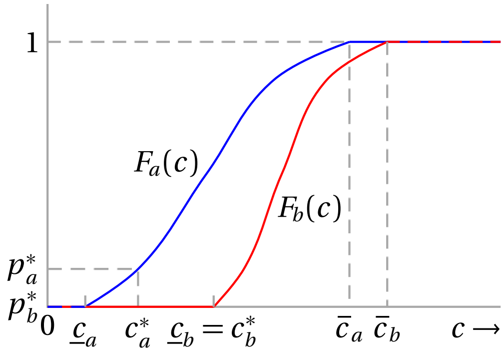


Figure 3.5 The thresholds and probabilities of voting in an equilibrium of a two-alternative voting game with uncertain voting costs in which individuals who favor b do not vote.

implies that there exists r^* such that for all $r > r^*$,

$$\max_{(p_a,p_b)} \{G_x(p_a, p_b, rk_a, rk_b) : p_a \geq \varepsilon \text{ or } p_b \geq \varepsilon\} < \delta.$$

In particular, if $F_a(c_a^*(r)) \geq \varepsilon$ then

$$G_a(F_a(c_a^*(r)), F_b(c_b^*(r)), rk_a, rk_b) < \delta.$$

This inequality is inconsistent with (3.2) given $F_a(c_a^*(r)) \geq \varepsilon > 0$ because $F_a(c_a^*(r)) \geq \varepsilon$ means that $c_a^*(r) > \underline{c}_a + \delta \geq \delta$. Thus for any $\varepsilon > 0$, there exists r^* such that for all $r > r^*$ such that $F_a(c_a^*(r)) < \varepsilon$, and similarly $F_b(c_b^*(r)) < \varepsilon$. That is, $c_a^*(r)$ and $c_b^*(r)$ converge to $\underline{c}_a$ and $\underline{c}_b$ as r increases without bound,

In an equilibrium, one of the alternatives may receive no votes: none of the individuals who favor that alternative may vote (see Figure 3.5). In such an equilibrium, the number of votes for the other alternative is zero with positive probability, so that the alternative that certainly receives no votes ties with positive probability. The alternative that receives votes may win with high probability, even if the number of individuals who favor it is much smaller than the number who favor the other alternative, as the following exercise shows.

Exercise 3.3: Equilibrium in which low-cost minority is likely to win

Let $\langle (N_a, N_b), \{a, b\}, (F_a, F_b) \rangle$ be a two-alternative voting game with uncertain voting costs. Suppose that $|N_a| = 2$ and let $w \in (\frac{1}{2}, 1)$. Find distributions of voting costs F_a and F_b such that the game has a symmetric Nash equilibrium in which individuals who favor b do not vote and a wins with probability w regardless of the number of individuals who favor b.

Exercise 3.4: Voluntary and mandatory voting

Compare the outcome when voting is mandatory with the symmetric equilibria of a two-alternative voting game with uncertain voting costs for the parameters in Exercise 3.2a and Exercise 3.3.

Note that the analysis of this section is limited to symmetric equilibria. The game may in addition have equilibria in which not all the individuals who favor a given alternative use the same strategy.

Note also, more fundamentally, that the notion of Nash equilibrium may be inappropriate as a solution concept for a model of an election. The interpretation of the notion of Nash equilibrium is most appealing for situations in which individuals repeatedly and anonymously interact. In such situations, it may be reasonable to assume that each individual's long experience playing the game allows her to form accurate beliefs about the actions the other individuals will take. Most elections do not fit into that category: they are unique events, and individuals have scant basis to form accurate beliefs about each other's strategies. The problem is particularly significant for a voting game with imperfect information, where the notion of Nash equilibrium requires that each individual form accurate beliefs about the action taken by every type (voting cost) of every other individual. In many elections, the source of the information an individual could use to form such beliefs is unclear.

3.3 The incentive to vote in a large population

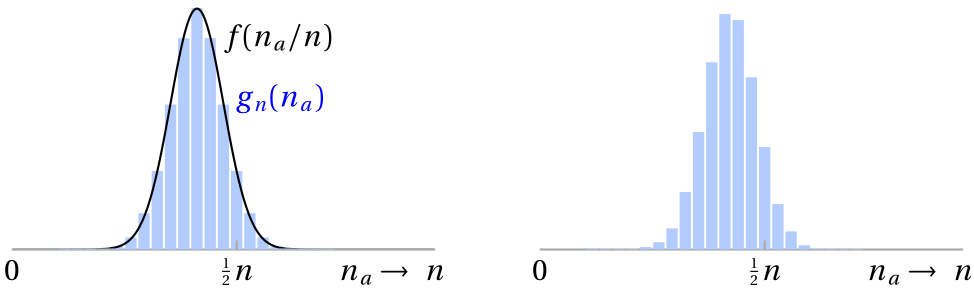
Is the turnout in elections consistent with a model in which individuals decide to vote by comparing the costs and benefits? Theory alone cannot answer this question, but it can provide a framework for thinking about it. I present a brief analysis of the rate of change of the probability that an individual's vote affects the outcome of an a election as the population increases.

An individual motivated to vote by the chance that her vote will affect the electoral outcome must form a belief about that probability. The basis of her belief is plausibly information about the voting intentions of the other individuals. Sources of such information include polls, reports in media, the individual's own observations, and other individuals. A model can impose discipline on these beliefs by requiring that in an equilibrium they are correct. For the purposes of this analysis, the model of the previous section, a two-alternative voting game with uncertain voting costs, seems inadequate, because it assumes that the number of individuals who favor each alternative is known; the only uncertainty in the model concerns the other individuals' voting costs. An essential ingredient in the formation of an individual's estimate of the probability that her vote will affect the outcome of an election appears to be uncertainty about the other individuals' preferences.

One model that incorporates such uncertainty assumes that each individual believes that the alternative favored by every other individual is drawn independently from a known distribution. But this assumption implies that in a large population the distribution of preferences, if not the preference of any one individual, is known almost with certainty. For example, if each individual believes that every other individual independently favors a with probability p and b with probability $1-p$, then if the number of individuals is large, she knows that the fraction who favor a is close to p and the fraction who favor b is close to $1-p$. In a large population, if $p > \frac{1}{2}$ then the probability that individuals who favor a are in a majority is close to 1 and if $p < \frac{1}{2}$ then the probability that they are in a minority is close to 1. Thus the model is hardly more appealing for a large population than one in which the fractions of individuals who favor the alternatives are known.

An alternative assumption, which is consistent with the survival in a large population of uncertainty about the majority, is that each individual believes that the probability p with which every other individual independently favors a is itself uncertain. One specific assumption is that every individual has the same prior belief about the distribution of p, and her only private information concerns her own preferences. Every individual views her own preference for a or b, as well as every other individual's preference, as being the outcome of a random draw: with probability p she has been assigned a preference for a, and with probability $1-p$ a preference for b. She treats her realized preference as evidence regarding the distribution of p. If she favors a, she concludes that this distribution is skewed towards large values, and if she favors b, she concludes that it is skewed towards small values. If, for example, the mean of the prior distribution of p is $\frac{1}{2}$, then the posterior probability density that an individual who favors a assigns to $p = \frac{1}{3}$ is (using Bayes' rule) $\frac{2}{3}$ of the prior density, whereas the density that an individual who favors b assigns to this value of p is $\frac{4}{3}$ of the prior density. The assumption that individuals view their own preferences as having been generated by a random process and that they make inferences from their preferences about this process seems odd, and its implication regarding the individuals' posterior beliefs seems implausible.

Assume instead that each individual starts with a prior directly over the number of individuals who favor each alternative, rather than deriving this prior from a model of the probabilistic determination of each individual's preferences. As-



(a) A common prior over the number of individuals who favor a, fitted to a continuous distribution.

(b) Posterior belief over the number of individuals who favor a for an individual who favors a.

Figure 3.6 Beliefs about the number of individuals who favor a.

sume that every individual has the same prior. We want to study how the individuals' beliefs change as the total number n of individuals increases, so assume specifically that each individual has in mind a continuous probability density f of the fraction q of individuals in the population who favor a, and f is independent of n. One possible source of this belief is a poll, which might suggest, for example, that 48% of the population favors a, with a margin of error of 2%. For any given population size n, each individual derives her prior belief about the distribution of the number of individuals who favor a by approximating f by a discrete distribution g_n over the numbers 1 through n, as illustrated in Figure 3.6a. She knows her own political preference, and derives her posterior belief about the distribution of the number of individuals who favor a in the standard way from g_n, using Bayes' rule. (If, for example, she favors a, then the information she uses to update her prior is that at least one individual favors a.) Under this assumption, when n is large, the posterior distribution for an individual who favors a differs only slightly from the prior distribution, and thus also only slightly from the posterior distribution for an individual who favors b. (Figure 3.6b shows the posterior for an individual who favors a given the prior in Figure 3.6a.)

If the number n of voters is odd, the vote of an individual who favors a makes a difference to the outcome if the number of other individuals who vote for a is $\frac{1}{2}(n-1)$; if n is even, it makes a difference if this number is $\frac{1}{2}(n-2)$. When n is large, the individual's posterior belief assigns a probability of approximately $f(\frac{1}{2})/n$ to each of these events. Thus the individual believes that the probability that her vote will make a difference to the outcome is proportional to $1/n$. When the population of voters doubles, for example, the probability that the individual's vote makes a difference to the outcome halves.

To get an idea of the magnitude of the probability, suppose that, based on a poll that reports that 48% of the individuals who intend to vote support a, with margin of error 2%, f is a (truncated) normal distribution with mean 0.48 and standard deviation $0.02/1.96 \approx 0.0102$. (I am taking the "margin of error" of 2% to mean that the 5th and 95th percentiles of f are 46% and 50%. The truncation of the normal distribution makes little difference, because the density of the distribution outside $[0,1]$ is close to zero.) For this distribution, $f(\frac{1}{2}) \approx 0.39$. Thus the probability that an individual's vote makes a difference to the outcome of an election in a population of n voters is approximately $0.39/n$, so that in an electoral district with 10,000 voters, it is approximately 0.000039. So to make voting worthwhile for an individual in such a district, the benefit to her from the election of her favorite candidate rather than the other candidate has to be about 25,000 ($= 1/0.000039$) times her voting cost.

Throughout this analysis I have assumed that every individual's voting cost is nonnegative. For an individual who has to travel far or to wait in a long line to vote, the cost may be significant. But for an individual who can vote online or by mail, or has only to walk a short distance to her polling station, the cost may be trivial. Further, an individual may derive satisfaction from endorsing a candidate she likes, or may feel good about carrying out a task that she believes is her duty, so that her cost is effectively negative. In a two-alternative voting game with uncertain voting costs, such an individual optimally votes regardless of her beliefs about the other individuals' behavior—and hence also regardless of the expected closeness of the election.

I have assumed also that every individual is self-interested; when deciding whether to vote, she considers only the change in her personal welfare that her vote might bring about. Suppose instead that individuals consider the benefit to society. The size of this benefit is plausibly proportional to the number n of individuals in the society, so that even if the probability that an individual's vote affects the outcome of the election is proportional to $1/n$, the expected benefit of voting may be large. Chapter 6 presents a different model of voting in which individuals are public-spirited. Under some conditions, the equilibria of this model entail positive turnout even in an arbitrarily large population.

3.4 Preferences with regret

A key assumption in the model of a two-alternative voting game with uncertain voting costs is that each individual's preferences over uncertain electoral outcomes are represented by the expected value of a payoff function over the possible deterministic outcomes. This function assigns a number to each outcome, independently of the lotteries among which the individual is choosing. This assumption may not be appropriate: an individual's evaluation of an outcome may depend not only on the outcome itself but also on the outcomes that would have occurred had she taken a different action. For example, if an individual who favors a abstains, she may experience regret if the outcome is a tie between a and b, because in that case her voting would have benefitted her. If the outcome is that a wins, she experiences no such regret. If the outcome is that a loses by a large margin, she may also experience no regret, but if it loses by a small margin, she may regret that she did not vote and did not do more to persuade her a-favoring friends to vote. Regret may be experienced not only by an individual who abstains, but also by one who votes: if an individual who favors a votes and the outcome is that a loses or wins by two votes or more, she may regret that she needlessly incurred the cost of voting.

An individual's knowing that she will experience regret for certain outcomes plausibly influences the action she chooses. For example, she may be inclined to vote if she knows she will deeply regret abstaining if the vote turns out to be a tie, but will only mildly regret voting if the outcome is far from a tie.

The assumption that each individual maximizes the expected value of a payoff function embodies another premise: each individual has a precise belief about the probabilities of the possible outcomes. In some elections, individuals have few sources of information about these probabilities. Consider, for example, the election of a legislator in a district in which the candidates have not previously competed against each other, and few opinion polls exist. When the basis for forming beliefs about the probabilities is unclear, individuals may use an alternative calculus. One option is for an individual to choose an action that minimizes the most she will regret from taking the action.

Consider an individual who favors a and has a cost c of voting that is less than $\frac{1}{2}$. Suppose that she abstains. If the vote among the other individuals is a tie, she obtains the payoff $\frac{1}{2}$, but could have obtained the higher payoff of $1 - c$ by voting for a, so her regret from abstaining is $1 - c - \frac{1}{2} = \frac{1}{2} - c$. If among the other individuals the winning margin for a is 1, then she obtains the payoff 1; no other action yields a higher payoff, so her regret from abstaining in this case is 0. Table 3.3 gives the regret for each action and each possible winning margin for a among the other individuals. The entry in the cell in row r and column w of the table is the difference between the largest entry in column w of Table 3.2 (which gives the individual's payoffs) and the entry in row r and column w of that table.

We see that the most the individual regrets from voting for a is c, which happens when it turns out that her vote makes no difference. The most she regrets from voting for b is 1, which happens when the other votes are tied, so that her vote makes a difference in the wrong direction, and the most she regrets from abstaining is $\frac{1}{2} - c$, which happens when her vote would have made a difference to the outcome in her favor. So given $c < \frac{1}{2}$, the action that minimizes her maximal regret is voting for a if $c < \frac{1}{2} - c$, or $c < \frac{1}{4}$, and abstaining if $c > \frac{1}{4}$. A similar analysis

	winning margin for a among other individuals				
	≥ 2	1	0	-1	≤ -2
vote for a	c	c	0	0	c
vote for b	c	$\frac{1}{2}+c$	1	$\frac{1}{2}$	c
abstain	0	0	$\frac{1}{2}-c$	$\frac{1}{2}-c$	0

Table 3.3 The regret for an individual who favors a and has voting cost $c < \frac{1}{2}$ as a function of her action and the winning margin for a among the other individuals. For each action, the highest regrets are highlighted.

for $c > \frac{1}{2}$ shows that in this case abstention always minimizes her maximal regret.

The logic can be stated succinctly as follows. If the individual abstains, the outcome for which she incurs the most regret for not having voted is a tie; if she votes for a, the outcome for which she incurs the most regret for having voted is a win for a by two votes or more. If $c < \frac{1}{4}$ then the amount of her maximal regret if she votes for a is less than the amount of her maximal regret if she abstains, so she votes for a. That is, if an individual acts to minimize her maximal regret, she votes (for her favorite alternative) if her voting cost is less than one-quarter of her payoff from her favored alternative, regardless of the number of individuals. Hence turnout is independent of the size of the population.

Exercise 3.5: Minmax regret individual with three alternatives

Suppose that there are three alternatives, a, b, and z, rather than two. An individual's payoffs to the alternatives are $u(a)=1$, $u(b)=k$, and $u(z)=0$, with $0 < k < 1$. If $k > \frac{1}{2}$ and $c < \frac{1}{2}k$, the maximum regret for each of the individual's actions is achieved for the following events. Vote for a: among the other individuals, z wins by 1 vote over b and a gets fewer votes than b. Vote for b, vote for z, or abstain: among the other individuals, a and z are tied and b gets at least two fewer votes than a and z. Which action minimizes the individual's maximum regret? The same action minimizes the individual's regret for all other parameter values. Do these conclusions make sense?

Notes

The methods of noncooperative game theory were first applied to the study of voting by Robin Farquharson in his doctoral thesis of 1958, published as Farquharson (1969). Section 3.1.2 draws on Palfrey and Rosenthal (1983, Proposition 1). Section 3.2 is based on Ledyard (1981) and Palfrey and Rosenthal (1985), and draws also on Krasa and Polborn (2009) and Taylor and Yildirim (2010). Proposition 3.3 is due to Ledyard (1981, Proposition 2) and Proposition 3.4 is due to Palfrey and Rosenthal (1985, Theorem 2). The first two models of the relation between turnout and population size discussed in Section 3.3 are analyzed by Chamberlain and Rothschild (1981) and Myatt (2015). Evren (2012) studies a model of the type mentioned at the end of the section, in which each individual considers the benefit to society from a change in the outcome that her vote might induce. Section 3.4 (including Exercise 3.5) is based on Ferejohn and Fiorina (1974). General models of decision-making under uncertainty that incorporate regret are explored by Loomes and Sugden (1982) and Bell (1982); Bleichrodt and Wakker (2015) discuss these theories.

Campbell (1999) (see also Taylor and Yildirim 2010, Proposition 2) explores the effect of differences in the distributions of voting costs on the winning alternative, an issue touched upon in Exercise 3.3.

Börgers (2004) and Krasa and Polborn (2009) explore the difference between voluntary and mandatory voting considered in Exercise 3.4.

Solutions to exercises

Exercise 3.1

An action profile is a Nash equilibrium if and only if it satisfies one of the following conditions, where the winning margin is the difference between the number of votes for the winner and the number of votes for the other alternative.

1. The winning margin is at least three votes.
2. The winning margin is two votes and every individual who votes for the winning alternative prefers that alternative to the other alternative.
3. The winning margin is one vote and every individual who either votes for the winning alternative or abstains prefers the winning alternative to the other alternative.
4. Each alternative receives the same number of votes, all individuals vote, and every individual votes for her favorite alternative.

(Case 4 is possible only if the number of individuals is even and each alternative is the favorite of exactly half of the individuals.)

Exercise 3.2

a. We have

$$\begin{aligned} P_a^0(F_a(c_a), F_b(c_b), 2, 2) &= B(1,0,F_a(c_a))B(2,0,F_b(c_b)) + B(1,1,F_a(c_a))B(2,1,F_b(c_b)) \\ &= (1 - F_a(c_a)) \times (1 - F_b(c_b))^2 + F_a(c_a) \times 2F_b(c_b)(1 - F_b(c_b)) \\ &= (1 - c_a)(1 - c_b)^2 + 2c_a c_b(1 - c_b) \end{aligned}$$

and

$$\begin{aligned} P_a^1(F_a(c_a), F_b(c_b), 2, 2) &= B(1,0,F_a(c_a))B(2,1,F_b(c_b)) + B(1,1,F_a(c_a))B(2,2,F_b(c_b)) \\ &= (1 - F_a(c_a)) \times 2F_b(c_b)(1 - F_b(c_b)) + F_a(c_a) \times (F_b(c_b))^2 \\ &= 2(1 - c_a)c_b(1 - c_b) + c_a c_b^2. \end{aligned}$$

After some algebra we get $G_a(F(c_a), F(c_b), 2, 2) = \frac{1}{2}(1 - c_a + 2c_a c_b - c_b^2)$, and similarly $G_b(F(c_a), F(c_b), 2, 2) = \frac{1}{2}(1 - c_b + 2c_a c_b - c_a^2)$. Thus condition (3.2) for a symmetric equilibrium in which the thresholds are c_a and c_b and some types of each individual vote and some abstain is

$$\begin{aligned} c_a &= \tfrac{1}{2}(1 - c_a + 2c_a c_b - c_b^2) \\ c_b &= \tfrac{1}{2}(1 - c_b + 2c_a c_b - c_a^2) \end{aligned}$$

or

$$\begin{aligned} 3c_a &= 1 + 2c_a c_b - c_b^2 \\ 3c_b &= 1 + 2c_a c_b - c_a^2. \end{aligned}$$

Subtracting the second equation from the first we get $3(c_a - c_b) = (c_a - c_b)(c_a + c_b)$, so that if $c_a \neq c_b$ then $c_a + c_b = 3$, which is not possible. Thus $c_a = c_b$. Denote the common value c. Then the condition for an equilibrium is $1 - 3c + c^2 = 0$ or $c = \frac{1}{2}(3 - \sqrt{5}) \approx 0.382$. (The other root of the equation is greater than 1.)

Thus the game has a symmetric Nash equilibrium in which each individual votes if her cost is less than $\frac{1}{2}(3 - \sqrt{5})$ and abstains if her cost is greater than $\frac{1}{2}(3 - \sqrt{5})$.

b. If no individual who favors b votes, then for an individual who favors a we have $P_a^0(F_a(c_a), F_b(c_b), 2, 2) = 1 - F_a(c_a)$ (the alternatives are tied if and only if the other individual who favors a does not vote) and $P_a^1(F_a(c_a), F_b(c_b), 2, 2) = 0$ (the winning margin for b is never positive). Thus $G_a(F_a(c_a), F_b(c_b), 2, 2) = \frac{1}{2}(1 - c_a)$, and hence each individual who favors a is indifferent between voting and abstaining if $c_a = \frac{1}{3}$.

Now consider an individual who favors b. If each individual who favors a votes if and only if her cost is less than c_a and the other individual who favors b does not vote, the probability assigned by an individual who favors b to a tie between a and b among the other voters is $(1-c_a)^2$ (neither of the individuals who favor a vote) and the probability that the winning margin for a is 1 is $2c_a(1-c_a)$ (exactly one of the individuals who favors a votes). Thus the expected gain from voting for an individual who favors b is $\frac{1}{2}[(1-c_a)^2 + 2c_a(1-c_a)] = \frac{1}{2}(1-c_a^2) = \frac{4}{9}$. Hence any distribution F_b for which the lower limit of the support is at least $\frac{4}{9}$ is consistent with an equilibrium in which no type of individual who favors b votes. In such an equilibrium, a wins with probability $\frac{5}{9}$ (the probability that neither of the individuals who favor a votes) and a and b tie with probability $\frac{4}{9}$.

Exercise 3.3

Denote by p_a the probability with which each individual who favors a votes and let $n_b = |N_b|$. Then

$$P_a^0(p_a, 0, 2, n_b) = 1 - p_a$$
$$P_a^1(p_a, 0, 2, n_b) = 0,$$

so that $G_a(p_a, 0, 2, n_b) = \frac{1}{2}(1-p_a)$, and

$$P_b^0(p_a, 0, 2, n_b) = (1-p_a)^2$$
$$P_b^1(p_a, 0, 2, n_b) = 2p_a(1-p_a),$$

so that $G_b(p_a, 0, 2, n_b) = \frac{1}{2}(1-p_a^2)$. Now, if neither of the individuals who favor a votes then the vote is a tie, so that a wins with probability $\frac{1}{2}$; otherwise a wins. So the probability that a wins is $\frac{1}{2}(1-p_a)^2 + 1 - (1-p_a)^2 = \frac{1}{2}(1+2p_a - p_a^2)$. Let p_a^* be the (unique) number such that this probability is w. Then $p_a^* \in (0,1)$. Construct F_a with the property that $F_a(\frac{1}{2}(1-p_a^*)) = p_a^*$ and F_b with the property that $F_b(\frac{1}{2}(1-(p_a^*)^2)) = 0$, as in Figure 3.7. Then the game has a threshold equilibrium in which the thresholds are $c_a^* = \frac{1}{2}(1-p_a^*)$ and $c_b^* = \frac{1}{2}(1-(p_a^*)^2)$ and a wins with probability w, regardless of the number n_b of individuals who favor b.

Exercise 3.4

For the game in Exercise 3.2a, the outcome of the symmetric Nash equilibrium is that each alternative is selected with probability $\frac{1}{2}$. If voting is mandatory, the outcome is the same, but every individual incurs the voting cost with certainty. Thus voluntary voting is better than mandatory voting.

In the symmetric equilibrium of the game in Exercise 3.3, a wins unless both individuals who favor a abstain, which occurs with probability $(1-p_a^*)^2$, in which case both a and b win with probability $\frac{1}{2}$. Thus the probability that b wins is $\frac{1}{2}(1-p_a^*)^2$. Hence each of the n_b individuals who favor b obtains the payoff 1 with probability $\frac{1}{2}(1-p_a^*)^2$ and the payoff 0 otherwise.

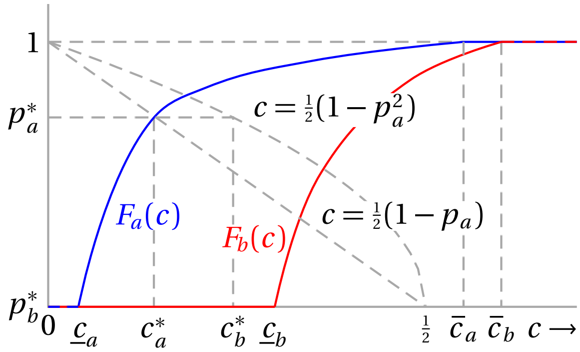


Figure 3.7 The thresholds and probabilities of voting in an equilibrium of a two-alternative voting game with uncertain voting costs in which individuals who favor b do not vote (Exercise 3.3).

If voting is mandatory, b wins with certainty and each individual i who favors b obtains the payoff $1 - c_i$. If $\bar{c}_b < 1$, then all of these payoffs are positive, so for n_b large enough the sum of the individuals' payoffs under mandatory voting exceeds the sum of the payoffs under voluntary voting (regardless of the value of p_a^*). Thus according to the utilitarian welfare criterion, mandatory voting is better than voluntary voting. However, the outcome under mandatory voting does not dominate the outcome under voluntary voting: the individuals who favor a are worse off under mandatory voting. If $\bar{c}_b > 1$ then some individuals who favor b are worse off under mandatory voting, so the comparison between voluntary and mandatory voting according to the utilitarian welfare criterion depends on the forms of the distributions F_a and F_b of voting costs.

Exercise 3.5

Given that the maximum regret for each of the individual's actions is achieved for the events given in the question, the individual's maximal regrets are

$$\begin{aligned}
\text{vote for } a &: \text{payoff } -c\text{; switch to } b \Rightarrow \text{payoff } \tfrac{1}{2}k - c, \text{ so regret } \tfrac{1}{2}k \\
\text{vote for } b &: \text{payoff } \tfrac{1}{2} - c\text{; switch to } a \Rightarrow \text{payoff } 1 - c, \text{ so regret } \tfrac{1}{2} \\
\text{vote for } z &: \text{payoff } -c\text{; switch to } a \Rightarrow \text{payoff } 1 - c, \text{ so regret } 1 \\
\text{abstain} &: \text{payoff } \tfrac{1}{2}\text{; switch to } a \Rightarrow \text{payoff } 1 - c, \text{ so regret } \tfrac{1}{2} - c.
\end{aligned}$$

Given $k < 1$, the action that minimizes the individual's maximal regret is to vote for a, her favorite alternative.

If an individual has no information about the probabilities of the voting behavior of the other individuals, and wants to choose an action that minimizes her regret, voting for her favorite alternative makes sense. One possibility is that the votes among the other individuals make voting for her second choice,

b, optimal: if among the other individuals b and z are tied, and a is two or more votes behind, then voting for a yields a payoff of $\frac{1}{2}k$ whereas voting for b yields a payoff of k. However, voting for b could lead to a bigger regret: if a and z are tied and b is trailing for two votes or more, voting for b yields a payoff of $\frac{1}{2}$ whereas voting for a yields a payoff of 1. Voting for a is a safer option for an individual who wants to minimize her regret, because it is guaranteed to generate a regret of at most $\frac{1}{2}k$.

Plausibly an individual who chooses the action that minimizes her maximal regret votes for her favorite alternative in an extension of the example to an arbitrary number of alternatives, but I do not know whether that is in fact the case.

4 Voting with many alternatives: plurality rule

4.1 Plurality rule voting games 130
4.2 Spatial model with concave payoff functions 138
4.3 Strategic and sincere voting: divided majority 139

One mechanism a group of individuals can use to select an alternative from a set of many alternatives is plurality rule: each individual votes for a single alternative, and the winner is the alternative that receives the most votes (or the set of alternatives that receive the most votes, in the case of a tie). This chapter analyzes models of this mechanism under the assumption that voting is costless.

Synopsis

The main model, a plurality rule voting game, is an extension to many alternatives of a two-alternative voting game with the restriction that all voting costs are zero. If there are three or more individuals, this game has many Nash equilibria. For example, for every alternative a, the strategy profile in which every individual votes for a is a Nash equilibrium. (No change in any individual's vote affects the outcome.) Further, restricting attention to actions that are not weakly dominated, which reduces the set of equilibria dramatically in two-alternative games, has little impact in many-alternative games. For an individual who is not indifferent among all the alternatives, voting for her least favored alternative is weakly dominated (by voting for her favorite alternative), but if there are at least four individuals, none of her others actions are weakly dominated (Proposition 4.1). Proposition 4.2 shows that every plurality rule voting game has a Nash equilibrium in which no individual's action is weakly dominated (which, despite the game's profusion of Nash equilibria and the limited number of weakly dominated actions, is not obvious.)

In some applications, the set of alternatives is assumed to be an interval of numbers and each individual's payoff function is assumed to be strictly concave. Section 4.2 considers such games. Proposition 4.3 shows that in any Nash equilibrium at most two alternatives tie for first place. The strict concavity of the payoff functions means that every individual's least favored alternative is an end-point of the interval of alternatives, so for any alternative a except these two, the game has a Nash equilibrium in which the sole winner is a and no individual's action is weakly dominated.

Among the many situations for which the model of a plurality rule voting game has many Nash equilibria, here is one that seems particularly problematic. There are three alternatives, a, b, and c. A majority of individuals prefer a and b to c, with about half of them preferring a to b and the remainder preferring b to a, while the remaining minority prefer c to both a and b, between which they are indifferent. The plurality rule voting game that models this situation has a Nash equilibrium in which every individual votes for her favorite alternative, so that c wins, even though it is the worst alternative for a majority of individuals. It also has Nash equilibria in which all the individuals who favor a or b vote for a, so that a wins, or all of these individuals vote for b, so that b wins. Is there any reason to think that one of these equilibria is more likely to occur than the others? Section 4.3 studies this question in a model in which each individual is uncertain of the other individuals' preferences. The analysis identifies circumstances in which, when the number of individuals is sufficiently large, in every equilibrium the individuals who favor a or b all vote for one of these two alternatives, so that it wins, as well as circumstances in which equilibria in which c wins persist even when the number of individuals is arbitrarily large.

4.1 Plurality rule voting games

Definition 4.1: Plurality rule voting game

The *plurality rule voting game* $\langle N, X, (u_i)_{i \in N} \rangle$, where

- N is a finite set (of individuals) with at least two members
- X is a set (of alternatives) that is either finite, with at least two members, or a nonempty interval of real numbers
- $u_i : X \to \mathbb{R}$ for each $i \in N$ (individual i's payoff function)

is the strategic game with the following components.

Players
The set N.

Actions
Each player's set of actions consists of *vote for a* for each $a \in X$ and *abstain*.

Payoffs

For any action profile x, denote by $W(x) \subseteq X$ the set of alternatives that receive the most votes (the *winning alternatives*): for each $a \in W(x)$, the number of players i for whom $x_i =$ *vote for a* is the same and, if $W(x) \neq X$, exceeds the number for whom $x_i =$ *vote for b* for every $b \in X \setminus W(x)$. The payoff of each player $i \in N$ for x is the average value of $u_i(w)$ over $W(x)$ (that is, $\sum_{w \in W(x)} u_i(w)/|W(x)|$ if $W(x)$ is finite, as it is unless no one votes and X is an interval of real numbers).

As for a two-alternative voting game, one rationale for the specification of the players' payoffs in the case of a tie for first place is that every alternative in the winning set is selected with the same probability, and each player evaluates a lottery according to its expected payoff.

Consider an action profile in a plurality rule voting game in which some alternatives are tied for first place, so that every individual's payoff is her average payoff for these alternatives. An individual who abstains or votes for an alternative that is not a winner can, by deviating to vote for a winner, cause that alternative to win outright. Thus in a Nash equilibrium every such individual is indifferent among the winners. Also, each individual who votes for a winner can, by switching her vote to another winner, cause that alternative to win outright. So in an equilibrium her payoff for each winning alternative for which she does not vote is at most her average payoff for the winning alternatives. Hence she either prefers the alternative for which she votes to every other winner or is indifferent among all the winners. These observations are stated in the following result.

Lemma 4.1: Nash equilibrium of plurality rule voting game

Let $\langle N, X, (u_i)_{i \in N} \rangle$ be a plurality rule voting game, let x be a Nash equilibrium of this game, and let $W(x)$ be the set of winning alternatives for x.

a. If for any $i \in N$ the action x_i is either a vote for an alternative outside $W(x)$ or *abstain*, then $u_i(w)$ is the same for all $w \in W(x)$ (i is indifferent among all winning alternatives).

b. If for some $a \in W(x)$ the action x_i is *vote for a*, then

$$u_i(b) \leq \sum_{w \in W(x)} u_i(w)/|W(x)| \quad \text{for all } b \in W(x) \setminus \{a\}$$

and hence either $u_i(a) > u_i(b)$ for all $b \in W(x) \setminus \{a\}$ (i prefers a to every other winning alternative) or $u_i(a) = u_i(w)$ for all $w \in W(x)$ (i is indifferent among all winning alternatives).

When the alternatives number three or more, a complete characterization of the set of Nash equilibria of a plurality rule voting game is complicated. The details are less significant than the message that if there are three or more individuals then regardless of the individuals' preferences, the game has many Nash equilibria. In particular, if there are three or more individuals then for any preference profile and *any* alternative, the game has Nash equilibria in which that alternative wins. For example, all individuals' voting for any given alternative is a Nash equilibrium, because no deviation by a single player affects the outcome.

Weakly dominated actions

When there are two alternatives, the only action of an individual in a plurality rule voting game that is not weakly dominated is a vote for her favorite alternative (Proposition 3.1). An application of the logic for this result shows that when there are three or more alternatives, an individual's voting for any alternative that she ranks lowest is weakly dominated (by her voting for one of her favorite alternatives). If there are four or more individuals, no other alternative is weakly dominated. To see why, suppose that z is an alternative that individual i ranks lowest and b is an alternative that she does not rank lowest. If b and z are tied for the highest number of votes among the other individuals and every other alternative has two or more fewer votes (which requires the total number of individuals to be at least four), then i's voting for b is better for her than voting for any other alternative or abstaining: if she votes for b, then b wins, whereas if she votes for any other alternative, then either z wins or b and z tie, and if she abstains, then b and z tie. Thus no action weakly dominates voting for b.

Proposition 4.1: Weak domination in plurality rule voting game

Let $\langle N, X, (u_i)_{i \in N} \rangle$ be a plurality rule voting game for which X contains three or more alternatives.

a. Let $i \in N$ be an individual who is not indifferent among all the alternatives. Individual i's abstaining and her voting for any alternative z that she likes least (that is, $u_i(z) \leq u_i(y)$ for all $y \in X$) are both weakly dominated by her voting for any of her favorite alternatives.

b. Suppose that the number of individuals is at least four. If an individual is indifferent among all the alternatives then none of her actions are weakly dominated. Otherwise, her only weakly dominated actions are votes for one of the alternatives she likes least.

Proof

a. Let a be one of i's favorite alternatives and z be one of the alternatives she likes least.

First consider an arbitrary list of actions of the individuals other than i. Denote by W^0 the set of winning alternatives given these actions when i abstains. If i switches to vote for a then the set of winning alternatives either remains W^0 or becomes $W^0 \cup \{a\}$ or $\{a\}$, depending on the other individuals' actions. Each of these outcomes is at least as good for i as W^0.

Now suppose that everyone but i abstains. Then if i abstains, the set of winning alternatives is the set X of all alternatives, and if she votes for a it is $\{a\}$. So her payoff is higher when she votes for a.

Thus i's voting for a weakly dominates her abstaining.

Now let W^0 be the set of winning alternatives given the actions of the individuals other than i when i votes for z. If she switches to vote for a then the set of winning alternatives either remains W^0, becomes $W^0 \cup \{a\}$, $W^0 \setminus \{z\}$, $W^0 \cup \{a\} \setminus \{z\}$, or $\{a\}$, or changes from $\{z\}$ to $\{z\} \cup Y$ for some set Y of alternatives. In each case, the outcome when i votes for a is as least as good for her as the outcome when she votes for z. If everyone but i abstains, the winning alternative is z if i votes for z and a if she votes for a, so her payoff when she votes for a is higher than it is when she votes for z. So i's voting for a weakly dominates her voting for z.

b. Let i be an individual who is not indifferent among all the alternatives, let z be an alternative that individual i likes least, and let b be an alternative that she prefers to z. I argue that i's voting for b is not weakly dominated.

Consider an action profile in which one individual votes for b, two vote for z, and the remainder, except for i, abstain. (Such an action profile is possible because there are four or more individuals.) If i votes for b the outcome is a tie between b and z and if she votes for any other alternative or abstains the outcome is z. Thus her payoff from voting for b exceeds her payoff from all her other actions, so that no action weakly dominates voting for b.

Exercise 4.1: Weak domination in plurality rule voting game with three individuals

Consider a plurality rule voting game with three individuals. Show that an individual's voting for an alternative b is weakly dominated if and only if she is not indifferent between all alternatives and her payoff for b is at most her payoff for a set of alternatives consisting of b, one of her favorite alternatives, and an alternative she likes least.

As I have argued, if there are three or more individuals then for any alternative a, a plurality rule voting game has a Nash equilibrium in which every individual votes for a. For which alternatives a does such a game have a Nash equilibrium in which the winner is a *and* no individual's action is weakly dominated? If a is ranked last by some individuals who are not indifferent among all the alternatives then if every individual votes for a, the actions of the individuals who rank a last are weakly dominated by Proposition 4.1a. However, if the number n of individuals is at least five and the number who rank a last is at most $\frac{1}{2}(n-3)$, the game does have a Nash equilibrium in which no individual's action is weakly dominated and the winner is a. In one such equilibrium, every individual who ranks a last votes for her favorite alternative and every other individual votes for a. The number of individuals who vote for a is at least three more than the number who vote for any other alternative, so that no change in any individual's vote affects the identity of the winner, and hence the action profile is a Nash equilibrium; by Proposition 4.1b, no individual's action is weakly dominated.

This argument shows that restricting individuals to actions that are not weakly dominated has no impact, in many plurality rule voting games, on the set of alternatives that are winners in some Nash equilibrium. Nevertheless, such a restriction seems sensible, simply because weakly dominated actions lack appeal. Imposing the restriction raises a question: does a plurality rule voting game with three or more alternatives necessarily have a Nash equilibrium game in which no individual's action is weakly dominated? The answer is affirmative.

Proposition 4.2: Existence of Nash equilibrium in weakly undominated actions of plurality rule voting game

Every plurality rule voting game has a Nash equilibrium in which no individual's action is weakly dominated.

For a game with two individuals, this result follows from Proposition 3.1. For a game with three individuals, you are asked to prove the result in the next exercise.

Exercise 4.2: Nash equilibrium in weakly undominated actions of plurality rule voting game with three individuals

Show that every plurality rule voting game with three individuals has a Nash equilibrium in which no individual's action is weakly dominated.

For a game with four or more individuals, each of whom has strict preferences, the proof is easy.

Proof of Proposition 4.2 for four or more individuals, strict preferences

Assume that no individual is indifferent between any two alternatives. Choose two alternatives arbitrarily; call them a and b. Consider the action profile in which each individual votes for the alternative in $\{a, b\}$ that she prefers. I argue that this action profile is a Nash equilibrium in which no individual's action is weakly dominated.

Suppose without loss of generality that the number of individuals who prefer a to b is at least the number who prefer b to a.

- If a receives three or more votes than b, no change in any individual's action affects the outcome.
- If a receives two votes more than b, the only change in an individual's action that affects the outcome is a switch by an individual voting for a to vote for b, which changes the outcome from a to a tie between a and b, and hence makes her worse off.
- If a receives one more vote than b, the only change in an individual's action that affects the outcome is a switch by an individual voting for a. If she switches to b, the outcome changes from a to b, so that she is worse off. If she switches to another alternative or to abstention, the outcome changes to a tie between a and b (given that the number of individuals is at least four), so that she is also worse off.
- If a and b receive the same number of votes, any change in an individual's action changes the outcome from a tie between a and b to a win for the alternative she likes less.

We conclude that the action profile is a Nash equilibrium. In the profile, no individual votes for the alternative she likes least, so that by Proposition 4.1b no individual's action is weakly dominated.

For a game with four or more individuals whose preferences are not necessarily strict, the only proof of which I am aware, due to Duggan and Sekiya (2009), is lengthy, and I omit it. This proof shows that the following procedure generates an equilibrium in which no individual's action is weakly dominated. At each step $t = 1, 2, \ldots$, an action profile x^t is defined. In the initial profile, x^1, every individual votes for an alternative selected arbitrarily from her set of favorite alternatives. If, at any step t, x^t is a Nash equilibrium, the procedure ends. Otherwise, some individual i, by voting for an alternative z different from x_i^t, can cause z to win or tie for first place and thereby increase her payoff, given the other individuals' actions. (If there is more than one such individual, select one arbitrarily.) Define x^{t+1} to be the profile derived from x^t by changing x_i^t to a vote for an alternative that yields i the highest payoff among such alternatives z. (If there is more than one such alternative, select one arbitrarily.) Duggan and Sekiya (2009, Theorem 1) show that this procedure terminates and generates a Nash equilibrium in which no individual's action is weakly dominated. (Their model does not allow abstention, but adding that option does not affect the result because a Nash equilibrium remains an equilibrium if the option to abstain is added, and if voting for some alternative is not weakly dominated by voting for another alternative then it is not weakly dominated by abstention.)

Exercise 4.3: Nash equilibrium in weakly undominated actions of plurality rule voting game

Use the procedure just described to find a Nash equilibrium in weakly undominated actions of the plurality rule voting game $\langle \{1,2,3,4,5,6\}, \{a,b,c,d\}, (u_i)_{i \in N} \rangle$, in which the payoffs are given as follows.

	1	2	3	4	5	6
payoff 9:	c	d	c	a	a	b
payoff 6:	d	b	b	d	c	a
payoff 0:	a, b	a, c	a, d	b, c	b, d	c, d

Suppose that we eliminate from a plurality rule voting game the actions of each individual that are weakly dominated. In the game that results, some actions may be weakly dominated. A game in which there are three alternatives, one of which, say c, is ranked last by every individual, provides a trivial example: after voting for c is removed as an option for each individual, each individual's voting for her less-preferred alternative among the two that remain is weakly dominated. In a less trivial example, there are three alternatives, a, b, and c, and three individuals, two of whom prefer a to b to c and one of whom prefers c to b to a. After voting for c is removed as an option for the first two individuals and voting for a is removed for the last individual, voting for b is weakly dominated by voting for a for the first individual if she prefers $\{a, b, c\}$ to $\{b\}$: if she switches from voting for b to voting for a, then either the outcome does not change, or it changes from $\{b\}$ to $\{a\}$, from $\{b\}$ to $\{a, b\}$, from $\{a, b\}$ to $\{a\}$, from $\{b, c\}$ to $\{a, c\}$, from $\{a, b, c\}$ to $\{a\}$, or from $\{b\}$ to $\{a, b, c\}$.

Exercise 4.4: Iterated elimination of weakly dominated actions in a plurality rule voting game

Consider a variant of a plurality rule voting game in which abstention is not an option and there are three alternatives and at least four individuals, all with strict preferences. Show that if more than two-thirds of the individuals rank the same alternative last then after weakly dominated actions are iteratively removed from the game, a single action profile remains.

Exercise 4.5: Voting game under proportional rule

Consider the following variant of a plurality rule voting game in which abstention is not an option, the alternatives are numbers, and the outcome is a vote-weighted average of the alternatives. The set of players (individuals) is $N = \{1, \ldots, n\}$ and the set of alternatives is $X = \{a_1, \ldots, a_k\}$, where each a_j is a number and $a_1 < a_2 < \cdots < a_k$. For an action profile x in which the number of votes for each alternative a_j is $v_j(x)$, the outcome is $O(x) = \sum_{j=1}^{k} v_j(x) a_j / n$ and the payoff of each individual i is $u_i(O(x))$, where each $u_i : \mathbb{R} \to \mathbb{R}_-$ is a single-peaked function.

a. Show that for any Nash equilibrium x^*, all individuals except those with favorite positions close to $O(x^*)$ vote for one of the extreme alternatives (a_1 or a_k). Specifically, every individual i whose favorite alternative is at most $O(x^*) - (a_k - a_1)/n$ votes for a_1 and every individual i whose favorite alternative is at least $O(x^*) + (a_k - a_1)/n$ votes for a_k.

b. Suppose that for some number z^* that is not the favorite alternative of any individual we have $z^* = [L(z^*)a_1 + G(z^*)a_k]/n$, where $L(z^*)$ is the number of individuals whose favorite alternatives are less than z^* and $G(z^*)$ is the number of individuals whose favorite alternatives are greater than z^*. Find a Nash equilibrium of the game in which every individual votes for one of the two extreme alternatives.

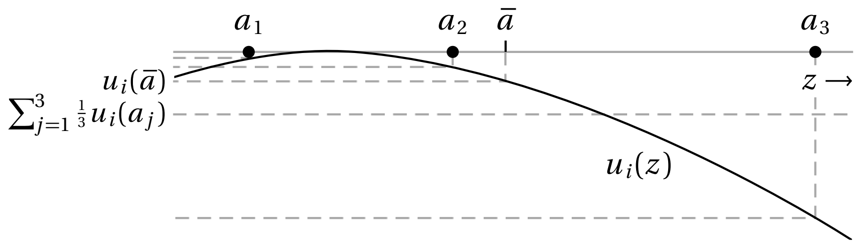


Figure 4.1 A strictly concave payoff function of individual i in a plurality rule voting game in which the alternatives are real numbers. If, when the individual votes for a_1, the set of winners is $\{a_1, a_2, a_3\}$, then her payoff is $\sum_{j=1}^{3} \frac{1}{3} u_i(a_j)$; if she switches her vote to a_2, the set of winners becomes $\{a_2\}$, yielding her the larger payoff $u_i(a_2)$.

4.2 Spatial model with concave payoff functions

In any Nash equilibrium of a plurality rule voting game in which the set of alternatives is a closed interval of real numbers and the individuals' payoff functions are strictly concave, the number of winning alternatives is at most two. To see why, suppose that the number of winning alternatives in a Nash equilibrium is three or more. Denote the average of the winning alternatives by $\bar{a}$. Either at least two alternatives are at least $\bar{a}$, or at least two alternatives are at most $\bar{a}$. Suppose the latter, as in Figure 4.1. Given the strict concavity of the individuals' payoff functions, no individual is indifferent among all the winning alternatives, so in every equilibrium every individual votes for one of the winning alternatives by Lemma 4.1*a* and prefers the alternative for which she votes to every other winning alternative by Lemma 4.1*b*. I claim that an individual, say i, who votes for the smallest winner, a_1, can increase her payoff by switching her vote to the next smallest winner, a_2, which causes that alternative to become the unique winner. The reason is that given $u_i(a_1) > u_i(a_2)$, the strict concavity of u_i, and the fact that $a_2 \le \bar{a}$, we have $u_i(a_2) \ge u_i(\bar{a})$, and given the strict concavity of u_i, $u_i(\bar{a})$ is greater than the average of i's payoffs to the winners, which is her payoff for the action profile. Part *a* of the next result states this conclusion and the remaining parts state other properties of the Nash equilibria.

Proposition 4.3: Nash equilibria of plurality rule voting game in spatial setting with strictly concave payoff functions

Let $G = \langle N, X, (u_i)_{i \in N} \rangle$ be a plurality rule voting game in which $X \subseteq \mathbb{R}$ is a (nonempty) closed interval and u_i is strictly concave for each $i \in N$.

a. In any Nash equilibrium of G the number of winning alternatives is at most two.

b. If N contains at least three individuals, for every alternative $a \in X$ the game G has a Nash equilibrium in which a is the sole winner. If N contains at least four individuals, for every alternative $a \in X$ other than the boundary points of X, G has a Nash equilibrium in which a is the sole winner and no individual's action is weakly dominated.

c. In a Nash equilibrium of G with two winning alternatives, each individual who is not indifferent between these alternatives votes for the one she prefers.

Proof

a. This result is proved in the text.

b. For any $a \in X$, the action profile in which every individual votes for a is a Nash equilibrium because no change in any individual's action has any effect on the outcome. If a is not a boundary point of X, it is not the lowest ranked alternative for any individual, and hence by Proposition 4.1*b* if there are at least four individuals then voting for a is not weakly dominated for any individual.

c. This result follows from parts a and b of Lemma 4.1.

Exercise 4.6: Variant of plurality rule voting game in spatial setting

Consider a variant of a plurality rule voting game satisfying the conditions in Proposition 4.3 in which each individual's preferences are lexicographic: she is primarily concerned with the set of winning alternatives, but among actions that yield the same set of winning alternatives (given the other individuals' actions) she prefers to abstain. What is the set of Nash equilibria of this game?

4.3 Strategic and sincere voting: divided majority

4.3.1 Basic model

When there are two alternatives, an individual's voting for her favorite alternative weakly dominates her voting for the other alternative (Proposition 3.1). In the argot of the field, her voting *sincerely* is a weakly dominant action. When there are three or more alternatives, voting sincerely is no longer a weakly dominant action: if, among the other individuals' votes, a and b are tied and every

	Type A n_A individuals	Type B n_B individuals	Type C n_C individuals
a	1	v	0
b	v	1	0
c	0	0	1
	majority		

Figure 4.2 The payoffs in a divided majority, with $\max\{n_A, n_B\} < n_C < n_A + n_B$ and $v \in (0,1)$.

other alternative gets at least two fewer votes, an individual whose favorite alternative is neither a nor b and who is not indifferent between these alternatives is better off voting for whichever of a and b she prefers, making that alternative the winner, than for her favorite alternative, which results in a tie between a and b. Such an individual may be perfectly sincere in the everyday sense of the word ("proceeding from genuine feelings"), but her action is called *strategic*. In an equilibrium when there are three or more individuals, are the individuals' votes sincere or strategic?

In this section I consider this question for an environment known as divided majority. There are three alternatives, say a, b, and c. A majority of individuals rank c last; among these individuals, n_A prefer a to b (type A) and n_B prefer b to a (type B). The remaining minority of individuals (type C, who number n_C) prefer c to both a and b, between which they are indifferent. Although types A and B together constitute a majority, each of these types separately is less populous than type C. That is, $\max\{n_A, n_B\} < n_C < n_A + n_B$. The payoffs are given in Figure 4.2.

The associated collective choice problem has a Condorcet winner: a if $n_A > n_B$, b if $n_A < n_B$, and both a and b if $n_A = n_B$. But if, in the associated plurality rule voting game, every individual votes sincerely, then c wins.

Voting for c and abstaining are weakly dominated for each individual of type A or B (by Proposition 4.1a) and voting for a or b and abstaining are weakly dominated for each individual of type C. So in any Nash equilibrium in which no individual uses a weakly dominated action, all individuals of type A or B vote for a or b and all individuals of type C vote for c. Such equilibria are of two types. In one type, the numbers of votes cast for a and for b are both at most $n_C - 2$, resulting in a win for c. Two such equilibria are illustrated in Figures 4.3a and 4.3b; in the first case each individual votes sincerely. In the other type of equilibrium, for either $x = a$ or $x = b$, $n_C + 1$ or more votes are cast for x, resulting in a win for x, as in Figures 4.3c and 4.3d.

If, for a given action profile, some change in an individual's vote would affect

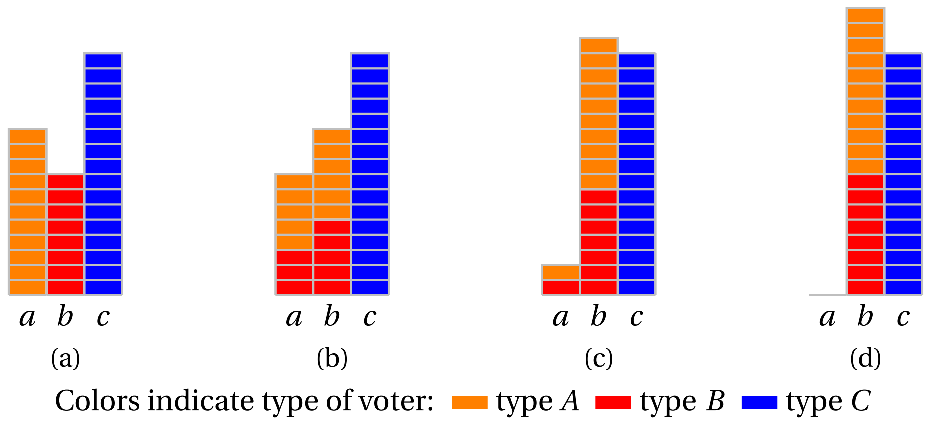


Figure 4.3 Some Nash equilibria in weakly undominated actions for a plurality rule voting game modeling a divided majority (Figure 4.2) in which $n_A > n_B$. Each block in each column represents an individual's vote for one of the alternatives. The colors of the blocks indicate the type of individual casting the vote: orange for type a, red for type b, and blue for type c. In panel a every individual votes sincerely, whereas in the other panels some individuals of types a and/or b vote strategically.

the outcome, we say that her vote is *pivotal* at that action profile. At the equilibrium shown in Figure 4.3c, where b gets one more vote than c, the vote of every individual of type A or B is pivotal: if an individual voting for a switches her vote to c or an individual voting for b switches her vote to a or switches to abstention then the outcome changes to a tie between b and c. (Also, if an individual voting for b switches her vote to c then the outcome changes to a win for c.) At none of the other equilibria shown in the figure is any individual's vote pivotal.

4.3.2 Model with uncertainty

In a Nash equilibrium of a strategic game, of which a plurality rule voting game is an example, each player's action is best for her given the other players' actions. The idea is that each player holds deterministic beliefs about the other players' actions, which, in equilibrium, are correct. For mass elections, the assumption that each individual has a deterministic belief about exactly how many votes each alternative will attract seems unreasonable. A more appealing formulation allows each individual to be uncertain of the other individuals' votes. In one such formulation, each individual knows her own type but has only probabilistic beliefs about the other individuals' types. She knows, in equilibrium, how each type of every other individual will cast her vote, but is uncertain about the configuration of votes that will be cast because she is uncertain about the distribution of types among the other individuals.

Even if an individual's uncertainty is minor, it may significantly affect her strategic calculations. Suppose, for example, that an individual of type B believes that the pattern of votes among the other individuals will certainly be the one shown in Figure 4.3a. Then her voting for a, b, or c, or abstaining, are all optimal; in each case, c wins. Now suppose instead that she believes that even though the pattern of votes is likely to be the one shown in Figure 4.3a, there is a small chance that enough of her fellow type B individuals will vote for a to make a and c tie, but no chance that enough type A individuals will vote for b to make b and c tie, and no chance of any other configuration of the other individuals' votes that will make her vote pivotal. Then her only optimal action is a vote for a, which results in a small chance that a wins rather than c.

The point is that only configurations of the other individuals' votes that make her vote pivotal have any impact on her decision; she should ignore configurations for which her vote is not pivotal. If for every configuration of the other individuals' votes that she believes will occur with positive probability and for which the outcome depends on whether she votes for x or y, she prefers the outcome when she votes for x, then voting for x is optimal for her, even if those configurations of the other individuals' votes are unlikely.

A model that captures the individuals' uncertainty about each other's types is a Bayesian game. A strategy of each individual in such a game assigns an action (a vote for a, b, or c, or abstention) to each of her possible types. A strategy profile is a Nash equilibrium if the action of each type of every individual i is optimal given the distribution of the other individuals' actions implied by their strategies and i's belief about the distribution of their types. One interpretation of such an equilibrium is that every individual knows every other individual's strategy, possibly from her long experience of voting in similar elections, and, given her probabilistic belief about the distribution of the other individuals' types in the population, chooses her vote optimally given her own type.

In the Nash equilibria of this Bayesian game, do individuals vote sincerely or strategically? The answer depends on the nature of the individuals' beliefs about each other's types. I start by assuming that for some numbers p_A, p_B, and p_C, each individual believes that the type of every other individual is t with probability p_t, for $t = A$, B, and C, independently of the type of every other individual. This assumption on the structure of the individuals' beliefs means that as the size of the population increases, every individual becomes increasingly certain of the proportions of the types in the population, and for this reason is not particularly plausible. However, it yields a result that is both striking and illuminating.

For simplicity, I assume that the number of individuals is odd and abstention is not an option for any individual.

Definition 4.2: Divided majority with independent beliefs

A *divided majority with independent beliefs* $\langle N, \{a, b, c\}, \{A, B, C\}, v, (p_A, p_B, p_C)\rangle$, where N is a finite set (of individuals) with a number of members that is odd and at least five, a, b, and c are alternatives, A, B, and C are preference types, $v \in (0, 1)$, and p_A, p_B, and p_C are positive numbers with $p_A + p_B + p_C = 1$ and $\max\{p_A, p_B\} < p_C < p_A + p_B$, is the following Bayesian game.

Players

The set N.

States

The set of states is the set of profiles $(t_j)_{j \in N}$ of preference types, with $t_j \in \{A, B, C\}$ for every player $j \in N$.

Actions

The set of actions of each player is $\{\textit{vote for } a, \textit{vote for } b, \textit{vote for } c\}$.

Signals

The signal function τ_i of each player i is given by $\tau_i((t_j)_{j \in N}) = t_i$ (every player knows only her own preference type).

Prior beliefs

Every player believes that the preference type of every individual is A with probability p_A, B with probability p_B, and C with probability p_C, independently of every other individual's preference type.

Payoffs

The Bernoulli function for each player's preferences regarding lotteries over the set of pairs of action profiles and states is defined as follows.

For each preference type $t \in \{A, B, C\}$ and each alternative $z \in \{a, b, c\}$, let $u_t(z)$ be the number given in the cell in column t and row z of Figure 4.2, and for any action profile x, denote by $W(x) \subseteq \{a, b, c\}$ the set of alternatives that receive the most votes. For any action profile x and any profile of preference types, the Bernoulli payoff for x of each player with each preference type $t \in \{A, B, C\}$ is $\sum_{w \in W(x)} u_t(w)/|W(x)|$.

4.3.3 Nash equilibria: sincere or strategic?

In a divided majority with independent beliefs, voting for c is weakly dominated for an individual of type A or type B and voting for a or b is weakly dominated for an individual of type C. So the main question is whether individuals of types

A and B vote for a or for b. Regardless of the values of p_A, p_B, and p_C, the game has an equilibrium in which all individuals of types A and B vote for a, and also one in which they vote for b. The argument is straightforward.

Proposition 4.4: Nash equilibria of divided majority with independent beliefs

A divided majority with independent beliefs has a Nash equilibrium in which every individual votes for a if her preference type is A or B and votes for c if her preference type is C, and also one in which every individual votes for b if her preference type is A or B and votes for c if her preference type is C.

Proof

For the strategy profile in which every individual of type A or B votes for a and every individual of type C votes for c, the set of winners is $\{a\}$ or $\{c\}$, depending on the realized distribution of preference types. (A tie is not possible because the number of individuals is odd and abstention is not allowed.) The action of an individual of type A or B is pivotal at this strategy profile if and only if the realized number of individuals of types A and B exceeds the realized number of individuals of type C by 1, an event with positive probability. For such a realization of types, the outcome of the strategy profile is a win for a. If an individual of type A or B changes her vote to b, the outcome changes to a tie between a and c (given that the number of individuals is at least five), and if she changes it to c, the outcome changes to a win for c. She prefers a win for a to each of these outcomes, so her voting for a is optimal.

Similar arguments apply to an individual of type C and to the strategy profile in which every individual of type A or B votes for b.

Is the strategy profile in which every individual votes sincerely also a Nash equilibrium of the game? We know that for some configurations of the other individuals' actions an individual is better off voting for an alternative different from her favorite, so that whether a vote for her favorite alternative is optimal depends on the probabilities of such configurations and her payoffs for the alternatives.

Denote the number of individuals by $n+1$. For any given individual i, the set of configurations of the other n individuals' preference types is illustrated in Figure 4.4. Each small disk represents one possible configuration; the three corners of the triangle represent configurations in which all n individuals have the same type, and the (brown) disk in the center represents the configuration in

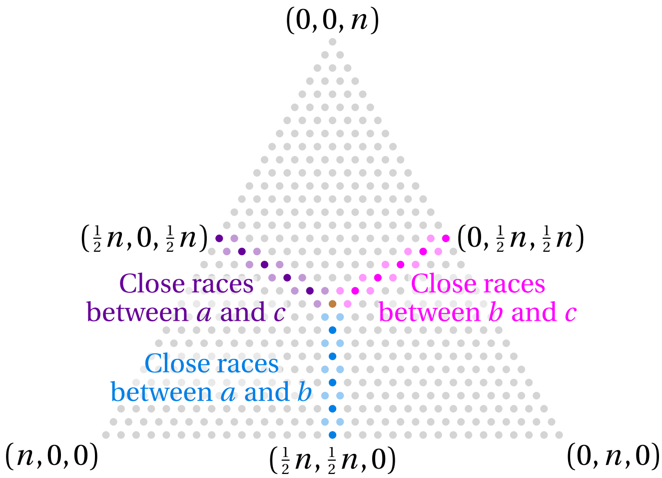


Figure 4.4 Possible configurations of the types of n individuals in a divided majority (Figure 4.2). Each disk represents one configuration (n_A, n_B, n_C). The corners of the triangle are the configurations in which all n individuals have the same type and the central disk (brown) is the configuration in which there are $\frac{1}{3}n$ individuals of each type. (The diagram assumes that n is divisible by 2 and by 3.) The colored disks represent close races when every type of every individual votes sincerely.

which the same number of individuals have each type. (The diagram assumes that n is divisible by both 2 and 3.)

Under the assumption that the other individuals vote sincerely, the disks in the diagram represent also the possible configurations of the other individuals' votes for the three alternatives. The alternative for which i optimally votes depends on the probabilities of vote configurations for the other individuals for which her vote is pivotal. These configurations are the ones in which there is a tie or near-tie (the margin of victory of the outright winner is one vote) for first place among the other individuals' votes. These close races are indicated in color in Figure 4.4.

Consider type A of individual i. If the close race is between a and b (blue disks) or between a and c (purple disks), then she optimally votes (sincerely) for a. But if it is between b and c (magenta disks) then, given that she prefers b to c, she optimally votes for b. To determine her optimal action we need to compare the probabilities of these events. The probability density over the configurations of types that the model generates for $n = 60$, $p_A = 0.25$, $p_B = 0.3$, and $p_C = 0.45$ is shown in Figure 4.5. In this case, if all of the n individuals other than i vote sincerely, close races between b and c are more likely than ones between a and b or between a and c, so there is a number v^* such that if $v \leq v^*$ then type A of individual i optimally votes sincerely, for a, and if $v \geq v^*$ then she optimally votes strategically, for b. (If $v = v^*$, both votes are optimal.) Whenever $p_A < p_B < p_C$,

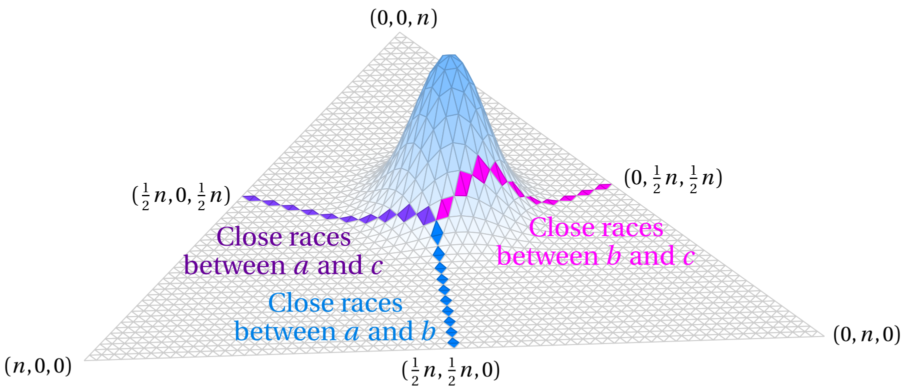
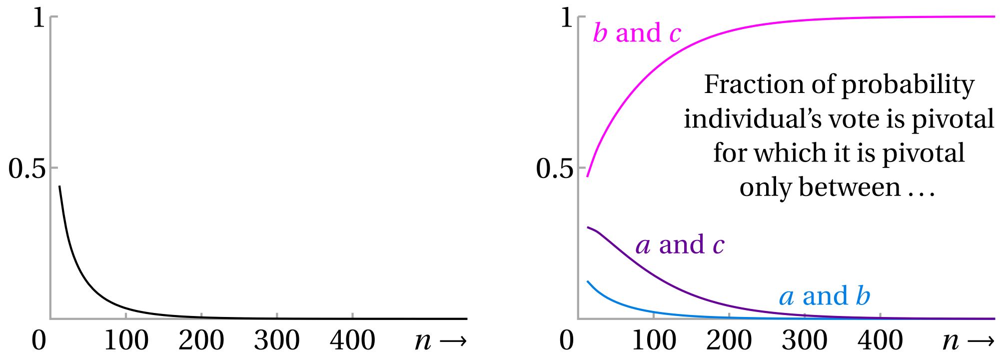


Figure 4.5 The probability density over the configurations of types for 60 individuals when each individual's type is a with probability 0.25, b with probability 0.3, and c with probability 0.45, independently of the types of the other individuals. (The distribution is discrete; its density is smoothed in the figure.) The regions indicated in dark blue, purple, and magenta are the closes races between two of the alternatives when every type of every individual votes sincerely.

the conclusion is the same.

Now suppose that the number of individuals increases. As it does so, the probability density over the configurations of types of the n individuals other than i becomes increasingly concentrated around (np_A, np_B, np_C), and the probability of a close race decreases (even if $p_A = p_B = p_C = \frac{1}{3}$). For $p_A = 0.25$, $p_B = 0.3$, and $p_C = 0.45$, Figure 4.6a shows this probability as a function of n; for $n > 300$ it is less than 0.001, and for $n > 430$ it is less than 0.0001. If $p_A < p_B < p_C$, then as the number of individuals increases, the proportion of the probability of a close race attributable to a close race between b and c increases, approaching one. Figure 4.6b shows the proportions attributable to close races between each of the three pairs of alternatives, as a function of n, when $p_A = 0.25$, $p_B = 0.3$, and $p_C = 0.45$. (A tie between all three alternatives is also possible; the probability of this event rapidly decreases to zero.) In this case, for $n > 320$ the proportion attributable to a close race between b and c exceeds 0.99. The proof of the general result that if $p_A < p_B < p_C$ then the probabilities of a tie or near-tie for the most populous type between A and B and between A and C become negligible compared with the probability of a tie or near-tie between B and C is routine but intricate; Lemma 2 of Palfrey (1989) is this result for a different tie-breaking rule. As a consequence of this result, for any given value of $v > 0$ there is a number N such that if $n > N$ then in a population of n individuals the strategy profile in which each type of each individual votes sincerely is not a Nash equilibrium, because almost every case in which the vote of an individual of type A is pivotal is a close race between b and c, so that such an individual optimally votes for B



(a) The probability that an individual's vote is pivotal when each of n other individuals votes for a with probability 0.25, for b with probability 0.3, and for c with probability 0.45, independently of the other individuals.

(b) The fraction of the probability that an individual's vote is pivotal under the conditions of panel (a) for which it is pivotal only between b and c (magenta curve), a and c (purple curve), and a and b (blue curve).

Figure 4.6

if each type of every other individual votes sincerely.

An implication of this result is that if $p_A < p_B < p_C$ then for any given value of $v > 0$, if the number of individuals is sufficiently large then in any Nash equilibrium in which all individuals of the same type vote for the same alternative and no type uses a weakly dominated action, all individuals of types A and B vote for the same alternative, either a or b, and all individuals of type C vote for c. In particular, in any such equilibrium only two of the three alternatives receive votes.

4.3.4 Discussion

These results concern the limit as the population size increases without bound. They do not mean that for any number that you or I might classify as large, the game has no equilibrium in which every type of every individual votes sincerely. The sufficiently large number depends on the parameters, and could be 100, 1 million, 1 billion, or any other number. Note also that the analysis implies that for any values of p_A, p_B, and p_C with $p_A < p_B < p_C$ and any given (finite) population size, there is a positive number v^* such that for $v < v^*$ the game has a Nash equilibrium in which every type votes sincerely. Finally, note that the analysis assumes that each individual chooses the alternative for which to vote by comparing the probabilities that votes for each of the alternatives change the outcome, even though these probabilities are minuscule in a large population. If her motivation for voting is, instead, expressive (see Section 6.2), then she may vote sincerely regardless of the population size.

The results depend on the model of the individuals' beliefs. The assumption that each individual believes that every other individual's type is t with probability p_t, independently of the other individuals' types, has two significant implications. First, when the number of individuals is large, every individual is almost certain of the proportions of the types in the population. Second, every individual's belief is the same. These implications seem implausible. Even in a large population—particularly in a large population?—each individual seems likely to be uncertain of the distribution of the other individuals' characteristics; different individuals get information from different sources, so their beliefs are likely to differ.

An alternative model assumes that the probabilities p_A, p_B, and p_C are themselves uncertain, so that the uncertainty about the proportion of individuals of each type in the population does not vanish as the population size increases without bound. In this case, every individual of each type may optimally vote sincerely if all the other individuals do so, regardless of the population size. If, for example, an individual of type A believes that a tie between b and c is more likely than a tie between a and c, but thinks that the difference between the likelihoods is not large, then the expected loss from voting for b rather than a in the event of a tie between a and c may outweigh the expected gain from doing so in the event of a tie between b and c. For specific models of the uncertainty regarding p_A, p_B, and p_C, we may be able to say more about the circumstances under which the strategy profile in which every type of every individual votes sincerely is a Nash equilibrium.

If the individuals' beliefs differ, then their strategic calculations differ. Suppose that each individual of type A believes that in expectation, individuals of type A outnumber those of type B, and the reverse is true for individuals of type B. Then if all the other individuals vote sincerely, an individual of type A may conclude that voting for a is optimal and an individual of type B may conclude that voting for b is optimal, so that again a strategy profile in which all individuals vote sincerely is a Nash equilibrium. If, instead, each individual of type A or type B believes that their type is outnumbered, then an equilibrium may exist in which every individual of type A votes for b and every individual of type B votes for a.

One way in which an individual may gather information about the voting intentions of the other individuals is through her interactions with those individuals. The next exercise asks you to analyze a model in which each individual bases her vote on the information obtained from a random sample of two other individuals.

Exercise 4.7: Sampling equilibrium in a divided majority

Consider the divided majority in Figure 4.2. Denote the fraction of the population consisting of individuals of type T by q_T for $T = A, B, C$. Each individual observes the voting intentions of two randomly-selected individuals. An individual of type A whose sample consists of one individual who intends to vote for B and one who intends to vote for C concludes that she should vote for B; for every other sample she votes for A. Symmetrically, an individual of type B whose sample consists of one individual who intends to vote for A and one who intends to vote for C concludes that she should vote for A, and for every other sample votes for B. An individual of type C votes for C regardless of her sample. For each type T, let $p_T(a)$, $p_T(b)$, and $p_T(c)$ be the fractions of the individuals of that type who vote for each alternative. In an equilibrium, $p_A(c) = p_B(c) = 0$, $p_C(c) = 1$, $p_A(b)$ is equal to the probability that the sample of an individual of type A consists of one individual who intends to vote for b and one who intends to vote for c, and $p_B(a)$ is equal to the probability that the sample of an individual of type B consists of one individual who intends to vote for a and one who intends to vote for c. Assume that the number of individuals is large enough that you can take q_T to be the probability that a given member of the sample of an individual i of type T is an individual of type T other than i. Find the equilibria. Why is there no equilibrium in which all individuals of types A and B vote for the same alternative?

Notes

Lemma 4.1 is based on Lemmas 1 and 2 of Feddersen et al. (1990). One source of Proposition 4.1b, which Duggan and Sekiya (2009, 879) say is well known, is Dhillon and Lockwood (2004, Lemma 1). Proposition 4.2 is due to Duggan and Sekiya (2009, Theorem 1). The proof of Proposition 4.3a is based on the proof of Lemma A.3 in Feddersen et al. (1990). The main part of Section 4.3 is based on Palfrey (1989). Myatt (2007) and Bouton et al. (2017) study models in which the uncertainty regarding the proportions of the various preference types does not vanish as the population size grows without bound.

Exercise 4.3 is taken from Duggan and Sekiya (2009). Exercise 4.4 is based on Dhillon and Lockwood (2004), who study the iterated elimination of weakly dominated actions in plurality rule voting games more generally. Exercise 4.5 is based on De Sinopoli and Iannantuoni (2007). Voting under proportional rule is explored further by Indriðason (2011) and Cho (2014). The notion of equilibrium in Exercise 4.7 is taken from Osborne and Rubinstein (2003).

Solutions to exercises

Exercise 4.1

Let i be an individual, let a be one of i's favorite alternatives, let z be an alternative she likes least, and let u_i be her payoff function over alternatives.

If i is indifferent between all alternatives, for any alternative b her voting for b is not weakly dominated by her voting for any other alternative.

Now suppose that i is not indifferent between all alternatives, so that $u_i(a) > u_i(z)$. I argue that i's voting for b is weakly dominated if and only if her payoff to $\{b\}$ is at most her payoff for the set of winners $\{a, b, z\}$, in which case $u_i(a) > u_i(b)$.

First, i's voting for b is weakly dominated if and only if it is weakly dominated by her voting for a. I now consider the conditions under which her voting for b is weakly dominated by her voting for a.

- If the other two individuals vote for the same alternative, i's vote does not affect the set of winners.
- If the other two individuals vote for different alternatives, say x and y, neither of which is a or b, the set of winners when i votes for a is $\{a, x, y\}$ and the set of winners when she votes for b is $\{b, x, y\}$, so her payoff is higher when she votes for a.
- If one of the other individuals votes for a and the other votes for an alternative, say x, other than a or b, then the set of winners is $\{a\}$ if i votes for a and $\{a, b, x\}$ if she votes for b, so her payoff is higher if she votes for a.
- If one of the other individuals votes for b and the other votes for an alternative, say x, other than a or b, then the set of winners is $\{a, b, x\}$ if i votes for a and $\{b\}$ if she votes for b, so her payoff is at least as high if she votes for a if and only if her payoff for $\{a, b, x\}$ is at least as high as her payoff for $\{b\}$.
- If one of the other individuals votes for a and the other votes for b then the set of winners is $\{a\}$ if i votes for a and $\{b\}$ if she votes for b, so her payoff is higher if she votes for a.

In the second, third, and fifth cases, i's payoff is higher when she votes for a than it is when she votes for b, and in the first case it is the same, so her voting for b is weakly dominated by her voting for a if and only if for every alternative x other than a and b her payoff for $\{a,b,x\}$ is at least as high as her payoff for $\{b\}$, which is the case if and only if her payoff for $\{a,b,z\}$ is at least as high as her payoff for $\{b\}$.

Exercise 4.2

For each individual $i \in N$, denote by T_i the set of i's favorite alternatives. For no alternative a in T_i is voting for a weakly dominated for individual i.

Thus if some alternative a is a member of T_1, T_2, and T_3, then the action profile in which each individual votes for a is a Nash equilibrium in which no individual's action is weakly dominated.

If for some alternatives a and b, a is a member of two of the sets T_1, T_2, and T_3, say T_i and T_j, and b is a member of the remaining set, say T_k, then the action profile in which i and j vote for a and k votes for b is a Nash equilibrium in which no individual's action is weakly dominated.

The remaining possibility is that the sets T_1, T_2, and T_3 have no alternative in common. In this case, select one alternative from each set, say a from T_1, b from T_2, and c from T_3.

- If the action profile in which 1 votes for a, 2 votes for b, and 3 votes for c is a Nash equilibrium, we are done.
- If not, one of the individuals can increase her payoff by changing her vote. Without loss of generality, assume that individual 1 can do so.
    - If individual 1 changes her vote to another alternative in T_1, her payoff does not change.
    - If individual 1 changes her vote to an alternative, say z, outside T_1 and $z \notin \{b,c\}$, then the set of winners changes from $\{a,b,c\}$ to $\{z,b,c\}$, so she is worse off.
    - Thus individual 1 must increase her payoff by changing her vote to either b or c, so that the set of winners becomes $\{b\}$ or $\{c\}$.
        - If b and c yield her the same payoff, then her payoff decreases.
        - Thus b and c yield her different payoffs. Let b be the one with the higher payoff. Given that individual 1 can increase her payoff by changing her vote, her payoff from $\{b\}$ is greater than her payoff from $\{a,b,c\}$. Thus the action profile in which individuals 1 and 2 vote for b and individual 3 votes for c is a Nash equilibrium (because the best deviation for individual 1 is to vote for a, which generates the set of winners $\{a,b,c\}$) and by the result

of Exercise 4.1 individual 1's action of voting for b is not weakly dominated.

Exercise 4.3

For the action profile x^1, in which every individual votes for her favorite alternative, the set of winners is $\{a, c\}$. This action profile is not a Nash equilibrium because individual 2, by changing her vote to b, can change the set of winners to $\{a, b, c\}$, thereby increasing her payoff from 0 to $\frac{1}{3} \cdot 6 = 2$. Individual 6 can also increase her payoff by changing her vote to a, which changes the set of winners to $\{a\}$ and hence increases her payoff from $\frac{1}{2} \cdot 6 = 3$ to 6.

If we select individual 2 then the list of votes in x^2 is (c, b, c, a, a, b) and the set of winners is $\{a, b, c\}$. This action profile is not a Nash equilibrium because individual 3, by changing her vote to b, can change the set of winners to $\{b\}$, thereby increasing her payoff from $\frac{1}{3} \cdot 9 + \frac{1}{3} \cdot 6 = 5$ to 6. Individual 5 can also increase her payoff by deviating to vote for c, and individual 6 can do so by deviating to vote for a.

If we select individual 3 then the list of votes in x^3 is (c, b, b, a, a, b) and the set of winners is $\{b\}$. This profile is a Nash equilibrium, so the procedure ends.

(If at the second step we choose individual 6 and set the list of votes in x^2 to be (c, d, c, a, a, a) then the set of winners is $\{a\}$. This profile is a Nash equilibrium, so the procedure ends. If at the third step we select individual 5, we reach the Nash equilibrium with votes (c, b, c, a, c, b), and if we select individual 6, we reach the Nash equilibrium with votes (c, b, c, a, a, a).)

Exercise 4.4

Denote the alternatives a, b, and c, and suppose that more than two-thirds of the individuals rank c below a and b. By Proposition 4.1, each individual's only weakly dominated action is a vote for her least preferred alternative. After eliminating this action for each individual, the set of winners generated by every remaining action profile is $\{a\}$, $\{b\}$, or $\{a, b\}$, because fewer than one-third of the individuals vote for c (given that abstention is not allowed). Thus each individual's voting for whichever of a and b she prefers weakly dominates her voting for the other remaining alternative. We are left with the action profile in which every citizen votes for her favorite alternative in $\{a, b\}$.

Exercise 4.5

a. Let i be an individual and let x be an action profile in which i votes for a_l. Suppose that $a_l > a_1$. If i switches her vote from a_l to a_1 then the outcome decreases from $O(x) = \sum_{j=1}^{k} v_j(x)a_j/n$ to

$$\sum_{j=1}^{k}(v_j(x)a_j - a_l + a_1)/n \geq \sum_{j=1}^{k} v_j(x)a_j/n - (a_k - a_1)/n = O(x) - (a_k - a_1)/n.$$

Thus if i's favorite alternative is at most $O(x) - (a_k - a_1)/n$ then she is better off voting for a_1 than for a_l and hence x is not a Nash equilibrium. Similarly if $a_l < a_k$ and i's favorite alternative is at least $O(x)+(a_k - a_1)/n$, she is better off voting for a_k than for a_l, so that x is not a Nash equilibrium.

b. The action profile in which every individual whose favorite alternative is less than z^* votes for a_1 and every individual whose favorite alternative is greater than z^* votes for a_k is a Nash equilibrium. The reason is that if an individual whose favorite alternative is less than z^* deviates to vote for an alternative other than a_1 then the outcome increases, which makes her worse off, and if an individual whose favorite alternative is greater than z^* deviates to vote for an alternative other than a_k then the outcome decreases, which makes her worse off.

Exercise 4.6

Every equilibrium of the modified game is an equilibrium of the original game, so by Proposition 4.3 in every equilibrium of the modified game the number of winning alternatives is at most two.

The modified game has no equilibrium with a single winning alternative unless every individual's favorite alternative is the same. To see why, consider an action profile that generates a single winning alternative, say a. Any individual who votes for an alternative other than a can switch to abstention without affecting the outcome, and if all votes are cast for a and more than one individual votes then any individual can switch to abstention without affecting the outcome. Thus in any equilibrium of the modified game one individual votes for a. But if some individual's favorite alternative b differs from a, an action profile with one vote for a is not an equilibrium because that individual can switch from abstention to voting for b and induce a tie between a and b, which she prefers to a.

Consider a two-alternative equilibrium of the original game in which at least one individual who votes is indifferent between the alternatives. If such an individual deviates to abstention then her payoff remains the same, so that such an equilibrium is not an equilibrium of the modified game.

Now consider a two-alternative equilibrium of the original game in which no individual who votes is indifferent between the alternatives. If an individual who votes deviates to voting for another alternative or to abstention, the resulting outcome is worse for her. Thus such an equilibrium is an equilibrium of the modified game.

Exercise 4.7

The equilibrium conditions are

$$p_A(b) = 2(q_A p_A(b) + q_B p_B(b))(1 - q_A - q_B)$$
$$p_B(a) = 2(q_A p_A(a) + q_B p_B(a))(1 - q_A - q_B).$$

The right-hand side of the first equation is the probability that an individual's sample consists of one individual who intends to vote for b and one who intends to vote for c, and the right-hand side of the second equation is the analogous expression for a and c. These equations have a unique solution,

$$p_A(b) = 2(1 - q_A)q_B - 2q_B^2$$
$$p_B(a) = 2(1 - q_B)q_A - 2q_A^2.$$

For the example in which $q_A = 0.25$, $q_B = 0.35$, and $q_C = 0.4$, $p_A(b) = 0.28$ and $p_B(a) = 0.2$.

The proportion of the population that votes for a in the equilibrium is

$$q_A p_A(a) + q_B p_B(a) = q_A(1 - 2(1 - q_A)q_B + 2q_B^2) + q_B(2(1 - q_B)q_A - 2q_A^2) = q_A$$

and the proportion that votes for b is q_B. Thus even though some individuals vote strategically, the total proportions who vote for each alternative are equal to the proportions of the types in the population.

There is no equilibrium in which all individuals of types A and B vote for the same alternative, say a, because for such a voting pattern an individual of type B who gets a sample consisting of two individuals who intend to vote for a is assumed to vote for b.

5 Sequential pairwise voting

5.1 Binary agendas 157
5.2 Successive agendas 168
5.3 Amendment agendas 171
5.4 Agenda formation 179
5.5 Optimistically sincere strategies, single-peaked preferences, convex agendas 180

One option for a group of individuals that has to choose an alternative from a set of many alternatives is to conduct a sequence of two-way votes. Such a procedure is called a binary agenda. An example when the alternatives are a, b, and c is illustrated in Figure 5.1. In this example, the individuals first vote on a motion to drop a from consideration. If that motion passes, they vote between b and c, and if it fails, they vote whether to choose c, with the failure of that motion leading to a vote between a and b. How does the outcome of a binary agenda depend on the sequence of choices and the individuals' preferences? Does the outcome have desirable properties?

Synopsis

We model a binary agenda as an extensive game, and apply to it the solution concept of subgame perfect equilibrium with the restriction that no player's vote at any point in the game is weakly dominated. The resulting strategy profiles are referred to as the outcomes of sophisticated voting. (The individuals' voting behavior in a subgame perfect equilibrium is sophisticated to the extent that when deciding how to cast her vote in any given ballot, an individual takes into account all individuals' future voting behavior.)

Say that a finite collective choice problem with an odd number of individuals, each of whose preference relations is strict, is odd-strict. As you might suspect, if an odd-strict collective choice problem has a strict Condorcet winner then that alternative is the unique outcome of sophisticated voting for any binary agenda (Proposition 5.1).

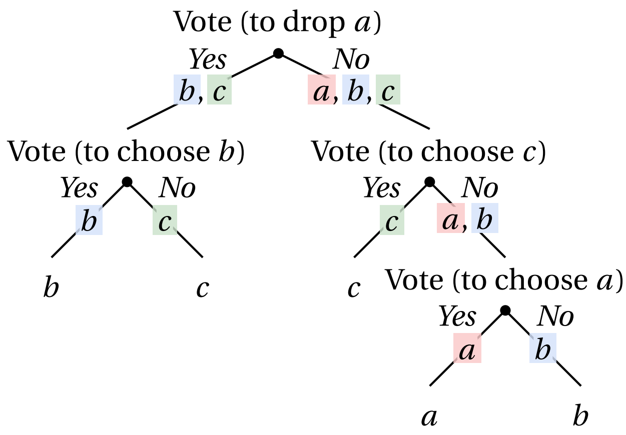


Figure 5.1 An example of a procedure for a binary agenda.

The outcomes of sophisticated voting for an odd-strict collective choice problem with no strict Condorcet winner are related to a set of alternatives called the top cycle set. Recall that a strict Condorcet winner beats every other alternative in two-way comparisons. Say that an alternative x *indirectly* beats another alternative y if for some alternatives $z_1, \ldots, z_k$, alternative x beats z_1, z_1 beats z_2, $\ldots$, z_{k-1} beats z_k, and z_k beats y. The top cycle set consists of the alternatives that beat or indirectly beat every other alternative. Proposition 5.3 shows that for an odd-strict collective choice problem, every outcome of sophisticated voting in any binary agenda is in the top cycle set, and for any alternative in the top cycle set there is a binary agenda for which that alternative is the outcome of sophisticated voting. The members of the top cycle set are not all reasonable outcomes of a collective choice problem. For example, a member of the top cycle set may be dominated: another alternative may be preferred by every individual (Example 5.1).

Sections 5.2 and 5.3 consider binary agendas that model the procedures used in some legislatures. Section 5.2 considers successive agendas, in which the alternatives are considered in some order. A vote is taken on whether to choose the first alternative or to drop it from consideration; if it is dropped, a vote is taken on whether to choose the second alternative or to drop it from consideration; and so forth. Proposition 5.5 shows that for an odd-strict collective choice problem, for every alternative in the top cycle set there is a successive agenda for which the alternative is the outcome of sophisticated voting.

By contrast, for the amendment agendas considered in Section 5.3, the set of outcome of sophisticated voting is a subset of the top cycle set. In an amendment agenda, as in a successive agenda, the alternatives are considered in some order. A vote is taken on whether to eliminate the first alternative or the second alternative from consideration; then a vote is taken on whether to eliminate the alternative not eliminated in the first round or the third alternative; and so forth. For amendment agendas, a subset of the top cycle set called the Banks set plays the role that the top cycle set plays for successive agendas. Unlike the top cycle set, the Banks set contains no alternative that is dominated (Exercise 5.5). Proposition 5.8 shows that for an odd-strict collective choice problem, the outcome of sophisticated voting in any amendment agenda is in the Banks set, and for every alternative in the Banks set there is an amendment agenda for which that alternative is the outcome of sophisticated voting.

Suppose that the agenda is chosen by someone with preferences over the alternatives. The results so far imply that if the agenda-setter is unrestricted in the type of agenda she can choose, or is restricted to successive agendas, then she can achieve, as an outcome of sophisticated voting, any alternative in the top cycle set, whereas if she is restricted to amendment agendas then she can achieve any alternative in the Banks set. Another possibility is that she is restricted to amendment agendas for which the first item—the status quo—is given, but is not restricted to include every alternative in the agenda. Proposition 5.9 shows that in this case she can achieve any alternative that either beats the first item or beats an alternative that beats the first item.

Section 5.5 returns to binary agendas for which the collective choice problem has a strict Condorcet winner. Say that an individual's strategy is optimistically sincere if, in every ballot, she votes *Yes* if the best outcome for her among those that could possibly occur following a majority vote of *Yes* is better than the best outcome for her that could possibly occur following a majority vote of *No*, and she votes *No* if the inequality is in the other direction. Such a strategy may appear to reflect naïve optimism, but Proposition 5.10 shows that if for some ordering of the alternatives the preferences profile is single-peaked and the outcomes possible following each vote are adjacent to each other in the ordering (the agenda is convex), then the strategy profile in which every individual's strategy is optimistically sincere is a subgame perfect equilibrium. The outcome of this equilibrium is the strict Condorcet winner of the collective choice problem. To implement an optimistically sincere strategy, an individual needs to know only the structure of the agenda and her own preferences, not the characteristics of the other individuals. The equilibrium thus achieves a desirable outcome even if the individuals have no information about each other.

5.1 Binary agendas

A binary agenda consists of a collective choice problem and a procedure for selecting an alternative that consists of a collection of sequences of two-way votes in which every alternative is the outcome of at least one possible sequence. Let $\langle N, X, \succsim \rangle$ be a collective choice problem. The structure of the choices in a binary agenda for selecting an alternative for this problem is modeled as a set H of sequences of ballot outcomes. In Figure 5.1, for example, H contains the empty sequence, (*Yes*), (*Yes*, *Yes*), (*Yes*, *No*), (*No*), and other sequences that start with *No*. The set H always contains the empty sequence, which represents the start of the procedure. Every element in every other member of H is either *Yes* or *No*; for every $h \in H$, either $(h, \textit{Yes}) \in H$ and $(h, \textit{No}) \in H$, as for $h = (\textit{Yes})$ in Figure 5.1, in which case h is nonterminal, or there is no value of x for which $(h, x) \in H$, as for $h = (\textit{Yes}, \textit{Yes})$ in Figure 5.1, in which case h is terminal. Denote the set of terminal members of H by Z. Let O be a function that associates a member of the set X of alternatives (an outcome) with every terminal member of H and suppose that every alternative is the outcome of some sequence of votes: for every $x \in X$ there exists $h \in Z$ for which $O(h) = x$.

The binary agenda generated by $\langle N, X, \succsim \rangle$, Z, and O is the extensive game with perfect information and simultaneous moves in which the set of players is N, the set of terminal histories is the set of sequences of vote profiles that generate sequences of majority decisions in Z, the player function assigns the set N of all players to every nonterminal history, the set of actions of each player after every nonterminal history is $\{\textit{Yes}, \textit{No}\}$, and each player i likes the terminal history t at least as much as the terminal history t' if and only if $O(h) \succsim_i O(h')$, where h is the sequence of ballot outcomes generated by t and h' is the sequence generated by t'.

In the example in Figure 5.1, the votes are given interpretations. For example, the first one is interpreted as a vote to drop a. These interpretations are not part of the formal description of the agenda. Everyone knows the structure of the agenda, and whenever a vote takes place, the options are to move down the branch on the left or to move down the branch on the right; these options may or may not have simple interpretations like dropping one of the alternatives from consideration or selecting one of the alternatives.

For simplicity I assume that the number of individuals is odd and that each individual's preferences are strict (no individual is indifferent between any two alternatives).

Definition 5.1: Odd-strict collective choice problem

An *odd-strict collective choice problem* is a finite collective choice problem in which the number of individuals is odd and every individual's preference relation is strict.

Definition 5.2: Binary agenda

Let $\langle N, X, \succsim \rangle$ be an odd-strict collective choice problem with $N = \{1, \ldots, n\}$. Let H be a set of sequences (of ballot outcomes) (*i*) that contains the empty sequence, (*ii*) in which every element is either *Yes* or *No*, and (*iii*) for which for every $h \in H$ either $(h, Yes) \in H$ and $(h, No) \in H$, in which case h is *nonterminal*, or there is no value of x for which $(h, x) \in H$, in which case h is *terminal*. Denote the terminal members of H by Z and let O be a function that assigns to each $h \in Z$ an alternative $O(h) \in X$. Assume that for every $x \in X$ there exists $h \in Z$ such that $O(h) = x$.

The *binary agenda* $\langle \langle N, X, \succsim \rangle, Z, O \rangle$ is the following extensive game with perfect information and simultaneous moves.

Players

The set of players is the set N (of individuals).

Terminal histories

For any profile $(v_1, \ldots, v_n)$ with $v_i \in \{Yes, No\}$ for all $i \in N$ (a *vote profile*), denote by $M(v_1, \ldots, v_n)$ the member of $\{Yes, No\}$ such that $\{i \in N : v_i = M(v_1, \ldots, v_n)\}$ is a majority of N.

Terminal histories are sequences of vote profiles. Specifically, a sequence $((v_1^1, \ldots, v_n^1), \ldots, (v_1^q, \ldots, v_n^q))$ of vote profiles, where $q \geq 1$ and $v_i^l \in \{Yes, No\}$ for $l = 1, \ldots, q$ and all $i \in N$, is a terminal history if and only if $(M(v_1^1, \ldots, v_n^1), \ldots, M(v_1^q, \ldots, v_n^q)) \in Z$.

Player function

The player function assigns the set N of all individuals to each terminal history.

Actions

Each player's set of actions after each nonterminal history is $\{Yes, No\}$.

Preferences

Each player i prefers the terminal history $((v_1^1, \ldots, v_n^1), \ldots, (v_1^q, \ldots, v_n^q))$ to the terminal history $((y_1^1, \ldots, y_n^1), \ldots, (y_1^q, \ldots, y_n^q))$ if and only if

$$O(M(v_1^1, \ldots, v_n^1), \ldots, M(v_1^q, \ldots, v_n^q)) \succ_i O(M(y_1^1, \ldots, y_n^1), \ldots, M(y_1^q, \ldots, y_n^q)).$$

Note that H is the set of possible sequences of ballot outcomes, not the set of histories in the game, and Z is the set of such sequences that are terminal. A history in the game is a sequence of vote profiles; a member of H is a sequence of majority decisions generated by one of these histories.

Equilibrium

Assume that whenever an individual votes, she is forward-looking: she votes for the option that, given her expectation of the outcomes of future votes (which, in equilibrium, is correct), leads to the better outcome for her. Specifically, I look for a subgame perfect equilibrium: a strategy profile for which no change in any vote of any individual leads (ultimately) to an outcome that is better for her, given the other individuals' strategies.

I further restrict each individual to weakly undominated strategies. If there are three or more individuals, then without this restriction, for every alternative $a \in X$ the game has a subgame perfect equilibrium in which the outcome is a. The reason is the same as the reason that for each alternative a in a two-alternative voting game the action profile in which every individual votes for a is a Nash equilibrium (Section 3.1.1). Take a sequence $h = (h^1, \ldots, h^q)$ of ballot results with $O(h) = a$ and suppose that for $j = 1, \ldots, q$ every individual's vote in the jth ballot is h^j. Then given that there are three or more individuals, no change in any individual's strategy has any effect on the outcome.

For brevity, I refer to an alternative that is the outcome of a subgame perfect equilibrium in which no individual's strategy in any subgame is weakly dominated as an outcome of sophisticated voting.

Definition 5.3: Sophisticated voting

An alternative a^* is an *outcome of sophisticated voting* in a binary agenda B if it is the outcome of a subgame perfect equilibrium s^* of B in which for no individual i is the restriction of s_i^* to any subgame Γ of B weakly dominated in Γ: i has no strategy s_i' that, for some strategies of the other individuals, yields an outcome in Γ that she prefers to the outcome when she uses s_i^* and, for all strategies of the other individuals, yields an outcome in Γ that she likes at least as much as the outcome when she uses s_i^*.

Suppose that if a majority of individuals vote *Yes* at the start of some subgame Γ then, given the individuals' strategies, the outcome of the game is ultimately a, whereas if a majority votes *No* then the outcome is ultimately b. Then it is reasonable that an individual who favors a votes *Yes* at the start of Γ and one who favors b votes *No*, so that the outcome of Γ is the member of $\{a, b\}$ preferred by a majority of individuals. The next result shows that any outcome of sophisticated voting has this property. It is expressed in terms of a game with a fictional single player whose preference relation is the majority relation for the collective choice problem: she prefers one outcome to another if and only if a majority of the individuals do so.

Lemma 5.1: Sophisticated voting and subgame perfect equilibrium

An alternative is an outcome of sophisticated voting in the binary agenda $\langle\langle N,X,\succsim\rangle,Z,O\rangle$ if and only if it is a subgame perfect equilibrium outcome of the variant of an extensive game with perfect information with possibly nontransitive preferences in which there is one player, the set of terminal histories is Z, the player function assigns the player to every nonterminal history, and the player's preference relation is the majority relation for $\langle N,X,\succsim\rangle$ (for any terminal histories $h \in Z$ and $h' \in Z$ she prefers h to h' if and only if $O(h) \succ_i O(h')$ for a majority of individuals $i \in N$).

The preferences of the single player in the game defined in this result are not transitive if, for example, the collective choice problem $\langle N,X,\succsim\rangle$ does not have a Condorcet winner. However, whenever the player chooses an action in the game she has exactly two options, so that she has a well-defined optimal action.

Proof of Lemma 5.1

Denote the binary agenda by B. The argument uses induction on the length of the game (i.e. the length of its longest history).

Let Γ be a subgame of B of length 1 (so that Γ is at the end of B); denote the two possible outcomes of Γ by a and b. By Corollary 3.1, in a Nash equilibrium of Γ in which no individual's action is weakly dominated, every individual votes for the option that leads to the alternative in $\{a,b\}$ that she prefers. Thus the outcome of sophisticated voting in the subgame, itself a binary agenda, is the member of $\{a,b\}$ preferred by a majority of individuals.

Now replace each subgame of length 1 with the outcome of sophisticated voting in the subgame. (See Figure 5.2 for an example.) In the resulting binary agenda, repeat the process. Continue in the same manner until reaching the start of the game.

This result makes the outcomes of sophisticated voting easy to find. Starting at the end of the game, we first find the alternative preferred by a majority of individuals among the outcomes of each subgame of length 1. Then we replace each of these subgames with the associated alternative and repeat the process for the resulting game, working our way to the start of the game.

If, for an agenda with the procedure shown in Figure 5.1, for example, a majority of individuals prefer a to b, a majority prefer b to c, and a majority prefer c to a, then the outcome of sophisticated voting in the subgame following the ballot outcome *Yes* is b and the outcome in the subgame following the sequence

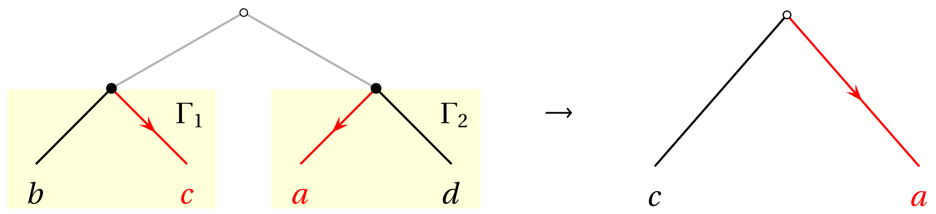


Figure 5.2 An illustration of the argument for Lemma 5.1 for a binary agenda in which a majority of individuals prefer c to b, a majority prefer a to d, and a majority prefer a to c. The agenda, shown on the left, has two subgames of length 1, Γ_1 and Γ_2. In the game shown on the right, each of these subgames is replaced by the outcome of sophisticated voting in the subgame.

(*No*, *No*) of ballot outcomes is a, so the outcome in the subgame following the ballot outcome *No* is c, and hence the outcome in the whole game is b.

An application of this procedure shows that for any binary agenda for a collective choice problem with a Condorcet winner (which is strict, given that the number of individuals is odd and their preferences are strict), this alternative is the only outcome of sophisticated voting.

Proposition 5.1: Sophisticated voting in binary agenda with strict Condorcet winner

If an odd-strict collective choice problem $\langle N, X, \succsim \rangle$ has a Condorcet winner then for any binary agenda $\langle \langle N, X, \succsim \rangle, Z, O \rangle$ this alternative is the only outcome of sophisticated voting.

Proof

Denote the Condorcet winner by a^*. Given that the collective choice problem is odd-strict, a^* is a strict Condorcet winner. Let Γ be a subgame of the binary agenda of length one (i.e. at the end of the game) for which at least one of the outcomes is a^*. Then a^* is the outcome of sophisticated voting in the subgame by Lemma 5.1 and the fact that for every other alternative a a majority of individuals prefer a^* to a. But then in the subgame of length two that includes Γ, the option that leads to Γ, and hence ultimately a^* (or the other option if that also leads to a^*), wins. Working back to the start of the game, we conclude that the only outcome of sophisticated voting is a^*.

For collective choice problems without Condorcet winners, things are more interesting. In particular, the outcome of sophisticated voting depends on the agenda. For example, for the collective choice problem of Example 1.5, which is a

Condorcet cycle in which a beats b beats c beats a, the outcome of sophisticated voting for the agenda on the left is a (b beats c, a beats b), whereas the outcome for the agenda on the right is c (a beats b, c beats a), and the outcome for the variant of this agenda in which c and b are interchanged is b.

In this example, every alternative is the outcome of sophisticated voting for some agenda. The same is not true for every collective choice problem. Recall that an alternative x is a Condorcet winner if it beats every other alternative y in two-way comparisons, in the sense that a majority of individuals prefer x to y. Say that x *indirectly* beats y if for some alternatives $z_1, \ldots, z_k$, alternative x beats z_1, z_1 beats $z_2, \ldots, z_{k-1}$ beats z_k, and z_k beats y. The set of alternatives that beat or indirectly beat every other alternative is called the top cycle set. A subsequent result, Proposition 5.3, shows that an alternative is the outcome of sophisticated voting for some binary agenda if and only if it is in this set.

Definition 5.4: Top cycle set

Let $\langle N, X, \succsim \rangle$ be an odd-strict collective choice problem and let $x \in X$ and $y \in X$ be alternatives. Then x *indirectly beats* y if for some $k \geq 1$ there are alternatives $z_1, \ldots, z_k$ such that x beats z_1, z_j beats z_{j+1} for $j = 1, \ldots, k-1$, and z_k beats y. The *top cycle set* of $\langle N, X, \succsim \rangle$ is the set of all alternatives x such that for every alternative $y \neq x$, x beats y either directly or indirectly.

For a collective choice problem with a strict Condorcet winner, the top cycle set consists solely of that alternative. For a collective choice problem without a Condorcet winner, it is nonempty and contains every Copeland winner, defined as follows.

Definition 5.5: Copeland winner

An alternative is a *Copeland winner* of a collective choice problem if it beats at least as many alternatives as does every other alternative.

Proposition 5.2: Nonemptiness of top cycle set and relation to Condorcet winner

The top cycle set of an odd-strict collective choice problem is nonempty and contains every Copeland winner. If the problem has a strict Condorcet winner then its top cycle set consists solely of that alternative.

Proof

Let $\langle N, X, \succsim\rangle$ be an odd-strict collective choice problem. By definition, its set of Copeland winners is nonempty. To show that every Copeland winner is in the top cycle set, I argue that if x is a Copeland winner and y is another alternative, then x either beats y or indirectly beats it in two steps. Suppose that x does not beat y, so that y beats x. If there is no alternative z such that x beats z and z beats y, then if x beats z, y beats z, so that y beats more alternatives than does x.

Now assume that $\langle N, X, \succsim\rangle$ has a strict Condorcet winner, x^*. Then no alternative beats x^* directly or indirectly, so no other alternative is in the top cycle set.

The top cycle set can alternatively be characterized as the smallest nonempty set of alternatives with the property that every member of the set beats every alternative not in the set.

Exercise 5.1: Characterization of top cycle set

Show that (*a*) every alternative in the top cycle set beats every alternative not in the set and (*b*) no proper subset of the top cycle set has this property. Deduce from (*a*) that any alternative that indirectly beats an alternative in the top cycle set is in the top cycle set.

For a problem without a strict Condorcet winner, the top cycle set contains at least three alternatives.

Exercise 5.2: Size of top cycle set

Show that for an odd-strict collective choice problem without a strict Condorcet winner, the top cycle set contains at least three alternatives.

For a Condorcet cycle, the top cycle set consists of all three alternatives (in Example 1.5, a beats b, b beats c, and c beats a). If we add an alternative to Example 1.5 that does not beat a, b, or c (for example, it could be ranked third by all individuals), then the top cycle set remains $\{a, b, c\}$.

The next result shows that the alternatives in the top cycle set may be ordered so that each alternative beats the next one and the last alternative beats the first one. This result justifies the word "cycle" in the name of the notion and is used in the proof of the next result.

Lemma 5.2: Top cycle set is a cycle

If the top cycle set of an odd-strict collective choice problem contains more than one alternative, then for some ordering $x_1, x_2, \ldots, x_k$ of its members, x_1 beats x_2 beats $\ldots$ beats x_k beats x_1.

Proof

Step 1 *For some subset $\{x_1, \ldots, x_p\}$ of the top cycle set with $p \geq 2$, x_1 beats x_2 $\ldots$ beats x_p beats x_1.*

Proof. Let x_1 and x_2 be members of the top cycle set, where x_1 beats x_2. Given that x_2 is in the set, it indirectly beats x_1. ◁

Let $C = \{x_1, \ldots, x_p\}$ be a largest subset of the top cycle set such that x_1 beats $x_2 \ldots$ beats x_p beats x_1 and suppose, contrary to the result, that C is not the whole top cycle set.

Step 2 *For each alternative y in the top cycle set outside C, either (i) y beats every member of C or (ii) every member of C beats y.*

Proof. If not, then for two consecutive members x_i and $x_i + 1$ of the sequence $(x_1, x_2, \ldots, x_p, x_1)$, x_i beats y and y beats x_{i+1}. But then the sequence can be extended by adding y between x_i and x_{i+1}, contradicting the maximality of C. ◁

Step 3 *Denote by C^+ the set of alternatives in (i) of Step 2 and by C^- the set in (ii). Both C^+ and C^- are nonempty.*

Proof. Either C^- or C^+ is nonempty. If C^- is empty, no member of C beats any member of C^+ directly or indirectly. If C^+ is empty, no member of C^- beats any member or C directly or indirectly. Thus both C^+ and C^- are nonempty. ◁

Now, take $y \in C^-$. Given that y is in the top cycle set and does not beat any alternative in C, it beats some alternative $z \in C^+$. But then C can be augmented by adding the alternatives y and z, contradicting its maximality.

To understand why every outcome of sophisticated voting in a binary agenda is in the top cycle set, look at Figure 5.3, in which the red branches indicate the outcomes of sophisticated voting in the game and its subgames. For the outcome

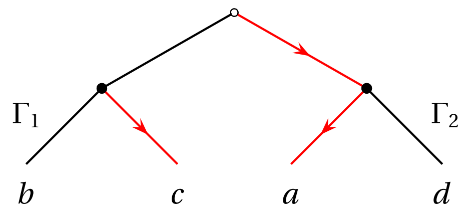


Figure 5.3 An illustration of the argument that an outcome of sophisticated voting beats every other alternative either directly or indirectly.

of sophisticated voting in the game to be a, this alternative must beat d, against which it is pitted in the subgame Γ_2. It must beat also the winner in the subgame Γ_1, namely c, which beats b in the subgame. Thus a must beat c and d directly and b indirectly (a beats c beats b).

The next proposition establishes the general result and also its converse: for any alternative in the top cycle set, there is a binary agenda for which that alternative is the outcome of sophisticated voting,

Proposition 5.3: Sophisticated voting in binary agenda and top cycle set

Let $\langle N, X, \succsim \rangle$ be an odd-strict collective choice problem.

a. For any binary agenda $\langle\langle N, X, \succsim\rangle, Z, O\rangle$, every outcome of sophisticated voting is in the top cycle set of $\langle N, X, \succsim \rangle$.

b. For every alternative x in the top cycle set of $\langle N, X, \succsim \rangle$ there is a binary agenda $\langle\langle N, X, \succsim\rangle, Z, O\rangle$ for which x is the outcome of sophisticated voting.

Proof

a. Let a be an outcome of sophisticated voting. By Lemma 5.1, a is the outcome of a subgame perfect equilibrium of the one-player game G^* in which the set of terminal histories is Z, the player function assigns the player to every nonterminal history, and for any terminal histories $h \in Z$ and $h' \in Z$ the player prefers h to h' if and only if $O(h) \succ_i O(h')$ for a majority of $i \in N$.

I use induction. The top cycle set is nonempty, and every member of it is the outcome of at least one terminal history of G^* (by the assumption that every alternative is the outcome of some terminal history). Thus the outcome of sophisticated voting in some subgame of G^* of length 1 is in the top cycle set of $\langle N, X, \succsim \rangle$.

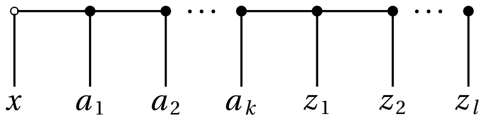


Figure 5.4 Binary agenda used in the proof of Proposition 5.3*b*.

Now let ℓ be a positive integer less than the length of the longest subgame of G^* and suppose that Γ is a subgame of G^* of length ℓ for which the outcome of sophisticated voting, say x, is in the top cycle set of $\langle N, X, \succsim \rangle$. Let Γ' be the subgame of G^* of length $\ell + 1$ that contains Γ. I argue that the outcome of sophisticated voting in Γ' is in the top cycle set. The player has the option to choose the action at the start of Γ' that leads to the subgame Γ, and hence to the outcome x. Thus the subgame perfect equilibrium outcome of Γ' is either x or an alternative y that beats x. By Exercise 5.1a, every alternative in the top cycle set beats every alternative outside the set, so y, like x, is in the top cycle set.

Thus by induction the outcome of sophisticated voting in G^* is in the top cycle set.

b. Denote the cycle among all the members of the top cycle set that is shown to exist by Lemma 5.2 by $(x, a_1, a_2, \ldots, a_k)$ and denote the remaining members of X (in an arbitrary order) by $(z_1, z_2, \ldots, z_l)$. The outcome of sophisticated voting for the binary agenda with the procedure shown in Figure 5.4 is x: a_k beats every z_j, a_i beats a_{i+1} for $i = 1, \ldots, k-1$, and x beats a_1.

Part *b* of this result says that for any alternative in the top cycle set, a binary agenda can be designed so that the outcome of sophisticated voting is that alternative. If every alternative in the top cycle set were a reasonable outcome of the collective choice problem, that might not be bad. But unfortunately the top cycle may be large and include alternatives that are dominated by other alternatives, as the following example (a generalization of a Condorcet cycle) shows.

Example 5.1: Large top cycle set, containing dominated alternatives

Suppose that $N = \{1, 2, 3\}$, $X = \{a_1, a_2, \ldots, a_k\}$, and

$$\begin{aligned}
&a_k \succ_1 a_1 \succ_1 a_2 \succ_1 \cdots \succ_1 a_{k-2} \succ_1 a_{k-1}\\
&a_1 \succ_2 a_2 \succ_2 a_3 \succ_2 \cdots \succ_2 a_{k-1} \succ_2 a_k\\
&a_2 \succ_3 a_3 \succ_3 a_4 \succ_3 \cdots \succ_3 a_k \quad \succ_3 a_1.
\end{aligned}$$

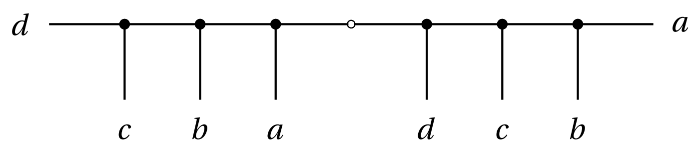


Figure 5.5 The procedure in the binary agenda in Exercise 5.3. (Note that the game starts in the middle of the diagram.)

Then a_i beats a_{i+1} for $i = 1, \ldots, k-1$ and a_k beats a_1, so that the top cycle set contains all k alternatives. However, all three individuals prefer a_2 to every alternative a_3, a_4, ..., a_{k-1}. (More generally, all individuals prefer a_i to a_j for $i = 2, \ldots, k-2$ and $j = i+1, \ldots, k-1$.)

Further, the outcome of sophisticated voting (a member of the top cycle set by Proposition 5.3a) may not respond positively to changes in the individuals' preferences, as you are asked to show in the following exercise.

Exercise 5.3: Outcome of sophisticated voting not positively responsive in binary agenda

Let $N = \{1,2,3\}$, $X = \{a,b,c,d\}$, $b \succ_1 a \succ_1 c \succ_1 d$, $c \succ_2 d \succ_2 b \succ_2 a$, and $d \succ_3 a \succ_3 c \succ_3 b$. Consider the binary agenda $\langle\langle N, X, \succsim\rangle, Z, O\rangle$ with the procedure shown in Figure 5.5, in which the individuals first vote on whether to decide in the order (a,b,c,d) or in the order (d,c,b,a), and then conduct three votes, first whether to select the first alternative in the order, then whether to select the second alternative, and then whether to select the third alternative. Find the outcome of sophisticated voting for this binary agenda and also for the binary agenda that differs only in that individual 1's preference between a and b is reversed. Check that the change is inconsistent with positive responsiveness.

5.2 Successive agendas

The procedures in many European and Latin American legislatures are approximated by a binary agenda known as a successive agenda. In a such an agenda the alternatives are considered in some order $(x_1, \ldots, x_k)$. First a vote is taken on whether to choose x_1 or to drop it from consideration; if it is dropped, a vote is taken on whether to choose x_2 or to drop it from consideration; and so forth. A successive agenda for four alternatives is shown in Figure 5.6.

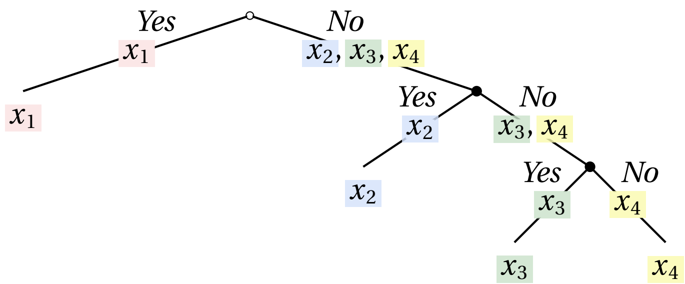


Figure 5.6 The procedure in a successive agenda for which $X = \{x_1, x_2, x_3, x_4\}$ and the ordering is (x_1, x_2, x_3, x_4).

Definition 5.6: Successive agenda

Let $\langle N, X, \succsim \rangle$ be an odd-strict collective choice problem, denote by k the number of alternatives (members of X), and let $(x_1, \ldots, x_k)$ be an ordering of the alternatives. The *successive agenda* $\langle \langle N, X, \succsim \rangle, (x_1, \ldots, x_k) \rangle$ is the binary agenda $\langle \langle N, X, \succsim \rangle, Z, O \rangle$ in which the set Z consists of two terminal sequences h' and h'' of length $k-1$, with $O(h') = x_{k-1}$ and $O(h'') = x_k$, and for each $l = 1, \ldots, k-2$, one terminal sequence h of length l, with $O(h) = x_l$.

We can find the outcome of sophisticated voting in a successive agenda by using backward induction, starting from the single subgame of length 1 (a choice between x_{k-1} and x_k). Let $x_k^* = x_k$. The choice in the single subgame Γ_1 of length 1 is between x_{k-1} and x_k^*, so, using Lemma 5.1, the outcome of sophisticated voting in the subgame is x_k^* if x_k^* beats x_{k-1}, and x_{k-1} if x_{k-1} beats x_k^*. Denote this alternative x_{k-1}^* and replace Γ_1 with it. Continue to the subgame of length 1 in the resulting game, where the choice is between x_{k-2} and x_{k-1}^*, and repeat the process. The sequence $(x_1^*, \ldots, x_k^*)$ thus created is called the sophisticated sequence for the agenda, and the conclusion of the argument is that x_1^* is the outcome of sophisticated voting in the agenda.

Definition 5.7: Sophisticated sequence for successive agenda

Let $\langle \langle N, X, \succsim \rangle, (x_1, \ldots, x_k) \rangle$ be a successive agenda. The *sophisticated sequence* for $\langle \langle N, X, \succsim \rangle, (x_1, \ldots, x_k) \rangle$ is the sequence $(x_1^*, \ldots, x_k^*)$ of alternatives defined iteratively as follows, starting with x_k^* and working backwards to x_1^*. First let $x_k^* = x_k$. Then for any j with $1 \leq j \leq k-1$ let

$$x_j^* = \begin{cases} x_j & \text{if } x_j \text{ beats } x_{j+1}^* \\ x_{j+1}^* & \text{otherwise.} \end{cases} \tag{5.1}$$

Proposition 5.4: Outcome of sophisticated voting in successive agenda

The outcome of sophisticated voting in a successive agenda is the first alternative in the sophisticated sequence for the agenda.

One implication of this result is that the outcome of sophisticated voting in a successive agenda is positively responsive: if the outcome is x and then x rises in the preferences of some individual, x continues to beat every alternative it beat previously, and hence remains the outcome of sophisticated voting.

The next exercise concerns two more implications of the result. First, the last alternative in a successive agenda is the outcome of sophisticated voting only if it is a strict Condorcet winner. Second, an alternative remains the outcome of sophisticated voting in a successive agenda if it is moved earlier in the agenda.

Exercise 5.4: Effect of order of alternatives on outcome of sophisticated voting in successive agenda

Let $B = \langle\langle N, X, \succsim\rangle, (x_1, \ldots, x_k)\rangle$ be a successive agenda. Use Proposition 5.4 to show that (*a*) if x_k is the outcome of sophisticated voting in B then it is a strict Condorcet winner of $\langle N, X, \succsim\rangle$, and (*b*) for any $l \in \{2, \ldots, k\}$, if x_l is the outcome of sophisticated voting in B then it is also the outcome of sophisticated voting in the successive agenda $\langle\langle N, X, \succsim\rangle, (y_1, \ldots, y_k)\rangle$ in which $(y_1, \ldots, y_k)$ differs from $(x_1, \ldots, x_k)$ only in that x_l and x_{l-1} are interchanged.

The agenda used to prove Proposition 5.3*b* is a successive agenda, so we have the following result.

Proposition 5.5: Sophisticated voting in successive agenda and top cycle set

Let $\langle N, X, \succsim\rangle$ be an odd-strict collective choice problem. For every alternative x in the top cycle set of $\langle N, X, \succsim\rangle$ there is a successive agenda $\langle\langle N, X, \succsim\rangle, (x_1, \ldots, x_k)\rangle$ for which x is the outcome of sophisticated voting.

Although the outcome of sophisticated voting in a successive agenda, unlike the outcome in a general binary agenda, is necessarily positively responsive, Proposition 5.5 means that it suffers from a drawback: for some collective choice problems, the top cycle set contains dominated alternatives (see Example 5.1), so the outcome of sophisticated voting in a successive agenda may be dominated.

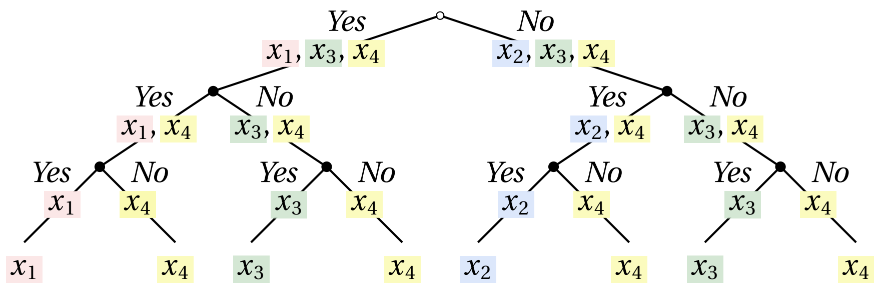


Figure 5.7 The procedure in an amendment agenda for $X = \{x_1, x_2, x_3, x_4\}$ and the ordering (x_1, x_2, x_3, x_4).

5.3 Amendment agendas

The procedures in a few European legislatures, as well as in Canada and the United States, are approximated by agendas that generate sets of sophisticated outcomes smaller than the top cycle set, with better properties. Let $(x_1, \ldots, x_k)$ be an ordering of the alternatives. In the amendment agenda for this ordering, a vote is taken to eliminate x_2 from consideration (*Yes*) or to eliminate x_1 (*No*); then a vote is taken whether to eliminate x_3 (*Yes*) or to eliminate whichever of x_1 or x_2 was retained on the first round (*No*); and so on, until the remaining alternative is pitted against x_k. An example for four alternatives is shown in Figure 5.7. One interpretation of this agenda is that x_4 is the status quo, x_3 is a bill, x_2 is an amendment, and $x_1$is an amendment to the amendment. The first vote determines which version of the amendment is considered, the second vote determines whether the bill or an amended version of it is considered, and the final vote determines whether the (possibly amended) bill passes.

Definition 5.8: Amendment agenda

Let $\langle N, X, \succsim \rangle$ be an odd-strict collective choice problem and let $(x_1, \ldots, x_k)$ be an ordering of the members of X. The *amendment agenda* $\langle \langle N, X, \succsim \rangle, (x_1, \ldots, x_k) \rangle$ is the binary agenda $\langle \langle N, X, \succsim \rangle, Z, O \rangle$ in which every terminal sequence of ballot outcomes (member of Z) has length $k-1$ and for any terminal sequence $(y^1, \ldots, y^{k-1})$ we have $O(y^1, \ldots, y^{k-1}) = x_{r+1}$, where r is the index of the last *No* in $(y^1, \ldots, y^{k-1})$, with $r = 0$ if $y^j = \textit{Yes}$ for all $j = 1, \ldots, k-1$.

The set of alternatives that are the outcomes of sophisticated voting in an amendment agenda is the Banks set, named for its originator, Jeffrey S. Banks (1958–2000). Recall that an alternative is in the top cycle set if it beats every other alternative either directly or indirectly. To qualify for membership in the Banks set, an alternative x must satisfy a more stringent requirement: there must exist a sequence $(z_1,\ldots,z_l)$ of alternatives such that (*i*) each z_j beats every subsequent member of the sequence, z_{j+1}, ..., z_l, (*ii*) x beats every member of the sequence, and (*iii*) no alternative beats x and every member of the sequence. For example, x is in the Banks set if it beats some alternative z and no alternative beats both x and z, or if it beats some alternatives z_1 and z_2, z_1 beats z_2, and no alternative beats x, z_1, and z_2.

Definition 5.9: Banks set

Let $\langle N, X, \succsim\rangle$ be an odd-strict collective choice problem. An alternative $x \in X$ is in the *Banks set* of $\langle N, X, \succsim\rangle$ if there is a sequence $(z_1,\ldots,z_l)$ of alternatives such that

- for $j = 1,\ldots,l-1$, z_j beats each of the alternatives z_{j+1}, ..., z_l
- x beats each of the alternatives z_1, ..., z_l
- no alternative beats each of the alternatives z_1, ..., z_l and x.

Suppose that x is in the Banks set, with the associated sequence $(z_1,\ldots,z_l)$. Then by the last of the three conditions, every alternative other than x or z_1, ..., z_l is beaten by either x or some z_j, and hence, given that x beats every z_j, is beaten by x either directly or indirectly. Thus the Banks set is a subset of the top cycle set. For a problem that has a strict Condorcet winner, say a^*, the Banks set is $\{a^*\}$, because a^* beats every other alternative. For problems without a strict Condorcet winner, the Banks set is nonempty.

Proposition 5.6: Banks set is nonempty subset of top cycle set

The Banks set of any odd-strict collective choice problem is a nonempty subset of the top cycle set. In particular, for a problem with a strict Condorcet winner the Banks set consists solely of that alternative.

Proof

An argument that the Banks set is a subset of the top cycle set is given in the text.

To prove the nonemptiness of the Banks set I use induction on the number of alternatives.

Let $\langle N, X, \succsim\rangle$ be an odd-strict collective choice problem. Let $X_2 = \{x_1, x_2\} \subset X$, where x_1 beats x_2. Then the Banks set of $\langle N, X_2, \succsim|_{X_2}\rangle$ is $\{x_1\}$

(where $\succsim|_{X_2}$ is the restriction of $\succsim$ to X_2).

Now let $l \geq 2$ and let $X_l \subset X$ be a set with l members. Assume that the Banks set of $\langle N, X_l, \succsim|_{X_l} \rangle$ is nonempty, and x_1 is a member of the set with the associated sequence $(z_1, z_2, \ldots, z_p)$. Let $X_{l+1} = X_l \cup \{x_{l+1}\}$ for some $x_{l+1} \in X \setminus X_l$. Either x_{l+1} beats x_1 and every z_j, in which case x_{l+1} is in the Banks set of $\langle N, X_{l+1}, \succsim|_{X_{l+1}} \rangle$ with the associated sequence $(x_1, z_1, z_2, \ldots, z_p)$, or some z_j or x_1 beats x_{l+1}, in which case x_1 is in the Banks set of $\langle N, X_{l+1}, \succsim|_{X_{l+1}} \rangle$ with the associated sequence $(z_1, z_2, \ldots, z_p)$. Thus the Banks set of $\langle N, X_{l+1}, \succsim|_{X_{l+1}} \rangle$ is nonempty.

The second part of Proposition 5.2 implies that for a problem with a strict Condorcet winner the Banks set consists solely of that alternative.

For a Condorcet cycle, the Banks set, like the top cycle set, consists of all three alternatives. (In Example 1.5, a sequence of alternatives supporting a, for example, is (b): a beats b and c does not beat b.) But for many collective choice problems the Banks set is smaller than the top cycle set. An example is the problem in Example 5.1, in which the Banks set contains only the alternatives that are not dominated.

Example 5.2: Banks set for Example 5.1

For the collective choice problem in Example 5.1, the Banks set is $\{a_1, a_2, a_k\}$. A sequence of alternatives supporting a_1 is $(a_2, \ldots, a_{k-1})$ (a_k does not beat a_{k-1}), a sequence supporting a_2 is $(a_3, a_4, \ldots, a_k)$ (a_1 does not beat a_k), and a sequence supporting a_k is (a_1) (no alternative beats both a_1 and a_k). No alternative a_i with $3 \leq i \leq k-1$ is in the Banks set because a_2 beats every such alternative a_i and all the alternatives that a_i beats.

In fact, no alternative in the Banks set of any odd-strict collective choice problem is dominated.

Exercise 5.5: No alternative in Banks set is dominated

Show that for any odd-strict collective choice problem, for no member a of the Banks set is there an alternative that every individual prefers to a.

To show that the set of outcomes of sophisticated voting for an amendment agenda is the Banks set, I first define a procedure for finding the outcome of sophisticated voting in any amendment agenda. This procedure involves generating a sequence of alternatives defined as follows.

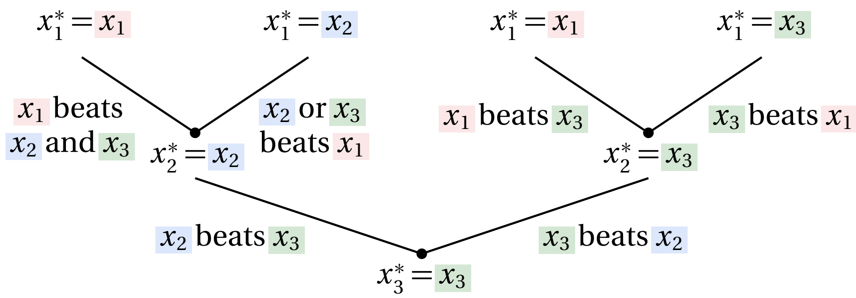


Figure 5.8 The sophisticated sequence for the amendment agenda for three alternatives and the ordering (x_1, x_2, x_3).

Definition 5.10: Sophisticated sequence for amendment agenda

The *sophisticated sequence* for the amendment agenda $\langle\langle N, X, \succsim\rangle, (x_1, \ldots, x_k)\rangle$ is the sequence $(x_1^*, \ldots, x_k^*)$ of alternatives defined iteratively as follows, starting with x_k^* and working backwards to x_1^*. First let $x_k^* = x_k$. Then for any j with $1 \leq j \leq k-1$ let

$$x_j^* = \begin{cases} x_j & \text{if } x_j \text{ beats } x_l^* \text{ for } l = j+1, \ldots, k \\ x_{j+1}^* & \text{otherwise.} \end{cases} \tag{5.2}$$

For $k = 2$, we have $(x_1^*, x_2^*) = (x_1, x_2)$ if x_1 beats x_2 and $(x_1^*, x_2^*) = (x_2, x_2)$ if x_2 beats x_1. An analysis of the case $k = 3$ is illustrated in Figure 5.8. The conclusion is

$$(x_1^*, x_2^*, x_3^*) = \begin{cases} (x_1, x_2, x_3) & \text{if } x_1 \text{ beats } x_2 \text{ and } x_3, \text{ and } x_2 \text{ beats } x_3 \\ (x_2, x_2, x_3) & \text{if } x_2 \text{ or } x_3 \text{ beats } x_1, \text{ and } x_2 \text{ beats } x_3 \\ (x_1, x_3, x_3) & \text{if } x_1 \text{ beats } x_3, \text{ and } x_3 \text{ beats } x_2 \\ (x_3, x_3, x_3) & \text{if } x_3 \text{ beats } x_1 \text{ and } x_2. \end{cases}$$

The next result shows that the outcome of sophisticated voting is the first alternative in the sophisticated sequence.

Proposition 5.7: Outcome of sophisticated voting in amendment agenda

The outcome of sophisticated voting in an amendment agenda is the first alternative in the sophisticated sequence for the agenda.

Proof

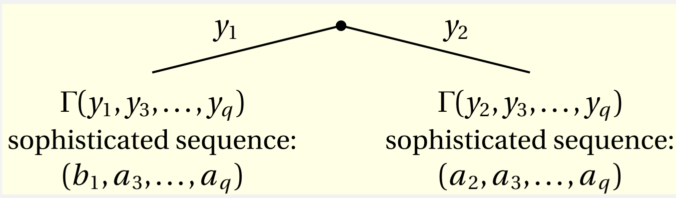
Denote by k the number of alternatives and by $\Gamma(y_1,\ldots,y_q)$, with $2 \le q \le k$, the subgame of the agenda in which the alternatives $y_1,\ldots,y_q$ remain, in that order. This subgame is itself an amendment agenda.

For $q=2$, from (5.2) the sophisticated sequence is (x_1^*, x_2^*) where $x_2^* = y_2$ and

$$x_1^* = \begin{cases} y_1 & \text{if } y_1 \text{ beats } y_2 \\ y_2 & \text{if } y_2 \text{ beats } y_1. \end{cases}$$

By Lemma 5.1, x_1^* is the outcome of sophisticated voting in $\Gamma(y_1, y_2)$.

Now suppose that the outcome of sophisticated voting in every subgame in which at most $q-1$ alternatives remain, with $3 \le q \le k$, is the first alternative in the sophisticated sequence for the subgame. I argue that the same is true for the subgame $\Gamma(y_1,\ldots,y_q)$. This subgame starts with a vote on whether to keep y_1 and eliminate y_2, moving to the subgame $\Gamma(y_1,y_3,\ldots,y_q)$, or to keep y_2 and eliminate y_1, moving to the subgame $\Gamma(y_2,y_3,\ldots,y_q)$. Denote the sophisticated sequence for $\Gamma(y_1,\ldots,y_q)$ by $(a_1,\ldots,a_q)$. Then the sophisticated sequence for $\Gamma(y_1,y_3,\ldots,y_q)$ is $(b_1,a_3,\ldots,a_q)$ for some alternative b_1 and the sophisticated sequence for $\Gamma(y_2,y_3,\ldots,y_q)$ is $(a_2,a_3,\ldots,a_q)$. Thus we can represent $\Gamma(y_1,\ldots,y_q)$ as



By assumption, the outcome of sophisticated voting in $\Gamma(y_1,y_3,\ldots,y_q)$ is b_1 and the outcome of sophisticated voting in $\Gamma(y_2,y_3,\ldots,y_q)$ is a_2, so the outcome of sophisticated voting in $\Gamma(y_1,\ldots,y_q)$ is

$$z = \begin{cases} b_1 & \text{if } b_1 \text{ beats } a_2 \text{ or } b_1 = a_2 \\ a_2 & \text{if } a_2 \text{ beats } b_1. \end{cases} \tag{5.3}$$

We need to show that z is equal to the value of a_1 given by (5.2):

$$a_1 = \begin{cases} y_1 & \text{if } y_1 \text{ beats } a_2, a_3, \ldots, a_q \\ a_2 & \text{if some member of } \{a_2, a_3, \ldots, a_q\} \text{ beats } y_1. \end{cases} \tag{5.4}$$

- Suppose that y_1 beats $a_2,\ldots,a_q$, so that $a_1 = y_1$ by (5.4). The fact that b_1 is the first member of the sophisticated sequence for $\Gamma(y_1,y_3,\ldots,y_q)$

and y_1 beats $a_3, \ldots, a_q$ implies, using (5.2), that $b_1 = y_1$. Since y_1 beats a_2, the value of z given by (5.3), namely b_1, is the value of a_1 given by (5.4).

- Suppose that a_j beats y_1 for some $j = 3, \ldots, q$, so that $a_1 = a_2$ by (5.4). The fact that b_1 is the first member of the sophisticated sequence for $\Gamma(y_1, y_3, \ldots, y_q)$ implies, using (5.2), that $b_1 = a_3$.
  - Suppose that y_2 beats $a_3, \ldots, a_q$. Then the fact that a_2 is the first member of the sophisticated sequence for $\Gamma(y_2, y_3, \ldots, y_q)$ implies, using (5.2), that $a_2 = y_2$. Given that y_2 beats a_3, $y_2 = a_2$, and $a_3 = b_1$, a_2 beats b_1. Thus the value of z given by (5.3), namely a_2, is the value of a_1 given by (5.4).
  - Suppose that a_j beats y_2 for some $j = 3, \ldots, q$. Then the fact that a_2 is the first member of the sophisticated sequence for $\Gamma(y_2, y_3, \ldots, y_q)$ implies, using (5.2), that $a_2 = a_3$. Given $b_1 = a_3$, we have $b_1 = a_2$, so that the value of z given by (5.3) is the value of a_1 given by (5.4).

We can now show that the outcome of sophisticated voting in an amendment agenda is in the Banks set, and for any alternative in the Banks set there is an amendment agenda for which the alternative is the outcome of sophisticated voting.

Proposition 5.8: Sophisticated voting in amendment agenda and Banks set

Let $\langle N, X, \succsim \rangle$ be an odd-strict collective choice problem.

a. For any amendment agenda $\langle \langle N, X, \succsim \rangle, (x_1, \ldots, x_k) \rangle$, the outcome of sophisticated voting is in the Banks set of $\langle N, X, \succsim \rangle$.

b. For every alternative a in the Banks set of $\langle N, X, \succsim \rangle$ there is an amendment agenda $\langle \langle N, X, \succsim \rangle, (x_1, \ldots, x_k) \rangle$ for which a is the outcome of sophisticated voting.

Proof

a. By Proposition 5.7, the outcome of sophisticated voting is the first alternative in the sophisticated sequence for the agenda. By definition, this alternative beats all other alternatives in the sequence, every alternative in the sequence beats every later alternative in the sequence, and every alternative not in the sequence is beaten by some alternative in the sequence. Thus the first alternative in the sequence is in the Banks set.

b. Let a be in the Banks set and let $(y_1,\ldots,y_l)$ be a sequence of alternatives that supports it. Let $z_1, \ldots, z_p$ be the remaining alternatives (other than a and $y_1, \ldots, y_l$); the order of these alternatives does not matter. I claim that a is the outcome of sophisticated voting for the amendment agenda $B = \langle\langle N, X, \succsim\rangle, (z_1,\ldots,z_p,a,y_1,\ldots,y_l)\rangle$. By the definition of $(y_1,\ldots,y_l)$, y_j beats $y_{j+1},\ldots,y_l$ for $j=1,\ldots,l-1$ and a beats every y_j. Also, every z_j is beaten by either a or some y_j. Thus the first alternative in the sophisticated sequence for B is a, so that by Proposition 5.7, a is the outcome of sophisticated voting in B.

Unlike the top cycle set, the Banks set has the property that no member of it is dominated (Exercise 5.5). In addition, the outcome of sophisticated voting in an amendment agenda, which is a member of the Banks set by Proposition 5.8, is positively responsive.

Exercise 5.6: Outcome of sophisticated voting in amendment agenda is positively responsive

Use Proposition 5.8 to show that the outcome of sophisticated voting in an amendment agenda is positively responsive.

Although the outcome of sophisticated voting in an amendment agenda has these desirable properties, such an agenda does not treat the alternatives equally. The last alternative on the agenda is the outcome of sophisticated voting only if it is a strict Condorcet winner, but an earlier alternative may be the outcome of sophisticated voting even if it is not a strict Condorcet winner, and an alternative is never disadvantaged by being moved to an earlier position in the agenda.

Exercise 5.7: Effect of order on outcome of sophisticated voting in amendment agenda

Show that the outcome of sophisticated voting in an amendment agenda satisfies the properties of the outcome of sophisticated voting in a successive agenda given in Exercise 5.4.

A set of alternatives known as the uncovered set is used in the next section.

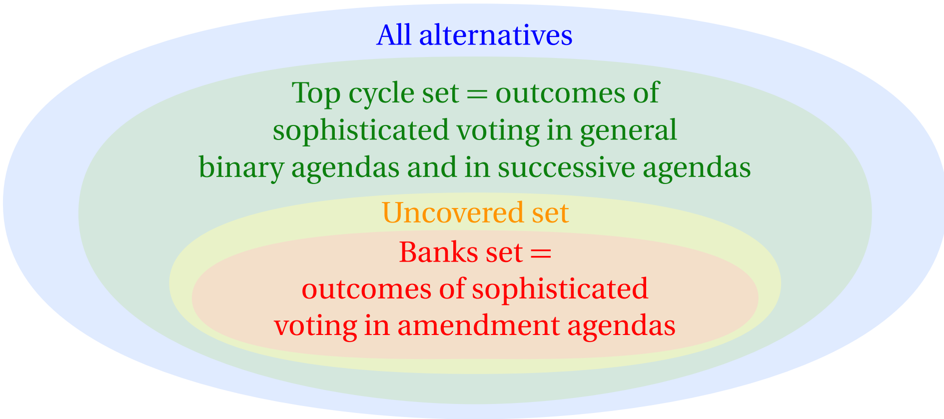


Figure 5.9 The relations among the top cycle set, uncovered set, Banks set, and the outcomes of sophisticated voting in general binary agendas, successive agendas, and amendment agendas.

Definition 5.11: Uncovered set

Let $\langle N, X, \succsim \rangle$ be an odd-strict collective choice problem and let $x \in X$ and $y \in X$. Then x *covers* y if (*i*) x beats y and (*ii*) every alternative that beats x also beats y. The *uncovered set* consists of the alternatives that are not covered by any other alternative.

The next exercise asks you to show that the uncovered set is a subset of the top cycle set and contains the Banks set.

Exercise 5.8: Uncovered set

Show that the uncovered set for an odd-strict collective choice problem (*a*) consists of the alternatives x such that for every other alternative y either x beats y or x beats y indirectly in two steps, so that it is a subset of the top cycle set, and (*b*) contains the Banks set.

The relations among the top cycle set, uncovered set, Banks set, and outcomes of sophisticated voting in general binary agendas, successive agendas, and amendment agendas, as established in Propositions 5.3, 5.5, 5.6, and 5.8, are illustrated in Figure 5.9. The set of Copeland winners, like the Banks set, is a subset of the uncovered set; it may be disjoint from the Banks set (Laffond and Laslier 1991).

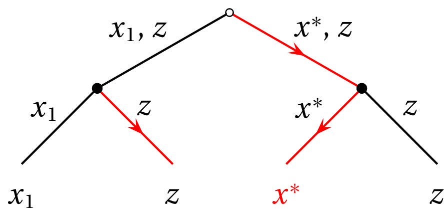


Figure 5.10 An amendment agenda for which x^* is the outcome of sophisticated voting if x^* beats z and z beats x_1.

5.4 Agenda formation

If the agenda is chosen by an individual with preferences over the alternatives, which one will she choose? If she is free to choose any binary agenda, then assuming that the alternative that results is the outcome of sophisticated voting, Proposition 5.3b says that she can obtain any alternative in the top cycle set of the associated collective choice problem, and thus can obtain her favorite alternative in this set. If she is restricted to amendment agendas, then Proposition 5.8b says that she can obtain any alternative in the Banks set for the collective choice problem, and thus can obtain her favorite alternative in this set.

In one situation that an amendment agenda $\langle\langle N, X, \succsim\rangle, (x_1, \ldots, x_k)\rangle$ models, x_1 is the status quo. The first vote either retains it or makes x_2 the new status quo, and each subsequent vote is a decision to retain the current status quo or replace it with a new alternative; the alternative chosen in the $(k-1)$th vote is the outcome. In such a situation, a reasonable assumption is that an agenda-setter cannot choose any amendment agenda, but rather is restricted to those for which the first alternative is the given status quo. At the same time, she may not be restricted to include every possible alternative—every member of X—in the agenda. Thus her problem may be to choose a subset $\{y_2, \ldots, y_l\}$ of $X \setminus \{x_1\}$ and an ordering $(y_2, \ldots, y_l)$ so that the outcome of sophisticated voting in the amendment agenda $\langle\langle N, X^*, \succsim|_{X^*}\rangle, (x_1, y_2, \ldots, y_l)\rangle$, where $X^* = \{x_1, y_2, y_3, \ldots, y_l\}$, is as high as possible in her preferences. In this case, she can obtain any alternative that is not covered by x_1: that either beats x_1 or beats an alternative that beats x_1. If x^* beats x_1, then she simply has to choose an agenda with a single stage in which x^* is pitted against x_1. If x^* beats z and z beats x_1, then she can choose the two-stage agenda in which the order of the alternatives is (x_1, x^*, z), shown in Figure 5.10. In this case, x^* beats z and z beats x_1 in the second vote, so that the first vote is a decision between x^* and z, which x^* wins. The next result shows that the alternatives not covered by x_1 are the only ones she can obtain.

Proposition 5.9: Outcome of sophisticated voting in amendment agenda with given first alternative

Let $\langle N, X, \succsim \rangle$ be an odd-strict collective choice problem and let $x^* \in X$ and $x_1 \in X$. There are alternatives y_2, y_3, $\ldots$, y_l in $X \setminus \{x_1\}$ such that x^* is the outcome of sophisticated voting in the amendment agenda $\langle \langle N, X^*, \succsim|_{X^*} \rangle, (x_1, y_2, y_3, \ldots, y_l) \rangle$, where $X^* = \{x_1, y_2, y_3, \ldots, y_l\}$, if and only if x_1 does not cover x^*.

Proof

A proof that if x_1 does not cover x^* then a suitable agenda exists is given in the text.

Now suppose that there is a sequence $(y_2, y_3, \ldots, y_l)$ of alternatives such that x^* is the outcome of sophisticated voting in the amendment agenda $\langle \langle N, X^*, \succsim|_{X^*} \rangle, (x_1, y_2, \ldots, y_l) \rangle$, where $X^* = \{x_1, y_2, \ldots, y_l\}$. Then by Proposition 5.7, x^* is the first member of the sophisticated sequence for the agenda, and hence x^* beats every member of the sophisticated sequence not equal to x^*. If x_1 were to cover x^*, it would thus beat x^* and all other members of the sophisticated sequence, hence becoming the first member of the sophisticated sequence, rather than x^*. So x_1 does not cover x^*.

5.5 Optimistically sincere strategies, single-peaked preferences, convex agendas

The focus of the previous analysis is the outcome of sophisticated voting for binary agendas that do not have a Condorcet winner. I now turn to equilibrium strategies, rather than outcomes. I limit the analysis to agendas for odd-strict collective choice problems that have a (strict) Condorcet winner; by Proposition 5.1, this alternative is the only outcome of sophisticated voting in these agendas.

I exhibit a class of binary agendas that have subgame perfect equilibria of a particularly simple form: each individual, when deciding how to vote after any history, looks at the sets of outcomes possible following a majority vote of *Yes* and a majority vote of *No*, and votes for the option that leads to the set she prefers. I restrict attention to agendas for which, after every history, these sets are disjoint, and assume that each individual prefers a set X_1 to a set X_2 whenever she prefers the best alternative in X_1 to the best alternative in X_2. That is, each individual's strategy is an optimistic version of sincere voting. The actions it specifies do not depend on the other individuals' strategies, so that it can be implemented by an individual who knows nothing about the other individuals.

To describe this class of binary agendas, I begin by introducing notation for the sets of alternatives to which a vote may lead. For any binary agenda $\langle\langle N, X, \succsim\rangle, Z, O\rangle$, any nonterminal sequence h of ballot outcomes, and $p \in \{Yes, No\}$, let $A(h, p)$ be the set of alternatives that are outcomes of sequences of ballot outcomes following a majority vote of p at h:

$$A(h, p) = \{O(z) : z = (h, p, h') \in Z \text{ for some sequence } h'\}. \tag{5.5}$$

For example, for a successive agenda with the procedure given in Figure 5.6, $A(\varnothing, Yes) = \{x_1\}$ and $A(\varnothing, No) = \{x_2, x_3, x_4\}$, and for an amendment agenda with the procedure given in Figure 5.7, $A(\varnothing, Yes) = \{x_1, x_3, x_4\}$ and $A(\varnothing, No) = \{x_2, x_3, x_4\}$.

For convenience, I consider only binary agendas for which after every nonterminal sequence of ballot outcomes, the set of outcomes possible following a majority vote of *Yes* is disjoint from the set of outcomes possible following a majority vote of *No*. Such agendas are called partitional. An agenda with the procedure given in Figure 5.6 is partitional, but one with the procedure given in Figure 5.7 is not—for example, x_3 and x_4 are members of both $A(\varnothing, Yes)$ and $A(\varnothing, No)$.

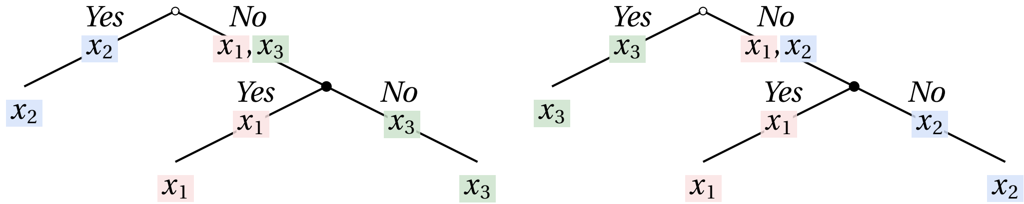
Definition 5.12: Partitional binary agenda

A binary agenda is *partitional* if for every nonterminal sequence h of ballot outcomes, $A(h, Yes) \cap A(h, No) = \varnothing$, where A is given in (5.5).

Here is the version of sincere voting used in the result that I present: in a partitional binary agenda for an odd-strict collective choice problem, an individual's vote for option p after a sequence h of ballot outcomes is optimistically sincere if the *best* outcome to which a majority vote for p at h may lead is better than the *best* outcome to which a majority vote for the other option may lead. (Given that the agenda is partitional and the collective choice problem is odd-strict, a tie is not possible.)

Definition 5.13: Optimistically sincere strategy

An individual's strategy in a partitional binary agenda for an odd-strict collective choice problem is *optimistically sincere* if for every nonterminal sequence h of ballot outcomes it assigns the option $p \in \{Yes, No\}$ for which $A(h, p)$ contains the alternative the individual likes best in $A(h, Yes) \cup A(h, No)$, where A is given in (5.5).



(a) In an agenda with this structure, if individual 1's preferences are given by $x_1 \succ_1 x_2 \succ_1 x_3$, half of the other individuals vote *Yes* and half vote *No* at the start, and all of them vote *No* after the ballot outcome *Yes*, then individual 1's optimistically sincere strategy is not optimal.

(b) A binary agenda that is convex with respect to the ordering $x_1 \lhd x_2 \lhd x_3$. If the preference profile is single-peaked with respect to this ordering and the strategies of individuals 2 and 3 are optimistically sincere then individual 1's optimistically sincere strategy is best for her.

Figure 5.11 The structures of two binary agendas.

In general, an optimistically sincere strategy is not an individual's best response to the other individuals' strategies, because these strategies may lead to an outcome that does not bear out the individual's optimism, regardless of her actions. Consider, for example, an agenda with the structure given in Figure 5.11a. If individual 1's preferences are given by $x_1 \succ_1 x_2 \succ_1 x_3$, then her optimistically sincere vote at the start is *No*. However, suppose that half of the remaining individuals vote *Yes* at the start and half vote *No*, and all of them vote *No* after the ballot outcome *No*. Then if individual 1 votes *Yes* at the start, the outcome is x_2, and if she votes *No* it is x_3 regardless of her subsequent vote, so that her best action is to vote *Yes*.

An optimistically sincere strategy is not in general an individual's best response to the other individuals' strategies even if these strategies are themselves optimistically sincere. Suppose, in the example, that there are three individuals, and individual 2's preferences are given by $x_2 \succ_2 x_3 \succ_2 x_1$ while individual 3's are given by $x_3 \succ_3 x_2 \succ_3 x_1$. Then at the start, individual 2's optimistically sincere vote is *Yes* while individual 3's is *No*, and after the ballot outcome *No* both of their optimistically sincere votes are *No*. Thus the pattern of their votes is the one specified in the previous paragraph, so that individual 1's best action at the start is *Yes* rather than her optimistically sincere vote of *No*.

However, for binary agendas that satisfy two further conditions, an optimistically sincere vote *is* an individual's best action whenever the other individuals' votes are optimistically sincere. The conditions are that there is a linear ordering of the alternatives with respect to which (*i*) the preference profile is single-peaked and (*ii*) the agenda is convex. Convexity means that for each nonterminal sequence h of ballot outcomes and each $p \in \{Yes, No\}$, whenever the set $A(h, p)$ of possible outcomes contains some alternatives x_r and x_s it contains also all alternatives between x_r and x_s according to the ordering. That is, at each stage, the possible outcomes of any majority decision are adjacent in the ordering. For example, if everyone's preferences are single-peaked with respect to the ordering (*low*, *medium*, *high*) for the funding level for a project, a vote between low funding on the one hand and medium or high funding on the other is possible, but one between medium funding on the one hand and low or high funding on the other is not.

Definition 5.14: Convex binary agenda

Let $\langle N, X, \succsim \rangle$ be an odd-strict collective choice problem and let $\trianglerighteq$ be a linear order on X. A binary agenda $\langle \langle N, X, \succsim \rangle, Z, O \rangle$ is *convex with respect to* $\trianglerighteq$ if for every nonterminal sequence h of ballot outcomes, each $p \in \{Yes, No\}$, and any alternatives x_r and x_s, if $A(h, p)$ contains x_r and x_s then it contains every alternative x_t with $x_r \lhd x_t \lhd x_s$, where A is given in (5.5).

The agenda in Figure 5.11a is not convex with respect to the ordering $x_1 \lhd x_2 \lhd x_3$, but the one in Figure 5.11b is. The successive agenda in Figure 5.6 is convex with respect to the ordering $x_1 \lhd x_2 \lhd \cdots \lhd x_4$, but the amendment agenda in Figure 5.7 is not (e.g. $A(\varnothing, Yes)$ contains x_1 and x_3 but not x_2).

Consider a binary agenda in which the set of alternatives is $\{x_1, x_2, \ldots, x_k\}$. Suppose that it is convex with respect to a linear order $x_1 \lhd x_2 \lhd \cdots \lhd x_k$. Then for some integer $r \in \{2, \ldots, k\}$ the vote at the start of the game leads to either the set of outcomes $\{x_1, x_2, \ldots, x_{r-1}\}$ or the set $\{x_r, x_{r+1}, \ldots, x_k\}$. In each case, each subsequent vote further splits the set of remaining outcomes into two, so that after every ballot the choice is between sets of the form $\{x_l, x_{l+1}, \ldots, x_{m-1}\}$ and $\{x_m, x_{m+1}, \ldots, x_n\}$ where $l < m \leq n$.

Now consider the binary agenda in Figure 5.11b. This agenda is convex with respect to the ordering $x_1 \lhd x_2 \lhd x_3$. Suppose that individual 1's preferences are given by $x_2 \succ_1 x_3 \succ_1 x_1$, so that her optimistically sincere vote at the start is *No*, but if a majority votes *Yes* after the ballot outcome *No* then she prefers the outcome of a majority vote of *Yes* at the start. Is it possible that individual 1's vote at the start is pivotal, so that her optimal vote there is *Yes*, rather than her optimistically sincere vote of *No*? If the other individuals' preferences are single-peaked with respect to the ordering of the alternatives and their strategies are optimistically sincere, the answer is no. For individual 1's vote to be pivotal at the start, one of the remaining individuals must vote *Yes* and one must vote *No*. If their strategies are optimistically sincere, one must thus prefer x_1 or x_2 to x_3 and the other must prefer x_3 to both x_1 and x_2. Given that their preferences are single-peaked with respect to the ordering $x_1 \lhd x_2 \lhd x_3$, the latter individual's preferences are $x_3 \succ_i x_2 \succ_i x_1$. Hence she votes *No* after the ballot outcome *No*, so that if individual 1 also votes *No*, the outcome is x_2. We conclude that given the other individuals' (optimistically sincere) strategies, individual 1's optimistically sincere strategy is best for her.

The next result generalizes this example. It shows, in addition, that the outcome of the profile of optimistically sincere strategies is the strict Condorcet winner of the collective choice problem. The argument for this second conclusion is simple. Given that the preference profile is single-peaked, the strict Condorcet winner of the collective choice problem is the median of the individuals' favorite alternatives (Proposition 1.4). Thus if every individual uses her optimistically sincere strategy then at the start of the game the winning option is the one that leads to the set of outcomes that contains the strict Condorcet winner. The same is true of the winning option after every subsequent sequence of ballot outcomes, so that the strict Condorcet winner is the ultimate outcome.

Proposition 5.10: Optimistically sincere voting in convex partitional binary agenda with single-peaked preferences

Consider a partitional binary agenda for an odd-strict collective choice problem. If for some linear order on the set of alternatives the preference profile for the collective choice problem is single-peaked with respect to the order and the agenda is convex with respect to the order then the strategy profile in which every individual's strategy is optimistically sincere is a subgame perfect equilibrium and the outcome of this equilibrium is the strict Condorcet winner of the collective choice problem.

Proof

Denote by k the number of alternatives and by $\trianglerighteq$ the linear order with respect to which the preferences are single-peaked and the agenda is convex. Label the alternatives so that $x_1 \triangleleft x_2 \triangleleft \cdots \triangleleft x_k$. Assume, contrary to the result, that i's optimal vote after some sequence h of ballot outcomes is not optimistically sincere. Specifically, suppose that i's optimistically sincere vote is *No* but she prefers the outcome of voting *Yes* to that of voting *No*, given that the other individuals' strategies are optimistically sincere. Because the agenda is partitional and convex, either $A(h, Yes) = \{x_q, x_{q+1}, \ldots, x_{r-1}\}$ and $A(h, No) = \{x_r, x_{r+1}, \ldots, x_s\}$ for some alternatives x_q, x_r, and x_s with $x_q \triangleleft x_r \triangleleft x_s$, or the roles of *Yes* and *No* are reversed. Suppose, without loss of generality, the former. (Refer to Figure 5.12.) The fact that i's optimistically sincere vote is *No* means that she prefers the best alternative in $\{x_r, x_{r+1}, \ldots, x_s\}$ to the best alternative in $\{x_q, x_{q+1}, , \dots, x_{r-1}\}$ and hence, given the single-peakedness of her preferences, prefers x_r to every alternative in $\{x_q, \dots, x_{r-1}\}$.

I argue that if, at h and later, i always votes for the option that contains x_r, the outcome is x_r, contradicting the supposition that i's voting *Yes* at h is optimal for her.

Given our assumption that i's voting *No* at h is not optimal for her, her switching from *No* to *Yes* must change the outcome. Thus the votes at h of the other individuals must be split equally between *Yes* and *No*. Given that these individuals' votes are by assumption optimistically sincere, the favorite alternatives of half of them are thus at most x_{r-1}.

Now suppose that i votes *No* at h, so that *No* wins. At the next vote, the possible outcomes are $\{x_r, x_{r+1}, , \dots, x_{t-1}\}$ and $\{x_t, x_{t+1}, \dots, x_s\}$ for some $t \in \{r+1, \dots, s\}$. Given that the favorite alternatives of half of the other individuals are at most x_{r-1}, at least half of them are at most x_{t-1}, so that given that these individuals' votes are optimistically sincere, at least half of them vote *Yes* at (h, No). Thus if i votes *Yes*, this option wins. Similarly, if i votes for the option for which x_r is a possible outcome in every subsequent ballot, that option wins, so that the ultimate outcome is x_r. But she prefers x_r to every alternative in $\{x_q, x_{q+1} \dots, x_{r-1}\}$, contradicting the supposition that her voting *Yes* at h is optimal.

I now argue that the outcome of the subgame perfect equilibrium is the strict Condorcet winner of the collective choice problem. At the start, the convexity of the agenda means that a majority of individuals vote optimistically sincerely for the option for which the strict Condorcet winner is a possible outcome. Thus this alternative is a possible outcome for one of the options in the next ballot, when the same argument implies that a majority votes for the option for which the strict Condorcet winner is a possible outcome. Repeating this process leads to the conclusion that the outcome of the subgame perfect equilibrium is the strict Condorcet winner.

Exercise 5.9: Pessimistic sincerity

Not everyone is an optimist. Is the variant of Proposition 5.10 in which "optimistically sincere" is replaced by "pessimistically sincere" true?

This result does not depend on the assumption that the agenda is partitional (Kleiner and Moldovanu 2017, Theorem 2). For a non-partitional agenda, the two possible ballot outcomes at any point may lead to sets of outcomes that intersect, like $\{x_1, x_2\}$ and $\{x_2, x_3\}$. The notion of optimistic sincerity may be extended to

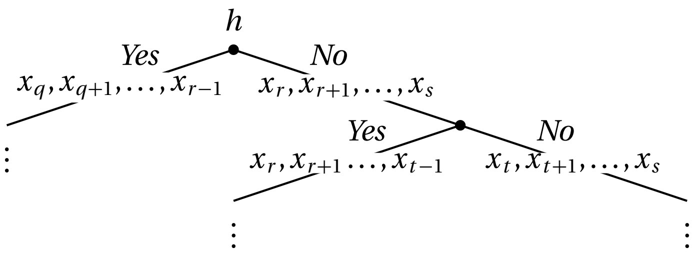


Figure 5.12 The part of an agenda following the history h used in the proof of Proposition 5.10.

deal with such cases by assuming that if an individual is indifferent between the best outcome in one set and the best outcome in the other, she makes a choice by comparing the second best outcomes in the sets, and if these second best outcomes are the same, she compares the third best outcomes, and so forth.

Given that an individual can implement an optimistically sincere strategy without knowing more than her own preferences and the structure of the procedure, the result may be seen as suggesting an advantage of a convex agenda: the subgame perfect equilibrium in which every individual uses an optimistically sincere strategy generates a desirable outcome and is not affected by the quality of the information the individuals have about each other.

Notes

The study of binary agendas was initiated by Black (1948a,b, 1958) and Farquharson (1969) (which was completed in 1958). Farquharson was the first person to study strategic behavior in agendas using tools from game theory; the names successive agenda and amendment agenda are his (Farquharson 1969, 61). My statements at the start of Sections 5.2 and 5.3 about the correspondence between agenda types and the parliamentary procedure in various countries are based on evidence assembled by Rasch (2000, Table 1). This evidence has led some people to refer to successive agendas as Euro-Latin and amendment agendas as Anglo-American. It has also been questioned: see Schwartz (2008, 368) and Horan (2021, 236–237).

The top cycle set was first used to analyze voting by Ward (1961) (who calls it the "majority set"). Lemma 5.2 is due to Camion (1959). Proposition 5.3a is due to McKelvey and Niemi (1978, Corollary 2). Proposition 5.5, and thus Proposition 5.3b, is due to Miller (1977, Proposition 6).

The Banks set, Proposition 5.6, and Proposition 5.8 are due to Banks (1985). Proposition 5.7 is due to Shepsle and Weingast (1984, Theorem 1); see also Moulin (1986, Theorem 3).

Section 5.4 is based on Shepsle and Weingast (1984); Proposition 5.9 is their Theorem 3.

Section 5.5 is based on Kleiner and Moldovanu (2017); Proposition 5.10 is due to them. The notion of an optimistically sincere strategy was introduced by Farquharson (1969, Chapter 3) for a general (not necessarily partitional) binary agenda. He uses the term "sincere", and others follow him; the term "lexicographically maximax" is used by some.

The uncovered set (Exercise 5.8) was independently suggested by Miller (1977) and Fishburn (1977) (C_9, p. 473). Duggan (2013) analyzes several closely related notions, some of which were suggested before the work of Miller and Fishburn.

For the rationale for naming the concept of a Copeland winner (Definition 5.5), see the discussion of Example 1.7 in the Notes for Chapter 1. Example 5.1 is taken from Moulin (1986, 274) (see also Fishburn 1977, 478). Exercise 5.3 is taken from Moulin (1986, 284).

Solutions to exercises

Exercise 5.1

Denote the top cycle set by T.

a. If $a \in T$ and b beats a, then b indirectly beats all alternatives that a indirectly beats, and hence $b \in T$. So if b is not in T then a beats it.

b. Now let T' be a proper subset of T with the property that every alternative in T' beats every alternative outside T'. Let $a \in T'$ and $b \in T \setminus T'$. Given that $b \in T$, b indirectly beats a; say b beats u_1 beats u_2 … beats u_{k-1} beats u_k beats a. Then $u_k \in T'$ because $a \in T'$ and hence beats every alternative outside T'. By the same argument, $u_{k-1} \in T'$, and hence $u_j \in T'$ for $j = 1, \ldots, k$. But then $b \in T'$ also, contrary to the assumption that $b \in T \setminus T'$. Hence no proper subset of T has the property that every alternative in it beats every alternative outside it.

Now let $a \in T$ and suppose that b beats a indirectly. Then for some alternatives $z_1, \ldots, z_p$, b beats z_1 beats … beats z_p beats a. If $z_p \notin T$ then by part (*a*), a beats it, so in fact $z_p \in T$. Similarly, $z_{p-1} \in T$ and hence … $z_1 \in T$ and hence $b \in T$.

Exercise 5.2

For a problem without a strict Condorcet winner, the top cycle set contains at least two alternatives. Suppose that x and y are both in the top cycle set. They cannot both beat each other (directly), so one must beat the other indirectly. Suppose that y beats z_1 beats … beats z_l beats x. Then every z_j beats

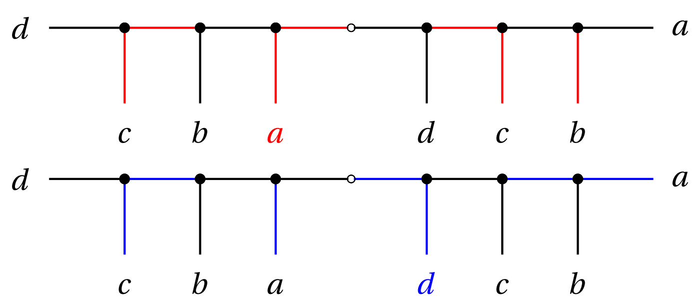


Figure 5.13 The agenda in Exercise 5.3. (Note that the start is in the middle of the diagram.) The outcomes of the votes for the original preference profile are shown at the top, in red. Those for the modified preference profile are shown at the bottom, in blue.

(indirectly) every alternative that x beats, so that they are all in the top cycle set. Thus the top cycle set contains at least one alternative in addition to x and y.

Exercise 5.3

The agenda is shown in Figure 5.13. For the original preference profile, the outcome of sophisticated voting is a. After a rises in individual 1's preferences, positive responsiveness requires that the outcome remains a, but it changes to d (which is worse for individual 1).

Exercise 5.4

a. By Proposition 5.4, if x_k is the outcome of sophisticated voting then every member of the sophisticated sequence is x_k, which means that x_k beats every other alternative, and hence is a strict Condorcet winner.

b. The fact that x_l is the outcome of sophisticated voting in B means that the sophisticated sequence for B takes the form $(x_l, x_l, \ldots, x_l, x^*_{l+1}, \ldots, x^*_k)$. In particular, x_l beats x_{l-1}. Now let $(y^*_1, \ldots, y^*_k)$ be the sophisticated sequence for the agenda $B' = \langle\langle N, X, \succsim\rangle, (y_1, \ldots, y_k)\rangle$. If x_{l-1} beats x^*_{l+1} then $y^*_l = x_{l-1}$, and given that x_l beats x_{l-1}, $y^*_{l-1} = x_l$, so that $y^*_j = x_l$ for all $j = 1, \ldots, l-1$. If x^*_{l+1} beats x_{l-1} then $y^*_l = x^*_{l+1}$ and hence again $y^*_j = x_l$ for all $j = 1, \ldots, l-1$. Thus in both cases the outcome of sophisticated voting in B' is x_l.

Exercise 5.5

If for some alternative b, every individual prefers b to a, then b beats a and every alternative that a beats, so that no sequence of alternatives satisfies the second and third conditions of Definition 5.9.

Exercise 5.6

Let a be the outcome of sophisticated voting for the amendment agenda $B = \langle\langle N, X, \succsim\rangle, (x_1, \ldots, x_k)\rangle$. By Proposition 5.8, a is in the Banks set, so that there exists a sequence $(x_1, x_2, \ldots, x_l)$ of alternatives satisfying the conditions in Definition 5.9. Let i be an individual, let $a \in X$, and let $\succsim'$ be a preference profile that differs from $\succsim$ only in that a is ranked higher by $\succsim'_i$ than it is by $\succsim_i$. Then for the problem $\langle N, X, \succsim' \rangle$, the sequence $(x_1, x_2, \ldots, x_l)$ satisfies the conditions in Definition 5.9, since a still beats every x_j, every x_j beats the same set of x_i's as it did for $\succsim$, and no alternative except possibly a beats any alternative that it did not beat for $\succsim$.

Exercise 5.7

a. By Proposition 5.7 and (5.2), if x_k is the outcome of sophisticated voting, then it beats every other alternative, and hence is the strict Condorcet winner.

b. Denote the sophisticated sequence for B by $(x_1^*, \ldots, x_k^*)$. Given that x_l is the outcome of sophisticated voting in B, by Proposition 5.7 we have $x_1^* = \cdots = x_l^* = x_l$.

	x_1	$\cdots$	x_{l-2}	x_{l-1}	x_l	x_{l+1}	$\cdots$	x_k
sophisticated sequence:	x_l	$\cdots$	x_l	x_l	x_l	x_{l+1}^*	$\cdots$	x_k^*

By (5.2), x_l beats every alternative x_{l+1}^*, ..., x_k^* and every alternative x_1, ..., x_{l-1} is beaten by some member of $\{x_l, x_{l+1}^* \ldots, x_k^*\}$.

Now use (5.2) to calculate the sophisticated sequence $(y_1^*, \ldots, y_k^*)$ for $\langle\langle N, X, \succsim\rangle, (y_1, \ldots, y_k)\rangle$. Given that $y_j = x_j$ for $j = l+1, \ldots, k$, we have $y_j^* = x_j^*$ for $j = l+1, \ldots, k$.

If x_{l-1} beats x_{l+1}^*, ..., x_k^*, so that $y_l^* = x_{l-1}$, then given that x_{l-1} is beaten by some member of $\{x_l, x_{l+1}^* \ldots, x_k^*\}$, x_l beats x_{l-1}, and consequently $y_j^* = x_l$ for $j = 1, \ldots, l-1$.

	x_1	$\cdots$	x_{l-2}	x_l	x_{l-1}	x_{l+1}	$\cdots$	x_k
sophisticated sequence	x_l	$\cdots$	x_l	x_l	x_{l-1}	x_{l+1}^*	$\cdots$	x_k^*

If x_{l-1} is beaten by one of the alternatives x_{l+1}^*, ..., x_k^*, then $y_l^* = x_{l+1}^*$, and again $y_j^* = x_l$ for $j = 1, \ldots, l-1$.

	x_1	$\cdots$	x_{l-2}	x_l	x_{l-1}	x_{l+1}	$\cdots$	x_k
sophisticated sequence	x_l	$\cdots$	x_l	x_l	x_{l+1}^*	x_{l+1}^*	$\cdots$	x_k^*

In both cases $y_1^* = x_l$, so that x_l is the outcome of sophisticated voting in $\langle\langle N, X, \succsim\rangle, (y_1, \ldots, y_k)\rangle$.

Exercise 5.8

a. If x is in the uncovered set then there is no alternative y such that y beats x and every alternative that beats y also beats x. Thus for every alternative $y \neq x$, either x beats y or there is an alternative z that beats y but is beaten by x. That is, for every alternative $y \neq x$, either x beats y or there is an alternative

z such that x beats z and z beats y.

b. Suppose that x is not in the uncovered set. Then some other alternative y beats both x and every alternative that x beats. Thus no sequence $z_1, z_2, \ldots, z_l$ satisfies the second and third points in the definition of the Banks set, so that x is not in the Banks set.

Exercise 5.9

No. Consider the binary agenda with the procedure shown in Figure 5.11b, which is convex with respect to the ordering $x_1 \lhd x_2 \lhd x_3$, and suppose that the individuals' preferences are given by $x_2 \succ_1 x_3 \succ_1 x_1$, $x_2 \succ_2 x_3 \succ_2 x_1$, and $x_1 \succ_3 x_2 \succ_3 x_3$, which are all single-peaked with respect to the ordering. Then individual 2's pessimistically sincere strategy is to vote *Yes* at the start and *No* after the ballot outcome *Yes*, and individual 3's pessimistically sincere strategy is to vote *No* at the start and *Yes* after the ballot outcome *Yes*. Individual 1's best response to these strategies is to vote *No* at the start, but her pessimistically sincere strategy is to vote *Yes*.

6 Ethical voting and expressive voting

6.1 Ethical voting 192
6.2 Expressive voting 197

Each individual in the models of voting in the previous chapters is motivated by the possibility that her vote will increase the probability that the outcome is one that she likes. This chapter explores two other motivations. The first model is one possible formulation of the idea that individuals' voting decisions are driven by ethical concerns. The second model retains the assumption that individuals are self-interested but assumes that they vote because they derive satisfaction from expressing their opinions, regardless of whether doing so has any effect on the outcome of the election.

In each of these models the motivation to vote is not related to the size of the electorate, so that turnout does not necessarily decline as the size of the electorate increases, as it does in the models in Sections 3.2 and 3.3.

Synopsis

In the model of ethical voting in Section 6.1, individuals differ both in the alternative they believe to be the best for society and in their voting cost. Each individual chooses a rule for casting a vote as a function of voting cost that, if adopted by all individuals who agree with her regarding the best alternative for society, would produce the best outcome for society, in her view, given the rules used by individuals with other views, taking into account everyone's voting cost. There is a continuum of individuals, so that no individual's vote affects the outcome of the election, but the individuals' ethical motivations lead a positive mass of individuals to vote in any equilibrium. Example 6.1 shows one possible way in which the equilibrium may vary with the parameters of the model.

The idea that people vote to express their beliefs and to affirm their political identity has been subject of little formal modeling. Section 6.2 briefly discusses the idea and presents a simple model of one facet of it: the benefit an individual derives from expressing her beliefs by voting for one candidate rather than another depends on the proportion of the population supporting that candidate.

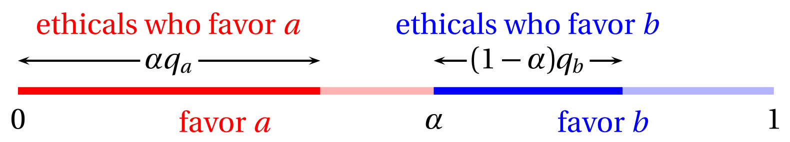


Figure 6.1 The structure of the set of individuals in a two-alternative plurality rule voting problem with ethical individuals. One realization of α (the fraction of individuals who favor a), q_a (the fraction of ethical individuals among a-individuals), and q_b (the fraction of ethical individuals among b-individuals), which are random variables, is shown.

6.1 Ethical voting

A society selects alternative a or alternative b by casting votes, with the majority determining the winner. Each individual can vote for one of the alternatives or abstain. Some individuals believe that a is the best alternative for society, while others believe that b is the best alternative. For brevity, call the former a-individuals and the latter b-individuals. Any individual who votes incurs a cost. Apart from differences in these costs, for each alternative z all z-individuals are identical. Each individual chooses a rule for casting a vote as a function of voting cost that, *if used by all individuals like her*, generates what she believes is the best outcome for society, given the other individuals' behavior.

For a model in which the individuals are self-interested, the discussion in Section 3.1.2 concludes that a deterministic model of costly voting is unlikely to have an equilibrium. Similar considerations suggest the same conclusion when individuals act ethically. So we incorporate uncertainty in the model: as for a two-alternative voting game with uncertain voting costs, we assume that while each individual knows her own voting cost, she knows only the distribution from which the other individuals' costs are drawn. This assumption generates uncertainty, but the amount it generates converges to zero as the number of individuals increases. We want to model a large population, and a convenient way to do so is to assume that there are infinitely many individuals, with the set of individuals equal to the unit interval $[0,1]$. In this case, the assumption that each individual knows the distribution from which the other individuals' costs are drawn means that she knows the actual distribution of costs in the population, and the problem of the nonexistence of an equilibrium reappears. To generate uncertainty that persists in a large population, we assume also that the fraction α of the population consisting of a-individuals is uncertain, and that within the sets of a-individuals and b-individuals, not all individuals behave ethically; the fractions q_a and q_b that do so are randomly-determined, with the remainder self-interested. (Refer to Figure 6.1.)

Given the continuum of individuals, no individual's vote affects the outcome of the election, and I assume that as a consequence none of the self-interested individuals vote. A voting rule assigns an action (vote for a, vote for b, abstain) to each possible value of an ethical individual's voting cost. For each alternative z, each ethical z-individual chooses the voting rule that, if used by all ethical z-individuals, generates what she believes is the highest social welfare, given the voting rules chosen by the individuals who believe the other alternative is best, taking into account *all* individuals' voting costs.

In every equilibrium of a two-alternative voting game with uncertain voting costs studied in Section 3.2, every individual i uses a threshold voting rule: there is a number $\bar{c}_i$ such that i votes for her favored alternative if her voting cost is at most $\bar{c}_i$ and abstains otherwise. The logic behind this conclusion applies also in the current context, and for simplicity I restrict individuals to such rules. That is, for each alternative z, each ethical z-individual chooses a number c_z such that if every ethical z-individual votes for z when her voting cost is at most c_z and abstains otherwise, then given the threshold chosen by the individuals who favor the other alternative, her evaluation of social welfare, taking into account the total cost of voting, is maximized.

Suppose that the fraction α of the population consisting of a-individuals is a draw from a distribution function H on $[0,1]$ and for each alternative z the fraction q_z of z-individuals who are ethical is an independent draw from a nonatomic distribution function G on $[0,1]$ that is also independent of α. Suppose also that for some number $\bar{c} > 0$, for each alternative z and each number $c \in [0,\bar{c}]$ the fraction of z-individuals with voting cost at most c is $F(c)$, where F is a nonatomic probability distribution function on $[0,\bar{c}]$ that has a density.

Under these assumptions, if the voting thresholds for ethical individuals are c_a and c_b, the probability that a wins is the probability that the fraction $\alpha q_a F(c_a)$ of individuals—those who favor a, are ethical, and have a voting cost at most c_a—is greater than the fraction $(1-\alpha)q_b F(c_b)$—those who favor b, are ethical, and have a voting cost at most c_b—plus half the probability that these two fractions are equal. If $F(c_a) = F(c_b) = 0$ then the fractions are equal, and otherwise the probability that they are equal is zero, so the probability that a wins when the thresholds are c_a and c_b is

$$\Pr(a \text{ wins} \mid c_a, c_b) = \begin{cases} \frac{1}{2} & \text{if } F(c_a) = F(c_b) = 0 \\ \Pr\left(\alpha q_a F(c_a) \geq (1-\alpha) q_b F(c_b)\right) & \text{otherwise.} \end{cases} \tag{6.1}$$

The expression for the probability that b wins is analogous.

How does an individual evaluate social welfare? Assume that for each alternative z, each ethical z-individual believes that the welfare of every individual is w_z if the outcome is z and 0 if it is the other alternative, minus the individual's voting cost if she votes. An individual may construct an index of social welfare from these individual welfares in various ways. Section 1.8 discusses social welfare orderings in general, and the leximin, utilitarian, and Nash orderings in particular. Here, following Feddersen and Sandroni (2006a), I assume that individuals use the utilitarian ordering, which ranks outcomes according to the sum of the individuals' welfares. Adapted to the current model, with a continuum of individuals of measure 1 and uncertainty, this assumption means that each ethical z-individual assigns to the pair (c_a, c_b) of voting thresholds the social welfare

$$u_z(c_a, c_b) = w_z \Pr(z \text{ wins} \mid c_a, c_b) - C(c_a, c_b), \tag{6.2}$$

where the probability is given by (6.1) and $C(c_a, c_b)$, the expected cost of voting, is the fraction $E(\alpha q_a)$ of individuals who are ethical and favor a times the expected cost of voting for these individuals, plus the analogous expression for individuals who favor b:

$$C(c_a, c_b) = E(\alpha q_a) \int_0^{c_a} c \, dF(c) + E((1-\alpha) q_b) \int_0^{c_b} c \, dF(c). \tag{6.3}$$

The theory is that each a-individual chooses c_a to maximize her evaluation $u_a(c_a, c_b)$ of social welfare ((6.2) for $z = a$) given c_b, and each b-individual chooses c_b to maximize $u_b(c_a, c_b)$ given c_a. That is, (c_a, c_b) is a Nash equilibrium of a two-player game in which the payoff functions are u_a and u_b. To specify the game precisely, I first collect the elements of the model in the following definition.

Definition 6.1: Two-alternative plurality rule voting problem with ethical individuals

A *two-alternative plurality rule voting problem with ethical individuals* $\langle [0,1], \{a, b\}, H, G, F, (w_a, w_b) \rangle$ consists of

- $[0,1]$ (the set of individuals)
- $\{a, b\}$ (the set of alternatives)
- H, a probability distribution function with support $[0,1]$ (the distribution of the fraction α of individuals who favor a)
- G, a nonatomic probability distribution function with support $[0,1]$ (the distribution of the fractions q_a of a-individuals and q_b of b-individuals who are ethical)
- F, a nonatomic probability distribution function on some interval $[0, \bar{c}]$, where $\bar{c} > 0$, that has a density (the distribution of the individuals' voting costs)

- w_a and w_b, positive numbers (the weights assigned to outcomes by each type of ethical individual).

The strategic game associated with a two-alternative plurality rule voting problem with ethical individuals is defined as follows.

Definition 6.2: Strategic game for two-alternative plurality rule voting problem with ethical individuals

Let $\langle [0,1], \{a,b\}, H, G, F, (w_a, w_b)\rangle$ be a two-alternative plurality rule voting problem with ethical individuals. The strategic game associated with this problem has the following components.

Players
The set of players is $\{a,b\}$.

Actions
The set of actions of each player is $[0,\bar{c}]$, the support of F (the set of possible cost thresholds for voting).

Payoffs
The players' payoff functions are u_a and u_b defined in (6.2), where $\Pr(z \text{ wins} \mid c_a, c_b)$, in which α, q_a, and q_b are independent draws from H, G, and G respectively, is given in (6.1), and the function C is given in (6.3).

Not every such game has a Nash equilibrium. One approach to finding conditions for the existence of an equilibrium appeals to the sufficient conditions for the existence of a Nash equilibrium in a general strategic game in Proposition 16.4: each player's set of actions is a nonempty compact convex subset of a Euclidean space and each player's payoff function is continuous and quasiconcave in her action for any given action of the other player. This approach is taken by Feddersen and Sandroni (2006a), who apply it to a transformation of the game in Definition 6.2 in which the strategic variables are the fractions $F(c_a)$ and $F(c_b)$ of ethical individuals of each type who vote rather than the thresholds c_a and c_b. Given that F is one-to-one, a pair (c_a^*, c_b^*) is an equilibrium of the game in Definition 6.2 if and only if $(F(c_a^*), F(c_b^*))$ is an equilibrium of the transformed game. Each player's set of actions in the transformed game is $[0,1]$, a nonempty compact convex set. Each player's payoff function, however, is not continuous at $(0,0)$. If $F(c_a) = F(c_b) = 0$ (no one votes) then the election is a tie. But if one of these numbers, say $F(c_a)$, is positive, while the other is zero, then given the assumptions about the distributions F, G, and H, the probability that some individuals favor a, are ethical, and vote is 1, so that the probability that a wins is 1.

Thus a straightforward application of Proposition 16.4 is not possible. One way to avoid the problem is to consider the existence of an equilibrium for a variant of the game in which the set of actions of each player is $[\varepsilon, 1]$ for some $\varepsilon > 0$ and then study the equilibria as ε approaches 0. The main remaining condition required by Proposition 16.4 is the quasiconcavity of each player's payoff function in her own action. Feddersen and Sandroni (2006a) show that if G, the common distribution function of q_a and q_b, is concave on its support, then this condition is satisfied, and this property of G is sufficient also for the existence of a Nash equilibrium in the game in which each individual's set of actions is $[0,1]$ rather than $[\varepsilon, 1]$.[1]

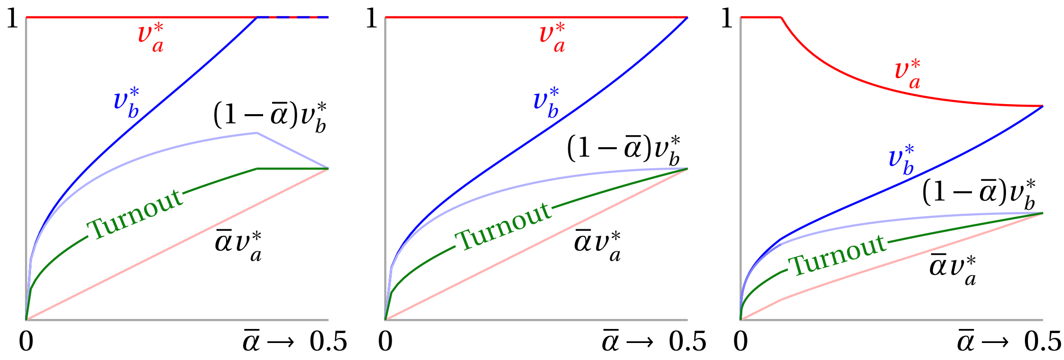
The fraction of the population that votes in an equilibrium depends on the distributions H, G, and F, and the numbers w_a and w_b. Here is an example.

Example 6.1: Voting problem with ethical individuals

Consider a two-alternative plurality rule voting problem with ethical individuals $\langle [0,1], \{a,b\}, H, G, F, (w_a, w_b) \rangle$ in which H (the distribution of the fraction of individuals favoring a) assigns probability 1 to one value, denoted $\overline{\alpha}$, G (the distribution of the fractions q_a and q_b) is uniform on $[0,1]$, F (the distribution of voting costs) is uniform on $[0, \overline{c}]$, and $w_a = w_b = w$. The strategic game associated with this problem has a unique Nash equilibrium, which can be calculated explicitly. The fractions v_a^* and v_b^* of a- and b-individuals who vote in this equilibrium are illustrated in Figure 6.2 as a function of $\overline{\alpha}$ for $\overline{\alpha} \in [0, 0.5]$ and various values of $\overline{c}/w$, the ratio of the upper limit of the cost of voting to the weight in the payoff function on the probability of winning. In the cases shown, $v_a^* \geq v_b^*$: the fraction of a-individuals (a minority of all individuals) who vote is at least the fraction of b-individuals who do so. However, the number (measure) $\overline{\alpha} v_a^*$ of a-individuals who vote is less than the number $(1-\overline{\alpha})v_b^*$ of b-individuals who do so if $\overline{\alpha} \in (0, 0.5)$, so that b wins despite the higher turnout rate among a-individuals. As the fraction $\overline{\alpha}$ of a-individuals in the population declines to 0, the turnout rate among b-individuals, and hence the overall turnout rate, approaches 0.

The properties of the equilibrium given for this example are shared by the equilibrium of the strategic game associated with any two-alternative plurality rule voting problem with ethical individuals (Feddersen and Sandroni 2005,

[1] Feddersen and Sandroni (2006a) study a more general model in which the distributions of q_a and q_b may differ. My claim follows from their Proposition 4 combined with the observation that if the distributions of q_a and q_b are the same, their Assumption A is satisfied if (and only if) G is concave.



(a) $w/\bar{c} = 1$. (b) $w/\bar{c} = 0.5$. (c) $w/\bar{c} = 0.25$.

Figure 6.2 The fractions v_a^* and v_b^* of a- and b-individuals who vote and the expected turnout for the unique Nash equilibrium of the strategic game associated with the two-alternative plurality rule voting problem with ethical individuals in Example 6.1, as a function of $\bar{\alpha}$, the fraction of individuals favoring a, for various values of $w/\bar{c}$.

Propositions 4 and 5). Other properties of the equilibria in the example, like the fact that turnout is increasing in $w/\bar{c}$, do not hold generally.

The model is one possible expression of the idea that individuals may adopt rules of behavior that they believe would produce the best outcome for society if everyone like them followed them. The example shows that its equilibrium may result in significant turnout, even though the population is large.

6.2 Expressive voting

In the words of Schuessler (2000a, 88), "at least for some voters, voting is a means to express political beliefs and preferences and, in doing so, to establish or reaffirm their own political identity". Even if the act of voting is not publicly observable, individuals may derive expressive benefits from wearing a campaign button, displaying a lawn sign, participating in campaign rallies, and letting other people know their views. Like a cheering spectator at a sporting event, an individual voter does not expect her action to affect the outcome; she votes to express her support for a candidate.

A simple model is based on the hypothesis that an individual's expressive benefit from voting for a candidate depends on the candidate's identity and the fraction of the population who support the candidate. The idea is that an individual's expressive benefit from supporting a candidate derives from being associated with the candidate's community of supporters and from being distinguished from the supporters of other candidates.

Suppose there are two candidates, 1 and 2, and denote individual i's (expressive) benefit from supporting (and voting for) candidate j by $u_i^j(q^j)$, where q^j is

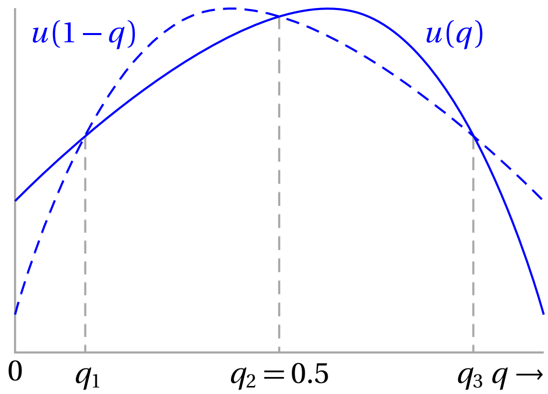


Figure 6.3 Equilibria in an example of a model of expressive voting. The value of $u(q)$ is the expressive benefit of each individual from supporting candidate 1 when the proportion of individuals who do so is q. Each individual i's cost c_i is assumed to be less than the smallest value of $u(q)$. If the proportion of individuals who support candidate 1 is q_1, q_2, or q_3, no individual has an incentive to switch her support to another candidate.

the fraction of the population that supports j. Individual i votes for candidate 1 if $u_i^1(q^1) > u_i^2(q^2)$ and $u_i^1(q^1)$ exceeds her voting cost c_i, with $q^2 = 1 - q^1$. In an equilibrium, q^1 is equal to the fraction of individuals for whom these conditions are satisfied:

$$q^1 = \text{fraction of individuals } i \text{ for whom } u_i^1(q^1) > u_i^2(1-q^1) \text{ and } u_i^1(q^1) > c_i.$$

As a simple example, suppose that the benefit from voting for a candidate j depends only on q^j, not on the identities of the candidate or the individual. Denote it $u(q^j)$. Then if $u(q^1) > u(1-q^1)$, all individuals vote for candidate 1, and if $u(q^1) < u(1-q^1)$ they all vote for candidate 2. Thus in an equilibrium $u(q^1) = u(1-q^1)$. In particular, regardless of the form of u, one equilibrium is $q^1 = 0.5$.

If an individual's expressive benefit from supporting a candidate derives from being identified with the other individuals supporting the candidate and distinguished from those supporting the other candidate, then the expressive benefit from supporting a candidate is small or nonexistent if no one supports the candidate or everyone does so. Thus we might expect $u(q^j)$ to initially increase as q^j increases from 0 and ultimately decrease as q^j approaches 1, as in Figure 6.3. In the example shown, in which each individual's cost c_i is assumed to be less than the smallest value of $u(q^j)$, the model has two asymmetric equilibria in addition to the equilibrium $q = 0.5$.

If an individual's expressive benefit from voting for a candidate depends on the identities of the individual and the candidate, as well as the proportion of the population supporting the candidate, then 0.5 may not be an equilibrium. If some or all of the costs c_i exceed $u(q)$ for some values of q, then in an equilibrium some individuals may not vote. But there is no reason for the proportion of such individuals to increase with the size of the population, as it does in the models in Sections 3.2 and 3.3.

This model captures only one facet of the idea that people are motivated to vote, at least in part, by the desire to express their beliefs. To date, other facets of the idea have not been expressed in formal models.

Notes

Harsanyi (1977b, Section 7) first explored a model in which individuals choose a voting rule that, if adopted by everyone, would be best for society. (See also Harsanyi 1977a, 1980.) Section 6.1 is based on Feddersen and Sandroni (2005, 2006a). Example 6.1 is the subject of Feddersen and Sandroni (2006b). Coate and Conlin (2004) study a closely related model. The main differences between their model and that of Feddersen and Sandroni are that they make a specific assumption about the distribution of the fraction of individuals who favor each alternative and assume that the payoff of each side includes only the voting costs borne by that side, not the costs borne by the individuals who favor the other alternative. Thus in their model each ethical individual chooses the voting rule that, if adopted by all members of her group, is best for her group, given the rule used by the other group.

Section 6.2 is based on Schuessler (2000a,b). Hamlin and Jennings (2011, 2019) discuss the idea of expressive voting informally.

7 Voting with asymmetric information

7.1 Strategic abstention 203
7.2 Unanimity rule 226
7.3 Majority rule with diverse preferences: an example 233

The individuals in the models of voting in the previous chapters disagree about the desirability of the alternatives. A central question is how the outcome of a voting system depends on their preferences. The individuals in the models in this chapter differ in their information about the state of nature, which determines the desirability of each alternative. A central question is how the outcome of a voting system depends on this information.

Synopsis

The model in Section 7.1.2 is intended to capture the idea that a poorly-informed individual may abstain because she thinks the decision is better left to well-informed individuals. In the model, there are two alternatives, a and b, and two states, α and β. Some individuals, called partisans, prefer one of the alternatives regardless of the state, and others, called independents, agree on the alternative best in each state: they prefer a in state α and b in state β. Among the independents, some individuals know the state and others do not. Proposition 7.1 shows that in any equilibrium, every partisan votes for the alternative she favors, every informed independent votes for the alternative she favors given the state, and uninformed independents vote so as to cancel out, as far as possible, the partisans' votes. If there are more than enough uninformed independents to completely cancel out the partisans' votes, then in some equilibria some of them abstain. Their behavior puts the decision in the hands of the informed independents as much as possible, and results in the same outcome as does an equilibrium of the variant of the game in which every individual is informed of the state. This result is sometimes expressed by saying that the equilibrium fully aggregates information.

The analysis in Section 7.1.3 shows that the implication of Proposition 7.1 that poorly-informed individuals may abstain does not depend on these individuals being completely uninformed and facing others who are perfectly informed. In the model, each individual observes a signal about the state, the quality of which is drawn randomly from a given distribution, independently of the qualities of the other individuals' signals. Each individual observes her own signal and its quality, but not the other individuals' signals or signal qualities. All individuals agree on the best alternative in each state. Proposition 7.2 shows that if there are two individuals and the states are equally likely, the game has an equilibrium in which each individual votes if and only if the quality of her signal is at least equal to some threshold. Thus an individual whose signal is informative but low-quality abstains, ceding the decision to the other individual, whose signal she expects to have higher quality. If the individuals were to make a decision by pooling their signals, taking into account the quality of each signal, then each individual would optimally contribute her signal. But instead they are making a decision by voting, which provides no means by which to convey the quality of a signal. An individual with an informative but low-quality signal does not vote because if she did her signal would effectively be given the same weight as the other individual's signal, the quality of which is likely to be higher than hers.

Section 7.2 studies unanimity rather than plurality rule. Consider the variant of the model in Section 7.1.2 in which the outcome is a unless all individuals vote for b. In this case, in every equilibrium every uninformed individual votes for b, because doing so is the only way she can hand the decision to the informed individuals: if she votes for a then the outcome is a regardless of the other individuals' votes. Similar considerations in a variant of the model in Section 7.1.3, in which the individuals are a priori identical, lead to the conclusion that when the number of individuals is large, the strategy profile in which every individual votes for the alternative that is more likely to be best according to her own signal is not an equilibrium. The reason is that when everyone else votes in this way, the remaining individual's vote affects the outcome only if all the other individuals vote for b, which happens only if all of them receive signals suggesting that b is the best outcome. But if all of them receive such signals, the probability that b is the best outcome is high even if the signal of the remaining individual suggests that a is best, so that the remaining individual should vote for b regardless of her signal. The general point is that when deciding how to cast her vote, an individual should pay attention only to the configurations of the other individuals' votes that make her vote pivotal, and should take into account the information implied by these configurations.

Section 7.3 illustrates the implications of this logic when the individuals, unlike those in the models in the previous sections, do not agree on the best alternative in each state. A group of individuals uses majority rule to choose between a policy whose outcome is known and one with many possible outcomes, each of which hurts one individual and benefits the rest. The payoffs are such that an individual who does not know the outcome of the second policy prefers that policy. Each individual is independently informed of the outcome of the second policy with a small positive probability. Even though the second policy would be chosen unanimously if no individual were informed, and would be chosen also by a majority if everyone were informed, there is an equilibrium in which an individual votes for the second policy only if she is informed of its outcome and that outcome benefits her, so that with high probability a majority vote selects the first policy. A key ingredient of each uninformed individual's decision is that if her vote is pivotal then the other individuals' votes must be evenly divided between the two policies, which means that half of them got a signal that the outcome of the second policy is good for them, making it likely that is it bad for the uninformed individual.

7.1 Strategic abstention

Faced in a voting booth with lists of choices for mayor, city councillor, and school superintendent, you realize that although you believe you know the best candidate for mayor and are reasonably confident about the merits of the candidates for city councillor, you have no idea about the candidates for school superintendent. As a consequence, you decide to abstain on the ballot for that position, hoping that other, better-informed, voters endorse the best person for the job. This section presents a model that captures this idea.

7.1.1 Example

Each of two individuals can vote for one of two alternatives or abstain. The society is in one of two states, α or β. The individuals agree that alternative a is best if the state is α and alternative b is best if the state is β. Each individual's payoff is 1 if the best alternative for the state is chosen, 0 if the other alternative is chosen, and $\frac{1}{2}$ if the alternatives tie.

The individuals differ in their information: individual 1 knows the state and individual 2 does not. Individual 2 believes that the state is α with probability 0.9 and β with probability 0.1. Each individual may vote for a, vote for b, or abstain. Neither individual incurs any cost when she votes.

An intuitive analysis suggests that individual 1, who knows the state, should vote for a in state α and b in state β. If individual 1 acts in this way, what should individual 2 do? If, absent individual 1, she were choosing an alternative by herself, she would choose a, because this alternative gives her an expected payoff of $0.9 \times 1 + 0.1 \times 0 = 0.9$, whereas b gives her an expected payoff of $0.9 \times 0 + 0.1 \times 1 = 0.1$. However, in the presence of individual 1, she can abstain, leaving the choice to individual 1, who votes for the best alternative in each state. In fact, abstention is her best option. If she votes for a then if the state is α, her vote makes no difference, and if the state is β, it changes the outcome from a win for b to a tie, making her worse off. By a similar argument, her voting for b also makes her worse off. In both cases, if her vote makes a difference, it leads to an outcome worse than abstention, an effect known as the swing voter's curse. So not only can she safely abstain, but she is better off doing so than voting.

We can model this situation as the following Bayesian game.

Players
The set of players is the set of individuals, $\{1, 2\}$.

States
The set of states is $\{\alpha, \beta\}$.

Actions
The set of actions of each individual is $\{\textit{vote for } a, \textit{vote for } b, \textit{abstain}\}$.

Signals
Individual 1 gets different signals in states α and β; individual 2 gets the same signal in both states.

Prior beliefs
Each individual assigns probability 0.9 to state α and probability 0.1 to state β.

Payoffs
The payoffs are given in the following tables, where ϕ stands for abstention and the actions a and b stand for voting for a and voting for b.

	a	b	ϕ
a	$1,1$	$\frac{1}{2},\frac{1}{2}$	$1,1$
b	$\frac{1}{2},\frac{1}{2}$	$0,0$	$0,0$
ϕ	$1,1$	$0,0$	$\frac{1}{2},\frac{1}{2}$

State α

	a	b	ϕ
a	$0,0$	$\frac{1}{2},\frac{1}{2}$	$0,0$
b	$\frac{1}{2},\frac{1}{2}$	$1,1$	$1,1$
ϕ	$0,0$	$1,1$	$\frac{1}{2},\frac{1}{2}$

State β

A player's strategy in a Bayesian game is a function that associates an action with each of her signals. So in this game, a strategy for player 1 specifies two actions: one associated with the signal generated by state α and one associated with the signal generated by state β. A strategy for player 2 is a single action (her signal conveys no information about the state). A Nash equilibrium of the game is a pair of strategies such that neither player has a strategy that increases

	a	b	ϕ
a	1,1	$\frac{1}{2},\frac{1}{2}$	1,1
b	$\frac{1}{2},\frac{1}{2}$	0,0	0,0
ϕ	1,1	0,0	$\frac{1}{2},\frac{1}{2}$

State α (prob. 0.9)

	a	b	ϕ
a	0,0	$\frac{1}{2},\frac{1}{2}$	0,0
b	$\frac{1}{2},\frac{1}{2}$	1,1	1,1
ϕ	0,0	1,1	$\frac{1}{2},\frac{1}{2}$

State β (prob. 0.1)

(a) Nash equilibrium in which no strategy is weakly dominated.

	a	b	ϕ
a	1,1	$\frac{1}{2},\frac{1}{2}$	1,1
b	$\frac{1}{2},\frac{1}{2}$	0,0	0,0
ϕ	1,1	0,0	$\frac{1}{2},\frac{1}{2}$

State α (prob. 0.9)

	a	b	ϕ
a	0,0	$\frac{1}{2},\frac{1}{2}$	0,0
b	$\frac{1}{2},\frac{1}{2}$	1,1	1,1
ϕ	0,0	1,1	$\frac{1}{2},\frac{1}{2}$

State β (prob. 0.1)

(b) Nash equilibrium in which the strategy of player 1 is weakly dominated.

Figure 7.1 The two Nash equilibria of the Bayesian game in Section 7.1.1.

her expected payoff, given the other player's strategy. The game has two Nash equilibria, illustrated in Figure 7.1.

a. Player 1 votes for a in state α and for b in state β, and player 2 abstains (highlighted in pink in Figure 7.1a).

If player 2 abstains, then player 1's voting for a in state α and for b in state β is better than any other strategy. If player 1 uses this strategy, then player 2's payoff to abstention is 1 whereas her payoff to voting for a is $0.9 \times 1 + 0.1 \times 0.5 = 0.95$ and her payoff to voting for b is $0.9 \times 0.5 + 0.1 \times 1 = 0.55$.

b. Player 1 abstains in state α and votes for b in state β, and player 2 votes for a (highlighted in blue in Figure 7.1b).

If player 2 votes for a, in state α player 1 can do no better than abstain (if she votes for a, her payoff remains 1, and if she votes for b, her payoff falls to $\frac{1}{2}$) and in state β she can do no better than vote for b. If player 1 abstains in state α and votes for b in state β then player 2's payoff to voting for a is $0.9 \times 1 + 0.1 \times 0.5 = 0.95$ whereas her payoff to abstention is $0.9 \times 0.5 + 0.1 \times 1 = 0.55$ and her payoff to voting for b is $0.9 \times 0 + 0.1 \times 1 = 0.1$.

In the second equilibrium, player 1's strategy is weakly dominated: her payoff from the strategy of voting for a in state α and b in state β is at least as high whatever strategy player 2 uses, and is higher if player 2 votes for b or abstains.

The first equilibrium reflects the earlier informal analysis. Player 1, who is fully informed, votes for the appropriate alternative in each state. Player 2, who is uninformed, abstains, leaving the decision to player 1; if she votes and her vote makes a difference then it affects the outcome adversely.

7.1.2 Model with many individuals

I now consider a more general model with two alternatives, two states, and many individuals. Some individuals, called partisans, prefer one of the alternatives regardless of the state: some always prefer a, and some always prefer b. Others, called independents, prefer a if the state is α and b if it is β. Although the independents all have the same preferences, they differ in their information: some are informed of the state, and some are not. The state is irrelevant to the partisans, and I assume that they are not informed of it.

Definition 7.1: Plurality rule voting game with two alternatives and asymmetric information

A *plurality rule voting game with two alternatives and asymmetric information* $\langle\{a,b\},(n_a,n_b,n_i,n_u),\{\alpha,\beta\},\pi,(v_a,v_b)\rangle$, where a and b are alternatives, n_a, n_b, n_i, and n_u are nonnegative integers with $n_a+n_b+n_i+n_u \geq 3$, α and β are states, $\pi \in (0,1)$, and v_a and v_b are positive numbers, is the following Bayesian game.

Players

A set with $n_a+n_b+n_i+n_u$ members; n_a players are *a-partisans*, n_b are *b-partisans*, n_i are *informed independents*, and n_u are *uninformed independents*.

States

The set of states is $\{\alpha,\beta\}$.

Actions

The set of actions of each player is $\{$*vote for a*, *vote for b*, *abstain*$\}$.

Signals

Each informed independent gets a different signal in each state; every other player gets the same signal in both states.

Prior beliefs

Each player assigns probability π to state α.

Payoffs

The payoff of player j for an action profile in which a majority of players vote for $x \in \{a,b\}$ and the state is $s \in \{\alpha,\beta\}$ is

$$j \text{ is an } a\text{-partisan:} \quad \begin{cases} v_a & \text{if } x=a \\ 0 & \text{if } x=b \end{cases}$$

$$j \text{ is a } b\text{-partisan:} \quad \begin{cases} 0 & \text{if } x=a \\ v_b & \text{if } x=b \end{cases}$$

$$j \text{ is an independent:} \quad \begin{cases} 1 & \text{if } (x,s)=(a,\alpha) \text{ or } (b,\beta) \\ 0 & \text{otherwise.} \end{cases}$$

(a) The Nash equilibrium σ^*, in which only informed independents vote, with the outcome a in state α and b in state β.

(b) The Nash equilibrium σ', in which informed and uninformed independents vote, with the outcome a in both states.

Figure 7.2 Two Nash equilibria of a plurality rule voting game with two alternatives and asymmetric information with no partisans and at least three more uninformed than informed independents .

Her payoff for an action profile in which a and b are tied is the average of her payoffs for profiles in which a wins and b wins.

We refer to a player in a Bayesian game who has received a given signal as a *type* of the player or a *player-type*. In a plurality rule voting game with two alternatives and asymmetric information, each informed independent has two types, one for the signal α and one for the signal β, and every other player has a single type. A strategy for a player is a function that assigns an action to each of her types. Thus a strategy of an informed independent is a pair of actions, one for state α and one for state β, and a strategy for every other player is a single action.

A strategy profile σ in a Bayesian game is a Nash equilibrium if, for each player i and each of her possible types t_i, the action $\sigma_i(t_i)$ that she takes when her type is t_i is optimal given the other players' strategies.

A plurality rule voting game with two alternatives and asymmetric information has many Nash equilibria, and even many in which no player uses a weakly dominated action. Suppose, for example, that there are no partisans ($n_a = n_b = 0$), some informed independents ($n_i \geq 1$), and at least three more uninformed independents than informed independents ($n_u \geq n_i + 3$). In one Nash equilibrium, every uninformed independent abstains and every informed independent votes for a in state α and b in state β, so that the outcome is a in state α and b in state β. This equilibrium, illustrated in Figure 7.2a, generalizes the equilibrium of the example in Section 7.1.1 in which no player's strategy is weakly dominated. Call the equilibrium σ^*.

Now consider the strategy profile σ' that differs from σ^* only in that every uninformed independent votes for a. This strategy profile is illustrated in Figure 7.2b. Like σ^*, it is a Nash equilibrium, but the outcome is a in both states. It is an equilibrium because no change in any player's strategy affects the outcome: uninformed independents outnumber informed independents by at least three, so for any deviation by a single player from σ', the alternative a still wins in both states. Further, no player's strategy in σ' is weakly dominated: every informed independent is voting for the alternative she prefers, given the state, and the strategy of every uninformed independent to vote for a is not weakly dominated because it would yield a payoff greater than that of the strategies of voting for b or abstaining if all other individuals were to abstain.

Even though σ' is a Nash equilibrium and no player's strategy is weakly dominated, the strategy of the uninformed independents to vote for a seems foolish. Suppose that the number of uninformed independents exceeds the number of informed independents by exactly 3, and put yourself in the shoes of one of the uninformed independents. If everyone votes according to σ', the outcome is a win for a in both states, regardless of your action. But if one of the other uninformed independents fails to vote despite her best intentions (perhaps she is ill on election day, or her bicycle chain breaks on the way to the polling station), then by switching your vote to b, you change the outcome in state β from a win for a to a tie between a and b, and do not affect the outcome in state α, so that you are better off. If one of the informed independents fails to vote in one of the states, then switching your vote from a to b does not affect the outcome. So if you assign equal and small probability to these events, and much smaller probability to events in which two or more of the other player-types fail to vote—which is reasonable if you believe that a player's failure to follow the prescription of σ' to vote is rare and independent of every other player's failure to do so—then switching your vote from a to b raises your expected payoff.

More generally, if the number of uninformed independents exceeds the number of informed independents by 3 or more, then a switch from voting for a to voting for b by an uninformed independent does not affect the outcome if at most $n_u - n_i - 3$ player-types fail to vote, improves it for her if $n_u - n_i - 2$ of the uninformed independents fail to vote (by changing it from a to a tie between a and b in state β), and does not affect it if the members of any other group of $n_u - n_i - 2$ player-types fail to vote.

These arguments lead to a definition of equilibrium that assumes that a player deviates from a strategy not only if she has another strategy that yields a higher payoff given the other players' strategies, but also if she has another strategy that yields the same payoff given the other players' strategies and a higher payoff if one of the other player-types whose strategy calls for her to vote fails to do so, or the same payoff in both cases but a higher payoff if two of the other player-types whose strategies call for them to vote fail to do so, or the same payoff in all these cases but a higher payoff if three of the other player-types whose strategies call for them to vote fail to do so, and so on. I define a deviation to be desirable if it satisfies these conditions, and an equilibrium to be a strategy profile from which no player has a desirable deviation.

Definition 7.2: Desirability of deviation from strategy profile in voting game with asymmetric information

Let σ be a strategy profile in a plurality rule voting game with two alternatives and asymmetric information, let j be a player, let t be a type of player j, let x_t be the action of t specified by σ, let x'_t be an action of t different from x_t, and let m be the number of other player-types for players other than j who, according to σ, vote. The *desirability* for t of a deviation from x_t to x'_t given σ is determined by the following iterative procedure.

Initialization

Set $k = 0$.

Step k

Assume that a randomly-determined set of exactly k of the other player-types whose strategies in σ call for them to vote fail to do so, with each such set equally likely. Then if the expected payoff generated by x'_t is

- less than that generated by x_t, a deviation to x'_t is *not desirable*
- more than that generated by x_t, a deviation to x'_t is *desirable*
- the same as that generated by x_t and $k = m$, a deviation to x'_t is *not desirable*
- the same as that generated by x_t and $k \leq m - 1$, continue to Step $k + 1$.

Definition 7.3: Equilibrium of voting game with asymmetric information

A strategy profile is an *equilibrium* of a plurality rule voting game with two alternatives and asymmetric information if no deviation by any player-type is desirable.

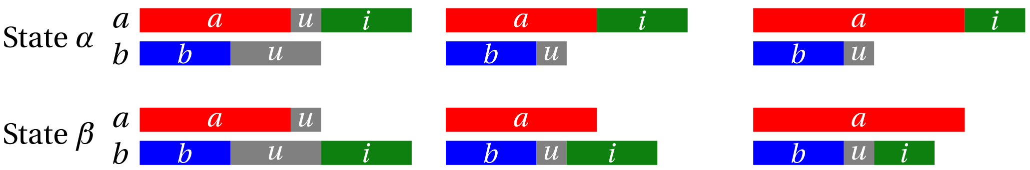
A strategy profile that is not a Nash equilibrium is not an equilibrium in this sense because any profitable deviation that rules it out as a Nash equilibrium is desirable at level $k = 0$. But as I have argued, a strategy profile that *is* a Nash equilibrium may also not be an equilibrium, even if no player's strategy is weakly dominated.

The notion of the desirability of a deviation, like the notion of weak domination, discriminates between some strategies the yield the same payoff given the other players' strategies. My arguments regarding the strategy profile σ' illustrated in Figure 7.2b, in which no player's strategy is weakly dominated, show that the ways in which they discriminate differ. For partisans and informed independents, however, both notions yield the same conclusion. For an informed independent whose signal indicates that the state is s or an s-partisan ($s = a$, b), voting for s weakly dominates abstaining or voting for the other alternative. Also, for any strategy profile σ in which such a player j abstains or votes for the other alternative, deviating to vote for s is desirable: for the smallest number of player-types whose deviations to abstention would cause j's deviation to affect the outcome, it either changes a loss for s in state s into a tie or changes a tie into a win for s. An implication is that if all players are partisans or informed independents, then in every equilibrium in the sense of Definition 7.3, as in every equilibrium in which no player's strategy is weakly dominated, every player votes for the alternative she favors (Proposition 3.1).

For uninformed independents the calculus, as I have argued, is different. The next result, illustrated in Figure 7.3, shows that in any equilibrium of a general plurality rule voting game with two alternatives and asymmetric information, every partisan (red and blue in the figure) votes for the alternative she favors, every informed independent (green) votes for the alternative she favors given the state, and uninformed independents (gray) vote so as to cancel out, as far as possible, the partisans' votes:

- if there are enough uninformed independents to cancel out the partisans' votes, uninformed independents vote so that in the absence of any votes by the informed independents, a and b would tie (in both states), and hence the margin in favor of a in state α among all votes is the same as the margin in favor of b in state β, equal to the number of informed independents (Figure 7.3a)

- if there are too few informed independents to cancel out the partisans' votes, all uninformed independents vote for the alternative favored by fewer partisans, maximizing the influence of the informed independents in case some partisans do not participate (Figures 7.3b and 7.3c).



(a) $n_a - n_b < n_u$. More than enough uninformed independents to cancel out partisans' votes.

(b) $n_u < n_a - n_b < n_i + n_u$. Not enough uninformed independents to cancel out partisans' votes, but enough independents in total to do so.

(c) $n_i + n_u < n_a - n_b$. Not enough independents to cancel out partisans' votes.

Figure 7.3 Equilibria of a plurality rule voting game with two alternatives and asymmetric information (Proposition 7.1) for $n_a > n_b$.

Proposition 7.1: Equilibrium of voting game with asymmetric information

Consider a plurality rule voting game with two alternatives and asymmetric information $\langle\{a,b\},(n_a,n_b,n_i,n_u),(\alpha,\beta),\pi,(v_a,v_b)\rangle$ with $n_i \geq 1$ (at least one player is an informed independent). A strategy profile is an equilibrium if and only if

a. every a-partisan votes for a and every b-partisan votes for b

b. every informed independent votes for a in state α and for b in state β

c. the number n_u^a of uninformed independents who vote for a and the number n_u^b who vote for b satisfy

$$\begin{aligned} n_u^a - n_u^b &= \min\{n_b - n_a, n_u\} \quad \text{if } n_a \leq n_b \\ n_u^b - n_u^a &= \min\{n_a - n_b, n_u\} \quad \text{if } n_a \geq n_b. \end{aligned} \tag{7.1}$$

If $n_a = n_b$ the outcome of an equilibrium is a in state α and b in state β, if $n_a > n_b$ it is

$$\begin{array}{ll} a \text{ in state } \alpha,\ b \text{ in state } \beta & \text{if } n_a - n_b < n_i + n_u \\ a \text{ in state } \alpha, \text{ tie in state } \beta & \text{if } n_a - n_b = n_i + n_u \\ a \text{ in both states} & \text{if } n_a - n_b > n_i + n_u, \end{array} \tag{7.2}$$

and if $n_a < n_b$ it is the variant of (7.2) in which a and b and interchanged and α and β are interchanged.

The outcome of an equilibrium is the same as the outcome of an equilibrium of the variant of the game in which every player is informed of the state.

Comments

- If $|n_a - n_b| < n_u$ (Figure 7.3a if $n_a > n_b$), some uninformed independents abstain in some equilibria; the only requirement on their equilibrium behavior is that the difference between the number who vote for b and the number who vote for a is $|n_a - n_b|$. In particular, if $n_a = n_b$ (the partisans are equally divided between a and b) then there is an equilibrium in which every uninformed independent abstains.

- If a-partisans outnumber b-partisans ($n_a > n_b$) and α is the more likely state, the alternative for which most uninformed independents vote is b, which according to their prior belief is the *wrong* alternative. By doing so they nullify, as much as possible, the partisans' votes, leaving the decision to the informed independents.

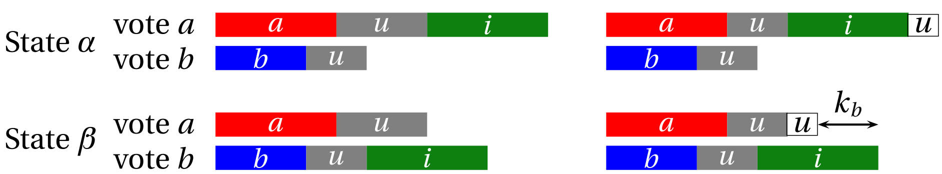
- The examples in Figures 7.3 illustrate the last part of the result, that the outcome is the same as outcome of the equilibrium in the variant of the game in which all players are informed, so that there are $n_i + n_u$ instead of n_i informed independents, and no uninformed independents. In Figures 7.3a and 7.3b the outcome is a in state α and b in state β, and in Figure 7.3c it is a in both states. One way to express this part of the result is to say that the equilibrium fully aggregates information.

- The equilibria do not depend on the values v_a and v_b that partisans attach to their favorite alternatives.

In an equilibrium, every partisan and informed independent votes for her favorite alternative because voting for the other alternative has no possible benefit, however the other players vote; voting for the other alternative can generate only a worse outcome, if not when all the other players adhere to their strategies then when some of them whose strategies call for them to vote fail to do so.

The argument that, if possible, uninformed independents vote so that a and b would tie (in both states) in the absence of any votes by informed independents, is more involved. Suppose that there are more than enough uninformed independents to cancel out the partisans' votes.

First consider a strategy profile in which the uninformed independents vote in such a way that, in the absence of any votes by informed independents, a



(a) Original strategy profile.

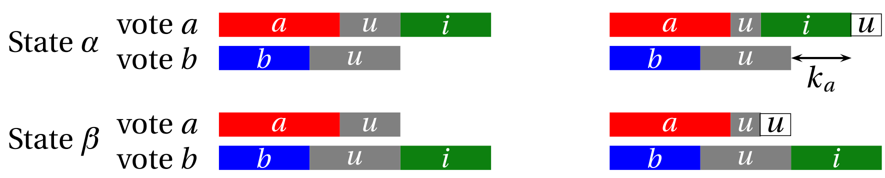
(b) Effect of uninformed independent's deviation to abstention.

Figure 7.4 The effect of an uninformed independent's deviation to abstention for a strategy profile in a plurality rule voting game with two alternatives and asymmetric information in which a would win (in both states) in the absence of any votes by informed independents.

would win (in both states), as in Figure 7.4a. Suppose that an uninformed independent j who is voting for a deviates to abstention, as illustrated in Figure 7.4b. Denote by k_b the original margin in favor of b in state β. The smallest set of player-types whose failure to vote causes this change in j's strategy to affect the outcome consists of k_b player-types whose strategies call for them to vote for b. If the players in such a set fail to vote, the deviation by j changes the outcome from a tie to a win for b in state β and does not affect the outcome in state α. If any other k_b player-types fail to vote, the deviation by j does not affect the outcome. Thus the deviation is desirable, so that the strategy profile is not an equilibrium.

Now consider a strategy profile in which the uninformed independents vote in such a way that, in the absence of any votes by informed independents, a and b would tie (in both states), as in Figure 7.5a. If an uninformed independent j who is voting for a deviates to abstention, the smallest number of player-types whose failure to vote causes the outcome to change is the new margin in favor of a in state α, say k_a, as illustrated in Figure 7.5b. If k_a player-types whose strategies call for them to vote for a fail to vote, then the deviation by j changes the outcome from a win for a to a tie in state α, and does not affect the outcome in state β. If any other k_a players fail to vote, the deviation by j does not affect the outcome. Thus the deviation is undesirable, so that the strategy profile is consistent with an equilibrium.

The arguments for the cases in which the there are enough independents to cancel out the partisans' votes, but not enough uninformed independents to do so, and in which there are not enough independents to cancel out the partisans' votes, are similar. The following proof has the details.



(a) Original strategy profile.

(b) Effect of uninformed independent's deviation to abstention.

Figure 7.5 The effect of an uninformed independent's deviation to abstention for a strategy profile in a plurality rule voting game with two alternatives and asymmetric information in which a and b would tie (in both states) in the absence of any votes by informed independents.

Proof of Proposition 7.1

I first argue that a strategy profile σ^* satisfying the conditions in the result is an equilibrium.

Step 1 *No change in the strategy in σ^* of an informed independent or partisan is desirable.*

Proof. If an informed independent changes her action in state α from a to either b or abstention, then regardless of how many other player-types fail to vote, the outcome in state α either does not change or changes to b or to a tie, so the deviation is not desirable. A similar argument applies to an informed independent who changes her action in state β and to a partisan who changes her action. $\lhd$

Step 2 *No change in the strategy in σ^* of an uninformed independent is desirable.*

Proof. Consider an uninformed independent, say j. Assume that $n_a \geq n_b$. There are three cases.

$n_a - n_b < n_i + n_u$

The outcome of σ^* is a in state α and b in state β. (For $n_a > n_b$, Figures 7.3a and 7.3b show examples.)

Suppose that j votes for a. If she deviates to abstention and the number of player-types who fail to vote is the smallest for which this deviation affects the outcome, then the outcome changes only in state α, where it becomes a tie between a and b rather than a, decreasing j's expected payoff. Thus the deviation is not desirable. By a similar argument, a deviation by j to vote for b is not desirable.

A similar argument shows that if j votes for b then any deviation is not desirable.

Now suppose that j abstains, which happens only if $n_a - n_b < n_u$ (Figure 7.3a). Suppose that she deviates to vote for a. If the number of player-types who fail to vote is the smallest for which this deviation affects the outcome, then the outcome changes only in state β, where it becomes a tie between a and b rather than a win for b, decreasing j's expected payoff. Thus the deviation is not desirable. A symmetric argument shows that also j's deviation to vote for b is not desirable.

$n_a - n_b > n_i + n_u$

Given $n_i \geq 1$, we have $n_a > n_b$, and the outcome of σ^* is a in both states (refer to Figure 7.3c). The strategy profile σ^* specifies that every uninformed independent, and in particular j, votes for b. If she deviates to abstention and the number of player-types who fail to vote is the smallest for which this deviation affects the outcome, then the outcome changes only in state β, where it becomes a win for a rather than a tie between a and b, decreasing j's expected payoff. Thus the deviation is not desirable.

A similar argument shows that also a deviation by j to vote for a is not desirable.

$n_a - n_b = n_i + n_u$

The outcome of σ^* is a in state α and a tie between a and b in state β. As in the previous case, every uninformed independent, and in particular j, votes for b. If she deviates to abstention or to vote for a, the outcome in state β changes to a win for a and the outcome in state α does not change, so the deviation is not desirable.

The argument for the case $n_a \leq n_b$ is symmetric with this argument. ◁

Step 3 *The strategy profile σ^* is an equilibrium.*

Proof. The result follows from Steps 1 and 2. ◁

I now argue that every equilibrium satisfies the conditions in the result.

Step 4 *In every equilibrium, every informed independent votes for a in state α and for b in state β.*

Proof. Let σ be a strategy profile and let j be an informed independent. Suppose that σ_j specifies a vote for b in state α. If j deviates to vote for a in state α and the number of player-types who fail to vote is the smallest for which this deviation affects the outcome, then the outcome changes only in state α, where it either becomes a win for a rather than a tie between a and b, or a win for a rather than a win for b, in both cases increasing j's expected payoff. Thus such a deviation is desirable.

A similar argument shows that if σ_j specifies abstention in state α then a deviation to vote for a is desirable.

Symmetric arguments show that if σ_j specifies a vote for a or abstention in state β then j has a desirable deviation. ◁

Step 5 *In every equilibrium, every a-partisan votes for a and every b-partisan votes for b.*

Proof. This conclusion follows from arguments like those in Step 4. ◁

Step 6 *In every equilibrium, the number n_u^a of uninformed independents who vote for a and the number n_u^b who vote for b satisfy (7.1).*

Proof. Suppose that $n_a \geq n_b$. There are two cases.

$n_a - n_b \geq n_u$
From (7.1), $n_u^b - n_u^a = n_u$, so $n_u^b = n_u$: all uninformed independents vote for b. Given $n_a - n_b \geq n_u$, if any uninformed independents exist we have $n_a > n_b$; Figures 7.3b and 7.3c are examples. Consider a strategy profile in which an uninformed independent j abstains. If she deviates to vote for b and the number of player-types who fail to vote is the smallest for which this deviation affects the outcome, then the outcome in state β changes either from a tie to a win for b, or from a win for a to a tie, and the outcome in state α remains a win for a (given $n_a > n_b$, $n_a - n_b \geq n_u$, and $n_i \geq 1$), so in both cases her expected payoff increases. A similar argument shows that if j's strategy calls for her to vote for a and she deviates to abstention then her expected payoff increases for any minimal set of player-types whose failure to vote affects the outcome. We conclude that in an equilibrium every uninformed independent votes for b, so that (7.1) is satisfied.

$n_a - n_b < n_u$
From (7.1), $n_u^b - n_u^a = n_a - n_b$. Figure 7.3a is an example. Consider a strategy profile in which $n_u^b - n_u^a > n_a - n_b$. Then $n_b + n_i + n_u^b > n_b +$

$n_u^a + n_a - n_b = n_a + n_u^a$ (given $n_i \geq 1$), so that b wins in state β, with a margin of victory of $n_b + n_i + n_u^b - n_a - n_u^a = n_i - (n_a - n_b) + (n_u^b - n_u^a) > n_i$.

- If a wins in state α, its margin of victory is $n_a + n_i + n_u^a - n_b - n_u^b = n_i + (n_a - n_b) - (n_u^b - n_u^a) < n_i$. Consider an uninformed independent j whose strategy calls for her to vote for b. If she deviates to abstention and the number of player-types who fail to vote is the smallest for which this deviation affects the outcome, then the outcome in state α changes from a tie to a win for a, and the outcome in state β, where the margin of victory is larger, does not change. Thus the deviation increases j's expected payoff, so the strategy profile is not an equilibrium.
- If b wins in state α, its margin of victory is $n_b + n_u^b - n_a - n_u^a - n_i$, which is less than b's margin of victory in state β, $n_b + n_u^b + n_i - n_a - n_u^a$. In this case also, consider an uninformed independent j whose strategy calls for her to vote for b. If she deviates to abstention and the number of player-types who fail to vote is the smallest for which this deviation affects the outcome, then the outcome in state α changes from a win for b to a tie, and the outcome in state β does not change. Thus the deviation increases j's expected payoff, so the strategy profile is not an equilibrium.

Now consider a strategy profile in which $n_u^b - n_u^a < n_a - n_b$. In this case, a wins in state α and the margin of victory of the winner in state β is less than the margin of victory of a in state α. By an argument symmetric with that in the previous case, an uninformed independent who switches from voting for a to abstention increases her expected payoff when the number of player-types who fail to vote is the smallest for which this deviation affects the outcome, so that the strategy profile is not an equilibrium.

The argument for the case $n_a \leq n_b$ is similar. $\lhd$

Step 7 *The outcome of an equilibrium is given in (7.2).*

Proof. If $n_a = n_b$ then by (7.1) we have $n_u^a = n_u^b$, so that given $n_i \geq 1$, a wins in state α and b wins in state β.

Now suppose that $n_a > n_b$. Then by the characterization of an equilibrium, in state α alternative a gets $n_a + n_i + n_u^a$ votes and alternative b gets $n_b + n_u^b$ votes.

- If $n_a - n_b < n_u$ then by (7.1) we have $n_b + n_u^b = n_a + n_u^a$, so that given $n_i \geq 1$, alternative a wins.
- If $n_a - n_b \geq n_u$ then $n_u^b = n_u$ and $n_u^a = 0$, so that a gets $n_a + n_i$ votes and b gets $n_b + n_u \leq n_a$ votes, so that again a wins.

In state β alternative a gets $n_a + n_u^a$ votes and alternative b gets $n_b + n_i + n_u^b$ votes.

- If $n_a - n_b < n_u$ then by (7.1) we have $n_u^b - n_u^a = n_a - n_b$, so that $n_b + n_i + n_u^b = n_a + n_u^a + n_i$ and hence b wins.
- If $n_a - n_b \geq n_u$ then $n_u^b = n_u$ and $n_u^a = 0$, so that a gets n_a votes and b gets $n_b + n_u + n_i$ votes, and hence a wins if $n_a - n_b > n_i + n_u$, a and b tie if $n_a - n_b = n_u + n_i$, and b wins if $n_a - n_b < n_i + n_u$.

The case $n_a < n_b$ is symmetric. ◁

Step 8 *The outcome of an equilibrium is the same as the outcome of an equilibrium of the variant of the game in which every player is informed of the state.*

Proof. From the characterization of an equilibrium, in a game in which the number of informed independents is $n_i + n_u$ and there are no uninformed independents, the number of votes for a is $n_a + n_i + n_u$ in state α and n_a in state β, and the number of votes for b is n_b in state α and $n_b + n_i + n_u$ in state β. Thus if $n_a = n_b$ then a wins in state α and b wins in state β, and if $n_a > n_b$ then a wins in state α and in state β it wins if $n_a - n_b > n_i + n_u$, the alternatives tie if $n_a - n_b = n_i + n_u$, and b wins if $n_a - n_b < n_i + n_u$, as when n_u of the independents are uninformed. ◁

The existence of equilibria in which some uninformed independents abstain (when $|n_a - n_b| < n_u$), leaving the decision to informed independents, rests intuitively on the two groups having the same preferences: a is best in state α and b is best in state β. If instead the preferences of some uninformed independents differ from those of the informed independents, abstention by those uninformed independents seems intuitively unappealing. The next exercise invites you to study an example in which the preferences of every uninformed independent are opposed to those of the informed independents: b is better than a in state α and a is better than b in state β. Section 7.3 presents another model in which the individuals' preferences are not aligned.

Exercise 7.1: Adversarial preferences in voting game with asymmetric information

Consider a variant of a plurality rule voting game with two alternatives and asymmetric information in which all uninformed independents prefer b to a in state α and a to b in state β, with payoffs of 1 if the outcome is b in state α or a in state β, and 0 otherwise. Assume that there are no partisans and $\pi > \frac{1}{2}$ (state α is more likely than state β). Find the equilibria of this game and show that if the number of uninformed independents exceeds the number of informed independents then the equilibrium outcomes differ from the equilibrium outcome of the game in which all the independents are informed.

7.1.3 Model with imperfectly-informed individuals

The result that some individuals abstain in equilibrium does not depend on these individuals being completely uninformed and facing individuals who are perfectly informed. Consider a variant of a plurality rule voting game with two alternatives and asymmetric information in which in each state the individuals are a priori identical. The processes generating signals for each individual i are shown in Figure 7.6. In state α each individual i gets the signal A with probability q_i and the signal B with probability $1-q_i$, and in state β she gets the signal B with probability q_i and the signal A with probability $1-q_i$, where q_i is a draw from a probability distribution with support $[\frac{1}{2}, 1]$ and a continuous density, independently of the draw of q_j for every other individual j. In particular, with probability 1 each individual's signal conveys some information. (Only the signal of an individual i for whom $q_i = \frac{1}{2}$, a value that occurs with probability 0, is completely uninformative.) Each individual i knows q_i and her signal, but not the value of q_j or the signal for any other individual j. To make the model symmetric, I assume that each individual believes that the prior probability of each state is $\frac{1}{2}$. Each individual votes for one of the alternatives or abstains; the outcome is the alternative that receives the most votes, and each individual's payoff is 1 if the outcome is a and the state is α or the outcome is b and the state is β, and is 0 otherwise.

Precisely, the model is the following Bayesian game. A state in this game, which captures all the uncertain features of the environment, is a triple consisting of the state of nature α or β, the signal qualities, and the signal realizations. A player's signal in the game, which captures all the information she obtains, is a pair consisting of either A or B and a signal quality. Although "state" and "signal" have these meanings in the following definition, outside the definition I continue

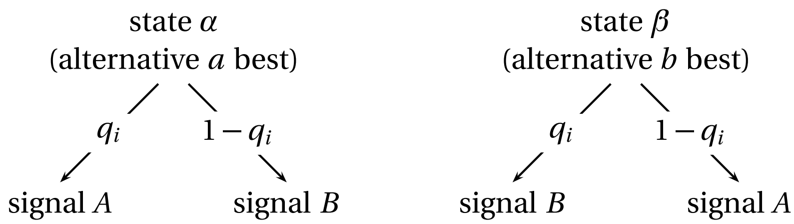


Figure 7.6 The processes generating signals for each individual i in a plurality rule voting game with two alternatives and uncertain signal qualities. The value of q_i is a draw from a distribution with support $[\frac{1}{2}, 1]$.

to use "state" to refer to α and β and "signal" to refer to A and B.

Definition 7.4: Plurality rule voting game with two alternatives and uncertain signal qualities

A *plurality rule voting game with two alternatives and uncertain signal qualities* $\langle\{a,b\}, n, (\alpha,\beta), \{A,B\}, F\rangle$, where a and b are alternatives, n is a positive integer, α and β are states of nature, A and B are signals, and F is a probability distribution function with support $[\frac{1}{2}, 1]$ and a continuous density is the following Bayesian game.

Players
A set N with n members.

States
The set of states is the set of triples $(\omega, (q_j)_{j\in N}, (s_j)_{j\in N})$ where $\omega \in \{\alpha, \beta\}$ (the state of nature), $q_j \in [\frac{1}{2}, 1]$ for each $j \in N$ (j's signal quality), and $s_j \in \{A, B\}$ for each $j \in N$ (j's signal).

Actions
For each player the set of actions is $\{$*vote for a*, *vote for b*, *abstain*$\}$.

Signals
For each player i the set of signals is $\{A, B\} \times [\frac{1}{2}, 1]$ and the signal function τ_i is given by $\tau_i(\omega, (q_j)_{j\in N}, (s_j)_{j\in N}) = (s_i, q_i)$ for each state $(\omega, (q_j)_{j\in N}, (s_j)_{j\in N})$.

Prior beliefs
Each player i believes that

- the value of ω is α with probability $\frac{1}{2}$ and β with probability $\frac{1}{2}$
- q_i is drawn randomly from F
- s_i is A with probability q_i and B with probability $1-q_i$ if $\omega = \alpha$,

and B with probability q_i and A with probability $1 - q_i$ if $\omega = \beta$ (as illustrated in Figure 7.6).

Every random draw is independent of every other random draw.

Payoffs

The payoff of each player for each pair consisting of an action profile and a state $(\omega, (q_j)_{j \in N}, (s_j)_{j \in N})$ is

$$\begin{cases} 1 & \text{if } \omega = \alpha \\ 0 & \text{if } \omega = \beta \end{cases} \quad \text{if more players vote for } a \text{ than for } b,$$
$$\begin{cases} 0 & \text{if } \omega = \alpha \\ 1 & \text{if } \omega = \beta \end{cases} \quad \text{if more players vote for } b \text{ than for } a,$$

and $\frac{1}{2}$ (regardless of the value of ω) if the numbers of players who vote for a and for b are the same.

The signal of an individual i is more informative the higher is q_i, so it is reasonable to think that the game has an equilibrium in which each individual i uses a strategy with the following form: for some number $q_i^* \in [\frac{1}{2}, 1]$, i's threshold, type (A, q_i) of any individual i votes for a if $q_i \geq q_i^*$ and abstains if $q_i < q_i^*$, and type (B, q_i) of any individual i votes for b if $q_i \geq q_i^*$ and abstains if $q_i < q_i^*$. I refer to a strategy of this form as a quality threshold strategy.

Definition 7.5: Quality threshold strategy in voting game with uncertain signal qualities

A strategy of any individual i in a plurality rule voting game with two alternatives and uncertain signal qualities $\langle \{a, b\}, n, (\alpha, \beta), \{A, B\}, F \rangle$ is a *quality threshold strategy* if there is a number $q_i^* \in [\frac{1}{2}, 1]$, i's *threshold*, such that the strategy is

$$\begin{cases} \textit{vote for a} & \text{if } i\text{'s signal is } A \text{ and } q_i \geq q_i^* \\ \textit{vote for b} & \text{if } i\text{'s signal is } B \text{ and } q_i \geq q_i^* \\ \textit{abstain} & \text{if } q_i < q_i^*. \end{cases}$$

Given the symmetry of the model, including the fact that the prior probability of each state is $\frac{1}{2}$, it is also reasonable to think that the game has a Nash equilibrium in which every individual uses a quality threshold strategy with the same threshold.

Definition 7.6: Quality threshold equilibrium of voting game with uncertain signal qualities

A strategy profile in a plurality rule voting game with two alternatives and uncertain signal qualities is a *quality threshold equilibrium* if it is a Nash equilibrium and every individual's strategy is a quality threshold strategy with the same threshold.

I show in the next result that if there are two individuals ($n = 2$), the game has a quality threshold equilibrium, and in every such equilibrium the threshold exceeds $\frac{1}{2}$, so that an individual with an informative but low-quality signal abstains. If each individual could report both her signal and its quality, and a decision were made by pooling this information, then each individual would optimally report her signal regardless of its quality. But voting provides no way for individuals to report the qualities of their signals. All they can do is vote or abstain, and an individual who votes based on a low-quality signal influences the outcome just as much as one who votes based on a high-quality signal. As a consequence, an individual with an informative but low-quality signal prefers to abstain than to vote, making the outcome depend on the vote of the other individual, who is likely to have received a more informative signal.

To understand in more detail why the equilibrium threshold exceeds $\frac{1}{2}$, consider the ingredients of an individual's decision of whether to abstain or vote. She should ponder the implications of her action for the outcome, given each possible action of the other individual, taking into account any information about the state that the other individual's action conveys.

Consider type (A, q_i) of individual i. Assume that the other individual, j, is using a quality threshold strategy with threshold q^*. Here are the considerations behind i's decision for each possible action of individual j.

j votes for a The outcome is the same whether i votes for a or abstains, so this possibility is irrelevant to i's decision.

j abstains If individual i votes for a then the outcome is a, and if she abstains then it is a tie, so her gain from voting for a rather than abstaining is $\frac{1}{2}$ if the state is α and $-\frac{1}{2}$ if the state is β.

j votes for b If individual i votes for a then the outcome is a tie and if she abstains then it is b, so her gain for voting for a rather than abstaining is $\frac{1}{2}$ if the state is α and $-\frac{1}{2}$ if the state is β (as when j abstains).

Thus the expected gain of type (A, q_i) of individual i from voting for a rather than

j's action	i's gain from voting for a rather than abstaining if state is α	if state is β	posterior prob. of state α
vote for a	0	0	irrelevant
abstain	$\frac{1}{2}$	$-\frac{1}{2}$	q^*
vote for b	$\frac{1}{2}$	$-\frac{1}{2}$	$< \frac{1}{2}$

Figure 7.7 The ingredients of the decision of type (A, q^*) of individual i regarding whether to vote for a or abstain.

abstaining is

$$\begin{aligned}\Pr(j \text{ abstains} \mid i\text{'s signal } A)\big[&\Pr(\text{state } \alpha \mid j \text{ abstains \& } i\text{'s signal } A)\cdot\tfrac{1}{2}\\ &+\Pr(\text{state } \beta \mid j \text{ abstains \& } i\text{'s signal } A)\cdot(-\tfrac{1}{2})\big]\\ +\Pr(j \text{ votes } b \mid i\text{'s signal } A)\big[&\Pr(\text{state } \alpha \mid j \text{ votes } b \text{ \& } i\text{'s signal } A)\cdot\tfrac{1}{2}\\ &+\Pr(\text{state } \beta \mid j \text{ votes } b \text{ \& } i\text{'s signal } A)\cdot(-\tfrac{1}{2})\big].\end{aligned} \tag{7.3}$$

Now, j abstains if and only if her signal q_j is less than q^*, so the probability that i assigns to her abstaining is $F(q^*)$ (independent of i's signal). The fact that she abstains conveys no information about the likelihood that the state is α or β, so Pr(state α | j abstains & i's signal A) is the probability of α conditional on one signal of A with quality q_i, which, using Bayes' rule, is q_i (given that the prior probability of each state is $\frac{1}{2}$). Thus the expected gain in (7.3) is

$$\begin{aligned}&\tfrac{1}{2}F(q^*)(q_i-(1-q_i))\\ &\quad+\Pr(j \text{ votes } b \mid i\text{'s signal } A)\big[\Pr(\text{state } \alpha \mid j \text{ votes } b \text{ \& } i\text{'s signal } A)\cdot\tfrac{1}{2}\\ &\qquad+\Pr(\text{state } \beta \mid j \text{ votes } b \text{ \& } i\text{'s signal } A)\cdot(-\tfrac{1}{2})\big].\end{aligned} \tag{7.4}$$

For a quality threshold equilibrium with threshold q^*, this expected gain must be zero for type (A, q^*): type (A, q^*) of individual i must be indifferent between voting for a and abstaining. Now, individual j votes for b only if her signal is B, which is more likely if the state is β than if it is α. Thus j's voting for b is evidence in favor of state β. Further, j votes for b only if her signal quality is at least q^*, and hence at least as high as i's signal quality. So j's voting for b provides stronger evidence about the state than i's signal A, and hence conditional on j's voting for b, the probability that i assigns to state α is less than $\frac{1}{2}$: Pr(state α | j votes b & i's signal A) $< \frac{1}{2}$ and Pr(state β | j votes b & i's signal A) $> \frac{1}{2}$. Thus for $q_i = q^*$ the second term in (7.4) is negative, so for the whole expression to be zero we need the first term to be positive, which requires $q^* > \frac{1}{2}$. The ingredients of i's reasoning that lead to this conclusion are summarized in Figure 7.7.

We conclude that in any quality threshold equilibrium, an individual with an informative but low-quality signal abstains. She does so because if her vote affects the outcome, then taking into account the information implied about the state by the other individual's action, her vote is more likely to change the outcome adversely than advantageously, given the superior quality of the other individual's signal. As in a plurality rule voting game with two alternatives and asymmetric information, in which some individuals are uninformed and others are perfectly informed, we can characterize her predicament by saying that she is subject to the swing voter's curse.

Proposition 7.2: Quality threshold equilibrium of voting game with uncertain signal qualities and two individuals

Let $\langle \{a, b\}, n, (\alpha, \beta), \{A, B\}, F \rangle$ be a plurality rule voting game with two alternatives and uncertain signal qualities for which $n = 2$. This game has a quality threshold equilibrium in which the threshold is in $(\frac{1}{2}, 1)$ and in every quality threshold equilibrium the threshold lies in this interval.

Proof

Denote the individuals by i and j, and suppose that j's strategy is a quality threshold strategy with threshold q^*. First note that if $q^* = 1$, so that j votes with probability zero, the outcome is a tie if i abstains and otherwise is the outcome for which she votes, so voting is optimal for i regardless of her signal quality. Thus the game has no quality threshold equilibrium with threshold 1.

Now suppose that $q^* < 1$ and consider type (A, q_i) of individual i. By the argument preceding the result, her expected gain from voting for a rather than abstaining is given in (7.4). To show that $q^* > \frac{1}{2}$ in any equilibrium, it is enough to show that the probability that type (A, q^*) of individual i assigns to state α in the case that j votes for b is less than $\frac{1}{2}$, as argued informally in the text preceding this result. But to show that an equilibrium exists, we need to study the expected gain in more detail.

We have

$$\begin{aligned}
&\Pr(j \text{ votes } b \mid i\text{'s signal } A)\Pr(\text{state } \alpha \mid j \text{ votes } b \ \&\ i\text{'s signal } A)\\
&\quad = \Pr(j \text{ votes } b \mid i\text{'s signal } A)\frac{\Pr(j \text{ votes } b \ \&\ i\text{'s signal } A \mid \text{state } \alpha)\Pr(\text{state } \alpha)}{\Pr(j \text{ votes } b \ \&\ i\text{'s signal } A)}\\
&\quad = \frac{\Pr(j \text{ votes } b \ \&\ i\text{'s signal } A \mid \text{state } \alpha)\Pr(\text{state } \alpha)}{\Pr(i\text{'s signal } A)}\\
&\quad = \frac{\Pr(j \text{ votes } b \mid \text{state } \alpha)\Pr(i\text{'s signal } A \mid \text{state } \alpha)\Pr(\text{state } \alpha)}{\Pr(i\text{'s signal } A)}\\
&\quad = \frac{\int_{q^*}^1 (1-q_j)\,dF(q_j)\cdot q_i \cdot \frac{1}{2}}{\frac{1}{2}q_i + \frac{1}{2}(1-q_i)}\\
&\quad = q_i \int_{q^*}^1 (1-q_j)\,dF(q_j).
\end{aligned}$$

By a similar argument,

$$\begin{aligned}
&\Pr(j \text{ votes } b \mid i\text{'s signal } A)\Pr(\text{state } \beta \mid j \text{ votes } b \ \&\ i\text{'s signal } A)\\
&\qquad\qquad = (1-q_i)\int_{q^*}^1 q_j\,dF(q_j).
\end{aligned}$$

Substituting these expressions into (7.4), we conclude that the expected gain of type (A, q_i) of individual i from voting for a rather than abstaining, given that j's strategy is a quality threshold strategy with threshold q^*, is

$$\begin{aligned}
G(q_i, q^*) &= \tfrac{1}{2}F(q^*)(2q_i - 1) + \tfrac{1}{2}q_i \int_{q^*}^1 (1-q_j)\,dF(q_j) - \tfrac{1}{2}(1-q_i)\int_{q^*}^1 q_j\,dF(q_j)\\
&= \tfrac{1}{2}(q_i - (1-q_i)F(q^*)) - \tfrac{1}{2}\int_{q^*}^1 q_j\,dF(q_j).
\end{aligned}$$

The expected gain of type (B, q_i) of individual i from voting for b rather than abstaining is given by the same expression.

Define the function $H : [\frac{1}{2}, 1) \to \mathbb{R}$ by $H(q) = G(q, q)$. The game has a quality threshold equilibrium with threshold q^* if and only if $H(q^*) = 0$. Now, the integral in the expression for $G(q_i, q^*)$ exceeds $q^*(1 - F(q^*))$, so $H(\frac{1}{2}) < 0$. Further, $H(q^*) \to \frac{1}{2} > 0$ as $q^* \to 1$ and H is continuous (given that F has a continuous density). Hence there is a number $q^* \in (\frac{1}{2}, 1)$ such that $H(q^*) = 0$ and every number q^* for which $H(q^*) = 0$ is in $(\frac{1}{2}, 1)$. That is, the game has a quality threshold equilibrium in which the threshold is in $(\frac{1}{2}, 1)$ and in every quality threshold equilibrium the threshold lies in this interval.

7.2 Unanimity rule

For a decision made by unanimity rather than plurality rule, the inference that an individual makes about the state from the fact that her vote is pivotal has a simple and striking implication. Assume, as before, that there are two alternatives, a and b. Alternative a is the default; the outcome is b if and only if every individual votes for b. For this rule the implication of abstention is the same as that of voting for a, so assume that the only actions available to each individual are *vote for* a and *vote for* b.

Suppose first, as in a plurality rule voting game with two alternatives and asymmetric information, that some individuals are perfectly informed and others are uninformed. The strategy profile in which every informed individual votes for a in state α and for b in state β and every uninformed individual votes for b is a Nash equilibrium of the Bayesian game. The reason is that, given the strategies of the other individuals, (*i*) a change in the action specified by any informed individual's strategy in state α either does not affect the outcome or, if only one individual is informed, changes it to b, while a change in the action in state β changes the outcome from b to a, and (*ii*) a change in any uninformed individual's strategy does not affect the outcome in state α (given the presence of at least one informed individual) and changes the outcome from b to a in state β. In fact, this strategy profile is the only equilibrium in the sense of Definition 7.3, as you are asked to show in the next exercise.

Exercise 7.2: Equilibria of unanimity rule voting game

Show that in the variant of a plurality rule voting game with two alternatives and asymmetric information in which the decision is made by unanimity rule, with a the default, the only equilibrium in the sense of Definition 7.3 is the strategy profile in which every informed individual votes for a in state α and b in state β and every uninformed individual votes for b.

The point is that under unanimity rule the only way an uninformed individual can hand the decision to the informed individuals is by voting for the non-default alternative, because if she votes for the default alternative then that alternative is the outcome regardless of the other individuals' votes.

If the qualities of the individuals' signals are less extreme, similar considerations lead to the conclusion that when the number of individuals is large, the strategy profile in which every individual votes for the alternative that is more likely to be best according to her signal is *not* a Nash equilibrium. Suppose that, as in the model in Section 7.1.3, the individuals are a priori identical. Every individual believes initially that the state is α with probability π and then receives

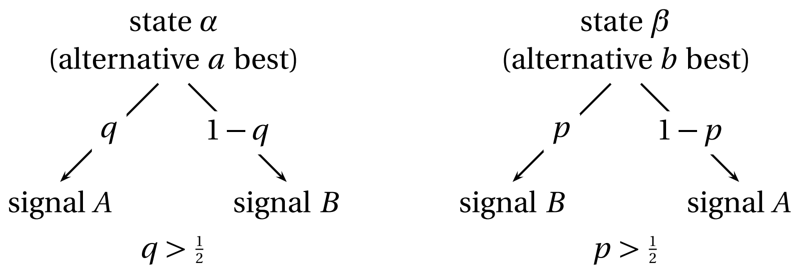


Figure 7.8 The processes generating signals in a model of unanimity rule in which each individual is a priori identical.

one of two signals, as shown in Figure 7.8. If the state is α, in which everyone agrees that alternative a is best, each individual independently gets the signal A with probability q and the signal B with probability $1-q$, where $\frac{1}{2} < q < 1$. If the state is β, in which everyone agrees that alternative b is best, each individual independently gets the signal B with probability p and the signal A with probability $1-p$, where $\frac{1}{2} < p < 1$. (The probabilities p and q are known, unlike the probabilities q_i in a plurality rule voting game with two alternatives and uncertain signal qualities.) Given the asymmetry of the alternatives, I allow the payoffs to be asymmetric: each individual's payoff is

$$\text{state } \alpha: \begin{cases} v_a & \text{if outcome } a \\ -w_b & \text{if outcome } b \end{cases} \qquad \text{state } \beta: \begin{cases} v_b & \text{if outcome } b \\ -w_a & \text{if outcome } a \end{cases}$$

where $v_a > 0$, $v_b > 0$, $w_a > 0$, and $w_b > 0$.

We can model this situation as the following Bayesian game. As for the model of a plurality rule voting game with two alternatives and uncertain signal qualities in Section 7.1.3, a state in the game, which captures all the uncertain features of the environment relevant to the individuals, includes a specification of the profile of signals that they receive. However, outside the definition I continue to refer to α and β as "states".

Definition 7.7: Unanimity rule voting game with two alternatives and asymmetric information

A *unanimity rule voting game with two alternatives and asymmetric information* $\langle\{a,b\},n,(\alpha,\beta),\{A,B\},\pi,(p,q),(v_a,v_b,w_a,w_b)\rangle$, where a and b are alternatives, $n \geq 2$ is an integer, α and β are states of nature, A and B are signals, $\pi \in (0,1)$, $p \in (\frac{1}{2},1)$, $q \in (\frac{1}{2},1)$, $v_a > 0$, $v_b > 0$, $w_a > 0$, and $w_b > 0$ is the following Bayesian game.

Players

A set N with n members.

States

The set of states is the set of pairs $(\omega, (s_j)_{j\in N})$ where $\omega \in \{\alpha, \beta\}$ (the state of nature) and $s_j \in \{A, B\}$ for each $j \in N$ (j's signal).

Actions

The set of actions of each player is $\{\textit{vote for } a, \textit{vote for } b\}$.

Signals

The set of signals that each player may receive is $\{A, B\}$ and the signal function τ_i of each player i is defined by $\tau_i(\omega, (s_j)_{j\in N}) = s_i$ for each $\omega \in \{\alpha, \beta\}$ and each profile $(s_j)_{j\in N}$ of signals.

Prior beliefs

For each $k \in \{1, \ldots, n\}$, every player assigns probability $\pi q^k (1-q)^{n-k}$ to each state $(\alpha, (s_j)_{j\in N})$ for which $s_j = A$ for k players and $s_j = B$ for the remaining $n-k$ players, and probability $(1-\pi)p^k(1-p)^{n-k}$ to each state $(\beta, (s_j)_{j\in N})$ for which $s_j = B$ for k players and $s_j = A$ for the remaining $n-k$ players.

Payoffs

The payoff of each player for an action profile in which all players vote for b is

$$\begin{cases} -w_b & \text{if the state of nature is } \alpha \\ v_b & \text{if the state of nature is } \beta \end{cases}$$

and her payoff for every other action profile is

$$\begin{cases} v_a & \text{if the state of nature is } \alpha \\ -w_a & \text{if the state of nature is } \beta. \end{cases}$$

I argue that if the number of individuals is sufficiently large, the strategy profile in which every individual votes for a if her signal is A and for b if her signal is B is not a Nash equilibrium of such a game. The reason derives from the fact that under unanimity rule, the vote of any individual i affects the outcome only if all the other individuals vote for b: if at least one of the other individuals votes for a, the outcome is a regardless of i's vote. Thus i's voting for b is optimal if and only if it yields her an expected payoff at least as high as the expected payoff from her voting for a, given the probabilities of the states implied by the fact that all the remaining individuals vote for b. Under the strategy profile we are considering, each remaining individual votes for b only if her signal is B, so conditional on all remaining individuals voting for b, the probability that the state is β is high if the number of individuals is large: given $p > \frac{1}{2}$ and $q > \frac{1}{2}$, the probability that every other individual receives a signal of B is larger if the state is β than if it is α, and the ratio of these probabilities approaches 1 as the number of individuals increases without bound. Thus when the number of individuals is large and the other individuals' votes are such that i's vote affects the outcome, then the state is likely to be β, so i should vote for b.

Proposition 7.3: Voting according to signal not Nash equilibrium of unanimity rule voting game

For each integer $n \geq 2$ let $\Gamma(n) = \langle\{a,b\}, n, (\alpha,\beta), \{A,B\}, \pi, (p,q), (v_a, v_b, w_a, w_b)\rangle$ be a unanimity rule voting game with two alternatives and asymmetric information. (The parameters other than n are fixed.) There is a number n^* such that if $n > n^*$ then the strategy profile in which every individual votes for a if she receives the signal A and for b if she receives the signal B is not a Nash equilibrium of $\Gamma(n)$.

Proof

Consider an individual i who receives the signal A. Her vote affects the outcome only if every other individual votes for b and hence only if every other individual receives the signal B. In this case, the outcome is a if she votes for a and b if she votes for b, so her gain from voting for a rather than b is $v_a + w_b$ if the state is α and $-v_b - w_a$ if the state is β. Thus her expected gain from voting for a rather than b is

$$\Pr(\text{state } \alpha \;\&\; n-1 \text{ other signals } B \mid i\text{'s signal } A)(v_a + w_b) \\ - \Pr(\text{state } \beta \;\&\; n-1 \text{ other signals } B \mid i\text{'s signal } A)(v_b + w_a).$$

Now,

$$\begin{aligned} &\Pr(\text{state } \alpha \;\&\; n-1 \text{ other signals } B \mid i\text{'s signal } A) \\ &\qquad = \frac{\Pr(\text{state } \alpha \;\&\; n-1 \text{ other signals } B \;\&\; i\text{'s signal } A)}{\Pr(i\text{'s signal } A)} \\ &\qquad = \frac{\Pr(n-1 \text{ other signals } B \;\&\; i\text{'s signal } A \mid \text{state } \alpha)\Pr(\text{state } \alpha)}{\Pr(i\text{'s signal } A)} \\ &\qquad = \frac{(1-q)^{n-1} q\pi}{q\pi + (1-p)(1-\pi)}, \end{aligned}$$

and similarly

$$\Pr(\text{state } \beta \text{ \& } n-1 \text{ other signals } B \mid i\text{'s signal } A) = \frac{p^{n-1}(1-p)(1-\pi)}{q\pi + (1-p)(1-\pi)}.$$

Hence i's expected payoff from voting for a is at least her expected payoff from voting for b if and only if

$$\frac{(1-p)(1-\pi)}{q\pi}\left(\frac{p}{1-q}\right)^{n-1} \leq \frac{v_a + w_b}{v_b + w_a}.$$

Given that $1-q < \frac{1}{2} < p$, the left-hand side of this inequality increases without bound as n increases. So for any given values of v_a, v_b, w_a, and w_b, for n sufficiently large, type A of individual i prefers to vote for b than for a if every other individual votes for a when her signal is A and for b when her signal is B.

A similar argument shows that the same is true for an individual who receives the signal B. Thus if n is sufficiently large then the strategy profile in which every individual votes for a if she receives the signal A and for b if she receives the signal B is not a Nash equilibrium of $\Gamma(n)$.

What *is* a Nash equilibrium of the game? If every individual votes for a regardless of her signal then a change in any individual's action has no effect on the outcome. Thus this strategy profile is a Nash equilibrium. Under some conditions, the strategy profile in which every individual votes for b regardless of her signal is also a Nash equilibrium.

Exercise 7.3: Nash equilibria of unanimity rule voting game

Find conditions under which a unanimity rule voting game with two alternatives and asymmetric information has a Nash equilibrium in which every individual votes for b regardless of her signal.

In addition, for some values of the parameters the game has a mixed strategy equilibrium in which type A of each individual votes for both a and b with positive probability, the same for each individual, and type B of each individual votes for b with probability 1.

Proposition 7.4: Symmetric mixed strategy equilibrium of unanimity rule voting game

Let $\langle\{a,b\},n,(\alpha,\beta),\{A,B\},\pi,(p,q),(v_a,v_b,w_a,w_b)\rangle$ be a unanimity rule voting game with two alternatives and asymmetric information. Let

$$\sigma^*(A)=\frac{p-(1-q)X(n)}{qX(n)-(1-p)} \qquad \text{where } X(n)=\left(\frac{v_a+w_b}{v_b+w_a}\frac{q}{1-p}\frac{\pi}{1-\pi}\right)^{1/(n-1)}.$$

If $0<\sigma^*(A)<1$ then the game has a mixed strategy equilibrium in which the strategy of type A of every individual votes for b with probability $\sigma^*(A)$ and the strategy of type B of every individual votes for b with probability 1. For some number $\hat{n}$, if $n>\hat{n}$ the game has no other equilibrium in which type A of every individual uses the same strategy, type B of every individual uses the same strategy, and at least one of these strategies assigns positive probabilities to both voting for a and voting for b.

Proof

Consider a mixed strategy equilibrium in which type A of every individual votes for b with the same probability and type B of every individual votes for b with the same probability. Denote these probabilities by $\sigma(A)$ and $\sigma(B)$.

An individual's vote affects the outcome only if all the other individuals vote for b, so the gain of an individual of type $T\in\{A,B\}$ from voting for a rather than b is

$$\begin{aligned}&\Pr(\text{state } \alpha \text{ \& } n-1 \text{ other individuals vote } b \mid i\text{'s signal } T)(v_a+w_b)\\&\quad-\Pr(\text{state } \beta \text{ \& } n-1 \text{ other individuals vote } b \mid i\text{'s signal } T)(v_b+w_a).\end{aligned}$$

Using the logic in the proof of Proposition 7.3 to transform the probabilities, this gain is positive or negative according to the following inequality:

$$\frac{\Pr(n-1 \text{ others vote } b \text{ \& } i\text{'s signal } T \mid \text{state } \beta)\Pr(\text{state } \beta)}{\Pr(n-1 \text{ others vote } b \text{ \& } i\text{'s signal } T \mid \text{state } \alpha)\Pr(\text{state } \alpha)} \lesseqqgtr \frac{v_a+w_b}{v_b+w_a}.$$

For $T=A$, the left-hand side of this inequality is

$$\frac{1-p}{q}\frac{1-\pi}{\pi}\left(\frac{(1-p)\sigma(A)+p\sigma(B)}{q\sigma(A)+(1-q)\sigma(B)}\right)^{n-1}, \tag{7.5}$$

and for $T = B$ it is

$$\frac{p}{1-q}\frac{1-\pi}{\pi}\left(\frac{(1-p)\sigma(A)+p\sigma(B)}{q\sigma(A)+(1-q)\sigma(B)}\right)^{n-1}. \tag{7.6}$$

Given that $p > \frac{1}{2}$ and $q > \frac{1}{2}$, the first of these expressions is smaller than the second. Thus if type A of an individual is indifferent between voting for a and voting for b then type B prefers to vote for b, and if type B of an individual is indifferent between voting for a and voting for b then type A prefers to vote for a. Hence in an equilibrium in which $0 < \sigma(B) < 1$, every individual of type A votes for a, so that $\sigma(A) = 0$, and in an equilibrium in which $0 < \sigma(A) < 1$, every individual of type B votes for b, so that $\sigma(B) = 1$.

If $\sigma(A) = 0$ then (7.6) is $(p/(1-q))((1-\pi)/\pi)(p/(1-q))^{n-1}$, which increases without bound as n increases, so that for any values of v_a, v_b, w_a, and w_b, type B prefers to vote for b when n is sufficiently large. Thus for sufficiently large values of n no equilibrium exists with $0 < \sigma(B) < 1$.

If $\sigma(B) = 1$ then the equality of (7.5) with $(v_a + w_b)/(v_b + w_a)$ implies that $\sigma(A) = \sigma^*(A)$.

The condition $0 < \sigma^*(A) < 1$ is equivalent to $(1-p)/q < X(n) < p/(1-q)$ and $X(n) > 1$, and the second of these conditions is equivalent to $q\pi(v_a + w_b) > (1-p)(1-\pi)(v_b + w_a)$. If this last condition is not satisfied, the game has an equilibrium in which all individuals vote for b regardless of their signals, as you know if you have done Exercise 7.3.

As n increases without bound, $X(n)$ approaches 1, so that $\sigma^*(A)$ approaches 1. Thus for large values of n, type A of each individual votes for b with high probability (and type B votes for b with probability 1). For $\pi = \frac{1}{2}$, $v_a = v_b = 0$, $w_a = 1 - w_b$, and $p = q$, Feddersen and Pesendorfer (1998) calculate for this equilibrium the limits of the probabilities that a is selected in state β and b is selected in state α as the number of individuals increases without bound, and show that these limits are positive. Thus for this example, even in a large population, in each state the wrong alternative is selected with positive probability.

Note that this result considers only mixed strategy equilibria in which every individual's strategy is the same. The game also may have mixed strategy equilibria in which the individuals' strategies differ.

Finally, my comments about the difficulty of interpreting Nash equilibria of voting games at the end of Section 3.2 apply with equal if not more force to mixed strategy equilibria. Individuals do not typically engage repeatedly in similar voting games, so that the steady state interpretation of equilibrium does not fit such games well, and good interpretations of mixed strategy equilibria of one-off games are lacking (see Section 3.2 of Osborne and Rubinstein (1994) for an extended discussion).

7.3 Majority rule with diverse preferences: an example

In the model in Section 7.1, independents are like-minded, so that uninformed individuals are happy to effectively delegate their votes to informed individuals. In this section, I briefly consider a model in which individuals disagree about the best policy in each state, so that an uninformed individual who knows how an informed individual votes wants to cast the opposite vote.

Suppose that a policy has several possible outcomes, each of which benefits a different group of people and harms the rest. Some people know which outcome will occur, but most do not. If you are among the uninformed and see people unlike you supporting the policy, you may reasonably infer that it benefits them and hence harms you, so that you should vote against it. But even if you do not observe others' support for the policy, you may still be able to reason that you should vote against it: you may be able to infer that for configurations of votes for which your vote is pivotal—the only ones that matter for your decision—informed individuals must be voting for it.

I present a model of a simple situation with these features. For a specific environment with diverse preferences, the model illustrates the implications of the type of reasoning underlying the equilibria of the models in the previous sections: when deciding how to vote, an individual should take into account the implications for the other individuals' information of her vote being pivotal.

Three individuals face a choice between policy a, which yields each of them a certain payoff of 0, and policy b, the outcome of which depends on an unknown state of nature. There are three equally-likely states, α_1, α_2, and α_3. Absent any additional information, each individual is better off in expectation under policy b than under policy a. However, in each state α_i, individual i's payoff is lower than it is under a while the other two individuals' payoffs are higher. Thus if no individual knows the state, they should all vote for b, and if they all know the state, two of them should vote for b, so that in both cases majority rule leads to the outcome b.

Suppose that each individual is independently informed of the state with the same probability λ. If λ is positive but small, the game has an equilibrium in which each individual i votes for b only if she is informed that the state differs from α_i, so that the probability that b is the outcome of a majority vote is small. The key part of an individual i's reasoning is that her vote makes a difference only if one of the other individuals votes for a and one votes for b, which means, given the strategy profile, that one of them is informed and got a signal that the state is one of the two that benefits her, and hence with probability $\frac{1}{2}$ harms i. Policy b is better than a for each individual if each state is equally likely, but if a probability of $\frac{1}{2}$ that the state is bad for i is enough for her to prefer a, the strategy profile is an equilibrium. The next exercise invites you to fill in the details for specific payoff functions.

Exercise 7.4: Voting between a safe alternative and a risky one with distributional consequences

Three individuals decide by majority vote (without the option to abstain) whether to implement policy a or policy b. There are three equally-likely states of nature, α_1, α_2, and α_3. Policy a yields each individual the payoff 0 regardless of the state, whereas in state α_i ($i = 1, 2, 3$) policy b yields the payoff -9 to individual i and 6 to each of the other two individuals. Each individual is independently informed of the state with probability λ.

(*a*) For (*i*) $\lambda = 0$ and (*ii*) $\lambda = 1$ and any state α_i, formulate this situation as a strategic game and find its Nash equilibria in which no individual's action is weakly dominated. (*b*) For $\lambda \in (0,1)$, formulate the situation as a Bayesian game and show that if $0 < \lambda \leq \frac{1}{3}$ then the game has a Nash equilibrium in which each individual i votes for b if she is informed that the state is α_j for $j \neq i$ and votes for a if she is informed that the state is α_i or is not informed, with the result that the probability that policy a is implemented is close to 1 when λ is close to 0.

Notes

Sections 7.1.1 and 7.1.2 are based on Feddersen and Pesendorfer (1996). The notion of equilibrium is a variant of the one in Osborne and Turner (2010), which differs from the one used by Feddersen and Pesendorfer but retains the same spirit. (Feddersen and Pesendorfer's results are asymptotic in the number of individuals, in contrast to the results here, which hold for any number of individuals.) The proof of Proposition 7.1 is taken (in parts verbatim) from Osborne and Turner (2010, 182–184). Section 7.1.3 is based on McMurray (2013), who studies a model in which the number of individuals is random.

Section 7.2 is based on Austen-Smith and Banks (1996) and Feddersen and Pesendorfer (1998). This work presents unanimity rule as the decision-making process for juries in some jurisdictions, with the default outcome acquittal. I do not give the model this interpretation because juries that use a version of unanimity rule treat conviction and acquittal symmetrically, with the outcome a retrial if unanimity is not achieved. In addition, deliberation appears to be an essential feature of a jury's decision-making process.

Section 7.3 is based on Ali et al. (2025), who study a general model and establish a general result. They interpret their result as giving conditions under which an equilibrium reflects "zero-sum thinking".

Exercise 7.1 is an example of a variant of the general model of Kim and Fey (2007).

Solutions to exercises

Exercise 7.1

The game has two equilibria. In both equilibria all informed independents vote for a in state α and for b in state β. In one equilibrium, σ^a, all uninformed independents vote for a, and in the other equilibrium, σ^b, they all vote for b. If $n_i > n_u$ then in both equilibria the outcome is a in state α and b in state β. If $n_i = n_u$ then the outcome of the first equilibrium is a in state α and a tie in state β, and the outcome of the second equilibrium is a tie in state α and b in state β. If $n_i < n_u$ then the outcome of the first equilibrium is a in both states and the outcome of the second equilibrium is b in both states.

Consider the strategy profile σ^a. The argument that no deviation from σ^a by an informed independent is desirable is the same as the argument in Step 1 in the proof of Proposition 7.1. Now suppose that an uninformed independent switches to vote for b or to abstain. If $n_i \geq n_u$ then for the smallest number of player-types for which this deviation affects the outcome, it changes a tie to a win for b in state β and does not affect the outcome in state α, so it is undesirable. If $n_i < n_u$ then for the smallest number of player-types for which this deviation affects the outcome, it changes a win for a in state β into a tie and does not affect the outcome in state α, so it is also undesirable. Thus σ^a is an equilibrium.

A similar argument shows that σ^b is an equilibrium.

I now argue that no strategy profile that differs from σ^a and σ^b is an equilibrium. First, by the argument in Step 4 in the proof of Proposition 7.1, in every equilibrium all informed independents vote for a in state α and for b in state β. Now let σ be a strategy profile for which all informed independents act in this way and some uninformed independents vote for a while others vote for b. If the outcome is a in both states then the margin of victory in state β is less than that in state α, so that if an uninformed independent switches from voting for b to voting for a then for the smallest number of player-types for which the outcome is affected, it changes from a tie to a win for a in state β and does not change in state α. Thus this deviation is desirable. By a similar argument, if the outcome is b in both states, a deviation by an uninformed independent from voting for a to voting for b is desirable. Finally, if the outcome is a in state α and b in state β then a deviation from voting for a to voting for b is desirable if the margin of victory for a in state α is at least the margin of victory for b in state β (using $\pi > \frac{1}{2}$ in the case that the margins of victory are the same), and otherwise a deviation from voting for b to voting for a is desirable.

If $n_u > n_i$ then the outcomes of the equilibria are a in both states and b in both states. If all the independents were informed, the outcome would, by contrast, be b in state α and a in state β.

Exercise 7.2

I first argue that the strategy profile is an equilibrium in the sense of Definition 7.3. A change in any uninformed individual's strategy or the action specified by any informed individual's strategy in state β changes the outcome in state β to a, making her worse off. Now consider the effect of a change in the action specified by an informed individual i's strategy in state α. If there are no other informed individuals, the outcome changes to b in that state, making i worse off. If there are other informed individuals, the outcome does not change. In this case, the smallest number of individuals whose failure to vote causes the outcome to change is the number of other informed individuals; their failure to vote would cause the change in i's strategy to change the outcome from a to b, making her worse off. Thus no change in any individual's strategy is desirable.

I now argue that no other strategy profile is an equilibrium.

Informed individual

Consider an informed individual who is voting for b in state α. Suppose she switches to voting for a. If no other individual is voting for a, the outcome improves from b to a. If some of the other individuals are voting for a then the smallest number of individuals whose failure to vote affects the outcome is the number of such individuals, and their failure to vote would mean that the change in the informed individual's action improves the outcome from b to a. Thus a change in the individual's action from b to a in state α is desirable.

By a similar argument, the change to vote for b for an informed individual voting for a in state β is desirable.

We conclude that in every equilibrium every informed individual votes for a in state α and for b in state β.

Uninformed individual

If an uninformed individual votes for a, the outcome is a in state β, and if she switches to vote for b the outcome does not get worse in either state, regardless of how many individuals (if any) fail to vote (given that the informed individuals vote for a in state α) and improves in state β if either no other individual votes for a or all the individuals planning to vote for a fail to vote.

Exercise 7.3

Suppose that every individual other than i votes for b regardless of her signal. Then i's vote determines the outcome (and the other individuals' votes convey no information about the state).

Suppose that i's signal is A. Then her expected payoff if she votes for a is

$$\Pr(\alpha \mid \text{signal } A)v_a - \Pr(\beta \mid \text{signal } A)w_a = \frac{q\pi v_a - (1-p)(1-\pi)w_a}{q\pi + (1-p)(1-\pi)}$$

(using Bayes' rule) and her expected payoff if she votes for b is

$$\Pr(\beta \mid \text{signal } A)v_b - \Pr(\alpha \mid \text{signal } A)w_b = \frac{(1-p)(1-\pi)v_b - q\pi w_b}{q\pi + (1-p)(1-\pi)},$$

so that she optimally votes for b if and only if $q\pi v_a - (1-p)(1-\pi)w_a \leq (1-p)(1-\pi)v_b - q\pi w_b$, or $q\pi(v_a + w_b) \leq (1-p)(1-\pi)(v_b + w_a)$.

Now suppose that i's signal is B. Then her expected payoff if she votes for a is

$$\Pr(\alpha \mid \text{signal } B)v_a - \Pr(\beta \mid \text{signal } B)w_a = \frac{(1-q)\pi v_a - p(1-\pi)w_a}{(1-q)\pi + p(1-\pi)}$$

and her expected payoff if she votes for b is

$$\Pr(\beta \mid \text{signal } B)v_b - \Pr(\alpha \mid \text{signal } B)w_b = \frac{p(1-\pi)v_b - (1-q)\pi w_b}{(1-q)\pi + p(1-\pi)},$$

so that she optimally votes for b if and only if $(1-q)\pi v_a - p(1-\pi)w_a \leq p(1-\pi)v_b - (1-q)\pi w_b$ or $(1-q)\pi(v_a + w_b) \leq p(1-\pi)(v_b + w_a)$.

Given that $p > \frac{1}{2}$ and $q > \frac{1}{2}$, the second inequality is satisfied whenever the first inequality is satisfied, so that i's voting for b regardless of her signal is optimal if and only if the first inequality is satisfied.

Thus the game has a Nash equilibrium in which every individual votes for b regardless of her signal if and only if $q\pi(v_a + w_b) \leq (1-p)(1-\pi)(v_b + w_a)$.

Exercise 7.4

a. The set of players is $N = \{1,2,3\}$ and the set of actions of each player is $\{\textit{vote for } a, \textit{vote for } b\}$.

For $\lambda = 0$, the payoffs are $(0,0,0)$ for each action profile in which two or more players vote for a and $(1,1,1)$ for each action profile in which two or more players vote for b (because $\frac{1}{3}(-9+6+6) = 1$). Each player's action of voting for b weakly dominates her action of voting for a, so the action profile in which each player's action is *vote for b* is the only Nash equilibrium in which no player's action is weakly dominated. The outcome is policy b.

For $\lambda = 1$ and state α_i, the payoffs are $(0,0,0)$ for each action profile in which two or more players vote for a, and z with $z_i = -9$ and $z_j = 6$ for $j \neq i$ for each action profile in which two or more players vote for b. Player i's action of voting for a weakly dominates her action of voting for b whereas each of the other player's action of voting for b weakly dominates her action of voting for a. So the game has a single Nash equilibrium in which no player's action is weakly dominated, and in this equilibrium player i votes for a and the other two players vote for b, so that the outcome is policy b.

b. Here is a Bayesian game that models the situation.

Players
The set $N = \{1,2,3\}$.

States
The set of states is the set of pairs $(\omega, (s_j)_{j\in N})$ where $\omega \in \{\alpha_1, \alpha_2, \alpha_3\}$ (the state of nature) and $s_j \in \{I, U\}$ for each $j \in N$ (j's signal), where I stands for "informed" and U stands for "uninformed".

Actions
The set of actions of each player is $\{\textit{vote for } a, \textit{vote for } b\}$.

Signals
The set of signals that each player may receive is $\{I, U\}$ and the signal function τ_i of each player i is defined for each $\omega \in \{\alpha_1, \alpha_2, \alpha_3\}$ and each profile $(s_j)_{j\in N}$ of signals by $\tau_i(\omega, (s_j)_{j\in N}) = \omega$ if $s_i = I$ and $\tau_i(\omega, (s_j)_{j\in N}) = U$ if $s_i = U$.

Prior beliefs
For each $k \in \{0,1,2,3\}$, every player assigns probability $\frac{1}{3}\lambda^k(1-\lambda)^{3-k}$ to each state $(\omega, (s_j)_{j\in N})$ for which $s_j = I$ for k players and $s_j = U$ for the remaining $3-k$ players.

Payoffs

The payoff of each player i for an action profile in which two or more players vote for a is 0, and her payoff for an action profile in which two or more players vote for b is -9 if the state of nature is α_i and 6 otherwise.

Consider the strategy profile in which each player i votes for b if she is informed that the state is α_j for $j \neq i$ and otherwise votes for a. If i is informed that the state is α_i then she optimally votes for a. Now suppose that she is uninformed. If the other two players both vote for a or both vote for b, player i's vote does not affect the outcome, so she should base her decision on the impact on the outcome of her vote if the other players follow their strategies and one of them votes for a and one votes for b, given the implications of these votes for the probabilities of the states of nature. Denote by E the event that one of the other players votes for a and one votes for b. What is the probability that the state is α_i conditional on E, given that the players follow their strategies? It is

$$\frac{\Pr(E \mid \alpha_i)\Pr(\alpha_i)}{\Pr(E)} = \frac{2\lambda(1-\lambda)\cdot\frac{1}{3}}{(2\lambda(1-\lambda)+2\lambda)\cdot\frac{1}{3}} = \frac{1-\lambda}{2-\lambda}.$$

Thus given that one of the other players votes for a and one votes for b, player i's expected payoff from voting for b is

$$\frac{1-\lambda}{2-\lambda}\cdot(-9) + \frac{1}{2-\lambda}\cdot(6) = \frac{-3+9\lambda}{2-\lambda}.$$

Her payoff from voting for a is 0, so she optimally votes for a if $\lambda \leq \frac{1}{3}$.

Hence if $\lambda \leq \frac{1}{3}$ then the strategy profile in which each player i votes for b if she is informed that the state is α_j for $j \neq i$ and otherwise votes for a is a Nash equilibrium. The probability that the outcome is policy b is thus the probability that at least two players are informed that the state is one of the two in which their payoff is 6, namely $3\cdot(\frac{2}{3}\lambda)^2(1-\frac{2}{3}\lambda) + (\frac{2}{3}\lambda)^3 = \frac{4}{3}\lambda^2(1-\frac{4}{9}\lambda)$, which converges to 0 as λ converges to 0.

III Electoral competition

8 Electoral competition: two office-motivated candidates 243
9 Electoral competition: two policy-motivated candidates 295
10 Electoral competition: endogenous candidates 331
11 Distributive politics 375
12 Money in electoral competition 413
13 Two-period electoral competition with imperfect information 439

8 Electoral competition: two office-motivated candidates

8.1 General model 247
8.2 Spatial model 253
8.3 Candidates uncertain of citizens' preferences 258
8.4 Costly voting 266
8.5 Citizens with preferences over candidates 274
8.6 Electing a legislature 282
8.7 Interpreting Nash equilibrium 287

Collective decisions in some societies are made by a legislature consisting of a relatively small number of individuals. Typically, some individuals are candidates for membership of the legislature and the members of a larger subset of the society, whom I call citizens, cast votes to determine which candidates are elected. Most of this chapter is devoted to models of a simple version of such a legislature, with a single member.

We have many options when designing a model. The set of candidates may be exogenous, or individuals may choose whether to be candidates; the citizens may or may not know the candidates' preferences; the candidates may or may not know the citizens' preferences; each candidate may or may not be able to commit to act in the legislature according to given preferences, which may differ from her own; the candidates may be motivated by a desire to win election, or they may have other motivations—for example, they may care about the policy ultimately chosen by the legislature; each candidate may make a decision without knowing the decisions of the remaining candidates, or the candidates may make their decisions sequentially, with each candidate observing the decisions of her predecessors.

When discussing models of electoral competition, I refer to an alternative as a *position*. A motivation for using this nomenclature is that legislatures decide multiple issues, some of which may be unknown at the time of an election. Candidates for legislative office may state principles that will guide their behavior if elected—political positions—rather than specifying the alternatives they will select if elected. However, although the language is different, the set of positions is simply a set, like the set of alternatives in the earlier chapters.

Here are the assumptions I start with in this chapter.

- The set of candidates
  - is given exogenously, distinct from the set of citizens
  - has two members.
- Each candidate
  - if elected, becomes the sole decision-maker (the legislature has a single member)
  - chooses a position, which she is committed to implement if she is elected
  - is motivated by the desire to win election
  - knows the citizens' preferences.
- Each citizen
  - cares about the position ultimately implemented
  - votes ("sincerely") for the candidate whose position she prefers.
- The candidate who receives the most votes is elected.

Subsequently I present variants of this model that retain the assumption of two exogenously given candidates who aim to win election. In the next chapter I consider models in which the candidates care about the policy ultimately implemented, rather than caring exclusively about winning election, and in Chapter 10 I present models in which individuals decide whether to become candidates. In Chapter 12 I present a model in which the candidates can affect their chances of winning by spending money on a campaign.

In this chapter and the next I restrict the analysis to models with two candidates not because most elections, or even most elections in which plurality rule determines the winner, involve two candidates, but because models with many candidates require a significantly different analysis.

In some of the models I discuss, the outcome of an election is uncertain. In these models, a candidate's aim to win election is operationalized by the assumption that she aims to maximize her probability of winning, which seems to be a fundamental objective of candidates for office. Two other possible assumptions are that each candidate aims to maximize her expected vote share or her expected plurality. I do not present models that make either of these assumptions mainly because they are generally inconsistent with the maximization of the probability of winning. Specifically, for some distributions α and β over electoral outcomes for which a candidate's expected vote share or expected plurality under α is greater than it is under β, the candidate's probability of winning under α is less than it is under β.

The originators of some of the models I present interpret the candidates as parties; because they are single decision-makers, I generally stick with the term "candidates".

Synopsis

In Section 8.1 I present a model with a finite number of citizens in which two candidates simultaneously choose positions from an arbitrary set and the candidate whose position is favored by more citizens wins. Proposition 8.1 shows that in any Nash equilibrium both candidates choose a Condorcet winner of the underlying collective choice problem, so that in particular the game has a Nash equilibrium only if the collective choice problem has a Condorcet winner. An implication of this result combined with Propositions 1.4 and 1.5 is that if the number of citizens is odd and the citizens' preferences are single-peaked or single-crossing then the game has a unique Nash equilibrium. In this equilibrium, each candidate's position is the median of the citizens' favorite positions with respect to the ordering of the alternatives if the preferences are single-peaked, and the favorite position of the median individual with respect to the ordering of the individuals if the preferences are single-crossing (Corollary 8.2). In particular, the candidates' positions are the same, a feature of the equilibria of most models in this chapter.

For the variant of this model in which the candidates move sequentially, Proposition 8.3 shows that for a collective choice problem that has a Condorcet winner, the outcome of a subgame perfect equilibrium is the same as the outcome of a Nash equilibrium of the simultaneous-move game, and for a collective choice problem without a Condorcet winner, the outcome of every subgame perfect equilibrium is that the second-mover wins.

Section 8.2 studies another variant of the model, in which the set of citizens is a continuum and the set of alternatives is an interval of real numbers. This variant is used as a component of several of the models in subsequent chapters. Proposition 8.4 shows that, as for the case in which the number of citizens is finite, the game has a unique Nash equilibrium, and in this equilibrium the position of each candidate is the median of the citizens' favorite positions.

In the models considered so far, the candidates know the citizens' preferences. Section 8.3 considers two models in which the candidates are uncertain of these preferences. In the first model, the candidates share a common belief about the distribution of the median of the citizens' favorite positions. The resulting game has a unique Nash equilibrium, in which the candidates' common position is the median of this distribution (Proposition 8.5). In the second model, each candidate gets a private signal about the median of the citizens' favorite positions. Proposition 8.6 characterizes any Nash equilibria that exist (there may be none).

An assumption common to the models in Sections 8.1 through 8.3 is that all citizens vote, even when the candidates' positions are the same, in which case no citizen's vote affects the outcome. Section 8.4 analyzes a model in which casting a vote is costly and abstention is an option. For simplicity, the model assumes there is one citizen, whose voting cost is known to her but not to the candidates, who also do not know her preferences over positions. For any pair of positions for the candidates, the citizen votes if her voting cost is less than a cutoff that depends on the difference between her payoffs for the positions of the two candidates. As a consequence, if one candidate's position becomes closer to the other candidate's position then the change in her probability of winning depends on the nature of the citizen's preferences and the distribution of her voting cost. Proposition 8.7 shows that in an equilibrium the candidates' positions are the same, and characterizes the common equilibrium position. Given the coincidence of the candidates' positions, the citizen votes only if her voting cost is zero, an event with probability zero.

Section 8.5 studies models in which the citizens have preferences over the candidates independently of the candidates' positions. In the main model, each candidate is uncertain of these preferences and is thus uncertain whether any given position will lead her to win, given the other candidate's position. Proposition 8.8 gives conditions under which in any equilibrium, if one exists, the candidates' positions are the same. No general result on the existence of an equilibrium is available, but an example shows that there are games for which an equilibrium exists.

Section 8.6 discusses models of legislatures with many members, each of whom is elected in a single district. Order the districts by the median of the favorite positions of the citizens in the district. Suppose that each candidate is associated with one of two parties and decisions in the legislature are made by the party whose candidates win the most districts. Then a model in which each party chooses a single position for all of its candidates has a unique Nash equilibrium, in which the position of each party is the median of the favorite positions of the citizens in the median district. The same is true for a model in which each candidate chooses her position independently and the position of a party is the average of its candidates' positions. However, if the first model is modified by assuming that the citizens' partisanships are uncertain, along the lines of the model in Section 8.5, and each party values winning an additional district even if doing so does not change its minority/majority status in the legislature, then equilibria in which the parties' positions are distinct may exist.

8.1 General model

8.1.1 Simultaneous decisions

The set of possible positions is an arbitrary set. Each of two candidates chooses a position not knowing the other candidate's position and then each of a finite set of citizens casts a vote for the candidate whose position she prefers. The electoral mechanism selects as the winner the candidate with the most votes. Each candidate cares only about whether she wins, preferring to win than to tie than to lose. One way to characterize these preferences is to say that each candidate is office-motivated.

I assume that if the candidates' positions are x_1 and x_2 and the number of citizens who prefer x_1 to x_2 is the same as the number who prefer x_2 to x_1 then the outcome of the election is a tie. This assumption is consistent with each citizen who is indifferent between the candidates' positions not voting, or splitting her vote, casting half a vote for each candidate. (The related assumption that each such citizen votes with equal probability for each candidate requires a formulation of the candidates' preferences regarding lotteries over outcomes.)

Definition 8.1: Electoral competition game with two office-motivated candidates

An *electoral competition game with two office-motivated candidates* $\langle \{1,2\}, \langle I, X, \succsim \rangle \rangle$, where $\langle I, X, \succsim \rangle$ is a collective choice problem in which the set I (of *citizens*) is finite, is the strategic game with the following components.

Players
$\{1,2\}$ (*candidates*).

Actions
The set of actions of each player is X (the set of possible *positions*).

Preferences
For each $(x_1, x_2) \in X \times X$, denote by $O(x_1, x_2)$ be the electoral outcome when each citizen (member of I) votes for the position in $\{x_1, x_2\}$ that

she prefers and the position that receives the most votes wins:

$$O(x_1,x_2) = \begin{cases} \textit{win for } 1 & \text{if } |\{i \in I : x_1 \succ_i x_2\}| > |\{i \in I : x_2 \succ_i x_1\}| \\ \textit{tie} & \text{if } |\{i \in I : x_1 \succ_i x_2\}| = |\{i \in I : x_2 \succ_i x_1\}| \\ \textit{win for } 2 & \text{if } |\{i \in I : x_1 \succ_i x_2\}| < |\{i \in I : x_2 \succ_i x_1\}|. \end{cases}$$

The preference relation $\trianglerighteq_j$ of each player j over pairs of positions satisfies

$$(w_1,w_2) \vartriangleright_j (y_1,y_2) \vartriangleright_j (z_1,z_2)$$

whenever $O(w_1,w_2) = \textit{win for } j$, $O(y_1,y_2) = \textit{tie}$, and $O(z_1,z_2) = \textit{win for } k$, where k is the other player.

The Nash equilibria of such a game and the Condorcet winners of the associated collective choice problem are closely related. In particular, if the collective choice problem has a Condorcet winner then in any Nash equilibrium of the game each candidate's position is a Condorcet winner and the outcome is a tie.

Proposition 8.1: Nash equilibrium of electoral competition game with two office-motivated candidates and Condorcet winner

Let $\langle\{1,2\}, \langle I, X, \succsim\rangle\rangle$ be an electoral competition game with two office-motivated candidates. The outcome of any Nash equilibrium of this game is a tie, and (x_1,x_2) is a Nash equilibrium if and only if both x_1 and x_2 are Condorcet winners of $\langle I, X, \succsim\rangle$, so that in particular the game has a Nash equilibrium if and only if $\langle I, X, \succsim\rangle$ has a Condorcet winner.

Proof

Suppose that (x_1,x_2) is a Nash equilibrium of the game. If $x_1 = x_2$, the outcome $O(x_1,x_2)$ is a tie. If $x_1 \neq x_2$ and either candidate deviates to the position of the other candidate, the outcome becomes a tie, so that $O(x_1,x_2)$ is at least as good as a tie for each candidate, and hence also is a tie.

Thus a pair (x_1,x_2) of positions is a Nash equilibrium if and only if

for all $x_1' \in X$ the outcome $O(x_1',x_2)$ is a tie or a loss for 1
for all $x_2' \in X$ the outcome $O(x_1,x_2')$ is a tie or a loss for 2,

conditions that are satisfied if and only if x_1 and x_2 are Condorcet winners of the collective choice problem $\langle I, X, \succsim\rangle$.

If an alternative is a strict Condorcet winner then it is the only Condorcet winner, so if the collective choice problem has a strict Condorcet winner then in any Nash equilibrium each candidate chooses that position.

Corollary 8.1: Nash equilibrium of electoral competition game with two office-motivated candidates and strict Condorcet winner

Let $\langle\{1,2\},\langle I,X,\succsim\rangle\rangle$ be an electoral competition game with two office-motivated candidates. If $\langle I,X,\succsim\rangle$ has a strict Condorcet winner x^* then (x^*,x^*) is the unique Nash equilibrium of the game.

If the number of citizens is odd and their preferences are single-peaked or single-crossing then the collective choice problem has a strict Condorcet winner. This position is the median of the citizens' favorite positions if the citizens' preferences are single-peaked (Proposition 1.4), and the favorite position of the median citizen if the citizens' preferences are single-crossing (Proposition 1.5), so the next result follows from Corollary 8.1.

Corollary 8.2: Median voter theorem for electoral competition game with two office-motivated candidates

Let $\langle\{1,2\},\langle I,X,\succsim\rangle\rangle$ be an electoral competition game with two office-motivated candidates in which the number of citizens (members of I) is odd.

- If $\langle I,X,\succsim\rangle$ has single-peaked preferences with respect to a linear order $\trianglerighteq$ on X, then the game has a unique Nash equilibrium, and in this equilibrium each candidate's position is the median with respect to $\trianglerighteq$ of the citizens' favorite positions.

- If $\langle I,X,\succsim\rangle$ has single-crossing preferences with respect to a linear order $\geq$ on I and the median individual with respect to $\geq$ has a unique favorite position, say x^*, then the game has a unique Nash equilibrium, and in this equilibrium each candidate's position is x^*.

A strict Condorcet winner is more than a Nash equilibrium action: it weakly dominates all other actions.

Proposition 8.2: Dominant actions in electoral competition game with two office-motivated candidates and strict Condorcet winner

Let $\langle\{1,2\},\langle I,X,\succsim\rangle\rangle$ be an electoral competition game with two office-motivated candidates. If $\langle I,X,\succsim\rangle$ has a strict Condorcet winner x^* then for each candidate the action x^* weakly dominates her other actions.

Proof

Suppose that candidate i chooses x^*. If the other candidate chooses x^*, the outcome is a tie, and if the other candidate chooses any other action, i wins. Now suppose that i chooses an action other than x^*. Then if the other candidate chooses x^*, i loses. No outcome is better for i than her winning, so x^* weakly dominates her other actions.

		other candidate: x^*	other candidate: $\neq x^*$
candidate i	x^*	tie	i wins
	$\neq x^*$	i loses	?

This result is significant because it means that a candidate's choosing the strict Condorcet winner is optimal for her regardless of the other candidate's action. The fact that (x^*,x^*) is a Nash equilibrium means that a candidate's choosing x^* is optimal for her if she believes that the other candidate will choose x^*; the fact that x^* weakly dominates all other actions means that her choosing it is optimal for her regardless of her belief about the other candidate's action. If the other candidate chooses an action different from x^*, the action x^* remains optimal for i even though other actions may also be optimal for her in that case.

8.1.2 Sequential decisions

If the candidates choose positions sequentially, we can model their interaction as an extensive game. In the following game, candidate 1 chooses a position, candidate 2 observes this position, and then candidate 2 chooses a position.

Definition 8.2: Sequential electoral competition game with two office-motivated candidates

A *sequential electoral competition game with two office-motivated candidates* $\langle\{1,2\},\langle I,X,\succsim\rangle\rangle$, where $\langle I,X,\succsim\rangle$ is a collective choice problem in which the set I (of *citizens*) is finite, is the extensive game with perfect information with the following components.

Players
$\{1,2\}$ (*candidates*).

Terminal histories
The set of sequences (x_1, x_2) where $x_i \in X$ for each $i \in \{1,2\}$.

Player function
The function P given by $P(\varnothing) = 1$ and $P(x_1) = 2$ for all $x_1 \in X$.

Preferences
The preference relation of each player over terminal histories (pairs of positions) satisfies the conditions in Definition 8.1.

The set of electoral outcomes in a sequential electoral competition game with two office-motivated candidates is finite (*win for candidate* 1, *tie*, *win for candidate* 2), so every such game has a subgame perfect equilibrium. If the underlying collective choice problem has a strict Condorcet winner, then the outcome of every subgame perfect equilibrium is that both candidates choose this alternative. The following result gives the subgame perfect equilibrium outcomes also for problems that do not have strict Condorcet winners.

Proposition 8.3: Subgame perfect equilibrium of sequential electoral competition game with two office-motivated candidates and Condorcet winner

Let $G = \langle \{1,2\}, \langle I, X, \succsim \rangle \rangle$ be a sequential electoral competition game with two office-motivated candidates.

a. If $\langle I, X, \succsim \rangle$ has no Condorcet winner then a pair (x_1, x_2) is the outcome of a subgame perfect equilibrium of G if and only if the electoral outcome of (x_1, x_2) is a win for candidate 2.

b. If $\langle I, X, \succsim \rangle$ has a Condorcet winner then a pair (x_1, x_2) is the outcome of a subgame perfect equilibrium of G if and only if x_1 is a Condorcet winner of $\langle I, X, \succsim \rangle$ and the electoral outcome of (x_1, x_2) is a tie.

c. If $\langle I, X, \succsim \rangle$ has a strict Condorcet winner x^* then the outcome of every subgame perfect equilibrium of G is (x^*, x^*).

Proof

a. The result follows from the observation that because $\langle I, X, \succsim \rangle$ has no Condorcet winner, for every alternative $x \in X$ there is an alternative $y \in X$ such that the electoral outcome $O(x, y)$ is a win for candidate 2.

b. First suppose that (x_1, x_2) is the outcome of a subgame perfect equilibrium of G. For any alternative chosen by candidate 1, the electoral outcome is a tie if candidate 2 chooses the same alternative, so the electoral outcome of (x_1, x_2) is either a tie or a win for candidate 2. If candidate 1 chooses a Condorcet winner of $\langle I, X, \succsim \rangle$, then for every alternative chosen by candidate 2 the electoral outcome is either a tie or a win for candidate 1. Thus the electoral outcome of (x_1, x_2) is a tie. Hence no alternative beats x_1, so that x_1 is a Condorcet winner of $\langle I, X, \succsim \rangle$.

Conversely, suppose that x_1 is a Condorcet winner of $\langle I, X, \succsim \rangle$ and the electoral outcome of (x_1, x_2) is a tie. Let (s_1^*, s_2^*) be the strategy pair in which $s_1^* = x_1$ and

$$s_2^*(z_1) = \begin{cases} x_2 & \text{if } z_1 = x_1 \\ z_1 & \text{if } z_1 \neq x_1 \text{ and } z_1 \text{ is a Condorcet winner} \\ y_2(z_1) & \text{if } z_1 \text{ is not a Condorcet winner} \end{cases}$$

where for every alternative z_1 that is not a Condorcet winner, $y_2(z_1)$ is an alternative that beats z_1. I argue that (s_1^*, s_2^*) is a subgame perfect equilibrium of G. The outcome of (s_1^*, s_2^*) is (x_1, x_2); by assumption $O(x_1, x_2)$ a tie. If candidate 1 deviates to another alternative z_1, then $O(z_1, s_2^*(z_1))$ is a tie if z_1 is a Condorcet winner and a win for candidate 2 otherwise. Thus s_1^* is optimal for candidate 1 given s_2^*. The action prescribed by candidate 2's strategy s_2^* after each history is optimal because if z_1 is a Condorcet winner then candidate 2 can do no better than tie and if z_1 is not a Condorcet winner then the electoral outcome of $(z_1, s_2^*(z_1))$ is a win for candidate 2.

c. This result follows from (*b*) because if x_1 is the strict Condorcet winner of $\langle I, X, \succsim \rangle$ then the only alternative x_2 for which the electoral outcome of (x_1, x_2) is a tie is $x_2 = x_1$.

The answer to Exercise 1.5 shows that in case (*b*), the alternative x_2 chosen by candidate 2 is not necessarily a Condorcet winner of the collective choice problem.

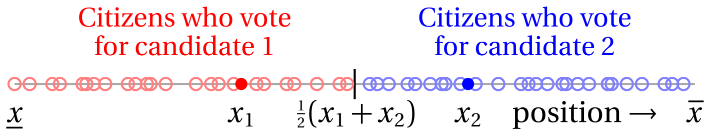


Figure 8.1 Two-candidate electoral competition in which the set of positions is an interval and each citizen's preference relation is symmetric about her favorite position. The favorite position of each citizen is indicated with a small circle, and the positions of the candidates are indicated with small disks.

8.2 Spatial model

8.2.1 One-dimensional set of positions

Suppose that the set X of positions is an interval of numbers, the number of citizens is odd, and the preference relation $\succsim_i$ of each citizen i is single-peaked with respect to $\geq$ (that is, if $x < y < x_i^*$ or $x_i^* < y < x$ then $x_i^* \succ_i y \succ_i x$, where x_i^* is i's favorite position). Then by Corollary 8.2 the electoral competition game with two office-motivated candidates has a unique Nash equilibrium, and in this equilibrium each candidate's position is the median of the citizens' favorite positions (with respect to $\geq$).

Now further assume that the preference relation of every citizen i is symmetric in the sense that $x_i^* - \delta \sim_i x_i^* + \delta$ for every $\delta > 0$. Then for any positions x_1 and x_2 for the candidates with $x_1 < x_2$, a citizen i votes for candidate 1 if $x_i^* < \frac{1}{2}(x_1+x_2)$ and for candidate 2 if $x_i^* > \frac{1}{2}(x_1+x_2)$. The division of votes between the candidates is illustrated in Figure 8.1, where the horizontal line represents the interval of positions and each citizen is identified with her favorite position. As x_1 increases, the dividing line $\frac{1}{2}(x_1+x_2)$ increases, so that the number of votes for candidate 1 increases and the number for candidate 2 falls.

In a variant of this model, the set of citizens is a continuum rather than being finite, the distribution of the citizens' favorite positions has a density, and the support of this distribution is an interval, so that the distribution has a unique median.

Definition 8.3: Median of distribution

Let X be an interval of real numbers and let $F : X \to [0,1]$ be a distribution function (a nondecreasing function with $F(\underline{x}) = 0$ for some $\underline{x} \in X$ and $F(\overline{x}) = 1$ for some $\overline{x} \in X$). A *median* of F is a number x such that $F(x) = \frac{1}{2}$.

This variant may be analyzed with a diagram like Figure 8.2. For any position x, the height of the curve represents the density of citizens with favorite position x; the area under the curve between any two positions x and y represents

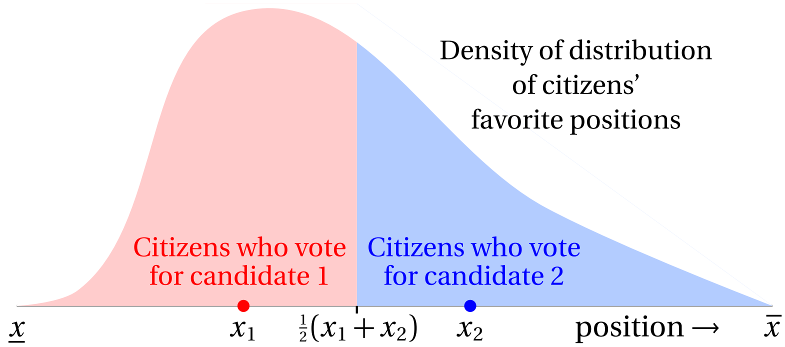


Figure 8.2 Two-candidate electoral competition in which the set of positions is the interval $[\underline{x}, \overline{x}]$ and there is a continuum of citizens.

the fraction of citizens with favorite positions between x and y. The fraction of citizens who prefer x_1 to x_2 is thus the area shaded pink and the fraction who prefer x_2 to x_1 is the area shaded blue.

The symmetry assumption on the citizens' preferences that this variant of the model entails is stronger than the assumptions of the model with finitely many citizens. But the graphical analysis that it permits is appealing, and several models analyzed in later chapters take it as a starting point, so I provide a detailed analysis of it.

The definition of the variant, unlike Definition 8.1, does not include citizens explicitly. Instead, the electoral outcome for any pair (x_1, x_2) of the candidates' positions and any nonatomic distribution F of the citizens' favorite positions with a unique median is assumed to be

$$O_F(x_1, x_2) = \begin{cases} \textit{tie} & \text{if } x_1 = x_2 \text{ or } \frac{1}{2}(x_1 + x_2) = \text{med}(F) \\ \textit{win for } j & \text{if } \begin{cases} \text{either } x_k < x_j \text{ and } \frac{1}{2}(x_1 + x_2) < \text{med}(F) \\ \text{or } x_k > x_j \text{ and } \frac{1}{2}(x_1 + x_2) > \text{med}(F), \end{cases} \end{cases} \tag{8.1}$$

where k is the player other than j and med(F) denotes the median of F. One rationale for this assumption is that each citizen's preference relation over positions is single-peaked and symmetric about her favorite position. A compact way to characterize O_F is that the winner is the candidate favored by the median voter.

Definition 8.4: Electoral competition game with continuum of citizens and two office-motivated candidates

An *electoral competition game with a continuum of citizens and two office-motivated candidates* $\langle \{1, 2\}, X, F \rangle$, where X is a closed interval of real numbers and F is a nonatomic distribution with support X (and hence a unique median), is the strategic game with the following components.

Players
$\{1,2\}$ (*candidates*).

Actions
The set of actions of each player is X (the set of possible *positions*).

Preferences
The preference relation $\trianglerighteq_j$ of each player j over $X \times X$ satisfies

$$(w_1, w_2) \vartriangleright_j (y_1, y_2) \vartriangleright_j (z_1, z_2)$$

whenever $O_F(w_1, w_2) = \textit{win for } j$, $O_F(y_1, y_2) = \textit{tie}$, and $O_F(z_1, z_2) = \textit{win for } k$, where k is the other player and O_F is given by (8.1).

Any such game has a unique Nash equilibrium, in which each candidate's position is the median of F. Further, as for a two-candidate electoral competition game for an arbitrary collective choice problem, the action of each candidate in a Nash equilibrium weakly dominates her other actions. These results do not follow from Corollary 8.2, because that result assumes a finite number of citizens, but the arguments are straightforward.

Proposition 8.4: Nash equilibrium of electoral competition game with continuum of citizens and two office-motivated candidates

Every electoral competition game with a continuum of citizens and two office-motivated candidates $\langle\{1,2\}, X, F\rangle$ has a unique Nash equilibrium, in which each candidate's position is the median of F, and for each candidate this action weakly dominates her other actions.

Proof

Denote the median of F by m. We have $O_F(m, m) = \textit{tie}$, and if either candidate deviates from m she loses, so (m, m) is a Nash equilibrium.

Now let $(x_1, x_2) \neq (m, m)$. If $\frac{1}{2}(x_1 + x_2) = m$ then $O_F(x_1, x_2) = \textit{tie}$, and either candidate can win by deviating to m. Otherwise one of the candidates loses, and by moving to m she at least ties. Thus (x_1, x_2) is not a Nash equilibrium.

If candidate i's position is m, the outcome is a tie if the other candidate's position is m and a win for i otherwise. If i's position differs from m, she loses if the other candidate's position is m. Thus i's action m weakly dominates her other actions.

Note that this result does not depend on the shape of the distribution F of the citizens' favorite positions. This distribution can be concentrated or dispersed, unimodal or multimodal, symmetric or skewed; in every case the action of choosing the median of the distribution weakly dominates all other actions, and the action pair in which both candidates choose the median is the only Nash equilibrium.

A notable feature of the unique Nash equilibrium is that both candidates choose the same position. Harold Hotelling, who in 1929 suggested that the model captures competition between parties, asserted that in the US at the time the convergence of positions was "strikingly exemplified" (Hotelling 1929, 54). Some observers of US politics claim that the exemplification of convergence is now less striking.

Exercise 8.1: Electoral competition with alienation

Consider a model that differs from an electoral competition game with a continuum of citizens and two office-motivated candidates only in that citizens whose favorite positions are more than some distance k from both candidates' positions do not vote. (Perhaps citizens' motivations for voting are expressive (Section 6.2), with alienation setting in when both candidates' positions are remote.) Characterize the Nash equilibria of the game when the distribution of the citizens' favorite positions is unimodal and has a differentiable density. Give an example in which the distribution of the citizens' favorite positions is not unimodal and a Nash equilibrium exists in which the candidates' positions differ.

The fact that a candidate's choosing m weakly dominates her other actions means that regardless of her beliefs about the other candidates, m is an optimal action for her. However, it does not mean that choosing a position different from m is unambiguously irrational. If, for example, one candidate believes that the other will certainly choose some given position $x > m$, then any position closer to m than x is optimal for her. But should a candidate not believe that the other candidate is rational? And also that the other candidate believes she is rational? And that the other candidate believes she believes that the other candidate is rational? And so forth .... For an electoral competition game with a finite set of possible positions, the next exercise asks you to show that the implication of this set of hypotheses is that a candidate chooses m.

Exercise 8.2: Rationalizable actions in two-candidate electoral competition

Consider a variant of an electoral competition game with a continuum of citizens and two office-motivated candidates in which the set of possible positions is finite: $\{z_1, \ldots, z_k\}$ with $z_1 < z_2 < \cdots < z_k$. Each citizen's favorite position is a member of the set and each candidate is restricted to choose a member of the set. Call a candidate's action *rational* if it is a best response to *some* belief about the other candidate's actions, or equivalently if it is not strictly dominated. (Take this equivalence as given; the argument for it is not simple.) Find a candidate's rational actions, her rational actions if she assumes that the other candidate is rational, her rational actions if she assumes that the other candidate assumes that she is rational, and so forth. An action that is rational under the union of these assumptions is *rationalizable*. Show that the only rationalizable action is the median of the citizens' favorite positions.

Just as an electoral competition game with a continuum of citizens and two office-motivated candidates is a variant of an electoral competition game with two office-motivated candidates, so we can define a variant with a continuum of citizens of a sequential electoral competition game with two office-motivated candidates. I postpone doing so until Section 10.3, where I study a version of the game with many office-seekers, each of whom has the option to become a candidate.

8.2.2 Two-dimensional positions

Proposition 8.1 shows that in a Nash equilibrium of an electoral competition game with two office-motivated candidates, each candidate's position is a Condorcet winner of the underlying collective choice problem. When the set of alternatives is two-dimensional, what do we know about the set of Condorcet winners? Section 1.6 shows that the answer depends on the nature of the individuals' preferences. Proposition 1.6 says that when the individuals have city block preferences, each component of a Condorcet winner is the median of the individuals' favorite values of that component, and Proposition 1.7 says that when the individuals have max preferences, a Condorcet winner is a component-wise median of a 45 degree rotation of the individuals' favorite positions. For the case in which the individuals have Euclidean preferences, Proposition 1.8 says that a Condorcet winner exists only if the individuals' favorite positions possess a specific symmetry: for some position x, half of these favorite positions lie on each

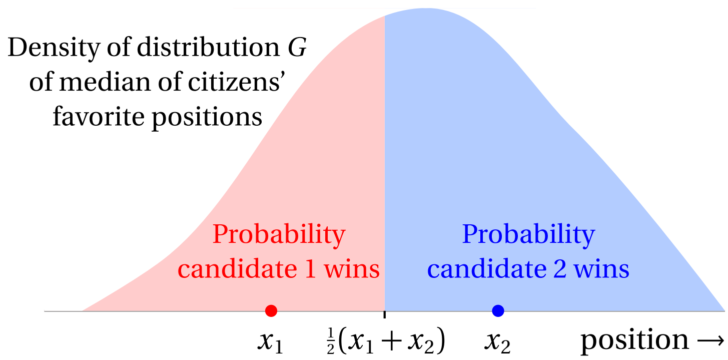


Figure 8.3 Two-candidate electoral competition in which the median of the citizens' favorite positions is uncertain.

side of x, in which case x is a Condorcet winner.

8.3 Candidates uncertain of citizens' preferences

In the models of electoral competition discussed so far, the candidates know the distribution of the citizens' preferences. I now present two models in which they are uncertain about this distribution.

8.3.1 Common information

The analysis in Section 8.2.1 shows that when the set of positions is an interval of real numbers, the key feature of the citizens' preferences is the median of their favorite positions. In the model I now present, the candidates' uncertainty about the citizens' preferences directly concerns this median. The candidates are assumed to share the belief that this median has a nonatomic distribution G. Denote the candidates' positions by x_1 and x_2. If $x_1 = x_2$ then each candidate wins with probability $\frac{1}{2}$, and if $x_1 < x_2$ then candidate 1 wins with probability $G(\frac{1}{2}(x_1+x_2))$, the probability that the median favorite position is at most $\frac{1}{2}(x_1+x_2)$, and candidate 2 wins with probability $1-G(\frac{1}{2}(x_1+x_2))$. (Refer to Figure 8.3.) Each candidate cares about her probability of winning.

Definition 8.5: Electoral competition game with two office-motivated candidates and uncertain median

An *electoral competition game with two office-motivated candidates and uncertain median* $\langle\{1,2\},X,G\rangle$, where X is a closed interval of real numbers and G is a nonatomic distribution with support X (and hence a unique median), is the strategic game with the following components.

Players

$\{1,2\}$ (candidates).

Actions

The set of actions of each player is X.

Preferences

The preferences of each player j over $X \times X$ are represented by the function $u_j : X \times X \to \mathbb{R}$ defined by

$$u_j(x_1, x_2) = \begin{cases} G(\frac{1}{2}(x_1 + x_2)) & \text{if } x_j < x_k \\ \frac{1}{2} & \text{if } x_1 = x_2 \\ 1 - G(\frac{1}{2}(x_1 + x_2)) & \text{if } x_j > x_k, \end{cases}$$

where k is the other player.

The interpretation of this game differs from that of an electoral competition game with a continuum of citizens and two office-motivated candidates, but formally the games are similar, and their analyses are also similar. The game has a unique Nash equilibrium, in which each candidate's position is the median of the distribution G of the median of the citizens' favorite positions.

Proposition 8.5: Nash equilibrium of electoral competition game with two office-motivated candidates and uncertain median

An electoral competition game with two office-motivated candidates and uncertain median $\langle\{1,2\}, X, G\rangle$ has a unique Nash equilibrium, in which each candidate's position is the median of G.

Proof

Denote the median of G by m. The outcome of the action pair (m, m) is that each candidate wins with probability $\frac{1}{2}$. If either candidate deviates to a position x, her probability of winning becomes $G(\frac{1}{2}(x+m)) < G(m) = \frac{1}{2}$ if $x < m$ and $1 - G(\frac{1}{2}(x+m)) < G(m) = \frac{1}{2}$ if $x > m$. Thus (m, m) is a Nash equilibrium.

Now let $(x_1, x_2) \neq (m, m)$. Assume without loss of generality that $x_1 \leq x_2$.

First suppose that $x_1 = x_2 < m$, so that each candidate's probability of winning is $\frac{1}{2}$. Let $x = x_1 = x_2$. Then either candidate can increase her probability of winning to $1 - G(\frac{1}{2}(x+m)) > \frac{1}{2}$ by deviating to m. A similar argument applies if $x_1 = x_2 > m$.

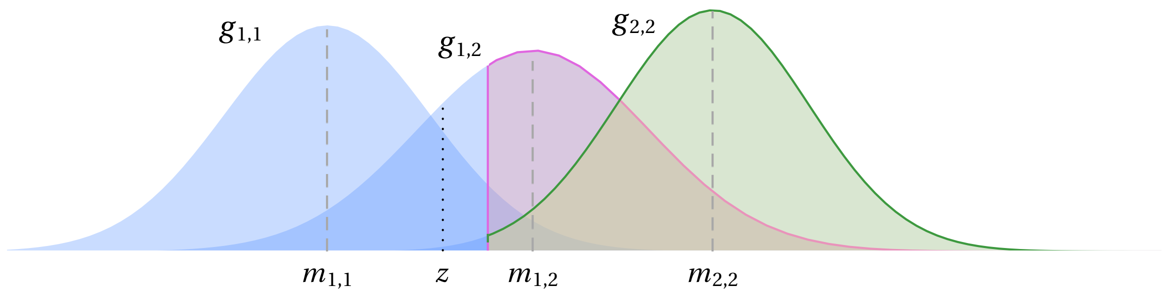


Figure 8.4 An example of the densities of the posterior distributions of the median of the citizens' favorite positions for each possible pair of signals in an electoral competition game with two office-motivated candidates privately informed about citizens in which the set of signals is $\{1,2\}$. Each function g_{t_1,t_2} is the density of the distribution function G_{t_1,t_2} of the median of the citizens' favorite positions when candidate 1's signal is t_1 and candidate 2's is t_2, and m_{t_1,t_2} is the median of G_{t_1,t_2}.

Now suppose that $x_1 < x_2$, so that candidate 1's probability of winning is $G(\frac{1}{2}(x_1 + x_2))$ and candidate 2's is $1 - G(\frac{1}{2}(x_1 + x_2))$. If $\frac{1}{2}(x_1 + x_2) < m$ then $G(\frac{1}{2}(x_1 + x_2)) < \frac{1}{2}$ and candidate 1 can increase her probability of winning to $\frac{1}{2}$ by deviating to x_2. If $\frac{1}{2}(x_1 + x_2) > m$ then similarly candidate 2 can increase her probability of winning by deviating to x_1. If $\frac{1}{2}(x_1 + x_2) = m$ then each candidate's probability of winning is $\frac{1}{2}$ and either candidate can increase this probability by deviating to m.

8.3.2 Private information

Now suppose that each candidate gets a private signal about the median of the citizens' favorite positions (perhaps from a poll she conducts). Assume that the candidates share a prior belief about the distribution of the median and that the probabilistic process that generates a signal for each candidate is the same, so that the posterior distribution function G_{t_1,t_2} of the median when candidate 1's signal is t_1 and candidate 2's is t_2 is the same as the distribution function G_{t_2,t_1} when candidate 1's signal is t_2 and candidate 2's is t_1. Denote by $P(t_1, t_2)$ the ("marginal") probability of the pair (t_1, t_2) of signals, given the prior distribution of the median and the processes that generate the signals. Given that the process that generates the signal of each candidate is the same, P is symmetric: $P(t_1, t_2) = P(t_2, t_1)$ for each $(t_1, t_2) \in T \times T$.

Assume that each candidate's signal is drawn from the same finite set T, and that each distribution function G_{t_1,t_2} has a density, g_{t_1,t_2}. An example of possible densities when $T = \{1,2\}$ is shown in Figure 8.4.

Definition 8.6: Electoral competition game with two office-motivated candidates privately informed about citizens

An *electoral competition game with two office-motivated candidates privately informed about citizens* $\langle\{1,2\}, X, T, P, (G_{t_1,t_2})_{t_1,t_2\in T}\rangle$, where

- $X \subset \mathbb{R}$ is a finite interval (the set of positions)
- T is a finite set (the set of signals each candidate may receive)
- P is a probability distribution over $T \times T$ with $P(t_1, t_2) > 0$ and $P(t_1, t_2) = P(t_2, t_1)$ for all $(t_1, t_2) \in T \times T$
- for each $t_1 \in T$ and $t_2 \in T$, G_{t_1,t_2} is a nonatomic probability distribution function for X that has a density and whose support is an interval, with $G_{t_1,t_2} = G_{t_2,t_1}$.

is a Bayesian game with the following components.

Players
$\{1,2\}$ (the candidates).

States
$T \times T$ (the set of pairs (t_1, t_2) with $t_i \in T$ for $i = 1, 2$).

Actions
The set of actions of each player is X (the set of positions).

Signals
For each player i, the set of signals is T and her signal function associates with each state (t_1, t_2) the signal t_i.

Prior beliefs
The players' common prior belief is that t_1 and t_2 are drawn from $T \times T$ according to P.

Payoffs
The payoff of each player j for the pair of actions (z_1, z_2) and state (t_1, t_2) is her probability of winning when the distribution of the median is G_{t_1,t_2}:

$$u_j((z_1, z_2), (t_1, t_2)) = \begin{cases} G_{t_1,t_2}(\frac{1}{2}(z_1 + z_2)) & \text{if } z_j < z_k \\ \frac{1}{2} & \text{if } z_j = z_k \\ 1 - G_{t_1,t_2}(\frac{1}{2}(z_1 + z_2)) & \text{if } z_j > z_k \end{cases} \qquad (8.2)$$

where k is the other player.

Given that the distributions G_{t_1,t_2} depend only on the pair of signals, not the identity of the candidate who receives each signal, it is natural to consider the possibility that the game has a Nash equilibrium (x_1, x_2) (where $x_i : T \to X$ for $i = 1, 2$) in which the candidates choose the same position when they receive the same signal: $x_1(t) = x_2(t)$ for all $t \in T$.

Suppose that there are two possible signals, with $T = \{1, 2\}$. For each $t_1 \in T$ and $t_2 \in T$, denote the median of G_{t_1,t_2} by m_{t_1,t_2} and assume that the higher signal is associated with a larger value of the median: $m_{1,1} < m_{1,2} < m_{2,2}$, as in the example in Figure 8.4.

I first argue that the game has no Nash equilibrium in which both types of each candidate choose the same position: $x_1(1) = x_2(1) = x_1(2) = x_2(2)$. For such a strategy pair, each type of each candidate wins with probability $\frac{1}{2}$. Denote the candidates' common position by z. Suppose that $z < m_{1,2}$, as in Figure 8.4. Consider type 2 of candidate 1. She believes that the density of the median is either $g_{1,2}$ (if candidate 2's signal is 1) or $g_{2,2}$ (if candidate 2's signal is 2). In both cases, her probability of winning increases if she deviates from z to $m_{1,2}$ (and candidate 2's position remains z). If candidate 2's signal is 1, this deviation causes 1's probability of winning to become equal to the area under $g_{1,2}$ for positions at least equal to the midpoint of z and $m_{1,2}$, shaded purple in Figure 8.4, and if candidate 2's signal is 2, the deviation causes the probability to become the area under $g_{2,2}$ for positions at least equal to the midpoint of z and $m_{1,2}$, shaded green in Figure 8.4. Given that $\frac{1}{2}(z + m_{1,2}) < m_{1,2} < m_{2,2}$, both of these probabilities exceed $\frac{1}{2}$. Thus type 2 of candidate 1 gains by deviating from z to $m_{1,2}$, and hence the game has no equilibrium in which both types of both candidates choose the same position less than $m_{1,2}$. By a symmetric argument, it has no equilibrium in which both types of both candidates choose the same position greater than $m_{1,2}$. Finally, suppose that both types of both candidates choose the position $m_{1,2}$. Then if type 2 of candidate 1 deviates to a slightly larger position, she slightly reduces her probability of winning when candidate 2's signal is 1 and discretely increases it when candidate 2's signal is 2, so that she has a deviation that increases her probability of winning.

I now consider the possibility that the game has a ("symmetric") Nash equilibrium in which type 1 of each candidate chooses the same position, as does type 2 of each candidate, but the types choose different positions. Consider a strategy pair (x_1, x_2) with $x_1(1) = x_2(1) = z_1$, $x_1(2) = x_2(2) = z_2$, and $z_1 \neq z_2$. For this strategy pair, type 1 of each candidate i ties with the other candidate, j, when j's signal is 1 (both candidates choose z_1) and wins with probability $G_{1,2}(\frac{1}{2}(z_1 + z_2))$ if $z_1 < z_2$ and with probability $1 - G_{1,2}(\frac{1}{2}(z_1 + z_2))$ if $z_1 > z_2$ when j's signal is 2 (i chooses z_1 and j chooses z_2). I argue that if this strategy pair is a Nash equilibrium then $z_1 = m_{1,1}$ and $z_2 = m_{2,2}$. Suppose that $z_1 > m_{1,1}$, as in Fig-

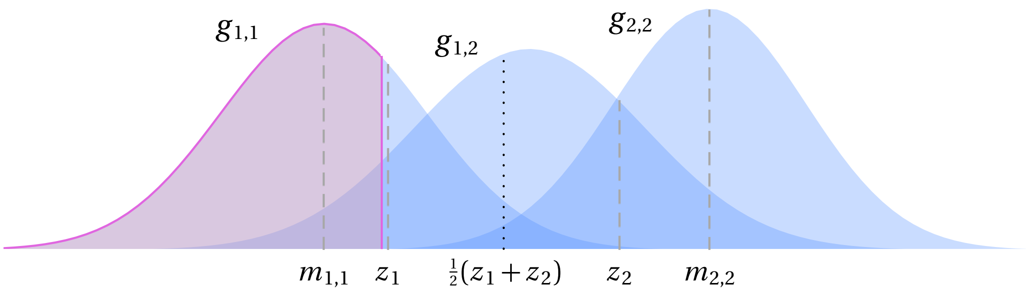


Figure 8.5 The effect of a deviation by type 1 of a candidate from a strategy pair (x_1, x_2) with $x_1(1) = x_2(1) = z_1$ and $x_1(2) = x_2(2) = z_2$ in an electoral competition game with two office-motivated candidates privately informed about citizens.

ure 8.5. Then if type 1 of a candidate deviates to a slightly smaller position, her probability of winning is discretely larger than $\frac{1}{2}$ when the other candidate's signal is 1 (it is an area like the one shaded purple in Figure 8.5) and is close to what it was when her position was z_1 when the other candidate's signal is 2 (if $z_2 > z_1$, as in Figure 8.5, it is slightly less than it was before, and if $z_2 < z_1$ then it is slightly more than it was before). Thus the deviation increases the candidate's probability of winning. A symmetric argument shows that if $z_1 < m_{1,1}$ then a candidate's deviation to a slightly larger position increases her probability of winning.

We conclude that if the game has a symmetric equilibrium, then the position of each type t of each candidate is the median $m_{t,t}$ of the distribution of the citizens' favorite positions when the other candidate's signal is also t. In a sense, such an equilibrium amplifies the candidates' signals: although a candidate who receives a signal t assigns positive probability to the other candidate's receiving each possible signal, the distribution of the citizens' favorite positions that determines her equilibrium position is the one associated with both candidates' signals being t; her position is more extreme than the expected value of the median of the citizens' favorite positions given her own signal.

Proposition 8.6: Nash equilibrium of electoral competition game with two office-motivated candidates privately informed about citizens

Let $\langle \{1,2\}, X, T, P, (G_{t_1,t_2})_{t_1,t_2 \in T} \rangle$ be an electoral competition game with two office-motivated candidates privately informed about citizens. Suppose that $T = \{1,2\}$ and $m_{1,1} < m_{1,2} < m_{2,2}$, where m_{t_1,t_2} is the median of G_{t_1,t_2} for each $t_1 \in T$ and $t_2 \in T$. In any Nash equilibrium of the game, the position of each type $t \in T$ of each candidate is the median $m_{t,t}$ of $G_{t,t}$.

Proof

In the text I argue that in any Nash equilibrium (x_1, x_2) in which $x_1(t) = x_2(t)$ for all $t \in T$ we have $x_1(1) = x_2(1) = m_{1,1}$ and $x_1(2) = x_2(2) = m_{2,2}$. To complete the proof, suppose that (x_1, x_2) is a Nash equilibrium in which $x_1 \neq x_2$. Then given the symmetry of the game (in particular, the symmetry of P), (x_2, x_1) is also a Nash equilibrium. Now, the game is strictly competitive (an outcome that is better for one candidate is worse for the other candidate), so its Nash equilibria are interchangeable (see for example Osborne 2004, Corollary 369.3). Thus if (x_1, x_2) and (x_2, x_1) are Nash equilibria then so are (x_1, x_1) and (x_2, x_2). The argument in the text shows that the game has at most one symmetric equilibrium, so we conclude that it has no asymmetric equilibria.

This result asserts only that if the game has an equilibrium then it takes a certain form, not that the strategy pair given is necessarily a Nash equilibrium. I now give an example in which the strategy pair is in fact a Nash equilibrium. Suppose that each distribution G_{t_1,t_2} is symmetric about its median and has the same form, differing only in its location, and the locations of $G_{1,1}$, $G_{1,2}$, and $G_{2,2}$ are equally spaced, as in Figure 8.6. That is, for some number $\alpha > 0$ and function $H : \mathbb{R} \to \mathbb{R}$ with a symmetric density we have $G_{1,1}(x) = H(x + \alpha)$, $G_{1,2}(x) = H(x)$, and $G_{2,2}(x) = H(x - \alpha)$ for each $x \in X$. Consider the strategy pair (x_1, x_2) for which $x_1(1) = x_2(1) = m_{1,1}$ and $x_1(2) = x_2(2) = m_{2,2}$. Suppose that type 1 of candidate i deviates to z_1. Denote the other candidate by j.

$z_1 < m_{1,1}$ or $z_1 > m_{2,2}$
: Type 1 of candidate i's probability of winning falls regardless of j's type.

$m_{1,1} < z_1 < m_{2,2}$
: If j's signal is 1, type 1 of candidate i's probability of winning decreases from $\frac{1}{2}$ to the area shaded purple in Figure 8.6, and if j's signal is 2, this probability increases from $\frac{1}{2}$ to the area shaded green in the figure. Given the symmetry of the distributions, the decrease in the first case is equal to the increase in the second case. Thus the deviation does not increase i's probability of winning if and only if the probability that j's signal is 1 given that i's signal is 1 is at least $\frac{1}{2}$.

$z_1 = m_{2,2}$
: If j's signal is 1, type 1 of candidate i's probability of winning falls from $\frac{1}{2}$ to $1 - G_{1,1}(\frac{1}{2}(m_{1,1} + m_{1,2})) = 1 - G_{1,1}(m_{1,2})$, and if j's signal is 2 it remains $\frac{1}{2}$.

Symmetric arguments apply to deviations by type 2 of a candidate, so we conclude that the strategy pair (x_1, x_2) is a Nash equilibrium in this example if and

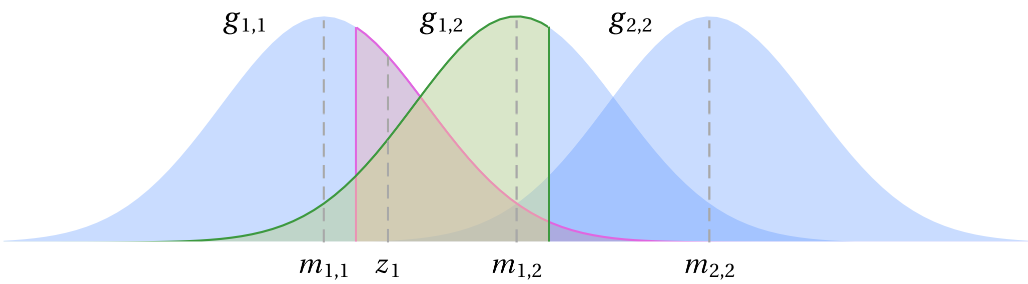


Figure 8.6 The effect of a deviation by type 1 of a candidate to z_1 from a strategy pair (x_1, x_2) with $x_1(1) = x_2(1) = m_{1,1}$ and $x_1(2) = x_2(2) = m_{2,2}$ in an electoral competition game with two office-motivated candidates privately informed about citizens

only if for each $t \in \{1,2\}$ the probability that one candidate's signal is t given that the other candidate's signal is t is at least $\frac{1}{2}$.

If the number of possible signals is arbitrary, an extension of this logic implies that the game has a Nash equilibrium only if the probability that one candidate's signal is extreme given that the other candidate's signal is extreme is at least $\frac{1}{2}$ (Bernhardt et al. 2009, Theorem 2). When the number of signals is large, this condition is particularly restrictive. If the condition is violated, then a version of the model has a mixed strategy equilibrium in which the strategy of each type t of each candidate with a relatively moderate assigns probability 1 to $m_{t,t}$, like the strategy of each candidate in Proposition 8.6, and the support of the strategy of type t of each remaining candidate is an interval consisting of positions more moderate than $m_{t,t}$ (Bernhardt et al. 2009, Theorem 3). Thus in this mixed strategy equilibrium the amplification of the candidates' private information in the strategy profile in Proposition 8.6 is tempered for candidates with extreme signals.

Exercise 8.3: Previous electoral outcomes as information sources

Candidates may obtain information about the citizens' preferences from the outcomes of previous elections, giving citizens an incentive to consider the effect of their votes not only on the outcome of the current election but also on the candidates' positions in future elections. Here is a simple example. Two candidates compete in a sequence of two elections, in each of which the set of possible positions is $[0,1]$. There is a single citizen, whose favorite position $\hat{x}$ is unknown to the candidates, who believe that its distribution function is H. For any outcomes x^1 in period 1 and x^2 in period 2, the citizen's payoff is $-|x^1 - \hat{x}| - |x^2 - \hat{x}|$. In the first period the candidates' positions are fixed at x_1 and x_2 with $x_1 < x_2$; the citizen chooses a cutoff position x^*, voting for candidate 1 if her favorite position is less than x^* and for candidate 2 if it is greater than x^*. If candidate 1 wins in the first period, then in the second period, in line with Proposition 8.5, the candidates both choose the median of the distribution of the citizen's favorite position conditional on this position being in $[0, x^*]$. If candidate 2 wins in the first period, then in the second period the candidates similarly both choose the median of the distribution of the citizen's favorite position conditional on this position being in $[x^*, 1]$. If $x_1 = \frac{1}{2}$, $x_2 = 1$, and H is the uniform distribution on $[0, 1]$, which position x^* does the citizen choose as her cutoff? How does this position compare with the one she would choose if there were no second period?

8.4 Costly voting

In the models of electoral competition I have discussed so far, every citizen votes even if the candidates' positions are the same. If voting is voluntary, the devotion to civic duty that this behavior requires seems excessive. In this section I discuss a model in which voting entails a cost, as in the models in Chapter 4, and a citizen votes only if she believes that the expected benefit of doing so outweighs this cost.

Assume that two candidates simultaneously choose positions and then citizens simultaneously vote. The citizens differ in their preferences over positions and their voting costs. Each citizen votes (for the candidate whose position she prefers) only if her expected benefit from doing so, given the probability that her vote affects the outcome, is at least her voting cost. Under this assumption, only if the candidates' positions differ do any citizens vote, and then only those whose voting costs are sufficiently small.

To simplify the analysis, assume that there is only one citizen. This assumption may seem extreme, but the model captures the main idea we want to study: that a citizen votes only if doing so sufficiently affects the outcome. In a model with many citizens, each citizen must consider the probability that her vote will affect the outcome, given the other citizens' behavior, which complicates the analysis. After establishing the main result, I argue that the presence of many citizens appears not to affect the key feature of an equilibrium.

I formulate the model as a Bayesian extensive game with observable actions. In such a game, each player may have many possible types. Each player knows her own type, but not the other players' types. Every player observes every other player's action, and holds the same probabilistic belief about the other players'

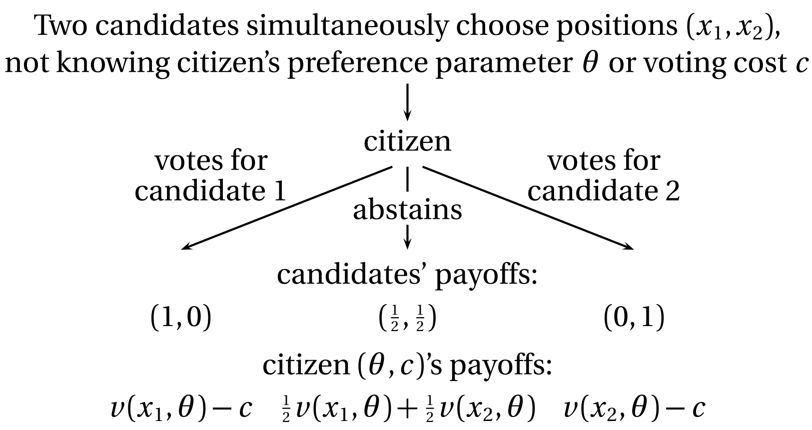


Figure 8.7 An electoral competition game with two office-motivated candidates and costly voting.

types.

Specifically, in the the model here, two candidates simultaneously choose positions in a finite interval X and then a single citizen either votes for one of the candidates, in which case that candidate wins, or abstains, in which case the candidates tie. The candidates' characteristics are known (formally, each candidate has a single type), but the citizen's preferences over X and voting cost are not known to the candidates. The citizen's type is a pair (θ, c), where $\theta \in \Theta$ parameterizes the citizen's preferences over X (the value of θ may be the citizen's favorite position, for example) and $c \in \mathbb{R}_+$ is her voting cost. The payoff of type (θ, c) of the citizen is $v(x_j, \theta) - c$ if she votes for candidate j (in which case j is the winner), where x_j is j's position, and $\frac{1}{2}(v(x_1,\theta) + v(x_2,\theta))$ if she abstains (in which case the outcome is a tie). The candidates' common belief about the citizen's type is given by a probability measure P over the set $\Theta \times \mathbb{R}_+$ of the citizen's types. Each candidate prefers to win than to tie than to lose. The game is illustrated in Figure 8.7.

Definition 8.7: Electoral competition game with two office-motivated candidates and costly voting

An *electoral competition game with two office-motivated candidates and costly voting* $\langle X, \Theta, P, v\rangle$, where

- $X \subset \mathbb{R}$ is a finite interval (the set of possible positions)
- $\Theta \subset \mathbb{R}$ (the set of possible preference parameters for the citizen)
- P is a probability measure on $\Theta \times \mathbb{R}_+$ (the candidates' common belief

about the citizen's preference parameter and voting cost)

- $v : X \times \mathbb{R} \to \mathbb{R}$ (the citizen's payoff function)

is a Bayesian extensive game with observable actions with the following components.

Players
$\{1,2\} \cup \{z\}$ (the candidates and a single citizen).

Terminal histories
The set of sequences $\{((x_1, x_2), b) : x_j \in X \text{ for } j = 1,2 \text{ and } b \in \{1,2,\phi\}\}$ (where x_j is the position of candidate j, $b \in \{1,2\}$ is the candidate for whom the citizen votes, and ϕ stands for abstention).

Player function
The function L with $L(\varnothing) = \{1,2\}$ (the candidates move (simultaneously) at the start of the game) and $L((x_1, x_2)) = \{z\}$ for each $(x_1, x_2) \in X \times X$ (the citizen moves after the candidates choose positions).

Actions
The set of actions of each candidate at the start of the game is X and the set of actions of the citizen following any pair of actions of the candidates is $\{1,2,\phi\}$ (a vote for one of the candidates or abstention).

Types
Each candidate has one possible type, known to the citizen. The set of types of the citizen is $\Theta \times \mathbb{R}_+$, the set of pairs consisting of a preference parameter and a nonnegative number (the citizen's cost of voting). The probability measure on the set $\Theta \times \mathbb{R}_+$ of the citizen's types is P.

Preferences
The preferences over lotteries over terminal histories of each candidate j (= 1, 2) are represented by the expected value of the function u_j defined by

$$u_j((\theta, c), ((x_1, x_2), b)) = \begin{cases} 1 & \text{if } b = j \\ \frac{1}{2} & \text{if } b = \phi \\ 0 & \text{if } b = k, \end{cases}$$

where k is the other candidate. The preferences over terminal histories of the citizen of type (θ, c) are represented by the function u defined by

$$u((\theta, c), ((x_1, x_2), b)) = \begin{cases} v(x_b, \theta) - c & \text{if } b \in \{1,2\} \\ \frac{1}{2}(v(x_1, \theta) + v(x_2, \theta)) & \text{if } b = \phi. \end{cases} \tag{8.3}$$

A strategy for a candidate in such a game is a position (member of X), and a strategy for the citizen is a function that assigns an action (member of $\{1, 2, \phi\}$) to each type $(\theta, c) \in \Theta \times \mathbb{R}_+$ and each pair of actions (x_1, x_2) for the candidates.

A standard notion of equilibrium for a Bayesian extensive game with observable actions is perfect Bayesian equilibrium. The definition of this notion of equilibrium has features that are irrelevant to a two-candidate electoral competition game with office-motivated candidates and costly voting because of the simple structure of such a game: each player moves only once, and the citizen is perfectly informed when she does so. The following notion of equilibrium suffices.

Definition 8.8: Equilibrium of electoral competition game with two office-motivated candidates and costly voting

An *equilibrium* of an electoral competition game with two office-motivated candidates and costly voting is a strategy profile in which, for each pair of the candidates' positions, the action of each type of the citizen maximizes that type's expected payoff, and each candidate's position maximizes her expected payoff given the other candidate's position and the citizen's strategy.

If $\Theta = X$ and the parameter θ is the citizen's favorite position, we can illustrate the citizen's optimal decision in a diagram. Figure 8.8 shows the candidates' positions, x_1 and x_2, and the function $v(\cdot, \theta_0)$. If the citizen's type is (θ, c) then she optimally votes for candidate 1 if

$$v(x_1, \theta) > v(x_2, \theta) \quad \text{and} \quad v(x_1, \theta) - c > \tfrac{1}{2}(v(x_1, \theta) + v(x_2, \theta)).$$

The second condition is equivalent to $c < \frac{1}{2}(v(x_1, \theta) - v(x_2, \theta))$. The length of the magenta line segment in the figure is this cutoff $\frac{1}{2}(v(x_1, \theta_0) - v(x_2, \theta_0))$ for a citizen with preference type θ_0. The red and blue lines in the figure indicate the cutoff for each preference type, so that the area shaded red is the set of types (θ, c) who optimally vote for candidate 1 and the area shaded blue is the set of types who optimally vote for candidate 2.

The effect of a change in candidate 1's position from x_1 to x_1', which reduces the difference between the candidates' positions, is illustrated in Figure 8.9. As for a game in which every type of the citizen is assumed to vote, this change causes some types of the citizen to switch from voting for candidate 2 to voting for candidate 1. But it affects also the set of types of the citizen that vote, because for almost all types it changes the expected gain from voting and thus the cost cutoff for voting rather than abstaining. For example, it reduces the cost cutoff for types of the citizen who prefer x_1 to x_1' to x_2 or who prefer x_2 to x_1' to x_1, and in-

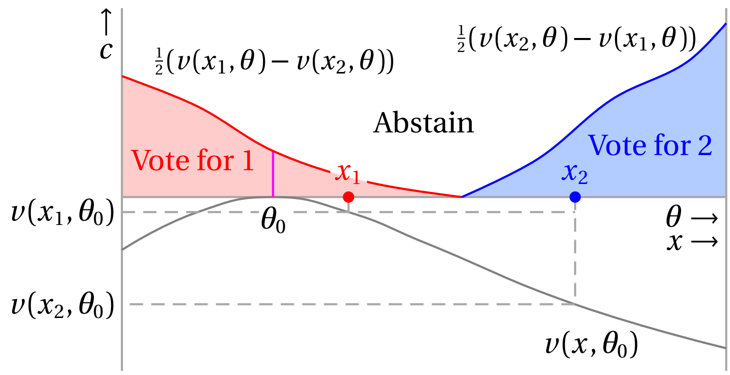


Figure 8.8 Payoffs in a two-candidate electoral competition with costly voting. The area shaded red is the set of types (θ, c) of the citizen who optimally vote for candidate 1 and the area shaded blue is the set of types who optimally vote for candidate 2. The citizen's function v is shown for the parameter value θ_0; the magenta lengths are equal.

creases the cost cutoff for types who prefer both x_1 to x_2 and x_1' to x_1. Informally, it reduces the motivation to vote for types of the citizen with relatively extreme preferences and increases this motivation for types whose favorite alternative is close to x_1'. In a model in which every citizen is assumed to vote, the net effect on candidate 1's probability of winning of her moving her position closer to that of candidate 2 is necessarily positive. In the model here, the net effect depends on the character of the citizen's preferences and the distribution of her types.

Consider the possibility that the game has an equilibrium in which the candidates' positions are the same, say equal to x^*. In such an equilibrium, the only types who vote are those for whom the voting cost is zero, and when the distribution of types is nonatomic, as the next result assumes, such types have measure zero. If one of the candidates, say i, deviates slightly from x^*, opening a small wedge between the positions, only citizen types with cost close to zero vote. For equilibrium, such a deviation must not increase i's payoff. That is, $x_i = x^*$ must locally maximize i's payoff, given that the other candidate's position is x^*. The next result shows that if for each value of the parameter $\theta \in \mathbb{R}$ the function $v(\cdot,\theta)$ is strictly concave and differentiable, and x^* is in the interior of X, then the condition for x^* to be a local maximizer implies that x^* maximizes the expected payoff, according to the candidates' belief, of the citizen types for which the voting cost is zero. Given the strict concavity of $v(\cdot,\theta)$ for each value of θ, there is only one such position x^*, so if the game has an equilibrium in which the candidates' positions are the same and in the interior of X then it has exactly one such equilibrium.

Now, for any given strategy of the citizen, the game between the candidates is strictly competitive (an outcome that is better for one candidate is worse for the other candidate), and the roles of the candidates are symmetric. So if (x_1, x_2) is

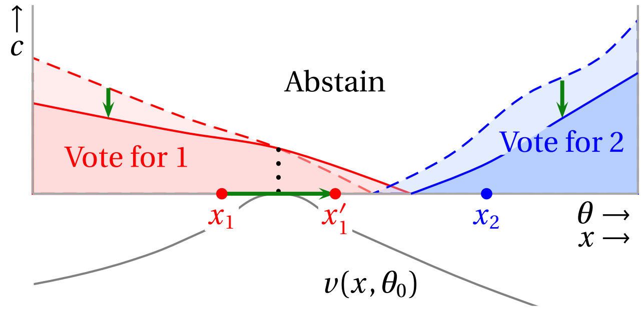


Figure 8.9 The decision to vote in a two-candidate electoral competition with costly voting. When candidate 1 moves from x_1 to x_1', the set of types who vote for each candidate changes from the areas shaded with light colors (and dashed boundaries) to those shaded with darker colors (and solid boundaries). For a citizen with preference type θ_0, who is indifferent between x_1 and x_1', the move does not affect the relative attractiveness of the candidates' positions.

an equilibrium pair of positions then so is (x_2, x_1), and hence by the interchangeability property of Nash equilibria of strictly competitive games (see for example Osborne 2004, Corollary 369.3), so are (x_1, x_1) and (x_2, x_2). So if the game has only one equilibrium in which the candidates' positions are the same then it has no equilibrium in which they differ.

Combining the arguments in the last two paragraphs, we conclude that if for each value of $\theta \in \mathbb{R}$ the function $v(\cdot, \theta)$ is strictly concave and differentiable and the game has an equilibrium in which the candidates' positions are in the interior of X then it has one such equilibrium, in which the candidates' positions are the same, equal to the position that maximizes the expected payoff of the citizen types for which the voting cost is zero.

Proposition 8.7: Equilibrium of electoral competition game with two office-motivated candidates and costly voting

Let $\langle X, \Theta, P, v\rangle$ be an electoral competition game with two office-motivated candidates and costly voting. If (*i*) P is nonatomic and has a continuous density, (*ii*) for each $\theta \in \mathbb{R}$ the function $v(\cdot, \theta)$ is strictly concave and differentiable, and (*iii*) for each $x \in X$ the function $v(x, \cdot)$ is continuous, then in every equilibrium of $\langle X, \Theta, P, v\rangle$ in which the candidates' positions are in the interior of X these positions are the same, equal to the solution of

$$\max_{x \in X} \int_{\Theta} v(x, \theta) g(\theta, 0)\, d\theta, \tag{8.4}$$

where g is the density of P, and the citizen abstains unless her cost is 0 (an event with probability zero).

Proof

First consider the citizen. For any pair (x_1, x_2) of the candidates' positions, type (θ, c) of the citizen chooses a solution of

$$\max_{b \in \{1,2,\phi\}} u((\theta, c), ((x_1, x_2), b)).$$

For $j = 1, 2$, denote by $T_j(x_1, x_2)$ the set of pairs (θ, c) such that j (i.e. *vote for* j) is a solution of this problem, and by $T_\phi(x_1, x_2)$ the set of pairs such that ϕ (*abstain*) is a solution.

Now, using (8.3), j is a solution of the problem only if $v(x_j, \theta) \geq v(x_k, \theta)$ and $v(x_j, \theta) - c \geq \frac{1}{2}[v(x_1, \theta) + v(x_2, \theta)]$, where k is the other candidate. Given $c \geq 0$, these conditions are equivalent to $c \leq \frac{1}{2}[v(x_j, \theta) - v(x_k, \theta)]$. So

$$T_j(x_1, x_2) = \{(\theta, c) \in \Theta \times \mathbb{R}_+ : c \leq \tfrac{1}{2}[v(x_j, \theta) - v(x_k, \theta)]\}. \tag{8.5}$$

Thus for any position x, we have $P(T_j(x, x)) = 0$ for $j = 1, 2$, given that P is nonatomic: if the candidates' positions are the same, the set of types of the citizen that vote has measure zero, so the outcome is a tie, and hence each candidate's payoff is $\frac{1}{2}$. So for an equilibrium in which both candidates choose the same position, we need the payoff of a candidate who deviates to be at most $\frac{1}{2}$. For the pair of positions (x_1, x_2), the probability that candidate j wins is $P(T_j(x_1, x_2))$ and the probability that she ties is $P(T_\phi(x_1, x_2))$, so her payoff is at most $\frac{1}{2}$ if and only if

$$P(T_j(x_1, x_2)) + \tfrac{1}{2} P(T_\phi(x_1, x_2)) \leq \tfrac{1}{2},$$

or, given that $P(T_1(x_1, x_2)) + P(T_2(x_1, x_2)) + P(T_\phi(x_1, x_2)) = 1$, $P(T_j(x_1, x_2)) - P(T_k(x_1, x_2)) \leq 0$, where k is the other candidate. So the game has an equilibrium in which both candidates' positions are x^* if and only if

$$P(T_1(x_1, x^*)) - P(T_2(x_1, x^*)) \leq 0 \text{ for all } x_1 \in X$$
$$P(T_2(x^*, x_2)) - P(T_1(x^*, x_2)) \leq 0 \text{ for all } x_2 \in X.$$

The left-hand side of each inequality is 0 for $x_1 = x_2 = x^*$, so equivalently x^* maximizes $P(T_1(x_1, x^*)) - P(T_2(x_1, x^*))$ and $P(T_2(x^*, x_2)) - P(T_1(x^*, x_2))$.

Now, from (8.5) we have

$$P(T_j(x_1, x_2)) = \int_\Theta \int_0^{\frac{1}{2}[v(x_j, \theta) - v(x_k, \theta)]} g(\theta, c)\, dc\, d\theta,$$

so that

$$P(T_1(x_1,x^*)) - P(T_2(x_1,x^*)) = \int_\Theta \left(\int_0^{\frac{1}{2}[v(x_1,\theta)-v(x^*,\theta)]} g(\theta,c)\,dc - \int_0^{\frac{1}{2}[v(x^*,\theta)-v(x_1,\theta)]} g(\theta,c)\,dc \right) d\theta.$$

A necessary condition for a position x^* interior to X to maximize this expression is that the derivative of the expression with respect to x_1 evaluated at x^* is zero, or

$$\int_\Theta \left(\tfrac{1}{2}v_1'(x^*,\theta)g(\theta,0) + \tfrac{1}{2}v_1'(x^*,\theta)g(\theta,0)\right) d\theta = \int_\Theta v_1'(x^*,\theta)g(\theta,0)\,d\theta = 0,$$

where v_1' denotes the derivative of v with respect to its first argument. Now, given that for any value of its second argument, v is strictly concave in its first argument,

$$\int_\Theta v_1'(x^*,\theta)g(\theta,0)\,d\theta = 0$$

if and only if x^* is the (unique) solution of (8.4). So for any equilibrium in which the positions of the candidates are the same and interior to X, the common position is the solution of this problem.

Now, the strategic game between the candidates, given the optimal action of each type of citizen for each pair of positions, is strictly competitive, and the candidates are symmetric. The argument in the text before the result shows that as a consequence the fact that the game has a unique equilibrium in which the candidates' positions are the same implies that it has no equilibrium in which they differ.

When there are many citizens, each of whom knows her own type but not the other citizens' types, the analysis is more complicated, because the last stage of the game is no longer simply a decision problem. However, an informal argument suggests that the candidates' equilibrium positions remain the same. When a candidate deviates slightly from a common position, only a vanishingly small measure of citizen types—those with very low voting cost—possibly find voting worthwhile, and their equilibrium actions are plausibly similar to the optimal action of the lone citizen in the model I have presented. Ledyard (1984, Theorem 1) shows that under some conditions an analog of Proposition 8.7 indeed holds.

The main qualitative assumption in the result is the strict concavity of the function $v(\cdot,\theta)$ for each preference type $\theta \in \mathbb{R}$. This assumption does not have any particular appeal; the assumption that this function is convex on each side of its maximizer (the type's favorite position), rather than concave, for example, seems equally plausible. As far as I know, no general analysis of the model without the concavity assumption exists, although under the (rather specific) assumptions of the following exercise, which include the symmetry of the citizen's payoff function and the uniformity of the probability measure on citizen types, the equilibria can be characterized.

Exercise 8.4: Electoral competition with costly voting

Consider an electoral competition game with two office-motivated candidates and costly voting $\langle X, \Theta, P, v \rangle$ in which the preferences of each type of the citizen have the same form, differing only in the favorite position. Specifically, assume that $\Theta = \mathbb{R}$ and $v(x, \theta) = \psi(x - \theta)$ for all $x \in X$ and $\theta \in \Theta$, where $\psi : \mathbb{R} \to \mathbb{R}$ is increasing for negative values of its argument and decreasing for positive values (so that its maximizer is 0) and $\psi(z) = \psi(-z)$ for all $z \in \mathbb{R}$ (so that it is symmetric about 0). Assume also that the measure P is uniform on $\Theta \times [0, \bar{c}]$ for some $\bar{c} > 0$. Show that the game has an equilibrium in which each candidate chooses the midpoint of X.

8.5 Citizens with preferences over candidates

The models so far in this chapter assume that each citizen cares about the policies proposed by the candidates, not about the candidates themselves. I now present a model in which each citizen cares directly about the candidates, as well as about the policies they propose. A citizen may simply like one candidate more than another, or the candidates may differ in their abilities at implementing given policies, for example.

8.5.1 Candidates know citizens' preferences

Consider a variant of an electoral competition game with two office-motivated candidates in which the preferences of each citizen i are represented by a function $u_i : X \times \{1, 2\} \to \mathbb{R}$, where X is the set of possible positions and $\{1, 2\}$ is the set of candidates. If $u_i(x_1, 1) > u_i(x_2, 2)$ then i prefers the outcome in which the winner is candidate 1 with position x_1 to the outcome in which the winner is candidate 2 with position x_2. Note that in this model a candidate does not necessarily have the option to tie with the other candidate by choosing the same position as she does, as was the case in the models discussed previously.

The following example illustrates possible forms for an equilibrium; I know of no general characterization of equilibria.

Example 8.1: Electoral competition when citizens have preferences over candidates

Suppose that the set X of possible positions is an interval of numbers and there are three citizens, with preferences represented by the functions shown in Figure 8.10. For each citizen there are two curves, a red one indicating the citizen's payoff as a function of the policy, x, if candidate 1 is elected, and a blue one indicating her payoff if candidate 2 is elected.

Is the pair (x^*, x^*) of positions a Nash equilibrium? For this pair of positions, candidate 2 wins, because citizens 2 and 3 vote for her; thus she has no deviation that generates an outcome she prefers. If candidate 1 deviates to a position $x < x^*$ then she becomes less desirable for citizens 2 and 3, and hence still loses. If she deviates to a position $x > x^*$ then she becomes less desirable for citizens 1 and 2 and more desirable for citizen 3. If for some such position x, citizen 3 prefers $(x, 1)$ to $(x^*, 2)$ and citizen 1 also still has this preference, then candidate 1 has a profitable deviation, and (x^*, x^*) is not a Nash equilibrium. In Figure 8.10, we need $x > x'$ for citizen 3 to prefer $(x, 1)$ to $(x^*, 2)$ and hence vote for candidate 1, and when $x = x'$ citizen 1 is indifferent between $(x, 1)$ and $(x^*, 2)$ if the graph of $u_1(x, 2)$ is the dashed curve.

Thus if the graph of $u_1(x, 2)$ lies above the dashed curve, then for no position does candidate 1 attract the votes of citizens 1 and 3: if she moves far enough right to attract the vote of citizen 3, she loses the vote of citizen 1. Hence in this case (x^*, x^*) is a Nash equilibrium.

If the graph of $u_1(x, 2)$ lies below the dashed curve, then for some position $x > x'$ with x close to x', candidate 1 attracts the votes of both citizen 1 and citizen 3, and hence wins, so that (x^*, x^*) is not a Nash equilibrium.

In both cases any pair (x_1, x_2) with $\underline{z} \leq x_2 \leq \bar{z}$ is a Nash equilibrium: for any value of x_1, citizens 2 and 3 prefer $(x_2, 2)$ to $(x_1, 1)$, so that candidate 2 wins and candidate 1 cannot stop her from doing so. (The game has also other Nash equilibria.)

Exercise 8.5: Electoral competition with an advantaged candidate

Suppose that every citizen prefers candidate 1 to candidate 2, in the sense that $u_i(x, 1) > u_i(x, 2)$ for all $x \in X$ for every citizen i. Assume specifically that (*i*) the set X of possible positions is an interval of numbers and (*ii*) for a single-peaked function $v : \mathbb{R} \to \mathbb{R}_-$ with $v(0) = 0$ and $v(z) < 0$ for all $z \neq 0$, a position $\hat{x}_i$ for each citizen i (i's favorite position), and a number $\delta > 0$,

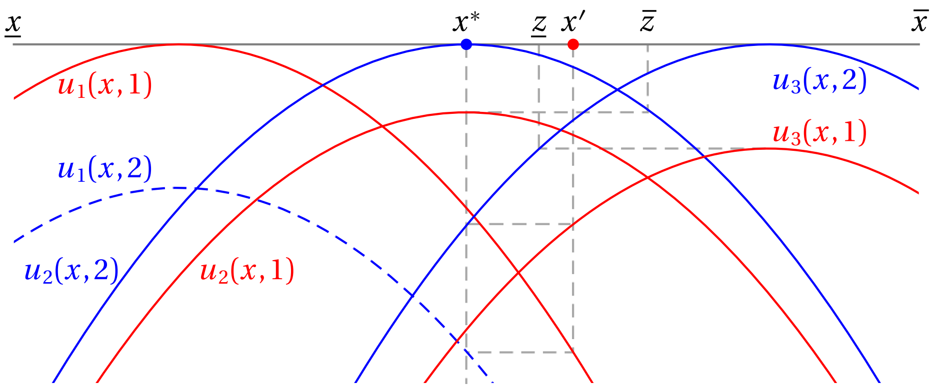


Figure 8.10 The functions u_i for the citizens in Example 8.1.

we have $u_i(x,1) = v(x - \hat{x}_i)$ and $u_i(x,2) = v(x - \hat{x}_i) - \delta$ for all $x \in X$ for each citizen i.

Assume that the number of citizens is finite and odd and that a citizen i for whom $u_i(x_1,1) = u_i(x_2,2)$, where x_1 and x_2 are the candidates' positions, votes for candidate 1. Denote by m the median of the citizens' favorite positions. Show that if $v(x_1 - m) \geq -\delta$ then (x_1, x_2) is a Nash equilibrium for any value of x_2. In any such equilibrium, candidate 1 wins.

Now suppose that each candidate's preferences are lexicographic: between two pairs of positions for which her probability of winning is the same, she prefers the one in which the number of votes she receives is larger. What can you say about the Nash equilibria in this case?

8.5.2 Candidates uncertain of citizens' preferences

Now suppose that the candidates are not perfectly informed about the citizens' preferences. Denote the set of possible positions by X and the payoff of any citizen i for the position x implemented by candidate j by $u_i(x,j)$. Assume that for each citizen i there is a function $v_i : X \to \mathbb{R}$ and numbers θ_i^1 and θ_i^2 such that

$$u_i(x,j) = v_i(x) + \theta_i^j \quad \text{for all } (x,j) \in X \times \{1,2\},$$

so that $u_i(x,1) \geq u_i(y,2)$ if and only if $\theta_i \leq v_i(x) - v_i(y)$ where $\theta_i = \theta_i^2 - \theta_i^1$. We can think of v_i as i's base payoff function and θ_i as her (positive or negative) bias towards candidate 2. Assume that each candidate knows each base payoff function v_i but not the bias θ_i, which she believes is drawn from a nonatomic distribution F_i independently of every $\theta_{i'}$ for $i' \neq i$. Under these assumptions, if candidate 1's position is x_1 and candidate 2's is x_2, each candidate believes that the probability with which any given citizen i votes for candidate 1 is $F_i(v_i(x_1) - v_i(x_2))$, the probability that i's bias θ_i towards candidate 2 is at most $v_i(x) - v_i(y)$, independently of the other citizens' votes.

Assume that the number n of citizens is odd. Then if the probability a candidate assigns to any citizen i's voting for candidate 1 is p_i, the probability that she assigns to a win by candidate 1, which occurs if the members of some set of more than $\frac{1}{2}n$ citizens all vote for candidate 1, is

$$P(p_1,\ldots,p_n) = \sum_{\{S\subseteq I:|S|>n/2\}} \left(\prod_{i\in S} p_i \prod_{i\in I\setminus S}(1-p_i)\right), \tag{8.6}$$

where I is the set of citizens. Thus the probability that she assigns to a win by candidate 1, as a function of the candidates' positions, is

$$\Pr(1 \text{ wins}) = P\big(F_1(v_1(x_1) - v_1(x_2)),\ldots,F_n(v_n(x_1) - v_n(x_2))\big).$$

Definition 8.9: Electoral competition game with two office-motivated candidates and uncertain partisanship

An *electoral competition game with two office-motivated candidates and uncertain partisanship* $\langle I, X, (v_i)_{i\in I}, (F_i)_{i\in I}, \{1,2\}\rangle$, where

- I is a finite set (of citizens) with an odd number of members
- X is a set (of positions)

and, for each $i \in I$,

- $v_i : X \to \mathbb{R}$ (i's base payoff function over positions)
- F_i is a nonatomic probability distribution function on $\mathbb{R}$ (the distribution of i's bias towards candidate 2)

is the strategic game with the following components.

Players
$\{1,2\}$ (candidates).

Actions
The set of actions of each player is X.

Preferences
Let $I = \{1,\ldots,n\}$ and let P be given by (8.6). Then the preference relation of player 1 over action profiles (x_1, x_2) is represented by the function w_1 defined by

$$w_1(x_1,x_2) = P\big(F_1(v_1(x_1) - v_1(x_2)),\ldots,F_n(v_n(x_1) - v_n(x_2))\big) \text{ for all } (x_1,x_2)$$

(the probability that player 1 wins) and the preference relation of player 2 is represented by the function w_2 defined by

$$w_2(x_1,x_2) = 1 - P\big(F_1(v_1(x_1) - v_1(x_2)), \ldots, F_n(v_n(x_1) - v_n(x_2))\big) \text{ for all } (x_1,x_2)$$

(the probability that player 2 wins).

The next result assumes that (i) the set X of positions is a convex subset of a Euclidean space, (ii) every base payoff function v_i is differentiable and strictly concave, (iii) the distribution F_i of citizen i's bias towards candidate 2 is differentiable and has a positive density in the interior of its support, and (iv) a bias large enough to cancel out the difference between the base payoffs for any pair of positions is possible. The result shows that in any Nash equilibrium in which each candidate's position is in the interior of X, (a) the candidates' positions are the same, and (b) if the distribution F_i of the bias of individual i is the same for every individual then the candidates' common equilibrium position maximizes the sum of $v_i(x)$ over all citizens.

Proposition 8.8: Nash equilibrium of electoral competition game with two office-motivated candidates and uncertain partisanship

Let $\langle I, X, (v_i)_{i\in I}, (F_i)_{i\in I}, \{1,2\}\rangle$ be an electoral competition game with two office-motivated candidates and uncertain partisanship. Assume that $I = \{1,\ldots,n\}$, X is a convex compact subset of a Euclidean space, each base payoff function v_i is differentiable and strictly concave, and the distribution F_i of each citizen i's bias towards candidate 2 is differentiable, with $F_i'(\theta) > 0$ for all θ in the interior of its support, and the support of F_i includes $[\min_{i\in I, x\in X, y\in X}(v_i(x) - v_i(y)) - \varepsilon, \max_{i\in I, x\in X, y\in X}(v_i(x) - v_i(y)) + \varepsilon]$ for some $\varepsilon > 0$.

If (x_1^*, x_2^*) is a Nash equilibrium and x_1^* and x_2^* are in the interior of X then

a. $x_1^* = x_2^*$

b. if the distribution F_i of the bias of citizen i towards candidate 2 is the same for all i then

$$x_1^* = x_2^* = \arg\max_{x\in X} \sum_{i=1}^{n} v_i(x).$$

Proof

a. For $j = 1, 2$, the position x_j^* of candidate j maximizes j's probability of winning, given the other candidate's position: x_1^* is a solution of

$$\max_{x_1 \in X} P(F_1(v_1(x_1) - v_1(x_2^*)), \ldots, F_n(v_n(x_1) - v_n(x_2^*))) \tag{8.7}$$

and x_2^* is a solution of

$$\max_{x_2 \in X} \left(1 - P(F_1(v_1(x_1^*) - v_1(x_2)), \ldots, F_n(v_n(x_1^*) - v_n(x_2)))\right). \tag{8.8}$$

If x_1^* is a solution of (8.7) in the interior of X then by Proposition 16.12 it satisfies the first-order condition

$$\sum_{i=1}^{n} P_i'(\pi(x_1^*, x_2^*)) F_i'(v_i(x_1^*) - v_i(x_2^*)) \nabla v_i(x_1^*) = 0, \tag{8.9}$$

where P_i' is the derivative of P with respect to its ith argument,

$$\pi(x_1^*, x_2^*) = \left(F_1(v_1(x_1^*) - v_1(x_2^*)), \ldots, F_n(v_n(x_1^*) - v_n(x_2^*))\right),$$

and ∇v_i is the gradient of v_i (the vector of its partial derivatives).

Define the function $W : X \to \mathbb{R}$ by

$$W(x) = \sum_{i=1}^{n} P_i'(\pi(x_1^*, x_2^*)) F_i'(v_i(x_1^*) - v_i(x_2^*)) v_i(x)$$

for all $x \in X$. Each function v_i is strictly concave and all the coefficients of $v_i(x)$ are positive ($P_i'(\pi(x_1^*, x_2^*))$ is the change in 1's probability of winning as citizen i becomes more likely to vote for her, and the values of F_i' are positive by the assumption about the support of F_i and the positivity of F_i' on the support), so W is strictly concave. Thus W has a unique maximizer and, by Proposition 16.13, (8.9) is necessary and sufficient for x_1^* to be a maximizer in the interior of X. Hence if x_1^* is in the interior of X then it maximizes W.

A solution x_2^* of (8.8) that is in the interior of X satisfies the same condition, (8.9), and hence also is the unique maximizer of W. Thus $x_1^* = x_2^*$.

b. Given $x_1^* = x_2^*$,

$$W(x) = \sum_{i=1}^{n} P_i'(F(0), \ldots, F(0)) F'(0) v_i(x),$$

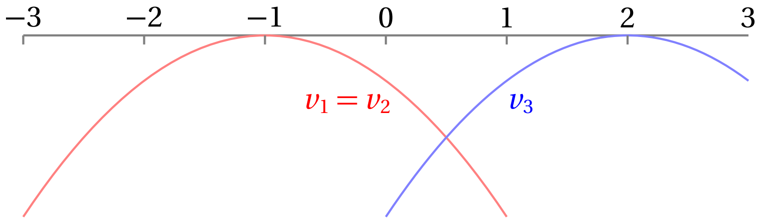


Figure 8.11 The functions v_i for the game in Example 8.2.

where F is the common distribution. Thus given $P_i'(p,\ldots,p) = P_k'(p,\ldots,p)$ for all p and all i and k, the maximizers of $W(x)$ are the maximizers of $\sum_{i=1}^n v_i(x)$.

Note that this result does not restrict the set X of positions to be one-dimensional. If this set *is* one-dimensional and for each citizen i there is a position $\hat{x}_i \in X$ such that $v_i(x) = -(x - \hat{x}_i)^2$ for all $x \in X$, then

$$\arg\max_{x \in X} \sum_{i=1}^{n} v_i(x) = \arg\max_{x \in X} \sum_{i=1}^{n} -(x - \hat{x}_i)^2 = \sum_{i=1}^{n} \hat{x}_i / n.$$

That is, in this case if the distributions of the citizens' biases towards candidate 2 are the same, the common position chosen by the candidates is the mean of the citizens' favorite positions.

Note also that the result does not assert that an equilibrium exists—only that *if* an equilibrium exists, it has certain properties. I am not aware of a result that provides sufficient conditions for an equilibrium to exist, but the following example shows that the result is not vacuous.

Example 8.2: Electoral competition with two office-motivated candidates and uncertain partisanship

Let $\langle I, X, (v_i)_{i\in I}, (F_i)_{i\in I}, \{1,2\}\rangle$ be a two-candidate electoral competition game with office-motivated candidates and uncertain partisanship. Suppose that there are three citizens, with $I = \{1,2,3\}$, and the set X of possible positions is the interval $[-3,3] \subset \mathbb{R}$. Citizens 1 and 2 have the same favorite position, $\hat{x}_1 = \hat{x}_2 = -1$, citizen 3 has favorite position $\hat{x}_3 = 2$, and $v_i(x) = -(x - \hat{x}_i)^2$ for all $x \in X$ for every citizen i. (Refer to Figure 8.11.) The functions F_1, F_2, and F_3 are all equal to F, which for some number $\alpha > 0$ is uniform on $[-\alpha, \alpha]$.

The median favorite position is -1 and the maximizer of $\sum_{i=1}^n v_i(x)$ is 0, the mean favorite position. Under what conditions is the pair $(0,0)$ of positions a Nash equilibrium? We need each candidate's probability of winning to be at most $\frac{1}{2}$ for every position, given that the other candidate's position is 0. If candidate 2's position is 0 and candidate 1's position is x then

$$\begin{aligned}\Pr(\text{citizen 1 votes for candidate 1}) &= \Pr(\theta < v_1(x) - v_1(0))\\ &= \Pr(\theta < -(x-(-1))^2 + (0-(-1))^2)\\ &= \Pr(\theta < -x(x+2))\\ &= [\max(-\alpha, \min(\alpha, -x(x+2))) - (-\alpha)]/(2\alpha).\end{aligned}$$

(Refer to Figure 8.12.) The probability that citizen 2 votes for candidate 1 in this case is the same, and a similar calculation yields

$$\begin{aligned}&\Pr(\text{citizen 3 votes for candidate 1})\\ &\qquad = [\max(-\alpha, \min(\alpha, -x(x-4))) - (-\alpha)]/(2\alpha).\end{aligned}$$

Using these expressions, we find that the probability a candidate wins (which happens if and only if two or three citizens vote for her) is given by the graphs in Figure 8.13 for various values of α. We see that $(0,0)$ is a Nash equilibrium if α is large enough (the cutoff is about 2.4); that is, if there is enough uncertainty.

This example does not satisfy the conditions in Proposition 8.8 because F is not differentiable at $-\alpha$ and α, and its support does not include all possible values of $v_i(x) - v_i(y)$. Suppose we modify F by slightly changing its values in $[-3, \alpha]$ and $[\alpha, 3]$ to satisfy these conditions. Then the probability of a candidate's winning when the other candidate's position is 0 is not affected if her position is close to 0 and is affected only slightly when her position is further from 0. Thus for any value of α for which $(0,0)$ is a strict Nash equilibrium in the original example, it is a Nash equilibrium in the modified example, and for any value of α for which it is not a Nash equilibrium in the original example, it is not a Nash equilibrium in the modified example. For the modified example, Proposition 8.8 implies that if $(0,0)$ is not a Nash equilibrium then the game has no Nash equilibrium.

The pair of positions $(-1,-1)$, in which both candidates choose the median of the citizens' favorite positions, is a Nash equilibrium in the variant of the example in which there is no uncertainty, but is not a Nash equilibrium for any $\alpha > 0$. If the position of each candidate is -1, in the presence of any uncertainty one candidate can increase her probability of winning by deviating to a position slightly greater than -1, because the impact of such a deviation on the prob-

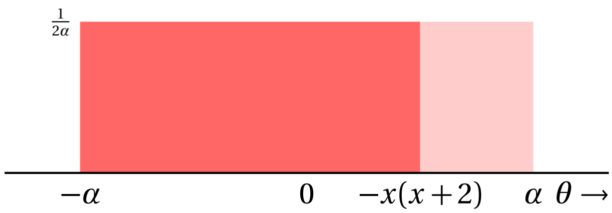


Figure 8.12 The values of θ for which citizen 1 votes for candidate 1 when candidate 1's position is x and candidate 2's position is 0 for the game in Example 8.2.

ability of her getting the votes of citizens 1 and 2, whose favorite positions are both -1, is almost zero, given that the functions u_1 and u_2 are differentiable, but the impact on the probability of her getting the vote of citizen 3, whose favorite position is 2, is bounded away from zero.

Exercise 8.6: Tent-shaped payoff functions

If the functions v_i are not differentiable, Proposition 8.8 does not apply. Suppose that X is an interval of numbers, the number n of citizens is odd, and for each citizen i there is a position $\hat{x}_i \in X$ such that $v_i(x) = -|x - \hat{x}_i|$ for all $x \in X$. Show that (*a*) the position x^* that maximizes $\sum_{i=1}^n v_i(x)$ is the median of the numbers $\hat{x}_i$ for $i \in I$ and (*b*) if each distribution F_i is the same, equal to F, and the density of F is symmetric about 0, then (x^*, x^*) is a Nash equilibrium of the game.

8.6 Electing a legislature

In the models of electoral competition that I have presented so far, a single candidate is elected. I now present a variant of an electoral competition game with a continuum of citizens and two office-motivated candidates in which the set of citizens is partitioned into an odd number of subsets, which may be interpreted as electoral districts. The distribution of the citizens' favorite positions may differ among districts. In each district, one candidate is elected; the elected candidates make up a legislature. I discuss two versions of the model.

8.6.1 Each party chooses one position for all its candidates

Symmetric parties Assume that each of two parties fields a candidate in each district. Each party chooses a single position. In each district, each citizen votes for the candidate who represents the party whose position she prefers. Denote the (odd) number of districts by l. Define a party's district tally to be the number

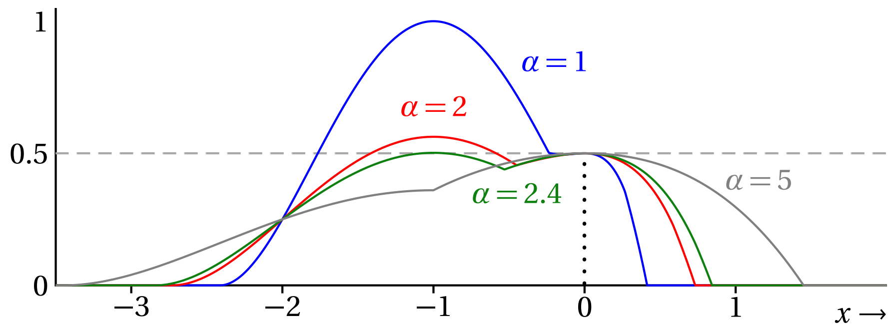


Figure 8.13 The probability of a candidate with position x winning when the other candidate's position is 0, for various values of α, for the game in Example 8.2.

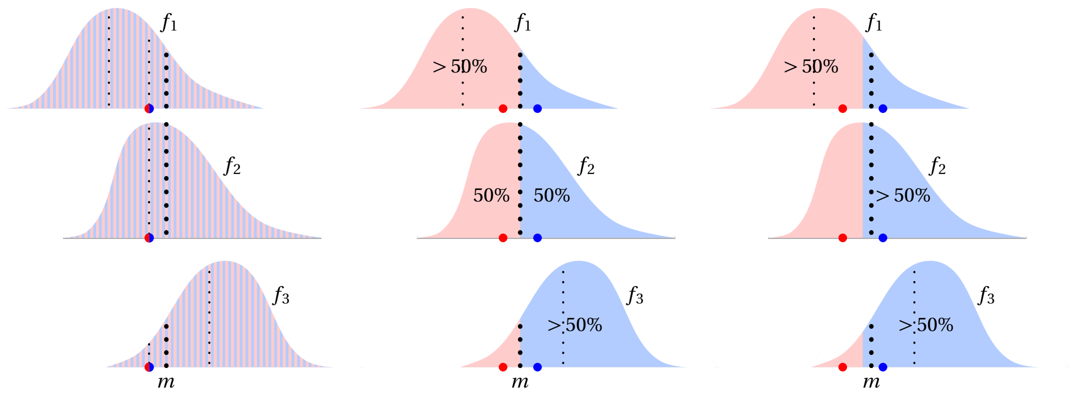
of districts in which it wins plus half the number in which it ties, and classify an outcome as a win (in the legislature) for the party if its district tally is more than $\frac{1}{2}l$, a tie if it is exactly $\frac{1}{2}l$, and a loss if it is less than $\frac{1}{2}l$. Each party prefers a win to a tie to a loss.

Denote the median of the favorite positions of the citizens in each district k by m_k. Order the districts so that $m_1 \leq m_2 \leq \cdots \leq m_l$ and denote the median of these medians by m. That is, m is the median favorite position among the citizens in the median district, $m_{(l+1)/2}$.

If the parties choose the same position ($x_1 = x_2$), as in the example in Figure 8.14a, then the outcome is a tie in every district, and hence a tie in the legislature. If the parties' positions are symmetric around m, as in the example in Figure 8.14b, then the outcome is a tie in the median district, a win for party 1 in half of the remaining districts, and a win for party 2 in the remaining half, also resulting in a tie in the legislature. Otherwise suppose that $x_i < x_j$ and $\frac{1}{2}(x_1+x_2) < m$, as in the example in Figure 8.14c (for $i = 1$ and $j = 2$). Then party j wins in the median district and hence in a majority of the districts. Thus for each pair (x_1, x_2) of the parties' positions the outcome is the same as the outcome of (x_1, x_2) in an electoral competition game with a continuum of citizens and two office-motivated candidates $\langle\{1,2\}, X, F\rangle$ in which the players are the parties, F is the distribution of the citizens' favorite positions in the median district, and *win for j* means a win in the legislature.

We conclude from Proposition 8.4 that the electoral competition between the parties has a unique Nash equilibrium, in which each party's position is m, the median favorite position among the citizens in the median district.

Asymmetric parties Consider a party that is currently winning a minority of the districts. In the model with symmetric parties, this party can induce a tie in the legislature, an outcome it prefers, by deviating to the same position as its rival. If



(a) $x_1 = x_2$; tie in every district.

(b) $\frac{1}{2}(x_1 + x_2) = m$; win for • in district 1, tie in district 2, and win for • in district 3.

(c) $x_1 < x_2$ and $\frac{1}{2}(x_1 + x_2) < m$; win for • in district 1 and win for • in districts 2 and 3.

Figure 8.14 Electoral outcomes in a model in which each of two parties fields a candidate in each of several districts. In this example, there are three districts; the density of the distribution of the favorite positions in district k is f_k for $k = 1, 2, 3$.

the parties' abilities to win are not the same, however, this deviation may not be desirable: it may induce an outcome that is worse for the party than a tie in the legislature, and, if the party attaches sufficient value to winning a minority of the districts, worse also than the current outcome.

I briefly discuss an example that shows that such a difference between the parties can generate equilibria in which the parties adopt different positions. Suppose, as in Section 8.5.1, that each citizen's payoff depends not only on the policy outcome, but also on the identity of the party that implements it. There are three districts, 1, 2, and 3. As before, denote by m_k the median of the favorite positions of the citizens in district k, and assume that $m_1 < m_2 < m_3$. The parties are perceived as differing in their competence in implementing policies, but the identity of the more competent party depends on events that occur between the time the parties choose their positions and the time of the election. At the time they choose their positions, each party believes that at the time of the election, for each position $\hat{x}$ the payoff of each citizen with favorite position $\hat{x}$ will be

$$\begin{cases} |x - \hat{x}| & \text{if } x \text{ is implemented by party 1} \\ |x - \hat{x}| + \lambda & \text{if } x \text{ is implemented by party 2,} \end{cases}$$

where $\lambda = h$ with probability q and $\lambda = -h$ with probability $1 - q$, where $0 < h < \min\{m_2 - m_1, m_3 - m_2\}$. Assume that $q > \frac{1}{2}$: party 2 is more likely to be viewed, at the time of the election, as the better party to implement any policy. Each party's payoff if it wins k districts is w_k, with $0 = w_0 < w_1 < \frac{1}{2} < w_2 < w_3 = 1$, and each party evaluates a lottery over outcomes by its expected payoff. (Winning

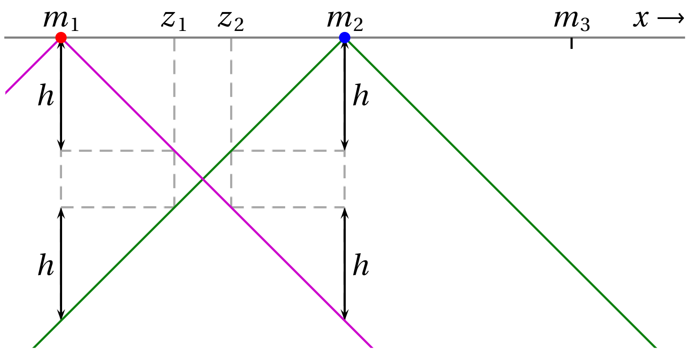


Figure 8.15 The payoff functions for citizens with favorite positions m_1 (violet) and m_2 (green) in an example of electoral competition between two parties in three districts.

one seat in the legislature might have value because the rules of the legislature confer some power upon a minority party.)

Suppose that party 1's position is m_1 and party 2's is m_2, as shown in Figure 8.15. I claim that if $w_1 > 1 - q$ and $w_2 > q$ then this pair of positions is an equilibrium. Its outcome is that party 1 wins district 1 and party 2 wins districts 2 and 3, regardless of the realization of λ (given $h < m_2 - m_1$), so that party 1's payoff is w_1 and party 2's is w_2.

Consider deviations by party 1. If it deviates to a position in (m_1, z_2), where $z_2 = m_2 - h$, the outcome remains the same: even if $\lambda = h$, the median voter in district 1 still prefers party 1. If it deviates closer to party 2, to a position in (z_2, m_2), then if $\lambda = h$ it receives no votes and hence wins no districts, and if $\lambda = -h$ it receives the vote of every citizen and hence wins in every district. Thus it wins no districts with probability q and all three districts with probability $1 - q$, so that its expected payoff is $1 - q$. If it deviates to a position greater than m_2, then by symmetric arguments the same two outcomes are possible: party 1 either wins a single district or wins no districts with probability q and all three with probability $1 - q$.

A similar analysis for deviations by party 2 shows that a deviation to a position in (z_1, m_2), where $z_1 = m_1 + h$, leaves the outcome the same, and a deviation closer to party 1 generates the outcome in which party 2 wins all three districts with probability q and no districts with probability $1 - q$, so that its payoff is q. A deviation to a position greater than m_2 either does not affect the outcome or reduces the number of districts that party 2 wins.

We conclude that the pair of positions (m_1, m_2) is an equilibrium if $w_1 > 1 - q$ and $w_2 > q$. By moving close to party 2, party 1 can win all three districts with positive probability, but if q is large then this probability is small, and as long as it attaches sufficient value to winning a single district with certainty, it prefers to stay put. In addition, party 2, by moving close to party 1, can increase its legislative majority from two to three seats with probability q, but faces the risk of a negative shock to its appeal, which will lead to its loss in every district. So if it attaches sufficient value to a legislative majority of two, it also prefers to stay put.

This example is a simple expression of the idea that an asymmetry between the parties, combined with payoffs that attach a sufficient value to a legislative minority, can generate equilibria in which the parties' positions differ. It is not, however, particularly robust to changes in the random process that gives one party an advantage or to changes in the character of the citizens' payoff functions. A much more elaborate example, which appears to be more robust, is studied by Bernhardt et al. (2020).

8.6.2 *Each candidate chooses a position independently*

Now assume that each candidate chooses a position independently, and each party's position is the average of its candidates' positions. Each citizen votes for the candidate (in her district) whose party's position is closest to her favorite position, and the party that wins a majority of districts and hence acquires a majority in the legislature implements its position.

Assume that each candidate's preferences are lexicographic. She is concerned primarily with whether her party wins a majority of districts, ties, or loses a majority of districts; among outcomes in which this outcome for her party is the same, she prefers to win than to tie than to lose in her own district.

Suppose that the set of possible positions for a candidate is the whole real line. Then for any given positions of the other candidates for her party, a candidate can induce any position for the party by choosing an appropriate position for herself. This fact, combined with the fact that each candidate cares primarily about her party's fortunes, means that the analysis of the game is very similar to the analysis of an electoral competition game with a continuum of citizens and two office-motivated candidates. In particular, a profile of positions for the candidates is a Nash equilibrium if and only if each party's position is the median favorite position among the citizens in the median district (as for the game with symmetric candidates in the previous section).

Exercise 8.7: Candidates who care about their own electoral fortunes

Consider a variant of the model in this section in which each candidate cares primarily about her own electoral fortune (rather than her party's fortune). Show that if the median of the citizens' favorite positions is not the same in every district then the game has no Nash equilibrium.

8.7 Interpreting Nash equilibrium

Is it reasonable to expect that the candidates in an electoral game will choose their Nash equilibrium actions? In a Nash equilibrium, each player's action is optimal for her, given the other players' actions. When choosing an action, a player does not know the other players' actions, and one way of conceiving her decision-making process assumes that she formulates a belief about those actions. An assumption implicit in the notion of Nash equilibrium is that this belief is correct.

Where does a player's belief about the other players' actions come from? According to the leading interpretation of the notion of Nash equilibrium, this belief is based on the player's experience playing the game against a variety of opponents. For each player, we imagine a large population of decision-makers; each time the game is played, one decision-maker is drawn randomly from each of these populations to take the role of one of the players in the game. Over time, each decision-maker learns the action chosen by each of the other players, but does not gather information on the actions chosen by any specific decision-maker. For example, whenever I have found myself walking straight towards another pedestrian, she has almost always stepped to the right (in the right-driving country in which I live). I do not know the history of any given pedestrian's actions, but my belief that the pedestrians I randomly encounter will invariably step to the right to avoid a collision is correct.

This interpretation does not fit many elections well. At least, it needs some stretching. Perhaps the participants in electoral competitions have observed many elections, or have advisors who have done so, and this experience has taught them how their opponents are likely to behave.

The difficulty with interpreting a Nash equilibrium in an electoral game gives added significance to results concerning stronger notions of equilibrium, like the ones for a electoral competition game with a continuum of citizens and two office-motivated candidates that show that the median of the citizens' favorite positions weakly dominates all other positions (Proposition 8.4) and is the only rationalizable position (Exercise 8.2).

Notes

The model in Section 8.1 has its origins in Hotelling (1929), which develops a model of competition between two spatially-separated firms and suggests (pp. 54–55) that it applies also to competition between political parties. The idea was taken up by Downs (1957, Chapter 8), who discusses the model informally. The version of the model in Definition 8.4 is sometimes called Hotelling's model or the Hotelling-Downs model, although neither Hotelling nor Downs formulated exactly this game.

Proposition 8.1 is a version of Theorem 7.1 (p. 257) in Austen-Smith and Banks (2005). Proposition 8.5 is due to Calvert (1985, Theorem 4). Section 8.3.2 is based on Bernhardt et al. (2007, 2009). The model and results in Section 8.4 are due to Ledyard (1984). Models like the one in Section 8.5 in which the candidates treat the citizens' actions as probabilistic date back to Hinich et al. (1972). Proposition 8.8 is due to Duggan (2014, Theorem 10). The full-information models in Section 8.6 are due to Austen-Smith (1984) and the example with uncertain partisanship is based on Bernhardt et al. (2020).

Exercise 8.1 is based on Brennan and Hamlin (1998). Exercise 8.3 is Example 1 in Meirowitz and Shotts (2009). The model in Exercise 8.5 is a variant of the ones studied by Ansolabehere and Snyder (2000) and Aragonès and Palfrey (2002).

Solutions to exercises

Exercise 8.1

Denote the distribution function of the citizens' favorite positions by F and its density by f. I claim that if F is unimodal then a pair (x_1, x_2) of positions is a Nash equilibrium if and only if $x_1 = x_2$ and

$$F(x^*) - F(x^* - k) = F(x^* + k) - F(x^*), \tag{8.10}$$

where $x^* = x_1 = x_2$.

Step 1 *Any pair of positions that satisfies (8.10) is a Nash equilibrium.*

Proof. If x^* satisfies (8.10) then $f(x^* - k) < f(x^*)$ and $f(x^* + k) < f(x^*)$. Hence a candidate who deviates from x^* loses. ◁

Step 2 *In any Nash equilibrium the candidates tie.*

Proof. For a pair of positions at which they do not tie, the losing candidate can move to the position of the other candidate and tie. ◁

Step 3 *In any Nash equilibrium the candidates' positions are the same.*

Proof. Suppose that (x_1, x_2) is a Nash equilibrium and $x_1 < x_2$. By Step 2, the candidates tie, and given that F is unimodal, either $f(x_1 - k) < f(\frac{1}{2}(x_1 + x_2))$ or $f(x_2 + k) < f(\frac{1}{2}(x_1 + x_2))$ (or both). The two cases are symmetric; assume the former.

If $x_1 + k < x_2 - k$ then $\frac{1}{2}(x_1 + x_2) \in [x_1 + k, x_2 - k]$, so that by increasing x_1 candidate 1 can increase her vote share without affecting candidate 2's vote share, and hence win, so that (x_1, x_2) is not a Nash equilibrium.

If $x_1 + k \geq x_2 - k$ then the difference between candidate 1's share of the votes and candidate 2's share is

$$F(\tfrac{1}{2}(x_1 + x_2)) - F(x_1 - k) - F(x_2 + k) + F(\tfrac{1}{2}(x_1 + x_2))$$
$$= 2F(\tfrac{1}{2}(x_1 + x_2) - F(x_1 - k) - F(x_2 + k).$$

The derivative of this expression with respect to x_1 is

$$f(\tfrac{1}{2}(x_1 + x_2) - f(x_1 - k),$$

which is positive. Thus by increasing x_1 slightly, candidate 1 wins rather than ties, contradicting the assumption that (x_1, x_2) is a Nash equilibrium. ◁

Step 4 *In any Nash equilibrium* (x^*, x^*)*, the position* x^* *satisfies (8.10).*

Proof. If $F(x^*) - F(x^* - k) > F(x^* + k) - F(x^*)$ then either candidate can win rather than tie by decreasing her position slightly, and if the inequality is reversed then either candidate can win rather than tie by increasing her position slightly. Thus (8.10) is satisfied. ◁

Here is an example of Nash equilibrium in which candidates' positions differ. Let $X = [-1, 1]$ and let $f(x) = \frac{3}{2}x^2$. For any $k \leq 1$, the pair $(-1 + k, 1 - k)$ of positions is a Nash equilibrium. An example in which $k > \frac{1}{2}$ is shown in Figure 8.16. In this case, if candidate 1 increases her position she loses to abstention a large number of citizens with favorite positions close to -1 and gains from candidate 2 a few with favorite positions close to 0. If she reduces her position then she loses to candidate 2 a few citizens with favorite positions close to 0. If $k < \frac{1}{2}$ then citizens with favorite positions close to 0 do not vote. In this case, if candidate 1 increases her position she loses a large number of citizens with favorite positions close to -1 and gains a few with favorite

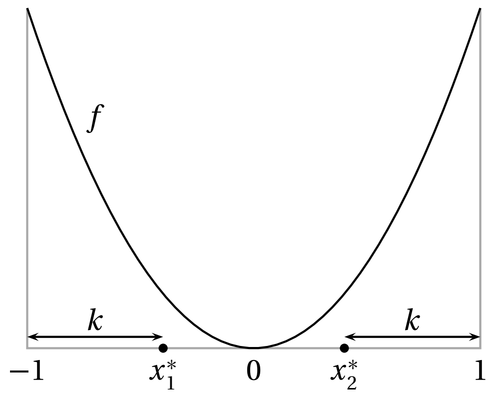


Figure 8.16 A Nash equilibrium of an electoral competition game with alienation in which the candidates' positions differ.

positions close to $-1+2k$, and if she reduces her position she loses a few citizens with favorite positions close to $-1+2k$. In both cases, candidate 2 is unaffected.

Note, however, that this example depends on the symmetry of the distribution f. If f is slightly asymmetric the game appears not to have a Nash equilibrium.

Exercise 8.2

Denote the median of the citizens' favorite positions by m. First suppose that a single position is furthest from m. Without loss of generality let this position be z_1. The following table gives the outcomes of a candidate's actions z_1 and m for all the actions possible for the other candidate. We see that m strictly dominates z_1.

	z_1	z_2	$\ldots$	z_{l-1}	$z_l = m$	z_{l+1}	$\ldots$	z_k
z_1	tie	lose	lose	lose	lose	lose	lose	lose
m	win	win	win	win	tie	win	win	win

No other position is strictly dominated because every other position leads to a win for the candidate if the other candidate's position is more extreme.

Thus every position except the one furthest from m is rational for the candidate.

If two positions are furthest from m, at exactly the same distance from it, then a similar argument shows that neither of them is rational for the candidate.

Now, the candidate's assuming that the other candidate is rational means that she assumes that the other candidate does not choose the position furthest from m. Under that assumption, the position second-furthest from m (or the two positions second-furthest from m, if there is a tie for that honor) is strictly dominated (by m) for the candidate.

Repeating this argument, we conclude that the only action that is rational if the candidate assumes that the other candidate is rational, that the other candidate assumes that she is rational, and so forth, is m.

Exercise 8.3

A citizen whose favorite position is x^* must be indifferent between voting for candidate 1 and for candidate 2 in the first period. If she votes for candidate 1 her payoff is $-\left|\frac{1}{2}-x^*\right|-\left|\frac{1}{2}x^*-x^*\right|$, and if she votes for candidate 2 her payoff is $-|1-x^*|-\left|\frac{1}{2}(x^*+1)-x^*\right|$. These payoffs are equal for $x^*=\frac{2}{3}$.

If there is no second period, the citizen votes for candidate 1 if her favorite position is less than $\frac{3}{4}$ and for candidate 2 if it is greater than $\frac{3}{4}$. Thus if the citizen's favorite position is between $\frac{2}{3}$ and $\frac{3}{4}$ she votes for candidate 1 if there is only one period but for candidate 2 if there are two periods and her vote is used as a signal about her favorite position. In the latter case, the citizen prefers candidate 1's first-period position to candidate 2's, but if she votes for candidate 1 in the first period then the candidates' common position in the second period is $\frac{1}{3}$ whereas if she votes for candidate 2 in the first period then the candidates' common position in the second period is $\frac{5}{6}$, and she prefers 1 followed by $\frac{5}{6}$ to $\frac{1}{2}$ followed by $\frac{1}{3}$.

Exercise 8.4

Let $X=[a,b]$ and $m=\frac{1}{2}(a+b)$. I argue that the game has an equilibrium in which each candidate's position is m, and it has no equilibrium in which either candidate chooses a position different from m.

Suppose that both candidates choose m. Then only types with zero cost vote; given that the distribution of types is nonatomic, these types have measure zero. Thus with probability 1 the candidates tie.

Now suppose that candidate 1 deviates to $x<m$. (Refer to Figure 8.17 for an example in which the function ψ is convex on each side of its maximizer.) Then given the symmetry of ψ, the set of types (θ,c) with $\theta\in[a,\frac{1}{2}(x+m)]$ who optimally vote for candidate 1 (shaded red in the figure) has the same measure as the set with $\theta\in[\frac{1}{2}(x+m),b-(m-x)]$ who optimally vote for candidate 2 (shaded dark blue in the figure). Candidate 2 gets, in addition, the votes of types with $\theta\in[b-(m-x),b]$ who optimally vote (the set shaded light blue in the figure); this set is nonempty because ψ is never constant, so that these types are not indifferent between the candidates. Thus candidate 1 wins with probability less than $\frac{1}{2}$, so that her deviation is not profitable.

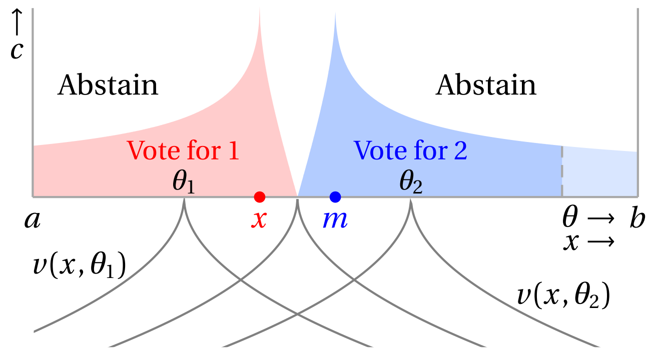


Figure 8.17 An example of a two-candidate electoral competition game with office-motivated candidates and costly voting that satisfies the assumptions of Exercise 8.4, with $\psi(z) = -(|z|)^{1/2}$. For clarity, the vertical scale above the x-axis is exaggerated relative to the vertical scale below the axis. The set of types (θ, c) who vote for 1 when candidate 1's position is x and candidate 2's is m is shaded red, and the set who vote for 2 is shaded blue. The subset shaded dark blue is the reflection in the line $\theta = \frac{1}{2}(x + m)$ of the set shaded red, so that the area of the blue set exceeds the area of the red set by the area of the set shaded light blue.

If candidate 1 deviates to $x > m$, the same argument applies, so that this deviation is also not profitable.

Thus the game has an equilibrium in which each candidate's position is m.

The same argument shows that the game has no equilibrium in which both candidates choose a position different from m. If the common position is $x < m$, for example, then the candidates tie, and either candidate can win with probability greater than $\frac{1}{2}$ by moving to m.

Finally, the argument in the last paragraph of the proof of Proposition 8.7 shows that the game has no equilibrium in which the candidates' positions differ.

Exercise 8.5

Suppose that $x_1 \leq m$ satisfies the condition $v(x_1 - m) \geq -\delta$. I argue that if $x_2 \leq x_1$ then all citizens with favorite positions at least m, a majority, vote for candidate 1, and if $x_2 > x_1$ then all citizens with favorite positions at most m, a majority, vote for candidate 1.

First suppose that $x_2 \leq x_1$. Then for any citizen i with favorite position $\hat{x}_i \geq m$ the payoff from x_1 is $u_i(x_1, 1) = v(x_1 - \hat{x}_i)$ and the payoff from x_2 is $u_i(x_2, 2) = v(x_2 - \hat{x}_i) - \delta \leq v(x_1 - \hat{x}_i) - \delta < v(x_1 - \hat{x}_i) = u_i(x_1, 1)$, so i votes for candidate 1.

Now suppose that $x_2 > x_1$. Then for any citizen i with favorite position $\hat{x}_i$ for which $x_1 \leq \hat{x}_i \leq m$ the payoff from x_1 is $u_i(x_1, 1) = v(x_1 - \hat{x}_i) \geq v(x_1 - m) \geq$

$-\delta$ and the payoff from x_2 is $u_i(x_2, 2) = v(x_2 - \hat{x}_i) - \delta \leq -\delta$, so i votes for candidate 1. For any citizen i with favorite position $\hat{x}_i < x_1$ the payoff from x_1 is $u_i(x_1, 1) = v(x_1 - \hat{x}_i)$ and the payoff from x_2 is $u_i(x_2, 2) = v(x_2 - \hat{x}_i) - \delta \leq v(x_1 - \hat{x}_i) - \delta$, so i votes for candidate 1.

Thus any pair (x_1, x_2) with $x_1 \leq m$ that satisfies the condition $v(x_1 - m) \geq -\delta$ is a Nash equilibrium. A symmetric argument shows that any pair (x_1, x_2) with $x_1 \geq m$ that satisfies the condition is a Nash equilibrium.

Now suppose that the candidates' preferences are lexicographic as described in the exercise. For any position of candidate 2, candidate 1 can obtain the vote of every citizen by choosing the same position, so in any best response to a position of candidate 2, candidate 1 obtains all the votes. Denote by $\underline{z}$ the smallest of the citizens' favorite positions and by $\bar{z}$ the largest. If there is a position x_1 such that $v(x_1 - \underline{z}) \geq -\delta$ and $v(x_1 - \bar{z}) \geq -\delta$, then for any such position the pair (x_1, x_2) is a Nash equilibrium for any position x_2. (For such a pair, candidate 1 gets all the votes and no position of candidate 2 yields candidate 2 any votes.) Otherwise, the game has no Nash equilibrium: a pair (x_1, x_2) in which candidate 1 gets all the votes is not an equilibrium because for some position of candidate 2, candidate 2 obtains some votes, and a pair (x_1, x_2) in which candidate 1 does not get all the votes is not an equilibrium because by deviating to x_2 candidate 1 can obtain all the votes.

Exercise 8.6 *a.* Let m be the median of the numbers $\hat{x}_i$ for $i \in I$. Divide I into three sets: a set M consisting of the citizens i for whom $\hat{x}_i = m$, a set L consisting of citizens i for whom $\hat{x}_i < m$, and a set R consisting of citizens i for whom $\hat{x}_i > m$, so that the sets L and R have the same number of members. Then if $x < m$ we have

$$v_i(x) - v_i(m) \begin{cases} \leq m - x & \text{if } i \in L \\ = -(m - x) & \text{if } i \in R \cup M, \end{cases}$$

so that, given that $R \cup M$ has more members than L,

$$\sum_{i \in I} v_i(x) - \sum_{i \in I} v_i(m) \leq -(m - x) < 0.$$

A symmetric argument shows the same result for $x > m$. Thus m is the unique maximizer of $\sum_{i \in I} v_i(x)$.

b. If the position of each candidate is x^*, then each candidates wins with probability $\frac{1}{2}$. Suppose that candidate 1 deviates to a position $x_1 < x^*$.

The number of citizens for whom $v_i(x^*) > v_i(x_1)$ is at least $\frac{1}{2}(n+1)$, and for every such citizen i with $\hat{x}_i \geq x^*$, of which there are $\frac{1}{2}(n+1)$, we have

$v_i(x^*) - v_i(x_1) = x^* - x_1$, so that the probability that each such citizen votes for candidate 2 is $F(x^* - x_1)$. Denote this probability by p^*. Given $x^* > x_1$ we have $p^* > \frac{1}{2}$.

The number of citizens for whom $v_i(x_1) > v_i(x^*)$ is at most $\frac{1}{2}(n-1)$. For each such citizen i with $\hat{x}_i \leq x_1$ we have $v_i(x_1) - v_i(x^*) = x^* - x_1$, so that the probability that each such citizen votes for candidate 1 is $F(x^* - x_1) = p^*$, and for any of these citizens for whom $x_1 < \hat{x}_i \leq x^*$ we have $v_i(x_1) - v_i(x^*) < x^* - x_1$, so that the probability that she votes for candidate 1 is less than p^*.

Thus of the citizens for whom $v_i(x^*) > v_i(x_1)$, $\frac{1}{2}(n+1)$ vote for candidate 2 with probability p^* and possibly additional citizens (with favorite positions between x_1 and x^*) vote for her with probability less than p^*. Of the citizens for whom $v_i(x_1) > v_i(x^*)$, who number at most $\frac{1}{2}(n-1)$, at most $\frac{1}{2}(n-1)$ vote for candidate 1 with probability p^* and any remaining citizens (with favorite positions between x_1 and x^*) vote for her with probability less than p^*. Thus the probability that candidate 2 wins exceeds $\frac{1}{2}$.

Symmetric arguments apply if candidate 1 deviates to a position greater than x^* or if candidate 2 deviates. Thus (x^*, x^*) is a Nash equilibrium.

Exercise 8.7

Because any candidate can cause her party's position to take any value (given the positions of the other candidates for the party) by choosing an appropriate position, the argument in the proof of Proposition 8.4 applied to a given district shows that in any Nash equilibrium the parties' positions are the same, equal to the median of the favorite positions of the citizens in the district. Thus if this median favorite position varies across districts, no Nash equilibrium exists.

9 Electoral competition: two policy-motivated candidates

9.1 Basic model 297
9.2 Uncertain median 308
9.3 Candidates privately informed about policies 314
9.4 Repeated elections 319

> "The peak of the campaign happened in Albuquerque, where a local reporter said to me, 'Dr. Commoner, are you a serious candidate or are you just running on the issues?' " (Barry Commoner, Citizens Party candidate in 1980 U.S. Presidential election, in interview in *New York Times*, 2007.6.19).

Some candidates for political office appear to be motivated by the possibility of implementing policies they like, rather than the possibility of winning per se, as the models in the previous chapter assume. We say that such candidates are policy-motivated.

Synopsis

Consider a variant of an electoral competition game with two office-motivated candidates in which each candidate cares only about the position of the winner of the election, not about being the winner herself. Proposition 9.1 shows that if the underlying collective choice problem has a strict Condorcet winner, then the action pair in which each candidate chooses that alternative is a Nash equilibrium. That is, the game has an equilibrium in which the candidates' actions are the same as they are in the unique Nash equilibrium of the game in which each candidate is office-motivated. The reason is simple: if either candidate deviates to a position different from the strict Condorcet winner then she loses, so that the outcome of the game remains the same.

Unlike the case in which the candidates are office-motivated, however, for some collective choice problems the game has Nash equilibria in which the policy outcome is not a Condorcet winner of the underlying collective choice problem. For example, if every citizen prefers a to b but both candidates prefer b

to a, then the action pair in which both candidates choose b is a Nash equilibrium. However, if the candidates are representative of the citizens in the sense that whenever all members of a majority of citizens prefer some position x to another position y, at least one candidate prefers x to y, then in any Nash equilibrium in which the candidates' positions are the same, their common position is a Condorcet winner of the underlying collective choice problem (Proposition 9.2). In Nash equilibria in which the candidates' positions differ, however, these positions are not necessarily Condorcet winners.

Section 9.1.2 considers an analogue of an electoral competition game with a continuum of citizens and two office-motivated candidates in which the candidates are policy-motivated. Proposition 9.3 shows that the action pair in which each candidate's position is the median of the citizens' favorite positions is a Nash equilibrium, and if the candidates are representative, that position is the policy outcome of every Nash equilibrium.

Suppose that we modify this model so that the candidates are uncertain about the median of the citizens' favorite positions. This model, unlike almost all of those I have discussed previously, robustly has equilibria in which the candidates' positions differ. Suppose that the candidates' common position is x, and candidate i's favorite position differs from x. Then if the probability that the median of the citizens' favorite positions lies between x and i's favorite position is positive, i can increase her payoff by deviating from x in the direction of her favorite position. If she does so, then with positive probability she wins, in which case she is better off, and the worst that can happen for her is that the other candidate wins, in which case the outcome remains x. Proposition 9.4 gives conditions under which the candidates' positions differ in every Nash equilibrium, and Proposition 9.5 gives conditions for the existence of a Nash equilibrium.

Section 9.3 explores models in which the candidates are better informed than the citizens about the desirability of the policies. In each case, there are two candidates and, for simplicity, a single citizen. In Section 9.3.1 the candidates receive information about the desirability of the policies after they are elected. The game in this case may have equilibria in which each candidate offers an interval of policies, with the winner choosing a specific policy after the uncertainty is resolved. In Section 9.3.2 the candidates are better informed than the citizen at the time of the election. In this case, a candidate's position is a signal of her information. When she considers changing her position, she needs to consider the impact of her move on the citizen's belief about the state of the world, which determines the appropriateness of her position. The game I present has equilibria in which in each state the candidates' positions differ, with a right-leaning candidate prevailing when the state favors policies on the left and a left-leaning candidate prevailing when it favors policies on the right. If the citizen were informed, in each state each candidate would have an incentive to move her position closer to that of her rival. When the citizen is uninformed, she does not benefit from such a move, because it changes the citizen's beliefs about the state in such a way that the citizen switches her vote to the other candidate, whose policy she now believes is more appropriate.

Section 9.4 studies models of repeated elections. The same pair of policy-motivated candidates compete in a series of elections, in each period observing the outcomes in all previous periods. If in each period each candidate is free to choose any position, the features of the subgame perfect equilibria depend on the nature of the candidates' preferences. If each candidate prefers a constant sequence of positions to any varying sequence with the same average, then only a repetition of the outcome in which both candidates choose the median m of the citizens' favorite positions is possible in a subgame perfect equilibrium. If each candidate prefers varying sequences to constant ones and the discount factor is close enough to 1 then a subgame perfect equilibrium exists in which the outcome in each period differs from m. Now suppose that rather than each candidate being allowed to choose any position in each period, unconstrained by her past positions, an incumbent in any period t is restricted to choose in period $t+1$ the policy she implemented in period t. Then even if the candidates prefer constant sequences to varying ones, the game has subgame perfect equilibria in which the winning policy differs from m in every period, and the candidates alternate as winners.

9.1 Basic model

9.1.1 General set of positions

As in the models of the previous chapter, two candidates select positions, which I refer to also as policies, from a set X, then each member of a set of citizens votes for the candidate whose position she prefers, and the candidate who receives the most votes wins. As before, the election is decided by the citizens who have a strict preference between the candidates; if the number of citizens who prefer the position x_1 of candidate 1 to the position x_2 of candidate 2 is the same as the number who prefer x_2 to x_1, then the outcome of the election is a tie. (A citizen who is indifferent between the candidates' positions does not vote, or splits her vote, casting half a vote for each candidate.)

Each candidate has preferences over electoral outcomes. For any position $x \in X$, denote the outcome in which either a candidate with position x wins outright or x is the position of both candidates by $\{x\}$, and the outcome in which the candidates' positions are x and $y \neq x$ and the candidates tie by $\{x, y\}$. How do the candidates evaluate ties? I assume that (*i*) if a candidate likes the outcome $\{x\}$ at least as much as $\{y\}$ then she likes $\{x\}$ at least as much as $\{x,y\}$ and likes $\{x,y\}$ at least as much as $\{y\}$, and (*ii*) if she prefers $\{x\}$ to $\{z\}$ and likes $\{y\}$ at least as much as $\{z\}$ then she prefers $\{x,y\}$ to $\{z\}$. That is, the preference relation $\succsim_j^*$ over electoral outcomes of each candidate j satisfies

$$\begin{aligned} \{x\} \succsim_j^* \{y\} \quad &\Rightarrow \quad \{x\} \succsim_j^* \{x,y\} \succsim_j^* \{y\} \\ \{x\} \succ_j^* \{z\} \text{ and } \{y\} \succsim_j^* \{z\} \quad &\Rightarrow \quad \{x,y\} \succ_j^* \{z\}. \end{aligned} \tag{9.1}$$

These assumptions are consistent with the outcome of a tie $\{x,y\}$ being the lottery in which x and y each occur with probability $\frac{1}{2}$ and the candidates' preference relations over lotteries are vNM preference relations.

Definition 9.1: Electoral competition game with two policy-motivated candidates

An *electoral competition game with two policy-motivated candidates* $\langle\{1,2\},\langle I,X,\succsim\rangle,(\succsim_1^*,\succsim_2^*)\rangle$, where $\langle I,X,\succsim\rangle$ is a collective choice problem in which the set I (of *citizens*) is finite and $\succsim_j^*$ for $j=1,2$ is a preference relation over subsets of X containing one or two alternatives that satisfies (9.1), is the strategic game with the following components.

Players

$\{1,2\}$ (*candidates*).

Actions

The set of actions of each player is X (the set of possible *positions*).

Preferences

The preference relation $\trianglerighteq_j$ of each player j over action pairs satisfies

$$(x_1,x_2) \trianglerighteq_j (y_1,y_2) \quad \Leftrightarrow \quad W(x_1,x_2) \succsim_j^* W(y_1,y_2),$$

where for each pair of positions $(x_1,x_2) \in X \times X$, $W(x_1,x_2)$ is the set of members of $\{x_1,x_2\}$ preferred by a majority of citizens:

$$W(x_1,x_2) = \begin{cases} \{x_1\} & \text{if } |\{i \in I : x_1 \succ_i x_2\}| > |\{i \in I : x_2 \succ_i x_1\}| \\ \{x_1,x_2\} & \text{if } |\{i \in I : x_1 \succ_i x_2\}| = |\{i \in I : x_2 \succ_i x_1\}| \\ \{x_2\} & \text{if } |\{i \in I : x_1 \succ_i x_2\}| < |\{i \in I : x_2 \succ_i x_1\}|. \end{cases}$$

Suppose that the collective choice problem $\langle I,X,\succsim\rangle$ has a strict Condorcet winner, x^*. If both candidates in the electoral competition game choose this position, it is the policy outcome. If either candidate deviates from x^*, she loses, so that x^* remains the policy outcome. Hence (x^*, x^*) is a Nash equilibrium of the game, as it is when the candidates are office-motivated (Proposition 8.1).

Proposition 9.1: Nash equilibrium of electoral competition game with two policy-motivated candidates

Let $\langle\{1,2\}, \langle I, X, \succsim\rangle, (\succsim_1^*, \succsim_2^*)\rangle$ be an electoral competition game with two policy-motivated candidates. If $\langle I, X, \succsim\rangle$ has a strict Condorcet winner, x^*, then (x^*, x^*) is a Nash equilibrium of the game.

Proof

We have $W(x^*, x^*) = \{x^*\}$ and $W(x, x^*) = \{x^*\}$ for any $x \neq x^*$, because x^* is a strict Condorcet winner of the collective choice problem. Thus neither candidate can profitably deviate from x^*.

If the number of citizens is odd and their preferences are single-peaked or single-crossing then the collective choice problem has a strict Condorcet winner and this position is the median of the citizens' favorite positions if the preferences are single-peaked (Proposition 1.4) and the favorite position of the median citizen if the preferences are single-crossing (Proposition 1.5), so the following result follows from Proposition 9.1. Note that this result is weaker than the corresponding result for the game with office-motivated candidates (Corollary 8.2) : it says only that a certain action pair is a Nash equilibrium, not that it is the only Nash equilibrium.

Corollary 9.1: Median voter theorem for electoral competition game with two policy-motivated candidates

Let $\langle\{1,2\}, \langle I, X, \succsim\rangle, (\succsim_1^*, \succsim_2^*)\rangle$ be an electoral competition game with two policy-motivated candidates for which the number of citizens (members of I) is odd.

- If $\langle I, X, \succsim\rangle$ has single-peaked preferences with respect to a linear order $\trianglerighteq$ on X, then the action pair (m, m) in which m is the median with respect to $\trianglerighteq$ of the citizens' favorite positions is a Nash equilibrium of the game.
- If $\langle I, X, \succsim\rangle$ has single-crossing preferences with respect to a linear order $\geq$ on I and the median individual with respect to $\geq$ has a unique favorite position, m, then the action pair (m, m) is a Nash equilibrium of the game.

When the candidates are office-motivated, a pair (x_1, x_2) is a Nash equilibrium if and only if both x_1 and x_2 are Condorcet winners (Proposition 8.1). The same is not true when the candidates are policy-motivated. If x^* is a Condorcet winner that is not strict then another alternative, say x, ties with x^*, so that a candidate who deviates from x^* to x induces a tie between x^* and x. If she prefers x to x^*, then by (9.1) she prefers a tie to x^* . So if at least one of the candidates prefers x to x^*, the action pair (x^*, x^*) is not a Nash equilibrium.

Further, if (x_1, x_2) is a Nash equilibrium, x_1 and x_2 are not necessarily Condorcet winners, even if $x_1 = x_2$. Consider a collective choice problem in which there are two alternatives, a and b, and all individuals prefer a to b, so that a is the strict Condorcet winner. If each candidate prefers b to a, then (b, b) is a Nash equilibrium of the associated electoral competition game with two policy-motivated candidates. Another example is based on the Condorcet cycle in Example 1.5, which has no Condorcet winner. Suppose that the preference relation $\succsim_j^*$ of each candidate j in the electoral competition game satisfies $\{a\} \succ_j^* \{b\} \succ_j^* \{c\}$. Then (a, a) is a Nash equilibrium: if either candidate deviates to b, she loses, so that the outcome remains $\{a\}$, and if either candidate deviates to c, she wins, so that the outcome changes to $\{c\}$, which is worse for her than $\{a\}$.

In these examples, the preferences of some sets of citizens containing a majority of individuals are not shared by any candidate: in the first case, all citizens prefer a to b but both candidates prefer b to a, and in the second example a majority of the citizens prefer c to a but both candidates prefer a to c. If, instead, for every majority of citizens who prefer some alternative x to some other alternative y there is a candidate with the same preference between x and y, I say that the candidates are representative.

Definition 9.2: Representative candidates in electoral competition game with two policy-motivated candidates

The candidates in an electoral competition game with two policy-motivated candidates are *representative* if, for all alternatives x and y, whenever every citizen in some set containing a majority of citizens prefers x to y, at least one candidate prefers $\{x\}$ to $\{y\}$.

For games in which the candidates are representative, their common position in any Nash equilibrium in which their positions are the same is a Condorcet winner of the collective choice problem.

Proposition 9.2: Nash equilibrium of electoral competition game with two representative policy-motivated candidates

Let $\langle \{1,2\}, \langle I, X, \succsim \rangle, (\succsim_1^*, \succsim_2^*) \rangle$ be an electoral competition game with two policy-motivated candidates in which the candidates are representative. In any Nash equilibrium in which the candidates' positions are the same, the common position is a Condorcet winner of $\langle I, X, \succsim \rangle$.

Proof

Let $x \in X$ and consider the action pair (x,x). If x is not a Condorcet winner of $\langle I, X, \succsim \rangle$ then for some position, say x', a majority of citizens prefer x' to x. Given that the candidates are representative, at least one candidate thus prefers x' to x. If that candidate deviates to x', she wins, so that the outcome is $\{x'\}$, which she prefers to $\{x\}$. Hence (x,x) is not a Nash equilibrium.

What about Nash equilibria in which the candidates' positions differ? The game can have such equilibria in which neither candidate's positions is a Condorcet winner. Consider again the Condorcet cycle in Example 1.5. Suppose that candidate 1's preferences satisfy $\{b\} \succ_1^* \{a\} \succ_1^* \{c\}$ and candidate 2's satisfy $\{c\} \succ_2^* \{a\} \succ_2^* \{b\}$. The action pair (b,a), for which neither action is a Condorcet winner, is a Nash equilibrium by the following argument. Candidate 2 wins (a beats b), so that the outcome is $\{a\}$. If candidate 1 deviates to a, the outcome remains $\{a\}$; if she deviates to c, she wins and the outcome changes to $\{c\}$, which is worse for her than $\{a\}$. If candidate 2 deviates to b, the outcome changes to $\{b\}$, which is worse for her than $\{a\}$; if she deviates to c, candidate 1 wins and the outcome again changes to $\{b\}$.

A small change in the candidates' preferences eliminates this equilibrium. Assume that each candidate cares mainly about the policy outcome, but slightly about winning. Precisely, for any position x, among action pairs that generate the outcome $\{x\}$, each candidate prefers those in which she wins to those in which she ties to those in which she loses. That is, her preferences are lexicographic: if two action pairs have different policy outcomes, candidate i (= 1, 2) prefers the one that is better according to $\succsim_i^*$, while if they have the same policy outcome, she prefers winning to tieing to losing. Under this assumption, the game has no Nash equilibrium in which one candidate loses, because if the losing candidate deviates to the position of the winning candidate then the policy outcome remains the same and the deviating candidate ties rather than loses.

However, even if the candidates are representative and their preferences put some weight on winning, an electoral competition game may have equilibria in which the outcome is a tie and one of the possible outcomes is not a Condorcet winner, as you are asked to show in the following exercise.

Exercise 9.1: Nash equilibrium with policy-motivated candidates

Consider the collective choice problem with four individuals in which three have the preferences of the individuals in the Condorcet cycle in Example 1.5 and the fourth prefers b to c to a. Show that a is not a Condorcet winner of the collective choice problem but for some preferences of the candidates the action pair (a,b) is a Nash equilibrium of the associated electoral competition game with two policy-motivated candidates, even if the candidates are representative and their preferences lexicographically value winning.

If the number of citizens is odd and their preferences are strict, a tie when the candidates choose different positions is not possible. So if in this case the candidates are representative and their preferences lexicographically value winning, then in any Nash equilibrium the candidates choose the same position, and hence, by Proposition 9.2, this position is a Condorcet winner.

9.1.2 One-dimensional positions

Consider a variant of an electoral competition game with two policy-motivated candidates in which, as in an electoral competition game with a continuum of citizens and two office-motivated candidates, the set of positions is an interval of numbers and the set of citizens is a continuum. This variant does not include an explicit specification of the set of citizens, but like its cousin includes an outcome function that may be rationalized by an assumption about the citizens' preferences. The outcome relevant to policy-motivated candidates is the position of the winner, so the outcome function in this case specifies that position (or those positions, in the case of a tie), rather than the identity of the winner.

Assume specifically that the preference relation $\succsim_i$ of each citizen i is single-peaked with respect to the linear order $\geq$ and symmetric about i's favorite position x_i^* ($x_i^* - \delta \sim_i x_i^* + \delta$ for every $\delta > 0$), and the distribution F of the citizens' favorite positions is nonatomic, with support an interval. Under these assumptions, F has a unique median, say m, and the policy outcome of the pair (x_1, x_2)

of the candidates' positions is

$$Y_F(x_1,x_2) = \begin{cases} \{x\} & \text{if } x_1 = x_2 = x \\ \{x_1,x_2\} & \text{if } x_1 \neq x_2 \text{ and } \frac{1}{2}(x_1+x_2) = m \\ \{x_j\} & \text{if } \begin{cases} \text{either } x_k < x_j \text{ and } \frac{1}{2}(x_1+x_2) < m \\ \text{or } x_k > x_j \text{ and } \frac{1}{2}(x_1+x_2) > m, \end{cases} \end{cases} \quad (9.2)$$

where $j \in \{1,2\}$ and k is the other candidate. (The function Y_F is the analogue of the function O_F defined in (8.1) for the model with office-motivated candidates.)

We assume that each candidate j has a preference relation $\succsim_j^*$ over policy outcomes (one- or two-member subsets of the set X of positions) that is single-peaked in the sense that

$$\text{for some } \hat{x}_j \in X: \quad x < y < \hat{x}_j \text{ or } \hat{x}_j < y < x \quad \Rightarrow \quad \{\hat{x}_j\} \succ_j^* \{y\} \succsim_j^* \{x\}. \quad (9.3)$$

(The position $\hat{x}_j$ is the favorite position of candidate j.)

Definition 9.3: Electoral competition game with continuum of citizens and two policy-motivated candidates

An *electoral competition game with a continuum of citizens and two policy-motivated candidates* $\langle \{1,2\}, X, F, (\succsim_1^*, \succsim_2^*) \rangle$, where X is a closed interval of real numbers, F is a nonatomic distribution with support X, and $\succsim_1^*$ and $\succsim_2^*$ are preference relations over subsets of X containing one or two positions that satisfy (9.1) and (9.3), is the strategic game with the following components.

Players
$\{1,2\}$ (*candidates*).

Actions
The set of actions of each player is X (the set of possible *positions*).

Preferences
The preference relation $\trianglerighteq_j$ of each player j over $X \times X$ satisfies

$$(x_1,x_2) \trianglerighteq_j (y_1,y_2) \quad \Leftrightarrow \quad Y_F(x_1,x_2) \succsim_j^* Y_F(y_1,y_2),$$

where Y_F is given by (9.2).

For the model with a finite number of citizens, the candidates are representative if whenever a majority of citizens prefer one position to another, so does at least one candidate (Definition 9.2). The analogue of this definition for the model with a continuum of citizens is that the candidates' preferences are symmetric about their favorite positions (like the citizens' preferences), with one candidate's favorite position on each side of the median of the citizens' favorite positions.

Definition 9.4: Representative candidates in electoral competition game with continuum of citizens and two policy-motivated candidates

Let $\langle\{1,2\}, X, F, (\succsim_1^*, \succsim_2^*)\rangle$ be an electoral competition game with a continuum of citizens and two policy-motivated candidates and denote by m the median of F, the distribution of the citizens' favorite positions. The candidates in $\langle\{1,2\}, X, F, (\succsim_1^*, \succsim_2^*)\rangle$ are *representative* if the preference relation $\succsim_i^*$ of each candidate i is symmetric about her favorite position (she is indifferent between positions equidistant from her favorite position), the favorite position of one candidate is at most m, and the favorite position of the other candidate is at least m.

The action pair in which each candidate's position is the median of the citizens' favorite positions is a Nash equilibrium, an analogue of Corollary 9.1. In addition, if the candidates are representative then the outcome of every Nash equilibrium is this position.

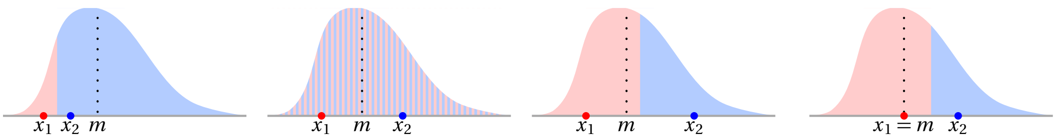
Proposition 9.3: Nash equilibrium of electoral competition game with continuum of citizens and two policy-motivated candidates

Let $\langle\{1,2\}, X, F, (\succsim_1^*, \succsim_2^*)\rangle$ be an electoral competition game with a continuum of citizens and two policy-motivated candidates. Denote the candidates' favorite positions, defined by (9.3), by $\hat{x}_1$ and $\hat{x}_2$. The action pair in which each candidate's position is the median m of the distribution F of the citizens' favorite positions is a Nash equilibrium of the game. If the candidates are representative then the outcome of every Nash equilibrium is $\{m\}$, and if in addition neither $\hat{x}_1$ nor $\hat{x}_2$ is equal to m, then (m,m) is the only Nash equilibrium.

Proof

The fact that the support of F is an interval means that it has a unique median m. The outcome of the action pair (m,m) is $Y_F(m,m) = \{m\}$. If either candidate deviates from m, she loses, and the outcome remains $\{m\}$. Thus (m,m) is a Nash equilibrium.

Now assume that the candidates are representative, and let (x_1, x_2) be



(a) $x_1 \le x_2 < m$ (b) $x_1 < m < x_2$ and $\frac{1}{2}(x_1 + x_2) = m$ (c) $x_1 < m < x_2$ and $\frac{1}{2}(x_1 + x_2) > m$ (d) $x_1 = m < x_2$

Figure 9.1 The four cases in the proof of Proposition 9.3.

a Nash equilibrium. Assume without loss of generality that $x_1 \le x_2$ and $\hat{x}_1 \le m \le \hat{x}_2$.

First suppose that $\hat{x}_1 < m < \hat{x}_2$. Refer to Figure 9.1 for illustrations of the following four cases.

- If $x_1 \le x_2 < m$ then $Y_F(x_1, x_2) = \{x_2\}$. If candidate 2 deviates to m, the outcome changes to $\{m\}$, which she prefers to $\{x_2\}$. A symmetric argument applies if $m < x_1 \le x_2$.
- If $x_1 < m < x_2$ and $\frac{1}{2}(x_1 + x_2) = m$ then $Y_F(x_1, x_2) = \{x_1, x_2\}$. If candidate 1 deviates to $x_1 + \varepsilon$ with $0 < \varepsilon < x_2 - x_1$ then she wins and the outcome changes to $\{x_1 + \varepsilon\}$, which she prefers if ε is sufficiently small. (Candidate 2 has an analogous profitable deviation.)
- If $x_1 < m < x_2$ and $\frac{1}{2}(x_1 + x_2) > m$ then $Y_F(x_1, x_2) = \{x_1\}$. If candidate 2 deviates to m, the outcome changes to $\{m\}$, which she prefers to $\{x_1\}$. A symmetric argument applies if $\frac{1}{2}(x_1 + x_2) < m$.
- If $x_1 = m < x_2$ then $Y_F(x_1, x_2) = \{x_1\}$. If candidate 1 deviates to $m - \varepsilon$ with $0 < \varepsilon < x_2 - m$ then she wins and the outcome changes to $\{m - \varepsilon\}$, which she prefers if ε is sufficiently small. A symmetric argument applies if $x_1 < m = x_2$.

Thus no pair (x_1, x_2) other than (m, m) is a Nash equilibrium.

Now suppose that $\hat{x}_1 = m \le \hat{x}_2$. If the outcome of (x_1, x_2) is not $\{m\}$, then $x_1 \ne m$ and candidate 1 can induce the outcome $\{m\}$, her favorite outcome, by deviating to m. Thus in any Nash equilibrium the outcome is $\{m\}$. A symmetric argument applies if $\hat{x}_1 \le m = \hat{x}_2$.

An example in which one of the candidates' favorite positions is m and the game has a Nash equilibrium different from (m, m) is easy to find: if candidate 1's favorite position is m and candidate 2's is greater than m, then the action pair $(m, \hat{x}_2)$ is a Nash equilibrium (for all positions of candidate 2, the outcome is

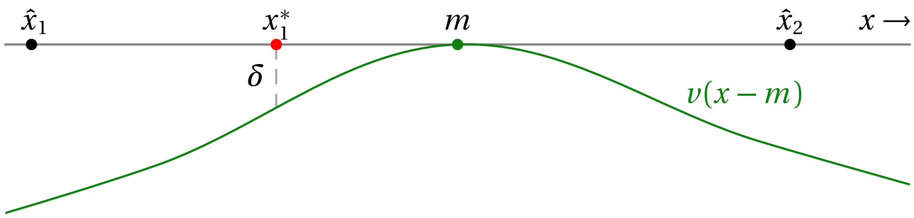


Figure 9.2 The policy x_1^* in the model in Exercise 9.3.

$\{m\}$). If we modify each candidate's preferences so that they lexicographically value winning in the way discussed at the end of Section 9.1.1, however, then (m, m) is the only Nash equilibrium.

Exercise 9.2: Candidates' favorite positions both less than the median

Find the Nash equilibria of an electoral competition game with a continuum of citizens and two policy-motivated candidates in which both candidates' favorite positions are less than the median of the citizens' favorite positions (so that in particular the candidates are not representative).

For the case in which the candidates are office-motivated, Section 8.5.1 presents a model in which the citizens have preferences over both candidates and policies. The next exercise asks you to study an analogue of this model for the case in which the candidates are policy-motivated.

Exercise 9.3: Electoral competition with an advantaged candidate

Suppose that the citizens have preferences over the candidates independently of the candidates' positions, as in Section 8.5.1. Specifically, consider the model that differs from the one in Exercise 8.5 only in that each candidate is policy-motivated rather than office-motivated, with preferences that satisfy (9.1) and (9.3). Denote the favorite position of each citizen i by $\hat{z}_i$, the median of these favorite positions by m, and the candidates' favorite positions by $\hat{x}_1$ and $\hat{x}_2$. Assume that $\hat{x}_1 < m < \hat{x}_2$ and $v(\hat{x}_1 - m) < -\delta$. Let x_1^* be the position for which $x_1^* < m$ and $v(x_1^* - m) = -\delta$, so that a citizen whose favorite position is m is indifferent between the candidates when candidate 1's position is x_1^* and candidate 2's is m, and hence votes for candidate 1 in this case, given the tie-breaking assumption. (Refer to Figure 9.2.) Show that (x_1^*, m) is a Nash equilibrium of the game.

The models in this section, like those in the previous chapter, assume that after a candidate is elected, she implements the policy she chose as her platform in

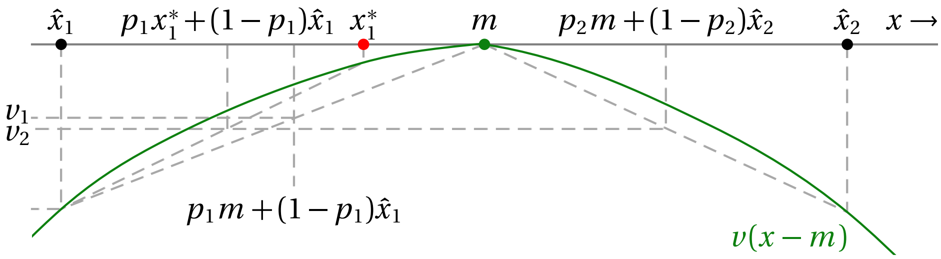


Figure 9.3 An example of the payoffs in the model at the end of Section 9.1 in which each candidate i is committed to her position with probability p_i. The pair (x_1^*, m) of positions is a Nash equilibrium.

the election. The need to periodically face re-election may provide office-holders with an incentive not to deviate from their stated policies, and the existence of parties may reinforce this incentive. But a candidate may be unable to fully commit that, if she wins, she will implement the policy she chose in the election. One way to model the inability to commit is to assume that if a candidate j who chooses a position x_j wins, then the outcome is x_j with some probability $p_j < 1$ and j's favorite policy with probability $1 - p_j$.

Suppose specifically that the number of citizens is odd, the citizens' favorite positions are distinct, and each citizen i has preferences on the set of lotteries over positions represented by the expected value of a function $u_i : X \to \mathbb{R}$ defined by $u_i(x) = v(x - \hat{z}_i)$, where v is a single-peaked function with maximizer 0, and $\hat{z}_i$ is i's favorite position. Suppose also that there are two candidates, with favorite positions $\hat{x}_1$ and $\hat{x}_2$ that satisfy $\hat{x}_1 < m < \hat{x}_2$, where m is the median of the citizens' favorite positions. Finally, suppose that given p_1 and p_2, the citizen with favorite position m prefers candidate 1 when both candidates choose m ($v_1 > v_2$ in Figure 9.3) and candidate 2 when the pair of positions is $(\hat{x}_1, m)$ ($v_2 > v(\hat{x}_1 - m)$), and votes for candidate 1 when indifferent between the candidates.

This model is closely related to the one in Exercise 9.3, and similar arguments lead to the conclusion that (x_1^*, m) is a Nash equilibrium, where x_1^* is the position in $(\hat{x}_1, m)$ for which the citizen with favorite position m is indifferent between the candidates:

$$p_1 v(x_1^* - m) + (1-p_1)v(\hat{x}_1 - m) = p_2 v(0) + (1-p_2)v(\hat{x}_2 - m).$$

(In Figure 9.3, v_2 is the common value of these payoffs.) In this equilibrium, candidate 1 wins (by the tie-breaking rule) and the outcome is thus x_1^* with probability p_1 and $\hat{x}_1$ with probability $1 - p_1$. Because candidate 1 wins when both candidates' positions are m, she can move her position closer to her favorite position and still win. In an equilibrium, she moves it to the point at which a citizen with favorite position m is indifferent between her and candidate 2.

9.2 Uncertain median

The conclusion of Proposition 9.3 that the outcome of any equilibrium is the citizens' median favorite position may seem surprising. The incentive for office-motivated candidates to cater to the median voter is clear, but you might think that a policy-motivated candidate faces a tradeoff: moving from her favorite position towards her rival's position increases her probability of winning, but results in a less desirable position if she wins. The models I have defined do not capture this tradeoff because they are deterministic: a candidate's probability of winning is either 0 or 1, or the outcome is a tie. I now specify and analyze a model that does capture the tradeoff.

Suppose that the candidates are uncertain about the citizens' preferences. Specifically, consider a variant of the game in Section 8.3.1 in which the candidates are policy-motivated rather than office-motivated. Each candidate believes that the median of the citizens' favorite positions has the distribution function G, which is nonatomic, with support an interval. Suppose that the candidates' positions are x_1 and x_2, with $x_1 < x_2$. Then if the median of the citizens' favorite positions is less than $\frac{1}{2}(x_1+x_2)$, an event with probability $G(\frac{1}{2}(x_1+x_2))$, candidate 1 wins and the policy outcome is x_1; if the median is greater than $\frac{1}{2}(x_1+x_2)$, an event with probability $1-G(\frac{1}{2}(x_1+x_2))$, candidate 2 wins and the policy outcome is x_2. Thus each candidate faces a lottery in which the outcome is x_1 with probability $G(\frac{1}{2}(x_1+x_2))$ and x_2 with probability $1-G(\frac{1}{2}(x_1+x_2))$.

Suppose that the preferences of candidate j $(= 1, 2)$ regarding probability distributions over positions are represented by the expected value of a single-peaked function u_j with favorite position $\hat{x}_j$. (Refer to Figure 9.4.) If $x_1 < x_2$ then candidate j's expected payoff is

$$G(\tfrac{1}{2}(x_1+x_2))u_j(x_1) + \big(1 - G(\tfrac{1}{2}(x_1+x_2))\big)u_j(x_2).$$

If $x_1 > x_2$ then $u_j(x_1)$ and $u_j(x_2)$ are interchanged in this expression, and if $x_1 = x_2 = x$ then j's payoff is $u_j(x)$.

Definition 9.5: Electoral competition game with two policy-motivated candidates and uncertain median

An *electoral competition game with two policy-motivated candidates and uncertain median* $\langle\{1,2\}, X, G, (u_1, u_2)\rangle$, where X is a closed interval of real numbers, G is a nonatomic distribution with a density and support X (so that it has a unique median), and $u_j : X \to \mathbb{R}$ for $j = 1, 2$ is a single-peaked function, is the strategic game with the following components.

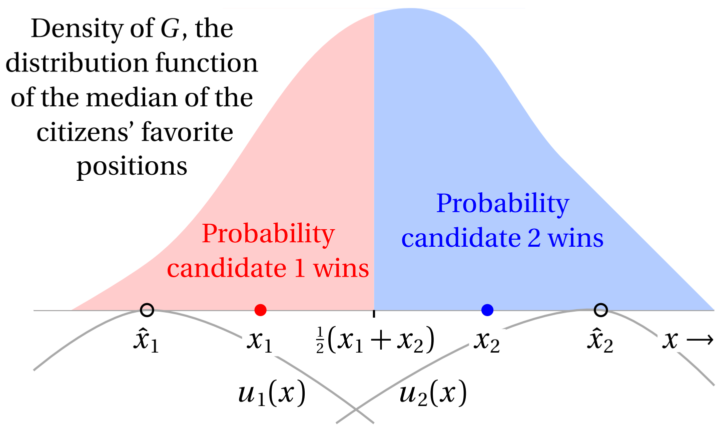


Figure 9.4 An illustration of the components of an electoral competition game with two policy-motivated candidates and uncertain median.

Players

$\{1, 2\}$ (*candidates*).

Actions

The set of actions of each player is X (the set of possible *positions*).

Preferences

The preferences of each player j are represented by the function v_j : $X \times X \to \mathbb{R}$ defined by

$$v_j(x_1, x_2) = G(\tfrac{1}{2}(x_1 + x_2))u_j(\min\{x_1, x_2\}) \\ + \big(1 - G(\tfrac{1}{2}(x_1 + x_2))\big)\, u_j(\max\{x_1, x_2\}).$$

If the candidates' positions in such a game are the same, equal to x, then the outcome is x. Suppose that the candidates' favorite positions, $\hat{x}_1$ and $\hat{x}_2$, differ. Then x is not the favorite position of at least one candidate, and a deviation by that candidate from x to her favorite position causes the outcome to change from x to one in which her favorite position occurs with positive probability and x occurs with the complementary probability. She prefers this outcome to x, so in any Nash equilibrium the candidates' positions differ. The next result shows also that these positions lie between the candidates' favorite positions, and each candidate's equilibrium position is closer to her favorite position than is the other candidate's equilibrium position.

Proposition 9.4: Nash equilibrium of electoral competition game with two policy-motivated candidates and uncertain median

Consider an electoral competition game with two policy-motivated candidates and uncertain median $\langle\{1,2\}, X, G, (u_1, u_2)\rangle$. Denote the candidates' favorite positions, the maximizers of u_1 and u_2, by $\hat{x}_1$ and $\hat{x}_2$, and suppose that $\hat{x}_1 < \hat{x}_2$. Then in every Nash equilibrium (x_1^*, x_2^*) we have $\hat{x}_1 \leq x_1^* < x_2^* \leq \hat{x}_2$. If G is differentiable, the density of G is positive on the interior of X, u_1 and u_2 are differentiable, and $\hat{x}_1$ and $\hat{x}_2$ are in the interior of X, then $\hat{x}_1 < x_1^* < x_2^* < \hat{x}_2$.

To prove this result, I first establish the following lemma, which is used also in the proof of a later result.

Lemma 9.1: Best responses in electoral competition game with two policy-motivated candidates and uncertain median

Consider an electoral competition game with two policy-motivated candidates and uncertain median $\langle\{1,2\}, X, G, (u_1, u_2)\rangle$. For any $x_2 \in X$, every best response of candidate 1 to x_2 is in $(x_2, \hat{x}_1]$ if $x_2 < \hat{x}_1$ and in $[\hat{x}_1, x_2)$ if $x_2 > \hat{x}_1$, where $\hat{x}_1$ is candidate 1's favorite position.

Proof

First suppose that $x_2 < \hat{x}_1$.

a. If $x_1 \leq x_2$ then candidate 1's payoff is at most $u_1(x_2)$. If she deviates to $\hat{x}_1$ then the outcome is x_2 with positive probability less than 1 and $\hat{x}_1$ with the complementary probability. Given that $u_1(\hat{x}_1) > u_1(x_2)$, the position x_1 is thus not a best response to x_2.

b. If $x_1 > \hat{x}_1$ then candidate 1's payoff is

$$G(\tfrac{1}{2}(x_1 + x_2))u_1(x_2) + \big(1 - G(\tfrac{1}{2}(x_1 + x_2))\big)\,u_1(x_1). \tag{9.4}$$

A deviation by candidate 1 to $\hat{x}_1$ changes her payoff to

$$G(\tfrac{1}{2}(\hat{x}_1 + x_2))u_1(x_2) + \big(1 - G(\tfrac{1}{2}(\hat{x}_1 + x_2))\big)\,u_1(\hat{x}_1).$$

The difference between this payoff and (9.4) is

$$\big(G(\tfrac{1}{2}(x_1+x_2))-G(\tfrac{1}{2}(\hat{x}_1+x_2))\big)(u_1(\hat{x}_1)-u_1(x_2)) \\ +\big(1-G(\tfrac{1}{2}(x_1+x_2))\big)(u_1(\hat{x}_1)-u_1(x_1)),$$

which is positive because $G(\frac{1}{2}(x_1+x_2)) > G(\frac{1}{2}(\hat{x}_1+x_2))$, $u_1(\hat{x}_1) > u_1(x_1)$, and $u_1(\hat{x}_1) > u_1(x_2)$. Thus x_1 is not a best response to x_2.

We conclude that if $x_2 < \hat{x}_1$ then every best response of candidate 1 to x_2 is in $(x_2, \hat{x}_1]$.

Now suppose that $x_2 > \hat{x}_1$. If $x_1 < \hat{x}_1$ or $x_1 \geq x_2$ then a deviation by candidate 1 to $\hat{x}_1$ increases her payoff; the argument for $x_1 < \hat{x}_1$ is symmetric with that for case *b* for $x_2 < \hat{x}_1$, and the argument for $x_1 \geq x_2$ is symmetric with that for case *a*.

Proof of Proposition 9.4

Lemma 9.1 implies that in every Nash equilibrium (x_1^*, x_2^*) we have $\hat{x}_1 \leq x_1^* < x_2^* \leq \hat{x}_2$. Now assume that G is differentiable, the density of G is positive on the interior of X, u_1 and u_2 are differentiable, and $\hat{x}_1$ and $\hat{x}_2$ are in the interior of X. We need to prove that $x_1^* \neq \hat{x}_1$ and $x_2^* \neq \hat{x}_2$. Given that u_1 is differentiable and $\hat{x}_1$ is in the interior of X, $u_1'(\hat{x}_1) = 0$, so the derivative of candidate 1's payoff with respect to x_1 at $(\hat{x}_1, x_2)$ for $x_2 > \hat{x}_1$ is

$$\tfrac{1}{2}G'(\tfrac{1}{2}(\hat{x}_1+x_2))(u_1(\hat{x}_1)-u_1(x_2)).$$

This expression is positive given that the density G' of G is positive on the interior of X. Thus candidate 1's best response to any position $x_2 > \hat{x}_1$ is greater than $\hat{x}_1$ and hence $x_1^* > \hat{x}_1$ in any Nash equilibrium. A symmetric argument applies to x_2^*.

If the candidates' payoff functions are not differentiable at their favorite positions then the game may have an equilibrium in which the candidates choose these positions.

Exercise 9.4: Nash equilibrium with policy-motivated candidates and uncertain median

Consider an electoral competition game with two policy-motivated candidates and uncertain median $\langle\{1,2\}, X, G, (u_1, u_2)\rangle$ in which $X = [-k, k]$ for some $k > 0$, G is uniform on X, and for some positions $\hat{x}_1 \in X$ and $\hat{x}_2 \in X$ with $\hat{x}_1 < \hat{x}_2$ we have $u_j(x) = -|x - \hat{x}_j|$ for $j = 1, 2$ for all $x \in X$. Show that this game has a Nash equilibrium in which the position of each candidate j is $\hat{x}_j$, her favorite position.

Proposition 9.4 does not assert that the game necessarily has a Nash equilibrium. The next result gives sufficient conditions for the existence of a Nash equilibrium when G and the payoff functions u_1 and u_2 are differentiable.

Proposition 9.5: Existence of Nash equilibrium for electoral competition game with two policy-motivated candidates and uncertain median

Consider an electoral competition game with two policy-motivated candidates and uncertain median $\langle\{1,2\}, X, G, (u_1, u_2)\rangle$ for which G is differentiable and the candidates' favorite positions, the maximizers of u_1 and u_2, differ and are in the interior of X. If $\ln G$ is concave and u_1 and u_2 are concave and twice-differentiable then the game has a Nash equilibrium.

Proof

Denote the favorite position of each candidate i by $\hat{x}_i$ and her payoff function by v_i, as in Definition 9.5. Assume without loss of generality that $\hat{x}_1 < \hat{x}_2$.

Step 1 *For any $x_2 \in [\hat{x}_1, \hat{x}_2]$, candidate 1 has a unique best response to x_2, which is in $[\hat{x}_1, \hat{x}_2]$.*

Proof. First suppose that $x_2 = \hat{x}_1$. If candidate 1 chooses the same position, then the outcome is $\hat{x}_1$ with certainty. If she chooses any other position x_1, then given that the support of G is X, the outcome is x_1 with positive probability and $\hat{x}_1$ with the complementary probability. Thus her unique best response to x_2 is $\hat{x}_1$.

Now suppose that $x_2 \in (\hat{x}_1, \hat{x}_2]$. By Lemma 9.1, every best response of candidate 1 to x_2 is in $[\hat{x}_1, x_2)$. Her payoff to a pair (x_1, x_2) with $x_1 < x_2$ is

$$G(\tfrac{1}{2}(x_1 + x_2))u_1(x_1) + \big(1 - G(\tfrac{1}{2}(x_1 + x_2))\big)\, u_1(x_2).$$

This function is differentiable in x_1, so a best response of candidate 1 to x_2, which is in the interior of X by Lemma 9.1 and the assumption that $\hat{x}_1$ is in the interior of X, satisfies

$$\tfrac{1}{2}G'(\tfrac{1}{2}(x_1 + x_2))u_1(x_1) + G(\tfrac{1}{2}(x_1 + x_2))u_1'(x_1) - \tfrac{1}{2}G'(\tfrac{1}{2}(x_1 + x_2))u_1(x_2) = 0$$

or

$$\tfrac{1}{2}G'(\tfrac{1}{2}(x_1+x_2))(u_1(x_1)-u_1(x_2))+G(\tfrac{1}{2}(x_1+x_2))u_1'(x_1)=0,$$

which implies

$$\frac{G'(\frac{1}{2}(x_1+x_2))}{2G(\frac{1}{2}(x_1+x_2))}=\frac{-u_1'(x_1)}{u_1(x_1)-u_1(x_2)}.$$

The left-hand side of this equation is the derivative of $\ln G(\frac{1}{2}(x_1+x_2))$, which is positive and nonincreasing in x_1 by the assumption that $\ln G$ is concave. The right-hand side of the equation is 0 for $x_1=\hat{x}_1$ and the sign of its derivative with respect to x_1 is the sign of

$$(u_1(x_1)-u_1(x_2))(-u_1''(x_1))+(u_1'(x_1))^2,$$

which is positive for $\hat{x}_1 < x_1 < x_2$ because $u_1(x_1) > u_1(x_2)$, $u_1''(x_1) \le 0$ by the concavity of u_1, and $u_1'(x_1) < 0$. Thus candidate 1's best response to x_2, which is less than x_2 by Lemma 9.1, is unique and greater than $\hat{x}_1$. ◁

Step 2 *The game has a Nash equilibrium.*

Proof. By Step 1 and the analogous result for candidate 2, each candidate has a unique best response to each position of the other candidate in $[\hat{x}_1,\hat{x}_2]$. So the continuity of the payoffs implies that the best response of each candidate is continuous in the other candidate's action. By Proposition 9.4, any Nash equilibrium of the game is a Nash equilibrium of the game that differs only in that each candidate's set of actions is $[\hat{x}_1,\hat{x}_2]$. Denote candidate i's best response function in this game by $B_i : [\hat{x}_1,\hat{x}_2] \to [\hat{x}_1,\hat{x}_2]$ for $i = 1, 2$. The action pair (x_1,x_2) is a Nash equilibrium of the game if and only if $x_1 = B_1(x_1,x_2)$ and $x_2 = B_2(x_1,x_2)$. Given the convexity and compactness of $[\hat{x}_1,\hat{x}_2]$ and the continuity of B_1 and B_2, Brouwer's fixed point theorem implies that these equations have a solution, so that the game has a Nash equilibrium. ◁

The model nicely captures the tradeoff a candidate faces when choosing a position: moving her position away from her favorite position, towards that of her rival, makes her worse off if she wins, but increases her probability of winning. If the model has an equilibrium, then the candidates' positions differ. A limitation of the model is that the conditions under which an equilibrium is known to exist are relatively restrictive.

9.3 Candidates privately informed about policies

Suppose that the outcome of each policy depends on a state of the world that the citizens do not know. In this section I discuss two models that differ in the time at which the candidates are informed of the state: in one case, the winning candidate is informed after the election, and in the other case both candidates are informed before they choose positions.

In each model, two candidates, 1 and 2, simultaneously choose positions and then a single citizen votes for one of them. The set of possible positions is denoted X and the set of states Θ. The preferences of each candidate j regarding probability distributions over X are represented by the expected value of a function $u_j : X \times \Theta \to \mathbb{R}$ and the preferences of the citizen are represented by the expected value of a function $v : X \times \Theta \to \mathbb{R}$.

9.3.1 Candidates privately informed after choosing positions

Suppose that the winning candidate is informed of the state after she is elected. Suppose also that it is not possible for a candidate to commit to a function that specifies the policy she will implement in each state. Instead, a candidate can commit only to a fixed set of policies; if she wins, she chooses the policy in the set that she likes best, given the state that occurs. That is, the model I analyze is the extensive game in which the candidates simultaneously choose subsets of X (*platforms*), the citizen votes for a candidate, and then the winning candidate observes the state and chooses a policy in her platform.

To start with an extreme example, suppose that the preferences of the candidates and the citizens are identical. Then the game has a subgame perfect equilibrium in which each candidate's platform is the set X of all positions. The citizen votes for either candidate and the winning candidate chooses the policy best for her given the state. Given that the candidates' and citizen's preferences are the same, neither candidate can do better by deviating to a subset of X. Following such a deviation, voting for the other (non-deviating) candidate remains optimal for the citizen. Further, in every equilibrium the winning candidate's platform contains every policy that is optimal in some state.

If the candidates' and citizen's preferences differ, then whether the candidates' equilibrium platforms contain one policy or many depends on the nature of the preferences and the distribution of the state. To get an idea of the factors involved, suppose that the set X of available policies is a compact interval $[\underline{x}, \overline{x}]$ that contains 0, the set Θ of states coincides with X, and there is a single-peaked function $u : \mathbb{R} \to \mathbb{R}$ with maximizer 0 and numbers b_1 and b_2 such that for each candidate j we have $u_j(x, \theta) = u(x - b_j - \theta)$ for all $(x, \theta) \in X \times \Theta$ and for the citi-

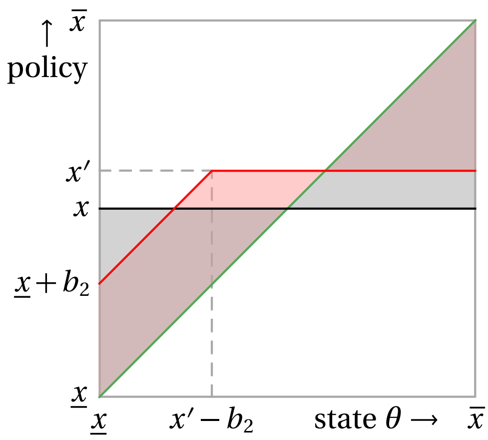


Figure 9.5 Policies as a function of the state in the model in Section 9.3.1. The policy optimal for the citizen in each state is given by the green line, and the policy induced when candidate 2 wins with the platform $[\underline{x}, x']$ is given by the red line.

zen we have $v(x,\theta) = u(x-\theta)$ for all $(x,\theta) \in X \times \Theta$. Under these assumptions, in state θ the policy optimal for the citizen is θ whereas the policies optimal for the candidates are $\theta + b_1$ and $\theta + b_2$.

Suppose that both candidates choose the platform $\{x\}$ consisting of the single policy x. Then the outcome is x regardless of the state. The difference between this outcome and the one best for the citizen, namely θ in each state θ, is indicated by the area shaded gray in Figure 9.5. Suppose that $b_2 > 0$ and that for some $x' \in X$ with $x' > x$ candidate 2 deviates to the platform $[\underline{x}, x']$. Then if candidate 2 wins, she chooses the policy $\theta + b_2$ if $\theta \in [\underline{x}, x' - b_2]$ and x' if $\theta \in [x' - b_2, \overline{x}]$, indicated by the solid red line in the figure. The difference between this outcome and the one best for the citizen is indicated by the area shaded pink. We see that the outcome when the winner is candidate 2 with the platform $[\underline{x}, x']$ is better for the citizen than the constant outcome x when the state is small or large, and is worse when the state takes intermediate values. Thus depending on the distribution of the state and the citizen's payoff function v, the citizen may prefer the outcome induced by candidate 2's deviation to the constant policy x. If she does, then she optimally votes for candidate 2, who prefers the resulting outcome in every state.

This argument suggests that for some specifications of the distribution of the state and the candidates' and citizen's preferences in which these preferences differ from each other, the game has an equilibrium in which each candidate's platform is an interval of policies. Kartik et al. (2017) study an example in which such equilibria exist.

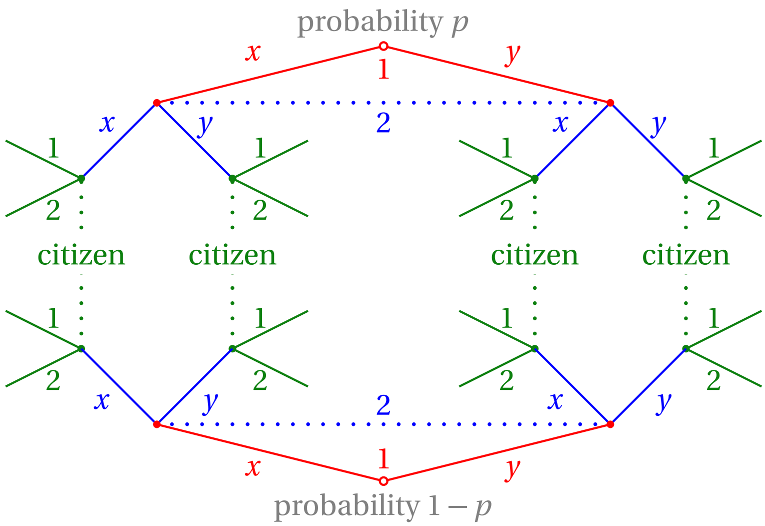


Figure 9.6 An example of the game in Section 9.3.2, in which the state is observed by the candidates before they choose positions. In this example, there are two possible policies, x and y, and two possible states. Candidate 1's actions are red, candidate 2's blue, and the citizen's green. Payoffs are not shown.

9.3.2 Candidates privately informed before choosing positions

Now assume that both candidates receive information about the desirability of the possible policies before they choose positions. In this case, the positions they choose potentially signal their information to the citizen.

Consider the extensive game with imperfect information in which chance determines the state, each candidate observes chance's move, the candidates simultaneously choose positions, and then the citizen observes the candidates' positions, but not the move of chance, and votes for one of the candidates. To make the structure of the game clear, an example in which there are two states and two possible policies is given in Figure 9.6. In this figure, the initial move of chance is not shown explicitly; instead, the small circles at the top and bottom indicate the two possible results of this move. (Payoffs are not shown in the figure.)

For the rest of the section I consider an example of the game in which, as in Figure 9.6, there are two states, but the set X of possible positions is the set of all real numbers. The states are -1 and 1: $\Theta = \{-1, 1\}$. The prior probability of state -1 is $p \in (0,1)$. For each candidate $j \in \{1,2\}$, the function $u_j : X \times \Theta \to \mathbb{R}$ whose expected value represents j's preferences is single-peaked in its first argument for each $\theta \in \Theta$. The favorite position of each candidate j in each state θ (i.e. the position x that maximizes $u_j(x, \theta)$) is denoted $\hat{x}_j(\theta)$. Assume that

$$\hat{x}_1(\theta) < \theta < \hat{x}_2(\theta) \text{ for each state } \theta \in \Theta.$$

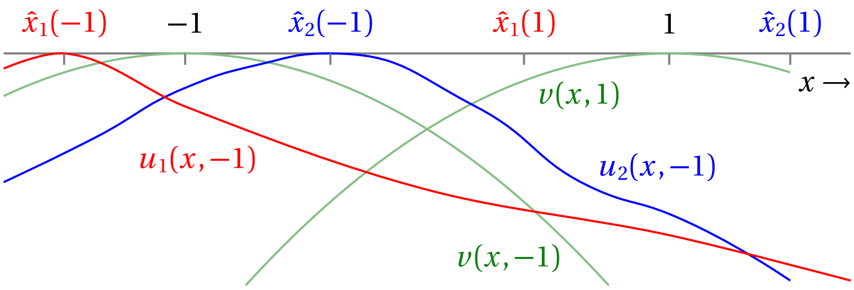


Figure 9.7 An example of the payoff functions for candidates 1 (red) and 2 (blue) in state -1 and for the citizen (green) in each state for the game in Section 9.3.2.

The function $v : X \times \Theta \to \mathbb{R}$ whose expected value represents the citizen's preferences is also single-peaked in its first argument for each $\theta \in \Theta$; in each state θ, the citizen's favorite position is θ. See Figure 9.7 for an example of payoff functions for the candidates in state -1 and for the citizen that satisfy these conditions. (For clarity, the payoff functions for the candidates in state 1 are not shown.) For any positions x_1 and x_2, denote by $I(x_1, x_2)$ the citizen's information set that is reached when candidate 1's position is x_1 and candidate 2's is x_2.

Like many extensive games with imperfect information, this game has many weak sequential equilibria. In some equilibria, in each state the candidates' positions are the same. These equilibria have an unappealing property. Suppose that in each state s the candidates' common position is x_s^*, which differs from candidate j's favorite position in state s. Then if candidate j deviates to her favorite position and there is any chance that she wins, she is better off. So any equilibrium in which in some state the candidates' positions are the same is vulnerable to small (or large) random perturbations in the citizen's strategy.

The game also has equilibria in which in each state the candidates' positions differ. I study one such equilibrium that has features of interest. Consider positions x_1^-, x_2^-, x_1^+, and x_2^+ for which $\hat{x}_1(-1) < x_1^- < -1 < x_2^- \leq \hat{x}_2(-1)$, $\hat{x}_1(1) \leq x_1^+ < 1 < x_2^+ < \hat{x}_2(1)$, and the citizen is indifferent between x_1^- and x_2^- in state -1 and between x_1^+ and x_2^+ in state 1, as shown in Figure 9.8. The game has a weak sequential equilibrium in which each candidate $i = 1, 2$ chooses x_i^- in state -1 and x_i^+ in state 1 and the citizen votes for candidate 2 at $I(x_1^-, x_2^-)$ and for candidate 1 at $I(x_1^+, x_2^+)$.

In every equilibrium in which the candidates' strategies take this form, the citizen votes for candidate 2 at $I(x_1^-, x_2^-)$. The reason is that if she votes for candidate 1 then candidate 2 can deviate in state -1 to a position $x_2 \in [-1, x_2^-)$, which induces the citizen to vote for her, making her better off, because the citizen prefers x_2 to x_1^- in both states. Similarly, in any equilibrium the citizen votes for candidate 1 at $I(x_1^+, x_2^+)$. Thus candidate 2, who favors a policy on the right,

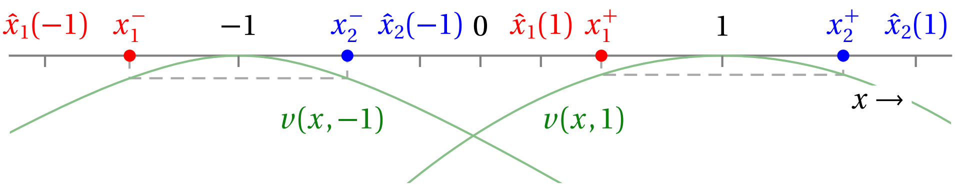


Figure 9.8 The candidates' positions in a weak sequential equilibrium of the game in Section 9.3.2. Candidate 1 chooses x_1^- in state -1 and x_1^+ in state 1 and candidate 2 chooses x_2^- in state -1 and x_2^+ in state 1.

wins in state -1, when the citizen favors a policy on the left, and candidate 1, who favors a policy on the left, wins in state 1, when the citizen favors a policy on the right.

If state -1 were the only state, candidate 1, who loses in this state, would have an incentive to move closer to -1, inducing the citizen to vote for her and thus increasing her payoff. This incentive is missing in the equilibrium of the model with two states because such a move changes the citizen's belief about the state. Specifically, if candidate 1 deviates to a position $x_1 \in (x_1^-, -1]$ in state -1, then the citizen continues to vote for candidate 2 because the deviation induces her to switch from assigning probability 1 to state -1 to assigning high probability to state 1, in which she prefers x_2^- to x_1. That is, she takes the move to the right by candidate 1 as a indicator that the state is likely to favor a policy on the right.

None of the other deviations by the candidates in state -1 are profitable even if they do not affect the citizen's belief that the state is -1. If candidate 1 deviates to a position less than x_1^-, then voting for candidate 2 remains optimal for the citizen, and the outcome does not change. If candidate 2 deviates to a position less than x_2^-, then voting for candidate 2 also remains optimal for the citizen, and the outcome is worse for candidate 2. If candidate 2 deviates to a position greater than x_2^-, then the citizen switches her vote to candidate 1, making candidate 2 worse off.

The next exercise invites you to fill in the details of an equilibrium under the assumption that $u_2(x_2^-,-1) \geq u_2(x_2^+,-1)$. This condition arises in the analysis of a deviation by candidate 2 in state -1 to x_2^+, leading to the information set $I(x_1^-, x_2^+)$, which is reached also by a deviation by candidate 1 in state 1 to x_1^-.

Exercise 9.5: Weak sequential equilibrium with dispersed positions in game with privately-informed candidates

Find a belief system and a strategy of the citizen that, combined with the strategy of candidate i that selects x_i^- in state -1 and x_i^+ in state 1, for $i = 1$, 2, where $\hat{x}_1(-1) < x_1^- < -1 < x_2^- \leq \hat{x}_2(-1)$ and $\hat{x}_1(1) \leq x_1^+ < 1 < x_2^+ < \hat{x}_2(1)$, as illustrated in Figure 9.8, is a weak sequential equilibrium of the game if $u_2(x_2^-, -1) \geq u_2(x_2^+, -1)$.

The general point that this model illustrates is that a change in a candidate's position may be interpreted by uninformed citizens as evidence regarding the likelihood that a certain policy is desirable, and the change in the citizens' beliefs that the candidate's move induces may work against the candidate.

9.4 Repeated elections

Political parties typically participate in sequences of elections, and have the opportunity to condition their positions in each election on the outcomes of previous elections. I present two models. The first is a standard repeated game, in which in each period each party may choose any position. In the second, a party's previous positions constrain its options in each period. The idea is that a party cannot credibly adopt a position completely at odds with its past positions. The model includes a simple version of a constraint: in the contest for its reelection, an incumbent's position is restricted to be the same as the one it implemented when in office. Individual candidates, as well as parties, may face sequences of elections, but the models in this section seem most fitting for parties, and that is the term I use for the players.

9.4.1 Repeated game

Suppose that two policy-motivated parties contest a sequence of elections. In each period $t = 1, 2, \ldots$, each party chooses a position in a strategic game G closely related to an electoral competition game with a continuum of citizens and two policy-motivated candidates. The players in G are the parties, 1 and 2, the set of actions of each party is a compact interval $X \subset \mathbb{R}$, and the preference relation of each party j over action pairs is represented by a function $v_j : X \times X \to \mathbb{R}$ given by

$$v_j(x_1, x_2) = \begin{cases} u_j(x_1) & \text{if } F(\frac{1}{2}(x_1 + x_2)) > \frac{1}{2} \\ \frac{1}{2}(u_j(x_1) + u_j(x_2)) & \text{if } F(\frac{1}{2}(x_1 + x_2)) = \frac{1}{2} \\ u_j(x_2) & \text{if } F(\frac{1}{2}(x_1 + x_2)) < \frac{1}{2}, \end{cases}$$

where F is a nonatomic probability distribution function with a density and support X (the distribution of the citizens' favorite positions) and $u_j : X \to \mathbb{R}$ is a single-peaked function. Denote by m the median of F and by $\hat{x}_j$ the maximizer of u_j, and assume that $\hat{x}_1 < m < \hat{x}_2$.

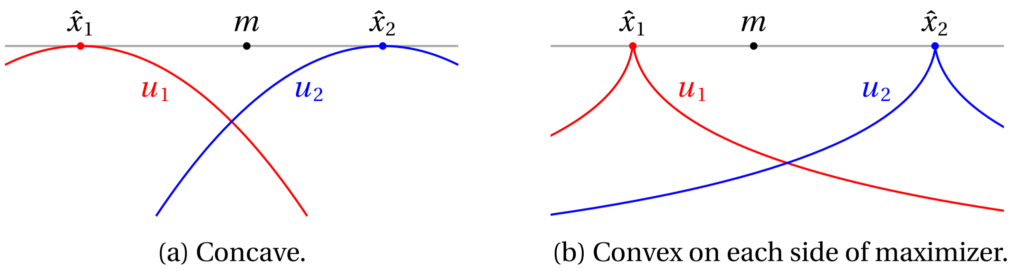


Figure 9.9 Payoff functions for policy-motivated parties.

I model the sequence of elections as an infinitely repeated game. In every period $t = 1, 2, \dots$, the parties choose positions in X after observing the positions chosen in every previous period. I assume that the preferences of each party j over sequences of outcomes are represented by the discounted average of the sequence of its payoffs according to the function v_j, with the same discount factor for both parties. That is, the payoff of each party j in the repeated game is $(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}v_j(x_1^t, x_2^t)$, for some $\delta \in (0,1)$, where (x_1^t, x_2^t) is the pair of positions chosen in period t.

Now, by choosing the position m in any period, a party guarantees that the outcome is m in that period. Thus in every Nash equilibrium of the infinitely repeated game the payoff of each party j is at least $u_j(m)$ in every period, so that its (discounted average) payoff in the repeated game is at least $u_j(m)$ (for any value of δ).

Further analysis of the equilibria depends on the nature of the parties' preferences over sequences of outcomes. If each party j prefers a constant sequence of positions to any varying sequence with the same average, then the function u_j that determines its payoff in the repeated game is concave, as in Figure 9.9a. In this case, for any value of δ, no sequence of positions yields a pair of payoffs in the repeated game for which each component j is larger than $u_j(m)$. Hence in every Nash equilibrium of the repeated game the outcome in every period is m and the payoff of each party j is $u_j(m)$.

If, instead, each party j prefers varying sequences of positions to a constant sequence with the same average, then each function u_j is convex on each side of j's favorite position, as in Figure 9.9b. In this case, payoff pairs (w_1, w_2) for which $w_1 > u_1(m)$ and $w_2 > u_2(m)$ exist in the repeated game. Suppose that $w = (w_1, w_2)$ is such a pair, with $w = (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}w^t$. For each $t = 1, 2, \dots$ let x^t be a pair of positions for which $(v_1(x^t), v_2(x^t)) = w^t$. Consider the strategy pair in the repeated game in which each party j chooses x_j^1 in period 1 and, in each period $t \geq 2$, chooses x_j^t after the history $(x^1, x^2, \dots, x^{t-1})$ and m after every other history. That is, each party adheres to the sequence $(x^1, x^2, \dots)$ as long as both parties do so, and switches to the position m otherwise. If the discount factor δ is close enough to 1, this strategy pair is a subgame perfect equilibrium of the repeated game. The game satisfies the condition in Proposition 16.9, so it suffices to show that the strategy profile satisfies the one-deviation property. If in any period t following a history in which each party has adhered to its strategy, a party j chooses a position other than x_j^t and subsequently adheres to its strategy, then, given the other party's strategy, the outcome in every subsequent period is (m, m), so that in each period after t its payoff is less than w_j, and hence for δ sufficiently close to 1 its payoff in the repeated game is less than w_j. In any period following any other history, both parties choose m, and no deviation by either party affects the policy outcome in any future period, given the other party's strategy.

Thus if each function u_j is convex on each side of $\hat{x}_j$ and the parties' common discount factor is close to 1, the repeated game has subgame perfect equilibria in which the outcome in each period differs from m. In some of these equilibria the outcome in every period is either $\hat{x}_1$ or $\hat{x}_2$. Such an outcome arises if, for example, in some periods party 1 chooses the position $\hat{x}_1$ and party 2 chooses a position more extreme than $2m - \hat{x}_1$, and in the remaining periods party 1 chooses a position more extreme than $2m - \hat{x}_2$ and party 2 chooses the position $\hat{x}_2$.

9.4.2 Persistent policies

The model of a repeated game assumes that a party can change its position arbitrarily from period to period; even a party that implements the policy x while in office in period t can commit to a position radically different from x in the election in period $t + 1$. Such a metamorphosis seems implausible: would citizens believe that a party that had previously espoused one policy is now committed to a wildly different one?

If we assume that parties can change their positions only when they are out of office, equilibria in which the winning policy in each period differs from the favorite position of the median voter are possible even if the parties' payoff functions are strictly concave (as in Figure 9.9a).

Consider an extensive game that differs from the repeated game specified in Section 9.4.1 in two respects. First, in each period, only one party is free to choose a position. In period 1, party 2's position is fixed; the idea is that party 2 was the incumbent in the previous, unmodeled, period. In every subsequent period, only the challenger is free to choose a position. The winner in any period t is constrained to adopt in period $t + 1$ the policy it implemented in period t: the set of actions available to a party i in any period $t+1$ following a period in which it won is $\{x_i^t\}$, where x_i^t is its policy in period t. Second, if the vote is tied in period 1 then party 1 is the winner, and if it is tied in any subsequent period then

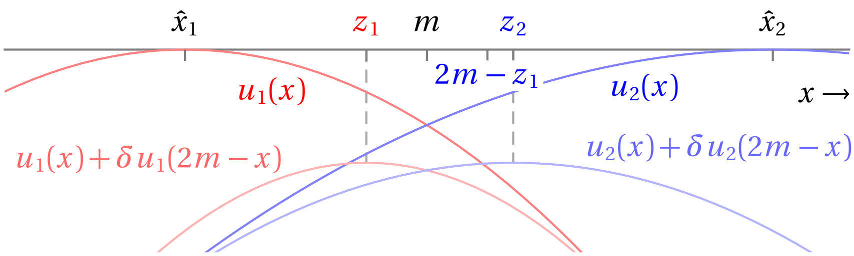


Figure 9.10 The positions relevant in a repeated election with persistent policies.

the challenger in that period is the winner.

Denote the game starting with party 2's position fixed at x_2^1 by $\Gamma(x_2^1)$. Assume that each payoff function u_j is strictly concave, and for simplicity assume that the set of positions from which party 1 can choose as a challenger is $[\hat{x}_1, m]$ and the set from which party 2 can choose as a challenger is $[m, \hat{x}_2]$. Let z_1 be the maximizer of $u_1(x_1) + \delta u_1(2m - x_1)$ for $x_1 \in [\hat{x}_1, m]$ and let z_2 be the maximizer of $u_2(x_2) + \delta u_2(2m - x_2)$ for $x_2 \in [m, \hat{x}_1]$. Given that $\delta < 1$, we have $z_1 < m$ and $z_2 > m$. Assume that $z_2 \geq 2m - z_1$. (Figure 9.10 shows an example.)

Let $x_2^* \in [m, 2m - z_1]$ and $x_1^* = 2m - x_2^*$, so that $x_1^* \in [z_1, m]$. I claim that the game $\Gamma(x_2^*)$ has a subgame perfect equilibrium in which the policy alternates between x_1^*, implemented by party 1, and x_2^*, implemented by party 2. In this equilibrium, if in any period the challenger deviates to a position x closer to m, the challenger in the next period reciprocates, choosing the position $2m - x$, and subsequently the outcome alternates between these two positions. Each party prefers an alternation between x_1^* and $x_2^* = 2m - x_1^*$ to one between x and $2m - x$, so such a deviation is not advantageous.

Specifically, the following strategy pair is a subgame perfect equilibrium of $\Gamma(x_2^*)$: party 1 chooses x_1^* in period 1, and in every other period t chooses its position in period $t - 1$ if it won in that period and $\max\{2m - y_2, z_1\}$ otherwise, where y_2 is the smallest position chosen previously by party 2, and party 2 chooses x_2^* in period 1, and in every other period t chooses its position in period $t - 1$ if it won in that period and $\min\{2m - y_1, z_2\}$ otherwise, where y_1 is the largest position chosen previously by party 1. You are invited to verify this claim in the next exercise.

Exercise 9.6: Repeated elections with persistent policies

Show that strategy pair specified in the text is a subgame perfect equilibrium of the variant of a repeated game with persistent policies defined in the text.

Notes

Austen-Smith and Banks (2005, Section 7.7) study a variant of the model in Section 9.1 in which each candidate cares slightly about winning and the set of alternatives is convex and compact. Versions of Proposition 9.3 are established by Wittman (1977, Proposition 5), Calvert (1985, Theorems 1 and 2), and Roemer (1994, Theorem 2.1). The model with imperfect commitment discussed at the end of Section 9.1 is due to Jean Guillaume Forand.

The model in Section 9.2 is due to Wittman (1983) and Calvert (1985, Section 4); Proposition 9.4 is based on Duggan (2014, Theorem 22). Proposition 9.5 is due to Roemer (1997, Theorem 3.2) and Duggan (2014, Theorem 22), who credits unpublished joint work with Mark Fey.

Section 9.3.1 is based on Kartik et al. (2017) and the model in Section 9.3.2 is a variant of the ones in Schultz (1996) and Martinelli and Matsui (2002).

The game with persistent policies discussed at the end of Section 9.4 is a variant of the one studied by Forand (2014).

Solutions to exercises

Exercise 9.1

Alternative a is not a Condorcet winner because a majority of individuals prefer c to a.

Suppose that candidate 1 prefers $\{a\}$ to $\{b\}$ to $\{c\}$ and candidate 2 prefers $\{b\}$ to $\{c\}$ to $\{a\}$, and also $\{a,b\}$ to $\{c\}$. These candidates are representative. The pair (a,b) is a Nash equilibrium of the game by the following argument.

- The outcome of (a,b) is a tie, $\{a,b\}$.
- If candidate 1 deviates to b, then she ties and the outcome is $\{b\}$, which she likes less than the outcome $\{a,b\}$ by the second part of (9.1).
- If candidate 1 deviates to c, then she loses and the outcome is $\{b\}$, which she likes less than $\{a,b\}$ by the second part of (9.1).
- If candidate 2 deviates to a, then she ties and the outcome is $\{a\}$, which she likes less than the outcome $\{a,b\}$ by the second part of (9.1).
- If candidate 2 deviates to c, then she wins and the outcome is $\{c\}$, which she likes less than the outcome $\{a,b\}$.

All of the deviations lead to outcomes different from the equilibrium outcome, so none of the arguments change if each candidate lexicographically favors winning.

Exercise 9.2

Assume without loss of generality that $\hat{x}_1 \leq \hat{x}_2$.

Denote the candidates' equilibrium positions by x_1 and x_2. I first argue that $x_2 \geq \hat{x}_2$. If $x_2 < \hat{x}_2$ then

- $x_1 < \hat{x}_2 \Rightarrow$ outcome is $\{\max\{x_1, x_2\}\}$; by moving to $\hat{x}_2$, candidate 2 changes the outcome to $\{\hat{x}_2\}$, which she prefers
- $x_1 = \hat{x}_2 \Rightarrow$ outcome is $\{\hat{x}_2\}$; by moving to $\hat{x}_1$, candidate 1 changes the outcome to $\{x_2\}$, which she prefers
- $x_1 > \hat{x}_2$ and candidate 1 wins $\Rightarrow$ outcome is $\{x_1\}$; by moving to $\hat{x}_2$, candidate 1 changes the outcome to $\{\hat{x}_2\}$, which she prefers
- $x_1 > \hat{x}_2$ and candidate 1 loses or ties for first place $\Rightarrow$ outcome is $\{x_2\}$ or $\{x_1, x_2\}$; by moving to $\hat{x}_2$, candidate 2 changes the outcome to $\{\hat{x}_2\}$, which she prefers.

I now argue that $x_2 \leq m$. If $x_2 > m$ then

- $x_1 < \hat{x}_2 \Rightarrow$ outcome is $\{x_1\}$, $\{x_2\}$, or $\{x_1, x_2\}$; by moving to $\hat{x}_2$, candidate 2 changes the outcome to $\{\hat{x}_2\}$, which she prefers
- $x_1 \geq \hat{x}_2$ and candidate 1 wins $\Rightarrow$ outcome is $\{x_1\}$; by reducing x_1 slightly, candidate 1 reduces the value of the winning position, which she prefers
- $\hat{x}_2 \leq x_1 < x_2$ and candidate 2 wins or ties $\Rightarrow$ outcome is $\{x_2\}$ or $\{x_1, x_2\}$; by moving to x_1, candidate 2 changes the outcome to $\{x_1\}$, which she prefers
- $x_1 \geq x_2 \Rightarrow$ candidate 2 wins or ties and outcome is $\{x_2\}$; by moving to m, candidate 2 changes the outcome to $\{m\}$, which she prefers.

Thus $\hat{x}_2 \leq x_2 \leq m$.

Suppose that $x_2 > \hat{x}_2$. Then $x_1 = x_2$, otherwise the winning candidate, if one candidate wins outright, or else the rightmost candidate who ties for first place, can increase her payoff by moving slightly to the left.

Now suppose that $x_2 = \hat{x}_2$. Then $x_1 \leq x_2$, otherwise candidate 1 can increase her payoff by moving to $\hat{x}_2$

Finally, any pair (x_1, x_2) for which $\hat{x}_2 \leq x_1 = x_2 \leq m$ or $x_1 \leq x_2 = \hat{x}_2$ is an equilibrium.

Thus for $\hat{x}_1 \leq \hat{x}_2$ the set of Nash equilibria is the set of pairs (x_1, x_2) such that $\hat{x}_2 \leq x_1 = x_2 \leq m$ or $x_1 \leq x_2 = \hat{x}_2$; in all of the equilibria the outcome is $\{x_2\}$.

Exercise 9.3

Every citizen i with favorite position at most m prefers x_1^* implemented by candidate 1 to m implemented by candidate 2, so the outcome of the pair (x_1^*, m) of positions is that candidate 1 wins and implements x_1^*. (The payoff for a citizen i with favorite position $\hat{z}_i \in [x_1^*, m]$ is $v(x_1^* - \hat{z}_i) \geq v(x_1^* - m) = -\delta$ for candidate 1's position x_1^* and $v(m - \hat{z}_i) - \delta \leq -\delta$ for candidate 2's position m.)

Consider a deviation by candidate 1. If she deviates to a position less than x_1^* then for some $\varepsilon > 0$ every citizen with favorite position at least $m - \varepsilon$ votes for candidate 2, so that candidate 2 wins. Candidate 1 prefers x_1^* to m, so the deviation makes her worse off. If she deviates to a position greater than x_1^* then she either wins, in which case she is no better off, or she loses and the position of the winner, candidate 2, is m, so she is also no better off.

Now consider a deviation by candidate 2. If she deviates to a position at most x_1^* then all citizens with favorite positions at least x_1^* vote for candidate 1, so that candidate 1 continues to win. If she deviates to a position greater than x_1^* then all citizens with favorite positions at most m vote for candidate 1, so that candidate 1 continues to win. Thus no deviation makes candidate 2 better off.

We conclude that (x_1^*, m) is a Nash equilibrium of the game.

Exercise 9.4

By Proposition 9.4 in any Nash equilibrium (x_1^*, x_2^*) we have $\hat{x}_1 \leq x_1^* < x_2^* \leq \hat{x}_2$.

Suppose that $\hat{x}_1 < x_1 \leq \hat{x}_2$. We have $G(z) = (z + k)/2k$ for $z \in [-k, k]$, so candidate 1's payoff at the pair of positions (x_1, x_2^*) is

$$\begin{aligned}
-(x_1 - \hat{x}_1)(\tfrac{1}{2}(x_1 + x_2^*) + k)/2k - (x_2^* - \hat{x}_1)(1 - (\tfrac{1}{2}(x_1 + x_2^*) + k)/2k) \\
= -(1/2k)\left[(x_1 - \hat{x}_1)(\tfrac{1}{2}(x_1 + x_2^*) + k) + (x_2^* - \hat{x}_1)(k - \tfrac{1}{2}(x_1 + x_2^*))\right] \\
= -(1/2k)\left[\tfrac{1}{2}x_1^2 + kx_1 + C\right],
\end{aligned}$$

where C is a constant (independent of x_1). This payoff is decreasing in x_1.

Thus candidate 1's best response to $\hat{x}_2$ is $\hat{x}_1$. The same argument with the roles of candidates 1 and 2 interchanged shows that $\hat{x}_1$ is a best response to $\hat{x}_2$. Thus $(\hat{x}_1, \hat{x}_2)$ is a Nash equilibrium.

Exercise 9.5

The assessment in which the strategies of the candidates are the ones described in the text (and illustrated in Figure 9.8) and the belief system and strategy for the citizen are given in Table 9.1 is a weak sequential equilibrium of the game if $u_2(x_2^-, -1) \geq u_2(x_2^+, -1)$. This condition is required so that in state -1 candidate 2 does not benefit from deviating to x_2^+. (Other belief systems and strategies for the citizen are consistent with equilibrium.) Here is

Information set	Prob. of state -1	Cand. chosen
$I(x_1^-, x_2^-)$	1	2
$I(x_1^+, x_2^+)$	0	1
$I(x_1^-, x)$ for $x < x_2^-$	0	2
$I(x_1^-, x)$ for $x > x_2^-$, $x \neq x_2^+$	1	1
$I(x, x_2^-)$ for $x \leq x_2^-$	0	2
$I(x, x_2^-)$ for $x > x_2^-$, $x \neq x_1^+$	1	2
$I(x_1^+, x)$ for $x < x_1^+$, $x \neq x_2^-$	0	1
$I(x_1^+, x)$ for $x \geq x_1^+$	1	1
$I(x, x_2^+)$ for $x < x_1^+$, $x \neq x_1^-$	0	2
$I(x, x_2^+)$ for $x > x_1^+$	1	1
$I(x_1^-, x_2^+)$	0	2
$I(x_1^+, x_2^-)$	0	1
$I(x, y)$ for $x \notin \{x_1^-, x_2^+\}$ and $y \notin \{x_1^-, x_2^+\}$	1	$c(x, y)$

Table 9.1 The probabilities assigned by the belief system to the citizen's information sets and the citizen's strategy in a weak sequential equilibrium of the game in Section 9.3.2 in which the candidates' strategies are the ones given in Exercise 9.5. The candidate $c(x, y)$ is 1 if $v(x, -1) \geq v(y, -1)$ and 2 if $v(x, -1) < v(y, -1)$.

the argument.

The belief system is weakly consistent with the strategy profile because at the two information sets of the citizen that are reached with positive probability given the strategies, $I(x_1^-, x_2^-)$ and $I(x_1^+, x_2^+)$, it assigns probabilities derived from the prior via Bayes' rule.

The citizen's strategy is sequentially rational because at her information sets $I(x_1^-, x_2^-)$ and $I(x_1^+, x_2^+)$ she is indifferent between the candidates' positions and at every other information set she votes for the candidate she prefers, given the belief system.

The candidates' strategies are sequentially rational given this belief system and the strategy for the citizen by the following arguments.

If candidate 1 changes her position in state -1 from x_1^- to a position other than x_1^+, the citizen continues to vote for candidate 2, so that the outcome does not change.

Similarly, if candidate 2 changes her position in state 1 from x_2^+ to a position other than x_2^-, the citizen continues to vote for candidate 1, so that the outcome does not change.

If candidate 1 changes her position in state 1 from x_1^+ to a position less than x_1^+ other than x_1^-, the citizen switches from voting for candidate 1 to voting for candidate 2, so that the outcome changes from x_1^+ to x_2^+, which is worse for candidate 1. If she changes her position in state 1 from x_1^+ to a position greater than x_1^+, the citizen continues to vote for her, so that she is worse off.

Similarly, if candidate 2 changes her position in state -1 from x_2^- to a position greater than x_2^- other than x_2^+, the citizen switches from voting for candidate 2 to voting for candidate 1, so that the outcome changes from x_2^- to x_1^-, which is worse for candidate 2. If she changes her position in state -1 from x_2^- to a position less than x_2^-, the citizen continues to vote for her, so that she is worse off.

If candidate 1 deviates in state -1 to x_1^+ then the citizen switches from voting for candidate 2 to voting for candidate 1, so that she is worse off. If she deviates in state 1 to x_1^- then the citizen switches from voting for candidate 1 to voting for candidate 2, which makes her worse off.

If candidate 2 deviates in state -1 to x_2^+ then the citizen continues to vote for her, so that she is not better off, given that if $u_2(x_2^-,-1) \geq u_2(x_2^+,-1)$. If she deviates in state 1 to x_2^- then the citizen continues to vote for candidate 1, so the outcome does not change.

Exercise 9.6

The game satisfies the condition in Proposition 16.9, so a strategy pair is a subgame perfect equilibrium if and only if it satisfies the one-deviation property.

First consider deviations by party 1 (in periods in which it is the challenger). For each of the following cases, the table gives the outcomes induced by party 1's adhering to its strategy and deviating from it in the first period of the subgame, given party 2's strategy.

Note that the function $u_1(x_1) + \delta u_1(2m - x_1)$ is concave in x_1, increasing up to z_1 and decreasing thereafter.

Subgame following history ending with $x_2 \in [m, 2m - z_1]$

	period 1	period $1+t$ $t \geq 1$ odd	period $1+t$ $t \geq 2$ even
adheres	$2m - x_2$	x_2	$2m - x_2$
deviates to $x_1 \in [\hat{x}_1, 2m - x_2)$	x_2	$2m - x_2$	x_2
deviates to $x_1 \in (2m - x_2, m]$	x_1	$2m - x_1$	x_1

It prefers $2m - x_2$ to x_2, so the first deviation makes it worse off. The second deviation makes it worse off because $u_1(x) + \delta u_1(2m - x)$ is decreasing in x for $x > z_1$.

The case $x_2 = 2m - x_1^*$ covers the subgame following the empty history (the start of the game).

Subgame following history ending with $x_2 \in (2m - z_1, \hat{x}_2]$

	period 1	period 2	period $1+t$ $t \geq 2$ even	period $2+t$ $t \geq 2$ even
adheres	z_1	$2m - z_1$	z_1	$2m - z_1$
deviates to $x_1 \in [\hat{x}_1, 2m - x_2)$	x_2	z_1	$2m - z_1$	z_1
deviates to $x_1 \in [2m - x_2, z_1)$	x_1	$2m - x_1$	z_1	$2m - z_1$
deviates to $x_1 \in (z_1, m]$	x_1	$2m - x_1$	x_1	$2m - x_1$

The first deviation makes it worse off because it prefers both z_1 and $2m - z_1$ to x_2, the second one does so because it prefers z_1 to x_1 and $2m - z_1$ to $2m - x_1$, and the third one does so because $u_1(x_1) + \delta u_1(2m - x_1)$ is decreasing in x_1 for $x_1 \geq z_1$.

Now consider deviations by party 2.

Subgame following history ending with $x_1 \in [z_1, m]$

	period 1	period $1+t$ $t \geq 1$ odd	period $1+t$ $t \geq 2$ even
adheres	$2m - x_1$	x_1	$2m - x_1$
deviates to $x_2 \in (2m - x_1, \hat{x}_2]$	x_1	$2m - x_1$	x_1
deviates to $x_2 \in [m, 2m - x_1)$	x_2	$2m - x_2$	x_2

It prefers $2m - x_1$ to x_1, so the first deviation makes it worse off. The second deviation makes it worse off because $x_2 < 2m - x_1 < z_2$ and $u_2(x) + \delta u_2(2m - x)$ is increasing in x for $x < z_2$.

Subgame following history ending with $x_1 \in [2m - z_2, z_1)$

	period 1	period 2	period $1+t$ $t \geq 2$ even	period $2+t$ $t \geq 2$ even
adheres	$2m - x_1$	z_1	$2m - z_1$	z_1
deviates to $x_2 \in (2m - x_1, \hat{x}_2]$	x_1	$2m - x_1$	z_1	$2m - z_1$
deviates to $x_2 \in (2m - z_1, 2m - x_1)$	x_2	z_1	$2m - z_1$	z_1
deviates to $x_2 \in [m, 2m - z_1]$	x_2	$2m - x_2$	x_2	$2m - x_2$

The first deviation makes it worse off because it prefers both $2m - x_1$ and z_1 to x_1, the second one does so because it prefers $2m - x_1$ to x_2 (given $x_2 < 2m - x_1 \leq z_2$), and the third one does so because it prefers $2m - x_1$ to x_2 and $u_2(x) + \delta u_2(2m - x)$ is increasing in x for $x < z_2$.

Subgame following history ending with $x_1 \in [\hat{x}_1, 2m - z_2)$

	period 1	period 2	period $1+t$ $t \geq 2$ even	period $2+t$ $t \geq 2$ even
adheres	z_2	z_1	$2m - z_1$	z_1
deviates to $x_2 \in (2m - x_1, \hat{x}_2]$	x_1	z_2	z_1	$2m - z_1$
deviates to $x_2 \in (z_2, 2m - x_1]$	x_2	z_1	$2m - z_1$	z_1
deviates to $x_2 \in [2m - z_1, z_2)$	x_2	z_1	$2m - z_1$	z_1
deviates to $x_2 \in [m, 2m - z_1)$	x_2	$2m - x_2$	x_2	$2m - x_2$

The first deviation makes it worse off because it prefers z_2, z_1, and $2m-z_1$ to x_1, the second and third ones do so because it prefers z_2 to x_2, and the fourth one does so because it prefers z_2 to x_2 and $u_2(x)+\delta u_2(2m-x)$ is increasing in x for $x < z_2$.

10 Electoral competition: endogenous candidates

10.1 Simultaneous entry with sincere voting 333
10.2 Simultaneous entry with strategic voting 336
10.3 Sequential entry 344
10.4 Citizen-candidates 350

In the models of electoral competition in the previous two chapters, the number of candidates is fixed, equal to two. In the models in this chapter, the number of candidates is determined as part of an equilibrium; each member of a set of politicians decides whether to become a candidate.

Synopsis

Section 10.1 presents a straightforward extension to many candidates of an electoral competition game with a continuum of citizens and two office-motivated candidates. There are three or more politicians, each of whom has the option to run as a candidate, and prefers to stay out of the competition than to enter and lose. I argue that for almost any distribution of the citizens' favorite positions, the resulting model has no Nash equilibrium, so that it is not a useful vehicle to study multicandidate electoral competition.

In that model, when two or more candidates choose the same position x, the votes of the citizens who prefer x to the position of every other candidate are split equally among the candidates choosing x. Section 10.2 explores a variant of the model in which the candidate for whom each citizen votes is instead specified as part of an equilibrium. That is, voting is "strategic". The model is an extensive game in which the politicians first simultaneously choose whether to run as candidates and if so the positions to take, and then the citizens simultaneously cast votes. Proposition 10.1 shows that the game has subgame perfect equilibria in which every candidate's position is the median m of the citizens' favorite positions and the citizens' votes are split equally between the candidates. If a candidate deviates to a position x different from m, all the citizens who prefer m to x vote for the *same* remaining candidate, who consequently wins outright, causing the deviating candidate to lose. In the model in Section 10.1, such a deviation splits the votes of the citizens who prefer m to x among the candidates remaining at m, causing the deviating candidate to win, but in these equilibria of the model in Section 10.2 it leads these citizens to rally around one of the remaining candidates. Proposition 10.1 shows also that if every citizen's payoff function is strictly concave then all subgame perfect equilibria in which every citizen's action in each subgame is weakly undominated takes this form.

Section 10.3 presents a variant of the model in Section 10.1 in which the politicians move sequentially rather than simultaneously. Proposition 10.4 shows that when there are three politicians, in the unique subgame perfect equilibrium the first one to move chooses the median m of the citizens' favorite positions, the second one stays out of the competition, and the third one, like the first one, chooses m. The reason the second politician stays out is that for every position x_2 at which she could enter, there is a position x_3 at which the third politician can win outright. If $x_2 = m$, then a position x_3 close to m is winning—the votes of the citizens who prefer m to x_3 are split between the first two entrants—and if x_2 differs from m then a position x_3 closer to m on the other side of m is winning. You may find yourself conjecturing that for the case of an arbitrary number n of politicians, the game has a unique subgame perfect equilibrium and in that equilibrium the first politician enters at m, the next $n-2$ stay out, and the last one enters at m. The veracity of that conjecture is not known.

The model in Section 10.4 differs more significantly from the one in Section 10.1. The set of politicians is the set of citizens: any citizen can become a candidate. A citizen who does so implements her favorite position if she is elected; she cannot commit to a different position. Each citizen cares about the position implemented by the winner of the election. In addition, if she becomes a candidate, she incurs a cost and, if she wins, she receives a benefit. I present two versions of the model, one in which voting is sincere (each citizen votes for the candidate whose favorite position she likes best) and one in which it is strategic. Both models have various equilibria that differ qualitatively from the equilibria of the models I have discussed previously. In one type of equilibrium, there are two candidates—citizens with favorite positions symmetric about the median, m, of all citizens' favorite positions (Proposition 10.7). In another type of equilibrium, a single citizen becomes a candidate; her favorite position is either m or close to m (Proposition 10.5). Depending on the benefit and cost of running as a candidate, the model with strategic voting also has equilibria in which many citizens with favorite position m enter as candidates, for much the same reason that the model in Section 10.2 has such equilibria (Proposition 10.6*c*). And for certain ranges of the parameter values, both models have equilibria in which three or more citizens run as candidates (Exercises 10.7 and 10.8).

10.1 Simultaneous entry with sincere voting

I begin by arguing that a straightforward extension to many candidates of an electoral competition game with a continuum of citizens and two office-motivated candidates is not a useful vehicle for exploring multicandidate elections because for most distributions of the citizens' favorite positions it has no Nash equilibrium.

Suppose that each of many politicians chooses whether to run as a candidate and, if she runs, the position to take. Assume that the set of possible positions is the real line and the distribution F of the citizens' favorite positions is nonatomic, with support an interval. I refer to a politician who chooses a position as a candidate. Each citizen votes for a candidate whose position is closest to her favorite position; if a position is occupied by several candidates, these candidates share equally the votes for that position. The candidates who obtain the largest fraction of the votes tie for first place and every other candidate loses. Each politician is motivated by the possibility of winning; she prefers to stay out of the election than to enter and lose, to win outright than to tie for first place, and to tie for first place with one other candidate than to stay out of the election.

The precise definition of the outcome of an action profile is a little involved. The set of winners for any given action profile is probably clear, so I relegate the precise definition to the appendix to this chapter.

Suppose that the number of politicians is at least three. Any Nash equilibrium of the game has the following properties.

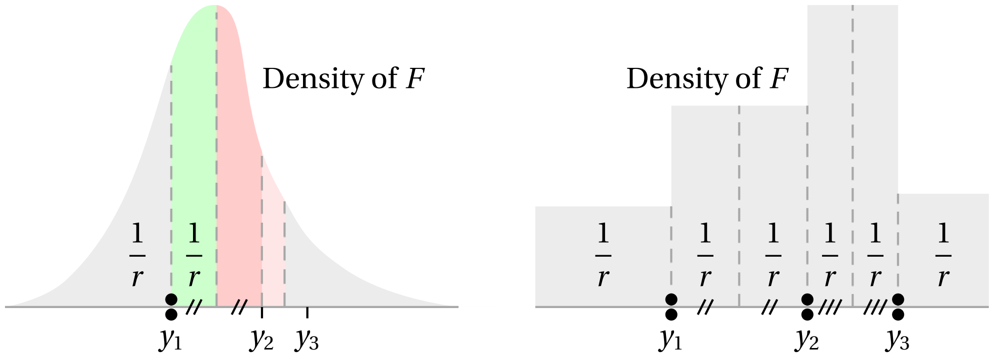
1. At least two politicians become candidates. If none do so, any one of them can enter and win outright, and if one does so, another one can enter at the same position and tie for first place.

2. All candidates tie. If not, one of them loses and is better off withdrawing.

3. At most two politicians choose each occupied position. If more than two choose the same position then they all tie for first place, by property 2, and any one of them can deviate slightly, obtaining a minimum of almost half of the votes for the position and hence winning outright.

4. Exactly two politicians choose the smallest occupied position. By property 3, the only other possibility is that one politician chooses this position, in which case she can increase her vote share and hence win rather than tie by increasing her position slightly. Similarly exactly two politicians choose the largest occupied position.

5. The number of candidates is at least four. This conclusion follows from property 4.



(a) If one candidate chooses y_2 then the mass of voters she attracts is the sum of the areas shaded dark and light pink.

(b) An example of a distribution F for which there is an equilibrium with six candidates.

Figure 10.1 The conditions required for an equilibrium with r candidates in a model with simultaneous entry and sincere voting. At most two politicians choose each occupied position, and exactly two choose the leftmost occupied position, y_1.

6. For any position y chosen by two politicians, the fraction of votes the position attracts from citizens with favorite positions less than y is equal to the fraction it attracts from citizens with favorite positions greater than y, and the common fraction is $1/r$, where r is the total number of candidates. By property 2, the candidates at y tie for first place. If the fractions differ then either of the candidates with position y can deviate slightly in the direction of the larger fraction and win outright rather than tying.

7. Exactly two politicians choose each occupied position. Denote by y_1 the smallest occupied position. By properties 4 and 6, we have $y_1 = F^{-1}(1/r)$, as in Figure 10.1a. The value y_2 of the next smallest occupied position is determined by the condition that y_1 attracts the fraction $2/r$ of the votes: the midpoint of $[y_1, y_2]$ must be $F^{-1}(2/r)$. By property 3, at most two candidates occupy y_2. Suppose that one does so. Then she attracts the votes of all citizens with favorite positions between the midpoint of $[y_1, y_2]$ and the midpoint of $[y_2, y_3]$, the mass of which is the sum of the areas shaded dark and light pink in the figure. But if she deviates to a position $y_1 + \varepsilon$ for some small number $\varepsilon > 0$, she attracts the votes of all citizens with favorite positions between $y_1 + \frac{1}{2}\varepsilon$ and the midpoint of $[y_1 + \varepsilon, y_3]$, the mass of which exceeds the area shaded green, and hence exceeds $1/r$, so that the configuration is not an equilibrium. Thus two candidates occupy y_2. Repeating the argument leads to the conclusion that two candidates occupy every occupied position.

Properties 6 and 7 applied to y_2 imply that the area shaded dark pink in Figure 10.1a is $1/r$. That is, $y_2 = F^{-1}(2/r) + (F^{-1}(2/r) - F^{-1}(1/r)) = F^{-1}(3/r)$, or

$F^{-1}(2/r) - F^{-1}(1/r) = F^{-1}(3/r) - F^{-1}(2/r)$. Applying the same argument to each occupied position, we conclude that an equilibrium with r candidates exists only if

$$F^{-1}((k+1)/r) - F^{-1}(k/r) = F^{-1}((k+2)/r) - F^{-1}((k+1)/r) \quad \text{for every odd number } k \text{ with } 1 \leq k \leq r-3. \tag{10.1}$$

An example of a distribution F for which these conditions are satisfied for $r = 6$ is given in Figure 10.1b. (The conditions are not sufficient for an equilibrium, but the configuration of positions shown in this figure is an equilibrium.)

If you try to construct a distribution that satisfies (10.1) for some integer $r \geq 4$, I think you will conclude that such distributions are few and far between. An implication of such a conclusion is that for most distributions the game has no Nash equilibrium.

A variant of the game in which the winner is determined by plurality rule with a runoff does have Nash equilibria, which you are invited to study in the next exercise.

Exercise 10.1: Nash equilibria of electoral competition game under plurality rule with runoff

Consider a variant of the game studied in this section in which the winner is determined by plurality rule with a runoff. In this system, there may be one or two rounds of voting. Assume that in each round, each citizen votes for the candidate whose position she likes best. If one candidate obtains the votes of more than half the citizens in the first round, she wins and there is no second round. Otherwise, the two candidates who obtain the most votes in the first round compete in a second round; the one who obtains the most votes in the second round wins. All ties are broken equiprobably. Assume that each candidate's payoff is her probability of winning. (Note that each candidate chooses her position before any voting takes place, and is not allowed to change it between the rounds of voting.)

Denote the number of politicians by n and the median of the citizens' favorite positions, which is assumed to be unique, by m. Are there values of k such that the game has a Nash equilibrium in which k politicians become candidates, choosing the position m, and the remaining politicians do not enter? If $n \geq 4$, are there values of k and $\delta > 0$ such that the game has a Nash equilibrium in which k candidates choose the position $m - \delta$, k choose $m + \delta$, and the remaining politicians do not enter?

10.2 Simultaneous entry with strategic voting

The model in the previous section assumes that if two or more candidates occupy the same position, the votes for that position are split equally among them. Each citizen is indifferent among candidates with the same position, so the optimality of the action of a citizen who prefers that position to every other occupied position requires only that she vote for one of these candidates. The conclusion that the candidates share the votes for the position equally follows, at least approximately, from the additional assumptions that the number of citizens is large and each citizen chooses the candidate for whom she votes randomly from those among whom she is indifferent, independently of all the other citizens. An alternative formulation, explored in this section, assumes that the candidate for whom each citizen votes is determined as part of the equilibrium. The electoral competition is modeled as a two-stage game, in which first each politician chooses whether to become a candidate, and if so the position to take, and then each citizen selects the candidate for whom to vote. Each candidate who receives the highest number of votes wins with the same probability. In an equilibrium, the action of each politician is optimal for her given the citizens' strategies, and the vote of each citizen is optimal for her given the candidates' positions.

The set of possible positions is the set of real numbers. Politicians and citizens are both finite in number. Each politician incurs a cost if she becomes a candidate and receives a benefit if she wins. Each citizen receives a payoff that depends on the position of the candidate who wins; her payoff function over positions is single-peaked. If no politician becomes a candidate, each citizen gets a fixed negative payoff.

Definition 10.1: Electoral competition game with office-motivated politicians and strategic voting

An *electoral competition game with office-motivated politicians and strategic voting* $\langle n, h, (u_1, \ldots, u_h), b, c, L\rangle$, where

- $n \geq 2$ is an integer (the number of politicians)
- $h \geq 3$ is an odd integer (the number of citizens)
- $u_i : \mathbb{R} \to \mathbb{R}_-$ for $i = 1, \ldots, h$ is a single-peaked function (citizen i's payoff function over positions)
- $b > 0$ (each politician's benefit from winning)
- $c > 0$ (each politician's cost of running as a candidate)

- $L \in \mathbb{R}$ (each citizen's loss if no politician runs as a candidate)

is an extensive game with perfect information and simultaneous moves with the following components.

Players
The set $N \cup I$, where $N = \{1,\ldots,n\}$ (politicians) and $I = \{1,\ldots,h\}$ (citizens).

Terminal histories
The set of sequences (x,v), where $x = (x_1,\ldots,x_n)$ and $v = (v_1,\ldots,v_h)$ with $x_j \in \mathbb{R} \cup \{Out\}$ for $j = 1,\ldots,n$ and $v_i \in \{j \in N : x_j \in \mathbb{R}\} \cup \{Abstain\}$ for $i = 1,\ldots,h$. (The value of x_j for $x_j \in \mathbb{R}$ is j's position; $v_i \in N$ is the candidate for whom i votes.)

Player function
The function P with $P(\varnothing) = N$ (every politician moves at the start of the game) and $P(x_1,\ldots,x_n) = I$ for all $(x_1,\ldots,x_n) \in (\mathbb{R} \cup \{Out\})^n$ (every citizen moves after the politicians have moved).

Preferences
For any terminal history (x,v), define $W(x,v)$ to be the set of winning candidates:

$$W(x,v) = \{j \in N : x_j \in \mathbb{R} \text{ and } |\{i \in I : v_i = j\}| \geq |\{i \in I : v_i = j'\}| \text{ for all } j' \in N \setminus \{j\}\}.$$

The preference relation of each politician $j \in N$ over terminal histories (x,v) is represented by the payoff function

$$\begin{cases} 0 & \text{if } x_j = Out \\ -c & \text{if } x_j \in \mathbb{R} \text{ and } j \notin W(x,v) \\ b/|W(x,v)| - c & \text{if } x_j \in \mathbb{R} \text{ and } j \in W(x,v). \end{cases}$$

The preference relation of each citizen $i \in I$ over terminal histories (x,v) is represented by the payoff function

$$\begin{cases} -L & \text{if } \{j \in N : x_j \in \mathbb{R}\} = \varnothing \\ \sum_{j \in W(x,v)} u_i(x_j)/|W(x,v)| & \text{if } \{j \in N : x_j \in \mathbb{R}\} \neq \varnothing. \end{cases}$$

In this game, a strategy for each politician is a position or *Out*, and a strategy for a citizen is a function that assigns to each possible list of actions for the politicians either one of the candidates (a politician whose action is a position) or *Abstain*. I argue that for any number $k \leq n$ with $1 \leq k \leq b/c$, the game has a subgame perfect equilibrium in which k politicians become candidates, each candidate chooses the median of the citizens' favorite positions, the candidates tie for first place, and the citizens' actions in every subgame are weakly undominated. Conversely, if the citizens' payoff functions are strictly concave and $b > c$ then in every subgame perfect equilibrium in which the citizens' actions in every subgame are weakly undominated, the number k of candidates satisfies $1 \leq k \leq b/c$, every candidate's position is the median of the citizens' favorite positions, and the outcome of the election is a tie among the candidates.

In these equilibria, no candidate can benefit from deviating to a slightly different position, because the citizens' strategies specify that after such a history, every citizen who prefers the median to the deviator's position votes for the *same* candidate. That is, after a history in which the position of every candidate but one is the median of the citizens' favorite positions, the citizens who prefer the median to the position of the remaining candidate coordinate their votes on one of the candidates whose position is the median. By contrast, the model in the previous section assumes that such a configuration of positions leads these citizens to divide their votes equally among the candidates whose position is the median, so that the candidate whose position differs from the median wins if her position is close enough to the median.

The condition $k \leq b/c$ is required because for a strategy profile in which k politicians become candidates, all choosing the median of the citizens' favorite positions and tying for first place, the payoff of each candidate is $b/k - c$, and if she deviates to *Out* then it becomes 0.

The citizens' behavior in some of the equilibria with $k = 1$ is plausible: every citizen votes for the single candidate; if a politician deviates to become a candidate at the same position, no citizen changes her vote (she has no positive incentive to do so), whereas if a politician deviates to become a candidate at a different position, each citizen votes for the candidate whose position she prefers. The citizens' behavior in equilibria with $k \geq 2$ is less plausible: the citizens' votes are split equally among the candidates, but if a politician deviates to become a candidate at a different position, the citizens rally around *one* of the candidates at the median. How such coordination could occur is unclear.

Why does the game have no equilibria in which the candidates' positions are dispersed when the citizens' payoff functions are strictly concave? Here are the key points of the argument.

- In an equilibrium, all candidates tie, otherwise one of them loses and can increase her payoff by deviating to *Out*. Consequently every citizen's vote is pivotal: any change in any citizen's vote changes the set of winners.

- Thus by Lemma 4.1 each citizen either votes for a candidate whose position she likes best among all the candidates' positions or abstains, and if she abstains, she is indifferent among all the candidates' positions.
- The strict concavity of the citizens' payoff functions means that by the argument for Proposition 4.3*a* at most two positions are occupied by candidates.
- In any equilibrium with two occupied positions, say y and y', only one candidate occupies each position: if j and j' both occupy y, a citizen who is voting for j can, by switching her vote to j', change the outcome from a tie among the candidates to an outright win for j', an outcome she prefers unless she is indifferent between y and y'. If all citizens who vote for a candidate with the position y are indifferent between y and y' then they all prefer positions between y and y' to both y and y', and any candidate who deviates from y or y' to a position in the interval (y, y') attracts all their votes and thus wins.
- If two positions are occupied, each by one candidate, each citizen who is not indifferent between the positions must vote for the position she prefers and the candidates must tie. But then either candidate can deviate to the median of the citizens' favorite positions and win, because in an equilibrium of the resulting subgame each citizen votes for the candidate whose position she prefers, and a majority prefer the median to any other position.

Proposition 10.1: SPE of electoral competition game with office-motivated politicians and strategic voting

Let $\langle n, h, (u_1, \ldots, u_h), b, c, L\rangle$ be an electoral competition game with office-motivated politicians and strategic voting in which $b > c$. For any positive integer $k \leq n$, let $X(k)$ be the set of lists $(x_1, \ldots, x_n)$ of the politicians' strategies for which the number of candidates $(|\{j \in N : x_j \in \mathbb{R}\}|)$ is k and the position x_j of every candidate j is the median of the citizens' favorite positions.

a. For any integer $k \leq n$ with $1 \leq k \leq b/c$ and any $(x_1, \ldots, x_n) \in X(k)$, the game has a subgame perfect equilibrium in which the list of the politicians' strategies is $(x_1, \ldots, x_n)$ and every citizen's action in each subgame is weakly undominated.

b. If each function u_i is strictly concave and has a maximizer (i's favorite position), then in every subgame perfect equilibrium in which every citizen's action in each subgame is weakly undominated, the list of the politicians' strategies is in $X(k)$ for some integer $k \leq n$ with $1 \leq k \leq b/c$.

In an equilibrium, all candidates occupy the same position, so that every citizen is indifferent between voting for any of them and abstaining. For equilibrium, the candidates must tie, so that the same number of citizens must vote for each of them. If the number of citizens is not divisible by the number of candidates, the number of votes for each candidate cannot be the same unless some citizens abstain. One possibility is that they all abstain; I use this equilibrium in the proof because it is easy to specify.

Proof

a. Let K be a set of politicians with k members, and denote the median of the citizens' favorite positions by m (which is unique, because the number of citizens is odd).

Consider a strategy profile in which the strategy x_j of each politician $j \in K$ is m, the strategy of each remaining politician is *Out*, and the citizens' actions after each history are given as follows, where $j^* \in K$ and, if $k \geq 2$, $\gamma(j) \in K \setminus \{j\}$ for each $j \in K$.

- History x. Every citizen chooses *Abstain*.
- History that differs from x only in that $x_j = Out$ for some $j \in K$. If $k = 1$, each citizen choose *Abstain*, her only option. If $k \geq 2$, every citizen votes for $\gamma(j)$.
- History that differs from x only in that $x_j \in \mathbb{R}$ with $x_j \neq m$ for some $j \in K$. If $k = 1$, each citizen votes for j. If $k \geq 2$, every citizen who prefers m to x_j votes for $\gamma(j)$ and every remaining citizen votes for j.
- History that differs from x only in that $x_j = m$ for some $j \notin K$. Every citizen votes for j^*.
- History that differs from x only in that $x_j \in \mathbb{R}$ with $x_j \neq m$ for some $j \notin K$. Every citizen who prefers m to x_j votes for j^* and every remaining citizen votes for j.
- History that differs from x in the actions of two or more politicians. By Proposition 4.2 the subgame that follows such a history has a Nash

equilibrium in which each citizen's action is weakly undominated. Select one of these equilibria arbitrarily; the citizens' actions are the ones specified in this equilibrium.

I claim that this strategy profile is a subgame perfect equilibrium in which every citizen's action in every subgame is weakly undominated.

In each case, the citizens' actions in each subgame constitute a Nash equilibrium. In the first, second, and fourth cases, the outcome is m regardless of the citizens' actions, so no citizen's action is weakly dominated. In the third and fifth cases, every citizen votes for a candidate whose position she likes best, so that her action is weakly undominated (Proposition 3.1).

To complete the proof, I argue that given these strategies of the citizens, no politician can increase her payoff by deviating. If all politicians follow their strategies, the k who become candidates tie, each receiving the payoff $b/k - c$, while each remaining politician receives the payoff 0. If $j \in K$ deviates to *Out*, then her payoff changes from $b/k - c$ to 0, so she is not better off. If $j \in K$ deviates to a position $z \neq m$, then if $k = 1$ she continues to be the winner, and is not better off, and if $k \geq 2$ then all citizens who prefer m to z, a majority, vote for $\gamma(j)$, one of j's ex-companions at m, so that she loses and her payoff changes to $-c$. If $j \notin K$ deviates to enter at m, then all citizens vote for the same member j^* of K, so that j loses and her payoff changes from 0 to $-c$. Finally, if $j \notin K$ deviates to enter at a position $z \neq m$, all citizens who prefer m to z, a majority, vote for j^*, so that j loses and her payoff changes to $-c$.

b. Let (x, v) be the terminal history generated by a subgame perfect equilibrium in which every citizen's action in every subgame is weakly undominated. Denote by I the set of citizens and by $C(x)$ the set of politicians who choose to become candidates for this terminal history: $C(x) = \{j \in N : x_j \in \mathbb{R}\}$.

Step 1 *At least one politician becomes a candidate ($|C(x)| \geq 1$).*

Proof. If no politician becomes a candidate, each politician's payoff is 0. Any politician can increase her payoff by deviating to run as a candidate at any position, in which case she wins (regardless of the citizens' strategies) and obtains the payoff $b - c > 0$. ◁

Step 2 *Every candidate receives the same number of votes.*

Proof. If not, then at least one candidate loses, obtaining the payoff $-c$. Such a candidate can increase her payoff to 0 by deviating to *Out*. ◁

Step 3 *In the subgame following x, a citizen who abstains is indifferent among all the candidates. The payoff of a citizen who votes for a candidate is at least her payoff from any of the other candidates' positions.*

Proof. The subgame following x is the plurality rule voting game $\langle I, C(x), (V_i)_{i \in I} \rangle$ where for each $i \in I$ the function $V_i : C(x) \to \mathbb{R}$ is defined by $V_i(j) = u_i(x_j)$ for all $j \in C(x)$. Given that by Step 2 every candidate is a winner, the result follows from Lemma 4.1 applied to this game. ◁

Step 4 *The number of distinct values of x_j for $j \in C(x)$ is at most two.*

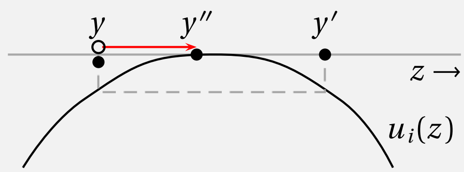
Proof. By Step 2, the set of winners of the voting subgame following x is the set $C(x)$ of candidates. The result follows from an argument like that in the proof of Proposition 4.3*a*. (That result does not apply directly because it assumes that the alternatives are distinct. Here, several candidates may choose the same position.) ◁

Step 5 *If two positions are occupied in x, exactly one candidate occupies each position.*

Proof. Suppose that the positions y and y' are occupied in x and two or more candidates occupy y. By Step 2 every candidate gets the same number of votes and hence is one of the winning candidates. Let i be a citizen who votes for a candidate at y. By Step 3, she either prefers y to y' or is indifferent between them.

If i prefers y to y' then her deviating to vote for another candidate at y induces the outcome in which that candidate wins outright. She prefers y to the outcome of (x, v), which includes y' with positive probability, so (x, v) is not a subgame perfect equilibrium.

Thus every citizen who votes for a candidate at y is indifferent between y and y'. By Step 3, every citizen who abstains is also indifferent between these positions. Now, by the strict concavity of each citizen's payoff function, positions between y and y' are better than y for every citizen who is indifferent between y and y'. Suppose that one of the candidates whose position is y deviates to a position y'' between y and y'.



By Proposition 4.1*a*, voting for the candidate at y'' is the only weakly undominated action of any citizen who is indifferent between y and y', so in the subgame following the candidate's deviation, every citizen who was voting for a candidate at y or abstaining votes for the candidate at y''. If one candidate occupies y', the candidate at y'' consequently wins outright. If more than one candidate occupies y', then by the same argument as for y, every citizen who votes for a candidate at y' is indifferent between y and y', and hence, like the citizens who were voting for a candidate at y, votes for the candidate at y'', so that in this case also the candidate at y'' wins outright. Hence (x, v) is not a subgame perfect equilibrium. ◁

Step 6 *The position of every candidate is the same, equal to the median m of the citizens' favorite positions.*

Proof. By Steps 4 and 5, the only other possibility is that the number of candidates is 2 and they choose different positions. In this case, by Step 2 the candidates tie and by Step 3 each citizen either votes for her favorite candidate or, if she is indifferent between the candidates, abstains. Thus neither candidate's position is m. Suppose that one of the candidates, say j, deviates to m. Then the only weakly undominated action of every citizen who prefers m to the position of the other candidate is to vote for j, so that j wins outright. Thus the game has no subgame perfect equilibrium in which two politicians become candidates.

If the position of every candidate is $x \neq m$, a candidate who deviates to m wins outright because by Proposition 4.1*a* the only weakly undominated action of every citizen who prefers m to x is to vote for her, and these citizens constitute a majority. ◁

Step 7 *The number of candidates is at most b/c.*

Proof. If the number of candidates exceeds b/c then each candidate's payoff is negative, so she is better off deviating to *Out*. ◁

Steps 1, 6, and 7 imply that in any subgame perfect equilibrium in which every citizen's action in every subgame is weakly undominated there is at least one and at most b/c candidates, and each candidate's position is m.

Exercise 10.2: Policy-motivated politicians and strategic voting

Consider a game that differs from an electoral competition game with office-motivated politicians and strategic voting $\langle n, h, (u_1, \ldots, u_h), b, c, L \rangle$ only in the payoff functions of the politicians, who value both policies and winning. Specifically, the payoff of each politician j to the strategy profile (x, v) is $-D < 0$ if no politician becomes a candidate and otherwise is her payoff in the game with office-motivated politicians plus $\sum_{l \in W(x,v)} U_j(x_l)/|W(x,v)|$, where $U_j : \mathbb{R} \to \mathbb{R}$ is a single-peaked function. Study the subgame perfect equilibria of this game in which every citizen's action in every subgame is weakly undominated.

10.3 Sequential entry

Section 8.1.2 presents a model of electoral competition in which two candidates act sequentially. I now present an extension of this model to many players that is analogous to the extension of an electoral competition game with a continuum of citizens and two office-motivated candidates in Section 10.1. There is an arbitrary finite number of politicians, each of whom chooses whether to become a candidate, and if so which position to take. The politicians move one at a time, and when choosing an action, each politician observes the actions chosen by her predecessors. In one environment that the model fits, the opportunity to act arises randomly for each individual; the individual selected in period i becomes politician i. The significant assumptions are that each individual gets one opportunity to act, and when it occurs she knows the actions of the individuals who moved before her.

As in the model of Section 10.1, the set of possible positions is the real line, and in the background is a continuum of citizens, whose votes determine the winning candidate(s). Each citizen has single-peaked preferences and votes for a candidate whose position is closest to her favorite position; if a position is occupied by several candidates, these candidates share equally the votes for that position. The winning candidates are the ones who tie for the most votes. Each politician prefers to be a candidate who ties with $k-1$ other candidates than to be one who ties with k other candidates, for any $k = 2, \ldots, n$, prefers to tie with all the other candidates than to stay out of the competition, and prefers to stay out of the competition than to lose.

Definition 10.2: Sequential electoral competition game with a continuum of citizens and office-motivated politicians

A *sequential electoral competition game with a continuum of citizens and office-motivated politicians* $\langle F, n, \trianglerighteq \rangle$, where

- n is a positive integer (the number of politicians)
- F is a nonatomic distribution with support an interval of real numbers (the distribution of the citizens' favorite positions)
- $\trianglerighteq$ is a preference profile over $(\mathbb{R} \cup \{Out\})^n$ (the profile of the politicians' preferences over action profiles)

is the following extensive game with perfect information.

Players
The set $N = \{1, \ldots, n\}$ (of politicians).

Terminal histories
The set of sequences $(x_1, \ldots, x_n)$ where $x_i \in \mathbb{R} \cup \{Out\}$ for each $i \in N$.

Player function
The function P given by $P(\varnothing) = 1$ and $P(x_1, \ldots, x_k) = k + 1$ for $k = 1, \ldots, n - 1$ and any history $(x_1, \ldots, x_k)$.

Preferences
For each player j and terminal history x, let $w_j(x)$ be the number of candidates (including j) with whom j ties for first place, as defined in (10.6), with a value of 0 meaning that j is not one of the candidates tied for first place. The preference relation $\trianglerighteq_j$ of each player j over terminal histories satisfies $x \triangleright_j y$ if any of the following conditions holds:

a. $1 \le w_j(x) < w_j(y)$ (tying with fewer candidates is preferred)

b. $w_j(x) = n$ and $y_j = Out$ (tying with all the other candidates is preferred to staying out of the competition)

c. $x_j = Out$ and $w_j(y) = 0$ (staying out of the competition is preferred to entering and losing).

Proposition 10.2: Existence of SPE of sequential electoral competition game

Every sequential electoral competition game with a continuum of citizens and office-motivated politicians has a subgame perfect equilibrium.

Proof

Denote the number of players by n. Each player's preferences are represented by a payoff function that takes at most $n+2$ values (for winning outright, tying with k of the other players for $k = 1, \ldots, n-1$, losing, and staying out of the competition). Thus the result follows from Proposition 16.8.

Every game with two politicians has a unique subgame perfect equilibrium outcome, in which both politicians enter at the median of the distribution of the citizens' favorite positions. Thus the outcome is the same as the outcome of the unique Nash equilibrium of the game in which they move simultaneously (Proposition 8.4). (This result is closely related to Proposition 8.3, given Proposition 1.4.)

Proposition 10.3: SPE of sequential electoral competition game with two politicians

Every subgame perfect equilibrium of a sequential electoral competition game with a continuum of citizens and two office-motivated politicians $\langle F, 2, \trianglerighteq \rangle$ generates the terminal history in which each politician is a candidate whose position is the median of F.

Proof

Denote the median of F by m. If player 1 chooses m then player 2's only optimal response is m, because she loses if she enters at any other position. If player 1 chooses a position other than m, then player 2 wins outright by entering at m, so all of player 2's optimal actions lead her to win outright and hence player 1 loses. Player 1 prefers to tie with player 2 than not to enter, so her only optimal action in a subgame perfect equilibrium is to enter at m, leading player 2 to do the same.

Note that although the two-politician game has only one subgame perfect equilibrium outcome, it has many subgame perfect equilibria. In the subgame following politician 1's entry at any position x different from m, any position closer to m than x results in politician 2's winning, and hence is an optimal action for her. Thus any strategy pair in which politician 1 chooses m and politician 2's strategy s_2 satisfies $s_2(m) = m$ and $|s_2(x) - m| < |m - x|$ for all $x \neq m$ is a subgame perfect equilibrium.

Exercise 10.3: Sequential electoral competition with two policy-motivated politicians

Consider a variant of a sequential electoral competition game with a continuum of citizens and office-motivated politicians in which the politicians are policy-motivated. Specifically, for each politician i there is a single-peaked function $U_i : \mathbb{R} \to \mathbb{R}$, and i's payoff to a terminal history in which the positions $x^1, \ldots, x^l$ are tied for first place is the expected value of U_i when each x^j occurs with the same probability, $1/l$. Assume that if no politician enters then the outcome is an arbitrary position x_0. Denote by m the median of the distribution F of the citizens' favorite positions. Assume that the favorite position of at least one politician is less than m and the favorite position of at least one politician is greater than m. Consider the game for $n = 2$. Some subgames of this game do not have (exact) subgame perfect equilibria. For any small $\varepsilon > 0$, find a strategy profile for which politician 1's strategy is optimal and politician 2 cannot increase her payoff by more than ε by changing her action after any history.

For a game with three politicians, the equilibrium outcome is more interesting: the first and last politicians to move enter at the median of the distribution of the citizens' favorite positions and the second one stays out. Thus the outcome is consistent with a much-studied claim, often called "Duverger's Law", that elections using plurality rule tend to involve only two candidates.

Proposition 10.4: SPE of sequential electoral competition game with three politicians

Every subgame perfect equilibrium of a sequential electoral competition game with a continuum of citizens and three office-motivated politicians $\langle F, 3, \trianglerighteq \rangle$ generates the terminal history in which the first and third politicians become candidates at the median of F and the second politician chooses *Out*.

Proof

Denote the median of F by m.

First consider the subgame following player 1's entry at m. In this subgame, for every position of player 2 there is a position of player 3 such that player 3 wins outright:

- if player 2 enters at m, player 3 wins outright at any position sufficiently close to m
- if player 2 enters at a position different from m, player 3 wins at a position closer to m on the opposite side of m.

So in every subgame perfect equilibrium of the subgame, player 2 stays out and player 3 enters at m, and hence player 1 ties for first place.

Thus by entering at m, player 1 ensures that she ties with one other player. Can she do better, or even at least as well, by entering at a position different from m? I argue that she cannot: for any such position x_1, the subgame following player 1's entry at x_1 has no subgame perfect equilibrium in which player 1 either wins outright or ties with one other player.

- The subgame has no subgame perfect equilibrium outcome in which players 2 and 3 choose *Out* because if player 2 chooses *Out* then player 3 optimally becomes a candidate at m and wins outright.
- The subgame has no subgame perfect equilibrium outcome in which player 1 ties only with player 3. In such an outcome, the position of player 3 is either x_1, in which case player 3 can deviate to m and win outright, or $2m - x_1$, in which case player 3 can deviate slightly closer to m and win outright.
- The subgame has no subgame perfect equilibrium outcome in which player 1 ties only with player 2. In such an outcome, the position of player 2 is either x_1, in which case player 3 can enter at m and win outright, or $2m - x_1$. If it is $2m - x_1$, then player 3 must lose wherever she enters. That is, for every position x_3, her vote share must be less than the vote shares of players 1 and 2. But then for some $\varepsilon > 0$, if player 2 deviates ε closer to m, player 3's vote share remains less than the votes shares of players 1 and 2 wherever she enters, so that she optimally stays out, and hence player 2 wins outright.

We conclude that the game has a unique subgame perfect equilibrium outcome, in which players 1 and 3 are candidates at m and player 2 stays out.

Exercise 10.4: Sequential electoral competition with three policy-motivated politician

Consider the variant with policy-motivated politicians of a sequential electoral competition game with a continuum of citizens and office-motivated politicians in Exercise 10.3 in which there are three players. Denote the favorite position of player i by $\hat{x}_i$ and assume that $\hat{x}_1 < m < \hat{x}_2 = \hat{x}_3$ and $F(\hat{x}_2) < \frac{2}{3}$. What are the subgame perfect equilibrium outcomes of the game?

What if the number of politicians exceeds three? Based on partial analyses of games with four or more players in collaboration with Amoz Kats, I conjectured in the mid-1980s that for any finite number n of politicians the game $\langle F, n, \trianglerighteq \rangle$ has a unique subgame perfect equilibrium outcome, in which the first politician enters at the median of the citizens' favorite positions, the next $n-2$ politicians choose *Out*, and the last politician joins the first one at the median. This conjecture remains unproved and uncontradicted.

A variant of the model has a unique subgame perfect equilibrium for any number of politicians. Suppose that if two or more candidates are tied for first place, the one who entered first wins outright, rather than each of them winning with the same probability. Then given the assumption that each politician prefers to stay out than to enter and lose, in any equilibrium at most one politician enters. The following exercise asks you to determine which one does so.

Exercise 10.5: Sequential electoral competition game with priority for early entrants

Consider a variant of a sequential electoral competition game with a continuum of citizens and office-motivated politicians in which each politician's preferences are represented by a function that assigns 1 to every terminal history in which she is the politician with the smallest index among those tied for the highest number of votes, -1 to every other terminal history in which she enters, and 0 to every terminal history in which she chooses *Out*. Find the subgame perfect equilibrium outcome (outcomes?) of this game.

The next exercise invites you to study the subgame perfect equilibrium outcomes of a game with three politicians in which the winner is determined by plurality rule with a runoff, rather than ordinary plurality rule.

Exercise 10.6: Sequential electoral competition with three office-motivated politicians and a runoff

Consider the variant of a sequential electoral competition game with a continuum of citizens and three office-motivated politicians in which the winner of the election is determined by plurality rule with a runoff, as described in Exercise 10.1. Show that in every subgame perfect equilibrium all three players enter as candidates.

10.4 Citizen-candidates

In the models I present in this section, the set of politicians is the set of citizens: any citizen can run as a candidate. The models differ in two other respects from those in the previous sections. First, each citizen chooses only whether to run for election; she does not choose a position. If she runs and is elected, she implements her favorite policy, which the voters know. One motivation for this assumption is that each citizen is self-interested and there is no mechanism by which she can commit to a policy different from her favorite policy. Second, citizens care both about policy and about winning as a candidate.

Each citizen first chooses whether to run as a candidate, and then votes for a candidate (one of the citizens who chose to run). The candidate receiving the most votes wins and implements her favorite policy. If several candidates are tied for the most votes, each of them is selected with the same probability to be the winner.

Regarding the citizens' voting behavior, one option is to assume that each citizen votes ("sincerely") for the candidate whose position she likes best, as in Section 10.1. The resulting model is a strategic game in which each citizen chooses only whether to run as a candidate. Another option is to assume that voting is "strategic". If the citizens are perfectly informed, the resulting model is an extensive game with perfect information and simultaneous moves in which each citizen first chooses whether to run as a candidate, then chooses the candidate for whom to vote, as in the model in Section 10.2.

In both models, the set of positions is the set of real numbers and the set of players is the set $N = \{1, \ldots, n\}$ of citizens. The preferences of each citizen $i \in N$ regarding lotteries over positions are represented by the expected value of a function u_i. I assume that the payoff $u_i(z)$ of each citizen i for any position z depends only on the distance between z and i's favorite position, and the form of the relationship is the same for all individuals. Specifically, for some decreasing function $u : \mathbb{R}_+ \to \mathbb{R}_-$ with $u(0) = 0$, for each citizen i the function $u_i : \mathbb{R} \to$

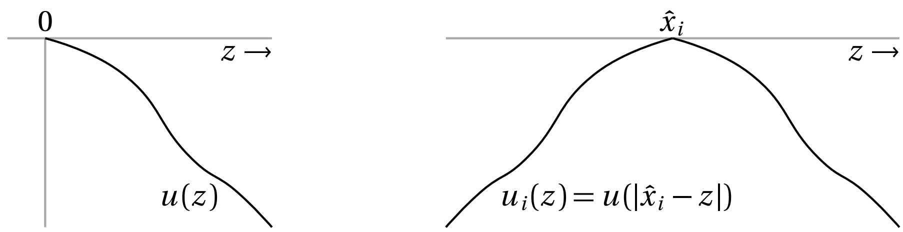


Figure 10.2 The Bernoulli payoff function of citizen i over positions in the citizen-candidate models in Section 10.4.

$\mathbb{R}_-$ is defined by $u_i(z) = u(|\hat{x}_i - z|)$ for all z, where $\hat{x}_i$ is i's favorite position, as illustrated in Figure 10.2.

A candidate is a winner if she obtains at least as many votes as every other candidate. In the model with sincere voting, a strategic game, each citizen votes for a candidate whose position is closest to her favorite position. If the positions of several candidates are equally close, the citizen's vote is divided equally among those candidates. (For simplicity, I allow fractional votes. You can alternatively imagine that a large number of citizens have any given favorite position z, and their votes are divided equally among the candidates whose positions are closest to z.) The actions available to each citizen are *Run* (become a candidate) and *Out*. Let $a = (a_1, \ldots, a_n)$ be an action profile for which $a_j = Run$ for at least one citizen $j \in N$. For any citizen $i \in N$, the set of candidates whose positions are closest to i's favorite position is

$$C(i,a) = \{j \in N : a_j = Run \text{ and } |\hat{x}_j - \hat{x}_i| \leq |\hat{x}_l - \hat{x}_i| \text{ for all } l \in N \text{ with } a_l = Run\}$$

and the fraction of i's vote that goes to any citizen j for whom $a_j = Run$ is

$$v_i(j,a) = \begin{cases} 1/|C(i,a)| & \text{if } j \in C(i,a) \\ 0 & \text{otherwise,} \end{cases}$$

so that the total number of votes obtained by citizen j is

$$V_j(a) = \sum_{i \in N} v_i(j,a)$$

and the set of winners of the election is

$$W(a) = \{i \in N : a_i = Run \text{ and } V_i(a) \geq V_j(a) \text{ for all } j \in N \text{ with } a_j = Run\}. \quad (10.2)$$

If $a_i = Out$ for all $i \in N$ (no citizen runs as a candidate), then $W(a) = \varnothing$.

In the model with strategic voting, an extensive game with perfect information and simultaneous moves, the citizens' strategies determine the number of votes for each candidate. The set of terminal histories consists of the action profile $(Out, \ldots, Out)$, in which no citizen runs as a candidate, and every sequence $((a_1, \ldots, a_n), (v_1, \ldots, v_n))$ for which $a_j \in \{Run, Out\}$ for each $j \in N$, $a_j = Run$ for at least one $j \in N$, and v_j is a citizen $i \in N$ for whom $a_i = Run$ (the candidate for whom j votes). For a terminal history (a, v), the set of winners of the election is the set of citizens who run as candidates and obtain at least as many votes as any other candidate:

$$\begin{aligned} W(a,v) = \{ & i \in N : a_i = Run \text{ and} \\ & |\{j \in N : v_j = i\}| \geq |\{j \in N : v_j = l\}| \text{ for all } l \in N\}. \end{aligned} \tag{10.3}$$

For the terminal history $a = (Out, \ldots, Out)$, let $W(a) = \varnothing$.

In both models, each citizen values the policy of the winner of the election. In addition, if she runs as a candidate she incurs a cost $c > 0$, and, if she wins election, obtains a benefit $b \geq 0$. These amounts appear as linear terms in her payoff. For example, it citizen i runs as a candidate and wins outright, her payoff is $u_i(\hat{x}_i) + b - c$ (which is equal to $b - c$ given that $u_i(\hat{x}_i) = u(0) = 0$). I assume that if no citizen runs as a candidate then a fixed position x_0 is realized.

Precisely, for any set $\mathcal{W} \subseteq N$ of winning candidates, the payoff of any citizen i is

$$V_i^O(\mathcal{W}) = \begin{cases} u_i(x_0) & \text{if } \mathcal{W} = \varnothing \\ \sum_{j \in \mathcal{W}} u_i(\hat{x}_j)/|\mathcal{W}| & \text{if } \mathcal{W} \neq \varnothing \end{cases} \tag{10.4}$$

if she does not run as a candidate and

$$V_i^R(\mathcal{W}) = \begin{cases} \sum_{j \in \mathcal{W}} u_i(\hat{x}_j)/|\mathcal{W}| - c & \text{if } i \notin \mathcal{W} \\ \left(\sum_{j \in \mathcal{W}} u_i(\hat{x}_j) + b\right)/|\mathcal{W}| - c & \text{if } i \in \mathcal{W} \end{cases} \tag{10.5}$$

if she does run (in which case $\mathcal{W} \neq \varnothing$).

Definition 10.3: Electoral competition game with citizen-candidates who vote sincerely

An *electoral competition game with citizen-candidates and sincere voting* $\langle n, (\hat{x}_1, \ldots, \hat{x}_n), u, b, c, x_0 \rangle$, where

- n is an odd positive integer (the number of citizens)
- $\hat{x}_i \in \mathbb{R}$ for $i = 1, \ldots, n$ (the favorite position of citizen i)
- $u : \mathbb{R}_+ \to \mathbb{R}_-$ is a decreasing function with $u(0) = 0$
- $b \in \mathbb{R}$ with $b \geq 0$ (the benefit from winning)
- $c \in \mathbb{R}$ with $c > 0$ (the cost of running as a candidate)

- $x_0 \in \mathbb{R}$ (the policy realized if no citizen runs as a candidate)

is a strategic game with the following components.

Players

The set $N = \{1, \ldots, n\}$ (citizens).

Actions

The set of actions of each citizen is $\{Run, Out\}$.

Preferences

The preferences of each citizen $i \in N$ over action profiles a are represented by the payoff function that assigns the payoff $V_i^O(W(a))$ if $a_i = Out$ and the payoff $V_i^R(W(a))$ if $a_i = Run$, where $W(a)$ is given by (10.2), V_i^O is given by (10.4), V_i^R is given by (10.5), and $u_i(z) = u(|\hat{x}_i - z|)$ for all z.

Definition 10.4: Electoral competition game with citizen-candidates who vote strategically

An *electoral competition game with citizen-candidates and strategic voting* $\langle n, (\hat{x}_1, \ldots, \hat{x}_n), u, b, c, x_0 \rangle$, where the variables have the same meanings as in Definition 10.3, is an extensive game with perfect information and simultaneous moves with the following components.

Players

The set $N = \{1, \ldots, n\}$ (citizens).

Terminal histories

The action profile $(Out, \ldots, Out)$ plus all sequences $((a_1, \ldots, a_n), (v_1, \ldots, v_n))$ with $a_j \in \{Run, Out\}$ for $j = 1, \ldots, n$, $a_j = Run$ for some $j \in N$, and $v_j \in \{i \in N : a_i = Run\}$ for $j = 1, \ldots, n$ (v_j is the citizen-candidate for whom j votes).

Player function

The function P with $P(\varnothing) = N$ and $P(a_1, \ldots, a_n) = N$ for all $(a_1, \ldots, a_n) \in \{Run, Out\}^n$ with $\{j \in N : a_j = Run\} \neq \varnothing$ (all citizens move simultaneously at the start of the game and again, as voters, after any profile of initial actions for which at least one citizen chooses to run).

Preferences

The preferences of each citizen $i \in N$ over terminal histories are represented by the payoff function that assigns to the terminal history $(Out, \ldots, Out)$ the payoff $V_i^O(\varnothing)$, and to any terminal history (a, v) with

$a_j = Run$ for some $j \in N$, the payoffs $V_i^O(W(a,v))$ if $a_i = Out$ and $V_i^R(W(a,v))$ if $a_i = Run$, where $W(a,v)$ is given by (10.3), V_i^O is given by (10.4), V_i^R is given by (10.5), and $u_i(z) = u(|\hat{x}_i - z|)$ for all z.

In the game with strategic voting, each voting subgame has many Nash equilibria (see Section 3.1). I restrict to equilibria in which each citizen's action in every voting subgame is weakly undominated.

Definition 10.5: Equilibrium of electoral competition game with citizen-candidates

In an electoral competition game with citizen-candidates and sincere voting, an *equilibrium* is a Nash equilibrium. In a game in which voting is strategic, an *equilibrium* is a subgame perfect equilibrium in which each citizen's vote in every subgame following the citizens' decisions to run as candidates is weakly undominated.

Suppose that no citizen enters as a candidate. Then the payoff of each citizen i is $u_i(x_0)$. If citizen i deviates to become a candidate, in both games she wins and obtains the payoff $b - c$. Thus an equilibrium in which no citizen enters as a candidate exists if and only if $b - c \leq u_i(x_0)$ for the citizen i whose favorite position is furthest from x_0. I now consider some more interesting equilibria.

One-candidate equilibria

For some ranges of the parameter values, both games have equilibria in which exactly one citizen runs as a candidate. In such equilibria, that citizen has to be no better off deviating to *Out*, and every other citizen has to be no better off deviating to *Run*. If the candidate's favorite position differs from the median m of the citizens' favorite positions then a citizen with favorite position m wins if she deviates to *Run*. (For the game with strategic voting, this conclusion follows from Corollary 3.1.) The entrant receives the payoff $b - c$, so in a one-candidate equilibrium the candidate's favorite position x satisfies $b - c \leq u(|m - x|)$, or equivalently $|m - x| \leq u^{-1}(b - c)$. In particular, if $b \geq c$ then no equilibrium with $x \neq m$ exists, because $u(z) < 0$ for every $z > 0$.

Under what conditions does a one-candidate equilibrium exist in which the candidate's position is m? The answers for the two games differ. Suppose that voting is sincere and another citizen's favorite position is m. If that citizen deviates to *Run* then she ties with the existing candidate and hence obtains the payoff $\frac{1}{2}b - c$ rather than 0. Consequently in this case a one-candidate equilibrium in which the candidate's position is m exists only if $b \leq 2c$. But if voting is strategic, the voting subgame following the entry of another citizen with favorite position m has an equilibrium in which the original candidate wins, because all citizens are indifferent between the her and the entrant. An entrant whose favorite position differs from m loses, so in this case a one-candidate equilibrium exists even if b is large; the only condition required is $u(|m - x_0|) \leq b - c$, so that the existing candidate does not prefer to deviate to *Out*.

Proposition 10.5: One-candidate equilibria of electoral competition game with citizen-candidates

Let $\langle n, (\hat{x}_1, \ldots, \hat{x}_n), u, b, c, x_0 \rangle$ be an electoral competition game with citizen-candidates and denote by m the median of the citizens' favorite positions.

a. Whether voting is sincere or strategic, in every equilibrium in which one citizen runs as a candidate, her position x is m if $b \geq c$ and satisfies $|m - x| \leq u^{-1}(b - c)$ if $b < c$.

b. If either voting is sincere and only one citizen has favorite position m or voting is strategic, the game has a one-candidate equilibrium in which the candidate's favorite position is m if and only if $u(|m - x_0|) \leq b - c$. If voting is sincere and more than one citizen has favorite position m, the game has such an equilibrium if and only if $u(|m - x_0|) \leq b - c$ and $b \leq 2c$.

Proof

Part *a* is proved in the text. To prove part *b*, first suppose that voting is sincere. Consider the action profile in which a citizen, say i, with favorite position m chooses *Run* and every other citizen chooses *Out*. Citizen i's payoff is $b - c$, and if she deviates to *Out* her payoff changes to $u(|m - x_0|)$. If a citizen whose favorite position differs from m deviates to *Run*, she loses, and hence is worse off. If another citizen with favorite position m deviates to *Run*, her payoff changes from 0 to $\frac{1}{2}b - c$. Thus if i is the only citizen with favorite position m, the action profile is an equilibrium if and only if $u(|m - x_0|) \leq b - c$, and if another citizen has favorite position m then it is an equilibrium if and only if $u(|m - x_0|) \leq b - c$ and $b \leq 2c$.

Now suppose that voting is strategic. Let s be a strategy profile in which a citizen, say i, with favorite position m runs as a candidate, every other citizen chooses *Out*, and the citizens vote as follows. In the subgame reached if the citizens adhere to s and in any subgame following the entry of i and another citizen with favorite position m, all citizens vote for i. In every other subgame, the citizens' vote profile is any Nash equilibrium of the subgame in which each citizen's action is weakly undominated, the existence of which is ensured by Proposition 4.2. (In particular, in any subgame in which two citizens run as candidates, by Corollary 3.1 each citizen who is not indifferent between the candidates votes for the candidate whose favorite position she prefers.) The outcome of s is a win for i, whose payoff is $b - c$.

The citizens' action profile in the subgame following the entry of i alone is the only one available to them. Their action profile in the subgame in which i and another citizen with favorite position m run as candidates is a Nash equilibrium in which each citizen's action is weakly undominated because no change in any citizen's vote affects the outcome. Their action profile in every other subgame is, by construction, a Nash equilibrium in which each citizen's action is weakly undominated.

Now consider changes in the citizens' actions at the start of the game. If i deviates to *Out*, her payoff changes from $b - c$ to $u(|m - x_0|)$, so she is no better off if $u(|m - x_0|) \le b - c$. If another citizen with favorite position m deviates to *Run*, she loses (because all the citizens continue to vote for i), and is thus worse off, and if a citizen with a favorite position different from m deviates to *Run* she also loses. Thus s is an equilibrium if $u(|m - x_0|) \le b - c$.

Now let s be a subgame perfect equilibrium in which each citizen's action in every voting subgame is weakly undominated, exactly one citizen, say i, runs as a candidate, and i's favorite position is m. Then i is no better off deviating from s_i to *Out* at the start of the game, so that $u(|m - x_0|) \le b - c$.

One-candidate equilibria of the game with strategic voting in which $b > 2c$ and the candidate's favorite position is m are vulnerable to uncertainty regarding the outcome of the election. In these equilibria, the deviation of another citizen to *Run* results in that candidate's losing. If the deviator's favorite position is m, then all citizens are indifferent between the candidates, and in the equilibrium a majority of them continue to vote for the original candidate. If the deviator's favorite position x differs from m, then every citizen votes for the candidate whose position she prefers (given the assumption that the citizens' voting strategies are weakly undominated), and hence the original candidate wins, because a majority of citizens prefer m to x. However, if x is close to m, the margin of victory of the original candidate is small. In fact, if only one citizen has favorite position m

and x is the closest favorite position to m, this margin of victory is exactly one vote.

Now suppose that with positive probability each citizen fails to vote, independently of every other citizen, and this probability is the same for every citizen. Then a deviator with favorite position $x \neq m$ wins with positive probability. Suppose specifically that only one citizen has favorite position m and x is the closest favorite position to m. Then the deviator wins if the number of citizens who vote among the $\frac{1}{2}(n-1)$ who prefer x to m exceeds the number who do so among the $\frac{1}{2}(n+1)$ who prefer m to x. The probability of this event is less than $\frac{1}{2}$, but if n is large it is close to $\frac{1}{2}$ (Eguia 2007, Theorem 1). Thus in this case if $b > 2c$, one-candidate equilibria do not exist in the model with strategic voting when the number of citizens is large, just as by part b of the result they do not exist when voting is sincere.

Agglomerated equilibria

Do the games have multi-candidate equilibria in which all the candidates' positions are the same? Neither game has such equilibria in which the common position differs from the median of the citizens' favorite positions. The reason is that such configurations are vulnerable to entry: an entrant whose favorite position is the median, for example, wins whether voting is sincere or strategic, because optimal strategic voting between two distinct alternatives is sincere (Corollary 3.1). The conclusions regarding equilibria in which the favorite position of every candidate is the median of the citizens' favorite positions differ between the games. Under sincere voting, if fewer than a third of the citizens' favorite positions are equal to the median then either an entrant whose favorite position is slightly less than the median or one whose favorite position is slightly greater than the median wins, because after her entry the votes of the citizens who prefer the median are split equally among the existing candidates. However, under strategic voting, the subgame following such entry has an equilibrium in which the votes of all the citizens who prefer the median go to one specific candidate among those whose favorite position is the median, so that the entrant loses. Thus if voting is strategic the game has an equilibrium in which two or more citizens run as candidates and all of their favorite positions are the median, but if voting is sincere and fewer than a third of the citizens' favorite positions are equal to the median then it has no such equilibrium.

Proposition 10.6: Agglomerated equilibria in electoral competition game with citizen-candidates

Let $\langle n, (\hat{x}_1, \ldots, \hat{x}_n), u, b, c, x_0 \rangle$ be an electoral competition game with citizen-candidates and denote by m the median of the citizens' favorite positions.

a. The game has no equilibrium in which two or more citizens run as candidates and all of their favorite positions are the same, different from m.

b. If voting is sincere and fewer than a third of the citizens have favorite positions of m then the game has no equilibrium in which two or more citizens run as candidates and all of their favorite positions are m.

c. For any $k \geq 2$, if voting is strategic, $b \geq kc$, and at least k citizens have favorite position m, then the game has an equilibrium in which k citizens run as candidates and all of their favorite positions are m.

Proof

Consider an action profile in which $k \geq 2$ citizens with favorite position z run as candidates and every other citizen chooses *Out*. If voting is sincere, the candidates tie. If voting is strategic, the action profile is consistent with equilibrium only if the candidates tie, because a candidate who loses is better off deviating to *Out*. In both cases each candidate's payoff is $b/k - c$. If a candidate deviates to *Out*, her payoff becomes 0, so for equilibrium we need $b \geq kc$.

a. If $z \neq m$ then whether voting is sincere or strategic, a citizen with favorite position m wins if she deviates to *Run*. (If voting is strategic, this conclusion follows from Corollary 3.1.) Her payoff is thus $b - c$ rather than $u(|m - z|) < 0$, so that for equilibrium we need $b < c$. This condition is inconsistent with $b \geq kc$, so no equilibrium exists in which $k \geq 2$ citizens with favorite position $z \neq m$ run as candidates and every other citizen chooses *Out*.

b. Now suppose that $z = m$ and voting is sincere. If fewer than a third of the citizens have favorite positions of m, then either more than a third of them have favorite positions less than m or more than a third have favorite positions greater than m (or both). The two cases are symmetric; suppose the former. Let j be a citizen whose favorite position is largest among the citizens with favorite positions less than m. If j deviates to *Run*, she obtains the votes of all citizens with favorite positions less than m, whereas each of the $k \geq 2$ original candidates gets an equal share of the votes of the remaining citizens. Thus j wins and obtains the payoff $b - c$ rather than $u(|m - \hat{x}_j|) < 0$. So for the original action profile with k candidates to be an equilibrium we need $b < c$, which is inconsistent with the requirement $b \geq kc$.

c. If $z = m$ and voting is strategic, the subgame following the deviation of any citizen j from *Out* to *Run* has an equilibrium in weakly undominated strategies in which all of the citizens who prefer m to j's position vote for the same candidate with position m, so that j loses. Thus if $b \geq kc$ and at least k citizens have favorite position m, the game has an equilibrium in which the candidates are k citizens with favorite position m.

The equilibria of the game with strategic voting in part *c* are vulnerable to uncertainty for the same reason that the one-candidate equilibria for $b > 2c$ in the previous result are vulnerable. Suppose as before that each citizen independently fails to vote with the same probability. Then if a citizen with favorite position x close to m deviates to *Run*, she wins with positive probability, making her deviation worthwhile if b is large enough.

Dispersed equilibria

Equilibria in which the candidates' positions are dispersed are possible in both models. In such an equilibrium with two candidates, the outcome must be a tie, because otherwise the loser can deviate to *Out* without affecting the outcome, saving the cost c. In a two-alternative voting subgame of the game with strategic voting, the only weakly undominated action of a citizen who is not indifferent between the alternatives is to vote sincerely, for her preferred candidate, so whether voting is sincere or strategic the candidates' equilibrium positions are symmetric about the median m of the citizens' favorite positions, say $m - \delta$ and $m + \delta$ for some $\delta > 0$. If a candidate deviates to *Out*, the other candidate wins, so for each value of the entry cost c there is a lower bound on δ for which an equilibrium of this type may exist.

To investigate the possibility of such an equilibrium further, consider deviations by citizens from *Out* to *Run*. First suppose that a citizen whose favorite position is outside $(m-\delta, m+\delta)$ deviates to *Run*. If voting is sincere, she loses. If voting is strategic, the resulting voting subgame has a Nash equilibrium in which every citizen votes as she would in the absence of the entrant, and in this case too the entrant loses. In this equilibrium no citizen votes for her least preferred candidate, so if the number of citizens is at least four then by Proposition 4.1*b* no citizen's action is weakly dominated.

Now consider the deviation to *Run* for a citizen whose favorite position is in $(m-\delta, m+\delta)$. If voting is sincere, then if δ is large enough such an entrant surely wins, and if she wins she may be better off. Part *b* of the next result gives a condition for such an entrant not to win, so that the configuration is an equilibrium. If voting is strategic, then no matter how large is δ, as long as the number of citizens with favorite position m is not too large the voting subgame following the entry of a candidate between $m-\delta$ and $m+\delta$ has an equilibrium in which the entrant loses. In this equilibrium, the citizens whose favorite positions are m vote for the entrant and every other citizen votes for the remaining candidate whom she prefers; no citizen votes for her least preferred candidate, so no citizen's action is weakly dominated. Thus in this case the model with strategic voting, unlike the one with sincere voting, has equilibria in which the separation between the candidates' positions is arbitrarily large.

Proposition 10.7: Two-candidate equilibria of electoral competition game with citizen-candidates

Let $\langle n, (\hat{x}_1, \ldots, \hat{x}_n), u, b, c, x_0 \rangle$ be an electoral competition game with citizen-candidates and denote the median of the citizens' favorite positions by m.

a. Whether voting is sincere or strategic, in any two-candidate equilibrium in which the candidates' positions differ, these positions are $m-\delta$ and $m+\delta$ for some $\delta > 0$ and either (*i*) $b > 2c$ or (*ii*) $b \le 2c$ and $\delta \ge \frac{1}{2}u^{-1}(b-2c)$.

b. Suppose that voting is sincere and the citizens' favorite positions are equally-spaced: for some $\Delta > 0$ we have $\hat{x}_i - \hat{x}_{i-1} = \Delta$ for $i = 2, \ldots, n$. If p is a positive integer such that (*i*) the conditions in part *a* are satisfied for $\delta = p\Delta$ and (*ii*) fewer than a third of the citizens' favorite positions are in $[m - \frac{1}{2}p\Delta, m + \frac{1}{2}p\Delta]$, then the game has a two-candidate equilibrium in which the candidates' positions are $m - p\Delta$ and $m + p\Delta$.

c. If voting is strategic and fewer than $\frac{1}{3}(n-4)$ of the citizens' favorite positions are m (which requires $n \ge 7$), then for every value of δ satisfying the conditions in part *a* for which citizens exist with favorite positions $m-\delta$ and $m+\delta$, the game has a two-candidate equilibrium in which the candidates' positions are $m-\delta$ and $m+\delta$.

Proof

a. The argument in the text shows that the candidates' positions in such an equilibrium are $m - \delta$ and $m + \delta$. (Note that while such positions are necessary for a tie, they are not sufficient if voting is sincere and m is the favorite position of more than one citizen.) The payoff of each candidate is $\frac{1}{2}b - c + \frac{1}{2}u(2\delta)$. If she deviates to *Out*, her payoff becomes $u(2\delta)$, so her entry is optimal if and only if $\frac{1}{2}b - c + \frac{1}{2}u(2\delta) \geq u(2\delta)$, or $b - 2c \geq u(2\delta)$, establishing part *a*.

b. Under condition (*i*), neither candidate is better off deviating to *Out*. Under condition (*ii*), any citizen who deviates from *Out* to *Run* loses and her entry either does not affect the outcome or changes it from a tie between the two existing candidates to a win for the one she likes less. Thus she is worse off.

c. Let s be a strategy profile in which two citizens, i with favorite position $m-\delta$ and j with favorite position $m+\delta$, run as candidates, every other citizen chooses *Out*, and the citizens vote in each subgame as follows. In each case I argue that the profile of votes is a Nash equilibrium of the subgame in which no citizen's action is weakly dominated.

Subgame that results if citizens adhere to s

Citizens with favorite positions less than m vote for i, citizens with favorite positions greater than m vote for j, and the votes of citizens with favorite position m (who are indifferent between the candidates) are split equally between i and j, so that the candidates tie.

This action profile is a Nash equilibrium in which every citizen with favorite position different from m votes for the candidate she prefers. Thus by Corollary 3.1 every citizen's action is weakly undominated.

Subgame in which one citizen runs as a candidate

Every citizen votes for the candidate (she has no choice).

Subgame in which candidates are i, j, and a citizen l with favorite position in $(m-\delta, m+\delta)$

Every citizen whose favorite position is m votes for l and every other citizen (including l) votes for i if her favorite position is less than m and for j if her favorite position is greater than m.

Denote by n_m the number of citizens with favorite position m. Then l receives n_m votes, and i and j each receives $\frac{1}{2}(n - n_m)$ votes. Thus given that $n_m < \frac{1}{3}(n-4)$, i and j tie for first place and each receives at least two more votes than does l. Hence no change in any citizen's vote causes l to win; every change in a citizen's vote either does not affect the outcome or causes it to deteriorate for the citizen. No citizen is voting for her least preferred candidate, so by Proposition 4.1*b* every citizen's action is weakly undominated.

Subgame in which candidates are i, j, and a citizen l with favorite position outside $(m-\delta, m+\delta)$

Every citizen votes for i if her favorite position is less than m and for j if her favorite position is greater than m. The votes of citizens whose favorite positions are m are split equally between i and j.

Any change in the vote of a citizen with favorite position m does not affect her payoff, while any change in the vote of any other citizen causes the winner to become her less preferred member of $\{i, j\}$. No citizen is voting for her least preferred candidate, so by Proposition 4.1*b* every citizen's action is weakly undominated.

Other subgames

Every citizen votes according to an arbitrary Nash equilibrium of the subgame in which every citizen's vote is weakly undominated. (Such a Nash equilibrium exists by Proposition 4.2.)

I now argue that no citizen can increase her payoff by deviating at the start of the game. By the argument for part *a*, neither candidate can increase her payoff by deviating to *Out*. Suppose that a citizen who is not a candidate deviates to *Run*. Given the equilibrium of the resulting voting subgame, such an entrant loses, so that the deviation makes her worse off.

Suppose that δ is large enough that a candidate with favorite position m beats candidates with favorite positions $m-\delta$ and $m+\delta$ if every citizen votes sincerely. Then the equilibrium in part *c* may be vulnerable to a deviation to *Run* by a citizen with favorite position m if the notion of equilibrium is modified to take into account deviations not only in the vote of a single citizen but also in the votes of groups of citizens.

In the model with sincere voting, equilibria with many dispersed candidates are also possible.

Exercise 10.7: Multi-candidate dispersed equilibria of electoral competition game with citizen-candidates and sincere voting

Let $\langle n, (\hat{x}_1, \ldots, \hat{x}_n), u, b, c, x_0 \rangle$ be an electoral electoral competition game with citizen-candidates and sincere voting. Under the assumption in part b of Proposition 10.7 that the citizens' favorite positions are equally-spaced, show that for any integer k with $3 \leq k \leq n$ and n divisible by k, if $b/k - c$ is large enough the game has a Nash equilibrium in which k citizens run as candidates.

If voting is strategic, Proposition 4.3a implies that equilibria in which winning candidates occupy more than two distinct positions are not possible if the payoff function u is strictly concave. However, if the cost c of running as a candidate is sufficiently small, equilibria with any number of candidates, two of whom win, exist. In these equilibria, no losing candidate is better off exiting because if she does so all citizens vote for the winner she likes least, causing that candidate to win outright. The citizens have no positive incentive to switch their votes in this way, but doing so is consistent with equilibrium and no citizen's action is weakly dominated. The next exercise invites you to establish this result.

Exercise 10.8: Multi-candidate dispersed equilibria of electoral competition game with citizen-candidates and strategic voting

Let $\langle n, (\hat{x}_1, \ldots, \hat{x}_n), u, b, c, x_0 \rangle$ be an electoral electoral competition game with citizen-candidates and strategic voting. Denote by m the median of the citizens' favorite positions. Show that for any integer $k \geq 4$, any number $\delta > 0$, and any positions x_1, x_2, $\ldots$, x_k with $x_1 = m - \delta$, $x_2 = m + \delta$, $x_i < x_1$ for some $i \geq 3$, $x_i > x_2$ for some $i \geq 3$, and $x_i \neq m$ for $i = 1, \ldots, k$, there exists $\underline{c} > 0$ such that if $c \leq \underline{c}$ then the game has an equilibrium in which k citizens, with positions x_1, x_2, $\ldots$, x_k, run as candidates, candidates 1 and 2 tie for first place, and all other candidates lose.

Comments

The character of the equilibria appears to survive if the set of positions is multi-dimensional. For example, if voting is sincere then a two-candidate equilibrium in which the candidates' positions differ exists if the difference is enough that neither candidate prefers to withdraw but not so much that an entrant can win.

If voting is strategic, such an equilibrium exists if the first of these two conditions holds, with the entry of a candidate leading all citizens to vote for the existing candidate the entrant likes less.

The citizen-candidate model has the merit of yielding tractable multi-candidate equilibria. The price is the assumption that each citizen is limited to running on her favorite position or not becoming a candidate—she cannot choose her position. An environment in which that restriction may be inappropriate is one in which candidates face a sequence of elections. In that case, a candidate may be able to credibly select a position different from her favorite position, knowing that voters can punish her in future elections if she deviates from the position while in office.

Appendix

The outcome of a profile of the politicians' actions in the models in Sections 10.1 and 10.3 is defined precisely as follows. Denote the set of politicians by $N = \{1, 2, \ldots, n\}$ and for any profile x of the politicians' actions let $C(x)$ be the set of politicians who become candidates by choosing positions:

$$C(x) = \{j \in N : x_j \in \mathbb{R}\}.$$

For any profile x, denote the *distinct* positions chosen by $z_1(x), \ldots, z_{k(x)}(x)$, with $z_1(x) < z_2(x) < \cdots < z_{k(x)}(x)$, and let $z(x) = (z_1(x), \ldots, z_{k(x)}(x))$. If, for example, $x = (Out, 1, -2, Out, -2, 0)$ then $C(x) = \{2, 3, 5, 6\}$ and $z(x) = (-2, 0, 1)$.

For any list y of distinct positions with $y_1 < y_2 < \cdots < y_p$, a citizen votes for one of the candidates with position y_1 if her favorite position is at most $\frac{1}{2}(y_1 + y_2)$, for one of the candidates with position y_l with $2 \le l \le p - 1$ if her favorite position is in $(\frac{1}{2}(y_{l-1} + y_l), \frac{1}{2}(y_l + y_{l+1})]$, and for one of the candidates with position y_p if her favorite position is greater than $\frac{1}{2}(y_{p-1} + y_p)$. (The first two cases are illustrated in Figure 10.3.) Thus the fraction $s_l(y)$ of citizens who vote for one of the candidates with position y_l is

$$s_l(y) = \begin{cases} F\big(\frac{1}{2}(y_1 + y_2)\big) & \text{if } l = 1 \\ F\big(\frac{1}{2}(y_l + y_{l+1})\big) - F\big(\frac{1}{2}(y_{l-1} + y_l)\big) & \text{if } 2 \le l \le p - 1 \\ 1 - F\big(\frac{1}{2}(y_{p-1} + y_p)\big) & \text{if } l = p, \end{cases}$$

where F is the distribution function of the citizens' favorite positions.

The votes that each position attracts are split equally among the candidates who choose that position, so that if politician j is a candidate then the fraction

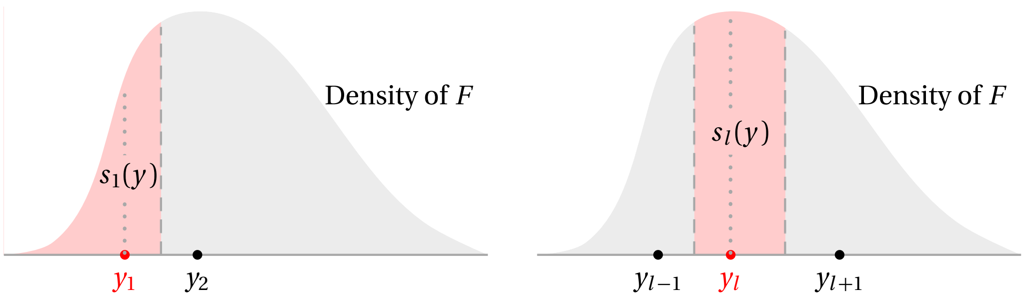


Figure 10.3 The area shaded pink is the fraction $s_l(y)$ of citizens who vote for one of the candidates whose position is y_l for $l = 1$ (left panel) and $2 \le l \le p-1$ (right panel).

of citizens who vote for her is

$$v_j(x) = \frac{s_l(z(x))}{|\{q \in C(x) : x_q = x_j\}|} \quad \text{where } l \text{ satisfies } x_j = z_l(x).$$

Finally, the number $w_j(x)$ of candidates who tie with j for first place, including j herself, is

$$w_j(x) = \begin{cases} 0 & \text{if } v_q(x) > v_j(x) \text{ for some } q \\ |\{q \in C(x) : v_q(x) = v_j(x)\}| & \text{otherwise,} \end{cases} \tag{10.6}$$

where $w_j(x) = 0$ means that j is not one of the winning candidates.

Notes

Section 10.1 is based on Osborne (1993). The model and results in Section 10.2 are due to Feddersen et al. (1990). The model in Section 10.4 was developed independently by Osborne and Slivinski (1996) (sincere voting) and Besley and Coate (1997) (strategic voting).

The much-discussed claim mentioned before Proposition 10.4 that plurality rule tends to lead to two-candidate electoral competitions appears to have been first stated explicitly in print by Droop (1881, 164) (see Riker 1982, 756). Riker writes (p. 754) "It is customary to call the law by Duverger's name, not because he had much to do with developing it but rather because he was the first to dare to claim it was a law." It appears in Duverger (1951, 247) (and in translation in Duverger 1964, 217).

Exercise 10.1 is based on Haan and Volkerink (2001) and Brusco et al. (2012). The model and result in Exercise 10.5 are due to Jeffrey S. Rosenthal and Phillip Morenz (personal communication, 1992).

Solutions to exercises

Exercise 10.1

Denote the option of not becoming a candidate by *Out*.

If all n politicians become candidates and choose the position m, each of them wins with probability $1/n$. If one deviates to *Out* she wins with probability 0, and if she deviates to a position different from m and gets into the second round with positive probability, then whenever she gets to the second round she loses (her opponent's position is m). Thus no deviation makes her better off, so that the action profile is a Nash equilibrium.

If k politicians choose the position m, with $k < n$, and the remainder choose *Out*, then any candidate who deviates from *Out* to the position m changes her probability of winning from 0 to $1/(k+1)$. Thus this action profile is not a Nash equilibrium.

Now let $\delta > 0$ and $1 \le k \le n$, and denote by $a(\delta, k)$ the action profile in which k politicians choose the position $m - \delta$, k choose $m + \delta$, and the remainder choose *Out*. For this action profile each candidate wins with probability $1/(2k)$.

If $k = 1$ then either candidate can deviate to m and win with probability 1, so $a(\delta, 1)$ is not a Nash equilibrium for any value of δ.

Now suppose that $k \ge 2$.

- First consider deviations by a candidate. If she deviates to *Out* then she wins with probability 0 and if she deviates to a position less than $m - \delta$ or greater than $m + \delta$ then she either does not make it to the second round or, if she does, loses in the second round. If she deviates to a position between $m - \delta$ and $m + \delta$ and δ is small enough that her vote share is less than the vote share of each remaining candidate, then she does not make it into the second round.
- Now consider deviations by a politician who is choosing *Out*. As in the previous point, if δ is small enough that when she enters as a candidate at any position between $m - \delta$ and $m + \delta$ her vote share is less than the vote share of each remaining candidate, then she does not make it into the second round.

We conclude that if $k \ge 2$ and δ is small enough that when a candidate deviates from $a(\delta, k)$ to a position between $m - \delta$ and $m + \delta$ her vote share is less than the vote share of every remaining candidate, then $a(\delta, k)$ is a Nash equilibrium.

For $k \geq 3$, if a candidate at $m-\delta$ deviates to a position between $m-\delta$ and $m+\delta$ then she gets into the second round only if she beats all the candidates remaining at $m-\delta$, because there are at least two such candidates. She does not beat them if the vote share of every position between $m-\delta$ and $m+\delta$ is at most $1/(2k)$, because then the vote share of each candidate at $m+\delta$ is less than $1/(2k)$ and hence the vote share of each candidate remaining at $m-\delta$ exceeds $1/(2k)$. A similar argument applies to a candidate at $m+\delta$ and to a politician choosing *Out*.

Thus for $k \geq 3$, if δ is small enough that when a candidate deviates from $a(\delta, k)$ to a position between $m-\delta$ and $m+\delta$ her vote share is less than $1/(2k)$, then $a(\delta, k)$ is a Nash equilibrium.

Exercise 10.2

Denote by m the median of the citizens' favorite positions. If a politician j has favorite position m and $U_j(m)+b-c \geq -D$ then the game has a subgame perfect equilibrium in which j is a candidate with position m, every other politician chooses *Out*, all citizens vote for j in every subgame in which she is the only candidate or every candidate's position is m, each citizen votes for the candidate she prefers in every subgame in which there are two candidates, and in any other subgame the profile of the citizens' actions is any Nash equilibrium in which the citizens' actions are weakly undominated (the existence of which is ensured by Proposition 4.2). For this strategy profile, j's payoff is $U_j(m)+b-c$ and the payoff of every other politician l is $U_l(m)$. The strategy profile is a subgame perfect equilibrium because if j deviates to another position she still wins, and is worse off; if she deviates to *Out* she gets the payoff $-D$; if another politician enters at m, she loses (all citizens continue to vote for j); and if another politician enters at a position other than m she loses.

If the favorite position of at least one politician is less than m, the favorite position of at least one politician is greater than m, and $b \geq c$, then the game has no subgame equilibrium with one candidate in which the candidate's position differs from m. To see why, let (x, v) be a strategy profile in which $x_j < m$ and $x_i = Out$ for all $i \neq j$. If a politician l with favorite position greater than m deviates to enter at m, she wins (because the only undominated action of each citizen in the resulting subgame is to vote for the candidate she prefers) and obtains the payoff $U_l(m)+b-c$, which exceeds $U_l(x_j)$.

The subgame perfect equilibria with two or more candidates who choose the position m are the same as the equilibria in the model with office-motivated candidates, and exist under the same conditions. In such an equilibrium with

k candidates, the payoff of candidate j is $U_j(m)+b/k-c$; if she deviates to *Out*, her payoff becomes $U_j(m)$, so that for equilibrium we need $k \leq b/c$. The payoff of a politician, say l, who is not a candidate is $U_l(m)$; if she deviates to become a candidate at m then the citizens all vote for one of the existing candidates and l loses, so that her payoff becomes $U_l(m)-c$.

Like the game with purely office-motivated candidates, if every citizen's payoff function is strictly concave and has a maximizer then the game appears to have no equilibrium with more than one position occupied. The logic in the proof of Proposition 10.1 rules out equilibria in which the candidates tie, and equilibria in which they do not tie are not possible because a citizen voting for a losing candidate benefits by shifting her vote to the winning candidate whom she likes best, so that losing candidates attract no votes and hence do not affect the outcome.

Exercise 10.3

For each player j, denote by $\hat{x}_j$ her favorite position.

In the subgame following player 1's choice of *Out*, if $x_0 \neq \hat{x}_2$ then the optimal action of player 2 is to choose $\hat{x}_2$ (and win), and if $x_0 = \hat{x}_2$ then $\hat{x}_2$ and *Out* are both optimal choices for her. In each case the outcome is $\hat{x}_2$.

Now consider the subgame following the entry of player 1 at m. If player 2 enters at m, the outcome is m. If player 2 enters at another position, the outcome is also m, because player 2 loses. Thus entering at m and *Out* are both optimal actions for player 2.

Now consider the subgame following the entry of player 1 at some position $x_1 < m$.

First suppose that player 1's favorite position, $\hat{x}_1$, is less than m. Then by assumption player 2's favorite position, $\hat{x}_2$, is greater than m.

- If $\hat{x}_2 - m < m - x_1$ then by entering at $\hat{x}_2$ player 2 wins and makes the policy outcome $\hat{x}_2$; she can do no better than that, so it is her optimal action.
- If $\hat{x}_2 - m \geq m - x_1$ then if player 2 enters at any position x_2 with $x_1 < x_2 < 2m - x_1$ she wins and induces the outcome x_2, which she likes better the closer x_2 is to $2m - x_1$. If she enters at $2m - x_1$ itself, she ties with player 1, which is worse for her than the outcome $2m - x_1$, and if she enters at a position greater than $2m - x_1$ she loses. Thus she has no optimal action, but for any $\varepsilon > 0$ there exists $\delta > 0$ such that no action yields her a payoff greater by more than ε than her payoff for any action in $[2m - x_1 - \delta, 2m - x_1)$.

Thus for any value of $\hat{x}_2$, player 2's optimal or approximately optimal action in response to player 1's entry at x_1 generates an outcome $\min\{\hat{x}_2, 2m - x_1 - \delta\}$ for some small $\delta > 0$, which player 1 likes less than m.

Now suppose that $\hat{x}_1 > m$, so that $\hat{x}_2 < m$. If $\hat{x}_2 \leq x_1$ then every position $x_2 \leq x_1$ is optimal for player 2; in each case, the policy outcome is x_1, and no position for player 2 generates a policy outcome closer to $\hat{x}_2$, because any position less than x_1 leads player 1 to win. If $\hat{x}_2 > x_1$ then player 2's optimal position is $\hat{x}_2$, which leads her to win, so that the policy outcome is $\hat{x}_2$.

We conclude that the action *Out* and every position $x_1 < m$ for player 1 causes every optimal or approximately optimal action of player 2 to generate an outcome that is worse for player 1 than m. A symmetric argument applies if $x_1 > m$. Thus player 1's optimal action is m; player 2 responds optimally by either also choosing m or by choosing *Out*. Hence every approximate subgame perfect equilibrium generates either the outcome in which both players choose m or the outcome in which player 1 chooses m and player 2 chooses *Out*.

Exercise 10.4

Suppose that player 1 enters at a position $x_1 < \hat{x}_2$. If player 2 enters at the same position then player 3 wins outright if she enters at $\hat{x}_2$ (she gets more than a third of the vote and players 1 and 2 split the remainder equally), generating the best possible outcome for both her and player 2. (Other positions for player 2 may have the same implications.) Thus every subgame perfect equilibrium of the subgame following x_1 generates the policy outcome $\hat{x}_2$.

If player 1 enters at the position $\hat{x}_2$ then players 2 and 3 can do no better than stay out, and if player 1 enters at a position $x_1 > \hat{x}_2$ then by entering at $\hat{x}_2$ player 2 ensures that the policy outcome is $\hat{x}_2$ (player 3 optimally either stays out or, if entering at $\hat{x}_2$ causes her to tie for first place with player 2, enters at that position).

If player 1 stays out, then by entering at $\hat{x}_2$ player 2 ensures that the policy outcome is $\hat{x}_2$; player 3 either enters at the same position or stays out.

Thus the policy outcome of every subgame perfect equilibrium is $\hat{x}_2$. In an equilibrium, player 1 either stays out or enters. If she enters, player 3 enters at $\hat{x}_2$ and if player 3 would not win outright unless player 2 entered, then player 2 enters at a position that causes player 3 to win outright.

Exercise 10.5

I begin by establishing that a player does not enter if, in the event no subsequent player enters, she would lose.

Claim *For any nonterminal history h, in any subgame perfect equilibrium of the subgame following h, the player who moves first chooses a position (rather than Out) only if she wins when every subsequent player (if any) chooses Out.*

Proof. I prove the claim by induction on the length of a history.

The claim is true for any history of length $n-1$, because after n moves the game ends, and n prefers *Out* to losing.

Now suppose that the claim is true for every history of length at least r, where $1 \leq r \leq n-2$. Consider a history h of length $r-1$. Suppose, contrary to the claim, that the subgame following h has a subgame perfect equilibrium in which player r chooses a position such that she loses if every subsequent player chooses *Out*. Then in this equilibrium, at least one subsequent player must enter (otherwise player r's entry is not optimal for her). Take the last player to do so, say player t. By the claim for histories of length $t-1$, player t wins. But then player r loses, so that the strategy profile is not a subgame perfect equilibrium. Thus the claim is true for every history of length $r-1$, and hence by induction for every history of any length. ◁

Now, if player 1 enters at m, no subsequent player can enter and win if no further players enter, because no position garners more votes than m. So by the claim, no subsequent player enters and hence player 1 wins. Thus the game has a subgame perfect equilibrium with the outcome in which player 1 enters at m and every other player chooses *Out*.

If player 1 enters at a position different from m then by entering at m player 2 ensures, by the claim, that no subsequent player enters, so that player 2 wins and player 1 loses.

Thus the game has a unique subgame perfect equilibrium outcome, in which player 1 enters at m and every other player chooses *Out*.

Exercise 10.6

I break the argument into steps.

Step 1 *In every subgame perfect equilibrium of the subgame following the entry of player* 1 *at m, players* 2 *and* 3 *enter at m.*

Proof. Suppose that player 1 enters at m. Then if player 2 enters at m, player 3's best action is to enter at m also, because her entry at another position either does not get her into a runoff or gets her into a runoff that she loses. Thus if player 2 enters at m then she wins with probability $\frac{1}{3}$, so that in every subgame perfect equilibrium of the subgame following player 1's entry at

m, player 2 enters. Suppose that she enters at a position $x_2 < m$. If her vote share in the event that player 3 does not enter is positive (that is, her position is not extreme), then for some $\varepsilon > 0$ player 3 gets into the runoff by entering at $m + \varepsilon$ and wins outright. If player 2's vote share in the event that player 3 does not enter is zero, then player 3's best action is to enter at m, in which case players 1 and 3 each win with probability $\frac{1}{2}$ and player 2 does not win. Thus for every position at which player 2 enters other than m, she ultimately loses.

Hence player 2's best action in the subgame following player 1's entry at m is to enter at m, in which case player 3 optimally enters at m and each of the three players wins with probability $\frac{1}{3}$. ◁

Step 2 *In every subgame perfect equilibrium of the game, player* 1 *enters and wins with probability at least* $\frac{1}{3}$.

Proof. By Step 1, if player 1 enters at m she wins with probability $\frac{1}{3}$, so in every subgame perfect equilibrium of the game she wins with probability at least $\frac{1}{3}$, and hence enters. ◁

Step 3 *In every subgame perfect equilibrium of the game, player* 2 *enters.*

Proof. Let s^* be a subgame perfect equilibrium strategy profile in which player 2 does not enter in the subgame following player 1's action s_1^*. By Step 2, s_1^* is entry at some position, so player 3's optimal action in the subgame following the history in which players 1 and 2 follow their strategies in s^* (and hence player 2 does not enter) is to enter: if player 1 enters at m then player 3's optimal action is to enter at m (at any other position she loses), and if player 1 enters at a position different from m then all of player 3's optimal actions, among which is entry at m, cause her to win outright. By Step 2, in any subgame perfect equilibrium player 1 wins with positive probability, so in s^* she must enter at m. That is, if the game has a subgame perfect equilibrium in which player 2 does not enter, player 1 enters at m. But by Step 1, if player 1 enters at m then players 2 and 3 enter at m, so the game has no subgame perfect equilibrium in which player 2 does not enter. ◁

Step 4 *In every subgame perfect equilibrium of the game, player* 3 *enters.*

Proof. By Step 3, in any subgame perfect equilibrium player 2 enters. By entering at the same position as player 2, player 3 ultimately wins with positive probability: she ties with player 2 in the first round and thus gets into the runoff with positive probability, and in the runoff she wins with the same probability as does player 2, which is positive because player 2 optimally enters. Thus in every subgame perfect equilibrium player 3 enters. ◁

Exercise 10.7

For any integer k with $3 \leq k \leq n$ and n divisible by k there is a set K of k citizens whose vote shares are equal when the set of candidates is K. If any citizen j deviates from *Out* to *Run* then her vote share becomes less than $1/k$ because the fraction of citizens with favorite positions between any two candidates is less than $1/k$. At least one other candidate's vote share remains $1/k$, so j loses. Her entry causes the vote shares of the candidate closest to her on the left (if any) and the one closest on the right (if any) to fall, so that they lose. Thus j's entry makes her worse off.

Now suppose that a candidate, citizen i, deviates from *Run* to *Out*. If her favorite position is between the favorite positions of two other candidates, then the outcome changes to a tie between these two candidates, and otherwise it changes to an outright win for the one candidate adjacent to her. Thus i's payoff changes from $b/k + \sum_{j \in K} u_i(\hat{x}_j)/k - c$ to at most $\max_{j \in K \setminus \{i\}} u_i(\hat{x}_j)$. Hence a sufficient condition for no candidate to be better off deviating to *Out* is

$$b/k - c \geq \max_{i \in K} \left(\max_{j \in K \setminus \{i\}} u_i(\hat{x}_j) - \sum_{j \in K} u_i(\hat{x}_j)/k \right).$$

Exercise 10.8

Suppose that the citizens vote as follows in each subgame.

Subgame following x

Every citizen with favorite position less than m votes for candidate 1, every citizen with favorite position greater than m votes for candidate 2, and the votes of citizens with favorite position m are split equally between candidates 1 and 2.

This action profile is a Nash equilibrium because a citizen who changes her vote causes the outcome to change from a tie between the candidates at $m-\delta$ and $m+\delta$ to a win for the member of this pair that the citizen likes less. No citizen votes for her least favorite candidate, so every citizen's action is weakly undominated.

Subgame following entry of new candidate

The citizens' votes remain the same.

The action profile remains a Nash equilibrium in which no citizen's vote is weakly dominated.

Subgame following deviation to *Out* by any candidate

If the candidate's position is less than m, all citizens vote for candidate 2, and if it is greater than m, all citizens vote for candidate 1.

No change in any citizen's vote affects the outcome, so the action profile is a Nash equilibrium. Neither candidate 1 nor candidate 2 is any citizen's least favorite candidate, because candidates exist whose positions are less than x_1 and greater than x_2, so every citizen's action is weakly undominated.

Other subgames

The citizens' vote profile is a arbitrary Nash equilibrium in which the citizens' actions are weakly undominated, at least one of which exists by Proposition 4.2.

I now consider deviations by citizens in the first stage of the game.

Citizen enters as a candidate

She loses (she obtains no votes) and does not affect the outcome, so that her payoff decreases.

Losing candidate i with position less than m deviates to *Out*

The outcome changes from a tie between candidates 1 and 2 to a win for candidate 2. Thus i's payoff changes from $\frac{1}{2}u_i(x_1)+\frac{1}{2}u_i(x_2)-c$ to $u_i(x_2)$, so she is no better off if $c \le \frac{1}{2}(u_i(x_1)-u_i(x_2))$.

Losing candidate i with position greater than m deviates to *Out*

By a similar argument, she is no better off if $c \le \frac{1}{2}(u_i(x_2)-u_i(x_1))$.

Candidate i with position x_1 deviates to *Out*

The outcome changes from a tie between candidates 1 and 2 to a win for candidate 2, so i's payoff changes from $\frac{1}{2}b+\frac{1}{2}u_i(x_2)-c$ to $u_i(x_2)$, and hence she is no better of if $c \le \frac{1}{2}(b-u_i(x_2))$.

Candidate i with position x_2 deviates to *Out*

Similarly, i is no better off if $c \le \frac{1}{2}(b-u_i(x_1))$.

We conclude that the strategy profile is an equilibrium if $c \le \frac{1}{2}(u_i(x_1)-u_i(x_2))$ for every losing candidate i whose position is less than m, $c \le \frac{1}{2}(u_i(x_2)-u_i(x_1))$ for every losing candidate i whose position is greater than m, $c \le \frac{1}{2}(b-u_i(x_2))$ for candidate i with position x_1, and $c \le \frac{1}{2}(b-u_i(x_1))$ for candidate i with position x_2.

11 Distributive politics

11.1 Two-candidate competition with exogenous incomes under uncertainty 377
11.2 Voting over transfer systems when income is endogenous 384
11.3 Voting over linear transfer systems 392
11.4 Coalitional bargaining over redistribution 400

How does political power affect the distribution of material resources among the members of a society? This chapter analyzes models that address this question for the political system of majority rule.

Synopsis

One way to start thinking about the question is to study a collective choice problem in which the set of alternatives is the set of all distributions of a fixed amount of a material resource and each individual cares only about the amount she is assigned. This problem has no Condorcet winner. Take any distribution a. Here is a distribution b that beats it: every member of the bare majority consisting of the individuals assigned the smallest amounts in a is assigned in b the amount she receives in a plus an equal share of the total amount assigned to the complementary minority, and each member of that minority is assigned zero. Then every member of the bare majority prefers b to a.

An implication of this observation is that a two-candidate electoral competition game with majority rule in which the set of positions is the set of all distributions of the resource has no Nash equilibrium: for any distribution proposed by one candidate, the other candidate can propose a distribution that is preferred by a majority of individuals.

One way to escape this conclusion is to assume that each citizen cares also about other (exogenous) features of the candidates' platforms, and the candidates are uncertain about these preferences. If the uncertainty is great enough, it turns out that there may exist a distribution of the resource with the property that each candidate believes that no other distribution increases her probability of winning, because the amount by which every other distribution raises the payoffs of a majority of citizens is not likely to outweigh the preference of these citizens for the other features of the other candidate's platform. A model of this type is analyzed in Section 11.1. Proposition 11.1 characterizes a Nash equilibrium, if one exists, and the rest of the section explores properties of the equilibria in some examples.

Another approach considers a collective choice problem in which the set of alternatives is the set of the individuals' favorite distributions rather than the set of all possible distributions. Individuals differ in their earning power, and a tax-subsidy system specifies transfers as a function of income (individuals with the same income pay/receive the same tax/subsidy). If individuals care only about their consumption, not their hours of work, an individual's favorite system imposes a 100% tax on individuals with lower earning power and equalizes the post-tax income of the remaining individuals. The favorite system of the individual with median earning power is not a Condorcet winner of the associated collective choice problem, but it comes close. It fails because each individual is indifferent between the favorite systems of all individuals with higher earning power, because they all give her zero consumption. A richer version of the model assumes that each individual cares about both her consumption and her hours of work. In this case, an individual with high earning power can, by choosing her hours of work appropriately, obtain the same income and hence pay/receive the same tax/subsidy as one with low earning power, which limits the taxes that the favorite system of an individual with low earning power can extract from one with high earning power. This model generates, for some parameters, favorite systems that assign positive consumptions to individuals with lower earning power, and the favorite system of an individual with median earning power may be a strict Condorcet winner of the associated collective choice problem.

Section 11.3 returns to a collective choice problem in which the set of alternatives is the set of all possible tax-subsidy systems, but restricts "possible" to mean linear in income. Proposition 11.2 shows that if individuals' pre-tax incomes when they choose their hours of work optimally are ordered in the same way for every tax-subsidy scheme (a strong condition) then for any finite set of linear transfer systems the associated collective choice problem has single-crossing preferences with respect to the ordering of the individuals by their pre-tax income. As a result, the favorite transfer system of the individual with the median pre-tax income is a Condorcet winner, and hence, by Proposition 8.1, the outcome of a Nash equilibrium of an electoral competition game with two office-motivated candidates.

Section 11.4 takes a different approach, modeling the tax system as the outcome of society-wide bargaining. The distribution of income that emerges in this model is a compromise molded by the possibility that a majority expropriates the remaining individuals, and these individuals, in response, destroy their endowments. In an equilibrium, these actions are not carried out, but the prospect that they could be shapes the outcome. Two solution concepts based on different principles, the Shapley value and the core, yield the same conclusion: each individual's income is taxed at the rate of 50% and the revenue is divided equally among all individuals (Propositions 11.3 and 11.4).

11.1 Two-candidate competition with exogenous incomes under uncertainty

Consider a model of a society in which the citizens' incomes are given (they do not depend on the citizens' actions) and two political candidates propose tax-subsidy schemes. Each citizen cares about both her post-tax income and the candidates' policies on issues other than redistribution, which are fixed. The candidates know the citizens' incomes but are uncertain about their preferences regarding other policies. Suppose, for example, that the society consists of three citizens with total income 1, and candidate 1 proposes a tax-subsidy scheme that generates the distribution of income $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. If candidate 2 proposes a scheme that generates the distribution $(\frac{1}{2}, \frac{1}{2}, 0)$ and each citizen is indifferent between the candidates' positions on other issues, then in the absence of uncertainty candidate 2 knows that she will obtain the votes of citizens 1 and 2, and hence win. But if the citizens are not neutral regarding the candidates' positions on other issues and candidate 2 is uncertain of their leanings, she may believe that the probability that the distribution $(\frac{1}{2}, \frac{1}{2}, 0)$ earns her the votes of citizens 1 and 2 is less than 1: with positive probability citizens 1 and 2 may like the non-distributional policies of candidate 1 enough to vote for candidate 1 even though candidate 2 offers them more post-tax income. In fact, candidate 2 may believe that the probability of her winning when she proposes the distribution $(\frac{1}{2}, \frac{1}{2}, 0)$ is less than her probability of winning when she proposes $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. For example, the former may result in only a slightly higher probability that citizens 1 and 2 vote for her and a sharply lower probability that citizen 3 does so. As a result, the action pair in which each candidate proposes the policy $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ may be a Nash equilibrium of the game.

I explore this idea using the model of two-candidate electoral competition in Section 8.5.2, in which the citizens care about both the policy chosen and the identity of the winning candidate, with the latter dependence reflecting characteristics of the candidate assumed to be fixed, like her competence or positions on issues other than distribution. Specifically, the model is an electoral competition game with two office-motivated candidates and uncertain partisanship in which the set of positions for each candidate is the set of distributions of consumption (post-transfer income) among the citizens.

In this game, two candidates simultaneously propose distributions of consumption among the citizens 1, ..., n, where, for convenience, n is odd. (You may equivalently think of the candidates proposing tax-subsidy schemes, given that the citizens' incomes are exogenous.) Each citizen i prefers the distribution $(c_1^1, \ldots, c_n^1)$ proposed by candidate 1 to the distribution $(c_1^2, \ldots, c_n^2)$ proposed by candidate 2 if and only if $g_i(c_i^1) > g_i(c_i^2) + \theta_i$, where g_i is an increasing function and θ_i is a number reflecting i's evaluation of candidate 2's advantage (positive or negative) over candidate 1 on non-distributional policies. The candidates know how the citizens evaluate consumption but not how they evaluate other aspects of the candidates' policies: they know the functions g_i but are uncertain about the values of the numbers θ_i. Each candidate believes that $\theta_1, \ldots, \theta_n$ are independent draws from nonatomic distributions $F_1, \ldots, F_n$, so that each citizen i votes for candidate 1 if $\theta_i < g_i(c_i^1) - g_i(c_i^2)$, independently of the votes of the other citizens, an event with probability $F_i(g_i(c_i^1) - g_i(c_i^2))$. Under this assumption, each candidate believes that candidate 1's probability of winning, the probability that at least $(n+1)/2$ citizens vote for candidate 1 (given that n is odd), is

$$P\big(F_1(g_1(c_1^1) - g_1(c_1^2)), \ldots, F_n(g_n(c_n^1) - g_n(c_n^2))\big),$$

where P is the function defined in (8.6). Similarly, each candidate believes that candidate 2's probability of winning is

$$1 - P\big(F_1(g_1(c_1^1) - g_1(c_1^2)), \ldots, F_n(g_n(c_n^1) - g_n(c_n^2))\big).$$

Definition 11.1: Two-candidate electoral competition game of redistribution with uncertain partisanship

A *two-candidate electoral competition game of redistribution with uncertain partisanship* $\langle I, (g_i)_{i\in I}, (y_i)_{i\in I}, (F_i)_{i\in I}, \{1,2\}\rangle$, where

- $I = \{1, \ldots, n\}$ for an odd integer n (the set of citizens)

and for each $i \in I$

- $g_i : \mathbb{R}_+ \to \mathbb{R}$ is increasing (the component of i's payoff function that relates to consumption (post-tax income))
- $y_i \geq 0$ (i's pre-tax income)
- F_i is a nonatomic probability distribution over $\mathbb{R}$

is an electoral competition game with two office-motivated candidates and uncertain partisanship $\langle I, X, (v_i)_{i\in I}, (F_i)_{i\in I}, \{1,2\}\rangle$ in which

$$X = \{(c_1,\ldots,c_n)\in\mathbb{R}^n : \textstyle\sum_{i=1}^n c_i = \sum_{i=1}^n y_i \text{ and } c_i \geq 0 \text{ for } i = 1,\ldots,n\}$$

and

$$v_i(c_1,\ldots,c_n) = g_i(c_i) \text{ for all } (c_1,\ldots,c_n)\in X \text{ and } i = 1,\ldots,n.$$

Proposition 8.8 gives conditions under which the candidates' positions are the same in an interior Nash equilibrium of an electoral competition game with two office-motivated candidates and uncertain partisanship, if one exists. This result is not applicable to the game defined here because no member of $X \subset \mathbb{R}^n$ is interior and the functions v_i are not strictly concave (each such function is constant in c_j for $j \neq i$). However, the following closely related result holds. Its proof runs parallel to that of Proposition 8.8.

Proposition 11.1: Nash equilibrium of two-candidate electoral competition game of redistribution with uncertain partisanship

Let $\langle I, (g_i)_{i\in I}, (y_i)_{i\in I}, (F_i)_{i\in I}, \{1,2\}\rangle$ be a two-candidate electoral competition game of redistribution with uncertain partisanship. Assume that $I = \{1,\ldots,n\}$, the payoff function g_i of each citizen i for consumption is continuously differentiable and strictly concave, and each function F_i is continuously differentiable, with $F_i'(\theta) > 0$ for all $\theta\in\mathbb{R}$. If the game has a Nash equilibrium (c^{1*}, c^{2*}) with $c_i^{j*} > 0$ for $j = 1, 2$ and $i = 1,\ldots,n$ then this equilibrium has the following properties.

a. The candidates' plans are the same ($c^{1*} = c^{2*}$) and for some number λ

$$P_i'(F_1(0),\ldots,F_n(0))F_i'(0)g_i'(c_i^*) = \lambda \text{ for citizens } i = 1,\ldots,n, \tag{11.1}$$

where $c^* = c^{1*} = c^{2*}$, P is given by (8.6), and P_i' is the derivative of P with respect to its ith argument.

b. If F_i is the same for all $i\in I$ then the common value of c^{1*} and c^{2*} is the solution of

$$\max_{c\in\mathbb{R}^n} \sum_{i=1}^n g_i(c_i) \text{ subject to } \sum_{i=1}^n c_i = \sum_{i=1}^n y_i \text{ and } c_i \geq 0 \text{ for } i = 1,\ldots,n.$$

Proof

a. By the definition of Nash equilibrium, for $j = 1, 2$ the action c^{j*} of candidate j maximizes j's probability of winning, given the other candidate's plan: $c^{1*} = (c_1^{1*}, \dots, c_n^{1*})$ is a solution of

$$\begin{aligned}&\max_{(c_1^1,\dots,c_n^1)} P\big(F_1(g_1(c_1^1) - g_1(c_1^{2*})), \dots, F_n(g_n(c_n^1) - g_n(c_n^{2*}))\big)\\ &\qquad\qquad \text{subject to } \sum_{i=1}^n c_i^1 = \sum_{i=1}^n y_i \text{ and } c_i^1 \geq 0 \text{ for } i = 1, \dots, n,\end{aligned} \tag{11.2}$$

and $c^{2*} = (c_1^{2*}, \dots, c_n^{2*})$ is a solution of

$$\begin{aligned}&\max_{(c_1^2,\dots,c_n^2)} \big(1 - P\big(F_1(g_1(c_1^{1*}) - g_1(c_1^2)), \dots, F_n(g_n(c_n^{*1}) - g_n(c_n^2))\big)\big)\\ &\qquad\qquad \text{subject to } \sum_{i=1}^n c_i^2 = \sum_{i=1}^n y_i \text{ and } c_i^2 \geq 0 \text{ for } i = 1, \dots, n.\end{aligned} \tag{11.3}$$

The derivatives of the equality constraint function in (11.2) with respect to the variables are all 1, and in particular are not all 0, so by Proposition 16.14 if c^{1*} is a solution of (11.2) with $c_i^{1*} > 0$ for $i = 1, \dots, n$ then there is a unique number λ such that

$$P_i'(\pi(c^{1*}, c^{2*}))F_i'(g_i(c_i^{1*}) - g_i(c_i^{2*}))g_i'(c_i^{1*}) = \lambda \text{ for } i = 1, \dots, n, \tag{11.4}$$

where the function $\pi : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}_+^n$ is defined by

$$\pi(c^1, c^2) = \big(F_1(g_1(c_1^1) - g_1(c_1^2)), \dots, F_n(g_n(c_n^1) - g_n(c_n^2))\big) \text{ for all } (c^1, c^2).$$

Now define the function $W : \mathbb{R}_+^n \to \mathbb{R}$ by

$$W(c) = \sum_{i=1}^n P_i'(\pi(c^{1*}, c^{2*}))F_i'(g_i(c_i^{1*}) - g_i(c_i^{2*}))g_i(c_i). \tag{11.5}$$

Each function g_i is strictly concave and all the coefficients of $g_i(c_i)$ in the definition of W are positive, so W is strictly concave. Consider the problem

$$\max_c W(c) \text{ subject to } \sum_{i=1}^n c_i = \sum_{i=1}^n y_i \text{ and } c_i \geq 0 \text{ for } i = 1, \dots, n. \tag{11.6}$$

Given that W is strictly concave, this problem has a unique solution, say $\hat{c}$. By Proposition 16.14, if $\hat{c}_i > 0$ for $i = 1, \dots, n$ then there is a unique number

λ such that (11.4) is satisfied by $c^{1*} = \hat{c}$, and by Proposition 16.15, if there is a number λ and a vector c^{*1} with $c_i^{*1} > 0$ for $i = 1, \ldots, n$ that satisfies (11.4) then, given the concavity of W and the linearity of the constraint in (11.6), $c^{*1} = \hat{c}$. Thus (11.2) has a unique solution, which is the solution of (11.6).

Now, if $c_i^{2*} > 0$ for $i = 1, \ldots, n$ then the fact that c^{2*} is a solution of (11.3) means that there is a unique number λ such that c^{2*} satisfies the same condition, (11.4). Hence c^{2*} is also the unique solution of (11.6) and thus $c^{1*} = c^{2*}$, so that (11.4) reduces to (11.1).

b. Given $c^{1*} = c^{2*}$,

$$W(c) = \sum_{i=1}^{n} P_i'(F_1(0), \ldots, F_n(0))F_i'(0)g_i(c_i) \text{ for all } c \in \mathbb{R}^n.$$

Thus if every distribution F_i is the same, then, given $P_i'(p, \ldots, p) = P_k'(p, \ldots, p)$ for all p and all i and k, c^* is a solution of (11.6) if and only if it is a maximizer of $\sum_{i=1}^{n} g_i(c_i)$ subject to the same constraints.

Like Proposition 8.8, this result does not assert that the game has an equilibrium; it only gives properties of an equilibrium if one exists. In the absence or near-absence of uncertainty, the game does not have an equilibrium, because for any distribution c proposed by one candidate, the other candidate can ensure that it wins with high probability by selecting the bare majority who are assigned the lowest total consumption in c and proposing to add to each of their allocations under c an equal share of the total consumption of the complementary minority. I know of no result that gives conditions under which the game necessarily has a Nash equilibrium, but some games do. Suppose, for example, that $n = 3$, $\sum_{i=1}^{3} y_i = 1$, and for $i = 1, 2, 3$ we have $g_i(z) = \sqrt{z}$ for all z and F_i is a normal distribution with mean 0 and standard deviation σ. My computations suggest that $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a Nash equilibrium of this game if σ exceeds 0.24.

Character of equilibrium

Proposition 11.1b says that if F_i is the same for every citizen then in any Nash equilibrium both candidates propose the distribution $(c_1^*, \ldots, c_n^*)$ that maximizes the sum of $g_i(c_i)$ across all citizens. Given the differentiability and concavity of each function g_i, $g_i'(c_i^*)$ is thus the same for all citizens. The derivative of g_i is a measure of the significance for i of consumption relative to a candidate's non-distributional policy. The smaller is the derivative, the larger is the increase in consumption needed to offset a given reduction in the appeal of the non-distributional policy. More loosely, the smaller is the derivative, the more the individual values non-distributional policy relative to consumption. Thus if, for every consumption level, citizen i attaches greater marginal value to non-distributional policy (in terms of consumption) than does citizen i', then in any equilibrium (if one exists), i is assigned less consumption than is i'.

Now suppose that g_i is the same for every citizen, equal to g, but F_i is not. First suppose that every F_i is symmetric about 0, with $F_i(x) = 1 - F_i(-x)$, and hence $F_i(0) = \frac{1}{2}$, and has a density f_i, so that $f_i(x) = f_i(-x)$. Then (11.1) is

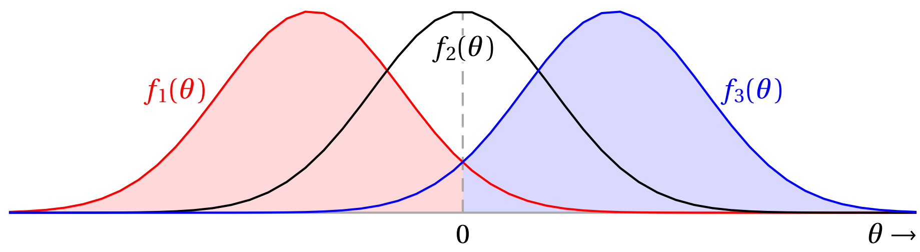
$$P_i'(\tfrac{1}{2}, \ldots, \tfrac{1}{2}) f_i(0) g'(c_i^*) = \lambda \text{ for } i = 1, \ldots, n.$$

The function P is symmetric in its arguments, so $P_i'(\frac{1}{2}, \ldots, \frac{1}{2})$ is the same for all i and hence $f_i(0)g'(c_i^*)$ is the same for all i. Given the strict concavity of g, c_i^* is thus large when $f_i(0)$ is large. That is, an individual for whom a small difference between the candidates' policies results in a large difference in the probability of voting for each candidate receives a larger amount of consumption than does an individual who is unlikely to be swayed by a small difference in the candidates' policies. Roughly speaking, individuals who are influenced more by policy than by the candidates' characteristics receive larger amounts of consumption in an equilibrium (if one exists).

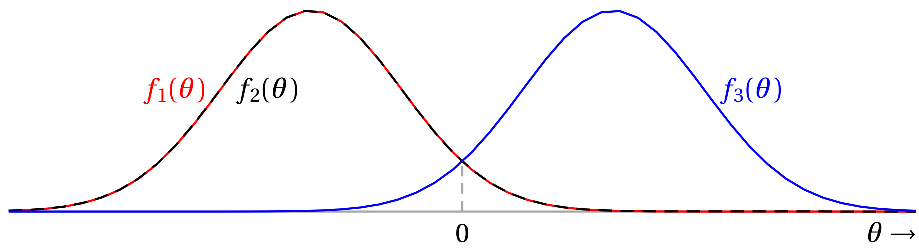
Now consider an example with three citizens in which the densities of the distributions F_1, F_2, and F_3 are the ones shown in Figure 11.1a. Citizen 1 is a partisan of candidate 1 in the sense that θ_1 is more likely to be negative than positive, so that if the amounts of consumption proposed for citizen 1 by the candidates are the same, citizen 1 is more likely to favor candidate 1. Similarly, citizen 3 is a partisan of candidate 2, and citizen 2 is neutral (she is equally likely to vote for each candidate if the candidates propose the same amount of consumption for her). To find the implications of (11.1) for this example, first note that for each citizen i we have $P_i'(p_1, p_2, p_3) = p_j(1-p_k) + p_k(1-p_j)$, where j and k are the citizens other than i. (The function P is defined in (8.6).) Now, $F_1(0)$ (the area shaded pink in Figure 11.1a) is equal to $1 - F_3(0)$ (the area shaded blue), so denoting their common value by p we have

$$P_2'(F_1(0), F_2(0), F_3(0)) = p^2 + (1-p)^2$$
$$P_1'(F_1(0), F_2(0), F_3(0)) = P_3'(F_1(0), F_2(0), F_3(0)) = \tfrac{1}{2}p + \tfrac{1}{2}(1-p) = \tfrac{1}{2}.$$

By Proposition 11.1a, if (c_1^*, c_2^*, c_3^*) is an equilibrium then $P_i'(F_1(0), F_2(0), F_3(0)) f_i(0) g'(c_i^*)$ is the same for every citizen i. Now, $f_1(0) = f_3(0) < f_2(0)$ and, given $p > \frac{1}{2}$, $p^2 + (1-p)^2 > \frac{1}{2}$, so $g'(c_1^*) = g'(c_3^*) > g'(c_2^*)$ and hence $c_1^* = c_3^* < c_2^*$. That is, citizens 1 and 3 are assigned the same amount of consumption, and this amount is less than the amount assigned to citizen 2. Intuitively, the fact that citizen 2 is equally likely to vote for each candidate if they both offer her the same amount of con-



(a) Citizen 1 is a partisan of candidate 1, citizen 2 is neutral, and citizen 3 is a partisan of candidate 2.



(b) Citizens 1 and 2 are partisans of candidate 1 and citizen 3 is a partisan of candidate 2.

Figure 11.1 Examples of densities for the distributions F_1, F_2, and F_3 in a two-candidate electoral competition game of redistribution with uncertain partisanship with three citizens.

sumption, whereas the other citizens have partisan leanings, means that a candidate's redistributing consumption from citizens 1 and 3 to citizen 2 increases the candidate's probability of winning.

Finally consider an example with three citizens in which the distributions F_1, F_2, and F_3 have the densities shown in Figure 11.1b. Denoting $F_1(0) = F_2(0) = 1 - F_3(0)$ by p, we have

$$\begin{aligned} P_1'(F_1(0), F_2(0), F_3(0)) = P_2'(F_1(0), F_2(0), F_3(0)) &= p^2 + (1-p)^2 \\ P_3'(F_1(0), F_2(0), F_3(0)) &= 2p(1-p). \end{aligned}$$

Given $p > \frac{1}{2}$, $p^2 + (1-p)^2 > 2p(1-p)$, so that since $f_1(0) = f_2(0) = f_3(0)$, $g'(c_1^*) = g'(c_2^*) < g'(c_3^*)$ and hence $c_1^* = c_2^* > c_3^*$. That is, citizens 1 and 2 are assigned the same amount of consumption, and this amount is greater than the amount assigned to citizen 3. A citizen's vote is pivotal only if the other two citizens vote for different candidates. Thus the votes of citizen 1 and citizen 2 are both likely to be pivotal, but that of citizen 3 is not. Hence the candidates find it advantageous to direct more consumption to citizens 1 and 2.

To summarize roughly, in the equilibria in these examples (if one exists) a citizen receives a larger amount of consumption the more sensitive is her payoff to her own consumption relative to the candidates' non-distributional policies, the less partisan she is, and the more likely her vote is to be pivotal.

11.2 Voting over transfer systems when income is endogenous

11.2.1 Main idea

Suppose that every individual's income is fixed, independently of her actions. Then an individual who cares only about her own consumption and can impose arbitrary taxes optimally expropriates all the income of the other individuals. If the tax she can impose on an individual may depend only on the individual's income, she is slightly constrained: she has to share the tax revenue with the individuals whose incomes are the same as hers. If the individuals' incomes are not fixed, but depend on the individuals' actions—like their choices of hours of work—then she is more constrained. In this case, the amount of tax she can extract from the other individuals is limited by the fact that some of these individuals may, by choosing appropriate hours of work, be able to earn the same income as she does, and hence pay the same tax or receive the same subsidy as she does. It is this case that I consider now.

The main model in this section assumes that each individual i has a given earning power w_i and chooses her hours (or intensity) of work $h_i \in [0,1]$, generating an income of $w_i h_i \in [0, w_i]$. She cares about both her hours of work and her consumption, which is equal to her post-transfer income. The number of individuals is finite, equal to n. I assume for convenience that n is odd and no two individuals have the same earning power; I index the individuals so that $w_1 < w_2 < \cdots < w_n$.

As a prelude, first assume that each individual i cares *only* about her consumption. Assume also that if two different values of her hours of work h_i generate the same amount of consumption, she chooses the larger value. A transfer system $T : \mathbb{R}_+ \to \mathbb{R}$ assigns to each possible income y a transfer $T(y)$, which makes the consumption of an individual with any income y equal to $y - T(y)$. Thus if $T(y) > 0$ the transfer is a tax and if $T(y) < 0$ it is a subsidy. Under these assumptions, for any transfer system T and any number w, any individual with earning power at least w has the option of consuming $w - T(w)$, by choosing her hours of work so that she earns the income w. Thus no transfer system can reduce the consumption of an individual j below the maximum of $w - T(w)$ for $w \le w_j$.

Among transfer systems that balance the budget, which one is best for individual i? Given that she cares only about her consumption, she chooses $h_i = 1$ and thus has a pre-transfer income of w_i. Denote the transfer system she chooses by T^i. Then her consumption is $w_i - T^i(w_i)$. There is no downside to her imposing a 100% tax on individuals with earning power lower than hers, so T^i has this feature. How should she treat individuals with higher earning power?

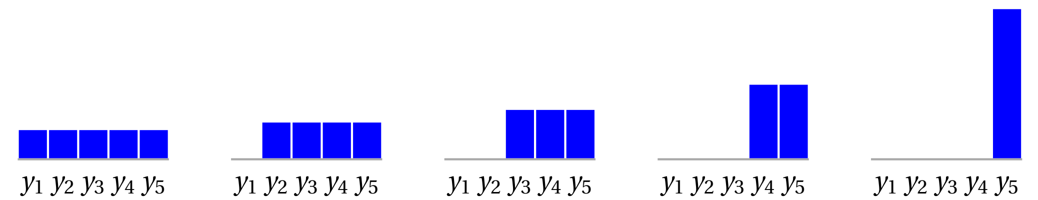


Figure 11.2 The levels of consumption assigned to the various income levels by the favorite transfer systems of the individuals in a model in which each individual can control her income. The society contains five individuals, who are ordered by their earning power, smallest to largest.

An individual j with $w_j > w_i$ can choose $h_j = w_i/w_j$, obtain the pre-transfer income w_i, and get the same consumption as i, namely $w_i - T^i(w_i)$. It is in i's interest for each individual j to choose $h_j = 1$, so that as much income as possible is generated. Thus given the assumption that if j is indifferent between two values of h_j she chooses the larger one, T^i should give her the same consumption when she chooses $h_j = w_i/w_j$ and $h_j = 1$, and not give her more for any value in between. Let $W = \sum_{j=1}^{n} w_j$, the total income when every individual j chooses $h_j = 1$. We conclude that T^i is such that the consumption of each individual j is zero if $w_j < w_i$ and W/k if $w_j \geq w_i$, where k is the number of individuals with earning power at least w_i, as in the example in Figure 11.2.

Now consider the collective choice problem for which the set of alternatives is the set of the individuals' favorite transfer systems, rather than the set of all feasible transfer systems. For $n \geq 5$ this problem has no Condorcet winner: each individual's favorite transfer system is beaten by the favorite transfer system of another individual, by the following argument. Denote by i^* the individual with the median earning power. For any $i < i^*$, every individual j with $j \geq i^*$ prefers T^{i^*} to T^i, so that T^{i^*} beats T^i. For any $i > i^*$, every individual j with $j \leq i^*$ prefers T^1 to T^i, so that T^1 beats T^i. And every individual j with $j > i^*$ prefers T^{i^*+1} to T^{i^*}, i^* has the reverse preference, and every individual j with $j < i^*$ is indifferent between these systems, so that if $n \geq 5$ then T^{i^*+1} beats T^{i^*}. (If you have done Exercise 1.10, you will recognize the preference profile.)

However, for a slight variant of the collective choice problem, T^{i^*} is a strict Condorcet winner: if T^{i^*} gives each individual $j < i^*$ a small positive amount of consumption, rather than none, then these individuals, as well as i^*, prefer T^{i^*} to T^i for $i > i^*$, so that T^{i^*} beats T^i.

Such a variant of the collective choice problem is generated by a model in which each individual cares not only about her income but also about her hours of work. In this model, an individual with earning power w who chooses to work at an intensity $h < 1$ can be imitated not only by individuals with greater earning power, but also by individuals with earning power between wh and w. Thus her optimal transfer scheme may need to assign positive consumption to individuals with earning power less than hers, otherwise they will choose a work intensity large enough that they earn the same income as she does, and hence will be assigned the same transfer as she is.

The impact on the individuals' favorite systems of this alternative assumption is greatest when the dispersion of the individuals' earning powers is small: an individual can imitate only those whose incomes she can obtain by working at a sufficiently high intensity, which requires their earning power to be not too much greater than hers. No general result is available, but I now briefly present a model and give a complete example.

11.2.2 Model and example

A society consists of a finite number of individuals, each endowed with one unit of time, which she divides between work and leisure. Each individual i's earning power is w_i; her income is $w_i h$ if she devotes the amount of time h to work. She uses her income to purchase a consumption good with price 1, and cares about the amount of her consumption and the amount of time she works. When she works for h units of time, her payoff is $u_i(w_i h, 1-h)$.

Definition 11.2: Society

A *society* $\langle N, (u_i)_{i\in N}, (w_i)_{i\in N}\rangle$ consists of

- a finite set N (of individuals, each endowed with one unit of time)

and for each $i \in N$

- a differentiable function $u_i : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ that is increasing in each of its arguments ($u_i(c,l)$ is i's payoff when her consumption is c and her amount of leisure is l)
- a number $w_i > 0$ (i's earning power).

A transfer system assigns to each possible income a number that is at most equal to the income. If the number is positive it is a tax, and if it is negative it is a subsidy. Note that the definition of a transfer system does not include a feasibility requirement: the net amount of income disbursed is not restricted by any budget. A feasibility condition is imposed later.

Definition 11.3: Transfer system

A *transfer system* is a function $T : \mathbb{R}_+ \to \mathbb{R}$ with $T(y) \le y$ for all y. For pre-transfer income y, post-transfer income is $y - T(y)$.

Consider a society $\langle N, (u_i)_{i\in N}, (w_i)_{i\in N}\rangle$. Given a transfer system T, if individual i works for h units of time her consumption is $w_i h - T(w_i h)$, so she chooses h to solve the problem

$$\max_h u_i(w_i h - T(w_i h), 1-h) \text{ subject to } 0 \le h \le 1. \tag{11.7}$$

Reformulating this problem facilitates its analysis. For each individual $i \in N$, define the function $v_i : [0, w_i] \times \mathbb{R}_+ \to \mathbb{R}$ by

$$v_i(y,c) = u_i(c, 1 - y/w_i) \text{ for all } (y,c), \tag{11.8}$$

so that $v_i(y,c)$ is i's payoff when she works enough hours to generate the (pre-transfer) income y and consumes c. Then individual i's optimization problem (11.7) may be formulated as

$$\max_{(y,c)} v_i(y,c) \text{ subject to } c = y - T(y) \text{ and } 0 \le y \le w_i. \tag{11.9}$$

Figure 11.3 illustrates the solution of an example of this problem. The orange curve gives the amount of consumption $y - T(y)$ that each amount y of pre-transfer income yields, given the transfer system. The transfer system assigns subsidies to values of y for which this curve is above the 45° line and taxes to values of y for which it is below the 45° line. The blue curve is a set of pairs (y,c) that yield i the same payoff—in economic jargon, an indifference set or indifference curve. The slope of the indifference curve through (y,c) at (y,c) is

$$-\frac{v'_{i,1}(y,c)}{v'_{i,2}(y,c)} = \frac{1}{w_i}\frac{u'_2(c, 1-y/w_i)}{u'_1(c, 1-y/w_i)} > 0,$$

where the subscripts 1 and 2 denote the index of the variable with respect to which the derivative is taken. The slope is positive because an individual obtains more income only by working longer, so that if $y' > y$ then we need $c' > c$ for (y,c) and (y',c') to yield the same payoff.

The pair (y_i, c_i) in Figure 11.3 is the solution of (11.9)—the pair chosen by the individual, given the transfer system. The length of the green line segment is the subsidy she receives, $-T(y_i)$.

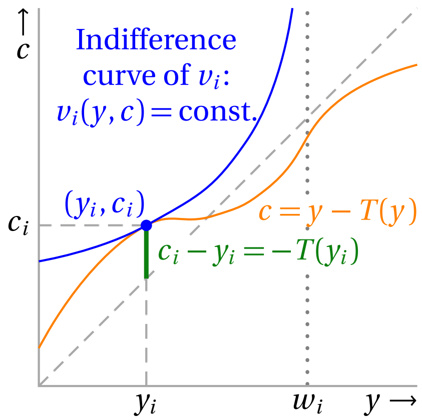


Figure 11.3 The pair (y_i, c_i) that solves individual i's optimization problem (11.9), given the transfer system T.

Individual's favorite transfer system

To be feasible, a transfer system must collect in taxes at least as much as it distributes in subsidies. Among feasible systems, the best one for any given individual is the one for which her payoff is highest, given that every individual (including her) chooses her hours of work optimally. I now present a convenient formulation of the problem of finding such a selfishly-optimal system.

Fix a transfer system T, and for each $i \in N$ let (y_i, c_i) be a solution of (11.9). If $w_i \geq y_j$, then individual i can obtain the income y_j and hence the consumption c_j by choosing $h = y_j / w_i$, so $v_i(y_i, c_i) \geq v_i(y_y, c_j)$ for every j with $y_j \leq w_i$. She also has the option of choosing $h = 0$, so $v_i(y_i, c_i) \geq v_i(0,0)$, given that $T(0) \leq 0$. That is, for any individual $i \in N$ we have

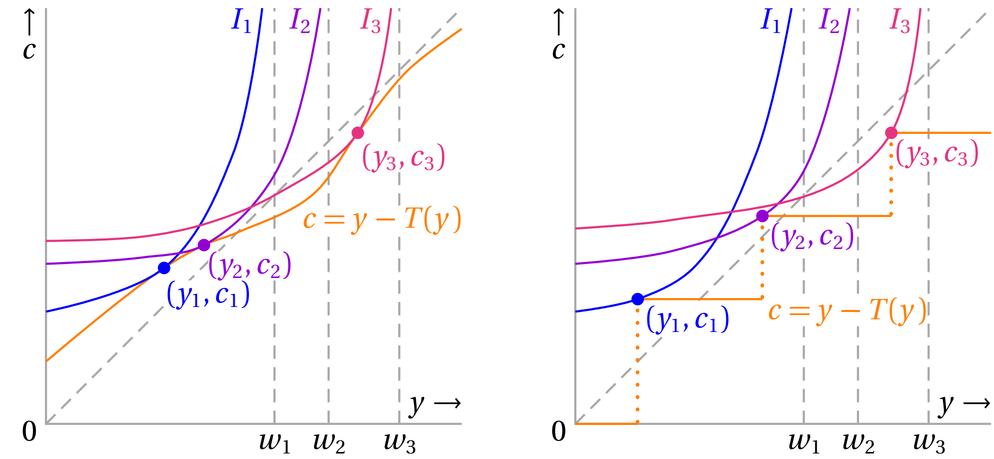
$$\begin{aligned} &v_i(y_i, c_i) \geq v_i(y_j, c_j) \text{ for all } j \in N \text{ with } y_j \leq w_i \\ &v_i(y_i, c_i) \geq v_i(0,0). \end{aligned} \tag{11.10}$$

Figure 11.4a illustrates these conditions for an example of a society containing three individuals.

Conversely, let $((y_i, c_i))_{i \in N}$ be a profile satisfying (11.10) with $y_1 < y_2 < \cdots < y_n$. Define the (discontinuous) transfer system T by

$$T(y) = \begin{cases} y & \text{if } y < y_1 \\ y - c_i & \text{if } y_i \leq y < y_{i+1} \text{ for } i = 1, \ldots, n-1 \\ y - c_n & \text{if } y_n \leq y. \end{cases}$$

An example for three individuals is given in Figure 11.4b. I claim that for this transfer system, for each individual $i \in N$ the pair (y_i, c_i) is a solution of (11.9).



(a) The pairs (y_i, c_i) optimal for three individuals given the transfer system T. The curves labeled I_1, I_2, and I_3 are indifference curves of the individuals.

(b) A transfer system T for which for each individual i, each member of the profile $((y_i, c_i))_{i \in N}$, which satisfies (11.10), is optimal.

Figure 11.4

The reason is that for any profile $((y_i, c_i))_{i \in N}$ satisfying (11.10) we have $c_j < c_k$ whenever $y_j < y_k$, otherwise k prefers (y_j, c_j) to (y_k, c_k). (The conclusion does not depend on the shape of the indifference curves, and in particular does not depend on the slope of each indifference curve increasing in y, as in Figure 11.4.)

In summary, $((y_i, c_i))_{i \in N}$ satisfies (11.10) if and only if for some transfer system T, for each $i \in N$ the pair (y_i, c_i) is a solution of (11.9) Thus rather than working with transfer systems we can work with transfer plans, defined as follows.

Definition 11.4: Transfer plan

A *transfer plan* is a profile $((y_i, c_i))_{i \in N}$ with $(y_i, c_i) \in \mathbb{R}_+ \times \mathbb{R}_+$ for all $i \in N$.

A selfishly-optimal transfer plan for any individual k maximizes k's payoff among the plans that satisfy (11.10) and raise at least as much in taxes as they distribute in subsidies.

Definition 11.5: Selfishly-optimal transfer plan for an individual

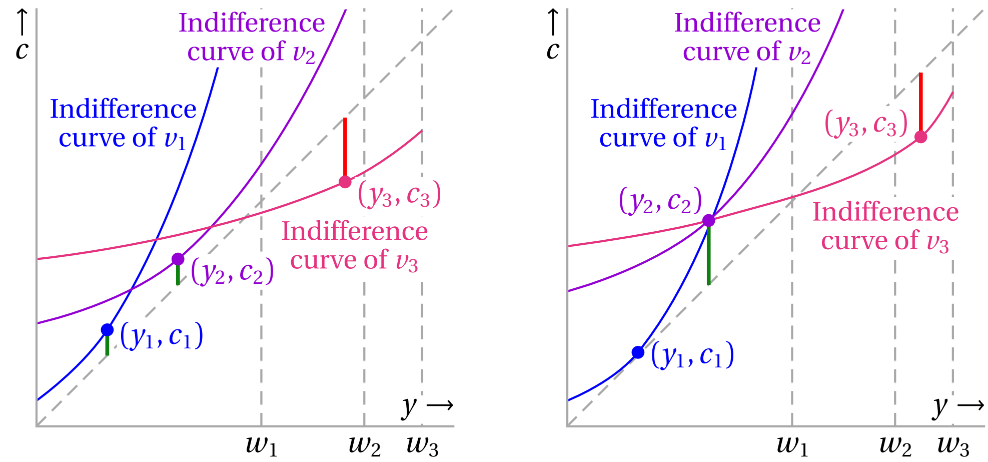
Let $\langle N, (u_i)_{i \in N}, (w_i)_{i \in N} \rangle$ be a society, for each $i \in N$ let v_i be the function defined in (11.8), and let $k \in N$. A transfer plan $((y_i, c_i))_{i \in N}$ is *selfishly-optimal* *for individual k* if it is a solution of the problem

$$\begin{gathered}\max_{((y_i,c_i))_{i\in N}} v_k(y_k,c_k) \text{ subject to} \\ v_i(y_i,c_i) \ge v_i(y_j,c_j) \text{ for all } i\in N \text{ and all } j\in N \text{ with } y_j \le w_i \\ v_i(y_i,c_i) \ge v_i(0,0) \text{ for all } i\in N \\ 0 \le y_i \le w_i \text{ and } c_i \ge 0 \text{ for all } i \in N \\ \textstyle\sum_{i\in N}(y_i - c_i) \ge 0.\end{gathered} \tag{11.11}$$

In the remainder of this section, I provide a diagrammatic analysis of the individuals' selfishly-optimal plans and an example of a society with three individuals in which the selfishly-optimal plan of the individual with the median earning power is a strict Condorcet winner of the collective choice problem in which the set of alternatives is the set of the individuals' selfishly-optimal plans.

My analysis is restricted to societies in which every individual's payoff function u_i is the same and has the property that the optimal amount of consumption for each individual in the absence of transfers is increasing in her earning power. That is, the value of c that maximizes $u_i(c, 1-c/w)$ is increasing in w. This condition is equivalent to the slope $-v'_{i,1}(y,c)/v'_{i,2}(y,c)$ of the indifference curve of v_i through (y,c) at (y,c) being smaller for larger values of w_i, for each value of (y,c), like those in Figure 11.4. (This equivalence is demonstrated in Mirrlees 1971, footnote 1, for example.) An interpretation of the condition is that the amount of additional consumption required to compensate for the extra work necessary to earn an additional unit of income is smaller for individuals with greater earning power.

Consider the transfer plan of individual 2 for the society with three individuals shown in Figure 11.5a. This plan satisfies the constraints in (11.11) for $k=2$: each individual i likes (y_i,c_i) better than (y_j,c_j) for $j\neq i$ and better than $(0,0)$, and the total tax paid (by individual 3), the length of the vertical red line segment, exceeds the total subsidy paid out (to individuals 1 and 2), the sum of the lengths of the green line segments. But this plan is not optimal for individual 2. First, the total tax exceeds the total subsidy, so that c_2 can be increased, making individual 2 better off, without violating the budget constraint or any other constraint. Second, individual 1 likes (y_1,c_1) better than (y_2,c_2), so that c_1 can be reduced while keeping (y_1,c_1) the best of the three pairs for her, which relaxes the budget constraint and allows c_2 to be increased. Similarly, c_3 can be reduced, increasing the tax on individual 3. Finally, the slope of individual 1's indifference curve through (y_1,c_1) and the slope of individual 3's indifference curve through (y_3,c_3) are both different from 1, so the pairs (y_1,c_1) and (y_3,c_3) can be moved along the



(a) A suboptimal plan.

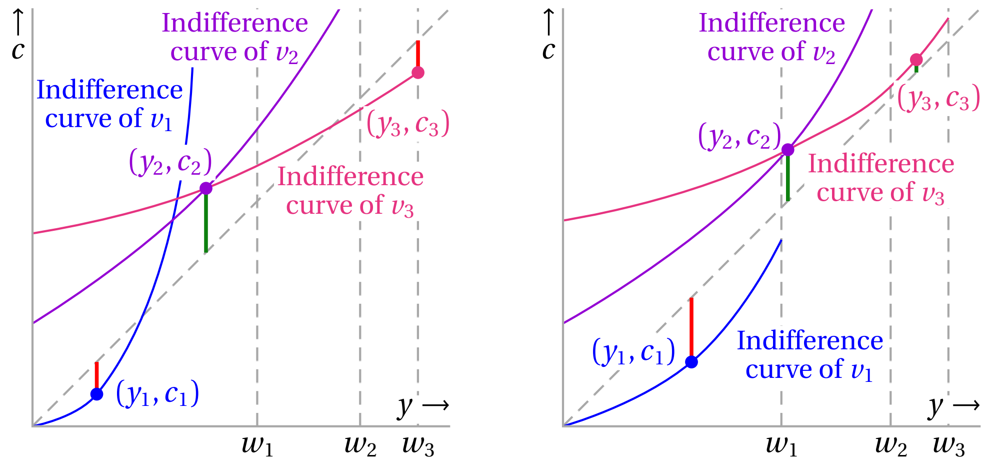
(b) A plan without suboptimal features of the plan in Figure 11.5a.

Figure 11.5 Transfer plans for individual 2 in a society with three individuals.

indifference curves to increase the tax or reduce the subsidy for each individual without affecting her payoff. Figure 11.5b shows a plan that cannot be improved by any such changes (although it might be improved by other changes). Similar considerations apply to the selfishly-optimal plans of the other individuals.

Figure 11.6 shows two examples of transfer plans for individual 2 in a society with three individuals that illustrate other possibilities for an optimal plan. In Figure 11.6a, the plan pushes individual 1 down to her lowest possible payoff: she is indifferent between (y_1, c_1) and $(0, 0)$. Although she is not indifferent between (y_1, c_1) and (y_2, c_2), the plan may still be optimal for individual 2. In Figure 11.6b, (y_2, c_2) is unattainable by individual 1 because $y_2 > w_1$. In this case, an optimal plan for individual 2 may not exist: plans in which y_2 is closer to w_1 may be better for her, but a plan in which $y_2 = w_1$ may be significantly worse, because (y_2, c_2) is then attainable for individual 1, who consequently has the option of receiving the same transfer as individual 2.

For a society with five individuals in which the values of w_i are large enough not to constrain individual 3's selfishly-optimal plan, Figure 11.7a shows a plan that satisfies the analogues of the necessary conditions for optimality for the plan of individual 2 in a society with three individuals illustrated in Figure 11.5b. The diagram suggests that, under the assumptions on preferences that I am making, an individual has more leverage in raising taxes from individuals whose earning powers are further from hers. As a consequence, under some conditions the payoffs in the individuals' selfishly-optimal plans plausibly take the form given in Figure 11.7b, so that the plan of the individual with the median earning power is a strict Condorcet winner of the collective choice problem in which the set of



(a) Individual 1 is indifferent between (y_1, c_1) and $(0, 0)$.

(b) Individual 1's earning power is insufficient to allow her to obtain (y_2, c_2).

Figure 11.6 Transfer plans for individual 2 for a society with three individuals.

alternatives is the set of the individuals' selfishly-optimal plans.

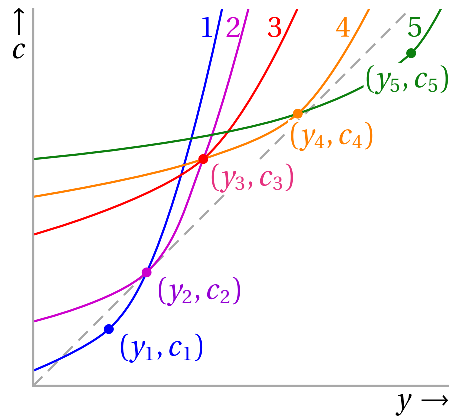
A computed example for a society with three individuals is shown in Figure 11.8. Each individual's preferences over the three selfishly-optimal plans are single-peaked with respect to the ordering $p_1^* < p_2^* < p_3^*$, where p_i^* denotes the selfishly-optimal plan of individual i, so that p_2^* is a strict Condorcet winner of the collective choice problem. No general result is available; see the Notes section at the end of the chapter for information about published results.

11.3 Voting over linear transfer systems

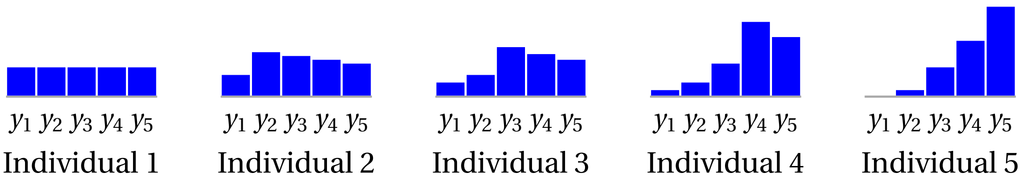
The model I now discuss differs in two main respects from the one in the previous section. First, the transfer system is restricted to be linear (more properly, affine): income is taxed at the constant rate t and every individual receives a fixed subsidy r. Second, the set of alternatives in the collective choice problem is the set of all linear transfer systems, not only the ones that are selfishly-optimal for some individual.

Definition 11.6: Linear transfer system

For any $(t, r) \in [0, 1] \times \mathbb{R}_+$ the *linear transfer system* (t, r) is the transfer system T for which $T(y) = ty - r$ for all $y \in \mathbb{R}_+$.



(a) A transfer plan of individual 3 for a society with five individuals.



(b) The payoffs assigned to the various income levels by the favorite transfer systems of the individuals in a model in which each individual can control her income.

Figure 11.7 Transfer plans for a society with five individuals.

11.3.1 *Exogenous incomes*

First suppose that the individuals' incomes are given; no individual makes a choice that affects her income. For budget balance we need $t\overline{y} = r$, where $\overline{y}$ is the individuals' average income. Then the after-tax income of an individual with income y is $y - t(y - \overline{y})$. This payoff is increasing in t if $y < \overline{y}$ and decreasing in t if $y > \overline{y}$, so that for $y \neq \overline{y}$ the individuals' preferences over t are single-peaked with respect to the ordering $\geq$ of t, with the favorite tax rate of an individual with income y equal to 1 if $y < \overline{y}$ and to 0 if $y > \overline{y}$. Suppose that the number of individuals is finite and none of them has an income of exactly $\overline{y}$. Consider a collective choice problem in which the set of alternatives is a finite set of feasible linear transfer systems that includes $(0,0)$ (no redistribution) and $(1,\overline{y})$ (complete equalization of incomes). Proposition 1.4 implies that the Condorcet winner of this problem is $(1,\overline{y})$ if the individuals' median income is less than the mean and $(0,0)$ if it is greater than the mean. An example of a distribution for which the median is less than the mean is shown in Figure 11.9.

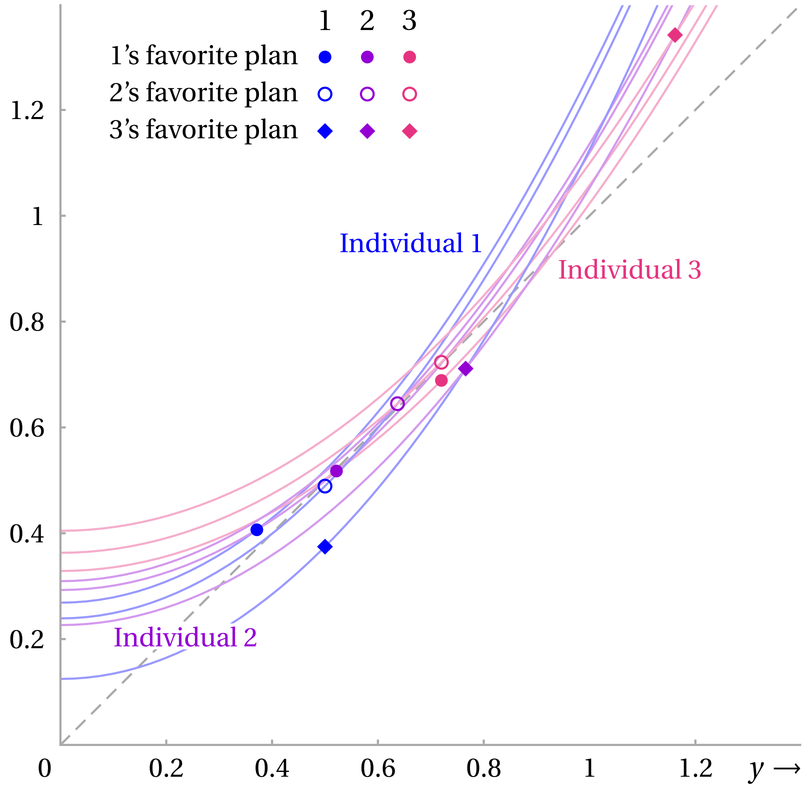


Figure 11.8 The selfishly-optimal transfer plans of the individuals in the society $\langle\{1,2,3\},(u,u,u),(w_1,w_2,w_3)\rangle$ where $w_1 = 2$, $w_2 = 2.2$, $w_3 = 2.4$, and $u(c,l) = c - 4(1-l)^2$ for all (c,l), so that $v_i(y,c) = c - 4(y/w_i)^2$ for $i = 1, 2, 3$.

11.3.2 Endogenous incomes and incentive effects

If an individual's income depends on the amount of time she works, an increase in the tax rate may cause her to devote less time to work, reducing the revenue from the tax. Thus an increase in the tax rate may not be desirable. In particular, a tax rate of 1 may no longer be preferred by a majority of individuals to any other rate, even if the distribution of earning power is skewed to the left.

Consider a society $\langle N,(u_i)_{i\in N},(w_i)_{i\in N}\rangle$. Suppose that the individuals' earnings are subject to a linear transfer system (t,r). If individual i chooses to work for h units of time then her payoff is $u_i((1-t)w_i h + r, 1-h)$. She chooses h to maximize this payoff. Denote by $h_i^*(t,r)$ a maximizer and by $V_i(t,r)$ her maximal payoff:

$$V_i(t,r) = \max_{h\in[0,1]} u_i((1-t)w_i h + r, 1-h) = u_i((1-t)w_i h_i^*(t,r) + r, 1-h_i^*(t,r)). \quad (11.12)$$

The next result applies to societies in which, independently of the transfer system, the individuals can be ordered by the pre-tax incomes generated when

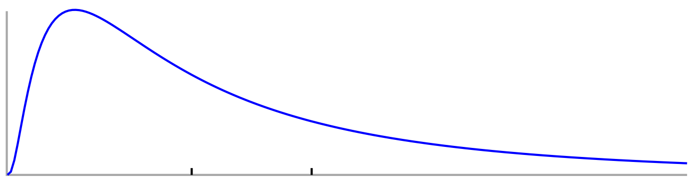


Figure 11.9 A distribution of income for which the median is less than the mean. (The distribution shown is a lognormal.)

they choose their hours of work optimally. That is, for any two individuals i and j, either $w_i h_i^*(t,r) > w_j h_j^*(t,r)$ for every linear transfer system (t,r) or $w_i h_i^*(t,r) < w_j h_j^*(t,r)$ for every such system.

Definition 11.7: Society with individuals ordered by pre-tax income

The individuals in a society $\langle N, (u_i)_{i\in N}, (w_i)_{i\in N}\rangle$ *can be ordered by pre-tax income* if for every linear transfer system (t,r) and every individual $i \in N$ the problem

$$\max_{h\in[0,1]} u_i((1-t)w_i h + r, 1-h)$$

has a unique solution $h_i^*(t,r)$, and for a linear order $\succeq$ on N

$$j \prec i \Leftrightarrow w_j h_j^*(t,r) > w_i h_i^*(t,r) \quad \text{for all } (t,r) \in [0,1]\times\mathbb{R}_+ \text{ with } h_i^*(t,r) > 0. \tag{11.13}$$

This condition is strong. Intuition suggests that in a diverse society the pre-tax incomes of some individuals are higher than those of other individuals under some transfer systems but lower under other transfer systems. Here are two examples.

Example 11.1: Cobb-Douglas payoff functions

Consider a society $\langle N, (u_i)_{i\in N}, (w_i)_{i\in N}\rangle$ for which for each $i \in N$ we have $u_i(y, 1-h) = y^{\beta_i}(1-h)^{1-\beta_i}$ for all y and h, where $\beta_i \in (0,1)$. Then $h_i^*(1,r) = 0$ and

$$w_i h_i^*(t,r) = w_i \max\left\{0, \beta_i - \frac{(1-\beta_i)r}{(1-t)w_i}\right\} \quad \text{if } t < 1.$$

Thus if w_i is the same for all $i \in N$, then $w_i h_i^*(t,r)$ is increasing in β_i when it is positive, so that the ordering of the individuals defined by $j \prec i$ if and only if $\beta_j > \beta_i$ satisfies a variant of (11.13) in which the second inequality is weak. Some indifference sets for $w_i = 1$ and various values of β_i

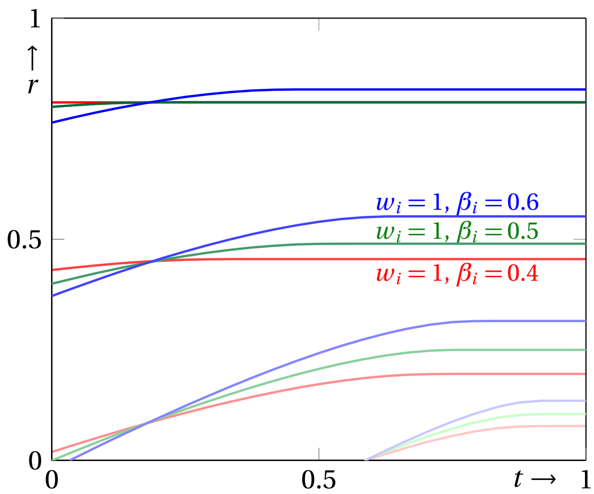


Figure 11.10 Indifference sets for the preferences over linear transfer systems of individuals with the payoff function specified in Example 11.1 for $w_i = 1$ and various values of β_i. The indifference sets for $\beta_i = 0.6$ are blue, those for $\beta_i = 0.5$ are green, and those for $\beta_i = 0.4$ are red; sets corresponding to higher payoffs are darker.

are shown in Figure 11.10. The function $w_i h_i^*(t, r)$ is not differentiable at any (t, r) for which $\beta_i = (1-\beta_i)r/((1-t)w_i)$, so that the function V_i is not differentiable at any such point. However, although the next result, Proposition 11.2, assume differentiability, it holds also if $w_i h_i^*(t, r)$ is piecewise differentiable, as it is in this example.

Alternatively, if β_i is the same for all $i \in N$, then $w_i h_i^*(t, r)$ is increasing in w_i when it is positive, so that again a variant of (11.13) is satisfied.

If the individuals differ in both w_i and β_i, (11.13) may be violated. For example, the pre-tax income of an individual with $w_i = 16$ and $\beta_i = 0.2$ is greater for the transfer system $(0.45, 0.25)$ than it is for the transfer system $(0.8, 0.8)$, but the pre-tax income of an individual with $w_i = 2$ and $\beta_i = 0.9$ is less for $(0.45, 0.25)$ than it is for $(0.8, 0.8)$.

Example 11.2: Quasilinear payoffs

Consider a society $\langle N, (u_i)_{i\in N}, (w_i)_{i\in N}\rangle$ for which for each $i \in N$ we have $u_i(y, 1-h) = y + v(1-h)$ for all y and h, where $v : [0,1] \to \mathbb{R}$ is an increasing concave differentiable function. Then

$$w_i h_i^*(t,r) = \begin{cases} 0 & \text{if } v'(1) \geq (1-t)w_i \\ w_i(1-(v')^{-1}((1-t)w_i)) & \text{if } v'(1) < (1-t)w_i < v'(0) \\ w_i & \text{if } v'(0) \leq (1-t)w_i \end{cases}$$

Given the concavity of v, this expression is nondecreasing in w_i. Thus a variant of (11.13) is satisfied by the ordering $\geq$ of the individuals defined by $j < i$ if and only if $w_j > w_i$.

Now consider a collective choice problem in which the alternatives are finitely many linear transfer systems and the preference relation of each individual i is represented by V_i, as defined in (11.12). If (t,r) and (t',r') are alternatives with $t' > t$ and $r' < r$, then every individual prefers (t,r) to (t',r'), so (t',r') can be eliminated from consideration. Thus we can assume that if (t,r) and (t',r') are alternatives with $t' > t$ then $r' \geq r$. For convenience, I make the stronger assumption that $t' > t$ if and only if $r' > r$.

The next result says that for a society in which the individuals can be ordered by pre-tax income, the collective choice problem has single-crossing preferences with respect to the ordering of the individuals by pre-tax income. Hence by Proposition 1.5, if each median individual according to the ordering has a unique favorite linear transfer system then the favorite linear transfer system of a median individual is a Condorcet winner of the problem. The key point in the argument is that the slope at any point (t,r) of i's indifference set for her payoff function V_i through (t,r) is her pre-tax income $w_i h_i^*(t,r)$, so that the assumption that the individuals can be ordered by pre-tax income implies that the slopes of their indifference sets are ordered independently of the transfer system.

Proposition 11.2: Single-crossing preferences over linear transfer systems

Let $\langle N, (u_i)_{i\in N}, (w_i)_{i\in N}\rangle$ be a society in which the individuals can be ordered by pre-tax income. For each $i \in N$ denote by $\succsim_i$ the preference relation represented by the function V_i defined in (11.12): $(t,r) \succsim_i (t',r')$ if and only if $V_i(t,r) \geq V_i(t',r')$. Assume that V_i is continuously differentiable and $V'_{i,2}(t,r) \neq 0$ for all $(t,r) \in [0,1] \times \mathbb{R}_+$.

Let $\mathscr{T}$ be a finite set of linear transfer systems such that if $(t,r) \in \mathscr{T}$ and $(t',r') \in \mathscr{T}$ then $t' < t$ if and only if $r' < r$. Then the collective choice problem $\langle N, \mathscr{T}, \succsim\rangle$ has single-crossing preferences with respect to the ordering of the individuals by pre-tax income.

As a consequence, if each median individual according to this ordering has a unique favorite alternative in $\mathscr{T}$ then

- if the number of individuals is even then each of these favorite alternatives is a Condorcet winner of $\langle N, \mathscr{T}, \succsim\rangle$
- if the number of individuals is odd then the favorite alternative of the (unique) median individual is the strict Condorcet winner of $\langle N, \mathscr{T}, \succsim\rangle$.

Proof

Let (t^1, r^1) be a linear transfer system, let $i \in N$, and consider i's indifference set that contains (t^1, r^1):

$$\{(t,r) \in [0,1] \times \mathbb{R}_+ : V_i(t,r) = V_i(t^1, r^1)\}.$$

By the implicit function theorem, the function g on $(0,1)$ defined by $V_i(t, g(t)) = V_i(t^1, r^1)$ (i's indifference set through (t^1, r^1)) is continuously differentiable and $g'(t^1) = -V'_{i,1}(t^1, r^1)/V'_{i,2}(t^1, r^1)$. By the envelope theorem

$$V'_{i,1}(t,r) = -w_i h_i^*(t,r) u'_{i,1}((1-t)w_i h_i^*(t,r) + r, 1 - h_i^*(t,r))$$
$$V'_{i,2}(t,r) = u'_{i,1}((1-t)w_i h_i^*(t,r) + r, 1 - h_i^*(t,r)),$$

so that

$$-\frac{V'_{i,1}(t,r)}{V'_{i,2}(t,r)} = w_i h_i^*(t,r). \tag{11.14}$$

That is, for any linear transfer system (t,r), the slope at (t,r) of i's indifference set through (t,r) is her pre-tax income when she chooses her hours of work optimally, given (t,r).

I argue that the collective choice problem $\langle N, \mathscr{T}, \succsim\rangle$ has single-crossing preferences with respect to the ordering $\geq$ of the individuals by pre-tax income. For any individual i and linear transfer system (t,r) with $h_i^*(t,r) > 0$, if $j < i$ then by (11.13) and (11.14) the slope of j's indifference set through (t,r) at (t,r) is greater than the slope of i's indifference set through (t,r) at (t,r), as for (t^1, r^1) in Figure 11.11. Thus j's indifference set through (t,r) lies above i's for all tax rates greater than t and below it for all tax rates less than t.

Now suppose that for $(t^1, r^1) \in \mathscr{T}$ and $(t^2, r^2) \in \mathscr{T}$ we have $t^1 < t^2$, so that $r^1 < r^2$, and $(t^1, r^1) \succsim_i (t^2, r^2)$. Then given $r^2 > r^1$, the slope at (t^1, r^1)

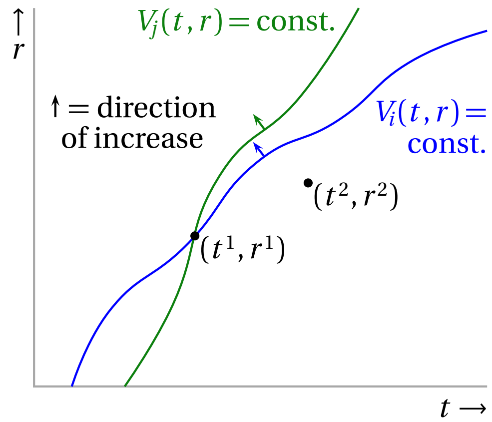


Figure 11.11 An illustration of the argument in the proof of Proposition 11.2. Individual i prefers (t^1, r^1) to (t^2, r^2), and so does any individual j whose indifference sets are more steeply sloped.

of i's indifference set through (t^1, r^1) is positive, so $h_i^*(t^1, r^1) > 0$ by (11.14). Thus given the property of the indifference sets in the previous paragraph, $(t^1, r^1) \succ_j (t^2, r^2)$, as in Figure 11.11. Similarly, if $t^1 < t^2$, $(t^2, r^2) \succsim_i (t^1, r^1)$, and $j > i$ then $w_j h_j^*(t, r) < w_i h_i^*(t, r)$, so that $(t^2, r^2) \succ_j (t^1, r^1)$. Thus the conditions for single-crossing are satisfied.

The claims about the Condorcet winners follow from Proposition 1.5.

Suppose that (11.13) is satisfied by an ordering of the individuals by earning power (w_i) and that the political system generates a transfer system that is a Condorcet winner among the alternatives. Then by Proposition 11.2 the linear transfer system that the political system generates is the favorite, among the alternatives, of the individual with median earning power. The slope of an individual's indifference set is her pre-tax income (see (11.14)), so if individuals with different pre-tax incomes disagree on the ordering of transfer systems, the individual with the lower pre-tax income prefers the system with a higher tax rate (and higher fixed component). Thus if, in a given society, the voting franchise is expanded among individuals with low earning power, so that the median individual's earning power decreases, the tax rate generated by the political system does not decrease, and may increase.

Now consider an electoral competition game with two office-motivated candidates. Proposition 8.1 says that in a Nash equilibrium of such a game the policy chosen by each candidate is a Condorcet winner of the underlying collective choice problem. Thus Proposition 11.2 implies that in such a Nash equilibrium each candidate selects the favorite tax system of the individual with median pre-tax income if the number of individuals is odd, and the favorite tax system of an individual whose pre-tax income is one of the medians if the number of individuals is even.

11.4 Coalitional bargaining over redistribution

The ideas underlying the analysis in this section are that the size and wealth of each group in society determine its power, and the distribution of power shapes the factors that determine taxes and subsidies, like the electoral system, the rules under which it operates (e.g. the rules on campaign spending), and the mechanisms by which a government makes decisions. However, the electoral system and the mechanisms of government decision-making are not modeled explicitly. Instead, the model aims to deduce directly from the distribution of power the distribution of payoffs that emerges.

The setting for the model is a society in which each individual is endowed with an amount of a consumption good; she does not have to work to obtain this good. Each individual's payoff is the amount of the good she ultimately obtains, after she pays the tax or receives the subsidy specified by the transfer system. For convenience, the number of individuals in the economy is assumed to be odd.

Definition 11.8: Endowed society

An *endowed society* $\langle N, (e_i)_{i \in S} \rangle$ consists of

- a finite set N (of individuals) with an odd number of members that is at least 3
- a number $e_i \geq 0$ for each $i \in N$ (the amount of a consumption good with which i is endowed).

A nonempty subset of N is a *coalition*. For any coalition S, $e(S) = \sum_{i \in S} e_i$, the total endowment of S.

The power of each coalition is delimited by two central assumptions:

- any majority has the option to expropriate any amount of the endowment of the complementary minority
- any individual has the option to destroy her endowment (the right to strike, if you like, although no one has to work to obtain their endowment).

The following definition includes the assumption that such actions exist, and also the assumption that for any distribution of the total endowment $e(N)$ and any coalition S, actions for S and its complement $N \setminus S$ exist that achieve that distribution.

Definition 11.9: Coalitional redistribution game

A *coalitional redistribution game* $\langle N, (e_i)_{i\in N}, (A_S)_{S\subseteq N}, (h_S)_{S\subseteq N}\rangle$ consists of an endowed society $\langle N, (e_i)_{i\in N}\rangle$ and, for each coalition S, a set A_S of actions and a payoff function $h_S : A_S \times A_{N\setminus S} \to \mathbb{R}_+$ such that

a. for every distribution of the society's total endowment $e(N)$ between S and $N \setminus S$ there are actions in A_S and $A_{N\setminus S}$ that achieve the distribution:

$$\{(h_S(\sigma_S, \sigma_{N\setminus S}), h_{N\setminus S}(\sigma_S, \sigma_{N\setminus S})) : (\sigma_S, \sigma_{N\setminus S}) \in A_S \times A_{N\setminus S}\} \\ = \{(\pi_S, \pi_{N\setminus S}) \in \mathbb{R}^2_+ : \pi_S + \pi_{N\setminus S} \le e(N)\}$$

b. if S is a majority (has more than $\frac{1}{2}|N|$ members) then there exists $\widehat{\sigma}_S \in A_S$ such that

$$h_S(\widehat{\sigma}_S, \sigma_{N\setminus S}) \ge e(S) \text{ and } h_{N\setminus S}(\widehat{\sigma}_S, \sigma_{N\setminus S}) = 0 \text{ for all } \sigma_{N\setminus S} \in A_{N\setminus S} \quad (11.15)$$

and $\widehat{\sigma}_{N\setminus S} \in A_{N\setminus S}$ such that

$$h_S(\sigma_S, \widehat{\sigma}_{N\setminus S}) \le e(S) \text{ and } h_{N\setminus S}(\sigma_S, \widehat{\sigma}_{N\setminus S}) = 0 \text{ for all } \sigma_S \in A_{S.} \quad (11.16)$$

(The action $\widehat{\sigma}_S$ may be interpreted as the expropriation of any endowment of $N \setminus S$ that $N \setminus S$ does not destroy, and $\widehat{\sigma}_{N\setminus S}$ may be interpreted as the destruction of $N \setminus S$'s endowment.)

In the solution concepts I use for a coalitional redistribution game, the *possibility* of each group's using its extreme actions (expropriation, destruction of endowment) determines the distribution of payoffs. No group takes those extreme actions; indeed, no coalition is singled out as the one that forms. But the compromise is shaped by the possibility of these actions.

To specify the solution concepts, first we derive an index of the strength of each coalition S by analyzing two-player games in which the players are S and $N \setminus S$. For reasons that will become apparent, I refer to these games as "threat games". Then we derive a distribution of payoffs that balances these strengths—a compromise. I now describe each of these components in detail.

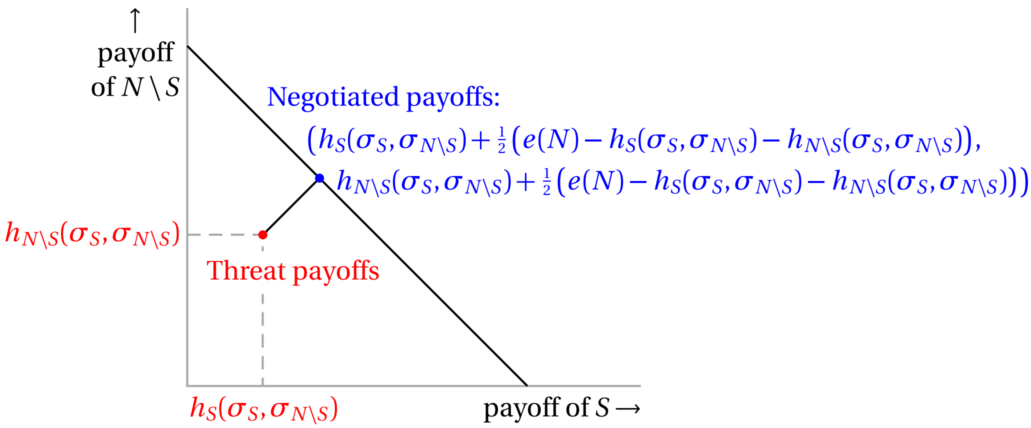


Figure 11.12 The negotiated payoffs in a threat game for a given pair of threats.

Threat games

Fix a coalition S and consider a two-player strategic game in which the players are S and its complement $N \setminus S$. The action chosen in this game by each player is interpreted as the action the player will take if negotiations break down—a threat. The sum of the payoffs of S and $N \setminus S$ when these threats are carried out is typically less than the total payoff available, $e(N)$. The model assumes that bargaining results in S and $N \setminus S$ splitting equally the difference between $e(N)$ and this sum. Each player knows that her negotiated payoff is determined in this way, and chooses her threat to maximize her negotiated payoff, given the threat chosen by the other player.

More precisely, suppose that the players choose the actions (threats) σ_S and $\sigma_{N\setminus S}$. If they carry out these threats, their payoffs are $h_S(\sigma_S,\sigma_{N\setminus S})$ and $h_{N\setminus S}(\sigma_S,\sigma_{N\setminus S})$, and hence the surplus they forego is $e(N)-(h_S(\sigma_S,\sigma_{N\setminus S})+h_{N\setminus S}(\sigma_S,\sigma_{N\setminus S}))$. In the negotiated outcome, this surplus is split equally between them, so that their negotiated payoffs are

$$u_S(\sigma_S,\sigma_{N\setminus S}) = h_S(\sigma_S,\sigma_{N\setminus S})+\tfrac{1}{2}(e(N)-h_S(\sigma_S,\sigma_{N\setminus S})-h_{N\setminus S}(\sigma_S,\sigma_{N\setminus S}))$$
$$u_{N\setminus S}(\sigma_S,\sigma_{N\setminus S}) = h_{N\setminus S}(\sigma_S,\sigma_{N\setminus S})+\tfrac{1}{2}(e(N)-h_S(\sigma_S,\sigma_{N\setminus S})-h_{N\setminus S}(\sigma_S,\sigma_{N\setminus S})).$$

These payoffs are illustrated in Figure 11.12.

Definition 11.10: Threat game between S and $N \setminus S$

Given a coalitional redistribution game $\langle N,(e_i)_{i\in N},(A_S)_{S\subseteq N},(h_S)_{S\subseteq N}\rangle$ and a coalition $S \subset N$ with more than $\frac{1}{2}|N|$ members, the *threat game* $\langle N,(e_i)_{i\in N},A_S,A_{N\setminus S},h_S,h_{N\setminus S}\rangle$ *between S and $N\setminus S$* is the following two-player strategic game.

Players

S and $N \setminus S$.

Actions

The sets of actions of S and $N \setminus S$ are A_S and $A_{N\setminus S}$.

Payoffs

The payoff functions $u_S : A_S \times A_{N\setminus S} \to \mathbb{R}$ of S and $u_{N\setminus S} : A_S \times A_{N\setminus S} \to \mathbb{R}$ of $N \setminus S$ are defined by

$$\begin{aligned} u_S(\sigma_S, \sigma_{N\setminus S}) &= \tfrac{1}{2}\big(e(N) + h_S(\sigma_S, \sigma_{N\setminus S}) - h_{N\setminus S}(\sigma_S, \sigma_{N\setminus S})\big) \\ u_{N\setminus S}(\sigma_S, \sigma_{N\setminus S}) &= \tfrac{1}{2}\big(e(N) - h_S(\sigma_S, \sigma_{N\setminus S}) + h_{N\setminus S}(\sigma_S, \sigma_{N\setminus S})\big) \end{aligned} \qquad (11.17)$$

for all $(\sigma_S, \sigma_{N\setminus S})$.

I assume that each player chooses her action (threat) in this game to maximize her payoff, given the other player's action. That is, the pair of actions is a Nash equilibrium; we may aptly characterize it as a pair of optimal threats. For every pair $(\sigma_S, \sigma_{N\setminus S})$ of actions in the game, the sum of the players' payoffs is the same, equal to $e(N)$, so that the game is strictly competitive and hence every Nash equilibrium yields the same pair of payoffs (Proposition 16.5). The next result shows that the pair $(\widehat{\sigma}_S, \widehat{\sigma}_{N\setminus S})$ of extreme actions given in Definition 11.9b is a pair of optimal threats, and calculates the resulting negotiated payoffs.

Lemma 11.1: Nash equilibrium of threat game

Let $\langle N, (e_i)_{i\in N}, (A_S)_{S\subseteq N}, (h_S)_{S\subseteq N}\rangle$ be a coalitional redistribution game and let $S \subset N$ be a coalition with more than $\frac{1}{2}|N|$ members. The pair $(\widehat{\sigma}_S, \widehat{\sigma}_{N\setminus S})$ of actions given in Definition 11.9b is a Nash equilibrium of the threat game $\langle N, (e_i)_{i\in N}, A_S, A_{N\setminus S}, h_S, h_{N\setminus S}\rangle$, and in every Nash equilibrium the payoff of S is $\frac{1}{2}\big(e(N) + e(S)\big)$ and that of $N \setminus S$ is $\frac{1}{2}\big(e(N) - e(S)\big) = \frac{1}{2}e(N \setminus S)$.

Proof

We have $h_S(\widehat{\sigma}_S, \widehat{\sigma}_{N\setminus S}) = e(S)$ and $h_{N\setminus S}(\widehat{\sigma}_S, \widehat{\sigma}_{N\setminus S}) = 0$ by (11.15) and (11.16), so

$$\begin{aligned} u_S(\widehat{\sigma}_S, \widehat{\sigma}_{N\setminus S}) &= \tfrac{1}{2}\big(e(N) + e(S)\big) \\ u_{N\setminus S}(\widehat{\sigma}_S, \widehat{\sigma}_{N\setminus S}) &= \tfrac{1}{2}\big(e(N) - e(S)\big), \end{aligned}$$

where u_S and $u_{N\setminus S}$ are the payoff functions in the threat game, given in

(11.17). Now, for any $\sigma_S \in A_S$ we have

$$u_S(\sigma_S, \widehat{\sigma}_{N\setminus S}) = \tfrac{1}{2}\big(e(N) + h_S(\sigma_S, \widehat{\sigma}_{N\setminus S}) - h_{N\setminus S}(\sigma_S, \widehat{\sigma}_{N\setminus S})\big) \leq \tfrac{1}{2}\big(e(N) + e(S)\big),$$

where the inequality follows from (11.16), and for any $\sigma_{N\setminus S} \in A_{N\setminus S}$ we have

$$u_S(\widehat{\sigma}_S, \sigma_{N\setminus S}) = \tfrac{1}{2}\big(e(N) - h_S(\widehat{\sigma}_S, \sigma_{N\setminus S}) + h_{N\setminus S}(\widehat{\sigma}_S, \sigma_{N\setminus S})\big) \leq \tfrac{1}{2}\big(e(N) - e(S)\big),$$

where the inequality follows from (11.15). Thus $(\widehat{\sigma}_S, \widehat{\sigma}_{N\setminus S})$ is a Nash equilibrium of the game.

For every $(\sigma_S, \sigma_{N\setminus S}) \in A_S \times A_{N\setminus S}$ we have $u_S(\sigma_S, \sigma_{N\setminus S}) + u_{N\setminus S}(\sigma_S, \sigma_{N\setminus S}) = e(N)$, so the threat game is strictly competitive. Hence it has a unique Nash equilibrium payoff pair (Proposition 16.5).

Compromise

The model uses the Nash equilibrium payoffs in the threat game between S and $N \setminus S$ as measures of the strengths of S and $N \setminus S$, for each coalition S. The coalitional game in which the worth of each coalition is its equilibrium payoff in the threat game is called the Harsanyi coalitional form of the redistribution game (after John C. Harsanyi, 1920–2000).

Definition 11.11: Harsanyi coalitional form of coalitional redistribution game

The *Harsanyi coalitional form* of the coalitional redistribution game $\langle N, (e_i)_{i\in N}, (A_S)_{S\subseteq N}, (h_S)_{S\subseteq N}\rangle$ is the coalitional game with transferable payoff $\langle N, v\rangle$ in which the worth $v(S)$ of each coalition S is the payoff of S in a Nash equilibrium of the threat game $\langle N, (e_i)_{i\in N}, A_S, A_{N\setminus S}, h_S, h_{N\setminus S}\rangle$ between S and $N \setminus S$.

The Harsanyi coalitional form of a coalitional redistribution game is given in the following result, which follows immediately from Lemma 11.1.

Lemma 11.2: Harsanyi coalitional form of coalitional redistribution game

Let $\langle N, (e_i)_{i\in N}, (A_S)_{S\subseteq N}, (h_S)_{S\subseteq N}\rangle$ be a coalitional redistribution game. The Harsanyi coalitional form of this game is the coalitional game $\langle N, v\rangle$ for which

$$v(S) = \begin{cases} \frac{1}{2}\big(e(N) + e(S)\big) & \text{if } |S| > \frac{1}{2}|N| \\ \frac{1}{2}e(S) & \text{if } |S| < \frac{1}{2}|N| \end{cases} \tag{11.18}$$

for each $S \subseteq N$.

Giving every coalition S its worth $v(S)$ in the Harsanyi coalitional form is not feasible: no distribution of the total endowment yields these payoffs. To see why, consider a distribution of the total endowment in which each individual $i \in N$ receives x_i, so that $\sum_{i \in N} x_i = e(N)$. For the total payoff of every coalition S to be its worth $v(S)$ in the Harsanyi coalitional form, we need $\sum_{i \in S} x_i = v(S)$ for every coalition S. Denote by $\mathscr{S}$ the set of coalitions that are bare majorities, with $k = \frac{1}{2}(n+1)$ members, and by l the number of such coalitions. For each coalition $S \in \mathscr{S}$ we have $v(S) = \frac{1}{2}(e(N) + e(S))$ by (11.18), so for the equality $\sum_{i \in S} x_i = v(S)$ to hold for all $S \in \mathscr{S}$ we need

$$\sum_{S \in \mathscr{S}} \sum_{i \in S} x_i = \tfrac{1}{2} \sum_{S \in \mathscr{S}} (e(N) + e(S)) = \tfrac{1}{2} l\, e(N) + \tfrac{1}{2} \sum_{S \in \mathscr{S}} \sum_{i \in S} e_i.$$

Reversing the order of each double summation and using the fact that each individual belongs to kl/n of the coalitions in $\mathscr{S}$, the left-hand side is

$$\sum_{i \in N} \sum_{\{S \in \mathscr{S} : i \in S\}} x_i = (kl/n) \sum_{i \in N} x_i = (kl/n) e(N)$$

and the right-hand side is

$$\tfrac{1}{2} l\, e(N) + \tfrac{1}{2} \sum_{i \in N} \sum_{\{S \in \mathscr{S} : i \in S\}} e_i = \tfrac{1}{2} l\, e(N) + \tfrac{1}{2}(kl/n) \sum_{i \in N} e_i = \tfrac{1}{2} l\, e(N) + \tfrac{1}{2}(kl/n) e(N).$$

Thus the two sides are equal if and only if $k = n$, which is not satisfied for any value of $n \geq 3$. Hence the distribution on which the individuals agree cannot give each coalition S its worth $v(S)$; it must entail compromise.

I present two models of compromise. One is the Shapley value, which assigns a payoff to each individual based on the impact of her membership of a coalition on the coalition's worth. Suppose that the individuals arrive in a given order. Let i be an individual, and let S be the set of individuals who arrive before i. Then i's arrival increases the worth of the set of individuals who have arrived by $v(S \cup \{i\}) - v(S)$. We can think of this amount as i's contribution for this order of arrival. The Shapley value assigns to each individual the average of her contributions over all orders. A property that imparts to it the flavor of a compromise is that the amount by which the payoff it assigns to any individual j decreases when any another individual i is excluded from the game is the same for all i and j (Proposition 16.11); no other solution concept has this property.

The next result says that the Shapley value of the Harsanyi coalitional form of a coalitional redistribution game involves a subsidy equal to half of the total endowment and a 50% tax rate.

Proposition 11.3: Shapley value of coalitional redistribution game

Let $\langle N,(e_i)_{i\in N},(A_S)_{S\subseteq N},(h_S)_{S\subseteq N}\rangle$ be a coalitional redistribution game and let $\langle N,v\rangle$ be its Harsanyi coalitional form. The Shapley value of $\langle N,v\rangle$ assigns the payoff

$$\tfrac{1}{2}(\overline{e}+e_i)$$

to each individual $i\in N$, where $\overline{e}=e(N)/|N|$, the average endowment.

Proof

Let $n=|N|$.

Step 1 *The Shapley value of $\langle N,v\rangle$ is the same as the Shapley value of the coalitional game $\langle N,q\rangle$ for which*

$$q(S)=\begin{cases} e(S) & \textit{if}\, |S|>\tfrac{1}{2}n \\ 0 & \textit{if}\, |S|<\tfrac{1}{2}n.\end{cases}$$

Proof. By Lemma 16.1 the Shapley value of $\langle N,q\rangle$ is equal to the Shapley value of the game $\langle N,q^{\#}\rangle$ where $q^{\#}(S)=q(N)-q(N\setminus S)$ for each coalition S, or

$$q^{\#}(S)=\begin{cases} e(N) & \text{if } |S|>\tfrac{1}{2}n \\ e(S) & \text{if } |S|<\tfrac{1}{2}n.\end{cases}$$

By Lemma 11.2 we have $v(S)=\tfrac{1}{2}q(S)+\tfrac{1}{2}q^{\#}(S)$ for each coalition S, so the additivity of the Shapley value implies the result. $\triangleleft$

Step 2 *The Shapley value of $\langle N,q\rangle$ assigns to each individual $i\in N$ the payoff $\tfrac{1}{2}(\overline{e}+e_i)$.*

Proof. The payoff of individual i in the Shapley value of $\langle N,q\rangle$ is the average, over all orderings R of N, of i's marginal contribution $q(S_i^R\cup\{i\})-q(S_i^R)$ in the ordering R, where S_i^R is the set of individuals who precede i in R (see (16.3)).

- If i's position in R is $\tfrac{1}{2}(n-1)$ or less, her marginal contribution is 0, because $q(S)=0$ if $|S|<\tfrac{1}{2}n$.

- If i's position in R is $\frac{1}{2}(n+1)$ (the middle position), her marginal contribution is $e(S_i^R \cup \{i\})$, because $q(S_i^R) = 0$ and $q(S_i^R \cup \{i\}) = e(S_i^R \cup \{i\})$. There are $(n-1)!$ orderings in which i is in this position, and each other individual comes before i in half of these orderings and after i in the other half, so the sum of i's marginal contributions over all the orderings is
$$(n-1)!\left[\tfrac{1}{2}e(N \setminus \{i\}) + e_i\right].$$

- If i's position in R is $\frac{1}{2}(n+1)+1$ or greater, her marginal contribution is e_i, because $q(S_i^R) = e(S_i^R)$ and $q(S_N^R \cup \{i\}) = e(S_i^R \cup \{i\}) = e(S_i^R) + e_i$. Thus the sum of i's marginal contributions over all the orderings in which she has a given position of $\frac{1}{2}(n+1)+1$ or greater is $(n-1)!e_i$. There are $\frac{1}{2}(n-1)$ such positions for her, so the sum of her marginal contributions over all orderings in which her position is $\frac{1}{2}(n+1)+1$ or greater is
$$\tfrac{1}{2}(n-1)(n-1)!\,e_i.$$

The average of these marginal contributions is
$$\frac{(n-1)!}{n!}\left[\tfrac{1}{2}e(N \setminus \{i\}) + (1 + \tfrac{1}{2}(n-1))\,e_i\right] = \frac{1}{2n}e(N \setminus \{i\}) + \frac{n+1}{2n}e_i = \tfrac{1}{2}\bar{e} + \tfrac{1}{2}e_i,$$
where $\bar{e}$ is the average endowment, $e(N)/n$. ◁

Another approach to modeling compromise is related to the solution concept of the core. The core of a coalitional game with transferable payoff $\langle N, v\rangle$ is the set of distributions of payoff among the players with the property that the total payoff of every coalition S is at least $v(S)$. The idea is that each coalition S can obtain $v(S)$ by itself, so that for a distribution of payoff among the members of N to be stable, each coalition S must be assigned a total payoff of at least $v(S)$. I have argued that for the Harsanyi coalitional form $\langle N, v\rangle$ of a coalitional redistribution game no payoff distribution satisfies this property, so the core of such a game is empty. For any payoff distribution y, we can view $v(S) - y(S)$, where $y(S) = \sum_{i \in S} y_i$, as the extent of S's dissatisfaction with y. Suppose that, in the absence of a distribution in which every coalition is satisfied, we look for a distribution that minimizes the dissatisfaction. Precisely, we look for a distribution for which the maximal dissatisfaction across all coalitions is minimal. It turns out that the only distribution with this property is the one for which $y_i = \frac{1}{2}(\bar{e} + e_i)$ for all $i \in N$, as in the Shapley value.

Proposition 11.4: Dissatisfaction-minimizing payoff distribution of coalitional redistribution game

Let $\langle N, (e_i)_{i\in N}, (A_S)_{S\subseteq N}, (h_S)_{S\subseteq N}\rangle$ be a coalitional redistribution game and let $\langle N, v\rangle$ be its Harsanyi coalitional form. Exactly one payoff distribution $(y_i)_{i\in N}$ minimizes $\max_{S\subseteq N}(v(S)-y(S))$, namely the one that assigns the payoff

$$\tfrac{1}{2}(\overline{e}+e_i)$$

to each individual $i \in N$, where $\overline{e} = e(N)/|N|$, the average endowment.

Proof

Let $n = |N|$ and $z_i = \frac{1}{2}(\overline{e}+e_i)$ for each $i \in N$. I first argue that the payoff distribution $(y_i)_{i\in N} = (z_i)_{i\in N}$ minimizes $\max_{S\subseteq N}(v(S)-y(S))$.

Using Lemma 11.2 we have

$$\begin{aligned} v(S)-z(S) &= \begin{cases} \frac{1}{2}\big(e(N)+e(S)-e(N)|S|/n-e(S)\big) & \text{if } |S| > \frac{1}{2}n \\ \frac{1}{2}\big(e(S)-e(N)|S|/n-e(S)\big) & \text{if } |S| < \frac{1}{2}n \end{cases} \\ &= \begin{cases} \frac{1}{2}e(N)(1-|S|/n) & \text{if } |S| > \frac{1}{2}n \\ -\frac{1}{2}e(N)|S|/n & \text{if } |S| < \frac{1}{2}n, \end{cases} \end{aligned}$$

so that the solutions of $\max_{S\subseteq N}(v(S)-z(S))$ are the coalitions of size $\frac{1}{2}(n+1)$ (a bare majority), and the maximum is $\frac{1}{2}e(N)(1-\frac{1}{2}(n+1)/n) = \frac{1}{4}(n-1)e(N)/n$.

Now suppose, contrary to the claim, that there is a payoff distribution $(y_i)_{i\in N}$ for which $v(S)-y(S) < \frac{1}{4}(n-1)e(N)/n$ for every coalition S. For a coalition with $\frac{1}{2}(n+1)$ members, this inequality is $\frac{1}{2}e(N)+\frac{1}{2}e(S)-y(S) < \frac{1}{4}(n-1)e(N)/n$, so that we need

$$y(S) > \tfrac{1}{2}e(S)+\tfrac{1}{4}(n+1)e(N)/n \text{ for every coalition } S \text{ with } |S| = \tfrac{1}{2}(n+1).$$

Denote by k the number of coalitions with $\frac{1}{2}(n+1)$ members. Each individual is a member of $\frac{1}{2}(n+1)k/n$ of these coalitions, so that adding the inequalities over all coalitions with $\frac{1}{2}(n+1)$ members we get

$$\begin{aligned} \tfrac{1}{2}(n+1)ky(N)/n &> \tfrac{1}{4}(n+1)ke(N)/n+\tfrac{1}{4}(n+1)ke(N)/n \\ &= \tfrac{1}{2}(n+1)ke(N)/n, \end{aligned}$$

which violates $y(N) = e(N)$, as required by the feasibility of $(y_i)_{i\in N}$.

Therefore the payoff distribution $(y_i)_{i \in N} = (z_i)_{i \in N}$ is a minimizer of $\max_{S \subseteq N}(v(S) - y(S))$. It is the only minimizer because if $(x_i)_{i \in N}$ is a payoff distribution that differs from $(z_i)_{i \in N}$ then $z(S) > x(S)$ for some bare majority S, so that $v(S) - x(S) > v(S) - z(S) = \frac{1}{4}(n-1)e(N)/n$, the value of $\max_{S \subseteq N}(v(S) - z(S))$.

Comments

- The analysis is limited to individuals with linear payoff functions, so that transferring endowment among individuals is equivalent to transferring payoff, which allows us to use the solution concepts of the Shapley value and the dissatisfaction-minimizing payoff distribution. If payoff is not transferable, designing appealing solution concepts that model compromise is challenging.
- The solution concept of the dissatisfaction-minimizing payoff distribution is closely related to that of the nucleolus, a standard solution concept for coalitional games with transferable payoff. The nucleolus consists of the payoff distributions $(y_i)_{i \in N}$ that minimize the largest dissatisfaction $v(S) - y(S)$ and, subject to doing so, minimize the second largest dissatisfaction, and so forth. (See, for example, Moulin 1988, Section 5.4 and Osborne and Rubinstein 1994, Section 14.3.3.)
- Every possible coalition is treated in the same way by both solution concepts. However, coalitions may differ in the likelihood that they exercise their bargaining power. For example, a coalition of poor (low endowment) individuals may be more likely than one consisting of a mix of rich and poor to do so, because of a shared identify or a more easily defined common purpose. In such cases, a different analysis may be appropriate.
- If we modify a coalitional redistribution game so that individuals cannot destroy their endowments, then every majority coalition can obtain the entire endowment of society, and an individual with a large endowment is no longer any more powerful than one with a small endowment. In this case, both solution concepts assign every individual the same payoff, equal to the average endowment.
- In an endowed society, incentive effects are absent: no individual chooses how much to work, so that taxation does not affect the total amount of payoff available.

Exercise 11.1: Variant of coalitional redistribution game in which wealth conveys power

Consider a variant of a coalitional redistribution game in which a coalition can expropriate its complement if it has a majority of the wealth rather than a majority of the votes. That is, the condition in part b of Definition 11.9 that S has more than $\frac{1}{2}|N|$ members is replaced by the condition that $e(S) > \frac{1}{2}e(N)$ (or equivalently $e(S) > e(N \setminus S)$). Assume that no coalition has exactly half of the total endowment $e(N)$ (to avoid having to specify the actions available to such a coalition). Arguments parallel to those in Step 1 of Proposition 11.3 lead to the conclusion that the Shapley value of the Harsanyi coalitional form $\langle N, v\rangle$ of this game is the Shapley value of the variant of the coalitional game $\langle N, q\rangle$ in that step in which the conditions $|S| > \frac{1}{2}n$ and $|S| < \frac{1}{2}n$ are replaced by $e(S) > \frac{1}{2}e(N)$ and $e(S) < \frac{1}{2}e(N)$. For an endowed society $\langle N, (e_i)_{i \in N}\rangle$ in which one individual has more than half of the total endowment, what is the Shapley value of this game? Which payoff distributions $(y_i)_{i \in N}$ minimize $\max_{S \subseteq N}(v(S) - y(S))$?

Notes

The model in Section 11.1 is due Coughlin (1986) and Lindbeck and Weibull (1987). (Much of the analysis in these papers concerns a model in which each candidate's objective is to maximize her expected vote share rather than her probability of winning. For the reasons discussed in the comment on page 244, I do not consider this model.)

The model and analysis in Section 11.2 are based on Röell (2012) (a revised version of a paper from 1996); my exposition draws also on Brett and Weymark (2017, 2020).[1]

[1] These papers state results that may appear to be stronger than the ones that I discuss. Note, however, the following points. 1. Röell's results rest on her Theorem 6, the proof of which is incomplete because it does not show that the variable Γ defined in equation (31) of the paper is nonzero. 2. The model in Brett and Weymark (2020) does not impose an upper bound on hours of work, so that some of the issues I discuss do not arise. 3. Both Röell (2012) and Brett and Weymark (2020) show only that the individuals' preferences satisfy a variant of single-peakedness in which the preference inequalities in (1.3) are weak (Theorem 7 in Röell 2012, Theorem 4 in Brett and Weymark 2020). This property, which is satisfied by the payoffs in Figure 11.2 for the model in which the individuals care only about their consumption, is not sufficient for the existence of a Condorcet winner (see Exercise 1.10), so that Theorem 5 in Brett and Weymark (2020) does not follow from Theorem 4. In personal correspondence, Weymark argues that the proof of Theorem 4 may be modified to show that each individual's preferences have a single plateau, with strict preferences on each side of the plateau, in which case at least one of the favorite alterna-

The model in Section 11.3.2 was first studied by Itsumi (1974) and Romer (1975), who consider whether individuals' preferences over linear tax systems are single-peaked. The single-crossing condition was developed by Rothstein (1990, 1991) (under the name order restricted preferences) and Gans and Smart (1996). Proposition 11.2 is based on Roberts (1977, Theorem 2) and Gans and Smart (1996, Proposition 1).

The model in Section 11.4 is a variant with finitely many individuals and transferable payoff of the one in Aumann and Kurz (1977); Proposition 11.3 is a version of their main result.

Solutions to exercises

Exercise 11.1

First consider the Shapley value. The variant of the coalitional game $\langle N, q\rangle$ in Step 1 of Proposition 11.3 is the coalitional game $\langle N, q'\rangle$ where

$$q'(S) = \begin{cases} e(S) & \text{if } e(S) > \frac{1}{2}e(N) \\ 0 & \text{if } e(S) < \frac{1}{2}e(N). \end{cases}$$

Let individual 1 be the one who has more than half of the total endowment $e(N)$. The marginal contribution $q'(S_i^R \cup \{i\}) - q'(S_i^R)$ of any individual $i \neq 1$ in an ordering R is 0 if she precedes individual 1 ($q'(S_i^R \cup \{i\}) = q'(S_i^R) = 0$) and her endowment e_i if she follows individual 1 ($q'(S_i^R \cup \{i\}) = e(S_i^R) + e_i$ and $q'(S_i^R) = e(S_i^R)$). Every individual $i \neq 1$ precedes individual 1 in half of the orderings and follows her in the other half, so the average of her marginal contributions, and hence the payoff she is assigned by the Shapley value, is $\frac{1}{2}e_i$. Thus the payoff assigned to individual 1 by the Shapley value is the remaining endowment, $e(N) - \frac{1}{2}e(N \setminus \{1\}) = e_1 + \frac{1}{2}e(N \setminus \{1\})$. That is, every individual $i \neq 1$ is taxed at the rate of 50% and the proceeds go to individual 1.

Now consider the dissatisfaction-minimizing payoff distributions. The Harsanyi coalitional form of the game is given by

$$v(S) = \begin{cases} \frac{1}{2}\big(e(N) + e(S)\big) & \text{if } 1 \in S \\ \frac{1}{2}e(S) & \text{if } 1 \notin S. \end{cases}$$

Denote by $(z_i)_{i \in N}$ the Shapley value payoff distribution: for each $i \in N$ let $z_i = \frac{1}{2}e_i$ for $i \neq 1$ and $z_1 = e(N) - \frac{1}{2}e(N \setminus \{1\})$. Notice that

$$v(S) - z(S) = 0 \text{ for all } S \subseteq N.$$

tives of an individual with median earning power is a Condorcet winner (see the text preceding Exercise 1.9).

Thus $(z_i)_{i \in N}$ minimizes $\max_{S \subseteq N}(v(S) - y(S))$ and is the only payoff distribution that does so.

12 Money in electoral competition

12.1 Mobilizing citizens to vote 415
12.2 Informing citizens of candidates' qualities 428

The models of electoral competition in the previous chapters have a glaring omission: money. In a mass election, a candidate may spend significant resources informing potential voters of her position and persuading them to cast their votes for her. She may try to make voting easier for her supporters and more difficult for everyone else; she may trumpet her accomplishments and impugn her opponents. Interested outside organizations may spend resources to engage in similar activities. Everyone may have a vote, but how they cast that vote may be affected by the campaign efforts of the candidates and outside organizations. In sum, a mass election may be less about the aggregation of the citizens' preferences and more about the manipulation of their votes by the wealthy members of society. And the policies adopted by the elected representatives may have less to do with the ones on which they campaigned and more to do with the preferences of wealthy lobbyists. In short, omitting money from the analysis of elections may be a critical flaw.

In fact, one perspective is that studying elections is the wrong place to start an investigation of the determinants of the policies societies adopt. In this view, these policies are determined by the distributions of wealth and power; the existence of elections and interest groups should not be treated as exogenous, but as an implication of the distributions of wealth and power. In some societies, elections act as one medium through which power is exercised, while in others the wealthy wield power more directly. Such a perspective underlies the model in Section 11.4, but the models I present in this chapter treat elections and interest groups as given. Even under this assumption, analyzing the issues is challenging. Much work in the field responds to the challenge by studying models in which the payoff functions and distributions involved have specific functional forms, making the generality of the conclusions hard to assess. I take a different approach, presenting some simple but relatively general models.

Synopsis

In the models I present there are two candidates and a single interest group. In Section 12.1 the set of positions is an interval of real numbers and there is a continuum of citizens, some of whom (*informed*) know the candidates' positions, and some of whom (*uninformed*) do not. Uninformed citizens vote only if prodded to do so by the interest group. The group's options, given its budget and technology, are characterized by a collection of sets of uninformed citizens; it can induce all the citizens in any one of these sets to vote. If its technology allows it to target citizens according to their preferences, for example, it may be able to mobilize all citizens with favorite positions in some interval. First the interest group selects a set of uninformed citizens to mobilize to vote, then the two candidates, who are office-motivated, simultaneously select positions, and finally each citizen who is either (*i*) informed or (*ii*) uninformed and mobilized votes for the candidate whose position she prefers. By Proposition 8.4, the subgame following the interest group's move has a unique subgame perfect equilibrium, in which the position of each candidate is the median of the voters' favorite positions. By judiciously selecting the set of citizens to mobilize, the interest group can move this median to a position it favors. If it is unable to target citizens with specific preferences, it can move the electoral outcome only closer to the median m of all the citizens' favorite positions, while if it is able to target citizens with specific preferences and its targeting ability is sufficiently precise, it can move the outcome away from m (parts c and d of Proposition 12.1).

Section 12.2 analyzes a model in which an interest group may provide citizens with verifiable information about the candidates' qualities. For the sake of tractability, there are only two possible policies, 0 and 1, two candidates, and a single citizen. Candidate 2's quality is known, whereas candidate 1 has two possible qualities, one better than candidate 2's and one worse. For any given candidate quality, the citizen prefers policy 0 to policy 1, but she prefers policy 1 implemented by a high-quality candidate 1 to policy 0 implemented by candidate 2. The interest group, by contrast, prefers policy 1 to policy 0. It observes candidate 1's quality and decides whether to offer to reveal this quality in exchange for candidate 1's choosing policy 1. If it makes this offer, candidate 1 decides whether to accept it. Candidate 2 is assumed to choose policy 0, because there is no advantage to her choosing policy 1, given the citizen's preferences. The citizen observes the policies chosen by the candidates, but not candidate 1's quality unless the interest group reveals it. Proposition 12.3 shows that this game has a weak sequential equilibrium in which if candidate 1's quality is high, the interest group offers to reveal it in exchange for candidate 1's choosing policy 1, and candidate 1 accepts this offer, but if candidate 1's quality is low, the interest group does not offer to reveal it. The outcome is that if candidate 1's quality is high she chooses policy 1 and is elected and if her quality is low, candidate 2, who chooses policy 0, is elected. This outcome is better for the citizen than the best outcome in the absence of the interest group if and only if candidate 1's quality is sufficiently unlikely to be high. The interest group identifies the quality of candidate 1, but the cost of its doing so is that candidate 1 chooses policy 1, so if candidate 1's quality is likely to be high, the presence of the interest group makes the citizen worse off.

12.1 Mobilizing citizens to vote

Model

In the model in this section, some citizens vote only if mobilized by an interest group. I refer to these citizens as uninformed, although the reason they need to be prodded to vote may not be lack of information. The setting for the model is a society defined as follows.

Definition 12.1: Society with informed and uninformed citizens

A *society with informed and uninformed citizens* $\langle X, I, N, U, F, G^N \rangle$ consists of

- $X = [\underline{x}, \overline{x}]$, a compact interval of real numbers (the set of possible positions)
- I, a compact interval of real numbers (the set of citizens), $N \subset I$ (the set of informed citizens), and $U \subset I$ (the set of uninformed citizens), where $N \cup U = I$ and $N \cap U = \varnothing$ (each point in I is a citizen's name)
- $F : X \to [0,1]$, an increasing differentiable function with $F(\underline{x}) = 0$ and $F(\overline{x}) = 1$ (for each $x \in X$, $F(x)$ is the fraction of citizens with favorite positions at most x)
- $G^N : X \to [0,1]$, an increasing differentiable function with $G^N(\underline{x}) = 0$ and $G^N(x) < F(x)$ for all $x > \underline{x}$ (for each $x \in X$, $G^N(x)$ is the fraction of citizens who are informed and have favorite positions at most x).

Figure 12.1 illustrates an example of such a society. The density of the favorite positions of the informed citizens is shown in green and that of the uninformed citizens is shown in orange. The sum of the two, represented by the upper boundary of the orange area, is the density of F, the distribution of all cit-

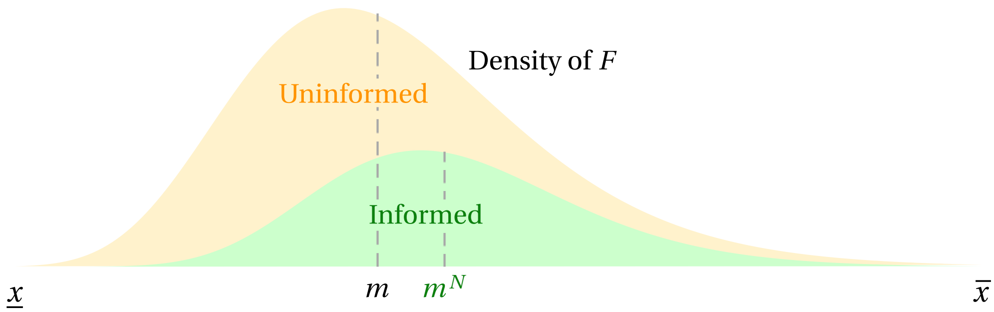


Figure 12.1 An illustration of a society with informed and uninformed citizens. The distribution of the favorite positions of informed citizens is shown in green and that of the uninformed citizens is shown in orange. The median of the informed citizens' favorite positions is m^N and that of all citizens' favorite positions is m.

izens' favorite positions. The median of all the citizens' favorite positions, which I denote by m, is the median of F. The fraction of all citizens who are informed is $G^N(\overline{x})$, so that the median of the informed citizens' favorite positions, which I denote m^N, is defined by the condition $G^N(m^N) = \frac{1}{2}G^N(\overline{x})$.

If only informed citizens vote, then by Proposition 8.4 the two-candidate electoral competition game with a continuum of citizens and office-motivated candidates has a unique Nash equilibrium, in which both candidates choose m^N.

Now suppose that before the candidates commit to positions, an interest group with preferences over positions can mobilize some of the uninformed citizens to vote. (Perhaps it does so by helping them to register to vote or by persuading them that the issues at stake are important enough to make their voting worthwhile.) Assume that the interest group's action affects only whether a citizen votes, not how she casts her vote; a citizen who is mobilized votes for the candidate she prefers. The group may be able to direct its efforts precisely to citizens with certain preferences, or may be able only to increase participation by uninformed citizens across the board. The model captures the limits the group faces by specifying a collection of sets of uninformed citizens, with the interpretation that, given the group's technology, its budget, and the citizens' characteristics, it is capable of mobilizing all the members of any one of the sets. For example, if the group knows the citizens' preferences and can precisely target its mobilization efforts, the collection may consist of all sets of uninformed citizens of at most a certain size with certain preferences.

After the group selects the set of uninformed citizens to mobilize, two candidates simultaneously choose positions and each citizen who is either informed or mobilized votes for her favorite candidate. The interest group cares about the position of the winner of the election, while each candidate is office-motivated, caring only about winning the election, not about the position of the winner.

Denote the collection of all the sets of uninformed citizens (subsets of U) that the interest group is capable of mobilizing by $\mathscr{S}$, and the favorite position of each citizen $i \in I$ by $\hat{x}_i$. If the group mobilizes $S \in \mathscr{S}$, for any position $x \in X$ the fraction of citizens who vote and have favorite positions at most x is

$$G_S(x) = G^N(x) + \phi(\{i \in S : \hat{x}_i \leq x\}),$$

where for any set $Z \subseteq I$ of citizens, $\phi(Z)$ is the fraction of citizens in Z. Thus for any position $x \in X$, the fraction of voters with favorite positions at most x is $G_S(x)/G_S(\overline{x})$. Define the function $F_S : X \to [0,1]$ by

$$F_S(x) = G_S(x)/G_S(\overline{x}) \text{ for all } x \in X. \tag{12.1}$$

If the candidates choose the positions x_1 and x_2, the outcome $O(S,(x_1,x_2))$ of the game is the winner of the election for the electorate $N \cup S$:

$$O(S,(x_1,x_2)) = O_{F_S}(x_1,x_2), \tag{12.2}$$

where F_S is given by (12.1) and O_{F_S} is the function defined in (8.1). Each candidate prefers to win than to tie than to lose, as in the game without the interest group.

Definition 12.2: Two-candidate electoral competition game with vote mobilization by an interest group

A *two-candidate electoral competition game with vote mobilization by an interest group* $\langle\langle X, I, N, U, F, G^N\rangle, \mathscr{S}, u_g\rangle$, where $\langle X, I, N, U, F, G^N\rangle$ is a society with informed and uninformed citizens and

- $\mathscr{S}$ is a collection of subsets of U that includes $\varnothing$ ($\mathscr{S}$ is the collection of all sets of citizens the interest group is able to mobilize, taking into account both its technological and budgetary constraints; one such set is the empty set)
- $u_g : X \to \mathbb{R}$ is a single-peaked function (which represents the interest group's preferences)

is an extensive game with perfect information and simultaneous moves with the following components.

Players
The set of players is $\{1, 2, g\}$ (1 and 2 are candidates and g is an interest group).

Terminal histories
A terminal history is a sequence $(S,(x_1,x_2))$ where $S \in \mathscr{S}$ and $(x_1,x_2) \in$

$X \times X$ (the set of citizens the interest group mobilizes, followed by the positions chosen by the candidates).

Player function

The player function P is defined by

- $P(\varnothing) = g$ (the interest group moves at the start of the game)
- $P(S) = \{1,2\}$ for every $S \in \mathscr{S}$ (the candidates move simultaneously after the interest group).

Actions

The set $A_i(h)$ of actions of each player i after the history h is given by $A_g(\varnothing) = \mathscr{S}$ and $A_1(S) = A_2(S) = X$.

Preferences

Each candidate prefers a terminal history $(S,(x_1,x_2))$ for which $O(S,(x_1,x_2))$, given in (12.2), is a win for her to one in which it is a tie to one in which it is a win for the other candidate.

The interest group's preferences are represented by the payoff function defined by

$$\begin{cases} u_g(x_j) & \text{if } O(S,(x_1,x_2)) = \textit{win for } j \ (=1,2) \\ \frac{1}{2}u_g(x_1) + \frac{1}{2}u_g(x_2) & \text{if } O(S,(x_1,x_2)) = \textit{tie}. \end{cases}$$

How much can the interest group affect the equilibrium?

How does the outcome of a subgame perfect equilibrium of this game depend on the collection $\mathscr{S}$ of sets of uninformed citizens that the interest group is capable of mobilizing? If the interest group mobilizes the set S then the set of voters is $N \cup S$, so that by Proposition 8.4 the subgame following the interest group's choice of S has a unique Nash equilibrium, in which each candidate chooses the median of the voters' favorite positions. As S varies, how does this median change? What are its smallest and largest possible values? That is, how far left and right is it possible for the interest group to move the equilibrium outcome? The next proposition, 12.1, answers these questions.

First suppose that the interest group is unable to target its efforts to citizens with specific preferences: every set it can mobilize is a random sample of uninformed citizens. If it can mobilize only small sets, the winning position it can induce is close to m^N, the median of the favorite positions of the informed individuals. As the size of the sets it can mobilize increases, the winning position it

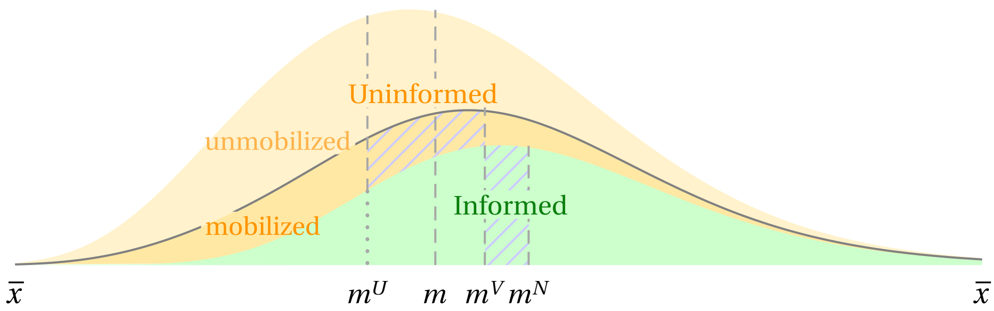


Figure 12.2 An example of distributions of informed and uninformed citizens in a two–candidate electoral competition game with vote mobilization by an interest group. In this example, the interest group can mobilize 30% of the uninformed citizens, generating the density of voters indicated by the black line, with median m^V. (The areas with blue hatching are equal.)

can induce moves towards m, the median of all the citizens' favorite positions. An example in which it can mobilize 30% of the uninformed citizens is shown in Figure 12.2.

Now suppose that it can target its mobilization efforts to citizens with specific preferences, and is in fact capable of mobilizing any set of uninformed citizens. Suppose that it starts mobilizing citizens with favorite positions on the far right, close to $\bar{x}$, and then gradually expands its efforts to citizens on the right with less extreme preferences. Then the median of the voters' favorite positions gradually moves to the right from m^N. It continues to do so until it reaches the value $\overline{m}$ for which the uninformed citizens who are mobilized are exactly those with favorite positions at least $\overline{m}$. At this point the density of the distribution of the voters' favorite positions is indicated by the black line in Figure 12.3 and $G^N(\overline{m}) = 1 - F(\overline{m})$. As more citizens, with favorite positions less than $\overline{m}$, are mobilized, the median of the voters' favorite positions decreases. Thus the largest possible value for the median of the voters' favorite positions is $\overline{m}$.

A symmetric argument shows that the smallest possible value of the median of the voters' favorite positions is achieved when the interest group mobilizes all uninformed citizens with favorite positions at most $\underline{m}$, where $F(\underline{m}) = G^N(\bar{x}) - G^N(\underline{m})$.

The next result states these conclusions precisely and shows that for every position $z \in [\underline{m}, \overline{m}]$ there is a set of uninformed citizens that, if mobilized, causes the median of the voters' favorite positions to be z.

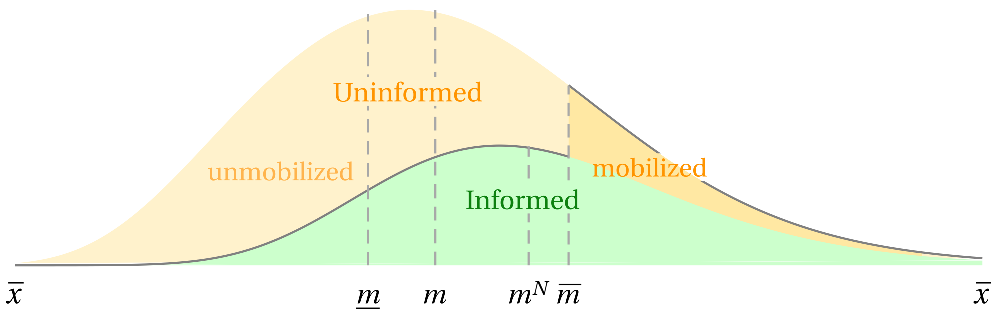


Figure 12.3 An example of distributions of informed and uninformed citizens in a two–candidate electoral competition game with vote mobilization by an interest group. If the interest group can mobilize arbitrary sets of uninformed citizens, the furthest to the right that it can move the median of the voters' favorite positions is $\overline{m}$ defined by $G^N(\overline{m}) = 1 - F(\overline{m})$, and the furthest to the left it can move it is $\underline{m}$ defined by $F(\underline{m}) = G^N(\overline{x}) - G^N(\underline{m})$.

Lemma 12.1: Properties of median of favorite positions of set of informed and mobilized citizens

Let $\langle\langle X, I, N, U, F, G^N\rangle, \mathscr{S}, u_g\rangle$ be a two-candidate electoral competition game with vote mobilization by an interest group, with $X = [\underline{x}, \overline{x}]$. Denote by m the median of F (the distribution of all citizens' favorite positions) and by m^N the position for which $G^N(m^N) = \frac{1}{2}G^N(\overline{x})$ (the median of the distribution of the informed citizens' favorite positions).

a. Suppose that mobilization cannot be targeted: let $\lambda \in [0,1]$ and suppose that $\mathscr{S}$ consists of every subset S of the set U of uninformed citizens that contains the fraction λ of U and for which the median of the favorite positions of its members is the median of the favorite positions of the members of U. Then for every $S \in \mathscr{S}$ the median of the favorite positions of the citizens in $N \cup S$ is the same; denote it $m^V(\lambda)$. The function m^V is continuous, with $m^V(0) = m^N$ and $m^V(1) = m$. If $m < m^N$ it is decreasing, and if $m > m^N$ it is increasing.

b. There is a unique position $\overline{m}$ such that $G^N(\overline{m}) = 1 - F(\overline{m})$ and a unique position $\underline{m}$ such that $F(\underline{m}) = G^N(\overline{x}) - G^N(\underline{m})$, and $\underline{m} < m < \overline{m}$ and $\underline{m} < m^N < \overline{m}$.

c. For any set $S \subset U$, the median of the distribution of the favorite positions of the citizens in $N \cup S$ is in $[\underline{m}, \overline{m}]$.

d. For any position z, let $\bar{S}(z)$ be the set of citizens in U with favorite positions at least z. For every position $x \in [m^N, \overline{m}]$ there exists a position $\bar{z}(x) \in [\overline{m}, \bar{x}]$ such that the median of the favorite positions of the citizens in $N \cup \bar{S}(\bar{z}(x))$ is x. We have $\bar{z}(m^N) = \bar{x}$ and $\bar{z}(\overline{m}) = \overline{m}$.

e. For any position z, let $\underline{S}(z)$ be the set of citizens in U with favorite positions at most z. For every position $x \in [\underline{m}, m^N]$ there exists a position $\underline{z}(x) \in [\underline{x}, \underline{m}]$ such that the median of the favorite positions of the citizens in $N \cup \underline{S}(\underline{z}(x))$ is x. We have $\underline{z}(\underline{m}) = \underline{m}$ and $\underline{z}(m^N) = \underline{x}$.

Proof

a. For $S \in \mathscr{S}$, the median $m^V(\lambda)$ of the favorite positions of the citizens in $N \cup S$ satisfies $G^N(m^V(\lambda)) + \lambda G^U(m^V(\lambda)) = \frac{1}{2}(G^N(\bar{x}) + \lambda G^U(\bar{x}))$, where G^U is the distribution function of the uninformed citizens, defined by $G^U(x) = F(x) - G^N(x)$ for all $x \in X$. Thus $m^V(\lambda)$ is independent of S and m^V is continuous, given that F and G^N are continuous. We have $\frac{1}{2}(G^N(\bar{x}) + \lambda G^U(\bar{x})) = G^N(m^N) + \lambda G^U(m^U)$, so $G^N(m^N) - G^N(m^V(\lambda)) = \lambda(G^U(m^V(\lambda)) - G^U(m^U))$. (That is, the areas of the regions with blue hatching in Figure 12.2 are equal.) Thus $m^V(0) = m^N$ and $m^V(1) = m$; if $m < m^N$ then $m^V(\lambda) \le m^N$ for all λ and m^V is decreasing, and if $m > m^N$ then $m^V(\lambda) \ge m^N$ for all λ and m^V is increasing.

b. The functions G^N and F are both continuous, with $G^N(\underline{x}) = F(\underline{x}) = 0$, $F(\bar{x}) = 1$, and $0 < G^N(\bar{x}) < 1$, so the equations $F(x) + G^N(x) = 1$ and $F(x) + G^N(x) = G^N(\bar{x})$ have solutions by the Intermediate Value Theorem. The solutions are unique because G^N and F are increasing, and $\underline{m} < \overline{m}$ because $G^N(\bar{x}) < 1$.

If $z \le m$ then $1 - F(z) \ge \frac{1}{2}$ and $G^N(z) < F(z) \le \frac{1}{2}$, and if $z \ge m$ then $F(z) \ge \frac{1}{2}$ and $G^N(\bar{x}) - G^N(z) < 1 - F(z) \le \frac{1}{2}$, so $\underline{m} < m < \overline{m}$.

If $z \le m^N$ then $1 - F(z) > G^N(\bar{x}) - G^N(z) \ge \frac{1}{2}G^N(\bar{x})$ and $G^N(z) \le \frac{1}{2}G^N(\bar{x})$, and if $z \ge m^N$ then $F(z) > \frac{1}{2}G^N(\bar{x})$ and $G^N(\bar{x}) - G^N(z) \le \frac{1}{2}G^N(\bar{x})$, so $\underline{m} < m^N < \overline{m}$.

c. Let x^V be the median of the favorite positions of the citizens in $N \cup S$. If S does not include all the uninformed citizens with favorite positions greater than x^V then adding such citizens to S increases the median, and if S includes citizens with favorite positions less than x^V then removing such citizens from S also increases the median. Thus a subset S of U for which the median x^V of the favorite positions of the citizens in $N \cup S$ is maximal consists of all members of U whose favorite positions are at least x^V. That is, $G^N(x^V) = 1 - F(x^V)$, and hence $x^V = \overline{m}$.

A symmetric argument shows that any set S for which the median of the favorite positions of the citizens in $N \cup S$ is minimal consists of all members of U whose favorite positions are at most $\underline{m}$.

d. For any position $z \in [\overline{m}, \overline{x}]$, let $\mu(z)$ be the median of the favorite positions of the citizens in $N \cup \bar{S}(z)$. We have $\mu(\overline{m}) = \overline{m}$ by the definition of $\overline{m}$ and $\mu(\overline{x}) = m^N$ because no citizen has a favorite position larger than $\overline{x}$. The result follows from the continuity of μ, which is a consequence of the continuity of F and G^N.

e. The argument is analogous to the argument for part *d*.

This result implies that in any subgame perfect equilibrium of the vote-mobilization game, the candidates' common position lies between $\underline{m}$ and $\overline{m}$. If the interest group is unable to target its mobilization efforts to citizens with specific preferences—if it can mobilize only random samples of uninformed citizens—then the winning positions it can induce lie between m and m^N and depend on the size of the set of citizens it can mobilize. If it has enough resources to mobilize all uninformed citizens then it can induce the position m, whereas if its resources allow the mobilization of only a small (random) subset of uninformed citizens then it can induce only positions close to m^N. If it can mobilize any set of citizens and can target its mobilization efforts precisely, then it can induce any position in between $\underline{m}$ and $\overline{m}$.

Proposition 12.1: Subgame perfect equilibrium of electoral competition game with vote mobilization by an interest group

Let $\langle\langle X, I, N, U, F, G^N\rangle, \mathscr{S}, u_g\rangle$ be a two-candidate electoral competition game with vote mobilization by an interest group, with $X = [\underline{x}, \overline{x}]$. Denote by m the median of F (the distribution of all citizens' favorite positions), by m^N the position for which $G^N(m^N) = \frac{1}{2}G^N(\overline{x})$ (the median of the distribution of the informed citizens' favorite positions), and by $\hat{x}_g$ the interest group's favorite position. Let $\underline{m}$ be the unique (by Lemma 12.1*b*) position for which $F(\underline{m}) = G^N(\overline{x}) - G^N(\underline{m})$ and $\overline{m}$ the unique position for which $G^N(\overline{m}) = 1 - F(\overline{m})$.

a. In every subgame perfect equilibrium the candidates choose the same position, the median of the favorite positions of the citizens in $N \cup S$, where S is the set chosen by the interest group at the start of the game. This position lies in $[\underline{m}, \overline{m}]$.

b. (No mobilization possible) If $\mathscr{S} = \{\varnothing\}$, the game has a unique subgame perfect equilibrium, in which each candidate's position is m^N.

c. (Mobilization cannot be targeted) Let $\lambda \in [0,1]$ and suppose that $\mathscr{S}$ consists of every subset S of the set U of uninformed citizens that contains at most the fraction λ of U and for which the median of the favorite positions of its members is the median of the favorite positions of the members of U. By Lemma 12.1*a* the median of the favorite positions of the members of $N \cup S$ for every $S \in \mathscr{S}$ that contains the fraction λ of U is independent of S; denote it $m^V(\lambda)$. If $m < m^N$ then in every subgame perfect equilibrium each candidate's position is

$$\begin{cases} m^V(\lambda) & \text{if } \hat{x}_g \le m^V(\lambda) \\ \hat{x}_g & \text{if } m^V(\lambda) < \hat{x}_g < m^N \\ m^N & \text{if } \hat{x}_g \ge m^N. \end{cases}$$

If $m^N < m$ then the common position satisfies conditions symmetric with these ones.

d. (Mobilization can be perfectly targeted) Suppose that for every $x \in X$ the subsets of U consisting of the citizens with favorite positions at least x and the citizens with favorite positions at most x are both in $\mathscr{S}$. Then in every subgame perfect equilibrium each candidate's position is

$$\begin{cases} \underline{m} & \text{if } \hat{x}_g \le \underline{m} \\ \hat{x}_g & \text{if } \underline{m} < \hat{x}_g < \overline{m} \\ \overline{m} & \text{if } \hat{x}_g \ge \underline{m}. \end{cases}$$

Proof

a. Let Γ be a two-candidate electoral competition game with a continuum of citizens and office-motivated candidates in which the set of citizens is $N \cup S$. The subgame of the game here following any action S of the interest group is equivalent to a variant of Γ in which the interest group is a third player, with no actions, and its analysis parallels that of Γ. In particular, by the arguments in the proof of Proposition 8.4, Γ has a unique Nash equilibrium, in which each candidate's position is the median of the favorite positions of the citizens in $N \cup S$.

By Lemma 12.1*c*, for every $S \subset U$ the median of the favorite positions of

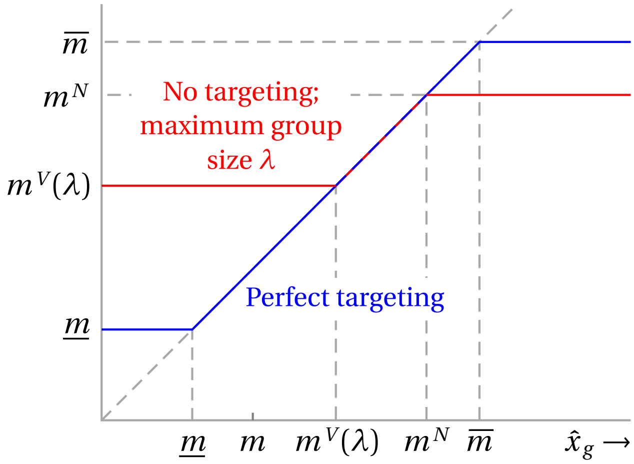


Figure 12.4 The electoral outcomes in a two-candidate electoral competition game with vote mobilization by an interest group as a function of the favorite position $\hat{x}_g$ of the interest group and the group's ability to target citizens with specific preferences, for an example in which $m < m^N$. (See Proposition 12.1*c* and *d*.)

the citizens in $N \cup S$ is in $[\underline{m}, \overline{m}]$, so in every outcome of a subgame perfect equilibrium the candidates' common position is in this interval.

b. The result follows from part *a* for $S = \varnothing$.

c. The result follows from part *a* and Lemma 12.1*a*.

d. First suppose that $\hat{x}_g \leq \underline{m}$. By Lemma 12.1*c*, the median of $N \cup S$ is at least $\underline{m}$ for every set S, so by part *a* the interest group can achieve no outcome better than the one in which both candidates choose $\underline{m}$. By Lemma 12.1*e*, the interest group can achieve this outcome (by choosing S to be the set of all uninformed citizens with favorite positions at most $\underline{m}$).

Now suppose that $\underline{m} < \hat{x}_g \leq m^N$. By Lemma 12.1*e* there is a position $z \in [\underline{x}, \underline{m}]$ such that when the interest group mobilizes the set S of uninformed citizens with favorite positions at most z, the median of the voter's favorite positions is $\hat{x}_g$. Thus by part *a*, when the interest group mobilizes S, both candidates choose the position $\hat{x}_g$. No outcome is better for the interest group, so it is the outcome of every subgame perfect equilibrium.

Similar analyses, using Lemma 12.1*d*, apply when $m^N \leq \hat{x}_g \leq \overline{m}$ and $\hat{x}_g \geq \overline{m}$.

Parts *c* and *d* of the result are illustrated in Figure 12.4. The result is a formalization of the idea that an interest group with resources available to mobilize citizens to vote can bend the outcome of an election in its favor. If it is unable to target citizens with specific preferences then it can move the electoral outcome only closer to the median m of all the citizens' favorite positions, while if it is able to target citizens with specific preferences and its targeting ability is sufficiently precise it can move the electoral outcome away from m.

Comments

Interest group cares about size of mobilized set In the game I have defined, the interest group cares only about the position of the winner of the election, not about the size of the set of citizens it mobilizes. If it cares about the size of the set, because, for example, it incurs a cost that increases in this size, it optimally balances the benefit of mobilizing more citizens (to change the candidates' common equilibrium position) with the cost of doing so.

Two interest groups Suppose that there are two interest groups rather than one, with favorite positions $\hat{x}_1$ and $\hat{x}_2 > \hat{x}_1$, and each group is able to mobilize any set of uninformed citizens (and is insensitive to the cost of doing so). Consider the game in which the interest groups choose their mobilization sets simultaneously before the candidates choose positions. Denote the median of the voters' favorite positions, given the interest groups' mobilization efforts, by m^V. Then in a subgame perfect equilibrium $m^V \in [\hat{x}_1, \hat{x}_2]$, because if m^V is outside this interval then the mobilization of some of the unmobilized uninformed citizens moves m^V into the interval, benefitting one or both interest groups. In any equilibrium in which $m^V \in (\hat{x}_1, \hat{x}_2)$, all uninformed citizens are mobilized, because if any uninformed citizens remain unmobilized, their mobilization moves m^V closer to the favorite position of one of the interest groups. Given that all citizens are mobilized, both candidates choose the position m in such an equilibrium.

Suppose that $\hat{x}_1 < m < \hat{x}_2$. If $m^V = \hat{x}_1$ then some uninformed citizens with favorite positions greater than m^V are not voting, and interest group 2 benefits from mobilizing them. Thus $m^V > \hat{x}_1$. Similarly, $m^V < \hat{x}_2$. Thus from the argument in the previous paragraph, in any subgame perfect equilibrium all uninformed citizens are mobilized and hence both candidates choose the position m. So the presence of interest groups with favorite positions on different sides of m that are able to mobilize any set of uninformed citizens moves the outcome from the median of the informed citizens' favorite positions to the median of all citizens' favorite positions, regardless of how extreme the interest groups' favorite positions are.

Now suppose that $m < \hat{x}_1 < \overline{m}$. If $m^V > \hat{x}_1$, interest group 1 benefits by mobilizing uninformed citizens with favorite positions less than m^V, because doing so reduces m^V, and if $m^V < \hat{x}_2$ interest group 2 benefits by reducing its mobilization efforts if they are positive. Thus in any subgame perfect equilibrium, interest group 2 mobilizes no one and interest group 1 mobilizes enough uninformed citizens to make $m^V = \hat{x}_1$, so that both candidates choose that position. So in this case, the less extreme of the interest groups does all the mobilization, and achieves its favorite outcome.

Interest group and candidates move simultaneously Consider the strategic game in which the interest group and the candidates move simultaneously: no player can commit to an action before the others move.

If the interest group cares only about the winning position, not about the size of the set it mobilizes, then every action profile (S, x_1, x_2) in which x_1 and x_2 are both equal to the median favorite position of the citizens in $N \cup S$ is a Nash equilibrium of this game. Given that the candidates' positions are the same, the set the interest group mobilizes has no effect on the winning position, and given this set, the only pair of mutually optimal positions for the candidates is the one in which both candidates choose the median favorite position of the citizens in $N \cup S$, by the arguments for the game without an interest group.

If the interest group cares about the size of the set of citizens it mobilizes and, for any given outcome, prefers to mobilize a small set than a large one, then the simultaneous move game has a single Nash equilibrium, in which the interest group mobilizes no one and the candidates' positions are both the median favorite position of the informed citizens. That is, the interest group has no effect.

Thus if no player can commit to an action before the others move, the model has no interesting equilibrium.

Interest group moves after candidates The subgame perfect outcome of the game survives in the variant in which the interest group moves after the candidates rather than before them, as you are asked to show in the next exercise.

Exercise 12.1: Mobilization game in which interest group moves after candidates

Let $(S^*, (x_1^*, x_2^*))$ be the outcome of a subgame perfect equilibrium of the game in which the interest group moves first, as described in Proposition 12.1. Consider the extensive game in which the candidates move (simultaneously) first, before the interest group. Show that if the interest group has an optimal action for every pair of the candidates' positions then this game has a subgame perfect equilibrium in which the candidates' positions are x_1^* and x_2^* and the interest group mobilizes S^* after this history.

Unlike the equilibria described in Proposition 12.1, however, such an equilibrium is sensitive to the assumption that the interest group cares only about the winning position, not about the size of the set of citizens it mobilizes. Suppose that the interest group incurs a cost $c(S)$ that increases with the size of the set S of citizens it mobilizes, and that its preferences are represented by $u_g(x_i) - c(S)$, where x_i is the winning position. (The cost of mobilizing citizens to vote may also reasonably depend on the candidates' positions—mobilizing citizens to vote when the positions are similar may require more effort than when they are far apart. Such a dependence reinforces the following argument.) Consider an equilibrium of the game in which the interest group moves first that has the form described in Proposition 12.1*d* with $m^N < x_1^* = x_2^* \leq \hat{x}_g$. If in the game in which the candidates move first they choose the positions x_1^* and x_2^* then the only optimal action of the interest group in the subsequent subgame is to mobilize no one, so that the median of the voters' favorite positions is m^N. Suppose that candidate 1 deviates to x_1 slightly less than x_1^*. Then if the interest group continues to mobilize no one, the outcome changes from x_1^* to x_1, which is worse for the interest group. To deter this deviation, in the subgame following (x_1, x_2^*) the interest group needs to mobilize enough citizens to move the median of the voters' favorite positions from m^N to a position greater than $\frac{1}{2}(x_1 + x_2^*)$. The cost of doing so may be significant, so that depending on the size of $u_g(x_1)$ relative to $u_g(x_1^*) = u_g(x_2^*)$ the interest group might optimally continue to mobilize no one, so that the deviation is not deterred. In this case, in no subgame perfect equilibrium is the pair of the candidates' positions (x_1^*, x_2^*).

The significance of the interest group's moving first is that it commits to a budget for mobilization before the candidates commit to positions. If it is better off in the equilibrium of the game in which it does so than in the one in which it does not, it has an interest in committing and may be able to do so by raising money early in the election campaign. That is, the interest group may be able to take actions that make it a first-mover, and may have an interest in so doing.

Interest group commits to position-contingent mobilization Another possible assumption about the timing of the players' actions is that the interest group first selects a candidate and a function that specifies the set of citizens the group will mobilize for each position the candidate chooses, and then the candidates choose positions. Let $(S^*, (x_1^*, x_2^*))$ be the outcome of a subgame perfect equilibrium of the game in which the interest group moves first, as described in Proposition 12.1. Then the game with position-contingent mobilization has a subgame perfect equilibrium in which the interest group proposes to candidate 1 that it will mobilize S^* regardless of the position the candidate chooses, and the candidates subsequently choose x_1^* and x_2^*. The game has no subgame perfect equilibrium with a better outcome for the interest group by the argument for the original game that S^* is optimal. (It has other subgame perfect equilibria with the same outcome, in which it proposes to mobilize S^* if candidate 1 chooses x_1^* and other sets if candidate 1 chooses a different position.)

Policy-motivated candidates If the candidates are policy-motivated and the distribution of the citizens' favorite positions is uncertain, as in a two-candidate electoral competition game with policy-motivated candidates and uncertain median, then in the absence of the interest group the candidates' equilibrium positions differ (Proposition 9.4). I conjecture that the addition of an interest group with mobilization options that moves either before or after the candidates shifts the candidates' equilibrium positions towards the interest group's favorite position, but I know of no formal results for this model.

12.2 Informing citizens of candidates' qualities

Suppose that interest groups can provide verifiable information about candidates that the candidates themselves are unable to provide. If the interest groups' preferences differ from those of the majority of citizens and they provide information in exchange for the candidates committing to positions that the interest groups favor, are the citizens better off or worse off than in their absence? The model in this section is intended to examine this question. The analysis of a suitable model rapidly increases in complexity and opacity as the number of players and the number of their possible actions increase. The model I present is one of the simplest that can address the issues.

There are two possible policies, 0 and 1, two candidates, and a single citizen. Each candidate chooses a policy and then the citizen votes for one of the candidates. Candidate 1 has two possible qualities, l and h; in line with standard terminology, I refer to l and h as the two possible *types* of candidate 1. Candidate 2's quality is known to be 0, which is between l and h. Candidate 1 knows her quality, but the citizen does not; the citizen believes that it is l with probability p and h with probability $1-p$. Each candidate is office-motivated: she prefers to win (that is, to obtain the citizen's vote), in which case her payoff is 1, than to lose, in which case her payoff is 0.

The citizen's payoff from policy x carried out by a candidate with quality q is $u(x,q)$. For each candidate quality, the citizen prefers policy 0 to policy 1:

$$u(0,q) > u(1,q) \text{ for } q \in \{l, 0, h\}.$$

For each policy, the citizen prefers a candidate of quality h to one of quality 0 to one of quality l:

$$u(x,h) > u(x,0) > u(x,l) \text{ for } x \in \{0,1\}.$$

Finally, to make it possible for candidate 1 to attract the citizen's vote if she chooses policy 1 and the interest group informs the citizen that her quality is h, the citizen prefers policy 1 carried out by a candidate of quality h to policy 0 carried out by a candidate of quality 0:

$$u(1,h) > u(0,0).$$

Given that candidate 2's quality is known and the citizen prefers policy 0 to policy 1 conditional on the quality of the candidate offering the policy, there is no advantage to candidate 2's selecting policy 1. Thus I assume that she selects policy 0. She has no other decisions to make, so she does not appear as a player in the games I analyze, although her existence affects the citizen's options.

In the main model, an interest group that prefers policy 1 to policy 0, in contrast to the citizen, can reveal candidate 1's quality in exchange for the candidate's committing to policy 1. To assess the interest group's impact, I first find the equilibrium outcomes in its absence and then analyze a model in which it is present.

12.2.1 Model without interest group

Consider the extensive game in Figure 12.5, in which no interest group is present. The game begins in the center, with a move of chance (c) that determines the quality of candidate 1. This quality is l with probability p and h with probability $1-p$. Candidate 1 observes this move—that is, she knows her quality—and then chooses a policy, 0 or 1. The citizen observes the policy chosen but not candidate 1's quality, and then selects (votes for) candidate 1 or candidate 2.

Definition 12.3: Policy game with candidate of uncertain quality

A *policy game with a candidate of uncertain quality* $\langle l,h,p,u\rangle$, in which the players are a citizen and candidate 1, where

- l and h are possible qualities of candidate 1
- $p \in (0,1)$ (the probability that candidate 1's quality is l)
- $u : \{0,1\} \times \{l,h\} \to \mathbb{R}$ with $u(0,q) > u(1,q)$ for $q = l$ and $q = h$, $u(x,h) > u(x,0) > u(x,l)$ for $x = 0$ and $x = 1$, and $u(1,h) > u(0,0)$ (the citizen's payoff function),

is the extensive game with imperfect information shown in Figure 12.5,

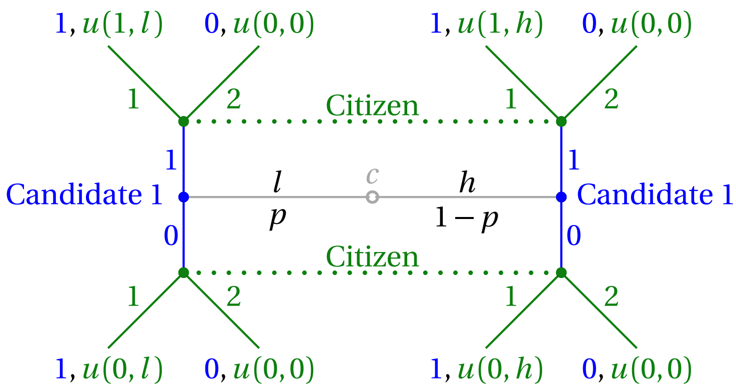


Figure 12.5 A policy game with a candidate of uncertain quality. The players are a citizen and candidate 1. The game begins with a move of chance (in the center of the figure), which determines the quality of candidate 1, either l or h. Candidate 1 observes this move and then chooses policy 0 or policy 1. The citizen does not observe candidate 1's quality. At each information set, the citizen selects (votes for) candidate 1 or candidate 2. For each terminal history, candidate 1's payoff is listed first and the citizen's second.

where for each terminal history candidate 1's payoff is listed first and the citizen's second.

I claim that in every weak sequential equilibrium of any such game $\langle l, h, p, u\rangle$ both types of candidate 1 (quality l and quality h) choose the same policy. Consider an assessment in which they choose different policies. Suppose that type h chooses policy 0 and type l chooses policy 1. Then for the citizen's beliefs to be consistent with the candidate's strategy, they assign probability 1 to the history $(h, 0)$ at her information set following the policy 0 (the lower of the two sets in Figure 12.5) and probability 1 to the history $(l, 1)$ in her information set following the policy 1. Consequently the citizen optimally votes for candidate 1 if candidate 1 chooses policy 0 and for candidate 2 if candidate 1 chooses policy 1. But then if type l of candidate 1 deviates and chooses policy 0, imitating type h, the citizen votes for her and she is better off, so the assessment is not a weak sequential equilibrium. A similar argument applies to an assessment in which type h chooses policy 1 and type l chooses policy 0.

Every game $\langle l, h, p, u\rangle$ has weak sequential equilibria in which both types of candidate 1 choose policy 0, and also ones in which both types of candidate 1 choose policy 1. For some games, candidate 1 loses in some of these equilibria. In these cases the citizen's belief leads her to vote for candidate 2 not only if candidate 1 adheres to her strategy but also if she deviates from it.

Proposition 12.2: Weak sequential equilibria of policy game with candidate of uncertain quality

Let $\langle l, h, p, u\rangle$ be a policy game with a candidate of uncertain quality.

a. In every weak sequential equilibrium of the game both types of candidate 1 choose the same policy.

b. The game has weak sequential equilibria in which both types of candidate 1 choose policy 0 and ones in which both types choose policy 1.

c. Let $x \in \{0,1\}$. In the weak sequential equilibria in which both types of candidate 1 choose policy x, the citizen votes for candidate 1 if $pu(x,l) + (1-p)u(x,h) > u(0,0)$ and for candidate 2 if $pu(x,l) + (1-p)u(x,h) < u(0,0)$. If $pu(x,l)+(1-p)u(x,h) = u(0,0)$ then equilibria exist in which the citizen votes for candidate 1 and in which she votes for candidate 2. In each of these equilibria the citizen's payoff is $\max\{u(0,0), pu(x,l)+(1-p)u(x,h)\}$.

Proof

A proof of part *a* is given in the text preceding the proposition.

To prove parts *b* and *c*, suppose that both types of candidate 1 choose policy 0. For the citizen's beliefs to be consistent with this strategy, at her lower information set the citizen assigns probability p to the history $(l,0)$ and probability $1-p$ to the history $(h,0)$. Thus at this information set she optimally votes for candidate 1 if $pu(0,l)+(1-p)u(0,h) > u(0,0)$, for candidate 2 if the reverse inequality holds, and for either candidate in the case of equality. Her payoff is thus $\max\{u(0,0), pu(0,l)+(1-p)u(0,h)\}$. If she votes for candidate 1 then candidate 1's strategy of choosing policy 0 is optimal for her regardless of the citizen's beliefs at her upper information set and her consequent optimal action. If she votes for candidate 2 then for each type of candidate 1 to be no better off deviating to policy 1, the citizen's belief at her upper information set must assign sufficient probability to the history $(l,1)$ to make a vote for candidate 2 optimal for her. Given that $u(1,l) < u(1,0) < u(0,0)$, such a belief exists, and given that this information set is not reached if the players follow their strategies, such a belief is consistent with equilibrium.

A similar argument applies if both types of candidate 1 choose policy 1.

12.2.2 Model with interest group

Now add an interest group to the model. The resulting game is shown in Figure 12.6, where the pair (x,q) attached to each terminal history consists of the position x of the chosen candidate and her quality q, not the players' payoffs as in Figure 12.5. As before, chance (c) first determines the quality of candidate 1. The interest group observes this quality and can offer to (convincingly) reveal it (action A, for "advertise") in exchange for candidate 1's selecting policy 1, which the interest group prefers to policy 0. Candidate 1 can accept this offer, in which case she selects policy 1 and the citizen is informed of her quality, or reject it, in which case she selects policy 0 and the citizen is not informed of her quality. If the interest group does not make an offer (action 0) then candidate 1 chooses policy 0 or policy 1 as before. The citizen observes the policy chosen by candidate 1 but does not observe her quality unless the interest group offers to reveal it and candidate 1 accepts this offer. Also, the citizen does not observe whether the interest group offered to reveal candidate 1's quality unless the interest group makes an offer and candidate 1 accepts it. That is, the four histories $(l,A,0)$, $(h,A,0)$, $(l,0,0)$, and $(h,0,0)$ are in the same information set, indicated by the dotted rectangle in Figure 12.6.

The preferences of the citizen and the candidates are the same as in the game without the interest group. In particular, for any given candidate quality, the citizen prefers policy 0 to policy 1. The interest group, by contrast, prefers policy 1 to policy 0, and does not care about the candidate's quality. (Perhaps the quality represents the candidate's policies on issues about which the interest group is not interested.) Specifically, if the policy of the winner of the election is x then the interest group's payoff is x if it does not advertise candidate 1's quality and $x-c$ if it does, where $0 \le c \le 1$.

Definition 12.4: Policy game with candidate of uncertain quality and interest group

A *policy game with a candidate of uncertain quality and an interest group* $\langle l,h,p,u,A,c\rangle$, in which the players are a citizen, candidate 1, and an interest group, where

- l and h are possible qualities for candidate 1
- $p \in (0,1)$ (the probability that candidate 1's quality is l)
- $u : \{0,1\} \times \{l,h\} \to \mathbb{R}$ with $u(0,q) > u(1,q)$ for $q=l$ and $q=h$, $u(x,h) > u(x,0) > u(x,l)$ for $x=0$ and $x=1$, and $u(1,h) > u(0,0)$ (the citizen's payoff function),

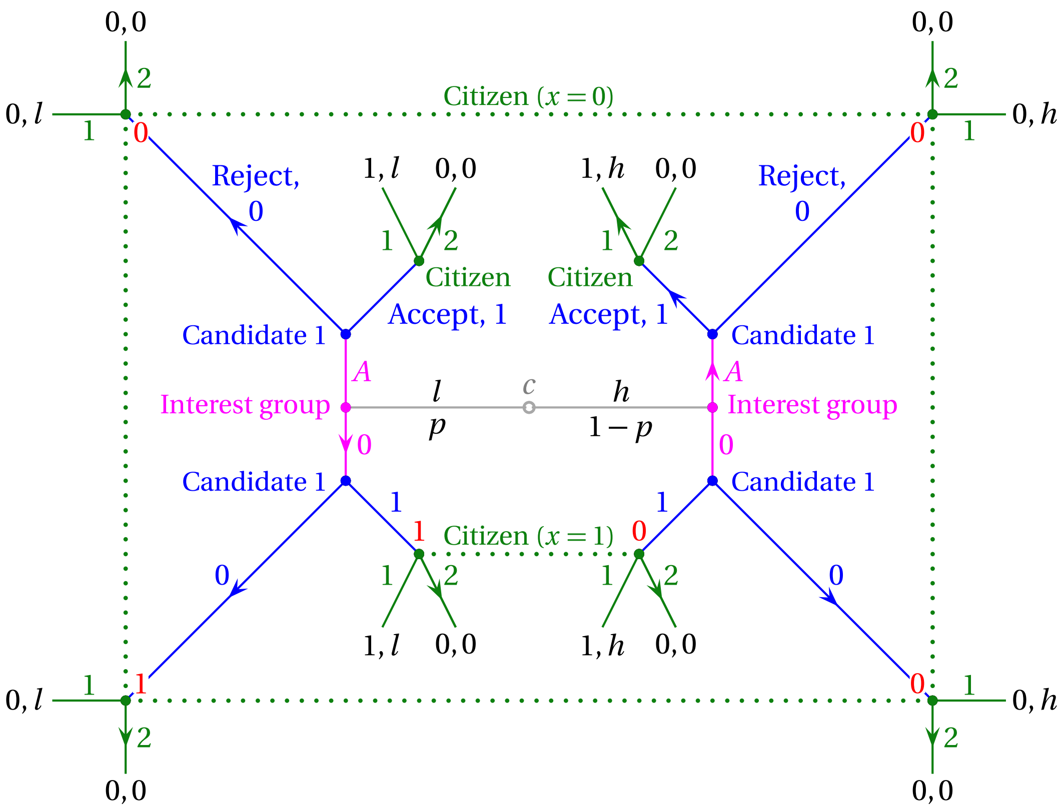


Figure 12.6 A policy game with a candidate of uncertain quality and interest group. The pair (x, q) attached to each terminal history consists of the position x of the chosen candidate and her quality q (*not the players' payoffs*). The arrows indicate the strategy profile and the numbers in red the belief system in one weak sequential equilibrium.

- A is the action of the interest group to advertise candidate 1's quality
- $c \in [0, 1]$ (the interest group's advertising cost),

is an extensive game with imperfect information with the terminal histories, player function, chance probabilities, and information partitions shown in Figure 12.6. The players' preferences are represented by payoff functions that assign the following payoffs to each terminal history (q, z, x, v), where $q \in \{l, h\}$ (the quality of candidate 1), $z \in \{A, 0\}$ (the action of the interest group, to offer to reveal candidate 1's quality or not), $x \in \{0, 1\}$ (the policy chosen by candidate 1, where 0 entails the rejection of the interest group's offer, if it made one, and 1 entails the acceptance of such an offer), and $v \in \{1,2\}$ (the candidate elected):

$$\begin{array}{rl} \text{Citizen} & \begin{cases} u(x,q) & \text{if } v=1 \\ u(0,0) & \text{if } v=2 \end{cases} \\ \text{Candidate 1} & \begin{cases} 1 & \text{if } v=1 \\ 0 & \text{if } v=2 \end{cases} \\ \text{Interest group} & \begin{cases} x-c & \text{if } z=A \text{ and } x=1 \\ x & \text{otherwise.} \end{cases} \end{array}$$

If the citizen's expected payoff from policy 0 carried out by candidate 1, evaluated according to the prior probabilities p and $1-p$, is at least her payoff $u(0,0)$ from choosing candidate 2, then this game has weak sequential equilibria that correspond to the equilibria of the game in which the interest group is absent: the interest group does not offer to reveal the quality of either type of candidate 1. Both types of candidate 1 choose policy 0 whether or not the interest group offers to reveal their types; at the information set reached when candidate 1 chooses policy 0, the belief assigns probability p to the history $(l,0,0)$ and probability $1-p$ to the history $(h,0,0)$ (as required by the consistency condition) and the citizen votes for candidate 1. The citizen votes for candidate 2 after the history $(l,A,1)$ and for candidate 1 after the history $(h,A,1)$. The belief at the information set $\{(l,0,1),(h,0,1)\}$ is arbitrary; the citizen's action at this set is the one that is optimal given her belief.

More interestingly, every policy game with a candidate of uncertain quality and an interest group has a weak sequential equilibrium in which if candidate 1's quality is h, the interest group offers to reveal this quality in exchange for candidate 1's selecting policy 1, and candidate 1 accepts this offer, but if candidate 1's quality is l then the interest group does not offer to reveal it. In this equilibrium the outcome is policy 0 implemented by candidate 2 (with quality 0) if candidate 1's quality is l and policy 1 implemented by candidate 1 if her quality is h. The arrows and beliefs (the red numbers) in Figure 12.6 indicate this equilibrium.

Proposition 12.3: Weak sequential equilibrium of policy game with candidate of uncertain quality and interest group

A policy game with a candidate of uncertain quality and an interest group $\langle l,h,p,u,A,c\rangle$ has a weak sequential equilibrium in which

- the interest group offers to reveal candidate 1's quality if it is h but not if it is l

- if the interest group offers to reveal candidate 1's quality, the candidate accepts the offer and chooses policy 1 if her quality is h, and rejects the offer and chooses policy 0 if her quality is l; if the interest group does not offer to reveal her quality, she chooses policy 0 regardless of her quality
- the citizen votes for candidate 1 if the interest group reveals that the candidate's quality is h and the candidate chooses policy 1, and otherwise votes for candidate 2
- at the information set reached after the interest group does not offer to reveal candidate 1's type and candidate 1 chooses policy 1, the citizen believes that the candidate's quality is l
- at the information set reached after candidate 1 chooses policy 0, the citizen believes that the candidate's quality is l and the interest group did not offer to reveal the candidate's quality.

The outcome is that (*a*) if candidate 1's quality is h, the interest group advertises this quality, candidate 1 chooses policy 1, and the citizen votes for candidate 1, and (*b*) if candidate 1's quality is l, the interest group does not offer to advertise this quality, candidate 1 chooses policy 0, and the citizen votes for candidate 2. The citizen's expected payoff in the equilibrium is $pu(0,l)+(1-p)u(1,h)$.

Proof

Strategy of interest group

After the history l, actions A and 0 both lead to the election of candidate 2 and hence policy 0, so that in particular the action 0 is optimal.

After the history h, the action A leads to the election of candidate 1, who chooses policy 1, and the action 0 leads to the election of candidate 2, who chooses policy 0. Given $c \leq 1$, the latter outcome is no better for the interest group than the former.

Strategy of candidate 1

After each of the histories (l,A), $(l,0)$, and $(h,0)$, the choice of either policy 0 or policy 1 leads to the election of candidate 2, so choosing policy 0 after each of these histories is optimal.

After the history (h,A), the choice of policy 1 leads to the election of candidate 1 and the choice of policy 0 leads to the election of candidate 2, so the former action is optimal.

Strategy of citizen

After the history $(l, A, 1)$, electing candidate 1 yields the payoff $u(1, l)$ whereas electing candidate 2 yields the payoff $u(0,0)$, so given $u(0,0) > u(0, l) > u(1, l)$, the citizen's electing candidate 2 is optimal.

After the history $(h, A, 1)$, electing candidate 1 yields the payoff $u(1, h)$ whereas electing candidate 2 yields the payoff $u(0,0)$, so given $u(1, h) > u(0,0)$, the citizen's electing candidate 1 is optimal.

At the information set $\{(l, 0, 1), (h, 0, 1)\}$, the citizen's belief assigns probability 1 to the history $(l, 0, 1)$, so that she believes that electing candidate 1 will yield the payoff $u(1, l)$ whereas electing candidate 2 will yield the payoff $u(0,0)$, and hence given $u(0,0) > u(0, l) > u(1, l)$, her electing candidate 2 is optimal.

At the information set $\{(l, A, 0), (l, 0, 0), (h, A, 0), (h, 0, 0)\}$ (the rectangle in Figure 12.6), the citizen's belief assigns probability 1 to the history $(l, 0, 0)$, so that she believes that electing candidate 1 will yield the payoff $u(0, l)$ whereas electing candidate 2 will yield the payoff $u(0,0)$, and hence given $u(0,0) > u(0, l)$, her electing candidate 2 is optimal.

Beliefs

Given the strategy profile, the probability of reaching the information set $\{(l, 0, 1), (h, 0, 1)\}$ is 0, so any belief at this information set is consistent with the strategy profile. The probability of the history $(l, 0, 0)$ conditional on reaching the information set $\{(l, A, 0), (l, 0, 0), (h, A, 0), (h, 0, 0)\}$ is 1, so the belief at this information set is consistent with the strategy profile.

Figure 12.7 shows the citizen's expected payoffs in the equilibria of the games without (Proposition 12.2) and with (Proposition 12.3) the interest group. In the latter equilibrium, the interest group identifies candidate 1's type, but the price of doing so is that if the type is h then the candidate chooses policy 1, which is worse for the citizen than policy 0. As a consequence, if the probability that candidate 1's quality is h is sufficiently large—if the probability that her type is l is less than $\overline{p}$ in the figure—the citizen is better off in the equilibrium without the interest group in which both types of candidate 1 choose policy 0.

This analysis of the citizen's payoff ignores the cost of advertising. If this cost is ultimately borne by the citizen then the changes in her payoff need to be adjusted appropriately.

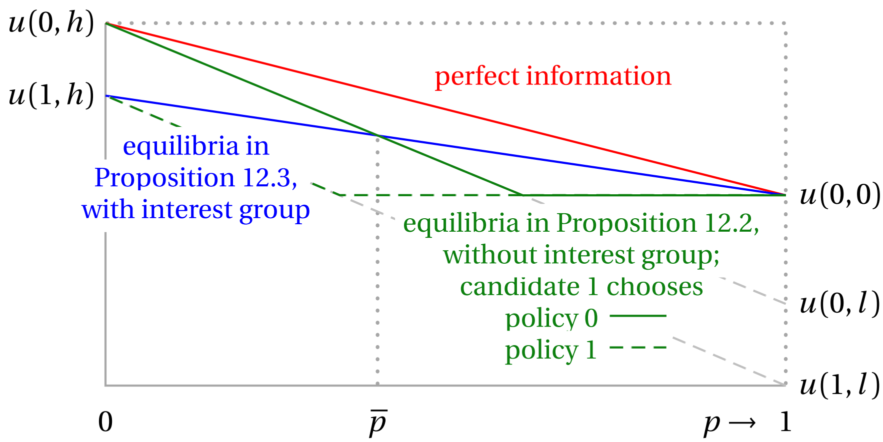


Figure 12.7 The citizen's expected payoffs in equilibria of policy games with a candidate of uncertain quality with and without an interest group, as a function of p, the probability that candidate 1's quality is l.

Notes

Section 12.1 is not closely based on any published model, but draws elements from the models of Baron (1994), Grossman and Helpman (1996), and Herrera et al. (2008).

Section 12.2 is based on Prat (2006), which takes elements from Coate (2004) and Ashworth (2006).

Solutions to exercises

Exercise 12.1

By Proposition 12.1a, x_1^* and x_2^* are both equal to the median of the favorite positions of the citizens in $N \cup S^*$. Given that $x_1^* = x_2^*$, the interest group is indifferent among all the sets of citizens it can mobilize, so that in particular S^* is optimal for it after the history (x_1^*, x_2^*). Now consider a history (x_1, x_2^*) with $x_1 \neq x_1^*$. If the interest group continues to choose S^*, then candidate 1 loses and the outcome remains $x_1^* = x_2^*$. The interest group benefits from changing the set of citizens it mobilizes only if doing so causes candidate 1 to win and it prefers x_1 to $x_1^* = x_2^*$. To cause candidate 1 to win, the interest group must mobilize a set S for which the median favorite position m^V of the set of voters is closer to x_1 than to x_2^*. However, in that case, in the game in which it moves first it could achieve the outcome m^V, which it prefers to $x_1^* = x_2^*$ (given that it prefers x_1 to $x_1^* = x_2^*$), by mobilizing S, contradicting the fact that $(S^*, (x_1^*, x_2^*))$ is the outcome of a subgame perfect equilibrium of the game in which it moves first. So for no value of x_1 can the interest group profitably deviate from S^*.

The same argument applies to histories (x_1^*, x_2) with $x_2 \neq x_2^*$.

If the interest group chooses S^* after all such histories, as well as after (x_1^*, x_2^*), then each candidate i faces the same electorate whether she chooses x_i^* or deviates from it, so that by the arguments in the proof of Proposition 8.4 for the strategic game in which the interest group is absent, neither candidate i can increase her payoff by deviating from x_i^*, given the position of the other candidate and the interest group's strategy.

We conclude that if the interest group has an optimal action for every pair (x_1, x_2) with $x_1 \neq x_1^*$ and $x_2 \neq x_2^*$ then the game in which the candidates move first has a subgame perfect equilibrium with the outcome $((x_1^*, x_2^*), S^*)$.

13 Two-period electoral competition with imperfect information

13.1 Selecting and controlling incumbents with unknown preferences 441
13.2 Inducing an incumbent to exert effort 447
13.3 Candidates who pander to voters 449

After serving a term in office, a politician stands for reelection. How does the possibility that she is reelected affect the policy she chooses in her first term? Does it induce her to choose a policy well-aligned with the citizens' preferences? Does it lead different types of incumbent to choose different policies, allowing citizens to reelect only ones who will implement policies favorable to them in the future? Could it lead a well-informed incumbent to choose a policy that the imperfectly-informed citizens believe is good for them even though it is not, as the incumbent knows?

Synopsis

The models I present are among the simplest that capture the main ideas that have been studied. They have two periods, a single citizen, and two possible policies, x and y. In the first period, the incumbent chooses a policy. The citizen has limited information: she may not know the incumbent's preferences; she may observe only the outcome of the policy chosen by the incumbent, not the policy itself; and she may not know the policy that is best for her. She either reelects the incumbent for a second term or elects a challenger. The second period is the last, so the officeholder in that period chooses the policy that maximizes her payoff in that period.

In the model in Section 13.1, the citizen prefers x to y. The incumbent and the second-period challenger are drawn from a pool of candidates in which the proportion π have the same preferences as the citizen (their preferences are *consonant* with the citizen's) and the proportion $1-\pi$ prefer y to x (their preferences are *dissonant* from the citizen's). The citizen does not know the incumbent's type. A candidate values both holding office and the policy chosen by the officeholder.

The model, like the ones in the subsequent sections, is an extensive game with imperfect information. I apply to these models the notion of weak sequential equilibrium.

If the benefit to the incumbent of holding office in the second period is large, the prospect of not being reelected induces behavior favorable to the citizen in the first period: Proposition 13.1 shows that the game has an equilibrium in which all incumbents choose x, the policy preferred by the citizen, in the first period. The cost of achieving this good outcome in the first period is that all incumbents, consonant and dissonant, are reelected, and dissonant ones choose y in the second period. In an extension of the model to many periods in which the future is accorded sufficient weight, the disadvantage of reelecting a dissonant candidate is absent, because a candidate faces the same incentive to choose x in the future as she does in the present. In this case, the game has an equilibrium in which the outcome is the best one possible for the citizen: x in every period.

If the benefit to the incumbent of holding office in the second period is small, Proposition 13.1 shows that the game has an equilibrium in which a consonant incumbent chooses x but a dissonant one chooses y. The first-period outcome is not so good for the citizen, but the incumbent's behavior allows the citizen to reelect only consonant incumbents, improving the second-period outcome.

Proposition 13.1 shows in addition that if the benefit to the incumbent of holding office in the second period is large, the game has an equilibrium in which all incumbents choose y, the citizen reelects an incumbent who chooses y and not one who chooses x, and the citizen believes that an incumbent who chooses x is more likely to be dissonant than she was originally. This belief may seem unreasonable, but could be an implication of the citizen's belief about what has happened when she observes the unexpected action x.

In the models in Section 13.2, all incumbents are dissonant. (Perhaps the social class from which they are drawn differs from that of the citizen.) Generating a good outcome for the citizen requires the incumbent to expend costly effort. The citizen observes only the outcome, which depends both on the effort and random factors out of the incumbent's control. A higher bar for reelection may induce an incumbent to exert more effort, improving the first-period outcome for the citizen. But it may alternatively make it not worthwhile for the incumbent to try to reach the bar, so that she reduces her effort. Proposition 13.2 shows that the game has equilibria in which the incumbent chooses low effort and, if the benefit to the incumbent of holding office in the second period is large, one in which the incumbent chooses high effort and the citizen reelects the incumbent only if the outcome is the one the citizen favors.

In the model in Section 13.3, the citizen is uncertain not only about the incumbent's preferences but also about the policy that is best for her, which depends on a state known to the incumbent but not to her. Proposition 13.3 shows that the model has a unique equilibrium, in which some types of the incumbent choose the policy that the citizen believes is most likely to be best for her, even if the incumbent knows that another policy is in fact best. As a consequence, under some conditions the citizen can do better by choosing a policy herself or by not allowing the incumbent to stand for reelection, so that in each period the policy is the favorite policy of a randomly-selected candidate.

13.1 Selecting and controlling incumbents with unknown preferences

A citizen prefers policy x to policy y. In the first period, an incumbent holds office. Her preferences are the same as the citizen's with probability π (in which case her type is said to be *consonant*) and the opposite of the citizen's with probability $1-\pi$ (the candidate's type is *dissonant*). Knowing her type, she chooses a policy, and the citizen observes this policy but not the incumbent's type. In the second period, the incumbent faces reelection. The challenger's type, like the incumbent's, is consonant with probability π and dissonant with probability $1-\pi$. The citizen votes for the incumbent or the challenger; whoever is elected chooses her favorite policy, because the second period is the last.

In each period in which a candidate is in office, she receives the payoff v' if the policy is the one she favors and v otherwise. She receives an additional payoff of $b > 0$ if she is the incumbent in the first period and is reelected in the second period. Her total payoff is the sum of her payoffs in the two periods.

This situation may be modeled as the extensive game with imperfect information in Figure 13.1, in which $\overline{v} = \pi v' + (1-\pi)v$ and $\underline{v} = \pi v + (1-\pi)v'$. (Ignore for the moment the arrows and numbers in red.) The players are the citizen and the incumbent (B); the citizen's action C is a vote for the challenger. The policy choice in the second period is not included as an action in the game, but is reflected in the payoffs.

Definition 13.1: Two-period electoral competition game with unobserved incumbent preferences

A *two-period electoral competition game with unobserved incumbent preferences* $\langle \pi, v, v', b \rangle$, where $\pi \in (0,1)$, $0 < v < v'$, and $b \geq 0$ is the extensive game with imperfect information shown in Figure 13.1, where $\underline{v} = \pi v + (1-\pi)v'$ and $\overline{v} = \pi v' + (1-\pi)v$.

Note that an incumbent who is induced by the carrot of reelection to select x in the first period when, in the absence of the incentive, she would not do

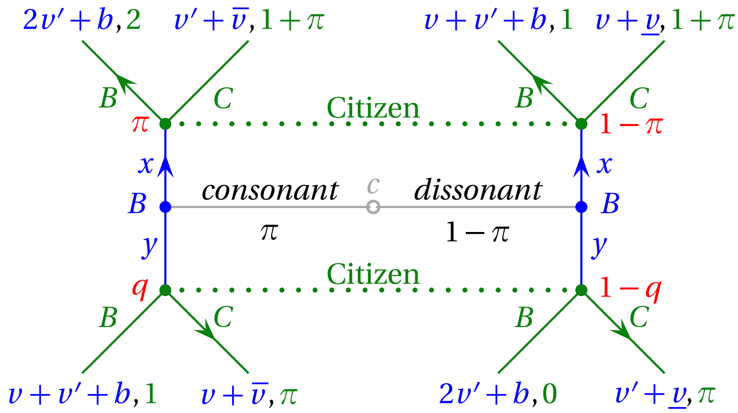


Figure 13.1 A two-period electoral competition game with unobserved incumbent preferences. The game begins at the small circle in the center of the figure, with a move of chance. The incumbent's payoff is listed first, and $\overline{v} = \pi v' + (1-\pi)v$ and $\underline{v} = (1-\pi)v' + \pi v$. If $b \geq (1-\pi)(v'-v)$ then for any $q \leq \pi$ the assessment for which the strategy profile is indicated by the arrows and the belief system is given by the numbers in red is a weak sequential equilibrium.

so, chooses y in the second period, which the citizen does not like. Thus if the threat to not reelect an incumbent who chooses y is effective in inducing all incumbents, consonant and dissonant, to choose x, making the first-period outcome better for the citizen, the second-period outcome is poor for the citizen if the incumbent is dissonant. Conversely, if the citizen's strategy is successful in selecting only consonant incumbents for reelection, then consonant and dissonant incumbents must choose different policies in the first period, generating a poor outcome for the citizen in the first period if the incumbent is dissonant. Specifically, if dissonant incumbents are induced to choose x in the first period, like consonant incumbents, and are thus reelected, the citizen's expected payoff is v' in the first period and $\pi v' + (1-\pi)v$ in the second period, whereas if consonant incumbents choose x and dissonant ones choose y in the first period, the citizen's payoffs are $\pi v' + (1-\pi)v < v'$ in the first period and $\pi v' + (1-\pi)(\pi v' + (1-\pi)v) > \pi v' + (1-\pi)v$ in the second period.

If the benefit b of holding office is sufficiently large, the game has a weak sequential equilibrium in which both types of incumbent choose x and the citizen reelects an incumbent if and only if she chooses x. In such an equilibrium, the belief system assigns probability π to the history $(consonant, x)$ at the upper information set and some probability $q \leq \pi$ to the history $(consonant, y)$ at the lower information set, as illustrated in Figure 13.1. This belief system is weakly consistent with the strategy profile: given the strategy profile, the history $(consonant, x)$ occurs with probability π and the history $(dissonant, x)$ oc-

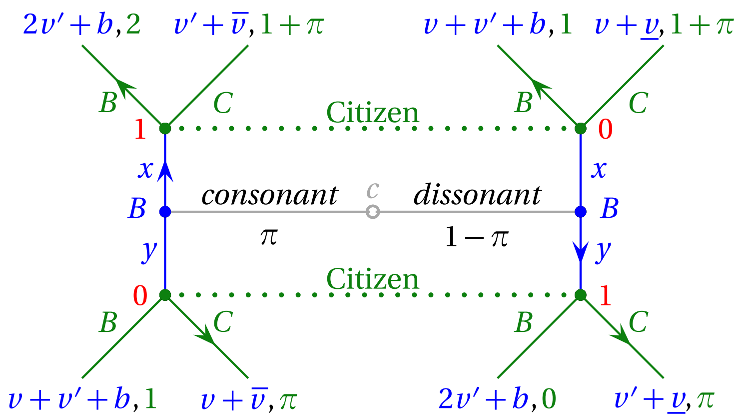


Figure 13.2 A weak sequential equilibrium of a two-period electoral competition game with unobserved incumbent preferences in which both types of incumbent choose the policy x.

curs with probability $1 - \pi$, and the bottom information set is not reached, so that weak consistency imposes no constraint on the beliefs there. At the top information set the citizen is indifferent between B and C, because observing the policy x conveys no information about the incumbent's type, and at the bottom information set she is no better off choosing B than C, given that $q \leq \pi$. A consonant incumbent decreases her payoff if she deviates to y and a dissonant one does not benefit from deviating to y if $b \geq (1 - \pi)(v' - v)$. The assessment is not a weak sequential equilibrium if $b < (1 - \pi)(v' - v)$ because then a dissonant incumbent gains by deviating to y in the first period, foregoing reelection but enjoying her favorite policy in the first period.

If $b \leq (1 - \pi)(v' - v)$ then the game has an equilibrium in which a consonant incumbent chooses x, a dissonant one chooses y, and the citizen reelects an incumbent if and only if she chooses x. In this equilibrium, the outcome is the same as it would be if the citizen knew the incumbent's type. An assessment with this strategy profile is shown in Figure 13.2. Given the belief system, the citizen's strategy is optimal: she prefers B to C at her top information set and C to B at her bottom information set. Also, given the citizen's strategy, the action of a consonant incumbent is optimal because $2v' + b > v + \overline{v}$, and the action of a dissonant incumbent is optimal if $v' + \underline{v} \geq v + v' + b$, or $b \leq (1 - \pi)(v' - v)$. Thus the assessment is a weak sequential equilibrium if this inequality is satisfied. The assessment is not a weak sequential equilibrium if $b > (1-\pi)(v'-v)$ because then a dissonant incumbent gains by deviating to x in the first period, and thus getting reelected in the second period. This maneuver hurts her in the first period but helps her in the second period, when she implements y rather than enduring the favored policy of a randomly-determined challenger.

The game has no weak sequential equilibrium in which a consonant incumbent chooses y and a dissonant one chooses x. In any assessment with these strategies the weak consistency of the belief system with the strategy profile requires the belief at the top information set to assign probability 1 to the history (*dissonant*, x) and the belief at the bottom information set to assign probability 1 to the history (*consonant*, y), so that the citizen optimally reelects an incumbent if and only if she chooses y. But then a dissonant incumbent gains by deviating to y (and getting reelected).

When b is large, the game does, however, have a weak sequential equilibrium in which both types of incumbent choose y, the citizen reelects an incumbent if and only if she chooses y, and the belief system assigns probability π to the history (*consonant*, y) at the bottom information set (as required by the weak consistency of the beliefs with the strategies) and at most π to the history (*consonant*, x) at the top information set. The citizen is indifferent between B and C at the bottom information set and does not benefit from deviating to B at the top information set. A consonant incumbent does not benefit from deviating to x if $v + v' + b \geq v' + \bar{v}$, or $b \geq \pi(v' - v)$, and a dissonant incumbent is worse off deviating to x.

Proposition 13.1: Weak sequential equilibria of two-period electoral competition game with unobserved incumbent preferences

Let $\langle \pi, v, v', b \rangle$ be a two-period electoral competition game with unobserved incumbent preferences. Each weak sequential equilibrium of this game has one of the following three forms.

a. (Pooling on x) If $b \geq (1 - \pi)(v' - v)$ then both types of incumbent choose x, the citizen reelects an incumbent who chooses x and not one who chooses y, and the belief system assigns probability π to the history (*consonant*, x) at the information set reached when the incumbent chooses x and, for some $q \leq \pi$, probability q to the history (*consonant*, y) at the information set reached when the incumbent chooses y.

b. (Separating) If $b \leq (1 - \pi)(v' - v)$ then a consonant incumbent chooses x and a dissonant one chooses y, the citizen reelects an incumbent who chooses x and not one who chooses y, and the belief system assigns probability 1 to the history (*consonant*, x) at the information set reached when the incumbent chooses x and probability 1 to the history (*dissonant*, y) at the information set reached when the incumbent chooses y.

c. (Pooling on y) If $b \geq \pi(v' - v)$ then both types of incumbent choose y, the citizen reelects an incumbent who chooses y and not one who chooses x, and for some $q \leq \pi$ the belief system assigns probability q to the history $(consonant, x)$ at the information set reached when the incumbent chooses x and probability π to the history $(consonant, y)$ at the information set reached when the incumbent chooses y.

In the equilibrium in part *c*, in which both types of incumbent choose y, if the citizen observes the policy x, then she believes that the incumbent is more likely to be dissonant than she was initially. Given that a history in which the policy x is chosen does not happen if the incumbent adheres to her strategy, the citizen has no basis on which to form a belief about the incumbent's type if she observes x. Thus her belief that the incumbent is more likely to be dissonant than she was originally is weakly consistent with the strategy profile. Is it reasonable?

Suppose that we interpret the equilibrium as a steady state: in each of a long sequence of periods, a citizen and an incumbent have been randomly selected from large populations, and in every period the selected incumbent has chosen y and has been reelected. Now, in the current period, the selected citizen observes the policy x. The citizen might conclude that the incumbent has made a mistake, or that something has changed—perhaps the incumbent's options, or her payoffs. Depending on the cause of the mistake or the nature of the change in the game, the citizen's belief that the incumbent is more likely to be dissonant that she was originally could be reasonable.

Another approach to thinking about the implications of the citizen observing x assumes that she regards the deviation as deliberate. In this case, she might reason that the incumbent must be consonant because deviating to x holds no possible advantage for a dissonant incumbent, since y gets her reelected and thus leads to the best possible outcome for her, but *is* advantageous for a consonant incumbent if it results in the citizen reelecting her. And if the citizen believes that the incumbent is consonant, then she *should* reelect her. The conclusion is that the equilibrium is unstable, because a deviation by the incumbent to x should persuade the citizen that the incumbent is consonant. I do not pursue this point further, but you might find it compelling.

The citizen's expected payoff in the equilibria in parts a and b of Proposition 13.1 is shown in Figure 13.3 as a function of π. This payoff is increasing in π except at the point at which the pooling equilibrium replaces the separating equilibrium.

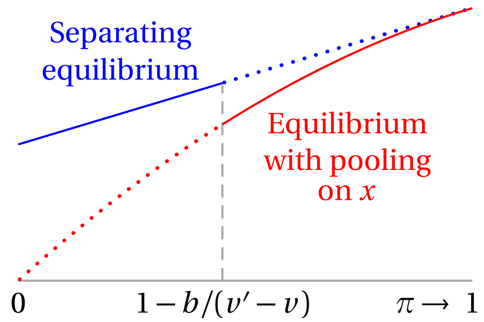


Figure 13.3 The expected payoff of the citizen in a separating equilibrium of a two-period electoral competition game with unobserved incumbent preferences, which exists if $\pi \le 1-b/(v'-v)$, and an equilibrium with pooling on x, which exists if $\pi \ge 1-b/(v'-v)$.

Long horizon

If, in the two-period model, the citizen's voting strategy induces a dissonant incumbent to choose x in the first period, then the outcome in the second period is undesirable if the incumbent is dissonant: such an incumbent, with no prospect of further reelection, chooses y. If the game lasts for more than two periods, the citizen may be able to generate a better outcome. Suppose, at the other extreme, that the horizon is infinite. In each period, the incumbent from the previous period competes with a challenger randomly chosen from a large set of candidates, the fraction π of which are consonant and the remainder dissonant. The set of candidates is large enough that the probability that an incumbent who is not reelected will subsequently be the challenger is negligible; I take it to be zero. A candidate obtains the payoff v' in every period in which the outcome is the one she favors (x if she is consonant, y if she is dissonant) and v in every period in which it is the other outcome, plus $b > 0$ in each period (including the first) in which she is the incumbent. The citizen obtains the payoff 1 in every period in which the outcome is x and 0 in every period in which it is y. Every player has the same discount factor, $\delta \in (0,1)$, and her total payoff is the discounted average ($1-\delta$ times the discounted sum) of her payoffs in each period.

Suppose that in each period the citizen reelects an incumbent if and only if she chose the policy x in the previous period, and every incumbent, consonant or dissonant, chooses policy x in every period regardless of the history. A dissonant incumbent then obtains $v+b$ in every period, and hence a discounted average payoff of $v+b$. If she deviates to y in any period, she is better off in that period, obtaining the payoff $v'+b$, and worse off in every subsequent period, obtaining v. Her discounted average payoff is thus $(1-\delta)(v'+b)+\delta v$. This payoff is less than $v+b$ if $v'-v$ is small, b is large, and/or δ is close to 1. Thus in these cases, the game has a subgame perfect equilibrium in which the citizen

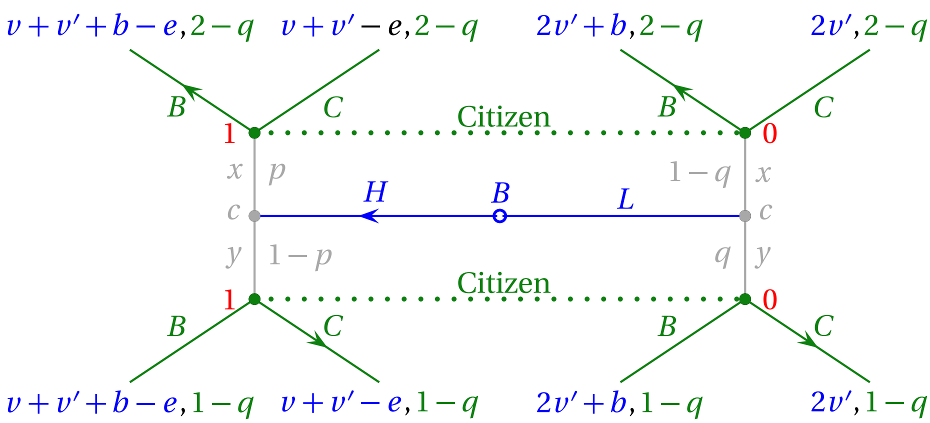


Figure 13.4 A two-period electoral competition game with unobserved incumbent actions. The game begins at the small circle in the center of the figure, with the choice of H or L by the incumbent. If $(p-(1-q))b \geq v' - v + e$ then the assessment for which the strategy profile is indicated by the arrows and the belief system is given by the numbers in red is a weak sequential equilibrium.

obtains the best possible outcome: x in every period. In this model, inducing an incumbent to choose x by not reelecting her if she chooses y does not doom the citizen to y in the future if the incumbent is dissonant, as it does in the two-period model, because the incumbent will face the same incentives in the future that she faces today.

13.2 Inducing an incumbent to exert effort

Uncertainty about the incumbent's motivation is only one of the possible limitations of the citizen's information. For example, the citizen may observe only the outcome of the incumbent's action, not the action itself. In this section I briefly discuss models that focus on this limitation in the citizen's information. In these models, the citizen knows the candidates' motivations.

In the first model, an incumbent interacts with a citizen as shown in Figure 13.4. The incumbent takes one of two actions, H and L, which may be interpreted as the effort she expends. Then chance determines the outcome, which is either x or y; it is x with probability $p > \frac{1}{2}$ if the action is H and with probability $1-q < \frac{1}{2}$ if the action is L. The citizen observes the outcome, but not the action, and either reelects the incumbent or elects a challenger. In both cases, the officeholder chooses L in the second period and the outcome is x with probability $1-q$ and y with probability q.

The citizen prefers x to y. Her payoff is 1 in each period in which the outcome is x and 0 in each period in which it is y, and her total payoff is the sum of her payoffs in the two periods. The incumbent's payoff depends on both her action and the outcome. She prefers L to H and, unlike the citizen, y to x. Specifically, her payoff in each period is v' if the outcome is y and $v < v'$ if it is x, minus $e > 0$ if her action is H and plus $b \geq 0$ if she is reelected. Given that she prefers y to x and L is more likely to yield y, she chooses L if she is reelected. For the same reason, a challenger also chooses L if elected.

Definition 13.2: Two-period electoral competition game with unobserved incumbent actions

A *two-period electoral competition game with unobserved incumbent actions* $\langle p, q, v, v', b, e\rangle$, where $p \in (\frac{1}{2}, 1)$, $q \in (\frac{1}{2}, 1)$, $0 < v < v'$, $b \geq 0$, and $e \geq 0$ is the extensive game with imperfect information shown in Figure 13.4.

By reelecting the incumbent only if the outcome is x, can the citizen induce the incumbent to choose H? The game has a weak sequential equilibrium with such a result if the benefit b to holding office is high enough relative to $v' - v + e$. This equilibrium is indicated in Figure 13.4. At each of her information sets, the citizen is indifferent between reelecting the incumbent and electing the challenger, because both of them will choose L in the second period. Thus both actions are optimal for her at each information set, regardless of her belief. Hence to check whether an assessment is a weak sequential equilibrium we need to check only the optimality of the incumbent's action. If the citizen reelects the incumbent only if the outcome is x then the incumbent's payoff to H is at least her payoff to L if and only if $(p-(1-q))b \geq v' - v + e$. Otherwise the incumbent's payoff to L is greater than her payoff to H. Thus the weak sequential equilibria of the game are given as follows.

Proposition 13.2: Weak sequential equilibria of a two-period electoral competition game with unobserved incumbent actions

Let $\langle p, q, v, v', b\rangle$ be a two-period electoral competition game with unobserved incumbent actions. Each weak sequential equilibrium of this game has one of the following forms.

a. If $(p-(1-q))b \geq v' - v + e$ then the incumbent chooses H, the citizen reelects an incumbent if the outcome is x and not if it is y, and the belief system assigns probability 1 to the history (H, x) at the information set reached when the outcome is x and probability 1 to the history (H, y) at the information set reached when the outcome is y.

b. The incumbent chooses L, the citizen (i) reelects the incumbent regardless of the outcome, or (ii) elects the challenger regardless of the outcome, or (iii) reelects the incumbent only if the outcome is y, or, if $(p-(1-q))b \leq v'-v+e$, (iv) reelects the incumbent only if the outcome is x, and the belief system assigns probability 1 to the history (L, x) at the information set reached when the outcome is x and probability 1 to the history (L, y) at the information set reached when the outcome is y.

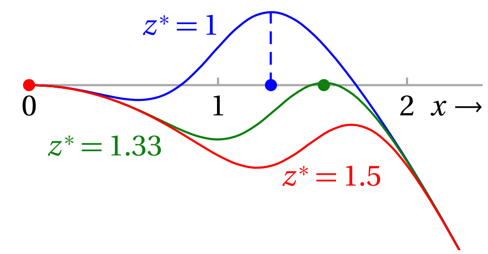
If the incumbent chooses L and the citizen raises the bar for reelection from y (that is, no bar at all) to x, then the incumbent's action either remains L, if the cost of meeting the bar outweighs the benefit, or changes from L to H. In a model with a richer set of actions, the tradeoff the citizen faces when choosing the reelection bar is smoother.

Suppose that the incumbent chooses an action (effort level) $x \in \mathbb{R}$. The outcome $z \in \mathbb{R}$ depends probabilistically on x; denote the probability that it is at most z when the incumbent's effort is x by $F(z, x)$. For each value of z, $F(z, x)$ is decreasing in x: more effort makes higher outcomes more likely. The citizen observes z and either reelects the incumbent or elects the challenger. The incumbent likes to be in office and dislikes effort. For simplicity, assume that her payoff does not depend on the policy chosen by the officeholder (unlike her payoff in the previous models in this chapter). If she is reelected her payoff is $b-c(x)$, and if she is not reelected it is $-c(x)$, where c is an increasing function.

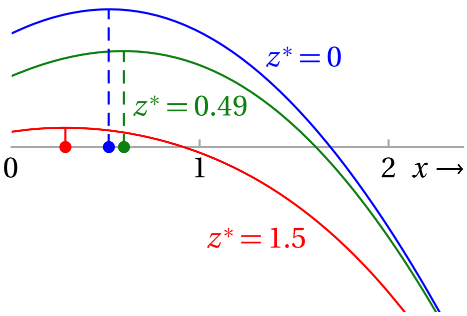
Assume that for some number z^*, the citizen reelects the incumbent if the outcome is at least z^*. Then the expected payoff of an incumbent as a function of her effort x is $(1-F(z^*, x))b-c(x)$. Examples are shown in Figure 13.5 for the case in which $F(\cdot, x)$ is a normal distribution with mean x, $b = 3$, and $c(x) = x^2$ for all x. Increasing the reelection standard z^* from 0 initially induces the incumbent to exert more effort, but at some point the cost of exerting enough effort that the outcome is likely to meet the standard overwhelms the benefit of reelection, and the incumbent optimally reduces her effort. In the examples, she reduces it to zero if the variance of the distribution of outcomes is small, and reduces it gradually if this variance is large. In these examples the citizen is better off when the outcome is a more precise signal of the incumbent's effort.

13.3 Candidates who pander to voters

A citizen may be uncertain not only of the candidates' preferences and actions, but also of the policy that is best for her. We can model this latter uncertainty by



(a) Variance of distribution of outcomes given effort: 0.0625. Reelection cutoff that induces most effort: $z^* = 1.33$.



(b) Variance of distribution of outcomes given effort: 1. Reelection cutoff that induces most effort: $z^* = 0.49$.

Figure 13.5 The payoff of an incumbent as a function of her effort x for various cutoffs z^* for reelection in two examples of a two-period electoral competition game with continua of possible actions and outcomes. In each example the distribution of outcomes when the incumbent's effort is x is a normal distribution with mean x, $b = 3$, and $c(x) = x^2$ for all x. The payoff-maximizing effort levels are indicated by small disks.

assuming that the policy best for the citizen depends on a state that she does not know. Assume that the candidates know this state and that their preferences may differ from those of the citizen, as in the model in Section 13.1, but the citizen, unlike the one in the model in Section 13.2, observes the policy chosen by the officeholder. In this environment, electing a representative to choose a policy may not be optimal for the citizen. On the one hand, given that the candidates are better informed than the citizen, having an elected representative choose a policy makes it possible that the chosen policy is better than the one the citizen would choose. On the other hand, the competition to be elected may induce a candidate to select the policy that she knows the citizen believes is most likely to be optimal, even though, as she knows, this policy is not best for the citizen. Depending on the balance of these effects, the citizen may be better off choosing a policy herself or selecting a candidate randomly in each period than electing a representative on the basis of the policies she chose as an incumbent.

Model

As previously, there are two periods, the participants are an incumbent, a challenger, and a citizen, and there are two possible policies, x and y. In the first period, the incumbent chooses a policy. The citizen observes this policy and either reelects the incumbent or elects the challenger. In the second period, the officeholder chooses a policy.

There are two possible states, called x and y like the policies. The policy

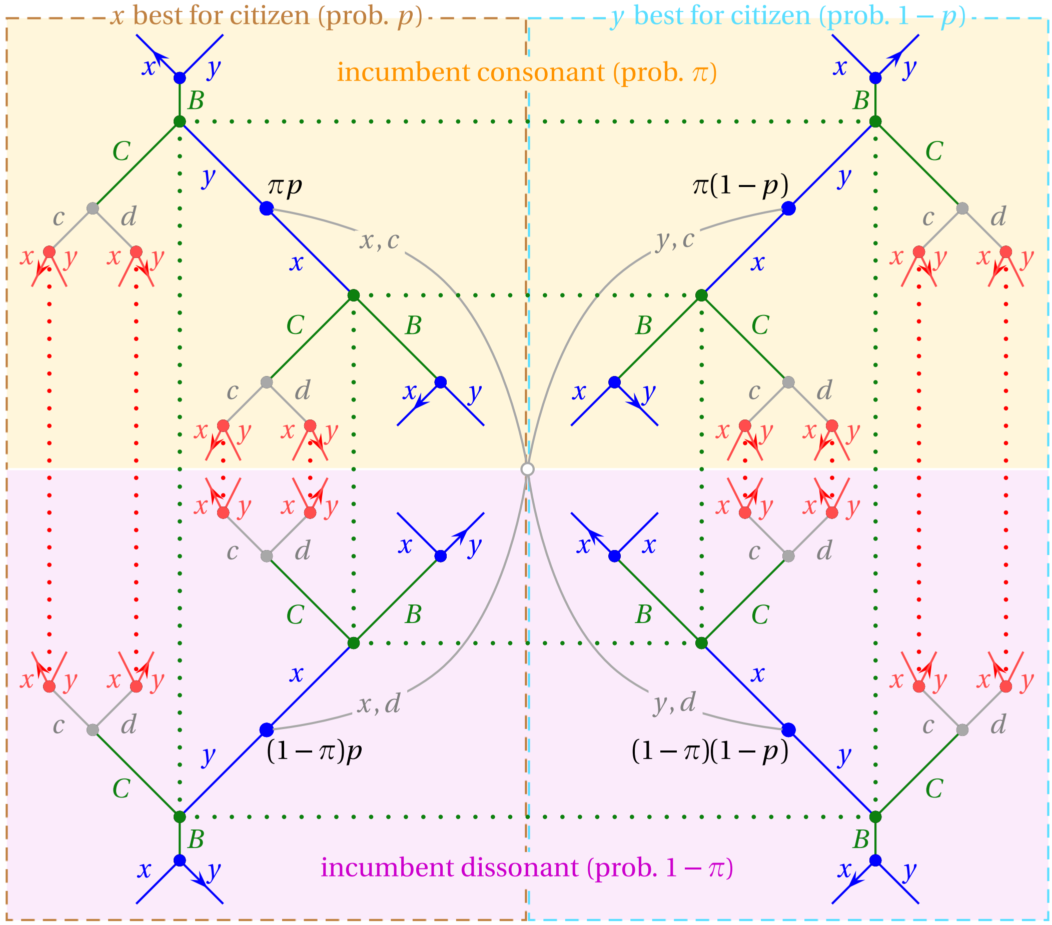


Figure 13.6 The structure of a two-period game of electoral competition in which the candidates, but not the citizen, know the best policy for the citizen. The game begins with the move of chance indicated by the small circle in the center. Blue elements belong to the incumbent, red elements to the challenger, green elements to the citizen, and gray elements to chance.

best for the citizen depends on the state: policy x is best in state x and policy y is best in state y. The incumbent knows the state, and hence the policy that is best for the citizen, but the citizen does not; she believes that the state is x with probability p and y with probability $1-p$. With probability π, a candidate (the incumbent or the challenger) has preferences that are consonant with the citizen's—she prefers x in state x and y in state y—and with probability $1-\pi$ she has preferences that are dissonant from the citizen's—she prefers y in state x and x in state y.

The structure of an extensive game with imperfect information that reflects these assumptions is shown in Figure 13.6. Play begins with the move of chance indicated by the small circle in the center of the figure. Chance independently determines the state (x with probability p, y with probability $1-p$) and the incumbent's preferences (consonant with the citizen's with probability π, dissonant from the citizen's with probability $1-\pi$). The branch labeled x, c, for example, corresponds to state x and preferences for the incumbent that are consonant with the citizen's, an event with probability πp. The incumbent observes the move of chance and chooses a policy (x or y) for period 1. The citizen observes the policy, but not the incumbent's type or the state, and thus has two information sets, indicated by the dotted squares. The inner information set is reached if the incumbent chooses policy x, and the outer one is reached if the incumbent chooses policy y. At each information set, the citizen selects the incumbent (B) or the challenger (C). If she selects the challenger, chance determines the challenger's type (c (consonant) with probability π and d (dissonant) with probability $1-\pi$). Finally, the candidate the citizen selected chooses a policy (x or y) for period 2. (The state remains the same in period 2 as it was in period 1.)

(An alternative and perhaps more natural specification assumes that the challenger's type, like the incumbent's, is chosen by chance initially, rather than after the citizen selects the challenger. This specification is more complicated to depict, but its analysis is the same. The important point is that the citizen does not know the challenger's type when she selects a candidate, which is true in both formulations.)

The citizen's payoff is the number of periods (0, 1, or 2) in which the policy matches the state. She does not receive this payoff until the game ends, and in particular cannot use the part due to the policy in period 1 to infer the state. (If the state were chosen independently in each period, we could allow the citizen to observe her first-period payoff before period 2. The game would then be more complicated, but the results would be the same.)

A candidate (incumbent or challenger) receives the payoff v' in each period in which the policy is the one she prefers and v in each period in which it is the other policy, with $v' > v > 0$. In addition, if she is in office in the second period then she receives an additional payoff $b \geq 0$, which reflects her degree of office-motivation. At the moment I assume that all candidates have the same degree of office-motivation; later I assume that they may differ in this degree. Thus, for example, if the state is x, the incumbent's type is c, the incumbent chooses policy x in the first period, is reelected, and chooses policy x again in the second period, then her payoff is $2v' + b$. For a history that differs only in that the challenger is elected in the second period and chooses the policy y, the incumbent's payoff is $v' + v$.

In any weak sequential equilibrium of the game, the officeholder in period 2 chooses her favorite action, as indicated by the arrows in Figure 13.6. Thus we

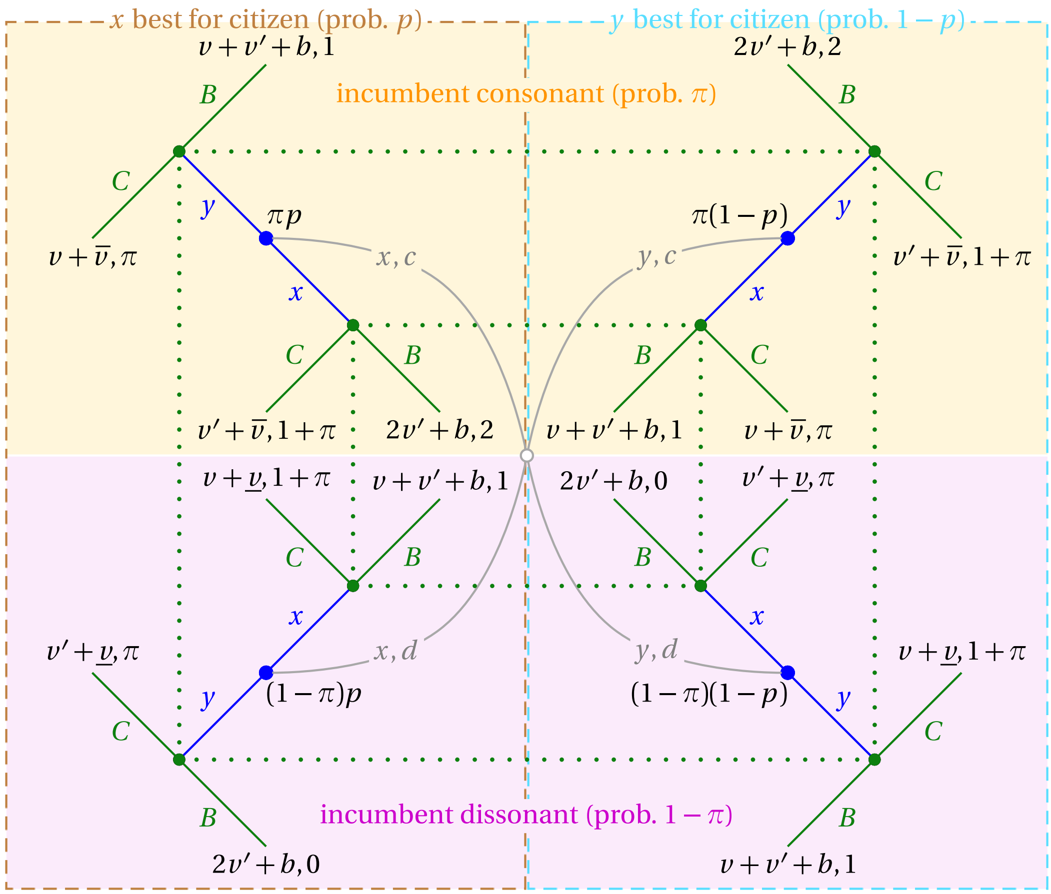


Figure 13.7 The game in Figure 13.6 with the second period replaced by the payoffs to the players' unique optimal actions in that period. The game begins with the move of chance indicated by the small circle in the center. The players are an incumbent (blue) and a citizen (green). The incumbent's payoff is listed first, and $\overline{v} = \pi v' + (1 - \pi)v$ and $\underline{v} = (1 - \pi)v' + \pi v$.

may find the weak sequential equilibria of the game by finding the weak sequential equilibria of the game in Figure 13.7, in which the players are the incumbent and the citizen (in addition to chance) and the part of each terminal history in period 2 in Figure 13.6 is replaced by the payoffs of the incumbent and the citizen when the second-period policy-maker chooses her favorite action. Subsequently I work with a variant of this reduced game.

In the variant, the incumbent has many possible degrees of office-motivation. For an incumbent to be willing to implement her less-preferred policy to get reelected, her office-motivation has to be sufficient that she prefers to endure her less-preferred policy in the first period and be reelected than to choose her favored policy in the first period and put up with the policy chosen by a random challenger in the second period. But if the office-motivation of every incumbent has such a magnitude, then in an equilibrium all incumbents choose the same policy, which has the unfortunate implication that the citizen's information set corresponding to the other policy is not reached. Consequently any belief regarding the history that led to this information set is consistent with the equilibrium strategy profile, leading to a multiplicity of equilibria. The model I specify avoids this problem by allowing the incumbent's degree of office-motivation to take any value in a finite set $\mathscr{B}$. Thus in this model, the incumbent's type is a pair (h, b), where h is the concordance of her preferences with those of the citizen (consonant or dissonant) and b is her degree of office-motivation. The subsequent result (Proposition 13.3) assumes that $\mathscr{B}$ contains two values, one high and one low. The optimal behavior of the incumbents with low office-motivation imposes discipline on the citizen's belief at the information set following the policy that is not chosen by any incumbent with high office-motivation.

In the model, chance starts by determining a state (x or y), the concordance of the incumbent's preferences with those of the citizens (*consonant* or *dissonant*), and the incumbent's degree of office-motivation (a member of $\mathscr{B}$). The incumbent observes these values and chooses a policy (x or y). Then the citizen, who observes the incumbent's policy but not the move of chance, selects either the incumbent or the challenger, who chooses her favorite policy in the second period. An example in which $\mathscr{B} = \{0, b\}$ is given in Figure 13.8.

Definition 13.3: Two-period electoral competition game with well-informed candidates with uncertain motivations

A *two-period electoral competition game with well-informed candidates with uncertain motivations* $\langle \{B, V, C\}, \{x, y\}, p, \pi, v', v, \mathscr{B}, \rho \rangle$, where

- B, V, and C are labels (B and V are the names of an incumbent and a citizen, and C represents the citizen's action of selecting a challenger rather than the incumbent)
- x and y are labels (the names both of two policies and two states)
- $p \in (0, 1)$ (the probability the state is x)
- $\pi \in (0, 1)$ (the probability that a candidate's preferences are the same as (consonant with) the citizen's preferences)
- $v' \geq 0$ (the component of the incumbent's payoff attributable to the policy in a period in which it is her favorite policy)

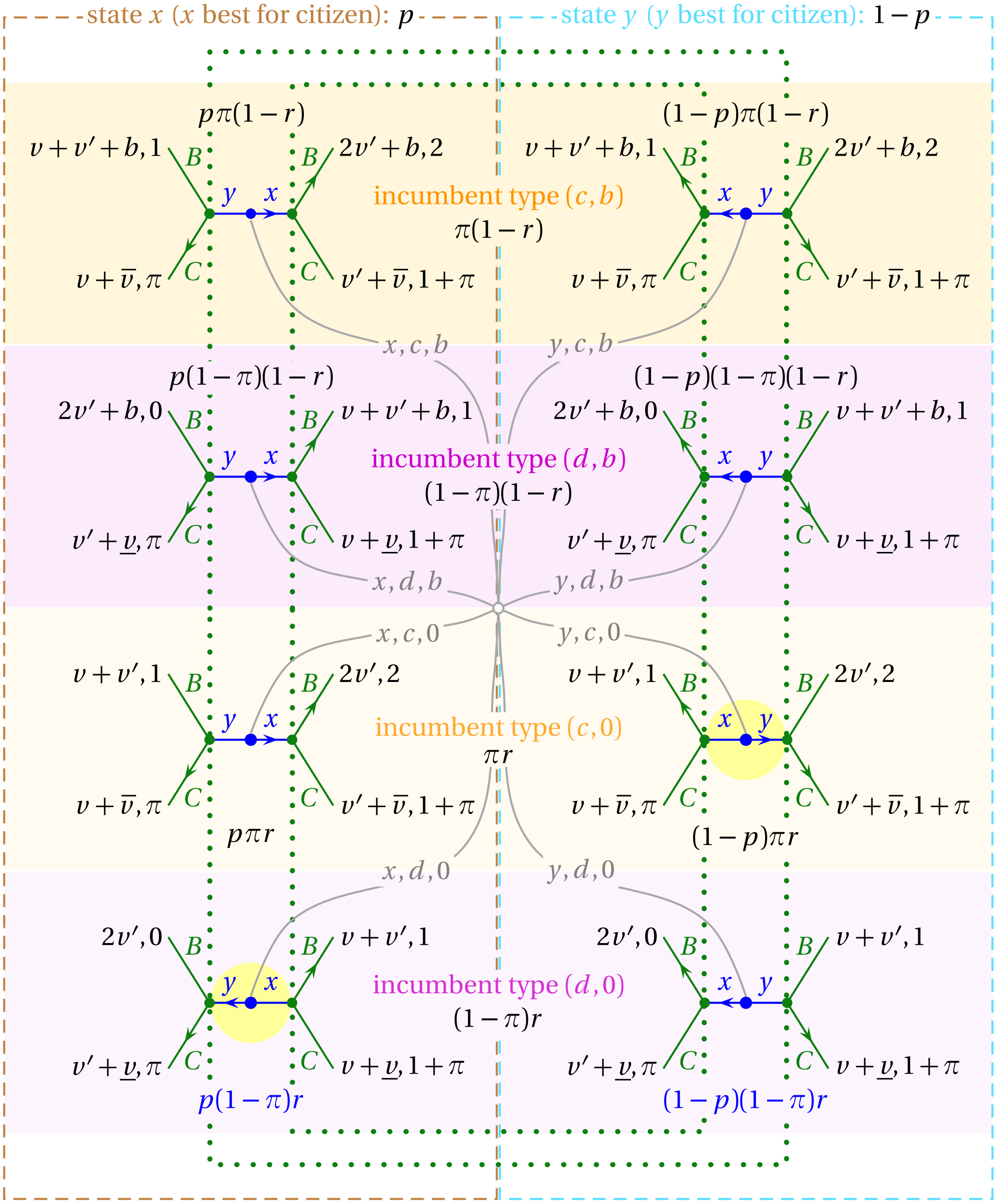


Figure 13.8 A two-period electoral competition game with well-informed candidates with uncertain motivations in which incumbents have two possible degrees of office–motivation, 0 (lower half of figure) and b (upper half of figure). Blue elements belong to the incumbent, green elements to the citizen, and gray elements to chance; probabilities are indicated in black. The incumbent's payoff is listed first; $\overline{v} = \pi v' + (1-\pi)v$ and $\underline{v} = (1-\pi)v' + \pi v$. The arrows indicate the actions chosen in any weak sequential equilibrium when $p > \frac{1}{2}$ and $v + b > \max\{\underline{v}, \overline{v}\}$, as given in Proposition 13.3. The cases in which the incumbent chooses y are highlighted.

- $v \geq 0$ with $v < v'$ (the component of the incumbent's payoff attributable to the policy in a period in which it is not her favorite policy)
- $\mathscr{B}$ is a finite set of nonnegative numbers (the possible values of the component of the incumbent's payoff attributable to her holding office)
- ρ is a probability distribution over $\mathscr{B}$ (the probabilities of the various degrees of office-motivation)

is an extensive game with imperfect information with the following components.

Players

B (an *incumbent*) and V (a *citizen*).

Terminal histories

The terminal histories are the sequences $((s,h,\beta),z,D)$ for $(s,h,\beta) \in \{x,y\} \times \{c,d\} \times \mathscr{B}$, $z \in \{x,y\}$, and $D \in \{B,C\}$.

Player function

The player function P is defined by

- $P(\varnothing) = chance$ (the game begins with a move of chance)
- $P(s,h,\beta) = B$ for each $(s,h,\beta) \in \{x,y\} \times \{c,d\} \times \mathscr{B}$ (the incumbent moves after chance determines the state and the incumbent's type)
- $P((s,h,\beta),z) = V$ for every $(s,h,\beta,z) \in \{x,y\} \times \{c,d\} \times \mathscr{B} \times \{x,y\}$ (the citizen moves after the incumbent chooses a policy).

Chance probabilities

At the initial history $\varnothing$ chance selects the state $s \in \{x,y\}$ (the best policy for the citizen), the concordance of the incumbent's preferences with the citizen's preferences (consonant (c) or dissonant (d)), and the incumbent's degree of office-motivation $\beta \in \mathscr{B}$, independently. The state s is x with probability p and y with probability $1-p$, the incumbent's preferences are consonant (c) with the citizen's with probability π and dissonant (d) from them with probability $1-\pi$, and the incumbent's degree of office-motivation is β with probability $\rho(\beta)$.

Information partitions

Player B's information partition consists of one set for each move of chance. Player V's information partition consists of the following two sets:

$$\{(s,h,\beta,x): s\in\{x,y\}, h\in\{c,d\}, \beta\in\mathscr{B}\}$$
$$\{(s,h,\beta,y): s\in\{x,y\}, h\in\{c,d\}, \beta\in\mathscr{B}\}.$$

Preferences

The preferences of each player over the set of lotteries over terminal histories are represented by the expected value of the following payoffs. For the terminal history $((s,h,\beta),z,D)$, the payoff of player B is the sum of two components: an amount attributable to the first-period policy,

$$\begin{cases} v' & \text{if } h=c \text{ and } s=z, \text{ or } h=d \text{ and } s\neq z \\ v & \text{if } h=c \text{ and } s\neq z, \text{ or } h=d \text{ and } s=z, \end{cases}$$

and an amount attributable to the second-period policy,

$$\begin{cases} v'+\beta & \text{if } D=B \text{ (citizen selects incumbent)} \\ \pi v'+(1-\pi)v & \text{if } D=C \text{ and } h=c \text{ (citizen selects challenger,} \\ & \qquad\qquad\text{incumbent is consonant)} \\ (1-\pi)v'+\pi v & \text{if } D=C \text{ and } h=d \text{ (citizen selects challenger,} \\ & \qquad\qquad\text{incumbent is dissonant).} \end{cases}$$

If player V, the citizen, selects the incumbent $(D=B)$ then her payoff is

$$\begin{cases} 0 & \text{if } z\neq s \text{ and } h=d \\ 1 & \text{if } z=s \text{ and } h=d, \text{ or } z\neq s \text{ and } h=c \\ 2 & \text{if } z=s \text{ and } h=c \end{cases}$$

and if she selects the challenger $(D=C)$ it is

$$\begin{cases} \pi & \text{if } z\neq s \\ 1+\pi & \text{if } z=s. \end{cases}$$

A strategy for the incumbent is a policy for each state and each of her types (h,β) (that is, for each move of chance). A strategy for the citizen is a function that assigns to each policy (x and y) either B (reelect incumbent) or C (elect challenger). A belief system assigns to each of the citizen's information sets a probability distribution over the histories in the set.

I now study the weak sequential equilibria of the game in which the incumbent has two possible degrees of office-motivation, 0 and b, with $v+b>\max\{\underline{v},\overline{v}\}$, where

$$\overline{v}=\pi v'+(1-\pi)v \quad \text{and} \quad \underline{v}=(1-\pi)v'+\pi v. \tag{13.1}$$

First consider an incumbent whose degree of office-motivation is 0 (types $(c,0)$ and $(d,0)$); her actions appear in the bottom two panels of Figure 13.8. For such an incumbent, choosing her favorite policy is unambiguously optimal. If she chooses the other policy, she sacrifices the payoff $v' - v$ in the first period and gains at most $v' - \max\{\underline{v}, \overline{v}\} < v' - v$ in the second period. She gains nothing from holding office per se, so there is no advantage in her choosing her less-favored policy in the first period in order to be reelected and thus choose her favorite policy in the second period, because if she chooses her favorite policy in the first period and is not reelected then with positive probability the challenger will choose her favorite policy in the second period. Thus in any weak sequential equilibrium, every incumbent whose degree of office-motivation is 0 chooses her favorite policy.

Now consider incumbents whose degree of office-motivation is b. Given that $v + b > \max\{\underline{v}, \overline{v}\}$, such an incumbent optimally chooses her less-favored policy in the first period if doing so gets her reelected, because the amount she thereby loses in the first period, $v' - v$, is less than the amount she gains in the second period, which is $v' + b - \overline{v}$ if her type is (c,b) and $v' + b - \underline{v}$ if her type is (d,b). Thus if, for example, the citizen reelects only incumbents who choose x, then an incumbent whose preferences are the same as the citizen's optimally chooses x even if she knows that the state is y.

The next result shows that in every weak sequential equilibrium all office-motivated incumbents choose the policy more likely to be best for the citizen, given the citizen's information that the state is x with probability p and y with probability $1 - p$, rather than the policy they know is best for the citizen given their knowledge of the state. Incumbents who behave in this way are sometimes said to "pander" to the citizen. The equilibrium is shown in Figure 13.8 for the case in which $p > \frac{1}{2}$, so that the policy more likely to be best for the citizen, given the citizen's information, is x. The policy x is chosen both by incumbents with positive office-motivation and by those with no office-motivation who prefer x, given the state (that is, type $(c,0)$ in state x and type $(d,0)$ in state y). If only the former chose x, then the citizen would be indifferent between reelecting the incumbent and electing the challenger following the policy x, because the probability that the incumbent shares her preferences is the same as the probability that the challenger does so. The fact that also the latter choose x tips the balance in favor of reelecting the incumbent, given that x is the more likely state.

Proposition 13.3: Equilibrium of two-period electoral competition game with well-informed candidates with uncertain motivations

Let $\langle\{B, V, C\}, \{x, y\}, p, \pi, v', v, \mathscr{B}, \rho\rangle$ be a two-period electoral competition game with well-informed candidates with uncertain motivations in which $\mathscr{B} = \{0, b\}$ with $b > 0$, and $\rho(0) = r$ and $\rho(b) = 1 - r$, with $r \in (0, 1]$. If $p > \frac{1}{2}$ (x is more likely than y to be best for the citizen) and $v + b > \max\{\underline{v}, \overline{v}\}$ (the incumbent's possible positive office-motivation is sufficiently high), where $\underline{v}$ and $\overline{v}$ are given in (13.1), then the game has a unique weak sequential equilibrium. In this equilibrium, each office-motivated incumbent (types (c, b) and (d, b)) chooses policy x in both state x and state y, and every incumbent with no office-motivation chooses her favorite policy.

Proof

Step 1 *In every weak sequential equilibrium the incumbent chooses x after the moves of chance $(x, c, 0)$ and $(y, d, 0)$, and y after the moves of chance $(y, c, 0)$ and $(x, d, 0)$.*

Proof. After the moves of chance $(x, c, 0)$ and $(y, d, 0)$, the incumbent's worst payoff if she chooses x, the policy she prefers, exceeds her best payoff if she chooses y, and after the moves of chance $(y, c, 0)$ and $(x, d, 0)$ her worse payoff if she chooses y, the policy she prefers, exceeds her best payoff if she chooses x. $\triangleleft$

Step 2 *The game has no weak sequential equilibrium in which the citizen chooses the same action at both of her information sets.*

Proof. Suppose that the citizen chooses the same action at both of her information sets. Then the incumbent's first-period policy does not affect whether she is reelected, so she optimally chooses x after the moves of chance (x, c, b) and (y, d, b) and y after the moves of chance (y, c, b) and (x, d, b). Given Step 1, the only belief that is consistent with the incumbent's strategy is given as follows. The probability that chance chooses $(x, c, 0)$ or (x, c, b) is $p\pi$, so at the citizen's inner information set (following the policy x) the belief assigns probability $p\pi/\Pi$ to the incumbent's being consonant and probability $(1 - p)(1 - \pi)/\Pi$ to her being dissonant, where Π is the sum of the numerators of these expressions. At her outer information set (following the policy y) it assigns probability $(1-p)\pi/(1-\Pi)$ to the incumbent's being consonant and probability $p(1-\pi)/(1-\Pi)$ to her being dissonant.

Given this belief and the incumbent's strategy, the citizen's payoffs to her actions at her inner information set are

$$B: 2p\pi/\Pi$$
$$C: [p\pi(1+\pi)+(1-p)(1-\pi)\pi]/\Pi.$$

The difference between these payoffs is $(2p-1)\pi(1-\pi)/\Pi$, which is positive given $p > \frac{1}{2}$ and $\pi \in (0,1)$. Thus the citizen's unique optimal action at her inner information set is B. The citizen's payoffs to her actions at her outer information set are

$$B: 2(1-p)\pi/(1-\Pi)$$
$$C: [(1-p)\pi(1+\pi)+p(1-\pi)\pi]/(1-\Pi).$$

The difference between these payoffs is $(1-2p)\pi(1-\pi)/(1-\Pi)$, which is negative. Thus the citizen's unique optimal action at her outer information set is C.

We conclude that the game has no weak sequential equilibrium in which the citizen chooses the same action at each of her information sets. ◁

Step 3 *In every weak sequential equilibrium the incumbent chooses the same policy after every move of chance (s,h,β) with $\beta=b$ (the ones in the top half of Figure 13.8).*

Proof. By Step 2, in every weak sequential equilibrium the citizen chooses B at one information set and C at the other. If she chooses B at her outer information set (following policy y) and C at her inner information set (following policy x), then given $v+b > \max\{\underline{v},\overline{v}\}$, the incumbent optimally chooses y after every move of chance (s,h,β) with $\beta=b$. If the citizen chooses C at her outer information set and B at her inner information set, then the incumbent optimally chooses x after every such move of chance. ◁

Step 4 *In every weak sequential equilibrium the incumbent chooses x after every move of chance (s,h,β) with $\beta=b$.*

Proof. By Step 3 the only other possibility is that the incumbent chooses y after every move of chance (s,h,β) with $\beta=b$. Then if the citizen's inner information set is reached, by Step 1 the citizen believes that the move of chance was $(x,c,0)$ with probability $p\pi r/\Lambda$ and $(y,d,0)$ with probability $(1-p)(1-\pi)r/\Lambda$, where Λ is the sum of the numerators. The difference between the citizen's expected payoffs to B and C at her inner information set is thus proportional to $p\pi r(2-(1+\pi))+(1-p)(1-\pi)r(-\pi) = r\pi(1-\pi)(2p-1) > 0$, so that the citizen optimally chooses B at the information set. But then after the move of chance (x,c,b) the incumbent is better off deviating from y to x, regardless of whether the citizen chooses B or C at her outer information set. $\lhd$

Step 5 *The game has a unique weak sequential equilibrium, and in this equilibrium types (c,b) and (d,b) of the incumbent choose x in both states, type $(c,0)$ chooses x in state x and y in state y, and type $(d,0)$ chooses y in state x and x in state y.*

Proof. By Steps 1 and 4 the incumbent's strategy has this form in every weak sequential equilibrium. Consider the assessment in which the incumbent's strategy takes this form, the citizen chooses B if the incumbent's policy is x and C if the incumbent's policy is y, and the belief system assigns to each history in each of the citizen's information sets the probability of the history occurring given the incumbent's strategy, conditional on the information set's being reached. This strategy profile is indicated by the arrows in Figure 13.8.

I argue that the assessment is a weak sequential equilibrium, the belief system is the only one that is consistent with the incumbent's strategy, and the citizen's strategy is the only one that is optimal given the belief system.

We have $\underline{v} \in (v,v')$, $\overline{v} \in (v,v')$, and $v+b > \max\{\underline{v},\overline{v}\}$, so that the strategy of each type of incumbent is optimal given the citizen's strategy.

At the citizen's inner information set, following the incumbent's choice of policy x, the citizen's expected payoff from choosing B, given the incumbent's strategy, is $[2p\pi(1-r)+(1-p)\pi(1-r)+p(1-\pi)(1-r)+2p\pi r]/\Delta$ and her expected payoff from choosing C is $[(1-r)(\pi+p)+p\pi r(1+\pi)+(1-p)(1-\pi)r\pi]/\Delta$, where Δ is the probability that chance selects (s,h,b) for some pair (s,h), $(x,c,0)$, or $(y,d,0)$. The difference between the numerators of these expressions is $(1-\pi)r\pi(2p-1)$, which is positive given $p > \frac{1}{2}$. Thus selecting B is the only optimal action for the citizen at her inner information set.

At the citizen's outer information set, following the incumbent's choice of policy y, the citizen's expected payoff from choosing B, given the incumbent's strategy, is $2(1-p)\pi r/(1-\Delta)$ and her expected payoff from choosing C is $[(1-p)\pi r(1+\pi)+p(1-\pi)r\pi]/(1-\Delta)$. The difference between the numerators of these expressions is $(1-\pi)r\pi(1-2p)$, which is negative given $p > \frac{1}{2}$. Thus selecting C is the only optimal action for the citizen at her outer information set.

Both information sets are reached with positive probability, and the beliefs at each information set are derived from the strategy profile using Bayes' rule, so they are the only ones weakly consistent with the strategy profile. ◁

The citizen's expected payoff in the equilibrium in this result is

$$(1-r)(p+\pi)+r\pi[1+\pi+2p(1-\pi)]. \tag{13.2}$$

If, instead of selecting a representative to choose a policy for her, the citizen chooses one herself, she optimally selects x, the policy more likely to be best for her, and consequently obtains the expected payoff $2p$. By doing so she does not benefit from the candidates' superior information, but at the same time is not subject to the risk that a candidate's preferences differ from hers. The payoff $2p$ exceeds the payoff given in (13.2) if the probability p that the best policy is x exceeds $\pi(1+\pi r)/[1+r-2\pi r(1-\pi)]$, which is less than 1 given $\pi < 1$. Thus however likely it is that a candidate's preferences are consonant with the citizen's, for p sufficiently high the citizen is better off choosing the policy herself than electing a representative to do it for her under the rules of the game we are considering. When p is less than this bound, so that the best policy is relatively uncertain, handing the decision to a representative is better even though the representative's preferences may differ from the citizen's and the representative may choose the policy the citizen believes is best, rather than the policy that is in fact best, in order to get reelected.

Another option for the citizen is to select a candidate randomly in each period, without the possibility of reelection. The citizen's expected payoff in this case is 2π. This payoff exceeds the payoff given in (13.2) if the probability p that the best policy is x is less than $\pi(1-\pi r)/[1-r+2\pi r(1-\pi)]$. This number is greater than $\frac{1}{2}$ if $\pi < \frac{1}{2}$. Thus when a candidate's preferences are likely to be discordant from the citizen's preferences and the identity of the best policy is sufficiently uncertain (p is close to $\frac{1}{2}$), the outcome for the citizen is better when the incumbent has no possibility of reelection than it is in the equilibrium of the game we are considering. In this case, the reelection incentive leads the incumbent to choose policies that are worse for the citizen, on average, than the favorite policy of a randomly chosen candidate.

Notes

The models in Sections 13.1 and 13.2 express in a two-alternative model the main ideas in the example studied by Fearon (1999), whose work draws in part on the infinite-horizon example studied by Ferejohn (1986). Section 3.3 of Duggan and Martinelli (2015) (of which Duggan and Martinelli 2017 is a shortened version) analyze much more general models. The approach to studying the implications of deviations on players' beliefs discussed in the paragraph preceding Proposition 13.1 was initiated by Cho and Kreps (1987); a large body of subsequent explores this idea. Section 13.3 is based on Maskin and Tirole (2004). Duggan and Martinelli (2020) study a general model.

IV Exercising power

14 Bargaining 467
15 Rulers threatened by rebellion 503

14 Bargaining

14.1 Bargaining game with voting 469
14.2 Recurrent distributive bargaining game with voting 496

Collective decision-making processes in which individuals sequentially make and vote on proposals are typically classified as bargaining. What factors shape the individuals' behavior in such processes?

Synopsis

I present two models. In Section 14.1, an outcome is an alternative. Negotiations may last many periods, but once agreement is reached, the game ends. In Section 14.2, an outcome is a sequence of alternatives, one in each period. The alternative in each period serves as the default in the next period: if the proposal in period t is not accepted, the alternative implemented in that period is the one from period $t-1$.

In both models, first an individual is selected randomly to make a proposal, which each individual then votes either for or against. In the model in Section 14.1, a bargaining game with voting, if a majority of individuals vote for the proposal, it is implemented and the game ends. Otherwise another individual is selected randomly to make a proposal, and a vote is again held. The pattern repeats until a proposal is accepted, when the game ends. In the model in Section 14.2, the acceptance of a proposal means that the proposal is implemented and its rejection means that the default alternative, from the previous period, is implemented. In both cases, negotiations remain open: in every future period a proposal may be made and voted upon.

Both models are extensive games. For a bargaining game with voting, the solution notion of subgame perfect equilibrium does not restrict the outcome: for every alternative x, the strategy profile in which every individual always proposes x and votes in favor of x and against any other alternative is a subgame perfect equilibrium (Proposition 14.1). In this equilibrium, no individual's vote makes a difference because all votes are unanimous; thus no individual has an incentive to change her behavior. However, an individual who is (unexpectedly) confronted with an alternative that she prefers to x has nothing to lose by voting for rather than against it as her equilibrium strategy mandates, and restricting individuals' strategies to not be weakly dominated in this sense holds the promise that it might eliminate some alternatives as equilibrium outcomes. If the individuals' votes are observable, however, it has little effect: for sets of alternatives and preferences of the type usually assumed, for most alternatives x the game has a subgame perfect equilibrium with undominated voting in which agreement on x is reached immediately (Proposition 14.2). The potential interest of these equilibria lies in the nature of the set of equilibrium strategies, although no results currently characterize this set.

If only the outcome of each vote, not the vote cast by each individual, is observable, then the outcomes of subgame perfect equilibria in which each individual's voting strategy is not weakly dominated depend on the nature of the set of alternatives and the individuals' preferences. If the alternatives are distributions among the individuals of a fixed amount of a good, there are at least five individuals, and each individual cares only about the amount she receives and is sufficiently patient, then for every alternative x the game has a subgame perfect equilibrium with undominated voting in which agreement on x is reached immediately (Proposition 14.3). If, however, the set of alternatives is an interval of numbers and each individual's preferences over this set are single-peaked, then if the individuals are sufficiently patient, the outcome of any subgame perfect equilibrium with undominated voting is close to the median of the individuals' favorite positions (Proposition 14.4).

A requirement much stronger than subgame perfect equilibrium with undominated voting is that each player's strategy is stationary, always making the same proposal and casting the same vote regarding any given proposal, independent of the history. The appeal of this requirement is questionable, but if the alternatives are distributions among the individuals of a fixed amount of a good, it leads to a unique equilibrium outcome (Proposition 14.5). In this outcome the player first selected proposes a particular distribution of the available good among a minimal majority, the members of this majority vote in favor, and the game ends.

Section 14.2 studies a game with recurrent bargaining: negotiations are always open. The alternatives are distributions among the individuals of a fixed amount of a good. In each period, one of these distributions is the default alternative. In the first period, it is the distribution in which no individual receives any of the good, and in each subsequent period it is the alternative implemented in the previous period. In each period, an individual is selected randomly. She can either pass, in which case the default alternative is implemented in the period, or she can propose another alternative. If a majority of individuals vote in favor of her proposal, it is implemented in the period, and if not, the default alternative from the previous period is implemented.

Even if the players' strategies are restricted to be stationary, this game, unlike one that ends once agreement is reached, has many subgame perfect equilibrium outcomes. Say that a player's strategy is stationary if (*i*) for any given default alternative, the strategy always makes the same proposal and (*ii*) for any given default alternative and proposal, the strategy always votes in the same way. The stationary equilibrium outcomes may involve waste and assign a positive amount to more than a bare majority of the players. I convey the main ideas by means of an example. In the example, for almost any distribution x, including ones in which some of the good is wasted, the game has an equilibrium in which every player's strategy is stationary and x is implemented in every period. The players whose shares of the good are relatively high in x constitute a minimal majority; none of them wants to deviate because doing so would reopen negotiations and expose her to the risk that the outcome in every future period is worse for her than x.

14.1 Bargaining game with voting

A finite set N of individuals faces a collective choice problem in which the set X of alternatives is a compact convex subset of a Euclidean space. The process by which the individuals may reach agreement takes place over a sequence of periods, $t = 1, 2, \ldots$. Agreement may be reached in any period, so that we need to specify each individual's preferences over the set consisting of pairs (x, t) in which $x \in X$ and t is the period in which agreement is reached together with the outcome in which agreement is never reached. I assume that the preference relation of each individual i is represented by a function with values of the form $\delta_i^{t-1} u_i(x)$ for pairs (x, t), where $\delta_i \in (0,1)$ and $u_i : X \to \mathbb{R}_+$, and the value 0 for the outcome in which agreement is never reached.

The model of the bargaining procedure is an extensive game with perfect information, simultaneous moves, and chance moves. The set of players is the set N of individuals; to avoid dealing with ties in votes, I assume that the number, n, of players is odd. First, chance selects one of the players. Each player i is chosen with probability $\rho_i \in (0,1)$. The selected player proposes a member x of X, and then all players simultaneously vote for or against x; every player observes the votes cast. If a majority of the players ($\frac{1}{2}(n+1)$ or more) vote in favor of x, the game ends with the outcome x. Otherwise, play moves to the next period, in which again a player chosen by chance makes a proposal and a vote is held.

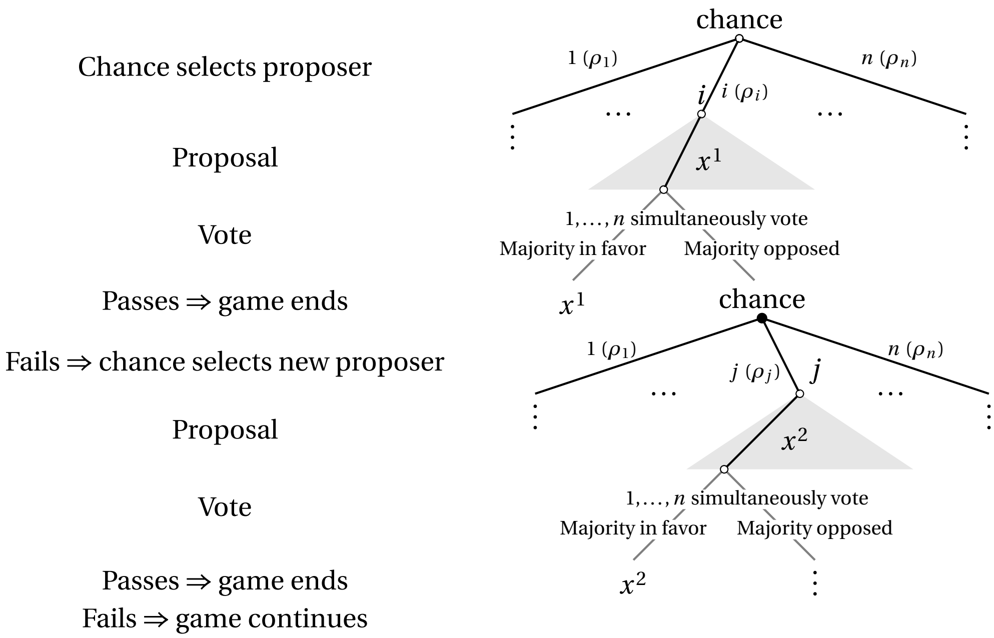


Figure 14.1 An illustration of the first two periods of a bargaining game with voting. Only one action of only one player in each period is shown.

Play continues in the same manner until a proposal is accepted, or goes on forever if all proposals are rejected. The first two periods of play are illustrated in Figure 14.1.

Definition 14.1: Bargaining game with voting

A *bargaining game with voting* $\langle N, X, (\rho_i)_{i\in N}, (\delta_i)_{i\in N}, (u_i)_{i\in N}\rangle$, where

- $N = \{1,\ldots,n\}$, where $n \geq 3$ is an odd integer (the set of individuals)
- X is a compact convex subset of a Euclidean space (the set of alternatives)
- $\rho_i \in (0,1)$ for each $i \in N$, with $\sum_{i\in N} \rho_i = 1$ (the probability that player i is chosen to make a proposal in any given period, her *recognition probability*)
- $\delta_i \in (0,1)$ for each $i \in N$ (player i's discount factor)
- $u_i : X \to \mathbb{R}_+$ is continuous, with $u_i(x) > 0$ for all $x \in \operatorname{int} X$ (u_i is player i's payoff function over alternatives)

is the extensive game with perfect information, simultaneous moves, and chance moves with the following components, where

$$V = \{(v_1, \ldots, v_n) : v_i \in \{\textit{for}, \textit{against}\} \text{ for } i = 1, \ldots, n\} \text{ (vote profiles)}$$
$$R = \{(v_1, \ldots, v_n) \in V : |\{i \in N : v_i = \textit{for}\}| < \tfrac{1}{2}(n+1)\} \text{ (vote profiles opposed)}$$
$$A = \{(v_1, \ldots, v_n) \in V : |\{i \in N : v_i = \textit{for}\}| \geq \tfrac{1}{2}(n+1)\} \text{ (vote profiles in favor).}$$

Players

The set N (of individuals).

Terminal histories

The set of terminal histories consists of

- $(i^1, x^1, v^1, i^2, x^2, v^2, \ldots, i^t, x^t, v^t)$ for any $t \geq 1$, $i^s \in N$ and $x^s \in X$ for $s = 1, \ldots, t$, $v^s \in R$ for $s = 1, \ldots, t-1$, and $v^t \in A$ (proposals through period $t-1$ are rejected and the proposal in period t is accepted)
- $(i^1, x^1, v^1, i^2, x^2, v^2, \ldots, i^t, x^t, v^t, \ldots)$ where $i^s \in N$, $x^s \in X$, and $v^s \in R$ for $s = 1, 2, \ldots$ (all proposals are rejected).

Player function

The player function P is defined as follows.

- $P(\varnothing) =$ chance and $P(i^1, x^1, v^1, i^2, x^2, v^2, \ldots, i^t, x^t, v^t) =$ chance for every $t \geq 1$ if $v^t \in R$ (chance moves at the start of the game and after a proposal is rejected).
- $P(i^1, x^1, v^1, i^2, x^2, v^2, \ldots, i^t) = i^t$ for every $t \geq 1$ (the player who moves (proposes an alternative) after chance is the one selected by chance)
- $P(i^1, x^1, v^1, i^2, x^2, v^2, \ldots, i^t, x^t) = N$ for every $t \geq 1$ (all players move (vote) after a player makes a proposal).

Actions

For any history h, the set $A_i(h)$ of actions of each player $i \in N$ after h is X if h ends with i's selection as the proposer and $\{\textit{for}, \textit{against}\}$ (a vote) if h ends with a proposal. The set of actions of chance at the beginning of the game and after any proposal is rejected is N.

Chance probabilities

For each player $i \in N$, chance selects i with probability ρ_i whenever it moves.

Preferences

The preference relation of each player $i \in N$ over lotteries over the set Z of terminal histories is represented by the expected value of the payoff function $U^i : Z \to \mathbb{R}_+$ given by

$$U^i(h^t) = \delta_i^{t-1} u_i(x^t)$$

for all $t \geq 1$ and all terminal histories h^t ending in period t with a proposal x^t followed by a vote $v^t \in A$ to accept x^t, and $U^i(h^\infty) = 0$ for all terminal histories h^∞ in which all proposals are rejected.

A strategy for any player i in this game is a function that associates an alternative with each history ending in a move of chance that selects i to be the proposer and either *for* or *against* with each history ending in a proposal. An outcome is a terminal history, which is either a history ending in a vote in which a majority of the players vote *for* the latest proposal or a history in which no proposal is approved by a majority of the players.

A variant of the model assumes that while negotiations are taking place, each player receives a payoff from the status quo, which may or may not be a member of X. Denoting the status quo by q, in this variant the (discounted average) payoff of individual i to an agreement on $x \in X$ in period t is $(1 - \delta_i^{t-1})u_i(q) + \delta_i^{t-1} u_i(x)$ (assuming that u_i is defined for q as well as for every member of X). A bargaining game with voting is thus equivalent to this variant if $u_i(q) = 0$ for every player i (that is, if each player's payoff from the status quo is the same as her payoff from perpetual disagreement).

14.1.1 Subgame perfect equilibrium

A bargaining game with voting has many subgame perfect equilibria. For example, for every alternative $x \in X$ it has a subgame perfect equilibrium in which agreement is reached on x immediately. The argument for this result is simple. Suppose that every player proposes x whenever she is selected and votes for x and against every other alternative regardless of the history. Then if a player deviates by proposing an alternative different from x, her proposal is voted down and the outcome is x in a future period, which she likes less than x immediately. If she deviates by voting against x or for another alternative after some history, the outcome is unaffected, given the other players' strategies.

Proposition 14.1: Subgame perfect equilibrium of bargaining game with voting

Let $\langle N, X, (\rho_i)_{i\in N}, (\delta_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a bargaining game with voting and let $x \in X$. For each player $i \in N$ let s_i^* be the strategy of player i in which she proposes x after every history that ends with her selection as the proposer and votes in favor of x and against every other proposal after every history that ends with a proposal. The strategy profile s^* is a subgame perfect equilibrium, the outcome of which is that x is proposed and accepted immediately.

Proof

The strategy profile s^* satisfies the one-deviation property, which requires that no player can increase her payoff in any subgame by changing only her action at the start of the subgame, given the other players' strategies. Consider player i. If, after any history that ends with i's selection as the proposer, she adheres to s_i^* and proposes x, then the outcome is x, whereas if she deviates from s_i^* and proposes $y \neq x$, then y is rejected and the outcome is x in the following period. If, after any history that ends with a proposal, i changes her vote, the outcome remains the same: every other player votes to accept x and to reject any other proposal. The game satisfies the condition in Proposition 16.9, so the fact that s^* satisfies the one-deviation property means that it is a subgame perfect equilibrium.

The strategy profile s^* in this result has an unattractive feature. Consider a history ending in a proposed alternative y that player j (and possibly other players) prefers to x. Every player's strategy in s^* calls for a vote against y, and if the players follow their strategies the outcome is x with one period of delay. Casting such a vote is optimal for every player, including player j, because no change in the vote of a single player affects the outcome, given the other players' strategies. For the same reason, voting *for* y also is optimal, given the other players' strategies. In fact, voting for y and voting against it are both optimal not only if the other players vote according to their strategies, but also if they cast any other votes, *unless* their votes are equally split between *for* and *against*, in which case j's voting *for* yields the outcome y and her voting *against* yields x with one period of delay (assuming that all players adhere to s^* in the following periods). Table 14.1 shows the outcome as a function of the players' votes. Player j prefers y to x, and hence to x with one period of delay, so her voting against y, as s_i^* specifies, is weakly dominated by her voting for y, given the behavior specified by s^* in

	number of other players voting against y						
	0	$\cdots$	$\frac{1}{2}(n-3)$	$\frac{1}{2}(n-1)$	$\frac{1}{2}(n+1)$	$\cdots$	$n-1$
j votes against y	y	$\cdots$	y	x^{+1}	x^{+1}	$\cdots$	x^{+1}
j votes for y	y	$\cdots$	y	y	x^{+1}	$\cdots$	x^{+1}

Table 14.1 The outcome in a bargaining game with voting following a history ending with the proposal y when all players adhere to the strategy s^* in the proof of Proposition 14.1 following the vote on y. The superscript $+1$ means that the outcome occurs with a period of delay.

other periods. I refer to a subgame perfect equilibrium in which no player's vote in any period is weakly dominated in this sense as a subgame perfect equilibrium with undominated voting. (Some authors say "subgame perfect equilibrium in stage-undominated voting strategies", to emphasize that the notion considers domination in each period separately.)

Definition 14.2: Subgame perfect equilibrium with undominated voting in bargaining game with voting

Let $\Gamma = \langle N, X, (\rho_i)_{i\in N}, (\delta_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a bargaining game with voting. For each strategy profile s and history h that ends with a proposal, define the strategic game $G(h,s)$ as follows.

Players
The set of players is N.

Actions
The set of actions of each player is {*for*, *against*} (the possible votes).

Preferences
The preferences of each player $i \in N$ over action profiles are represented by the function that assigns to the action profile v the payoff of player i in Γ for the history consisting of h followed by v followed by the sequence of action profiles in Γ generated by s after the history (h, v).

A strategy profile s in Γ is a *subgame perfect equilibrium with undominated voting* if it is a subgame perfect equilibrium of Γ and for every player $i \in N$ and every history h of Γ ending with a proposal, the action $s_i(h)$ is not weakly dominated in the game $G(h,s)$.

Does the restriction to undominated voting reduce the set of equilibrium outcomes? Roughly speaking, not much. The next result shows that every alternative in any subset of X that satisfies a certain condition is the outcome of a subgame perfect equilibrium with undominated voting. For some games, subsets

	number of other players voting against y							
	0	$\cdots$	$\frac{1}{2}(n-3)$	$\frac{1}{2}(n-1)$	$\frac{1}{2}(n+1)$	$\cdots$	$n-2$	$n-1$
j votes against y	y	$\cdots$	y	x^{+1}	x^{+1}	$\cdots$	x^{+1}	x^{+1}
j votes for y	y	$\cdots$	y	y	x^{+1}	$\cdots$	x^{+1}	$\gamma^j(x)^{+1}$

Table 14.2 The outcome in a bargaining game with voting following a history ending with the proposal y when all players' strategies specify the proposal $x \neq y$, a vote for x and against every other alternative, and a switch to the proposal $\gamma^j(x)$ if a single player j deviates and votes if favor of an alternative different from x. The superscript $+1$ means that the outcome occurs with a period of delay.

consisting of almost all members of X satisfy the condition. The idea behind the equilibrium is that if, when the players' strategies call for them to propose x, y is proposed and some player j votes in favor of it while every other player adheres to her strategy and votes against it, all other players react by switching from proposing x to subsequently proposing an alternative $\gamma^j(x)$ that is worse for j than x and voting in favor of this alternative. Then even if j prefers y to x, voting for y does not weakly dominate voting against it, because if all the other players vote against it then voting for y leads to the outcome $\gamma^j(x)$ in the next period whereas voting against it, which causes no change in the other players' behavior, leads to the outcome x in the next period. The outcomes as a function of the players' votes are given in Table 14.2.

To construct a subgame perfect equilibrium along these lines, we need to ensure that the switch to proposing $\gamma^j(x)$ is optimal for every player, given the strategies of the remaining players. We do so by specifying that, just as a player's vote in favor of an alternative that she is supposed to vote against induces a switch to a regime in which she is punished, so a single player's deviation from the new regime by voting for a proposal that she is supposed to vote against induces a switch to a regime in which *she* is punished.

These strategies may conveniently be modeled as automata. We specify a set of states, the actions taken by each strategy in each state, and a rule for transitions between states. The set of states in this case is X. For any $\sigma \in X$, in state σ a player's strategy proposes σ and votes for σ and against every other proposal. The state is initially x and remains x unless an alternative different from x is proposed and exactly one player, j, votes in favor of this proposal, in which case the state changes to $\gamma^j(x)$, an alternative that is worse for j than x.

The states that can be reached are x, $\gamma^j(x)$ for each $j \in N$, $\gamma^k(\gamma^j(x))$ for each $k \in N$ and $j \in N$, and so on. For each of these states σ, we need $\gamma^j(\sigma)$ to be worse than σ for j. A sufficient condition for the existence of such alternatives is that x

is a member of a subset X^* of X with the property

$$\text{for all } i \in N \text{ and all } y \in X^* \text{ there exists } \gamma^i(y) \in X^* \text{ with } u_i(\gamma^i(y)) < u_i(y). \quad (14.1)$$

Suppose, for example, that X is the one-dimensional interval $[\underline{x}, \overline{x}]$ and each player's payoff function is single-peaked. For each player j, let $\hat{x}_j$ be j's favorite alternative. Then the set $X^* = (\underline{x}, \overline{x})$ consisting of all of X except its endpoints satisfies (14.1), with

$$\gamma^j(y) = \begin{cases} \frac{1}{2}(y + \underline{x}) & \text{if } y \leq \hat{x}_j \\ \frac{1}{2}(y + \overline{x}) & \text{if } y > \hat{x}_j. \end{cases}$$

Proposition 14.2: Subgame perfect equilibrium with undominated voting in bargaining game

Let $\langle N, X, (\rho_i)_{i \in N}, (\delta_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a bargaining game with voting and let X^* be a subset of X that satisfies (14.1). Then for any $x \in X^*$ the game has a subgame perfect equilibrium with undominated voting in which at the start of the game each player proposes x and all players vote in favor of x.

Proof

For each player $i \in N$, let s_i^* be the strategy defined by the automaton in Figure 14.2. Note that given that $x \in X^*$, every state that can be reached is in X^*. According to the strategy profile s^*, every player proposes x if she is selected to make a proposal in the first period and votes in favor of x if it is proposed in this period, so the outcome of s^* is immediate agreement on x.

I claim that s^* is a subgame perfect equilibrium with undominated voting . The game satisfies the condition in Proposition 16.9, so to show that s^* is a subgame perfect equilibrium it suffices to show that it satisfies the one-deviation property.

Proposal in any state σ

If player i adheres to s_i^*, she proposes σ, which is accepted (all players vote in favor). If she deviates to propose $y \neq \sigma$, then all players vote against y and the state remains σ, so that the outcome is σ with one period of delay, which is no better for i than immediate agreement on σ.

Vote in any state σ regarding proposal y

First suppose that $y = \sigma$. Then all players vote in favor of σ, and the

	State x (initial)	State $\gamma^j(y)$ ($j \in N$, $y \in X^*$)
i proposes	x	$\gamma^j(y)$
i votes	for x against any $y \neq x$	for $\gamma^j(y)$ against any $z \neq \gamma^j(y)$
Transitions	If, in any state σ, $y \neq \sigma$ is proposed and exactly one player, k, votes for it, go to state $\gamma^k(\sigma)$.	

Figure 14.2 The strategy of each player i in the proof of Proposition 14.2. For each player j and each alternative $\sigma \in X^*$, $\gamma^j(\sigma)$ is the alternative in X^* for which $u_j(\gamma^j(\sigma)) < u_j(\sigma)$ given in (14.1). The state remains the same unless the condition in the transition cell is satisfied.

outcome is σ. If player i deviates and votes against σ, the outcome remains σ, because the remaining players constitute a majority. Thus i's voting for σ is optimal.

I now argue that her voting for σ is undominated. If $\frac{1}{2}(|N|-1)$ of the other players vote for σ, her voting for σ leads to σ's being accepted, whereas her voting against σ leads to its being rejected, in which case the state remains σ and hence the outcome is σ with one period of delay. Given that $\sigma \in \operatorname{int} X$ we have $u_i(\sigma) > 0$, so that $\delta_i u_i(\sigma) < u_i(\sigma)$. Thus for these votes of the other players, player i's payoff from voting against σ is less than her payoff from voting for σ. Thus her voting for σ is not weakly dominated.

Now suppose that $y \neq \sigma$. Then all players vote against y, the state remains σ, and the outcome is thus σ with one period of delay. If player i deviates from s_i^* and votes for y while the other players adhere to s^* and vote against y, the state changes to $\gamma^i(\sigma)$, so that the outcome is $\gamma^i(\sigma)$ with one period of delay, which is worse for i than σ with one period of delay. Thus i's voting against y is optimal and undominated.

The equilibrium strategies in the proof of this result depend on each player's vote being observable, so that if a single player deviates from her strategy and votes in favor of a proposal different from the one the equilibrium specifies, given the history, she can be identified and punished. If only the numbers of votes for and against a proposal are observable, not each individual's vote, such punishments cannot be implemented. Under this assumption, the set of equilibrium outcomes depends on the form of the set of alternatives and the individuals' preferences over this set. I present two cases.

Bargaining over distribution when individuals' votes are unobserved

Suppose that the individuals have one unit of a good to distribute among themselves, some of which may be wasted, and the payoff of each individual is the amount of the good that she receives. The model of a bargaining game with voting was first explored under this assumption, with the interpretation that the individuals are legislators and the alternatives are bills that distribute benefits among the legislators' districts.

Definition 14.3: Distributive bargaining game with voting

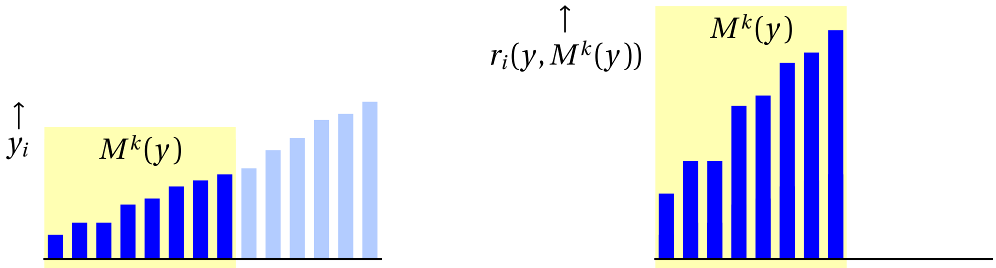
A bargaining game with voting $\langle N, X, (\rho_i)_{i\in N}, (\delta_i)_{i\in N}, (u_i)_{i\in N}\rangle$ is *distributive* if

$$X = \{(x_1, \ldots, x_n) \in \mathbb{R}^n_+ : \textstyle\sum_{i=1}^n x_i \le 1\},$$

where $n = |N|$, and $u_i(x) = x_i$ for all $i \in N$ and all $x \in X$.

If in such a game the players number at least five and each player's discount factor is close enough to 1, then for any alternative $x \in X$, immediate agreement on x is the outcome of a subgame perfect equilibrium with undominated voting in which the strategies do not rely on the observability of the individuals' votes. That is, the conclusion that any, or almost any, alternative is the outcome of an equilibrium does not depend on each individual's vote being observable, as Proposition 14.2 assumes. (The identity of the individual who makes each proposal does, however, need to remain observable.)

In such an equilibrium, proposing x is optimal for every player—even those whose shares x_i are small—because any other proposal, say y, is rejected, and in the subgame that is reached, the player who proposed y, say j, obtains the payoff 0. A set of players excluding j that (*i*) has $\frac{1}{2}(n+1)$ members (a bare majority) and (*ii*) has the lowest total payoff under y among all such sets, gangs up on j. Every member of this set votes against y, proposes a distribution z in which j's payoff is 0 and all of their payoffs are at least as high as they are under y (accounting for the fact that z is received with a period of delay), and votes for z and against every other proposal. (The fact that z is received with a period of delay is the reason the players' discount factors need to be sufficiently close to 1.) Why do these players engage in this behavior? For each of them, voting against y is not weakly dominated by voting in favor of it because all of them are better off when the outcome is z with a period of delay than when it is y immediately. Also, if any of them makes a proposal different from z, the remaining players gang up on *her* in the same way, reducing her payoff to 0.



(a) A subset $M^k(y)$ of $N \setminus \{k\}$ with $\frac{1}{2}(n+1)$ members for which $\sum_{i \in M^k(y)} y_i$ is smallest. (In this case, the only such subset.)

(b) The payoffs of the players in the proposal $r(y, M^k(y))$.

Figure 14.3 For a proposal y and player k, the payoffs of the remaining players ordered by y_i.

Proposition 14.3: Subgame perfect equilibrium with undominated voting in distributive bargaining game with unobserved individual votes

Let $\langle N, X, (\rho_i)_{i\in N}, (\delta_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a distributive bargaining game with voting and let $n = |N|$. If $n \geq 5$ and $\delta_i \in \big(\frac{1}{2}(n+1)/(n-1), 1\big)$ for all $i \in N$ then for every $x \in X$ the game has a subgame perfect equilibrium with undominated voting in which every player's action after every history depends only on the sequence of proposals and outcomes of the votes (majority in favor, majority opposed) and every player proposes x at the start of the game and votes in favor of this proposal.

Proof

I define a strategy profile s^* and argue that it is a subgame perfect equilibrium with undominated voting. In s^*, every player proposes x at the start of the game and all players vote in favor. If a player deviates to a different proposal, this proposal is rejected and the other players penalize her. To specify the penalty, for any proposal $y \in X$ and nonempty subset S of N, let $r(y, S)$ be the proposal that allocates all the payoff to the members of S, proportionally to their payoffs in y: for each $i \in N$,

$$r_i(y,S) = \begin{cases} y_i / \sum_{j\in S} y_j & \text{if } i \in S \text{ and } \sum_{j\in S} y_j > 0 \\ 1/|S| & \text{if } i \in S \text{ and } \sum_{j\in S} y_j = 0 \\ 0 & \text{if } i \in N \setminus S. \end{cases} \tag{14.2}$$

Now, for each player $k \in N$ and each proposal $y \in X$, let $M^k(y)$ be a subset of $N \setminus \{k\}$ with $\frac{1}{2}(n+1)$ members for which the total payoff $\sum_{i \in M^k(y)} y_i$ is smallest (refer to Figure 14.3a). If player k proposes $y \neq x$, then the members of $M^k(y)$, a majority, vote against y, and the next player to make a proposal chooses $r(y, M^k(y))$ (refer to Figure 14.3b). This behavior penalizes player k because $r_k(y, M^k(y)) = 0$ given that $k \notin M^k(y)$.

The strategy s_i^* may conveniently be described more precisely as the automaton shown in Figure 14.4. The states are x and $p^j(y)$ for each $j \in N$ and $y \in X$. The initial state is x. In this state, s_i^* proposes x and votes in favor of x. If, in this state, player j proposes $y \neq x$, then s_i^* votes against the proposal if and only if $i \in M^j(y)$, and hence y is rejected. The state remains x if x is proposed, and changes to $p^j(y)$ if player j proposes $y \neq x$ and this proposal is rejected (as it is according to s^*). In state $p^j(y)$, j is penalized: s_i^* proposes $r(y, M^j(y))$ and every member of $M^j(y)$ votes in favor of it. If a different alternative, say z, is proposed by some player l, then i votes against it if and only if $i \in M^l(z)$. Thus in state $p^j(y)$, $r(y, M^j(y))$ is accepted and every other proposal is rejected. If, in this state, some player l proposes $z \neq r(y, M^j(y))$ and z is rejected (as it is according to s^*), the state changes to $p^l(z)$.

The outcome of the strategy profile s^* is that the player chosen to move first proposes x, and all players vote in favor.

I now argue that s^* has undominated voting and satisfies the one-deviation property. The game satisfies the condition in Proposition 16.9, so the fact that s^* satisfies the one-deviation property means that it is a subgame perfect equilibrium.

Proposal in state x

If player i adheres to her strategy and proposes x then her payoff is x_i. If she proposes $y \neq x$, then every player in $M^i(y)$, a majority, votes against the proposal and the state becomes $p^i(y)$, in which $r(y, M^i(y))$ is proposed and accepted (because all members of $M^i(y)$ vote in favor), resulting in a payoff for i of 0. Thus i's payoff if she proposes x is at least her payoff if she makes any other proposal.

Vote in state x regarding proposal y

First suppose that $y = x$. If every player follows her strategy then x is accepted and i's payoff is x_i.

A player's vote regarding x affects the outcome generated by s^* only if the other players' votes are split equally between *for* and *against*. In this case, if i votes *against*, then x is rejected, the state remains x, and x is accepted with one period of delay, so that i's payoff is $\delta_i x_i$. If i votes *for*, then x is accepted immediately and her payoff is x_i. The latter payoff is at least the former, so i's voting for x is optimal and undominated. (If $x_i = 0$, i's voting *for* and *against* yield her the same payoff regardless of the other players' votes.)

Now suppose that $y \neq x$, and let j be the player who proposes y. If every player adheres to her strategy, y is rejected, the state changes to $p^j(y)$, and each player i obtains $r_i(y, M^j(y))$ with one period of delay, which is worth $\delta_i r_i(y, M^j(y))$ to her.

As in the previous case, a player's vote regarding y affects the outcome generated by s^* only if the other players' votes are split equally between *for* and *against*.

Consider a player $i \notin M^j(y)$. If she deviates from her strategy and votes against y, then if the other players' votes are split equally between *for* and *against*, y is rejected and her payoff changes from y_i to $r_i(y, M^j(y)) = 0$. Thus i's voting for y is optimal and undominated.

Now consider a player $i \in M^j(y)$. If she deviates from her strategy and votes for y, then if the other players' votes are split equally between *for* and *against*, her payoff changes from $\delta_i r_i(y, M^j(y))$ to y_i. If $y_i = 0$ then certainly $y_i < \delta_i r_i(y, M^j(y))$, so assume that $y_i > 0$. Then, given (14.2),

$$\frac{y_i}{r_i(y, M^j(y))} = \sum_{l \in M^j(y)} y_l.$$

This sum, the total payoff of a subset of $N \setminus \{j\}$ with $\frac{1}{2}(n+1)$ players that has the smallest total payoff under y, is at most $\frac{1}{2}(n+1)/(n-1)$. (The maximum is achieved when $y_i = 1/(n-1)$ for all $i \in N \setminus \{j\}$.) By assumption, this number is less than δ_i. Thus $y_i < \delta_i r_i(y, M^j(y))$, and hence i's voting against y is optimal and undominated.

Proposal in state $p^j(y)$

If i adheres to her strategy and proposes $r(y, M^j(y))$ then the outcome is $r(y, M^j(y))$ and her payoff is $r_i(y, M^j(y))$.

If she proposes $z \neq r(y, M^j(y))$ then this proposal is rejected, the state changes to $p^i(z)$, and her payoff is $r_i(z, M^i(z)) = 0$. So i is not better off deviating from her strategy than adhering to it.

Vote in state $p^j(y)$ regarding proposal z

First suppose that $z = r(y, M^j(y))$. If every player follows her strategy

	State x (initial)	State $p^j(y)$ ($j \in N$, $y \in X$)
i proposes	x	$r(y, M^j(y))$
i votes	for x against $y \neq x$ proposed by j $\Leftrightarrow i \in M^j(y)$	for $r(y, M^j(y)) \Leftrightarrow i \in M^j(y)$ against $z \neq r(y, M^j(y))$ proposed by l $\Leftrightarrow i \in M^l(z)$
Transitions	if j proposes $y \neq x$ and y is rejected, go to $p^j(y)$	if l proposes $z \neq r(y, M^j(y))$ and z is rejected, go to $p^l(z)$

Figure 14.4 The strategy of player i in the proof of Proposition 14.3. The state remains the same unless the condition in the transition cell is satisfied. The function r is defined in (14.2) and $M^j(y)$ is a subset S of $N \setminus \{j\}$ with $\frac{1}{2}(n+1)$ members for which $\sum_{i \in S} y_i$ is smallest.

then z is accepted and i's payoff is $r_i(y, M^j(y))$.

As in state x, a player's vote affects the outcome only if the other players' votes are split equally between *for* and *against*. In this case, if i votes *against*, then z is rejected, the state remains $p^j(y)$, and z is accepted with one period of delay, so that i's payoff is $\delta_i r_i(y, M^j(y))$. If i votes *for*, then z is accepted immediately and her payoff is $r_i(y, M^j(y))$. If $i \in M^j(y)$ then the latter payoff is at least the former and if $i \notin M^j(y)$ then both payoffs are zero. So in both cases the action specified by i's strategy is optimal and undominated.

Now suppose that $z \neq r(y, M^j(y))$. If every player follows her strategy then z is rejected and the state changes to $p^l(z)$, where l is the player who proposed z, with the outcome $r(z, M^l(z))$.

Again, a player's vote affects the outcome only if the other players' votes are split equally between *for* and *against*. In this case, if i votes *against*, then z is rejected, the state changes to $p^l(z)$, and $r(z, M^l(z))$ is accepted with one period of delay, so that i's payoff is $\delta_i r_i(z, M^l(z))$. If i votes *for*, then z is accepted immediately and her payoff is z_i. If $i \notin M^l(z)$ then $r_i(z, M^l(z)) = 0$, so the latter payoff is at least the former. If $i \in M^l(z)$ then $z_i < \delta_i r_i(z, M^l(z))$ by the argument for the vote of a player in $M^j(y)$ regarding $y \neq x$ in state x. So in both cases the action specified by i's strategy is optimal and undominated.

Exercise 14.1: Distributive bargaining game with three players

Why is the strategy profile in the proof of the result not a subgame perfect equilibrium for $n = 3$?

Bargaining over one-dimensional policy when individuals' votes are unobserved

Now suppose that X is an interval of numbers and each individual's payoff function u_i is concave (and hence single-peaked). In this case, the preferences have a degree of commonality absent in a distributive bargaining game, where reducing one individual's payoff allows the payoffs of all the remaining individuals to be raised. If X is an interval of numbers and the individuals' preferences are single-peaked, an alternative that is bad for one individual is bad also for individuals with similar favorite positions, so that punishing one individual for a deviation means imposing a low payoff on other individuals, making it difficult to provide them with an incentive to take part in the punishment.

I argue that if in this case the individuals' discount factors are close to 1, only alternatives close to the median x^* of the individuals' favorite alternatives are outcomes of subgame perfect equilibria with undominated voting in which every individual's action after every history depends only on the proposals in the history (and not, for example, the margins by which these proposals were rejected). (After the proof of the next result, I comment on the case in which individuals may condition their action on the vote totals, which is perhaps a more common possibility.)

A key step in the argument is that if any player proposes x^* then a majority of individuals vote in favor of it. If x^* is voted down, the outcome is a lottery over alternatives in future periods that by assumption does not depend on the margin by which x^* loses or the identity of the individuals who voted against it. Denoting this lottery α, each individual j faces the situation in Table 14.3 when x^* is proposed. Given that each function u_i is concave, every such lottery is worse than x^* for a majority of individuals, so for a majority of individuals, voting for x^* weakly dominates voting against it—and hence x^* passes.

Now let $\underline{x}$ and $\overline{x}$ be the smallest and largest alternatives that are outcomes of subgame perfect equilibria of the type we are considering, and suppose that a majority of individuals prefer $\underline{x}$ to $\overline{x}$. I argue that $\overline{x}$ is close to x^*, and hence so too is $\underline{x}$. To pass, $\overline{x}$ needs the vote of at least one individual whose favorite position is at most x^* and hence prefers x^* to $\overline{x}$. The vote of such an individual i makes a difference to the outcome only if the votes of the remaining individuals are split equally between *for* and *against*. If in this case i votes in favor of $\overline{x}$ then the out-

	number of other players voting against x^*						
	0	$\cdots$	$\frac{1}{2}(n-3)$	$\frac{1}{2}(n-1)$	$\frac{1}{2}(n+1)$	$\cdots$	$n-1$
j votes against x^*	x^*	$\cdots$	x^*	α	α	$\cdots$	α
j votes for x^*	x^*	$\cdots$	x^*	x^*	α	$\cdots$	α

Table 14.3 The outcome of an individual's voting for and against a proposal in a bargaining game with voting following a history ending with the proposal x^*, where α is the lottery (over alternatives in future periods) that results if x^* is voted down, for a strategy profile in which every individual's action after every history depends only on the proposals in the history.

come is $\bar{x}$, whereas if she votes against $\bar{x}$ then (*a*) with positive probability she is selected to be the proposer and can, by proposing x^*, obtain the outcome x^* with one period of delay, given the previous argument, and (*b*) with the remaining probability the outcome is at worst $\bar{x}$ with one period of delay. Thus if her discount factor is close to 1, her voting against $\bar{x}$ weakly dominates her voting in favor of $\bar{x}$ unless $\bar{x}$ is close to x^*.

Proposition 14.4: Subgame perfect equilibrium with undominated voting in bargaining game over policy with unobserved individual votes

Let $\langle N, X, (\rho_i)_{i\in N}, (\delta_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a bargaining game with voting for which X is a compact interval of numbers and each function u_i is concave (and hence single-peaked). Denote by x^* the median of the players' favorite alternatives. For any $\varepsilon > 0$ there exists $\bar{\delta} < 1$ such that if $\delta_i \geq \bar{\delta}$ for all $i \in N$ then a proposal x passes in a subgame perfect equilibrium with undominated voting in which every player's action after every history depends only on the sequences of proposals and outcomes of the votes (pass, fail) only if $x \in [x^* - \varepsilon, x^* + \varepsilon]$.

Proof

Let s^* be a subgame perfect equilibrium with undominated voting in which every player's action after every history depends only on the sequences of proposals and outcomes of the votes.

Step 1 *After a history that ends with the proposal x^*, for a majority of players i the strategy s_i^* votes in favor.*

Proof. If a majority votes against x^*, the outcome is a lottery over agreements in later periods (and no agreement at all), independent of the margin by which x^* loses and the identity of the players who vote for and against it, by the assumption on the form of the strategies. Thus each player j faces the situation in Table 14.3, where α is the lottery that results if x^* is voted down. Given that each function u_i is concave and that each δ_i is less than 1, x^* is preferred to α by a majority of players. Thus the requirement that each player's vote in each period be undominated means that a majority of players vote for x^*. ◁

Denote by $\underline{x}$ the smallest alternative that passes in a subgame perfect equilibrium of the type described in the result and by $\overline{x}$ the largest such alternative. (I assume that such alternatives exist; if they do not, choose alternatives close to the infimum and supremum of the set of alternatives that pass.) Given Step 1, we have $\underline{x} \leq x^* \leq \overline{x}$. Suppose, without loss of generality, that a majority of players prefer $\underline{x}$ to $\overline{x}$, so that, in particular, $x^* < \overline{x}$.

Step 2 *For any* $\varepsilon > 0$ *there exists* $\overline{\delta} < 1$ *such that if* $\delta_i \geq \overline{\delta}$ *for all* $i \in N$ *then* $\overline{x} < x^* + \varepsilon$ *and* $\underline{x} > x^* - \varepsilon$.

Proof. Consider a subgame perfect equilibrium in which $\overline{x}$ passes. Given that the set of players who vote for it is a majority, this set includes at least one player, say i, whose favorite position is at most x^*, so that $u_i(x^*) > u_i(\overline{x})$. Her vote makes a difference to the outcome only if the other players' votes are split equally between *for* and *against*, in which case her voting *for* leads to the outcome $\overline{x}$ and her voting *against* leads to a period of delay followed by an outcome that, given Step 1, is at least as good for her as x^* if she is selected to propose, and is no worse than $\overline{x}$ otherwise (because no equilibrium outcome is worse for her than that). Thus if her discount factor is close enough to 1 then unless $\overline{x}$ is close to x^* her action of voting against $\overline{x}$ weakly dominates her action of voting for it, so that she optimally votes against it, contrary to our assumption.

Finally, the fact that a majority of players prefer $\underline{x}$ to $\overline{x}$ means that a player with favorite position x^* does so, and hence given the continuity of each payoff function, if $\overline{x}$ is close to x^* then so too is $\underline{x}$. ◁

Cho and Duggan (2015, 934) argue that Theorem 1 in Cho and Duggan (2009) implies that the conclusion of Proposition 14.4 holds also for equilibria in which the players' actions may depend on the margins by which the previous proposals failed, not only on the proposals themselves.

14.1.2 Equilibrium in stationary strategies

A restriction much stronger than the ones I have discussed so far is stationarity, which requires that each player's strategy be history-independent. That is, each player makes the same proposal whenever she is selected to be the proposer and casts the same vote regarding any given proposal regardless of the history that preceded the proposal.

Definition 14.4: Stationary strategy in bargaining game with voting

A strategy σ_i of a player i in a bargaining game with voting $\langle N, X, (\rho_i)_{i\in N}, (\delta_i)_{i\in N}, (u_i)_{i\in N}\rangle$ is *stationary* if the actions it prescribes after any two histories h and h' are the same if the subgames following h and h' are the same: if the histories h and h' both end with chance selecting i to be the proposer or if they both end with the same proposal then $\sigma_i(h) = \sigma_i(h')$.

The results I have presented so far rely on strategies that are not stationary: their essence is that they react to the players' past actions. No such reaction is possible for a player using a stationary strategy. As Austen-Smith and Banks (2005, 249) write,

> stationary strategies rule out much of interest in many political interactions: reciprocity, both positive and negative, is impossible without strategies being sensitive to the realized decision history.

If we interpret the actions specified by a player's strategy after a deviation to be the other players' beliefs about her future actions, then the stationarity of a player's strategy means that the other players' beliefs do not change when a deviation occurs. As Rubinstein (1991, 912) writes,

> assuming passivity of beliefs eliminates a great deal of what sequential games are intended to model: namely, the changing pattern in players' behavior and beliefs, as they accumulate experience.

Nevertheless, significant effort has been devoted to the analysis of equilibria in stationary strategies, which I now briefly discuss.

The result I present characterizes the subgame perfect equilibria with undominated voting of distributive bargaining games with voting in which every player is identical (same recognition probability and same discount factor) and every player's strategy is stationary. The result shows that in every such equilibrium the payoff profile is the same and agreement is reached immediately, so that each player's expected payoff is $1/n$ (given that the players are identical). Thus if a proposal is rejected and then the players adhere to an equilibrium, each player receives the payoff $1/n$ with one period of delay, which is worth δ/n to her, where δ is the common discount factor. Hence for every player i, voting in favor of a proposal x with $x_i > \delta/n$ weakly dominates voting against it, and voting against a proposal x with $x_i < \delta/n$ weakly dominates voting in favor of it. So to induce a majority to vote in favor of a proposal, it is enough for a player to offer δ/n to each member of a set containing $\frac{1}{2}(n-1)$ of the other players, assembling a minimal majority (including herself). (A minimal majority is sometimes called a minimal winning coalition.) In a stationary equilibrium, every player i assembles the same minimal majority S^i whenever she is selected to make a proposal. As far as i is concerned, the identity of the members of S^i does not matter, but for all the players' payoffs to be equal, as is the case in every equilibrium, every player j has to be a member of exactly $\frac{1}{2}(n+1)$ of these sets. I call a collection of minimal majorities with this property balanced.

Definition 14.5: Balanced collection of minimal majorities

Let $\langle N, X, (\rho_i)_{i\in N}, (\delta_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a bargaining game with voting and let $n = |N|$. A *minimal majority* is a subset of N with $\frac{1}{2}(n+1)$ members. A collection $\{S^j\}_{j\in N}$ of n minimal majorities is *balanced* if each player is a member of $\frac{1}{2}(n+1)$ of them.

For every (odd) value of n, a balanced collection of minimal majorities exists. One such collection consists of

$$S^j = \{j, (j+1)(\text{mod } n), \ldots, (j+\tfrac{1}{2}(n-1))(\text{mod } n)\} \quad \text{for all } j \in N,$$

so that $S^1 = \{1, 2, \ldots, \frac{1}{2}(n+1)\}$, $S^2 = \{2, 3, \ldots, \frac{1}{2}(n+3)\}$, and so forth.

Proposition 14.5: Stationary subgame perfect equilibria of distributive bargaining game with voting

Let $\langle N, X, (\rho_i)_{i\in N}, (\delta_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a distributive bargaining game with voting and let $n = |N|$. Suppose that $\rho_i = 1/n$ for all $i \in N$ and for some $\delta \in (0,1)$ we have $\delta_i = \delta$ for all $i \in N$. A strategy profile is a subgame perfect equilibrium with undominated voting in which every player's strategy is stationary if and only if for some balanced collection $\{S^j\}_{j\in N}$ of minimal majorities with $j \in S^j$ for all $j \in N$ the strategy of each player $i \in N$ makes the proposal x for which

$$x_j = \begin{cases} 1 - \frac{1}{2}\delta(n-1)/n & \text{if } j = i \\ \delta/n & \text{if } j \in S^i \setminus \{i\} \\ 0 & \text{otherwise} \end{cases} \tag{14.3}$$

after every history ending with the selection of i as the proposer and votes *for* if and only if $y_i \geq \delta/n$ after every history ending with the proposal y. Each player's expected payoff in every equilibrium is $1/n$.

Proof

I first argue that a strategy profile s^* satisfying the conditions in the result is a subgame perfect equilibrium with undominated voting. The game satisfies the condition in Proposition 16.9, so that s^* is a subgame perfect equilibrium if and only if it satisfies the one-deviation property. To show that it satisfies this property, I consider each type of subgame in turn.

Subgame starting with proposal by player i

If i adheres to s_i^*, her proposal is accepted and her payoff is $1 - \frac{1}{2}\delta(n-1)/n$.

If she makes a proposal y with $y_i > 1 - \frac{1}{2}\delta(n-1)/n$ then it is rejected, because the total payoff to the other players is less than $\frac{1}{2}\delta(n-1)/n$ and hence fewer than $\frac{1}{2}(n-1)$ of the other players receive at least δ/n. With probability $1/n$ she is selected to be the proposer again in the next period, in which case her payoff is $1 - \frac{1}{2}\delta(n-1)/n$ (with one period of delay), and with probability $1/n$ each of the other players is selected to be the proposer, in which case her payoff is either δ/n or 0 (with one period of delay). Given that $\delta < 1$, we have $\delta/n < 1 - \frac{1}{2}\delta(n-1)/n$, so that i's payoff is less than $1 - \frac{1}{2}\delta(n-1)/n$.

If she makes a proposal y with $y_i < 1 - \frac{1}{2}\delta(n-1)/n$ then either it is accepted, in which case her payoff is y_i, or it is rejected, in which case her payoff is less than $1 - \frac{1}{2}\delta(n-1)/n$ by the argument for the previous case.

Subgame starting with vote regarding proposal y

If y is rejected, the outcome, after one period of delay, is given in the following table for any player $i \in N$.

proposer	i's payoff	probability
i	$1 - \frac{1}{2}\delta(n-1)/n$	$1/n$
j with $i \in S^j$	δ/n	$\frac{1}{2}(n-1)/n$
j with $i \notin S^j$	0	$\frac{1}{2}(n-1)/n$

Thus every player's expected payoff if y is rejected is

$$\delta\left[(1/n)\big(1 - \tfrac{1}{2}\delta(n-1)/n\big) + \tfrac{1}{2}\big((n-1)/n\big)(\delta/n)\right] = \delta/n.$$

Hence if $y_i = \delta/n$ and $\frac{1}{2}(n-1)$ of the remaining players vote *for* (as s^* prescribes), so that i's vote is pivotal, then i's expected payoffs if she votes *for* (as s_i^* prescribes) or *against* are the same. If some other number of the other players vote *for*, i's vote makes no difference to the outcome, so that her payoffs to voting *for* and *against* are also equal. If $y_i > \delta/n$ then i is better off voting *for* than *against* if the other players adhere to their strategies (in which case her vote is pivotal), and if $y_i < \delta/n$ then she is better off voting *against* than *for*. Thus the voting behavior prescribed by s_i^* is optimal given that the other players adhere to s^* and is also undominated.

Now let s be a subgame perfect equilibrium with undominated voting in which each player's strategy is stationary. I argue that s takes the form given in the result.

Step 1 *For the strategy profile s and each $i \in N$, the expected payoffs of player i at the start of the game and at the start of each subgame following the rejection of a proposal are all the same.*

Proof. The conclusion follows from the definition of a stationary strategy and the fact that the game and every such subgame are identical. ◁

Denote by $V_i(s)$ the expected payoff of player i at the start of the game (and hence by Step 1 at the start of each subgame beginning with the selection of a proposer) for the strategy profile s.

Step 2 *The proposal that s specifies for each player at the start of any subgame following her selection as the proposer is accepted and hence $\sum_{i\in N} V_i(s) = 1$.*

Proof. Suppose that the proposal that s_j specifies for player j is rejected. Then play moves to the next period, at the start of which the expected payoff of each player i is $V_i(s)$. Now suppose that j deviates from s_j and proposes that the share of each player i be $V_i(s)/\sum_{k=1}^n V_k(s)$. Given that $\sum_{k=1}^n V_k(s) \le 1$, this share is at least $V_i(s)$, and hence is greater than $\delta V_i(s)$. Thus every player votes for the proposal, yielding j a payoff of at least $V_j(s)$, which is more than the payoff $\delta V_j(s)$ she gets if she makes a proposal that is rejected.

If $\sum_{i\in N} V_i(s) < 1$ then any player can increase the amount assigned to herself in her proposal without changing the amounts assigned to the other players. ◁

Step 3 *Let x be a proposal. If $x_j > \delta V_j(s)$ then j's strategy s_j votes in favor of x, and if $x_j < \delta V_j(s)$ then s_j votes against x.*

Proof. If x fails then j's payoff is $\delta V_j(s)$, so her voting in favor of x weakly dominates her voting against it if $x_j > \delta V_j(s)$ and her voting against it weakly dominates her voting for it if $x_j < \delta V_j(s)$. ◁

Step 4 *For every player $i \in N$, $V_i(s) > 0$.*

Proof. Suppose that i is selected to be the proposer. For some set S containing half of the remaining players, $\sum_{j\in S} V_j(s) \le \frac{1}{2}$, so that there is a proposal x^i with $x_i^i > 0$ and $x_j^i > V_j(s)$ for all $j \in S$, and by Step 3 the strategy of every player $j \in S$ votes in favor of x^i. ◁

Step 5 *Let x^i be the proposal made by s^i whenever i is selected to be the proposer. If for some player $j \neq i$ we have $x_j^i = \delta V_j(s)$ then j's strategy s_j votes in favor of x^i.*

Proof. Suppose that $x_j^i = \delta V_j(s)$. By Step 4, $x_j^i > 0$ and by Step 2, x^i passes. Suppose that j's strategy s_j votes against x^i. If $x_k^i < \delta V_k(s)$ for some player k then her voting against x^i weakly dominates her voting for it, so $x_k^i \ge \delta V_k(s)$ for every player k who votes in favor of x^i. Then i can increase her payoff by deviating to a proposal y^i in which $y_j^i = 0$ and $y_l^i > x_l^i$ for every player $l \neq j$, which passes because, by Step 3, every player who voted for x^i optimally votes for y^i. So s_j votes in favor of x^i. ◁

Step 6 *$V_i(s)$ is the same for every player $i \in N$.*

Proof. An optimal proposal of any player minimizes the amount allocated to the other players among the proposals that a majority of players vote in favor of. By Steps 3 and 5, every player j votes in favor of a proposal that gives her at least $\delta V_j(s)$ and against one that gives her less than $\delta V_j(s)$, so every optimal proposal of player i gives $\delta V_j(s)$ to each member j of a minimal majority S^i for which $\sum_{j\in S^i\setminus\{i\}} V_i(s)$ is smallest, and nothing to the players outside S^i.

Suppose that the values of $V_j(s)$ are not the same for all $j \in N$.

If the number of players tied for the smallest value of $V_j(s)$ is at least $\frac{1}{2}(n+1)$, then none of the remaining players are included in any set S^i, so that every such player, including the ones for which $V_j(s)$ is highest, obtains a positive payoff only if she is the proposer. Thus the payoff of every such player k is $V_k(s) = (1/n)\big(1 - \delta \sum_{l \neq k} V_l(s)\big) < 1/n$. Hence $V_j(s) < 1/n$ for every player, contradicting $\sum_{j \in N} V_j(s) = 1$ (from Step 2).

If the number of players tied for the smallest value of $V_j(s)$ is at most $\frac{1}{2}(n-1)$, then all of these players are included in every set S^i. Thus for every such player k, we have $V_k(s) = (1/n)\big(1 - \delta \sum_{l \neq k} V_l(s)\big) + ((n-1)/n)\delta V_k(s)$, so that, given $\sum_{l \neq k} V_l(s) = 1 - V_k(s)$, we have $V_k(s) = 1/n$. That is, the smallest value of $V_j(s)$ is $1/n$, and hence $V_j(s) = 1/n$ for all $j \in N$, contradicting the assumption that the value of $V_j(s)$ are not the same for all $j \in N$. ◁

Step 7 *$V_i(s) = 1/n$ for all $i \in N$.*

Proof. By Step 2, $\sum_{j \in N} V_j(s) = 1$, so the result follows from Step 6. ◁

Step 7 and the optimality of s_i implies that the proposal made by s_i gives δ/n to each member of a set of $\frac{1}{2}(n-1)$ players and hence $1 - \frac{1}{2}(n-1)\delta/n$ to i. Thus the expected payoff of each player j under s is

$$(1/n)\big(1 - \tfrac{1}{2}(n-1)\delta/n\big) + (K/n)\delta/n,$$

where K is the number of players who include j in the set to whose members they give δ/n. For this expected payoff to equal $1/n$, as it must, $K = \frac{1}{2}(n-1)$. Hence the collection of minimal majorities to which the players' proposals assign positive payoffs is balanced, so that s takes the form in the result.

Comments

Equilibrium outcome The outcome of an equilibrium is that the player selected to make the first proposal offers δ/n to each member of a set containing $\frac{1}{2}(n-1)$ of the other players, which, together with her, makes up a minimal majority; she takes the remainder of the pie. All members of the minimal majority vote in favor of this proposal, and hence the game ends.

Payoffs A player's equilibrium payoff in a subgame that starts with her making a proposal, $1 - \frac{1}{2}\delta(n-1)/n$, exceeds her equilibrium payoff in a subgame that starts with another player making a proposal, δ/n or 0. If δ is close to 1 then the former payoff is close to $(n+1)/2n$, which is close to $\frac{1}{2}$ when n is large: the proposer gets almost half of the pie when the players are patient and numerous.

In a generalization of the model in which the payoff functions are not restricted to be linear, the stationary equilibrium payoffs may not be unique (Banks and Duggan 2000, Example 4).

Heterogeneous discount factors and recognition probabilities If the players' discount factors or recognition probabilities differ, the game may have no subgame perfect equilibrium in which each player proposes the same payoff distribution whenever she is the proposer. However, it has stationary equilibria in which the strategies are mixed (Banks and Duggan 2000, Theorem 1). In each such equilibrium, each player chooses the same probability distribution over proposals whenever she is selected to be the proposer and uses the same voting rule after every history ending in a proposal, and the payoff profile is the same (Eraslan 2002, Theorem 5).

These equilibria have features in common with those in Proposition 14.5. Denote by $(v_1, \ldots, v_n)$ the (unique) equilibrium payoff profile. Then each proposal that is assigned positive probability by an equilibrium strategy of any player i assigns the payoff $\delta_j v_j$ to each member $j \neq i$ of a minimal majority S^i that contains i and 0 to every player outside S^i. When the players' equilibrium payoffs are the same, every minimal majority serves its purpose equally well. But when these payoffs differ, a proposer optimally assigns positive probability only to minimal majorities for which the total equilibrium payoff of the members other than her is as small as possible. Thus in an equilibrium, for any minimal majority S^i assigned positive probability by i's strategy, the payoff of no player outside S^i is less than that of any member of $S^i \setminus \{i\}$. In particular, if no two players have the same equilibrium payoff, then every player's strategy assigns probability 1 to a single proposal—the one that assigns $\delta_j v_j$ to each of the $\frac{1}{2}(n-1)$ players j with the smallest values of v_j, the remainder of the pie to her, and the payoff 0 to every other player. A final property of an equilibrium is that each player j votes in favor of a proposal in which her payoff is at least $\delta_j v_j$ and against a proposal in which it is lower, so that agreement is reached in the first period of every subgame.

A distributive bargaining game with voting in which all discount factors and all recognition probabilities are the same has equilibria in which each player uses a strategy of this more general form, in addition to the equilibria given in Proposition 14.5. For each player i and each minimal majority S^i that contains i, let $x(S^i)$ be the proposal defined in (14.3). Then in one such equilibrium, each player i assigns equal probability to $x(S^i)$ for every minimal majority S^i that contains her.

Now suppose that the players' discount factors may differ, but their recognition probabilities are the same. Assume specifically that $\delta_1 < \delta_2 < \cdots < \delta_n$, and denote the stationary equilibrium payoff of each player i by v_i. Is it possible that $v_1 < v_2 < \cdots < v_n$: more patient players receive higher payoffs? In this case, the equilibrium strategy of each player j assigns probability 1 to the proposal that offers $\delta_j v_j$ to the $\frac{1}{2}(n-1)$ remaining players j with the smallest values of v_j and nothing to the other players, maximizing the amount left over for her. (The former players are then indifferent between voting *for* and *against*; in an equilibrium they vote *for*.) So in particular, players $k = \frac{1}{2}(n+1)$ and n, when proposing, offer the same amounts to the members of the same subset of the remaining players, namely $\delta_j v_j$ to each player $j = 1, \ldots, \frac{1}{2}(n-1)$. Thus when proposing, they receive the same payoff. But player n receives no payoff in the proposal of any other player, whereas player k obtains $\delta_k v_k$ when players $1, \ldots, k-1$ propose. Thus player k's payoff exceeds player n's, contrary to the assumption that $v_k < v_n$. We conclude that players' payoffs are not ordered by their discount factors.

If the discount factors are all sufficiently close to 1 and the recognition probabilities may differ but are sufficiently similar, then independent of the exact values of the recognition probabilities, the equilibrium payoff of each player i is $1/\big(\delta_i \sum_{j=1}^{n}(1/\delta_j)\big)$, or equivalently $(1/n)H(\delta_1,\ldots,\delta_n)/\delta_i$, where $H(\delta_1,\ldots,\delta_n)$ is the harmonic mean of $\delta_1, \ldots, \delta_n$ (Kawamori 2005, Theorem 1). Thus under these conditions, a more patient player receives a *smaller* equilibrium payoff, and a player's payoff *decreases* as she becomes more patient. For a three-player game in which the recognition probabilities are equal, a sufficient condition for the players' equilibrium payoffs to take this form is that each player's discount factor is at least $\frac{2}{3}$ (Kawamori 2005, Example 1). Figure 14.5 shows an example of player 1's equilibrium payoff in such a game as a function of her discount factor and that of player 2, given that player 3's discount factor is 0.8.

Now suppose that the players' discount factors are the same but their recognition probabilities may differ. Consider the effect of a small change in the recognition probabilities away from equality. When every player's recognition probability is the same, every player's equilibrium payoff is $1/n$. In an equilibrium, each player's payoff when she is the proposer, an event with probability $1/n$, is thus $1-\frac{1}{2}\delta(n-1)/n$, so her expected payoff when she is not the proposer is $(1/n)\frac{1}{2}\delta(n-1)/n$. Now increase player i's recognition probability slightly. One possibility that seems superficially plausible is that player i's equilibrium payoff increases slightly and the other players' equilibrium payoffs decrease slightly. But if the payoffs change in this way, then i does not receive a positive payoff in any proposal of any other player, because, given that her equilibrium payoff is higher than every other player's, she is not a member of any minimal majority to which any other player assigns positive probability. So she obtains a positive payoff only when she is the proposer, and given that the other players' payoffs have decreased only slightly, this payoff is about the same as the payoff she received when she was the proposer in the original game. Thus she receives roughly $1-\frac{1}{2}\delta(n-1)/n$ with a probability slightly greater than $1/n$, which gives

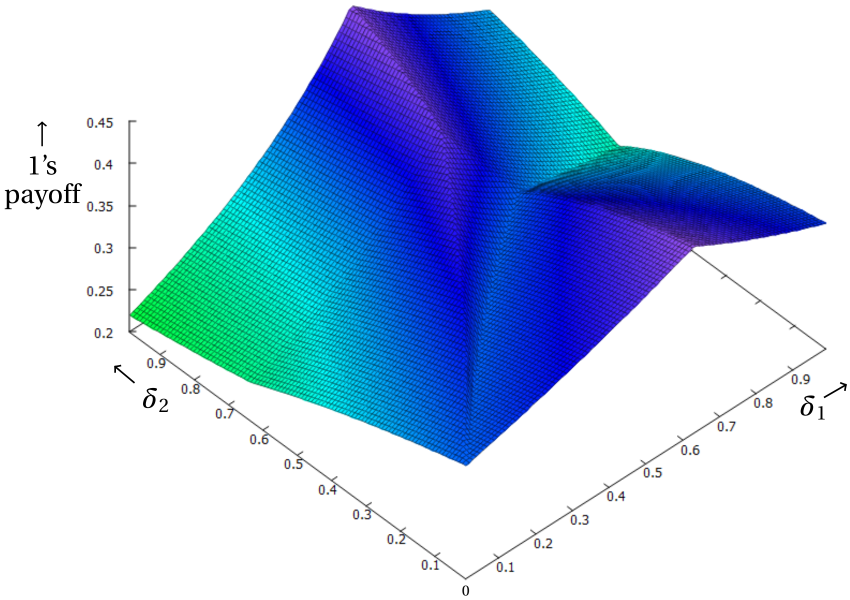


Figure 14.5 The payoffs of player 1 in a stationary subgame perfect equilibrium with undominated voting of a three-player distributive bargaining game with voting in which each player's recognition probability is $\frac{1}{3}$, as a function of δ_1 and δ_2 for $\delta_3 = 0.8$. (Constructed from the calculations in Kawamori 2004; see also Kawamori 2005.)

her an expected payoff that, far from being more than her original equilibrium payoff, is less than it by approximately $(1/n)\frac{1}{2}\delta(n-1)/n$, her expected payoff in the original game when she was not the proposer. So perhaps instead of her payoff increasing slightly and every other player's payoff decreasing slightly, it's the other way around? That is, her payoff decreases slightly and every other player's payoff increases slightly. But then she is a member of *every* minimal majority to which a player's proposal assigns positive probability, and an analogous argument leads to the contradictory conclusion that her equilibrium payoff increases substantially. The only remaining possibility is that her payoff remains the same: for any common discount factor, the players' equilibrium payoffs are independent of their recognition probabilities around the point where these probabilities are equal.

Further away from the point of equality of the recognition probabilities, a player benefits from an increase in her probability, and if i's recognition probability is at least as high as j's, then i's stationary equilibrium payoff is as least as high as j's (Eraslan 2002, Corollary 1). Figure 14.6 shows the limit, as the common discount factor approaches 1, of the stationary equilibrium payoffs of player 1 in a three-player distributive bargaining game with voting as a function of the recognition probabilities. In this case, a player's equilibrium payoff is constant

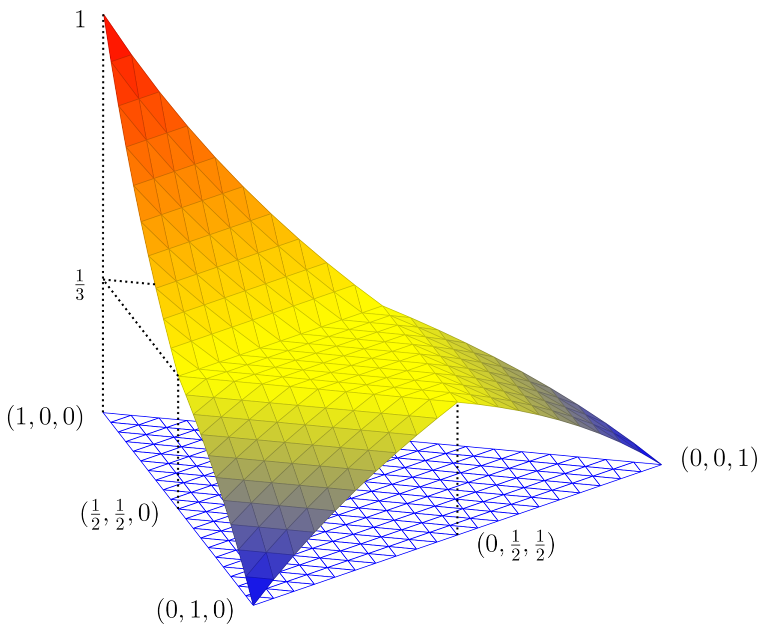


Figure 14.6 The limit of the payoffs of player 1 in a stationary subgame perfect equilibrium with undominated voting of a three-player distributive bargaining game with voting as the players' common discount factor approaches 1, as a function of the vector (ρ_1, ρ_2, ρ_3) of recognition probabilities. (Constructed from Table 1 (p. 12) of Imai and Salonen 2012.)

(equal to $\frac{1}{3}$) whenever every player's recognition probability is at most $\frac{1}{2}$.

Voting rules The characterization in Proposition 14.5 can be generalized to a game in which the votes of q players are needed for a proposal to be accepted, where q may differ from $\frac{1}{2}(n+1)$.

If $q \leq n-1$, the discount factors are close enough to 1, and the recognition probabilities are not too dissimilar, then the equilibrium payoff of each player i is $1/(\delta_i \sum_{j=1}^{n}(1/\delta_j))$, independent of q (Kawamori 2005, Theorem 1). When q is larger, a player proposes a smaller amount for herself, but obtains more when she is a responder, and the changes exactly offset each other.

If $q = n$, so that every player's vote is needed for acceptance, and every player has the same discount factor, then each player's expected payoff in a stationary subgame perfect equilibrium with undominated voting is her recognition probability. Here is an argument for this result. Assume that the game has unique stationary equilibrium payoffs and that agreement is reached immediately in any stationary equilibrium. Denote the equilibrium payoff of each player i by v_i. If i is selected to be the proposer, she offers δv_j to every other player j to induce her to vote in favor of her proposal, leaving $1 - \delta \sum_{j \in N \setminus \{i\}} v_j = 1 - \delta(1 - v_i)$ for herself, and if another player is selected to be the proposer, that player offers δv_i to player i. Thus player i's expected payoff at the start of the game is

$$v_i = \rho_i(1-\delta(1-v_i)) + (1-\rho_i)\delta v_i.$$

Solving this equation for v_i we obtain $v_i = \rho_i$. Note that the result implies a stark difference between a player's equilibrium payoff under majority rule and her payoff under unanimity rule for some profiles of recognition probabilities. For example, if $n = 3$ and the recognition probabilities are $(\varepsilon, \frac{1}{2}(1-\varepsilon), \frac{1}{2}(1-\varepsilon))$ for some small $\varepsilon > 0$, then for δ close to 1, player 1's payoff is ε under unanimity rule and close to $\frac{1}{3}$ under majority rule (see Figure 14.6).

More generally, if $q = n$ and the players' discount factors and recognition probabilities differ, then an argument that extends the one in the previous paragraph shows that the expected equilibrium payoff of each player i is

$$\frac{\rho_i}{1-\delta_i}\frac{1}{\sum_{j=1}^{n}(\rho_j/(1-\delta_j))}.$$

So given the recognition probabilities, player i obtains a higher payoff than does player j if and only if $\delta_i > \delta_j$.

Exercise 14.2: Game with deterministic rotating recognition and voting

Consider a variant of a distributive bargaining game with voting with three players in which, following a proposal by any player i, the remaining players vote sequentially. First player $(i+1)(\text{mod } 3)$ votes. If she votes in favor, the proposal is accepted. Otherwise, player $(i+2)(\text{mod } 3)$ votes. If she votes in favor, the proposal is accepted, and otherwise there is a period of delay and then player $(i+1)(\text{mod } 3)$ makes a proposal. Play continues in this fashion until a proposal is accepted. Show that the game has a stationary subgame perfect equilibrium in which each player's proposal gives her all the pie. (This equilibrium is the only stationary subgame perfect equilibrium of the game.)

14.2 Recurrent distributive bargaining game with voting

A bargaining game with voting ends when the individuals reach agreement on an alternative; no player has an opportunity to reopen the negotiations. In this section I briefly consider a model in which negotiations are always open.

Time is discrete, starting with period 1. In each period, one unit of a good is available to a finite set N of individuals; it may be distributed in any way among the individuals, and some of it may be wasted. An outcome is a sequence $(x^1, x^2, \ldots)$ of distributions of the good among the individuals, where $x_i^t \in [0,1]$ for $t = 1, 2, \ldots$ and all $i \in N$, and $\sum_{i \in N} x_i^t \leq 1$ for all t.

In each period, there is a default distribution of the good. In period 1 this default is $x^0 = (0, \ldots, 0)$ and in every subsequent period t it is x^{t-1}, the distribution from the previous period. At the start of each period t, an individual is selected by chance. She either chooses *pass*, in which case $x^t = x^{t-1}$ and the period ends, or proposes a distribution different from x^{t-1}, in which case all individuals vote simultaneously for or against the proposal. If a majority of the individuals vote for the proposal, it becomes the distribution x^t in period t, and otherwise the distribution in period t is x^{t-1}.

For each individual $i \in N$, let $\delta_i \in (0,1)$ and let $u_i : [0,1] \to \mathbb{R}$ be an increasing function. The preferences of individual i regarding lotteries over outcomes (sequences of distributions) are represented by the expected value of the discounted average of the sequence $(u_i(x_i^1), u_i(x_i^2), \ldots)$ for the discount factor δ_i, namely

$$(1 - \delta_i) \sum_{t=1}^{\infty} \delta_i^{t-1} u_i(x_i^t).$$

Note that if $x_i^t = x_i$ for all t then this discounted average is $u_i(x_i)$, because $\sum_{t=1}^{\infty} \delta_i^{t-1} = 1/(1 - \delta_i)$. I refer to a game defined in this way as a *recurrent distributive bargaining game with voting.*

When an individual makes a proposal in such a game, the default distribution for the period is relevant, because it will be the distribution in the current period if the individual's proposal is voted down; when an individual casts a vote, both the default distribution and the proposal are relevant for the same reason. Thus for this game I define a stationary strategy of any individual i to be a pair (s_i^p, s_i^v) of functions, with $s_i^p : X \to \{pass\} \cup X$ specifying i's proposal as a function of the default distribution, whenever she is selected to be the proposer, and $s_i^v : X \times X \to \{for, against\}$ specifying her vote, as a function of the default distribution and the proposal, whenever a ballot is held.

A recurrent distributive bargaining game with voting has many subgame perfect equilibria with undominated voting in which every player's strategy is stationary, and in some of these equilibria some of the available good is wasted. These features are demonstrated well by the following example.

Let $N = \{1, 2, 3\}$ and suppose that each player has the same recognition probability ($\rho_1 = \rho_2 = \rho_3 = \frac{1}{3}$) and discount factor ($\delta_1 = \delta_2 = \delta_3 = \delta$), and, for all $i \in N$, $u_i(x_i) = x_i$ for all $x_i \in [0,1]$. The strategy profile I define is based on three distributions,

$$y^1 = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{6}), y^2 = (\tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{3}), \text{ and } y^3 = (\tfrac{1}{3}, \tfrac{1}{6}, \tfrac{1}{3}).$$

Let $Y = \{y^1, y^2, y^3\}$. Here is the strategy of player i.

Proposal of player i

$$s_i^p(x) = \begin{cases} pass & \text{if } x \in Y \\ y^i & \text{if } x \notin Y. \end{cases}$$

Vote of player i

$$s_i^v(x,z) = \begin{cases} for & \text{if } x \in Y \text{ and } x_i = \frac{1}{6} \\ & \quad \text{or } x \notin Y,\, z \in Y, \text{ and } z_i \geq (1-\delta)x_i + \frac{5}{18}\delta \\ & \quad \text{or } x \notin Y,\, z \notin Y, \text{ and } z_i \geq x_i \\ against & \text{otherwise.} \end{cases}$$

Notice that some of the good is wasted in the distributions y^1, y^2, and y^3: in each case, only $\frac{5}{6}$ of the unit is assigned to the individuals. Notice also that if the default distribution is y^i then a proposal z that gives each player an amount of the good that is larger than the amount she gets in y^i is rejected because the two players whose share in y^i is $\frac{1}{3}$, say i and j, vote against it. Why do they do that? Because the acceptance of z would reopen negotiations. The player k chosen to be the proposer in the next period would propose y^k, and, if δ is close enough to 1, this proposal would be accepted, which means that rather than being assured of the share $\frac{1}{3}$ in every future period, as they are when z is rejected and the players subsequently follow their strategies, i and j would face a future in which each of their shares is $\frac{1}{3}$ in every future period with probability $\frac{2}{3}$ but only $\frac{1}{6}$ in every future period with probability $\frac{1}{3}$. Given that a proposal that improves upon y^i for every player is rejected, a player who makes such a proposal reaps no benefit from doing so.

In the outcome of this strategy profile, the player chosen in period 1 to be the proposer, say i, selects y^i. This alternative gives her and one of the other players, say j, the fraction $\frac{1}{3}$ of the good. The default amount x_k^0 for each player k in period 1 is 0 and $\frac{1}{3} > \frac{5}{18}$, so players i and j vote for y^i and thus $x^1 = y^i$. In every subsequent period the player chosen to be the proposer passes, so that the distribution remains y^i. Thus the outcome of the strategy profile is the lottery in which the distribution is y^1 in every period, or y^2 in every period, or y^3 in every period, each with probability $\frac{1}{3}$. I denote this lottery by ξ. The payoff of each player for ξ is

$$\tfrac{2}{3} \cdot \tfrac{1}{3} + \tfrac{1}{3} \cdot \tfrac{1}{6} = \tfrac{5}{18}.$$

I now argue that the strategy profile is a subgame perfect equilibrium with undominated voting if $\delta \geq \frac{12}{13}$. The game satisfies the condition in Proposition 16.9, so that the strategy profile is a subgame perfect equilibrium if and only if it satisfies the one-deviation property. Here is an argument that it satisfies this property and that no player's vote after any history is weakly dominated.

Subgame following selection of player i when default distribution is x

Suppose that $x \in Y$. If i follows her strategy, she passes, and the distribution is x in every subsequent period (so that i's payoff in the remainder of the game is x_i). If i deviates from her strategy and makes a proposal, then only the player for whom $x_i = \frac{1}{6}$ votes *for*, so x remains the distribution in every subsequent period, as it does if she follows her strategy.

Suppose that $x \notin Y$. If i follows her strategy, she proposes y^i, which assigns $\frac{1}{3}$ to her and to one of the other players. For at least two players j, we have $x_j \le \frac{1}{2}$, so that $(1-\delta)x_j + \frac{5}{18}\delta \le \frac{1}{2} - \frac{4}{18}\delta \le \frac{1}{2} - \frac{4}{18}\frac{12}{13} = \frac{23}{78} < \frac{1}{3}$. Thus at least two players vote for y^i, so that it becomes the default distribution and hence, if all players follow their strategies subsequently, the distribution in every subsequent period. If i deviates from her strategy and proposes a member of Y different from y^i, at least two players vote for it by the same argument, so that it becomes the distribution in every subsequent period. Thus i does not benefit from the deviation. Finally, if i deviates from her strategy and proposes a distribution z not in Y, whether or not it is accepted the outcome subsequently is the lottery ξ. If she adheres to her strategy she receives $\frac{1}{3}$ in every future period, so regardless of the value of z, she is worse off if she deviates.

Subgame starting with vote on z when default distribution is x

Suppose that $x \in Y$ and $x_i = \frac{1}{6}$. If i follows her strategy, she votes for z. The other two players vote against z, so the distribution in the period is x, and that distribution persists in every future period. If i deviates from her strategy and votes against z and the other players' votes are such that her vote affects the outcome, i's deviation changes the distribution in the period from z to x. If it is x, then it remains x in every future period, so that i's payoff in every future period is $\frac{1}{6}$. Suppose it is z. Then if $z \in Y$, the outcome remains z in every future period, and otherwise the outcome is y^1, y^2, and y^3 in every future period, each with probability $\frac{1}{3}$. Thus i's voting for z is optimal and undominated.

Suppose that $x \in Y$ and $x_i = \frac{1}{3}$. If i follows her strategy, she votes against z, as does one of the other players, so the distribution in the period is x, which persists in every future period. An argument analogous to the one for the case in which $x_i = \frac{1}{6}$ shows that i is made worse off by a deviation that affects the outcome, so that her voting against z is optimal and undominated.

Suppose that $x \notin Y$ and $z \in Y$. If z is accepted, the distribution is z in every future period, so that i receives the amount z_i in every period and hence the payoff z_i. If it is rejected, then i receives x_i in the current period and then the lottery ξ, and hence the payoff $(1-\delta)x_i + \frac{5}{18}\delta$. Thus i's voting for z is optimal and undominated if $z_i \geq (1-\delta)x_i + \frac{5}{18}\delta$ and her voting against it is optimal and undominated if the reverse inequality holds.

Suppose that $x \notin Y$ and $z \notin Y$. If z is accepted, the distribution is z in the current period and then the lottery ξ. If it is rejected, then i receives x_i in the current period and then the lottery ξ. Thus i's voting for z is optimal and undominated if $z_i \geq x_i$ and her voting against it is optimal and undominated if $z_i < x_i$.

This equilibrium is a member of a large class of equilibria: the strategy profile obtained by replacing y^1, y^2, and y^3 with $(\alpha_1, \alpha_2, \beta_3)$, $(\beta_1, \alpha_2, \alpha_3)$, and $(\alpha_1, \beta_2, \alpha_3)$ for any $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$ with $\alpha_i \in [0,1]$, $\beta_i \in [0,1]$, $\alpha_i > \beta_i$ for $i = 1$, 2, 3, and $\alpha_1 + \alpha_2 + \beta_3 \leq 1$, $\beta_1 + \alpha_2 + \alpha_3 \leq 1$, and $\alpha_1 + \beta_2 + \alpha_3 \leq 1$ is an equilibrium for values of δ sufficiently close to 1. These equilibria differ qualitatively from the equilibrium of a distributive bargaining game with voting, as characterized in Proposition 14.5, in following respects.

- The set of players among whom the pie is shared may be larger than a minimal majority. If $\beta_i > 0$ for $i = 1, 2, 3$, the player chosen to be the first proposer offers a positive amount to every player, and this proposal is accepted.
- Some of the pie may be wasted. In the equilibria in which $\alpha_i = \frac{1}{6}$ and $\beta_i = \frac{1}{3}$ for $i = 1$, 2, 3, for example, the player chosen to be the first proposer distributes only $\frac{5}{6}$ of the pie.
- The game has multiple stationary equilibrium outcomes. In fact, for any distribution (x_1, x_2, x_3) with $x_i > 0$ for $i = 1, 2, 3$, there is a number $\delta^* \in (0,1)$ such that if $\delta \geq \delta^*$ then the game has a stationary equilibrium in which the first player selected by chance proposes (x_1, x_2, x_3), which remains the alternative in every future period.

Anesi and Seidmann (2015) show that the example may be extended to games with more than three players and to voting rules that require more than a bare majority of the players to vote in favor for a proposal to be accepted, as long as the number of votes required is less than the number of players. They show also that under unanimity rule, by contrast, the game has a unique stationary equilibrium payoff, which coincides with the stationary equilibrium payoff of a distributive bargaining game with voting.

Notes

The model of a bargaining game with voting is due to Baron and Ferejohn (1989) and Banks and Duggan (2000). Proposition 14.2 is based on Cho and Duggan (2015), Propositions 14.3 and 14.5 are based on Baron and Ferejohn (1989), and Proposition 14.4 is based on Cho and Duggan (2009). My exposition draws also on Austen-Smith and Banks (2005, Section 6.2) and Eraslan (2002)[1]. The model in Section 14.2 was first studied by Baron (1996). My presentatiois based on Anesi and Seidmann (2015); the example I present is their Example 1.

Exercise 14.2 is based on Ali et al. (2019, Example 3).

Solutions to exercises

Exercise 14.1

If $x \neq (1,0,0)$ and player 3 proposes $y = (1,0,0)$ in state x, then $M^3(y) = \{1,2\}$, and player 1's strategy calls for her to vote against y. But her doing so is not optimal: if she votes for y then her payoff is 1 and if she votes against y then her payoff is at most $\delta_1 < 1$. (If there are at least five players, $x \neq (1,0,\ldots,0)$, and player $k \neq 1$ proposes $y = (1,0,\ldots,0)$, then every set $M^k(y)$ contains only players whose payoffs in y are 0.)

Exercise 14.2

Consider the strategy profile in which whenever a player is chosen to be the proposer, she proposes that she get the entire pie, and whenever she votes on a proposal, she votes in favor, regardless of the proposal. By the following argument, this strategy profile satisfies the one-deviation property, so that by Proposition 16.9 it is a subgame perfect equilibrium. Consider player 1.

Subgame starting with proposal

Player 1 cannot propose that she gets more; a proposal to get less is accepted, and player 1 is worse off.

Subgame following a proposal x by player 3

If player 1 accepts x then she gets x_1. If she rejects x then player 2 accepts it, and player 1 still gets x_1. So player 1 optimally accepts x.

Subgame following rejection by player 3 of proposal x by player 2

If player 1 accepts x then she gets x_1. If she rejects x then player 3 proposes $(0,0,1)$, which is accepted, so that player 1 gets 0. Thus player 1

[1] Note that Corollary 2 in Eraslan (2002), which is stated also on p. 217 of Austen-Smith and Banks (2005), is incorrect, as the argument on page 492 and the results of Kawamori (2004, 2005) show.

optimally accepts x.

The arguments for players 2 and 3 are the same.

15 Rulers threatened by rebellion

15.1 Threat of revolt 504
15.2 Coordinating rebellion 517

The models in the previous chapters concern collective choice in societies ruled by a fixed set of individuals. This chapter analyzes models of societies in which the set of rulers is not fixed. Individuals currently outside the set of rulers have the option to collectively seize power; the rulers have the option to preempt such a seizure by handing over power voluntarily. Under what circumstances will rulers choose to cede power? If a takeover requires the currently-unenfranchised individuals to coordinate their actions, how can they do so?

Synopsis

Section 15.1 considers a society consisting of two unitary actors, *Rich* and *Poor*, who interact over an infinite sequence of periods. Initially, *Rich* controls the distribution of wealth. In each period, it decides how much wealth to give to *Poor*. In some periods, the environment is conducive to revolt. In those periods, *Rich* has the option to hand over control to *Poor*—to “democratize”, in the terminology of the model; if it does not do so, then *Poor* has the option to revolt.

Proposition 15.1 shows that if the probability that the environment is conducive to revolt is small enough then the game has a unique subgame perfect equilibrium. In this equilibrium, *Rich* hands over control when the environment is conducive to revolt and gives no wealth to *Poor* when the environment is not conducive to revolt. If, when the environment is conducive to revolt, *Rich* were not to democratize, *Poor* would revolt. When the probability that the environment is conducive to revolt is low, *Rich*'s expected payoff if it follows this strategy is high and is independent of the strategy of *Poor*, putting an upper bound on the payoff of *Poor* in any subgame perfect equilibrium. So when the environment is conducive to revolt, *Poor* optimally takes that opportunity, knowing that if it does not then even if *Rich* gives it all the wealth in that period, its best payoff in every equilibrium of the remainder of the game is low.

The model gives *Rich* no means by which to commit to redistribute wealth to *Poor* when the environment is not conducive to revolt. Does this limitation account for the fact that it democratizes the first time the environment is conducive to revolt? If we modify the model to give it that option, then when the probability that the environment is conducive to revolt is low, it does not take the option in any equilibrium (see Section 15.1.3). Thus the fact that it democratizes in the original model cannot be ascribed to its inability to commit to redistribution when the environment is not conducive to revolt.

Section 15.2 models *Poor* not as a unitary actor, but as a large set of individuals. It studies the coordination problem these individuals face when deciding whether to seize power. In the model, in each period in an infinite sequence a dictator chooses an action and then each member of a large set of citizens independently decides whether to rebel. The probability that a rebellion is successful is an increasing function of the fraction of citizens who rebel. If the citizens observe the dictator's action, they can coordinate their actions. In this case, Proposition 15.2 describes some equilibria and specifies the highest payoff the citizens can achieve in any equilibrium. If the citizens observe only a noisy signal of the dictator's action, then perfect coordination is not possible. Proposition 15.3 specifies the best equilibria for the citizens in this case, which are worse than the best equilibria when the dictator's action is observed. The remainder of the section analyzes a variant of the model in which a dictator has the option to call an election. Her decision to exercise this option may be used by the citizens to coordinate their actions and in doing so achieve a payoff as high as in the best equilibrium for them in the model in which they observe the dictator's action.

15.1 Threat of revolt

A society consists of a rich elite and a mass of poor individuals. Initially, the rich elite rules. The poor individuals have the option to stage a revolution, after which they will have access to the entire wealth of the society minus the amount destroyed during the revolution. Two options of the elite that may forestall a revolution are the transfer of wealth to the poor individuals and the transfer of the control of the distribution of wealth to these individuals, an action I refer to as democratization. Under what circumstances will the elite choose each of these strategies? The model I present is designed to study the idea that if environments conducive to revolt are rare, the poor have an incentive to take such opportunities when they arise, inducing rulers to forestall revolt by preemptively democratizing.

15.1.1 Model

A society consists of two players: *Rich* and *Poor*. One unit of wealth is available, the distribution of which between *Rich* and *Poor* is initially controlled by *Rich*. *Rich* may offer some wealth to *Poor*, in which case *Poor* may accept the offer or rebel, or may hand over control of the distribution of wealth to *Poor*. If *Poor* rebels, the fraction v of wealth is lost and the remainder is taken by *Poor*. If *Rich* hands over control to *Poor*, *Rich* gets the fraction d of wealth and *Poor* gets the remainder. The numbers v and d are parameters; the model does not contain mechanisms by which they are determined. One reason that d might be positive is that *Rich* owns all the capital, which it does not forfeit when it democratizes.

An extensive game with perfect information that models this society is given in Figure 15.1. To find its subgame perfect equilibria, first note that *Poor* optimally accepts a proposal $(z, 1-z)$ if $z < v$ and revolts if $z > v$; if $z = v$ then both acceptance and revolt are optimal. Now consider the best action for *Rich*, given that *Poor* uses a strategy with these properties. If *Rich* proposes $(z, 1-z)$ with $z < v$ then *Poor* accepts, and the payoff of *Rich* is z; if it proposes $(z, 1-z)$ with $z > v$ then *Poor* revolts, and the payoff of *Rich* is 0; and if it chooses *Democratize* then its payoff is d. Thus if $d > v$ then the best action for *Rich* is *Democratize*, and hence in every subgame perfect equilibrium *Rich* chooses this action. Now suppose that $d < v$. For any number $z < v$, *Rich* obtains the payoff z by proposing $(z, 1-z)$. If it proposes $(v, 1-v)$, then both *Accept* and *Revolt* are optimal responses of *Poor*. The fact that *Accept* is optimal means that the game has a subgame perfect equilibrium in which *Rich* proposes $(v, 1-v)$ and *Poor* accepts this proposal. The game has no subgame perfect equilibrium in which *Poor* revolts in response to the proposal $(v, 1-v)$, because if it does so then the payoff of *Rich* is 0, and by deviating to a proposal $(z, 1-z)$ with $d < z < v$, which *Poor* accepts, *Rich* obtains the payoff z.

In summary, in every subgame perfect equilibrium, *Rich* chooses *Democratize* if $d > v$ and proposes $(v, 1-v)$, which *Poor* accepts, if $d < v$. If $d = v$ then in every subgame perfect equilibrium *Rich* either chooses *Democratize* or proposes $(v, 1-v)$, which *Poor* accepts. That is, in a subgame perfect equilibrium *Rich* democratizes only if the payoff of *Poor* under democracy is at most its payoff under revolution. However, at least in the case that this inequality is strict, the model seems to artificially restrict the actions of *Poor*. If it is better off under revolution than under democracy, why does it not have the option to revolt under democracy? If it has this option, it will take it, and in the only subgame perfect equilibrium of the resulting game, *Rich* offers *Poor* $1-v$, which it accepts. That is, only in the singular case of $v = d$ does the modified game have an equilibrium in which *Rich* democratizes; the game fails to capture the idea that *Rich* might

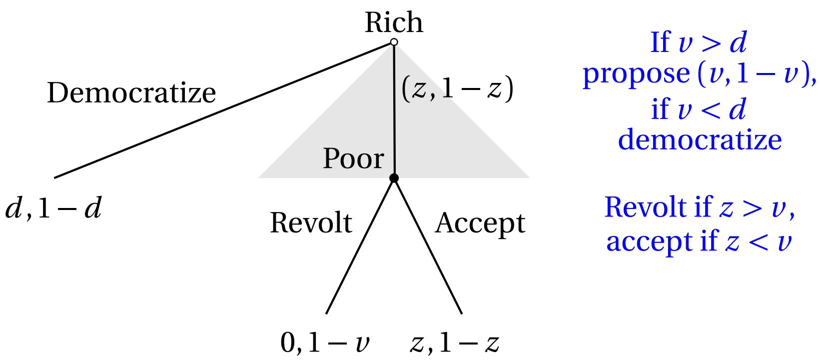


Figure 15.1 An extensive game with perfect information in which *Rich* initially controls the distribution of wealth. At the start of the game, its options are to democratize, in which case it receives the amount d, and to offer some amount to *Poor*, which may accept the offer or revolt, in which case the amount v is destroyed and *Rich* obtains nothing. (The diagram shows only one of the possible amounts *Rich* can offer to *Poor*.) In each payoff pair, the first number is the payoff of *Rich*. Conditions satisfied by any subgame perfect equilibrium are indicated on the right.

democratize to stave off revolt.

An extension of the game addresses this shortcoming. In the new model, play takes place over an infinite sequence of periods $t=1,2,\dots$, rather than in a single period, and in some periods the environment is conducive to revolt while in others it is not. The payoff of *Poor* after a revolt is assumed to be less than its payoff under democracy, so it has no incentive to revolt once democracy is established.

The game is illustrated in Figure 15.2. In each period, with probability $1-q$ the environment is not conducive to revolt (bad, b). In this case, *Rich* chooses a distribution of wealth in the period and play moves to the next period. I assume that for some number $\overline{z}\in[0,1]$, *Rich* is constrained to keep at most the fraction $\overline{z}$ of the wealth for itself. (If $\overline{z}=1$, it is effectively unconstrained.) With probability q, the environment is conducive to revolt (good, g), in which case *Rich* has two options. One is to democratize (D), resulting in the payoff pair $(d,1-d)$ in every subsequent period and no further strategic options for either player. The other is to propose a distribution $(y^t,1-y^t)$ of wealth in the period, where $y^t\in[0,1]$; *Poor* can accept this proposal or revolt. If *Poor* accepts the proposal then the payoff pair in the period is $(y^t,1-y^t)$ and play continues to the next period. If *Poor* revolts, then the payoff pair is $(0,1-v)$ in every subsequent period and neither player has any further strategic options. I assume $v>d$: *Poor* is better off under democracy than under revolution. Each player has the same discount factor, $\delta\in(0,1)$.

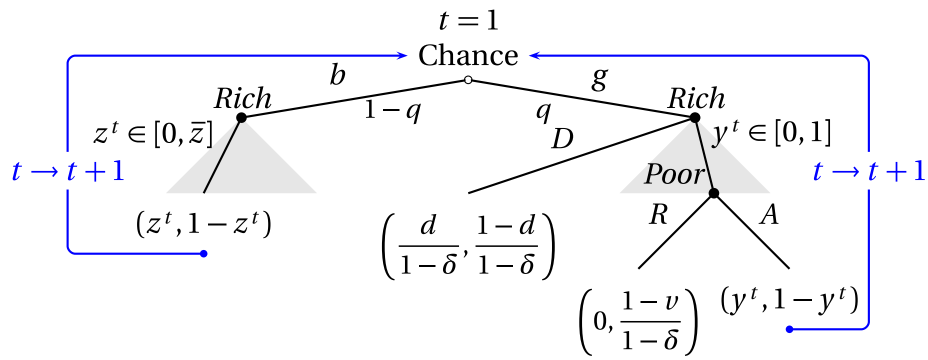


Figure 15.2 A dynamic game of revolt (see Definition 15.1). In each payoff pair, the first number is the payoff of *Rich*.

Definition 15.1: Dynamic game of revolt

A *dynamic game of revolt* $\langle\{Rich, Poor\}, q, \delta, v, d, \bar{z}\rangle$, where

- $q \in (0,1)$ (the probability that the environment is conducive to revolt)
- $\delta \in (0,1)$ (the discount factor)
- $v \in [0,1)$ (the amount of per-period wealth destroyed by a revolt)
- $d \in [0, v)$ (the amount of per-period wealth obtained by *Rich* each period under democracy)
- $\bar{z} \in [0,1]$ (the upper limit on the fraction of wealth *Rich* can retain when the environment is not conducive to revolt)

is an extensive game with perfect information and chance moves with the following components.

Players

Rich and *Poor*.

Terminal histories

For every $\tau \geq 1$, let $e^\tau = (b, z^\tau)$ or (g, y^τ, A) for some $z^\tau \in [0, \bar{z}]$ and $y^\tau \in [0,1]$. The set of terminal histories is the set of all sequences

- (g, D) and $(e^1, e^2, \ldots, e^t, g, D)$ for all $t \geq 1$
- (g, y^1, R) for all $y^1 \in [0,1]$ and $(e^1, e^2, \ldots, e^t, g, y^{t+1}, R)$ for all $t \geq 1$ and all $y^{t+1} \in [0,1]$
- $(e^1, e^2, \ldots)$.

Player function

Chance is assigned to the initial history and to every history ending with (b, z^t) for some $z^t \in [0, \bar{z}]$ or (g, y^t, A) for some $y^t \in [0, 1]$. *Rich* is assigned to every history ending with b or g, and *Poor* is assigned to every history ending with (g, y^t) for some $y^t \in [0, 1]$.

Chance probabilities

Chance selects g with probability q and b with probability $1 - q$, independent of history, whenever it moves.

Preferences

The preferences of *Rich* over terminal histories h are represented by the payoff function

$$\begin{cases} \dfrac{d}{1-\delta} & \text{if } h = (g, D) \\ \sum_{\tau=1}^{t} \delta^{\tau-1} x^\tau + \delta^t \dfrac{d}{1-\delta} & \text{if } h = (e^1, e^2, \ldots, e^t, g, D) \text{ for } t \geq 1 \\ 0 & \text{if } h = (g, y^1, R) \\ \sum_{\tau=1}^{t} \delta^{\tau-1} x^\tau & \text{if } h = (e^1, e^2, \ldots, e^t, g, y^{t+1}, R) \text{ for } t \geq 1 \\ \sum_{\tau=1}^{\infty} \delta^{\tau-1} x^\tau & \text{if } h = (e^1, e^2, \ldots) \end{cases}$$

and those of *Poor* are represented by the payoff function

$$\begin{cases} \dfrac{1-d}{1-\delta} & \text{if } h = (g, D) \\ \sum_{\tau=1}^{t} \delta^{\tau-1}(1 - x^\tau) + \delta^t \dfrac{1-d}{1-\delta} & \text{if } h = (e^1, e^2, \ldots, e^t, g, D) \text{ for } t \geq 1 \\ \dfrac{1-v}{1-\delta} & \text{if } h = (g, y^1, R) \\ \sum_{\tau=1}^{t} \delta^{\tau-1}(1 - x^\tau) + \delta^t \dfrac{1-v}{1-\delta} & \text{if } h = (e^1, e^2, \ldots, e^t, g, y^{t+1}, R) \text{ for } t \geq 1 \\ \sum_{\tau=1}^{\infty} \delta^{\tau-1}(1 - x^\tau) & \text{if } h = (e^1, e^2, \ldots) \end{cases}$$

where $x^\tau = z^\tau$ if $e^\tau = (b, z^t)$ and $x^\tau = y^\tau$ if $e^\tau = (g, y^\tau, A)$.

For convenience, I say that after a history ending in b, the game is "in state b", and after a history ending in g, it is "in state g".

15.1.2 Subgame perfect equilibrium

If (*i*) the amount of wealth destroyed by a revolt, v, is less than $\delta \bar{z}$, (*ii*) *Rich* receives a positive payoff under democracy ($d > 0$), and (*iii*) environments conducive to revolt are rare (q is small), the game has a unique subgame perfect equilibrium, in which *Rich* democratizes in state g and takes as much of the wealth as it can ($\bar{z}$) in state b, and *Poor* revolts whenever *Rich* does not democratize. The outcome is that *Rich* democratizes on the first occurrence of an environment favorable for revolt.

In this equilibrium, unlike the equilibria of the static model, *Rich* democratizes to stave off revolt. Given that *Rich* gives *Poor* as little wealth as possible in state b, even if it offers *Poor* all the wealth in state g, *Poor* optimally revolts, because the expected time before the state is g again is large. As a consequence, whenever the state is g, *Rich* optimally democratizes to avoid being impoverished by revolution.

Here is the argument that the game has no other subgame perfect equilibrium. If *Rich* takes as much wealth as possible in state b and democratizes in state g, its payoff is independent of the strategy of *Poor*, so this payoff is a lower bound for its payoff in a subgame perfect equilibrium. This payoff is relatively large when the probability q of an environment good for revolt is small. Now, in each period the total wealth available is at most 1, so the lower bound for the equilibrium payoff of *Rich* implies an upper bound for the equilibrium payoff of *Poor*, which is thus relatively small when q is small. As a consequence, if q is small, *Poor* revolts regardless of how much wealth *Rich* offers it in a period in which the state is g, because even if *Rich* offers it all the wealth in such a period its expected payoff from accepting the offer and then getting the discounted value of its highest payoff in the game is less than its payoff if it revolts. Finally, given that *Poor* revolts even if *Rich* offers it all the wealth in state g, *Rich* optimally democratizes and, because its payoff is thus independent of the strategy of *Poor*, takes as much of the wealth as possible whenever the state is b.

Before stating the result characterizing the subgame perfect equilibrium when q is small, it is useful to establish bounds on the payoff of *Rich* in a subgame perfect equilibrium.

Lemma 15.1: Bounds on payoff of *Rich* in subgame perfect equilibrium of dynamic game of revolt

The payoff V^R of *Rich* in any subgame perfect equilibrium of a dynamic game of revolt $\langle\{Rich, Poor\}, q, \delta, v, d, \bar{z}\rangle$ with $d \leq v$ satisfies

$$\frac{(1-q)(1-\delta)\bar{z}+qd}{(1-\delta)(1-\delta(1-q))} \leq V^R \leq \frac{(1-\delta)(1-q)\bar{z}+qv}{(1-\delta)(1-\delta(1-q))}. \tag{15.1}$$

Proof

In a subgame following a history ending in g, *Poor* gets a payoff of at least $(1-v)/(1-\delta)$ by choosing R after every history ending in (g,y) for any value of y (given $1-d > 1-v$). Thus its payoff in any subgame perfect equilibrium is at least V^P where

$$V^P = (1-q)(1-\bar{z}+\delta V^P) + q(1-v)/(1-\delta)$$

or

$$V^P = \frac{(1-\delta)(1-q)(1-\bar{z}) + q(1-v)}{(1-\delta)(1-\delta(1-q))}.$$

The sum of the players' payoffs in each period is at most 1, so that the sum of their total payoffs in the game is at most $1/(1-\delta)$, and hence the payoff of *Rich* in any subgame perfect equilibrium is at most

$$\frac{1}{1-\delta} - \frac{(1-\delta)(1-q)(1-\bar{z}) + q(1-v)}{(1-\delta)(1-\delta(1-q))} = \frac{(1-\delta)(1-q)\bar{z} + qv}{(1-\delta)(1-\delta(1-q))}.$$

If *Rich* uses the strategy that chooses $\bar{z}$ following every history ending in b and D after every history ending in g, its expected payoff V^R satisfies

$$V^R = (1-q)(\bar{z}+\delta V^R) + q\frac{d}{1-\delta},$$

regardless of the strategy of *Poor*, so that V^R is at least the expression on the left-hand side of (15.1).

The next result characterizes the subgame perfect equilibrium a dynamic game of revolt when q is small.

Proposition 15.1: Subgame perfect equilibrium of dynamic game of revolt

Let $\langle\{Rich, Poor\}, q, \delta, v, d, \bar{z}\rangle$ be a dynamic game of revolt with $v < \delta\bar{z}$, and let

$$q^* = \frac{(1-\delta)(\delta\bar{z}-v)}{\delta((1-\delta)\bar{z}+v-d)}.$$

If $q \le q^*$ then the following strategy pair is a subgame perfect equilibrium, and if $d > 0$ and $q < q^*$ is the only subgame perfect equilibrium.

- The strategy of *Rich* assigns $\bar{z}$ to every history ending in b and D to every history ending in g. (*Rich* takes as much wealth as possible whenever the environment is bad for revolt and democratizes whenever it is good for revolt.)

- For every number $y \in [0, 1]$, the strategy of *Poor* assigns R to every history ending in (g, y). (If, when the environment is good for revolt, *Rich* does not democratize, *Poor* revolts regardless of how much wealth *Rich* offers it.)

The payoff of *Rich* in this equilibrium is given by the left-hand side of (15.1).

Proof

The payoffs satisfy the condition in Proposition 16.9, so a strategy pair is a subgame perfect equilibrium if and only if it satisfies the one-deviation property.

I first argue that the strategy pair in the result satisfies the one-deviation property. That is, neither player can increase its payoff by changing its action after any history, given that it follows its strategy subsequently and the other player follows its strategy.

Action of *Rich* after history ending in b

If *Rich* deviates from $\bar{z}$ to any other number then it obtains less in the period of its deviation and the same in every subsequent period, so that it is worse off.

Action of *Rich* after history ending in g

If *Rich* deviates from D to any number y then its payoff in the resulting subgame is 0 rather than $d/(1-\delta)$, so it is no better off.

Action of *Poor* after history ending in (g, y)

If *Poor* follows its strategy and chooses R, its payoff in the resulting subgame is $(1-v)/(1-\delta)$. If it deviates and chooses A, its payoff is $1 - y + \delta V^P$, where V^P is its payoff from the strategy pair at the start of the game, so

$$V^P = (1-q)(1-\bar{z}+\delta V^P) + q\frac{1-d}{1-\delta},$$

and hence

$$V^P = \frac{(1-\delta)(1-q)(1-\bar{z}) + q(1-d)}{(1-\delta)(1-\delta(1-q))}.$$

Thus *Poor* is no better off deviating than choosing R if and only if

$$1-y+\frac{\delta(1-\delta)(1-q)(1-\bar{z})+\delta q(1-d)}{(1-\delta)(1-\delta(1-q))} \leq \frac{1-v}{1-\delta}.$$

This condition is satisfied for all values of y if and only if $q \leq q^*$.

I now argue that if $q < q^*$ then the game has no other subgame perfect equilibrium.

Step 1 *The expected payoff of Poor in every subgame perfect equilibrium is at most*

$$\frac{(1-\delta)(1-q)(1-\bar{z})+q(1-d)}{(1-\delta)(1-\delta(1-q))}.$$

Proof. The sum of the players' payoffs in each period is at most 1, so that the sum of their total payoffs in the game is at most $1/(1-\delta)$. The result follows from Lemma 15.1. ◁

Step 2 *If $q < q^*$ then in every subgame perfect equilibrium the strategy of Poor chooses R after every history ending in (g, y) for any $y \in [0,1]$.*

Proof. If *Poor* chooses A after such a history, its payoff in the subgame following the history is at most $1+\delta M^P$, where M^P is its highest payoff in a subgame perfect equilibrium of the game. If it chooses R, its payoff is $(1-v)/(1-\delta)$. Thus using the expression for M^P in Step 1, its choosing R is optimal if

$$\frac{1-v}{1-\delta} > 1+\delta\frac{(1-\delta)(1-q)(1-\bar{z})+q(1-d)}{(1-\delta)(1-\delta(1-q))},$$

which is equivalent to $q < q^*$. ◁

Step 3 *If $q < q^*$ then in every subgame perfect equilibrium the strategy of Rich chooses D after every history ending in g.*

Proof. By Step 2, if *Rich* chooses any value of y after a history ending in g its payoff is 0, whereas if it chooses D after such a history its payoff is $d/(1-\delta) > 0$. ◁

Step 4 *If $q < q^*$ then in every subgame perfect equilibrium the strategy of Rich chooses $\bar{z}$ after every history ending in b.*

Proof. Suppose that *Rich* chooses a value of z less than $\bar{z}$ after a history ending in b. If it deviates to $\bar{z}$, its payoff in the period increases and by Step 3 its payoff in every future period is unaffected. Thus the deviation increases its payoff. ◁

If good environments are sufficiently common (q is large), the game has a subgame perfect equilibrium in which democratization does not occur. In this equilibrium, *Rich* staves off a revolution by redistributing wealth in environments good for revolt, even though it redistributes as little wealth as possible in environments bad for revolt.

Exercise 15.1: Subgame perfect equilibrium of dynamic game of revolt without democratization

Show that the following strategy pair, where

$$y^* = \frac{v - \delta(1-q)\bar{z}}{1 - \delta(1-q)},$$

is a subgame perfect equilibrium of a dynamic game of revolt $\langle\{Rich, Poor\}, q, \delta, v, d, \bar{z}\rangle$ if $0 < d < v < \bar{z}$ and $q \geq 1 - v/(\delta\bar{z})$.

- The strategy of *Rich* assigns $\bar{z}$ to every history ending in b and y^* to every history ending in g. (*Rich* takes as much wealth as possible whenever the environment is bad for revolt and takes y^* whenever it is good for revolt.)
- The strategy of *Poor* assigns R to every history ending in (g, y) with $y > y^*$ and A to every history ending in (g, y) with $y \leq y^*$. (If, when the environment is good for revolt, *Rich* does not democratize, *Poor* revolts if *Rich* takes more than y^* and otherwise accepts the amount *Rich* offers it.)

The payoff of *Rich* in this equilibrium is given by the right-hand side of (15.1).

The upper bound q^* on q in Proposition 15.1 is less than the lower bound $1 - v/(\delta\bar{z})$ in Exercise 15.1, given that $d < v$. If q lies between these bounds then the game has no stationary subgame perfect equilibrium in pure strategies. That is, it has no subgame perfect equilibrium in which *Rich* chooses the same action after every history ending in g and the same action after every history ending in b, and for each $y \in [0, 1]$, *Poor* acts in the same way after every history ending in (g, y). For such values of q it has a stationary subgame perfect equilibrium in mixed strategies: *Rich* chooses $\bar{z}$ after a history ending in b and chooses both D and $y = 0$ with positive probability and all other values of y with probability 0 after a history ending in g, and *Poor* chooses both R and A with positive probability after a history ending in $(g, 0)$, and R with probability 1 after a history ending in (g, y) for any $y > 0$. For $q \in [q^*, 1 - v/(\delta\bar{z}))$ there is one such equilibrium. In this

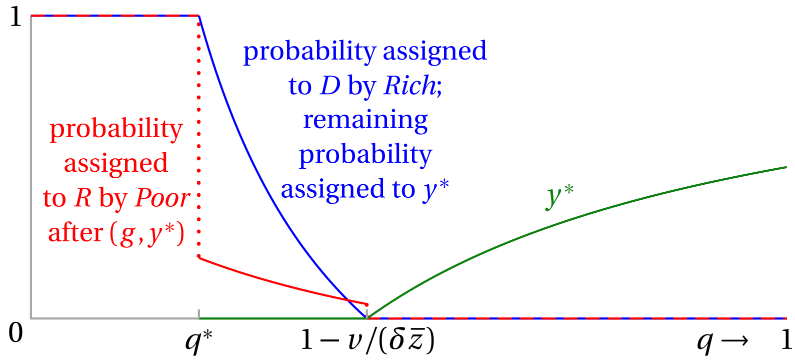


Figure 15.3 An example of subgame perfect equilibrium in a dynamic game of revolt $\langle\{Rich, Poor\}, q, \delta, v, d, \bar{z}\rangle$ for $\delta = 0.9$, $v = 0.5$, $d = 0.4$, and $\bar{z} = 1$, as a function of q, where $(y^*, 1 - y^*)$ is the distribution proposed by *Rich* after a history ending in g. The equilibrium is the one described in Proposition 15.1 for $q < q^*$, in Exercise 15.1 for $q \geq 1 - v/(\delta\bar{z})$, and in the discussion following Exercise 15.1 for $q^* < q \leq 1 - v/(\delta\bar{z})$.

equilibrium, after a history ending in g, *Rich* is indifferent between choosing D and offering *Poor* all the wealth, and hence its payoff is given by the left-hand side of (15.1). For $q = 1 - v/(\delta\bar{z})$ there is a continuum of such equilibria. In each of these equilibria, after a history ending in g, *Rich* offers *Poor* all the wealth and then *Poor* chooses R with probability at least $(v-d)(1-\delta)/(v(1-\delta)+(\delta - v/\bar{z})d)$. *Poor* is indifferent between R and A, and the payoff of *Rich* lies in the interval given in (15.1).

An example of the subgame perfect equilibria characterized in these results, as a function of q, is illustrated in Figure 15.3.

15.1.3 Commitment to redistribution

Consider a dynamic game of revolt $\langle\{Rich, Poor\}, q, \delta, v, d, \bar{z}\rangle$ with $0 < d < v < \delta$ in which $\bar{z} = 1$, so that *Rich* is unconstrained in the amount of wealth it can take in state b. If q is small, then by Proposition 15.1 in any subgame perfect equilibrium *Rich* democratizes the first time the state is g. Can this action be ascribed to the inability of *Rich* to commit to redistribution in state b?

If *Rich* were to commit to redistribute wealth in state b, *Poor* might choose to accept an offer of all the pie rather than revolting in state g, so that *Rich* would optimally make that offer rather than democratizing. However, this commitment has a cost: *Rich* has to give wealth to *Poor* whenever the state is b. Specifically, to be persuaded not to revolt in state g, *Poor* has to be assured of a future average per-period payoff of at least $1 - v$, in which case the average per-period payoff of *Rich* is at most v. Thus *Rich* has a choice between getting all the wealth in every period as long as the state is b and then the fraction d in every future

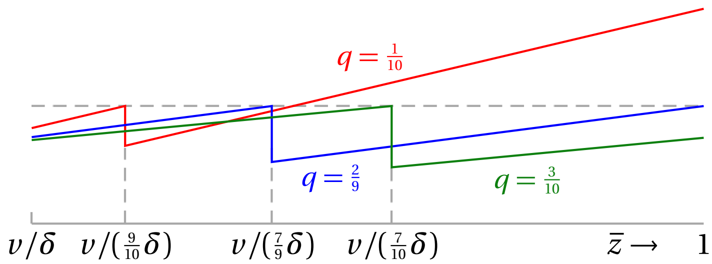


Figure 15.4 The payoff of *Rich* in subgame perfect equilibria of a dynamic game of revolt $\langle\{Rich, Poor\}, q, \delta, v, d, \bar{z}\rangle$ for $\delta = 0.9$, $v = 0.5$, and $d = 0.4$, as a function of $\bar{z}$ for $q = \frac{1}{10}$, $\frac{2}{9}$, and $\frac{3}{10}$. The equilibrium depicted is the one described in Proposition 15.1 for $\bar{z} > z^*$ (the only equilibrium in this case), in Exercise 15.1 for $\bar{z} < v/(\delta(1-q))$, and in the discussion following Exercise 15.1 for $v/(\delta(1-q)) \leq \bar{z} < z^*$.

period starting in the first period in which the state is g, or a per-period average of v in the whole game. If v is not much larger than d and q is small, so that the expected number of periods before the state is g is large, the former payoff is higher than the latter, and hence *Rich* prefers not to take the option of committing to redistribution when the state is b.

We can give a more precise answer to the question by considering the game in which *Rich* first chooses $\bar{z}$, and then *Rich* and *Poor* engage in a dynamic game of revolt $\langle\{Rich, Poor\}, q, \delta, v, d, \bar{z}\rangle$. By Proposition 15.1, if $q \leq q^*$, or equivalently

$$\bar{z} \geq \frac{(1-\delta(1-q))v - \delta q d}{\delta(1-\delta)(1-q)} = z^*$$

(which exceeds $v/(\delta(1-q))$), then the payoff of *Rich* in the unique subgame perfect equilibrium of this game is given by the left-hand side of (15.1). If $\bar{z} \in \big(v/\delta, v/(\delta(1-q))\big)$ then the game has the equilibrium described in Exercise 15.1, in which the payoff of *Rich* is given by the right-hand side of (15.1). If $\bar{z}$ lies between $v/(\delta(1-q))$ and z^* then the game has a stationary subgame perfect equilibrium in which each player randomizes between two actions, as discussed in the paragraph following Exercise 15.1. In this equilibrium, like the equilibrium for $\bar{z} \geq z^*$, the payoff of *Rich* is given by the left-hand side of (15.1).

Thus the equilibrium payoff of *Rich* is given by the right-hand side of (15.1) for $\bar{z} \in \big(v/\delta, v/(\delta(1-q))\big)$ and by the left-hand side if $\bar{z} > v/(\delta(1-q))$. For $\bar{z} = v/(\delta(1-q))$, the set of equilibrium payoffs is the interval defined in (15.1), and for $\bar{z} < v/\delta$, the equilibrium payoff is at most the right-hand side of (15.1). Figure 15.4 illustrates the resulting form of the equilibrium payoff as a function of $\bar{z}$ for the values of δ, v, and d used for Figure 15.3 and $q = \frac{1}{10}$, $\frac{2}{9}$, and $\frac{3}{10}$.

Thus in the game in which *Rich* chooses $\bar{z}$ and then *Rich* and *Poor* engage in the dynamic game of revolt for that value of $\bar{z}$, *Rich* chooses either $\bar{z} = 1$ or

$\bar{z} = v/(\delta(1-q))$. The former is better if and only if

$$\frac{(1-q)(1-\delta)+qd}{(1-\delta)(1-\delta(1-q))} > \frac{(1-\delta)v/\delta+qv}{(1-\delta)(1-\delta(1-q))},$$

or

$$q < \frac{(\delta-v)(1-\delta)}{\delta(v-d+1-\delta)}.$$

The right-hand side of this inequality is the value of q^* for $\bar{z}=1$ (which is $\frac{2}{9}$ for the parameter values in Figure 15.4). Thus for all values of q for which a dynamic game of revolt has a pure strategy equilibrium in which, without the ability to commit to redistribution in state b, *Rich* democratizes in state g, *Rich* optimally does not commit if it has the option to do so. Hence for such values of q, the fact that *Rich* democratizes in state g cannot be ascribed to its inability to commit to redistribution in state b. For values of q for which the dynamic game of revolt has a mixed strategy equilibrium when $\bar{z}=1$ ($q^* \leq q \leq 1-v/\delta$), like $q=\frac{3}{10}$ for the parameter values in Figure 15.4, *Rich* does prefer to commit to redistribute wealth in state b rather than selecting $\bar{z}=1$. In this case, by choosing $\bar{z}=v/(\delta(1-q))$, *Rich* induces a dynamic game of revolt that has a subgame perfect equilibrium in which *Poor* accepts the proposal in which it receives all the wealth (and rejects every other proposal).

15.1.4 Coups

In a dynamic game of revolt, the action D of *Rich* is an endpoint; once it is chosen, there is no going back. Tying the hands of *Rich* in this way seems artificial: why does it not have the option to re-take power?

Suppose that we modify the game by giving *Rich* the option during democracy of carrying out a coup, which returns it to power. Specifically, consider the game illustrated in Figure 15.5, an extension of a dynamic game of revolt in which, for simplicity, $\bar{z}=1$. In a period in which *Rich* chooses D, the payoff pair is $(d, 1-d)$, and in each subsequent period the state is either conducive to a coup (g') or not (b'). In state g', *Rich* can mount a coup (C) or acquiesce (A). In a period in which it mounts a coup, some wealth is destroyed—each player receives the fraction ϕ of its payoff under democracy—and *Rich* takes over control of the distribution of wealth.

If *Rich* never mounts a coup, it obtains the payoff d in every future period, so its payoff in every subgame perfect equilibrium is at least its payoff in every subgame perfect equilibrium of a dynamic game of revolt. As a consequence, the inequality goes in the other direction for *Poor*, so that whenever R is optimal for *Poor* in the original game it is optimal in the game in Figure 15.5. In particular, by

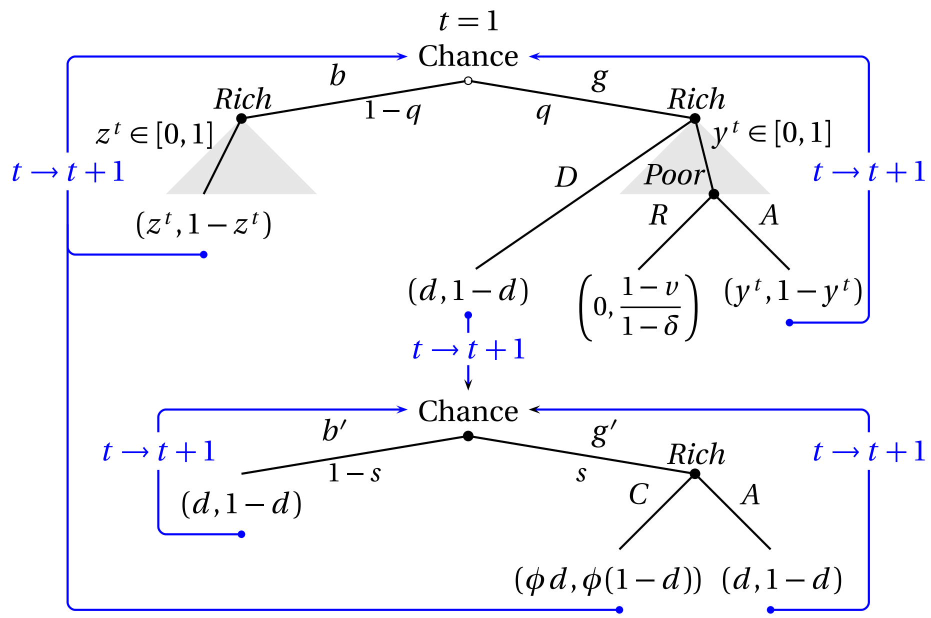


Figure 15.5 A variant of a dynamic game of revolt in which *Rich* has the option, during democracy, of carrying out a coup. In each payoff pair, the first number is the payoff of *Rich*.

Proposition 15.1, if $q \leq q^*$ then *Poor* optimally chooses R for all values of y^t. In any subgame perfect equilibrium in which *Poor* uses this strategy, *Rich* optimally chooses $z = 1$ after every history ending in b and D after every history ending in g.

If $q \leq q^*$, does the game have a subgame perfect equilibrium in which the players' strategies have these features and *Rich* never mounts a coup? The answer turns out to be negative. Even if $\phi = 0$, the condition $d < v$ implies that *Rich* can increase its payoff by deviating from A to C after a history ending in g'. However, if ϕ is large enough the game *does* have a subgame perfect equilibrium in which the players' strategies have these features and *Rich* mounts a coup after every history ending in g'. To see why, note that if $\phi = 1$ then mounting a coup is advantageous: it does not diminish the payoff of *Rich* in the period in which it takes place, and in restoring its dictatorial power over the distribution of wealth benefits it until the state is g again. In this equilibrium, periods of democracy are interspersed with periods of autocratic rule by *Rich*.

15.2 Coordinating rebellion

A dynamic game of revolt models the poor as a unitary actor, which decides whether to accept a proposal or rebel. In this section I present a model with a large number of citizens, who face a coordination problem: rebellion is worthwhile for a citizen only if it is successful, which requires that sufficiently many citizens rebel. The ruler's choice of how much wealth to assign to the masses is the trigger for a rebellion. If the masses observe only a noisy signal of this choice then their ability to coordinate their actions is limited and the most they receive in an equilibrium is less than the most they receive when the dictator's choice is perfectly observable (Proposition 15.2, Proposition 15.3). Giving the dictator the option of calling an election, the exercise of which (or lack thereof) is perfectly observed, facilitates coordination and restores an equilibrium in which the masses receive as much as they do when they perfectly observe the dictator's choice of how much wealth to assign them (Section 15.2.3).

The structure of the model is depicted in Figure 15.6. In each of an infinite sequence of periods, one of a countably infinite set of oligarchs interacts with a continuum of citizens. In each period, the ruling oligarch—the current dictator—chooses how much of a pie of size w to devote to public goods, and consumes the remainder. (One interpretation of the public good is that it represents the effort the dictator expends in governing the society.) In each period, each citizen's economic fortune depends on her *state*, which is random. When the dictator chooses y in some period t, each citizen's state in that period is y^t with probability α and 0 with probability $1-\alpha$, independently of every other citizen's state. Each citizen observes her state in each period; her payoff depends on this state in a way that I specify subsequently.

In one version of the model, each citizen observes the dictator's action in each period, and in another version, she does not. If she does not, then when she observes the state 0 she does not know whether the dictator chose (*i*) $y=0$ or (*ii*) a positive value of y and she was unlucky. In both cases, she rebels or acquiesces; the citizens make their choices simultaneously. The more citizens rebel, the more likely the rebellion is to succeed. Specifically, when the fraction z^t of citizens rebel in period t, the probability that a rebellion succeeds in that period is $G(z^t)$, where G is an increasing function with $G(0)=0$ and $G(1)=1$ (rebellion surely fails if no citizen participates and surely succeeds if every citizen does so). If the rebellion succeeds, the dictator is ousted and replaced by the next oligarch in line. Otherwise the currently ruling oligarch continues as dictator. At the end of the period, all oligarchs and citizens observe the fraction of citizens who rebelled and the outcome of the rebellion (succeeded, failed).

In any given period t an oligarch obtains the payoff $w-y^t$ if she is the dictator in that period and chooses y^t, and the payoff 0 if she is not the dictator in that period. Her total payoff is the discounted sum of her payoffs in all periods, with discount factor $\delta \in (0,1)$. If she is the dictator in every period and chooses y in every period, for example, her total payoff is $(w-y)/(1-\delta)$.

Each citizen's payoff in any given period depends on her action in the period,

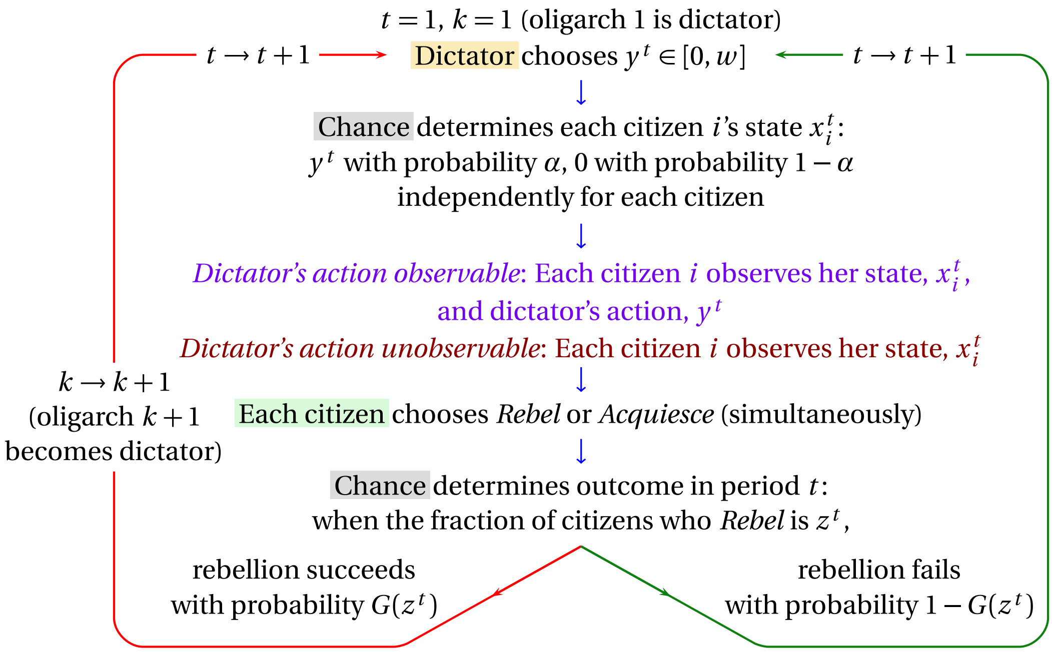


Figure 15.6 The structure of the interaction between the oligarchs and the citizens in an oligarchic society.

the fraction of citizens who rebel in the period, and the outcome of the rebellion, as specified in Table 15.1. Suppose that citizen i rebels in period t. Then if the rebellion succeeds, her payoff in the period is $x_i^t + b - L$, where x_i^t is her state in the period, $b > 0$ represents her benefit, material or psychological, of participating in a successful rebellion, and $L > 0$ represents her loss due to the disruption of the rebellion. If the rebellion fails, her payoff is $x_i^t - c - L$ if a positive fraction of citizens rebel and $x_i^t - c$ if the fraction of citizens who rebel is zero (isolated rebellious actions are individually costly, but do not cause significant aggregate damage), where $c > 0$. If citizen i acquiesces in period t, her payoff in the period is $x_i^t - L$ if a positive fraction of citizens rebel (regardless of whether the rebellion succeeds or fails) and x_i^t if the fraction of citizens who rebel is zero (in which case the rebellion surely fails).

The model in which each citizen observes only her own state in each period, not the dictator's action (the more interesting of the two cases), is captured by an extensive game with imperfect information with a set of players that is the union of a countably infinite set (the oligarchs) and a continuum (the citizens). Defining this game, and an appropriate solution concept for it, entails technical challenges. Rather than doing so, I define an oligarchic society as a list of parameters, without specifying the actions available to the members of the society, each member's information when she takes an action, or the members' payoffs,

Rebel			*Acquiesce*		
Rebellion succeeds	Rebellion fails		Rebellion succeeds	Rebellion fails	
	$z^t = 0$	$z^t > 0$		$z^t = 0$	$z^t > 0$
$x_i^t + b - L$	$x_i^t - c$	$x_i^t - c - L$	$x_i^t - L$	x_i^t	$x_i^t - L$

Table 15.1 The payoff of citizen $i \in I$ in any given period t in an oligarchic society $\langle I, K, w, b, c, L, \alpha, G, \delta \rangle$, where x_i^t is the citizen's state in period t and z^t is the fraction of citizens who choose *Rebel* in period t.

and subsequently define a notion of equilibrium for such a society that is motivated by the structure of interaction shown in Figure 15.6 and the considerations in game-theoretic notions of equilibrium, but is not cast as a specific type of equilibrium of a specific game.

Definition 15.2: Oligarchic society

An *oligarchic society* $\langle I, K, w, b, c, L, \alpha, G, \delta \rangle$ consists of

- I, a nonempty interval (the set of citizens, a continuum)
- $K = \{1, 2, \ldots\}$ (oligarchs, each of whom is a potential dictator)
- $w > 0$ (the amount of wealth available in each period)
- $b > 0$ (the increment to the payoff of a citizen who participates in a successful rebellion)
- $c > 0$ (the decrement to the payoff of a citizen who participates in an unsuccessful rebellion)
- $L > 0$ (the decrement to the payoff of each citizen if a positive fraction of citizens rebel)
- $\alpha \in [0, 1]$ (when the dictator chooses y, any given citizen's state is y with probability α and 0 with probability $1 - \alpha$, independently of the other citizens' states)
- $G : [0, 1] \to [0, 1]$, with $G(0) = 0$ and $G(1) = 1$, increasing on $(0, 1)$ ($G(z)$ is the probability a rebellion succeeds if the fraction of citizens who participate in it is z)
- $\delta \in (0, 1)$ (the oligarchs' common discount factor).

The notions of equilibrium that I define are intended to capture the considerations in a stationary equilibrium of a game in which every oligarch uses the same strategy and every citizen uses the same strategy: the behavior of each oligarch and each citizen is optimal, given the other players' actions, is unvarying over time, and for any given period depends only on the events within that period.

15.2.1 Dictator's action observable

I start with the case in which the citizens observe the dictator's action before choosing whether to rebel. I define a (stationary) equilibrium as a pair (y^*, σ^*), where $y^* \in [0, w]$ is the action of each oligarch in every period in which she is the dictator and $\sigma^* : [0, w] \to [0, 1]$ defines the probability $\sigma^*(y)$ with which each citizen rebels in any given period when the ruling oligarch chooses y in that period. (I do not include the citizen's state as an argument of σ^* because, as I argue subsequently, the form of the citizens' payoff functions means that each citizen's optimal action is independent of her state.)

First consider the optimality of y^*. Given σ^*, the fraction of citizens who rebel in a period in which the dictator chooses y is $\sigma^*(y)$, so the probability that rebellion succeeds is $p^*(y) = G(\sigma^*(y))$. Thus the total payoff $V(y)$ of the oligarch who rules in period 1 and chooses y in every period, regardless of history, is equal to $w - y + (1 - p^*(y))\delta V(y)$, so that

$$V(y) = \frac{w - y}{1 - \delta(1 - p^*(y))}.$$

For the oligarchs' common action y^* to be optimal, we need $V(y^*) \geq V(y)$ for all $y \in [0, w]$.

Now consider the optimality of σ^*. A citizen reasonably bases her decision on whether to rebel in any given period on her estimate of the probability that a rebellion will succeed. Suppose that her estimate of this probability is p. Then if her state is x and a positive fraction of citizens rebel, her payoff is

$$\begin{cases} p(x + b - L) + (1 - p)(x - c - L) & \text{if she rebels} \\ x - L & \text{if she acquiesces,} \end{cases}$$

so that, independently of her state, she optimally rebels if $p > c/(b + c)$, optimally acquiesces if $p < c/(b + c)$, and is indifferent between the two if $p = c/(b + c)$. Let $\widehat{p} = c/(b + c)$. Then the probability she assigns to rebelling is optimal if and only if it is 0 if $p < \widehat{p}$ and 1 if $p > \widehat{p}$; every probability is optimal if $p = \widehat{p}$. As for standard notions of equilibria in games, I assume that her belief about the probability p that a rebellion will succeed is correct. (Perhaps she has inferred this probability from her long experience playing the game or observing other people play the game or similar games.) That is, in an equilibrium in which every citizen chooses σ, she believes that if the dictator chooses y then the fraction $\sigma(y)$ of her compatriots will rebel, and hence the probability that a rebellion will succeed is $G(\sigma(y))$. (Given that the set of citizens is a continuum, her participation in the rebellion has no effect on its success.) These considerations lead to the following definition of an observable-action equilibrium.

Definition 15.3: Observable-action equilibrium of oligarchic society

An *observable-action equilibrium* of an oligarchic society $\langle I, K, w, b, c, L, \alpha, G, \delta\rangle$ is a pair (y^*, σ^*), where $y^* \in [0, w]$ (each oligarch's action) and $\sigma^* : [0, w] \to [0, 1]$ ($\sigma^*(y)$ is each citizen's probability of rebelling when the ruling oligarch chooses y), such that

$$V(y^*) \geq V(y) \quad \text{for all } y \in [0, w] \tag{15.2}$$

and

$$\sigma^*(y) = \begin{cases} 0 & \text{if } p^*(y) < \widehat{p} \\ 1 & \text{if } p^*(y) > \widehat{p} \end{cases} \quad \text{for all } y \in [0, w] \tag{15.3}$$

where

$$V(y) = \frac{w - y}{1 - \delta(1 - p^*(y))} \quad \text{for all } y \in [0, w]$$

(the payoff of an oligarch when she is in power and chooses y),

$$p^*(y) = G(\sigma^*(y)) \quad \text{for all } y \in [0, w]$$

(the probability of a successful rebellion when the ruling oligarch chooses y), and $\widehat{p} = c/(b + c)$.

The observability of the ruling oligarch's action allows the citizens to perfectly coordinate their actions, so that for any $y^* \in [0, \delta w]$ an oligarchic society has an observable-action equilibrium in which each citizen rebels if and only if $y < y^*$, and hence rebellion certainly succeeds if $y < y^*$ and certainly fails otherwise, and each oligarch devotes the amount y^* to public goods. (The restriction $y^* \leq \delta w$ ensures that the dictator cannot do better by deviating to $y = 0$, in which case she is in power for one period and obtains the payoff w.) There is also an observable-action equilibrium in which $y^* = 0$ and every citizen rebels whatever the ruling oligarch does.

Proposition 15.2: Observable-action equilibria of oligarchic society

Let $\langle I, K, w, b, c, L, \alpha, G, \delta\rangle$ be an oligarchic society.

a. Define $\sigma^0 : [0, w] \to [0,1]$ by $\sigma^0(y) = 0$ for all $y \in [0, w]$ (each citizen acquiesces for all values of y) and $\sigma^1 : [0, w] \to [0,1]$ by $\sigma^1(y) = 1$ for all $y \in [0, w]$ (each citizen rebels for all values of y). Both $(0, \sigma^0)$ and $(0, \sigma^1)$ are observable-action equilibria of the society.

b. Let $y^* \in [0, \delta w]$ and define $\sigma^* : [0, w] \to [0,1]$ by

$$\sigma^*(y) = \begin{cases} 0 & \text{if } y \geq y^* \\ 1 & \text{if } y < y^* \end{cases}$$

(each citizen rebels if and only if $y < y^*$). Then (y^*, σ^*) is an observable-action equilibrium of the society.

c. In every observable-action equilibrium (y, σ) of the society we have $y \leq \delta w$.

Proof

a. First consider $(0, \sigma^0)$. In the notation of Definition 15.3 we have $y^* = 0$ and $p^*(y) = 0$ for all y, so that $V(y) = (w - y)/(1 - \delta)$ for all y, and hence (15.2) is satisfied. Given $\widehat{p} > 0$, (15.3) is also satisfied. Now consider $(0, \sigma^1)$. We have $p^*(y) = 1$ for all y, so that $V(y) = w - y \leq V(0)$ for all y, so that (15.2) is satisfied. Given $\widehat{p} < 1$, (15.3) is also satisfied.

b. We have

$$p^*(y) = G(\sigma^*(y)) = \begin{cases} G(0) = 0 & \text{if } y \geq y^* \\ G(1) = 1 & \text{if } y < y^*. \end{cases}$$

Thus

$$V(y) = \begin{cases} (w - y)/(1 - \delta) & \text{if } y \geq y^* \\ w - y & \text{if } y < y^*. \end{cases}$$

Hence $V(y^*) \geq V(y)$ for all $y \in [0, w]$ if and only if $(w - y^*)/(1 - \delta) \geq w$, or $y^* \leq \delta w$. Thus (15.2) is satisfied.

Now, $\widehat{p} \in (0,1)$, so if $p^*(y) < \widehat{p}$ then $y \geq y^*$, so that $\sigma^*(y) = 0$, and if $p^*(y) > \widehat{p}$ then $y < y^*$, so that $\sigma^*(y) = 1$. Thus (15.3) is satisfied.

c. We have $V(0) = w/(1 - \delta(1 - p^*(0))) \geq w$, so by (15.2) for any observable-action equilibrium (y, σ) we have $V(y) \geq w$ and hence $y \leq \delta w(1 - p^*(y)) \leq \delta w$.

15.2.2 Dictator's action unobservable

Now consider the more interesting case in which the citizens do not observe the dictator's action. When the dictator's action is y, each citizen's state is y with probability α and 0 with probability $1-\alpha$, and each citizen observes only her state. I define a (stationary) equilibrium to be a pair (y^*, ρ^*), where $y^* \in [0, w]$ is the action of each oligarch in every period in which she is the dictator and $\rho^* : [0, w] \to [0, 1]$ defines the probability $\rho^*(x)$ with which each citizen rebels in any given period when her state is x.

The fraction of citizens who rebel when the dictator chooses y is now $\alpha\rho^*(y) + (1-\alpha)\rho^*(0)$. Thus the probability of a successful rebellion when the dictator chooses y is $G(\alpha\rho^*(y) + (1-\alpha)\rho^*(0))$, which I denote $q^*(y)$, and the total payoff $V(y)$ of the oligarch who rules in period 1 and chooses y in every period, regardless of history, is equal to $w - y + (1 - q^*(y))\delta V(y)$, so that

$$V(y) = \frac{w-y}{1-\delta(1-q^*(y))}.$$

As before, for the oligarchs' common action y^* to be optimal, we need $V(y^*) \geq V(y)$ for all $y \in [0, w]$.

Now consider the optimality of ρ^*. If a citizen's state is $x \in (0, w]$, she knows that the ruling oligarch has chosen x (and, in particular, if $x \neq y^*$ she knows that the ruling oligarch has deviated from y^*), and hence she knows that the state is x for the fraction α of citizens and 0 for the fraction $1-\alpha$. Thus given the citizens' common strategy ρ^*, she knows that the fraction of citizens who will rebel is $\alpha\rho^*(y) + (1-\alpha)\rho^*(0)$ and hence the probability of a successful rebellion is $q^*(y)$. If a citizen's state is 0, she has no information about the ruling oligarch's action, and in particular has no reason to believe that the ruling oligarch has deviated from y^*. In the spirit of the assumption of the consistency of beliefs with strategies in the standard solution concepts for extensive games with imperfect information, I assume that in this case she believes that the fraction of citizens who will rebel is $\alpha\rho^*(y^*) + (1-\alpha)\rho^*(0)$ and hence the probability of a successful rebellion is $q^*(y^*)$. Given that a citizen optimally rebels if she believes that the probability of success exceeds $\widehat{p}$ $(= c/(b+c))$, optimally acquiesces if she believes that this probability is less than $\widehat{p}$, and is indifferent between the two actions if she believes that this probability is $\widehat{p}$, we are led to the following definition of an equilibrium.

Definition 15.4: Unobservable-action equilibrium of oligarchic society

An *unobservable-action equilibrium* of a oligarchic society $\langle I, K, w, b, c, L, \alpha, G, \delta \rangle$ is a pair (y^*, ρ^*), where $y^* \in [0, w]$ (each oligarch's action) and $\rho^* : [0, w] \to [0, 1]$ (each citizen's probability of rebelling as a function of the state), such that

$$V(y^*) \geq V(y) \quad \text{for all } y \in [0, w], \tag{15.4}$$

$$\rho^*(0) = \begin{cases} 0 & \text{if } q^*(y^*) < \widehat{p} \\ 1 & \text{if } q^*(y^*) > \widehat{p}, \end{cases} \tag{15.5}$$

and

$$\rho^*(x) = \begin{cases} 0 & \text{if } q^*(x) < \widehat{p} \\ 1 & \text{if } q^*(x) > \widehat{p} \end{cases} \quad \text{for all } x \in (0, w], \tag{15.6}$$

where

$$V(y) = \frac{w - y}{1 - \delta(1 - q^*(y))} \quad \text{for all } y \in [0, w] \tag{15.7}$$

(the payoff of an oligarch when she is in power and chooses y),

$$q^*(y) = G(\alpha\rho^*(y) + (1-\alpha)\rho^*(0)) \quad \text{for all } y \in [0, w]$$

(the probability of a successful rebellion when the ruling oligarch chooses y), and $\widehat{p} = c/(b+c)$.

Analogues of the patterns of behavior in Proposition 15.2*a* are unobservable-action equilibria: if every citizen acquiesces regardless of her state, no rebellion ever occurs and the ruling oligarch optimally chooses 0 and stays in power indefinitely; if every citizen rebels regardless of her state, a rebellion succeeds in every period, and the ruling oligarch optimally chooses 0 and remains in power for only one period. Dismal outcomes, but equilibria nonetheless.

The analogue of the pattern of behavior in Proposition 15.2*b*, however, is not generally an equilibrium. Suppose that for some number x^* every citizen rebels when her state is less than x^* and acquiesces when it is at least x^*. Then if the dictator chooses x^*, the fraction $1-\alpha$ of citizens, whose state is 0, rebel, so that rebellion succeeds with probability $q^*(x^*) = G(1-\alpha)$. Thus if $G(1-\alpha) < \widehat{p}$ then rebelling is not optimal for the citizens whose state is 0 ((15.5) is violated) and if $G(1-\alpha) > \widehat{p}$ then acquiescing is not optimal for the citizens whose state is x^* ((15.6) is violated). Only if the parameters happen to satisfy $G(1-\alpha) = \widehat{p}$ is the pattern of behavior an equilibrium.

The fact that citizens whose state is 0 do not know the value of y chosen by the dictator hampers the disciplining of the dictator with rebellion. In an equilibrium in which the dictator chooses a positive value of y, a deviation by the dictator to $y = 0$ must increase the probability of rebellion, otherwise it would be advantageous. Such a deviation causes every citizen's state to become 0, so the probability of rebellion for a citizen with state 0 must be positive. As a consequence, rebellion occurs with positive probability also when the dictator does not deviate, because even then some citizens' states are (randomly) 0. Thus the impact of a deviation to 0 on the dictator's payoff is less than it is when her action is observable, and hence the maximum amount of wealth the dictator gives to the citizens in an equilibrium is also less than it is when her action is observable.

Further, there is a discrete difference between the case in which the dictator's action is observable and that in which it is observed by most but not all citizens (α is close to 1). The reason is that, independently of the value of α, for the probability of rebellion for a citizen with state 0 to be positive, as required in an equilibrium, the citizen's payoff to rebelling must be at least her payoff to acquiescing, which means that if the dictator adheres to the equilibrium and chooses $y = y^*$, the probability of success of a rebellion must be at least $\widehat{p}$. When α is close to 1, that means that the probability that a citizen with state y^* rebels is, at a minimum, not much less than the number r for which $G(r) = \widehat{p}$. By contrast, when the dictator's action is observable, no one rebels when the dictator adheres to her equilibrium strategy.

This argument leads to the conclusion that the largest value of y possible in an unobservable-action equilibrium is discretely less than δw, the value that Proposition 15.2b shows can be achieved in an observable-action equilibrium. Suppose that (y^*, ρ^*) is an equilibrium with $y^* > 0$. If, when the dictator chooses y^*, a successful rebellion occurs with probability less than $\widehat{p}$ then no one optimally rebels, and if a successful rebellion occurs with probability greater than $\widehat{p}$ then everyone optimally rebels; in both cases the dictator can increase her payoff by deviating to $y = 0$. Thus in an equilibrium (y^*, ρ^*) with $y^* > 0$, a successful rebellion occurs with probability $\widehat{p}$, and hence the dictator's payoff is

$$V(y^*) = \frac{w - y^*}{1 - \delta(1 - \widehat{p})}.$$

If she deviates to $y = 0$, her payoff is

$$V(0) = \frac{w}{1 - \delta(1 - G(\rho^*(0)))}.$$

For equilibrium we need $V(y^*) \geq V(0)$, which implies

$$y^* \leq \frac{\delta w(G(\rho^*(0)) - \widehat{p})}{1 - \delta(1 - G(\rho^*(0)))}, \tag{15.8}$$

so that regardless of the value of $\rho^*(0)$, the upper limit on y^* is less than δw.

In the equilibria defined in the result, all oligarchs choose some number y^* and if $G(1 - \alpha) \leq \widehat{p}$ then every citizen whose state is less than y^* rebels and some of those whose state is at least y^* do so, while if $G(1 - \alpha) > \widehat{p}$ then some of those whose state is 0 rebel, all of those whose state is between 0 and y^* do so, and none of those whose state is at least y^* do so.

Proposition 15.3: Unobservable-action equilibria of oligarchic society

Let $\langle I, K, w, b, c, L, \alpha, G, \delta \rangle$ be an oligarchic society and let $\widehat{p} = c/(b+c)$.

a. Define $\rho^0 : [0, w] \to [0, 1]$ by $\rho^0(x) = 0$ for all $x \in [0, w]$, and $\rho^1 : [0, w] \to [0, 1]$ by $\rho^1(x) = 1$ for all $x \in [0, w]$. Both $(0, \rho^0)$ and $(0, \rho^1)$ are unobservable-action equilibria of the society.

b. In every unobservable-action equilibrium (y^*, ρ^*) with $y^* > 0$,

$$G(\alpha\rho^*(y^*) + (1-\alpha)\rho^*(0)) = \widehat{p}$$

(the probability of a successful rebellion is $\widehat{p}$) and y^* satisfies (15.8).

c. Suppose that $G(1-\alpha) \leq \widehat{p}$. Let $y^* \in (0, \delta w(1-\widehat{p})]$, let r be the unique number satisfying $G(\alpha r + 1 - \alpha) = \widehat{p}$, and define $\rho^* : [0, w] \to [0, 1]$ by

$$\rho^*(x) = \begin{cases} 1 & \text{if } 0 \leq x < y^* \\ r & \text{if } y^* \leq x \leq w. \end{cases} \tag{15.9}$$

The pair (y^*, ρ^*) is an unobservable-action equilibrium of the society, and in every equilibrium (y, ρ) we have $y \leq \delta w(1-\widehat{p})$.

d. Suppose that $G(1-\alpha) > \widehat{p}$. Let y^* satisfy

$$0 < y^* \leq \frac{\delta w(G(r) - \widehat{p})}{1 - \delta(1 - G(r))}, \tag{15.10}$$

where r is the unique number satisfying $G((1-\alpha)r) = \widehat{p}$, and define $\rho^* : [0, w] \to [0, 1]$ by

$$\rho^*(x) = \begin{cases} r & \text{if } x = 0 \\ 1 & \text{if } 0 < x < y^* \\ 0 & \text{if } y^* \leq x \leq w. \end{cases} \tag{15.11}$$

The pair (y^*, ρ^*) is an unobservable-action equilibrium of the society, and in every equilibrium (y, ρ) the value of y is at most the right-hand side of (15.10).

Proof

a. First consider $(0,\rho^0)$. In the notation of Definition 15.4, we have $q^*(y) = G(0) = 0$ for all $y \in [0,w]$, so $V(y) = (w-y)/(1-\delta)$ for all $y \in [0,w]$. Thus (15.4) is satisfied. Given that $q^*(x) = 0$ for all $x \in [0,w]$, (15.5) and (15.6) are also satisfied. Hence $(0,\rho^0)$ is an unobservable-action equilibrium of the society.

Now consider $(0,\rho^1)$. We have $q^*(y) = G(1) = 1$ for all $y \in [0,w]$, so $V(y) = w - y$ for all $y \in [0,w]$. Thus (15.4) is satisfied, and (15.5) and (15.6) are also satisfied. Hence $(0,\rho^1)$ is an unobservable-action equilibrium of the society.

b. Let $q^*(y) = G(\alpha\rho^*(y) + (1-\alpha)\rho^*(0))$ for all $y \in [0,w]$, the probability of a successful rebellion when the dictator chooses y, as given in Definition 15.4. If $q^*(y^*) < \widehat{p}$ then $\rho^*(0) = \rho^*(y^*) = 0$ by (15.5) and (15.6), so that $q^*(0) = q^*(y^*) = 0$. If $q^*(y^*) > \widehat{p}$ then $\rho^*(0) = \rho^*(y^*) = 1$ by (15.5) and (15.6), so that $q^*(0) = q^*(y^*) = 1$. Using (15.7), given $y^* > 0$, in both cases we thus have $V(0) > V(y^*)$, contradicting (15.4). Thus $q^*(y^*) = \widehat{p}$.

The conclusion regarding the upper bound on y^* follows from the argument in the text.

c. I first argue that (y^*,ρ^*) is an unobservable-action equilibrium. Given (15.9), we have

$$q^*(x) = \begin{cases} 1 & \text{if } 0 \le x < y^* \\ G(\alpha r + 1 - \alpha) = \widehat{p} & \text{if } x \ge y^*, \end{cases}$$

so that (15.5) and (15.6) are satisfied. Also

$$V(y) = \begin{cases} w - y & \text{if } 0 \le y < y^* \\ \dfrac{w-y}{1-\delta(1-\widehat{p})} & \text{if } y \ge y^*, \end{cases}$$

so that (15.4) is satisfied, given $y^* \le \delta w(1-\widehat{p})$.

The right-hand side of (15.8) is increasing in $G(\rho^*(0))$, and hence attains its maximum when $G(\rho^*(0)) = 1$; this maximum is $\delta w(1-\widehat{p})$, so the last claim follows from part *b*.

d. I first argue that (y^*,ρ^*) is an unobservable-action equilibrium. We have

$$q^*(x) = \begin{cases} G(r) & \text{if } x = 0 \\ G(\alpha + (1-\alpha)r) & \text{if } 0 < x < y^* \\ G((1-\alpha)r) = \widehat{p} & \text{if } x \ge y^*, \end{cases}$$

so that (15.5) and (15.6) are satisfied. Also

$$V(y) = \begin{cases} \dfrac{w}{1-\delta(1-G(r))} & \text{if } y = 0 \\ \dfrac{w-y}{1-\delta(1-G(\alpha+(1-\alpha)r))} & \text{if } 0 < y < y^* \\ \dfrac{w-y}{1-\delta(1-\widehat{p})} & \text{if } y \geq y^*. \end{cases}$$

Now,

$$\frac{w}{1-\delta(1-G(r))} > \frac{w-y}{1-\delta(1-G(\alpha+(1-\alpha)r))}$$

for $y > 0$, so that (15.4) is satisfied if

$$\frac{w-y^*}{1-\delta(1-\widehat{p})} \geq \frac{w}{1-\delta(1-G(r))},$$

which is true given (15.10).

The right-hand side of (15.8) is increasing in $G(\rho^*(0))$, and hence attains its maximum at the largest value of $\rho^*(0)$ that is consistent with $q^*(y^*) = G(\alpha\rho^*(y^*) + (1-\alpha)\rho^*(0)) = \widehat{p}$. Given that $G(1-\alpha) > \widehat{p}$, this value is the number r such that $G((1-\alpha)r) = \widehat{p}$. Thus the right-hand side of (15.8) is equal to the right-hand side of (15.10).

If α is close to 0, then $G(r)$ is close to $\widehat{p}$, so by part d the largest value of y in an equilibrium is close to 0, as one would expect: if almost no one observes the dictator's action, rebellion is a blunt tool. The largest value of y in an equilibrium is also close to 0 if the benefit b that citizens receive from participating in a successful rebellion is small, or the cost c they incur from participating in an unsuccessful rebellion is large, both of which cause $\widehat{p}$ to be close to 1.

If α is close to, but less than, 1, then by part c the largest equilibrium value of y is at most $\delta w(1-\widehat{p})$, which if $b > 0$ is discretely less than its largest equilibrium value of δw in an observable-action equilibrium. That is, as I argued informally before the statement of the proposition, the presence of a small amount of imperfect information significantly reduces the largest equilibrium value of y.

In the equilibria in parts c and d of the result, a positive fraction of citizens rebel in every period. Thus each citizen's payoff in each period is $\alpha y^* - L$. In the equilibrium $(0, \rho^0)$ in part a, in which rebellion never occurs, each citizen's payoff is 0 in each period. Thus each citizen is better off in an equilibrium (y^*, ρ^*) as specified in part c or d than in the equilibrium $(0, \rho^0)$ if and only if $\alpha y^* > L$.

15.2.3 Using dictator's action as coordination device

To induce the oligarchs to choose values of y higher than the ones possible in unobservable-action equilibria, the citizens need to coordinate their rebellious impulses. One way for them to do so is simply to communicate: if the citizens who observe the value of y chosen by the dictator inform their compatriots, then an observable-action equilibrium can be implemented.

Another natural possibility is that the citizens condition their actions not only on events in the current period, but also on events in previous periods. That is, we could consider the possibility of an equilibrium in nonstationary strategies. To do so, we need to formulate a game and a solution concept precisely, which is challenging. Plausibly such a game has an equilibrium in which each oligarch chooses a number, say y^{**}, larger than the largest value of y in an unobservable-action equilibrium, with some citizens rebelling whenever their state is less than y^{**} and all citizens rebelling if the fraction of citizens who rebelled in the previous period was large enough to imply that the oligarch deviated and chose a value of y smaller than y^{**} in that period.

An additional possibility is that each citizen has an opportunity to take an action before choosing whether to rebel, and the number of citizens choosing the action is publicly observable. For example, each citizen might have the opportunity to demonstrate or to vote in a referendum. I describe such a model, adopting the latter interpretation.

Assume that in each period t, after each citizen i observes her state x_i^t, the ruling oligarch can give the citizens the opportunity to vote (simultaneously) to allow her to rule for another period. Suppose that the outcome of every vote is observed by all citizens. If the ruling oligarch does not give the citizens the opportunity to vote, or if she does so, a majority favors her resignation, and she does not resign, the citizens have the option to rebel. The structure of the interaction is shown in Figure 15.7.

I argue that this model has an equilibrium in which each oligarch chooses δw in every period, as in the observable-action equilibrium of an oligarchic society that is best for the citizens. That is, even though the citizens cannot observe the actions of the ruling oligarch, the model has an equilibrium in which their payoffs are as high as possible in an equilibrium of the model in which they can observe these actions.

A formal model that captures this interaction, like one that captures the interaction in Figure 15.6 when the dictator's action is unobservable, is an extensive game with imperfect information. A citizen whose state is 0 does not know the value of y that the dictator chose, and when deciding the action to take has to form a belief about this value. A full precise specification of the game is com-

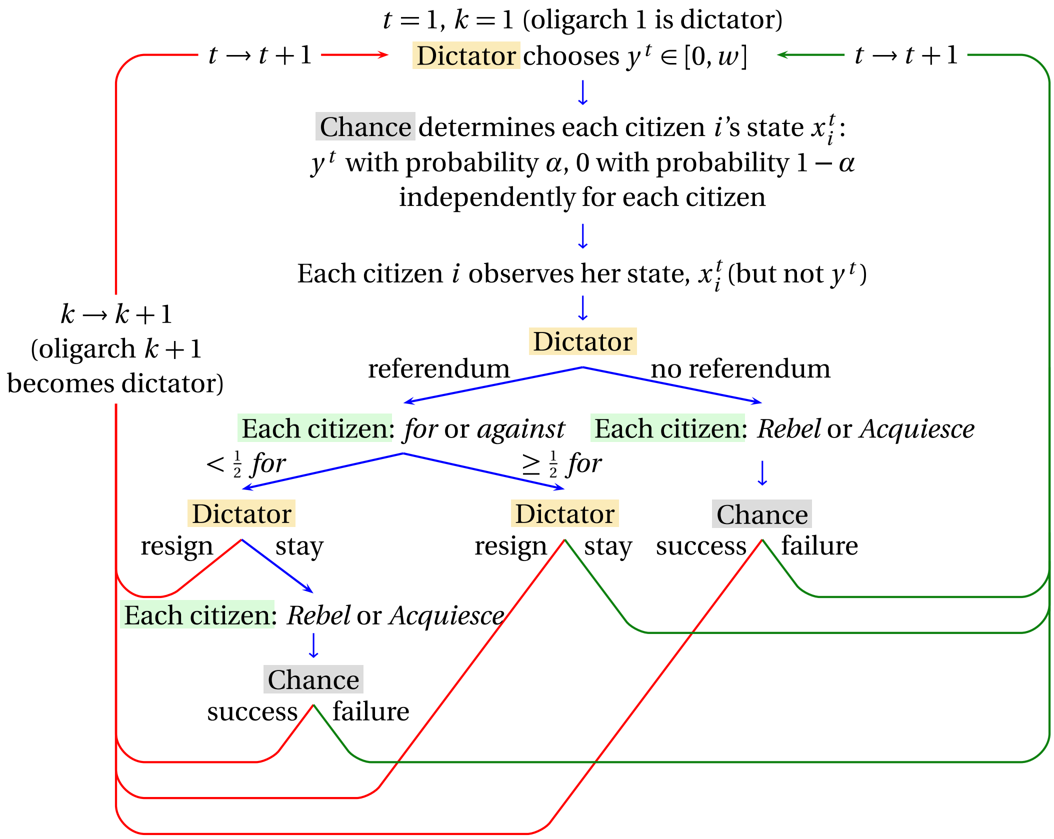


Figure 15.7 The structure of a dynamic game of rebellion with the option of a referendum. Whenever the citizens move, they do so simultaneously.

plex, and as for the earlier model, I do not provide one. In fact, in this case I do not specify a formal model beyond Figure 15.7, but merely sketch an informal analysis of equilibrium.

I argue that the (unformulated) game has an equilibrium in which after every history in which an oligarch is in power, she

- chooses δw
- holds a referendum
- resigns if and only if less than half of the citizens vote in her favor

and every citizen i

- rebels after any history that ends with the ruling oligarch's (*i*) not holding a referendum or (*ii*) holding one, losing the vote, and choosing to stay in power

- whenever her state in any period t is x_i^t,

$$\text{if } \alpha > \tfrac{1}{2} \quad \begin{cases} \text{votes } \textit{against} & \text{if } x_i^t < \delta w \\ \text{votes } \textit{for} & \text{if } x_i^t \geq \delta w \end{cases}$$

$$\text{if } \alpha \leq \tfrac{1}{2} \quad \begin{cases} \text{votes } \textit{against} \text{ with probability } \tfrac{1}{2} & \text{if } x_i^t = 0 \\ \text{votes } \textit{against} & \text{if } 0 < x_i^t < \delta w \\ \text{votes } \textit{for} & \text{if } x_i^t \geq \delta w. \end{cases}$$

Here is my argument.

Oligarchs

If a ruling oligarch adheres to her strategy, she chooses δw in every period, so that the state is δw for the fraction α of citizens and 0 for the fraction $1-\alpha$. The oligarch holds a referendum, which she wins: if $\alpha > \frac{1}{2}$ she obtains the fraction α of the votes, and if $\alpha \leq \frac{1}{2}$ she obtains the fraction $\alpha + \frac{1}{2}(1-\alpha) = \frac{1}{2} + \frac{1}{2}\alpha$. The oligarch remains in power and obtains $w - \delta w$ in every future period, so her payoff in the game is $(w - \delta w)/(1-\delta) = w$.

If a ruling oligarch deviates from her strategy by choosing δw in some period t but then not holding a referendum, all citizens rebel and the oligarch is thus removed from power. Her payoff in the game starting in period t in this case is $w - \delta w$, which is less than w.

If a ruling oligarch deviates from her strategy by choosing a number greater than δw in some period t, she is worse off in period t and, depending on her subsequent actions in the period, either remains in power or is removed from power, so that her payoff in the game starting in period t is less than $w - \delta w + \delta(w - \delta w)/(1-\delta) = w$.

Suppose that a ruling oligarch deviates from her strategy by choosing $y^t < \delta w$ in some period t. If she does not hold a referendum, all the citizens rebel and she is removed from power. If she does hold a referendum, she loses: if $\alpha > \frac{1}{2}$ then all citizens vote against her and if $\alpha \leq \frac{1}{2}$ then the fraction $\alpha + \frac{1}{2}(1-\alpha) = \frac{1}{2} + \frac{1}{2}\alpha$ does so. Having lost, she either resigns or stays and is removed from power by a rebellion. Thus she obtains the payoff $w - y^t$ in the game starting in period t. Among such deviations the best one for her is thus $y^t = 0$, which yields her the payoff w, equal to her payoff if she adheres to her strategy.

Citizens

If citizen i follows her strategy, she obtains x_i^t in every period t.

If citizen i deviates from her strategy by acquiescing rather than rebelling after a history that ends with the ruling oligarch's (*i*) not holding a referendum or (*ii*) holding one, losing the vote, and choosing to stay, her payoff in the period changes from $x_i^t + b - L$ to $x_i^t - L$ (see Table 15.1), so she is worse off.

If a citizen deviates from her strategy by changing her vote in some period, the outcome of the vote remains the same, given that there is a continuum of citizens.

Note that although the fact that no citizen's vote affects the outcome follows immediately from the assumption that there is a continuum of citizens, it remains true if the number of citizens is finite and large, because the margin of victory or loss is positive for all (positive) values of α.

Although the action that the dictator allows the citizens to take before deciding whether to rebel in this model is called voting in a referendum, any binary action could play the same role. The dictator could allow the citizens to participate in a demonstration, or prohibit them from doing so, and if she allows them to do so and more than half of them participate, they could all rebel if she does not resign. Or the dictator could allow them to take any other specific action, or prohibit them from doing so, and if she allows them to do so and more than half of them take the action, they could all rebel if she does not resign.

The action the citizens may be allowed to take acts as coordinating device: the fact that all citizens costlessly observe whether the action is allowed and the fraction of citizens who take it allows them to coordinate their actions. The original game, in which no such action is available, may have a nonstationary equilibrium in which a citizen's rebelling in one period is a costly signal of her state that is used by the citizens to coordinate rebellion in the following period, although the analysis of such equilibria seems complex.

Notes

Section 15.1 is based on Acemoglu and Robinson (2000) and Acemoglu and Robinson (2001). (Note that Proposition 1 in the first of these papers is incorrect; see Acemoglu and Robinson 2017. See Osborne 2024 for corrections of a different type.) Section 15.2 is based on Fearon (2011).

Solutions to exercises

Exercise 15.1

The payoffs satisfy the condition in Proposition 16.9, so a strategy pair is a subgame perfect equilibrium if and only if it satisfies the one-deviation property.

I now argue that the strategy pair in the result satisfies the one-deviation property. Given that $v < \bar{z}$, the value of y^* is increasing in q. It is zero for $q = 1 - v/(\delta\bar{z})$ and v for $q = 1$, and hence is in $[0,1]$ given the assumption that $q \geq 1 - v/(\delta\bar{z})$.

Action of *Rich* after history ending in b

If *Rich* deviates from $\bar{z}$ to any other amount then it obtains less in the period of its deviation and the same in every subsequent period, so that it is worse off.

Action of *Rich* after history ending in g

If *Rich* deviates from y^* to $y < y^*$ then it obtains less in the period of its deviation and the same in every subsequent period, so that it is worse off.

If *Rich* deviates to $y > y^*$ then its payoff in the resulting subgame is 0, so it is no better off (and is worse off if $y^* > 0$).

If *Rich* deviates to D then its payoff in the resulting subgame is $d/(1-\delta)$, so for the deviation not to increase its payoff we need

$$\frac{d}{1-\delta} \leq y^* + \delta V^R,$$

where V^R is its payoff from the strategy pair at the start of the game, so that

$$V^R = (1-q)(\bar{z} + \delta V^R) + q(y^* + \delta V^R),$$

and hence

$$V^R = \frac{(1-q)\bar{z} + qy^*}{1-\delta}.$$

Thus the condition for the deviation not to increase the payoff of *Rich* is

$$\frac{d}{1-\delta} \leq y^* + \frac{\delta(1-q)\bar{z} + \delta q y^*}{1-\delta}$$

or

$$d \leq \delta(1-q)\bar{z} + (1-\delta(1-q))y^*$$

or

$$y^* \geq \frac{d - \delta(1-q)\bar{z}}{1-\delta(1-q)},$$

which is satisfied, given the assumption that $d < v$.

Action of *Poor* after history ending in (g, y)

The payoff of *Poor* from accepting y is $1 - y + \delta V^P$, where V^P is its payoff from the strategy pair at the start of the game, so that

$$V^P = (1-q)(1-\bar{z} + \delta V^P) + q(1 - y^* + \delta V^P),$$

and hence

$$V^P = \frac{(1-q)(1-\bar{z}) + q(1-y^*)}{1-\delta}.$$

Thus its payoff from accepting y is

$$1 - y + \frac{\delta(1-q)(1-\bar{z}) + \delta q(1-y^*)}{1-\delta}.$$

If it chooses R, its payoff is $(1-v)/(1-\delta)$, so it optimally accepts $(y, 1-y)$ if

$$1 - y + \frac{\delta(1-q)(1-\bar{z}) + \delta q(1-y^*)}{1-\delta} \geq \frac{1-v}{1-\delta}$$

and optimally rejects it if the inequality is reversed.

Thus we need

$$1 - y^* + \frac{\delta(1-q)(1-\bar{z}) + \delta q(1-y^*)}{1-\delta} = \frac{1-v}{1-\delta},$$

which is satisfied given the value of y^*.

V Appendix

16 Preferences, profiles, games, and optimization 539

16 Preferences, profiles, games, and optimization

16.1 Preferences and payoffs 539
16.2 Sets of individuals and profiles 544
16.3 Brouwer's fixed-point theorem 545
16.4 Strategic games 545
16.5 Bayesian games 549
16.6 Extensive games 552
16.7 Coalitional games 562
16.8 Optimization 566

This appendix provides definitions of most of the formal concepts used in the book. It includes only brief explanations and discussions; it is intended only to remind you of the definitions. For detailed explanations and discussions using the same terminology and notation, see Osborne and Rubinstein (2023), Osborne (2004), and Osborne and Rubinstein (1994). (The first and last of these books, and several chapters of the second, are freely available in electronic form.)

16.1 Preferences and payoffs

A binary relation on a set X specifies, for each ordered pair (x, y) with $x \in X$ and $y \in X$, whether x and y are related in a certain way. For example, "at least", usually denoted $\geq$, is a binary relation on the set of real numbers: for every pair (x, y) of real numbers, $x \geq y$ means that x is at least y. Two other binary relations on the set of real numbers are "greater than" ($>$) and "equals" ($=$). Formally, a binary relation B on a set X is a subset of $X \times X$, the set of all ordered pairs (x, y) with $x \in X$ and $y \in X$; if $(x, y) \in B$ then x is related to y, and if $(x, y) \notin B$ then it is not. However, we usually write $x\,B\,y$ rather that $(x, y) \in B$, and commonly use a symbol resembling $\geq$ or $>$, like $\succsim$, $\succ$, $\trianglerighteq$, or $\triangleright$, rather than a letter, for a relation.

Definition 16.1: Binary relation

For any set X, a *binary relation on* X is a subset of $X \times X$, the set of ordered pairs (x, y) with $x \in X$ and $y \in X$. If $\succsim$ is a binary relation, we write $x \succsim y$ to mean $(x, y) \in \succsim$.

Here are two key properties that a binary relation may or may not possess.

Definition 16.2: Properties of binary relation

For any set X, a binary relation $\succsim$ on X is

- *complete* if for every $x \in X$ and $y \in X$ (with x and y not necessarily distinct) we have either $x \succsim y$ or $y \succsim x$ (or both)
- *transitive* if for every $x \in X$, $y \in X$, and $z \in X$ with $x \succsim y$ and $y \succsim z$ we have $x \succsim z$.

The binary relation $\geq$ on the set of real numbers is complete, whereas $>$ and $=$ are not; all three are transitive.

We model an individual's preferences over a set of alternatives as a complete transitive binary relation, say $\succsim$, interpreting $x \succsim y$ to mean that the individual likes x at least as much as y.

Definition 16.3: Preference relation

For any set X, a *preference relation on* X is a complete transitive binary relation on X.

Given any preference relation, two associated binary relations are defined as follows.

Definition 16.4: Binary relations associated with preference relation

Let X be a set and $\succsim$ a preference relation on X. The *strict preference relation* and *indifference relation* associated with $\succsim$ are the binary relations $\succ$ and $\sim$ defined by

$$\begin{aligned} x \succ y \quad &\Leftrightarrow \quad x \succsim y \text{ and not } y \succsim x \\ x \sim y \quad &\Leftrightarrow \quad x \succsim y \text{ and } y \succsim x. \end{aligned}$$

The strict preference relation $\succ$ and indifference relation $\sim$ associated with any preference relation $\succsim$ are transitive, and if $x \succsim y$ and $y \succ z$ then $x \succ z$. If $\succsim$ is an individual's preference relation then $x \succ y$ may be interpreted to mean that the individual prefers x to y, and $x \sim y$ may be interpreted to mean that she regards x and y as equally appealing. Whenever I introduce a preference relation denoted $\succsim$, I implicitly introduce also the strict preference relation and indifference relation associated with $\succsim$, which I denote by $\succ$ and $\sim$, and whenever I introduce a preference relation denoted $\trianglerighteq$, I implicitly introduce the strict preference relation associated with $\trianglerighteq$, which I denote $\triangleright$.

If a complete transitive binary relation $\succsim$ on a set X has the property that $x \sim y$ only if $x = y$, so that the members of X may be arranged in order according to $\succsim$ with no distinct members at the same position in the ordering (no members of X are tied), then it is called a linear order. ("Linear" because we can order the members along a line.)

Definition 16.5: Linear order

For any set X, a *linear order on* X is a complete transitive binary relation $\succsim$ on X with the property that whenever $x \in X$, $y \in X$, $x \succsim y$, and $y \succsim x$ we have $x = y$.

An example of a linear order on the set of real numbers is $\geq$ (at least). An example of a linear order on the set of n-vectors is the lexicographic order, defined as follows. ("Lexicographic" because for languages written using an alphabet, words in a dictionary are ordered in this way.)

Definition 16.6: Lexicographic order on $\mathbb{R}^n$

For any positive integer n, the *lexicographic order* on $\mathbb{R}^n$ is the binary relation $\geq_L$ defined by $x \geq_L y$ if and only if either $x_1 > y_1$ or for some $k \in \{1, \ldots, n-1\}$ we have $x_i = y_i$ for $i = 1, \ldots, k$ and $x_{k+1} > y_{k+1}$. This binary relation is complete and transitive, and is hence a preference relation.

If a preference relation is a linear order then it coincides with the strict preference relation associated with it. For convenience, in this case I say that the preference relation is strict.

Definition 16.7: Strict preference relation

For any set X, a preference relation $\succsim$ on X is *strict* if it is a linear order.

When analyzing an individual's behavior, it is often convenient to work with a function that attaches a number to each alternative and represents the individual's preference relation in the sense that the number it attaches to alternative x is larger than the number it attaches to alternative y if and only if the individual prefers x to y.

Definition 16.8: Payoff function that represents preference relation

Let X be a set and let $\succsim$ be a preference relation on X. The function $u : X \to \mathbb{R}$ *represents* $\succsim$ if

$$x \succsim y \text{ if and only if } u(x) \geq u(y).$$

Not every preference relation can be represented by a payoff function. For example, a lexicographic preference ordering cannot be so represented. A sufficient condition for a preference relation to be represented by a payoff function is that it is continuous.

Definition 16.9: Continous preference relation

Let X be a subset of a Euclidean space. A preference relation $\succsim$ on X is *continuous* if for every sequence $(x^j, y^j)_{j=1}^{\infty}$ for which $x^j \in X$, $y^j \in X$, $x^j \succsim y^j$ for all j, and $\lim_{j\to\infty} x^j$ and $\lim_{j\to\infty} y^j$ exist, we have $\lim_{j\to\infty} x^j \succsim \lim_{j\to\infty} y^j$. Equivalently, for every $x^* \in X$ the sets $\{x \in X : x \succsim x^*\}$ and $\{x \in X : x^* \succsim x\}$ are closed.

Proposition 16.1: Payoff function that represents preference relation

Let X be a subset of a Euclidean space and let $\succsim$ be a continuous preference relation on X. Then there is a continuous function $u : X \to \mathbb{R}$ that represents $\succsim$.

For a proof of a more general result see Debreu (1954), and for a proof of a special case see Mas-Colell et al. (1995, 47–49).

In models in which a decision-maker has preferences over the numbers in an interval, it is sometimes assumed that these preferences are represented by a single-peaked function, defined precisely as follows.

Definition 16.10: Single-peaked function

Let X be a convex subset of $\mathbb{R}$. A function $f : X \to \mathbb{R}$ is *single-peaked* if it is continuous and strictly quasiconcave. Equivalently, f is continuous and satisfies one of the following conditions.

- f is increasing
- f is decreasing
- there exists a number $x^* \in X$ such that f is increasing on $\{x \in X : x \leq x^*\}$ and decreasing on $\{x \in X : x \geq x^*\}$.

Uncertainty

A lottery over a set is a function that assigns a positive number to each of a finite number of members and 0 to the remaining members, with the sum of the numbers equal to 1. The numbers are interpreted as probabilities.

Definition 16.11: Lottery

For any set Z, a *lottery* over Z is a function $p : Z \to \mathbb{R}$ for which $p(z)$ is positive for a finite number of members of Z and 0 for all other members, with $\sum_{z \in Z} p(z) = 1$. The lottery p with $p(z_k) = p_k$ for $k = 1, \ldots, K$ and $p(z) = 0$ otherwise is denoted $p_1 \cdot z_1 \oplus p_2 \cdot z_2 \oplus \cdots \oplus p_k \cdot z_k$ and the lottery p for which $p(z) = 1$ for a single alternative $z \in Z$ is denoted $[z]$.

Whenever a preference relation on a set of lotteries over a set is used in this book, I assume that it may be represented by the expected value of a real-valued function on the set. Such preference relations were first systematically studied by von Neumann and Morgenstern (1944). A preference relation on a set of lotteries over a set is called a vNM preference relation if and only if it is continuous and has the independence property, defined as follows.

Definition 16.12: Continuous preference relation on set of lotteries

For any set Z, a preference relation $\succsim$ on the set of lotteries over Z is *continuous* if for any $a \in Z$, $b \in Z$, and $c \in Z$ such that $[a] \succ [b] \succ [c]$ there is a number α with $0 < \alpha < 1$ such that $[b] \sim \alpha \cdot a \oplus (1 - \alpha) \cdot c$.

Definition 16.13: Independence property for preference relation on set of lotteries

For any set Z, a preference relation $\succsim$ on the set of lotteries over Z satisfies the *independence property* if for any $K \geq 1$, any $z_1, \ldots, z_K$, a, and b in Z, any probabilities $\alpha_1, \ldots, \alpha_K$, and β, and any $k \in \{1, \ldots, K\}$ we have

$$[z_k] \succsim \beta \cdot a \oplus (1 - \beta) \cdot b$$

$$\Leftrightarrow$$

$$\alpha_1 \cdot z_1 \oplus \cdots \oplus \alpha_k \cdot z_k \oplus \cdots \oplus \alpha_K \cdot z_K \succsim \alpha_1 \cdot z_1 \oplus \cdots \oplus \alpha_k \cdot (\beta \cdot a \oplus (1 - \beta) \cdot b) \oplus \cdots \oplus \alpha_K \cdot z_K.$$

Definition 16.14: vNM preference relation

Let Z be a finite set. A preference relation $\succsim$ on the set $L(Z)$ of lotteries over Z is a *vNM preference relation* if there is a function $u : Z \to \mathbb{R}$ such that $\succsim$ is represented by the function $U : L(Z) \to \mathbb{R}$ defined by

$$U(p) = \sum_{z \in Z} p(z)u(z) \quad \text{for all } p \in L(Z).$$

Such a function u is called a *Bernoulli function* for $\succsim$.

Proposition 16.2: Representation of preferences on set of lotteries by expected value of payoff function

Let Z be a finite set. A preference relation $\succsim$ on the set $L(Z)$ of lotteries over Z is continuous and satisfies the independence property if and only if it is a vNM preference relation. If u is a Bernoulli function for $\succsim$, then $v : Z \to \mathbb{R}$ is also a Bernoulli function for $\succsim$ if and only if there is a number α and a positive number β such that $v(z) = \alpha + \beta u(z)$ for all $z \in Z$.

For a proof of the first part this result, see Propositions 3.1 and 3.2 of Osborne and Rubinstein (2023). For a proof of the second part, see Proposition 6.B.2 of Mas-Colell et al. (1995).

16.2 Sets of individuals and profiles

In some of the models in this book the set of individuals is identified with an interval of real numbers, to capture situations in which the number of individuals is large enough that each individual's behavior is insignificant relative to the totality of all individuals. In these cases, I sometimes need to refer to the size of a subset of individuals. For a subset of the interval that is a countable union of disjoint intervals, we can use the sum of the lengths of these intervals for this purpose. But extending this notion of size to all subsets while maintaining appealing properties for it is impossible. (See, for example, Royden 1968, Chapter 3.) I follow standard practice and restrict attention to Lebesgue-measurable subsets, taking the size of such a subset to be its Lebesgue measure (which, roughly, is the smallest total length of a collection of intervals whose union contains the subset). In particular, whenever I refer to a set of individuals in such a model, I mean a Lebesgue-measurable subset of the interval.

Given a set N of individuals, I use the term *profile* to refer to a collection of objects, one for each individual $i \in N$. For example, if each individual $i \in N$ has a preference relation $\succsim_i$, then the *preference profile* for the set of individuals is the collection of these preference relations; if each individual $i \in N$ is associated with the action a_i, then the *action profile* for the set of individuals is the collection of these actions. We can think of any such collection as a function that associates an object with each individual $i \in N$.

Definition 16.15: Profile

For a set N of individuals and any set S, a *profile* of members of S is a function that associates with each $i \in N$ a member of S.

One way to denote the profile that associates with each $i \in N$ the member s_i of S is $(s_i)_{i \in N}$. For any profile $(s_i)_{i \in N}$, (x_i, s_{-i}) denotes the profile that differs from $(s_i)_{i \in N}$ only in that the element for individual i is x_i rather than s_i (so that, in particular, $(s_i, s_{-i}) = (s_i)_{i \in N}$).

The notation $(s_i)_{i \in S}$ is most appealing if N is countable, but I use it also when N is uncountable (for example, an interval).

16.3 Brouwer's fixed-point theorem

The following result, due to Luitzen Egbertus Jan Brouwer (1881–1966), is used in some proofs that models have equilibria. For a proof of the result, see Smart (1974, Theorem 2.1.11).

Proposition 16.3: Brouwer's fixed point theorem

Let X be a compact convex subset of a Euclidean space and let $f : X \to X$ be a continuous function. Then f has a fixed point: there exists $x \in X$ with $f(x) = x$.

16.4 Strategic games

A strategic game is a model of interaction among the members of a set of decision-makers. Each decision-maker, called a player, chooses an action and cares about the actions chosen by all decision-makers.

Definition 16.16: Strategic game

A *strategic game* $\langle N, (A_i)_{i \in N}, (\succsim_i)_{i \in N} \rangle$ consists of

players
a set N

and for each player $i \in N$

actions

a set A_i

preferences

a preference relation $\succsim_i$ over the set $\times_{j \in N} A_j$ of *action profiles*.

For every $i \in N$, a function $u_i : \times_{j \in N} A_j \to \mathbb{R}$ that represents $\succsim_i$ is a *payoff function* for player i.

A Nash equilibrium of a strategic game is an action profile with the property that no player is better off choosing a different action, given the actions of the remaining players. One interpretation of a Nash equilibrium is that it corresponds to a steady state in an environment in which each decision-maker plays the game many times against other decision-makers chosen randomly from populations of potential players. No decision-maker observes the identity of any particular player, so no decision-maker can condition her action in any play of the game on the actions chosen previously by any other particular decision-maker. But every decision-maker knows, from her long experience playing the game, the actions that the other players will take in any occurrence of the game. (For more discussion, see Osborne 2004, 21–22.)

Definition 16.17: Nash equilibrium of strategic game

A *Nash equilibrium* of a strategic game $\langle N, (A_i)_{i \in N}, (\succsim_i)_{i \in N} \rangle$ is an action profile $(a_i)_{i \in N} \in \times_{i \in N} A_i$ for which for every player $i \in N$

$$(a_i, a_{-i}) \succsim_i (x_i, a_{-i}) \text{ for all } x_i \in A_i.$$

The following result gives sufficient conditions for a strategic game to have a Nash equilibrium. For a proof of the result, see Osborne and Rubinstein (1994, Proposition 20.3).

Proposition 16.4: Existence of Nash equilibrium in strategic game

A strategic game $\langle N, (A_i)_{i \in N}, (\succsim_i)_{i \in N} \rangle$ in which N is finite has a Nash equilibrium if for every $i \in N$

- the set A_i of actions of player i is a nonempty compact convex subset of a Euclidean space

and the preference relation $\succsim_i$

- may be represented by a continuous payoff function

- is quasiconcave on A_i:

$$\{a_i \in A_i : (a_i, a^*_{-i}) \succsim_i a^*\} \text{ is convex for every } a^* \in \times_{j \in N} A_j.$$

A strategic game may have more than one Nash equilibrium, but if the game has two players and their interests are opposed, the players' payoffs in every Nash equilibrium are the same.

Definition 16.18: Strictly competitive strategic game

A two-player strategic game $\langle\{1,2\}, (A_i)_{i\in\{1,2\}}, (\succsim_i)_{i\in\{1,2\}}\rangle$ is *strictly competitive* if $a \succsim_1 b$ if and only if $b \succsim_2 a$ for all $a \in A_1 \times A_2$ and $b \in A_1 \times A_2$.

Proposition 16.5: Unique Nash equilibrium payoffs in strictly competitive strategic game

Every Nash equilibrium of a strictly competitive strategic game yields the same pair of payoffs.

For a proof of this result, see Osborne and Rubinstein (1994, Proposition 22.2).

An action a'_i of player i in a strategic game weakly dominates an action a_i if, regardless of the other players' actions, i likes the outcome in which she chooses a'_i at least as much as the outcome in which she chooses a_i, and for at least one collection of actions of the other players she prefers the outcome when she chooses a'_i to the outcome when she chooses a_i.

Definition 16.19: Weak domination in strategic game

Let $\langle N, (A_i)_{i\in N}, (\succsim_i)_{i\in N}\rangle$ be a strategic game. The action $a'_i \in A_i$ of player $i \in N$ *weakly dominates* her action $a_i \in A_i$ if

$$(a'_i, x_{-i}) \succsim_i (a_i, x_{-i}) \text{ for every action profile } x$$
$$(a'_i, x_{-i}) \succ_i (a_i, x_{-i}) \text{ for some action profile } x.$$

Mixed strategy equilibrium, a notion closely related to Nash equilibrium, models the set of options of each individual as the set of probability distributions over a set of actions. The outcome of such an equilibrium is a probability distribution over action profiles, so that to define such an equilibrium we need to include in the description of the game the players' preferences over such probability distributions, not only their preferences over deterministic action profiles. We assume that these preference relations are vNM preference relations.

Definition 16.20: Strategic game with vNM preferences

A *strategic game with vNM preferences* $\langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ consists of

players
a set N

and for each player $i \in N$

actions
a set A_i

preferences
a function $u_i : \times_{j \in N} A_j \to \mathbb{R}$ (whose expected value represents individual i's preferences regarding lotteries over the set $\times_{j \in N} A_j$ of action profiles).

To define the notion of mixed strategy equilibrium, it is convenient to first define the notion of a mixed strategy.

Definition 16.21: Mixed strategy of player in strategic game with vNM preferences

Let $\langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game with vNM preferences. For any $i \in N$, a *mixed strategy* of player i is a probability distribution over A_i.

Definition 16.22: Mixed strategy equilibrium of strategic game with vNM preferences

Let $\langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game with vNM preferences for which N and each set A_i are finite. For each $i \in N$, denote by M_i the set of mixed strategies of player i and define the function $U_i : \times_{j \in N} M_j \to \mathbb{R}$ by

$$U_i(x) = \sum_{a \in A} \left(\prod_{j \in N} x_j(a_j) \right) u_i(a) \quad \text{for every } x \in \times_{j \in N} M_j,$$

where $A = \times_{j \in N} A_j$, the set of action profiles, and $x_j(a_j)$ is the probability that j's mixed strategy x_j assigns to the action a_j. (Thus $U_i(x)$ is i's expected payoff for x.) A *mixed strategy equilibrium* of the game is a profile α of mixed strategies for which

$$U_i(\alpha_i, \alpha_{-i}) \geq U(x_i, \alpha_{-i}) \text{ for all } x_i \in M_i.$$

The next result asserts that every finite strategic game has a mixed strategy equilibrium.

Proposition 16.6: Existence of mixed strategy equilibrium in finite strategic game with vNM preferences

Every strategic game with vNM preferences in which the set of players and set of actions of each player are finite has a mixed strategy equilibrium.

For a proof of this result, see Proposition 33.1 of Osborne and Rubinstein (1994).

One interpretation of a mixed strategy equilibrium is an extension of the interpretation of a Nash equilibrium that I mention before Definition 16.17: it corresponds to a *stochastic* steady state in an environment in which each decision-maker plays the game many times against other decision-makers chosen randomly from populations of potential players. This interpretation and others are discussed in Section 3.2 of Osborne and Rubinstein (1994).

Consider a mixed strategy equilibrium in which some player's mixed strategy assigns positive probabilities to two actions, say a and b. Her expected payoff if she chooses a must equal her expected payoff if she chooses b, given the equilibrium mixed strategies of the other players, because if these payoffs differ then she can increase her expected payoff by increasing the probability she assigns to the action that yields the higher payoff. That is, in the equilibrium she is indifferent between a and b, and has no positive incentive to choose them with the probabilities required by the equilibrium. The equilibrium probabilities are determined not by her optimization process, but by the equilibrium requirement that the *other* players must be indifferent between the actions to which their mixed strategies assign positive probability. This property of a mixed strategy equilibrium makes it more difficult to interpret than a strict Nash equilibrium, for which a deviation by any player from her equilibrium action decreases her payoff.

16.5 Bayesian games

We may model players' uncertainty about each other's characteristics by using the notion of a Bayesian game. The uncertainty is modeled by specifying a set of states. For each state, each player observes a signal; for any given signal, she cannot distinguish among the states that generate the signal. A player who observes the same signal for every state, for example, has no information about the state, and a player who observes a different signal for every state has perfect information. Each player has a prior belief about the probability of each state, which she updates, using Thomas Bayes' eponymous rule, after observing her signal. (This formulation follows Osborne and Rubinstein 1994 in taking the players' prior beliefs as primitive and deriving from them the posterior belief following the observation of a signal; Osborne 2004 takes the latter as primitive.)

Definition 16.23: Bayesian game

A *Bayesian game* $\langle N, \Omega, (A_i)_{i\in N}, (T_i)_{i\in N}, (\tau_i)_{i\in N}, (p_i)_{i\in N}, (u_i)_{i\in N}\rangle$ consists of

players
a set N

states
a set Ω

and for each player $i \in N$

actions
a set A_i

signals
a set T_i and a function $\tau_i : \Omega \to T_i$ that associates a signal with every state

prior beliefs
a probability measure p_i on Ω with $p_i(\tau_i^{-1}(t_i)) > 0$ for all $t_i \in T_i$

payoffs
a function $u_i : (\times_{i\in N} A_i) \times \Omega \to \mathbb{R}$ (whose expected value represents i's preferences over pairs consisting of an action profile and a state).

A player who receives the signal t is said to have *type* t.

The condition $p_i(\tau_i^{-1}(t_i)) > 0$ for all $t_i \in T_i$ on the prior beliefs says that each signal of every player is possible: each player assigns positive probability to the set of states that generate each of her signals. (If a signal of a player is not possible, we can omit it from the description of the game.)

A player's strategy assigns an action to each of her possible signals (types).

Definition 16.24: Strategy of player in Bayesian game

Let $\langle N, \Omega, (A_i)_{i\in N}, (T_i)_{i\in N}, (\tau_i)_{i\in N}, (p_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a Bayesian game. A *strategy of player* $i \in N$ in this game is a function $\sigma_i : T_i \to A_i$ that associates an action with each of i's signals.

?

We define a Nash equilibrium of a Bayesian game with reference to an associated strategic game G^* that has one player for each type of each player in the Bayesian game. Precisely, let $\langle N, \Omega, (A_i)_{i\in N}, (T_i)_{i\in N}, (\tau_i)_{i\in N}, (p_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a Bayesian game. The set of players in the associated strategic game G^* is the set of pairs (i, t_i) for $i \in N$ and $t_i \in T_i$. The set of actions of each player (i, t_i) is A_i. (That is, for each player i, A_i is the set of actions of every type of i.) To specify the payoffs for G^*, consider player (i, t_i). The set of states that generate the signal t_i for player i in the Bayesian game is $\tau_i^{-1}(t_i)$, so the probability that (i, t_i) assigns to each state ω, derived from her prior belief using Bayes' rule, is

$$\Pr(\omega \mid t_i) = \begin{cases} p_i(\omega)/p_i(\tau_i^{-1}(t_i)) & \text{if } \omega \in \tau_i^{-1}(t_i) \\ 0 & \text{otherwise.} \end{cases} \tag{16.1}$$

Let a^* be an action profile in G^*. That is, a^* assigns a member of A_i to each pair (i, t_i) with $i \in N$ and $t_i \in T_i$. Denote the action assigned by a^* to (i, t_i) by $a^*(i, t_i)$ (rather than $a^*_{(i,t_i)}$, for readability). Then the action that player i takes in the Bayesian game when the state is ω is $a^*(i, \tau_i(\omega))$ and hence the expected payoff of player (i, t_i) in G^* for the action profile a^* is

$$u^*_{(i,t_i)}(a^*) = \sum_{\omega \in \Omega} \Pr(\omega \mid t_i) u_i((a^*(j, \tau_j(\omega)))_{j \in N}, \omega), \tag{16.2}$$

where $\Pr(\omega \mid t_i)$ is given by (16.1).

Definition 16.25: Nash equilibrium of Bayesian game

Let $G = \langle N, \Omega, (A_i)_{i\in N}, (T_i)_{i\in N}, (\tau_i)_{i\in N}, (p_i)_{i\in N}, (u_i)_{i\in N} \rangle$ be a Bayesian game and let $G^* = \langle N^*, (A^*_j)_{j\in N^*}, (u^*_j)_{j\in N^*} \rangle$ be the strategic game with vNM preferences for which

players

$N^* = \{(i, t_i) : i \in N \text{ and } t_i \in T_i\}$

and for all $(i, t_i) \in N^*$

actions

$A^*_{(i,t_i)} = A_i$

payoffs

$u^*_{(i,t_i)}$ is given by (16.2).

A *Nash equilibrium* of G is a strategy profile σ of G for which the action profile a of G^* defined by $a(i, t_i) = \sigma_i(t_i)$ for all $i \in N$ and all $t_i \in T_i$ is a Nash equilibrium of G^*.

Like a Nash equilibrium of a strategic game, a Nash equilibrium of a Bayesian game may be interpreted as a steady state in an environment in which the decision-makers interact anonymously. In this interpretation, we assume that from her long experience playing the game, each decision-maker knows the action of each type of every other decision-maker.

16.6 Extensive games

An extensive game is a model of interaction among decision-makers that includes a specification of the sequential structure of the decision-making. At the start of the game, the members of a subset of the players (consisting possibly of a single player) simultaneously choose actions. This list of actions determines the subset of players who move next. Play continues in the same manner until the game ends. The resulting sequence of lists of actions is called a *terminal history*. The structure of the decision-making in the game is specified by the set of possible terminal histories. (This formulation follows Osborne 2004 in taking terminal histories as primitive and deriving histories from them; Osborne and Rubinstein 1994 takes histories as primitive and derives terminal histories from them.)

A sequence of lists of actions is a *history*. For any finite history $h = (a^1, \ldots, a^k)$, the *subhistories* of h are $\varnothing$ (the empty sequence) and all sequences of the form $(a^1, a^2, \ldots, a^m)$ with $1 \le m \le k$. (Note that h is a subhistory of itself.) A sequence $(a^1, a^2, \ldots, a^m)$ with $m \le k - 1$ is a *proper subhistory* of h. For any infinite history $h = (a^1, a^2, \ldots)$, the subhistories are $\varnothing$, all sequences of the form $(a^1, a^2, \ldots, a^m)$ with $m \ge 1$ (proper subhistories), and h itself.

16.6.1 Extensive game with perfect information

I start with the most general definition of an extensive game with perfect information, which allows for both simultaneous moves and chance moves.

Definition 16.26: Extensive game with perfect information, simultaneous moves, and chance moves

An *extensive game with perfect information, simultaneous moves, and chance moves* $\langle N, Z, P, (A_i(h))_{\{(i,h): i \in P(h)\}}, (q^h)_{\{h: c = P(h)\}}, (\succsim_i)_{i \in N} \rangle$ has the following components.

players
A set N.

terminal histories
A set Z of sequences with the property that no member of Z is a proper subhistory of any other member of Z; the set of all subhistories of members of Z, proper or not, is the set H of *histories*, and $H \setminus Z$ is the set of *nonterminal histories*.

player function
A function P that assigns to each nonterminal history either c (*chance*) or a subset of N.

actions

For each nonterminal history h with $P(h) \subseteq N$ and each player $i \in P(h)$, a set $A_i(h)$ (the set of *actions* available to player i after the history h).

chance probabilities

For each nonterminal history h for which $P(h) = c$, a probability measure q^h on $\{a : (h,a) \in H\}$, with each such measure independent of every other such measure (q^h gives the probabilities with which chance selects actions after the history h).

preferences

For each player $i \in N$, a preference relation $\succsim_i$ on the set of lotteries over the set Z of terminal histories.

The set H of histories, player function P, and sets $(A_i(h))_{\{(i,h):i \in P(h)\}}$ of actions are required to be consistent in the sense that for every (nonterminal) history $h \in H \setminus Z$ for which $P(h) \subseteq N$ we have $\{a : (h,a) \in H\} = \times_{i \in P(h)} A_i(h)$.

Special cases in which chance moves are absent and/or no players ever move simultaneously are defined as follows.

Definition 16.27: Special cases of extensive game with perfect information, simultaneous moves, and chance moves

An extensive game with perfect information, simultaneous moves, and chance moves $\langle N, Z, P, (A_i(h))_{\{(i,h):i \in P(h)\}}, (q^h)_{\{h:c=P(h)\}}, (\succsim_i)_{i \in N} \rangle$ is

- an *extensive game with perfect information* if P assigns a single player (member of N) to every nonterminal history
- an *extensive game with perfect information and chance moves* if P assigns either c (chance) or a single player (member of N) to every nonterminal history
- an *extensive game with perfect information and simultaneous moves* if P assigns a subset of N to every nonterminal history.

In the first two cases, $P(h)$ is a singleton for all h, and I write $P(h) = i$ rather than $P(h) = \{i\}$. For such games, for each history h and player $i \in P(h)$, the consistency condition says that $\{a : (h,a) \in H\} = A_i(h)$, so that the set $A_i(h)$ of actions available to i after the history h does not need to be specified explicitly as part of the description of the game.

An extensive game with perfect information that has finitely many terminal

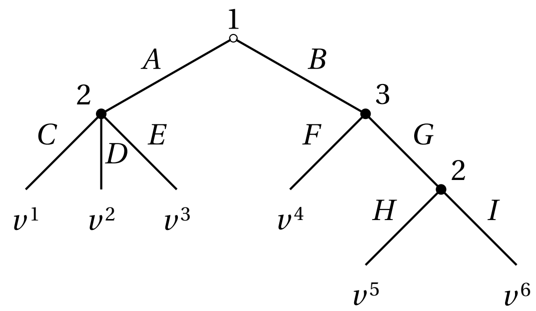


Figure 16.1 An example of an extensive game with perfect information. The start of the game (the empty history) is indicated by the small circle. Each line segment represents an action. The number near the empty history and the end of each nonterminal history is the player who moves after that history. The symbols $v^1, \ldots, v^6$ are profiles of payoffs that represent the players' preferences over terminal histories.

histories, each of finite length, may be represented in a diagram. An example is given in Figure 16.1. In this diagram, the start of the game is indicated by a small circle. Here, the start is located at the top; for some games, it is conveniently located at another position.

A key concept in the analysis of an extensive game is that of a strategy. The definition of a strategy is straightforward, but its interpretation is not. For discussions of the interpretation, see Osborne and Rubinstein (1994, Section 6.1.2), Osborne (2004, Section 5.2.1), or Osborne and Rubinstein (2023, Section 16.2).

Definition 16.28: Strategy in extensive game with perfect information, simultaneous moves, and chance moves

Let $\langle N, Z, P, (A_i(h))_{\{(i,h):i\in P(h)\}}, (q^h)_{\{h:c=P(h)\}}, (\succsim_i)_{i\in N}\rangle$ be an extensive game with perfect information, simultaneous moves, and chance moves. For any $i \in N$, a *strategy of player i* is a function that assigns a member of $A_i(h)$ to each history $h \in H$ for which $i \in P(h)$.

A Nash equilibrium of an extensive game is the analogue of the notion for a strategic game: a strategy profile such that no player prefers the terminal history that results from any change in her strategy, given the other players' strategies. This notion of equilibrium does not restrict the actions of players following histories that are inconsistent with the equilibrium. The main solution concept for an extensive game with perfect information, subgame perfect equilibrium, does restrict these actions: it requires that each player's strategy is optimal in the remainder of the game whenever the player moves, given the other players' strategies. To define this solution concept, I first define the subgame following any history to be the part of the game that remains after the history has occurred.

Definition 16.29: Subgame of extensive game with perfect information, simultaneous moves, and chance moves

Let $\langle N, Z, P, (A_i(h))_{\{(i,h):i\in P(h)\}}, (q^h)_{\{h:c=P(h)\}}, (\succsim_i)_{i\in N}\rangle$ be an extensive game with perfect information, simultaneous moves, and chance moves. For any nonterminal history h, the *subgame following* h is the extensive game with perfect information, simultaneous moves, and chance moves with the following components.

Players
The set N.

Terminal histories
The set of all sequences h' such that $(h,h')\in Z$.

Player function
The player assigned to each proper subhistory h' of a terminal history is $P(h,h')$.

Actions
For all sequences h' such that $P(h,h')\subseteq N$, the set of actions of each player $i\in P(h,h')$ is $A_i(h,h')$.

Chance probabilities
For all sequences h' such that $P(h,h') = c$, the probability measure that determines the action selected by chance after h' is $q^{h,h'}$.

Preferences
Each player $i \in N$ prefers the lottery l over sequences h' such that $(h,h') \in Z$ to the lottery l' over such sequences if and only if according to $\succsim_i$ she prefers the lottery over Z generated by h followed by l to the lottery generated by h followed by l'.

Definition 16.30: Subgame perfect equilibrium of extensive game with perfect information, simultaneous moves, and chance moves

A *subgame perfect equilibrium* of an extensive game with perfect information, simultaneous moves, and chance moves $\langle N, Z, P, (A_i(h))_{\{(i,h):i\in P(h)\}}, (q^h)_{\{h:c=P(h)\}}, (\succsim_i)_{i\in N}\rangle$ is a strategy profile s^* such that for every player $i\in N$ and every nonterminal history h with $i\in P(h)$,

$$L_h(s^*) \succsim_i L_h(r_i, s^*_{-i}) \text{ for every strategy } r_i \text{ of player } i,$$

where for every strategy profile s, $L_h(s)$ is the lottery over Z that assigns to each terminal history (h, h') the probability assigned to h' by the lottery over the terminal histories of the subgame following h that results when the players follow the prescriptions of s in the subgame.

In a subgame perfect equilibrium, for every subgame no player can generate an outcome in the subgame that she prefers by changing her strategy in the subgame. In particular, for every subgame the player who moves first cannot generate an outcome in the subgame that she prefers by changing her action at the start of the subgame. This second feature is called the one-deviation property.

Definition 16.31: One-deviation property

A strategy profile in an extensive game with perfect information, simultaneous moves, and chance moves satisfies the *one-deviation property* if, for each player i and each history h after which i moves, i does not prefer any lottery over terminal histories generated by changing only her action at the start of the subgame following h, given the other players' strategies, to the lottery over terminal histories generated by the strategy profile in the subgame.

For a game in which the number of players is finite and every terminal history is finite, a strategy profile is a subgame perfect equilibrium if and only if it satisfies this property. This result reduces considerably the complexity of checking that a strategy profile is a subgame perfect equilibrium. For a proof of the result, see Osborne and Rubinstein (1994, Lemma 98.2, Exercise 102.1, and Exercise 103.3).

Proposition 16.7: Subgame perfect equilibrium of finite horizon extensive game and the one-deviation property

A strategy profile in an extensive game with perfect information, simultaneous moves, and chance moves in which the number of players is finite and every terminal history is finite is a subgame perfect equilibrium if and only if it satisfies the one-deviation property.

An implication of this result is that for such a game, a subgame perfect equilibrium may be found (if one exists) by using the procedure of backward induction. We find a Nash equilibrium of the last subgame in each terminal history, replace the subgame with the outcome of the Nash equilibrium, and then repeat the process for the resulting game, working back to the start of the game. (For a precise description of the procedure for a game without simultaneous or chance moves, see Osborne and Rubinstein 2023, Section 16.3.) For a game without simultaneous moves in which the player who moves at the start of each subgame has an optimal action, the procedure of backward induction generates at least one strategy profile, so that such a game has a subgame perfect equilibrium. For a game without simultaneous or chance moves, a sufficient condition for each player to have an optimal action whenever she moves is that her preference relation over terminal histories is represented by a payoff function that takes finitely many values, implying the next result. (A stronger condition is that the number of terminal histories is finite.)

Proposition 16.8: Existence of subgame perfect equilibrium for finite extensive game

Every extensive game with perfect information in which the number of players is finite, every terminal history is finite, and every player's preference relation over terminal histories is represented by a payoff function that takes finitely many values has a subgame perfect equilibrium.

A version of Proposition 16.7 holds for games in which the terminal histories are not finite if for every player the difference between the payoffs of pairs of terminal histories whose first t components coincide converges to zero as t increases without bound. For a proof of this result see Theorem 4.2 of Fudenberg and Tirole (1991, 110).

Proposition 16.9: Subgame perfect equilibrium of extensive game with perfect information, simultaneous moves, and chance moves and the one-deviation property

Let G be an extensive game with perfect information, simultaneous moves, and chance moves in which the set of players is finite. Denote the set of terminal histories by Z and suppose that the preferences of each player i over lotteries over Z are represented by the expected value of a function u_i for which

$$\lim_{t\to\infty} \sup_{h,\tilde{h}\in Z} \{|u_i(h) - u_i(\tilde{h})| : h^t = \tilde{h}^t\} = 0,$$

where for any terminal history h, h^t consists of the first t components of h. A strategy profile in G is a subgame perfect equilibrium if and only if it satisfies the one-deviation property.

16.6.2 Bayesian extensive game with observable actions

An extensive game with perfect information, simultaneous moves, and chance moves models a situation in which each player knows the structure of the interaction (who moves when, and which actions they can choose) and all the players' characteristics. A Bayesian extensive game with observable actions models a situation in which each player knows the structure of the interaction but does not know the other players' characteristics. For each player i there is a set Θ_i of possible types and a probability measure p_i over this set. The probability measures $(p_i)_{i \in N}$ are independent; the profile of the players' types is drawn according to these measures. Each player knows her own type, but not the type of any other player. The type profile $\theta \in \times_{j \in N} \Theta_j$ determines each player's payoff function over terminal histories. We can think of the game as one in which chance first determines a type profile, then the players engage in an extensive game with perfect information and simultaneous moves in which the payoffs are determined by the type profile.

Definition 16.32: Bayesian extensive game with observable actions

A *Bayesian extensive game with observable actions* $\langle N, Z, P, (A_i(h))_{\{(i,h): i \in P(h)\}}, (\Theta_i)_{i \in N}, (p_i)_{i \in N}, (u_i)_{i \in N} \rangle$ consists of

players
a set N

terminal histories
a set Z of sequences with the property that no member of Z is a proper subhistory of any other member of Z; the set of all subhistories of members of Z, proper or not, is the set H of *histories*, and $H \setminus Z$ is the set of *nonterminal histories*

player function
a function P that assigns a subset of N to every nonterminal history

actions
for each nonterminal history h and each player $i \in P(h)$, a set $A_i(h)$ (the set of *actions* available to player i after the history h)

and for each player $i \in N$

types
a set Θ_i

probabilities
a probability measure p_i on Θ_i with $p_i(\theta_i) > 0$ for all $\theta_i \in \Theta_i$ and ev-

ery measure p_i independent of every other measure p_j ($p_i(\theta_i)$ is the probability that i's type is θ_i)

payoff function

a function $u_i : \Theta \times Z \to \mathbb{R}$ (whose expected value represents i's preferences regarding the set of lotteries over the set $\Theta \times Z$ of pairs consisting of a profile of types and a terminal history).

The set H of histories, player function P, and sets $(A_i(h))_{\{(i,h):i\in P(h)\}}$ of actions are required to be consistent in the sense that for every (nonterminal) history $h \in H \setminus Z$ we have $\{a : (h,a) \in H\} = \times_{i\in P(h)} A_i(h)$.

A strategy of each player i in a Bayesian extensive game with observable actions specifies, for each type $\theta_i \in \Theta_i$, a strategy for i in the extensive game with perfect information and simultaneous moves. That is, a strategy of player i is a function that associates with each type $\theta_i \in \Theta_i$ a function that assigns to each (nonterminal) history $h \in H \setminus Z$ for which $i \in P(h)$ a member of $A_i(h)$. A notion of equilibrium may be defined for a general Bayesian extensive game with observable actions (see Definition 232.1 in Osborne and Rubinstein 1994), but for the specific model I analyze (see Definition 8.7) a simpler notion, which I specify in Definition 8.8, suffices.

16.6.3 Extensive game with imperfect information

An extensive game with imperfect information allows for the possibility that each player, when choosing an action, does not know the actions chosen previously by the other players. This lack of information is modeled by assuming that each player i, when choosing an action, knows only that the history is a member of some set, called an information set. We assume that following every history in a given information set, the set of actions available to the player who moves is the same, so that the set of actions available to a player gives her no information about the history that led to the information set. In a diagrammatic representation of a game, I connect the ends of all the histories in each information set with a dotted line, as in Figure 16.2.

Definition 16.33: Extensive game with imperfect information

An *extensive game with imperfect information* $\langle N, Z, P, (q^h)_{\{h:c=P(h)\}}, (\mathscr{I}_i)_{i\in N}, (\succsim_i)_{i\in N}\rangle$ consists of

players

a set N

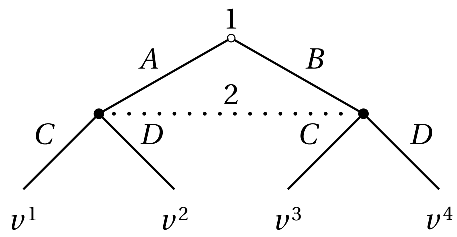


Figure 16.2 An example of an extensive game with imperfect information. The dotted line indicates that the histories A and B are in the same information set; when player 2 moves, she does not know whether player 1 chose A or B at the start of the game.

terminal histories
: a set Z of sequences with the property that no member of Z is a proper subhistory of any other member of Z; the set of all subhistories of members of Z, proper or not, is the set H of *histories*, $H \setminus Z$ is the set of *nonterminal histories*, and for any nonterminal history h, the set of actions available following h is $A(h) = \{a : (h, a) \in H\}$

player function
: a function P that assigns either c (*chance*) or a member of N to every nonterminal history

chance probabilities
: for each nonterminal history h for which $P(h) = c$, a probability measure q^h on $A_c(h) = \{a : (h, a) \in H\}$, with each such measure independent of every other such measure ($q^h(a)$ is the probability with which chance selects a after the history h)

information partitions
: for each player $i \in N$ a partition $\mathcal{I}_i$ of $\{h \in H : P(h) = i\}$ with the property that for any $I_i \in \mathcal{I}_i$, $A(h) = A(h')$ whenever $h \in I_i$ and $h' \in I_i$; the common value of $A(h)$ for all $h \in I_i$ is denoted $A(I_i)$ ($\mathcal{I}_i$ is the *information partition* of player i, and each member of $\mathcal{I}_i$ is an *information set*)

preferences
: for each player $i \in N$ a preference relation $\succsim_i$ on the set of lotteries over the set Z of terminal histories.

The solution concept I use for an extensive game with imperfect information allows for the possibility that players' actions are probabilistic. Specifically, a behavioral strategy assigns to each of a player's information sets a probability distribution over the set of actions available at that information set, with the probability distribution for each information set independent of the probability distribution for all of the player's other information sets.

Definition 16.34: Behavioral strategy in extensive game with imperfect information

A *behavioral strategy* of player $i \in N$ in an extensive game with imperfect information $\langle N, Z, P, (q^h)_{\{h:c=P(h)\}}, (\mathscr{I}_i)_{i\in N}, (\succsim_i)_{i\in N}\rangle$ is a function that assigns to each of i's information sets $I_i \in \mathscr{I}_i$ a probability distribution over the actions in $A(I_i)$ with the property that the probability distribution for any given information set I_i of player i is independent of the distributions for all her other information sets.

We assume that each player's choice at each of her information sets is based on her belief about the history in the information set that has occurred. For an information set reached with positive probability given the strategy profile, this belief may be derived from the strategy profile via Bayes' law, but the same is not true for an information set reached with probability zero given the strategy profile. We finesse this issue by making each player's belief part of the equilibrium, with the restriction that for information sets reached with positive probability given the strategy profile, the probability assigned by the belief to each history in the information set is the one derived from the strategy profile using Bayes' law.

Definition 16.35: Belief system in extensive game with imperfect information

A *belief system* in an extensive game with imperfect information is a function that assigns to each information set a probability distribution over the histories in that set.

Definition 16.36: Assessment in extensive game with imperfect information

An *assessment* in an extensive game with imperfect information is a pair consisting of a profile of behavioral strategies and a belief system.

The solution concept that I use imposes two conditions on an assessment. First, for each information set of each player, the player's strategy is required to be optimal in the remainder of the game, given the other players' strategies and the probabilities assigned by the belief system to the histories in the information set. Second, for each information set reached with positive probability given the strategy profile, the belief system is required to assign to each history in the information set the probability that the history occurs conditional on the information set's being reached, given the strategy profile. This requirement is called weak consistency of beliefs with strategies; the word "weak" honors the fact that the requirement puts no restriction on the probabilities assigned by the belief system to histories in information sets that are not reached if the players adhere to the strategy profile.

This solution concept is called weak sequential equilibrium. (The concept of sequential equilibrium, which I do not use in this book, imposes an additional condition on an assessment. The notion of weak sequential equilibrium is sometimes called weak perfect Bayesian equilibrium, although no related notion of perfect Bayesian equilibrium is defined for the class of all extensive games with imperfect information.)

Definition 16.37: Weak sequential equilibrium

An assessment (β,μ) in an extensive game with imperfect information $\langle N, Z, P, (q^h)_{\{h:c=P(h)\}}, (\mathscr{I}_i)_{i\in N}, (\succsim_i)_{i\in N}\rangle$, where β is a behavioral strategy profile and μ is a belief system, is a *weak sequential equilibrium* if it satisfies the following two conditions.

Sequential rationality

For each player $i \in N$ and each information set $I_i \in \mathscr{I}_i$,

$$O_{I_i}(\beta,\mu) \succsim_i O_{I_i}((\gamma_i,\beta_{-i}),\mu) \text{ for each behavioral strategy } \gamma_i \text{ of player } i,$$

where for any profile σ of behavioral strategies, $O_{I_i}(\sigma,\mu)$ is the probability distribution over terminal histories conditional on play reaching I_i, given σ and μ.

Weak consistency of beliefs with strategies

For each player $i \in N$ and every information set $I_i \in \mathscr{I}_i$ reached with positive probability given the strategy profile β, the probability assigned by the belief system μ to each history $h^* \in I_i$ is

$$\frac{\Pr(h^* \text{ according to } \beta)}{\sum_{h\in I_i} \Pr(h \text{ according to } \beta)}.$$

16.7 Coalitional games

A coalitional game with transferable payoff models a situation in which the total payoff that each group of players can obtain is independent of the behavior of the remaining players, and may be distributed in any way among the members of the group. A group of players is called a coalition, and the total payoff available to it is called its worth. I restrict attention to games in which the set of players is finite.

Definition 16.38: Coalitional game with transferable payoff

A *coalitional game with transferable payoff* $\langle N, v\rangle$ consists of a finite set N (of players) and a function v that assigns a real number $v(S)$ (the *worth* of S) to every nonempty subset S of N (*coalition*). A *payoff profile* for the game is a profile $(x_i)_{i\in N}$ of real numbers; it is *feasible* if $\sum_{i\in N} x_i = v(N)$.

16.7.1 Core

One solution concept for coalitional games with transferable payoff is the core, the set of feasible payoff profiles with the property that no coalition can by itself make all its members better off.

Definition 16.39: Core of coalitional game with transferable payoff

Let $\langle N, v\rangle$ be a coalitional game with transferable payoff. A coalition S can *improve upon* the payoff profile $(x_i)_{i\in N}$ if $\sum_{i\in S} x_i < v(S)$. The *core* of $\langle N, v\rangle$ is the set of feasible payoff profiles upon which no coalition can improve.

The core of a coalitional game with transferable payoff may be empty. For example, the game $\langle N, v\rangle$ with $N = \{1,2,3\}$, $v(\{1,2,3\}) = v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = 1$, and $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$ (in which a majority rules) has an empty core: for any feasible payoff profile $(x_i)_{i\in N}$ we have $x_j > 0$ for some $j \in N$, so that the coalition $S = N \setminus \{j\}$ can improve upon $(x_i)_{i\in N}$ because $v(S) = 1$ and $\sum_{k\in S} x_k < 1$.

16.7.2 Shapley value

Another solution concept for coalitional games with transferable payoff is the Shapley value (due to Lloyd S. Shapley, 1923–2016). Unlike the core, the Shapley value assigns a single payoff profile to every game. I first define a value to be a solution of this type.

Definition 16.40: Value for coalitional games with transferable payoff

A *value* for coalitional games with transferable payoff is a function that assigns a unique feasible payoff profile to every such game.

The Shapley value assigns to a game the payoff profile in which each player's payoff is the average of the amounts by which the player's presence increases the worth of the coalition that precedes her in a random ordering of the players.

Definition 16.41: Shapley value of coalitional game with transferable payoff

Let $\langle N, v\rangle$ be a coalitional game with transferable payoff and let n be the number of members of N. The *Shapley value* assigns to $\langle N, v\rangle$ the payoff profile $(x_i)_{i\in N}$ for which

$$x_i = \frac{1}{n!}\sum_{R\in\mathscr{R}}(v(S_i^R \cup \{i\}) - v(S_i^R)) \quad \text{for each } i \in N, \tag{16.3}$$

where $\mathscr{R}$ is the set of all $n!$ orderings of N, S_i^R is the set of players who precede i in the ordering R, and $v(\varnothing) = 0$.

The Shapley value is the only value that satisfies the following properties.

Definition 16.42: Properties of value of coalitional game with transferable payoff

A value ψ for coalitional games with transferable payoff is

symmetric
if $\psi_i(N, v) = \psi_j(N, v)$ for every game $\langle N, v\rangle$ and all players $i \in N$ and $j \in N$ for which $v(S\cup\{i\}) = v(S\cup\{j\})$ for every coalition S that includes neither i nor j

null-consistent
if $\psi_i(N, v) = 0$ for every game $\langle N, v\rangle$ and player $i \in N$ for which $v(S\cup\{i\}) = v(S)$ for every coalition S of which i is not a member

additive
if $\psi_i(N, w) = \psi_i(N, v) + \psi_i(N, v')$ for all games $\langle N, v\rangle$ and $\langle N, v'\rangle$ and every player $i \in N$, where $w(S) = v(S) + v'(S)$ for every coalition S.

Proposition 16.10: Axiomatic characterization of Shapley value

Let N be a finite set. The Shapley value is the only value for coalitional games with transferable payoff with player set N that is symmetric, null-consistent, and additive.

For a proof of this result, see Osborne and Rubinstein (1994, Proposition 293.1).

Suppose that the players' payoffs in a coalitional game with transferable payoff $\langle N, v\rangle$ are determined by the value ψ. If player i leaves the game, the payoff of any player $j \neq i$ changes from $\psi_j(N, v)$ to $\psi_j(N \setminus \{i\}, v^{N\setminus\{i\}})$, where $v^{N\setminus\{i\}}$ is the restriction of v to coalitions in $N \setminus \{i\}$ (i.e. $v^{N\setminus\{i\}}(S) = v(S)$ for every nonempty set $S \subseteq N \setminus \{i\}$). That is, j loses $\psi_j(N, v) - \psi_j(N \setminus \{i\}, v^{N\setminus\{i\}})$ when i departs. The Shapley value has the property that for all players i and j, this amount is equal to the amount that i loses when j departs. Further, it is the only value with this property. One motivation for the property is that for every objection of a certain type by one player to the payoff profile there is a valid counterobjection by another player (Osborne and Rubinstein 1994, 290–291).

Proposition 16.11: Characterization of Shapley value in terms of balanced contributions

The Shapley value is the only value for coalitional games with transferable payoff that satisfies the condition

$$\psi_i(N, v) - \psi_i(N \setminus \{j\}, v^{N\setminus\{j\}}) = \psi_j(N, v) - \psi_j(N \setminus \{i\}, v^{N\setminus\{i\}})$$

for every coalitional game $\langle N, v\rangle$ and all $i \in N$ and $j \in N$.

For a proof of this result, see Osborne and Rubinstein (1994, Proposition 291.3).

The following result is used in the proof of Proposition 11.3.

Lemma 16.1: Shapley value of dual of coalitional game with transferable payoff

Let $\langle N, v\rangle$ be a coalitional game with transferable payoff and let $v^{\#}(S) = v(N) - v(N \setminus S)$ for each nonempty subset S of N. The Shapley value assigns the same payoff profile to $\langle N, v^{\#}\rangle$ as it does to $\langle N, v\rangle$.

Proof

Let R be an ordering of N, let R' be the reverse ordering, and let $i \in N$. Then the set of individuals who come before i in R' is the set of individuals who come after i in R: $S_i^{R'} = N \setminus (S_i^R \cup \{i\})$ in the notation of Definition 16.41. Thus

$$\begin{aligned} v^{\#}(S_i^{R'} \cup \{i\}) - v^{\#}(S_i^{R'}) &= (v(N) - v(N \setminus (S_i^R \cup \{i\}))) - (v(N) - v(N \setminus S_i^R)) \\ &= v(S_i^R \cup \{i\}) - v(S_i^R). \end{aligned}$$

Hence by (16.3), i's payoff in the Shapley value of $\langle N, v^{\#}\rangle$ is equal to her payoff in the Shapley value of $\langle N, v\rangle$.

16.8 Optimization

The results in this section are used in Propositions 8.8 and 11.1.

Proposition 16.12: Necessary conditions for solution of unconstrained optimization problem

Let $S \subseteq \mathbb{R}^n$ and let $f : S \to \mathbb{R}$. If a point x^* in the interior of S is a local maximizer or minimizer of f and the partial derivative f'_j of f with respect to its jth argument exists at x^* then $f'_j(x^*) = 0$. In particular, if all the partial derivatives of f exist at x^* then

$$f'_i(x^*) = 0 \text{ for } i = 1, \ldots, n.$$

For a proof, see Sydsæter (1981, Theorem 5.7).

Proposition 16.13: Conditions under which first-order conditions are necessary and sufficient for solution of unconstrained optimization problem

Let $S \subseteq \mathbb{R}^n$ be convex and let $f : S \to \mathbb{R}$ be differentiable.

- If f is concave then a point x^* in the interior of S is a (global) maximizer of f in S if and only if it is a stationary point of f (i.e. $f'_i(x^*) = 0$ for $i = 1, \ldots, n$).
- If f is convex then a point x^* in the interior of S is a (global) minimizer of f in S if and only if it is a stationary point of f (i.e. $f'_i(x^*) = 0$ for $i = 1, \ldots, n$).

For a proof, see Sydsæter (1981, Theorem 5.18). (Sydsæter's result assumes that f is continuously differentiable, but that assumption is stronger than necessary because every differentiable concave or convex function is continuously differentiable by Rockafellar 1970, Corollary 25.5.1.)

Proposition 16.14: Necessary conditions for solution of optimization problem with equality constraint

Let $S \subseteq \mathbb{R}^n$, let $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ be continuously differentiable, let $c \in \mathbb{R}$, and let x^* be an interior point of S that solves the problem

$$\max_{x \in S} f(x) \text{ subject to } g(x) = c$$

or the problem

$$\min_{x \in S} f(x) \text{ subject to } g(x) = c$$

or is a local maximizer or minimizer of $f(x)$ subject to $g(x) = c$. Suppose also that $g_i'(x^*) \neq 0$ for some $i \in \{1, \ldots, n\}$.

Then there is a unique number λ such that

$$f_i'(x^*) - \lambda g_i'(x^*) = 0 \text{ for } i = 1, \ldots, n.$$

In addition, $g(x^*) = c$.

Proposition 16.15: Conditions under which first-order conditions are sufficient for solution of optimization problem with equality constraint

Let $S \subseteq \mathbb{R}^n$ be open and convex, let $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ be differentiable, and let $c \in \mathbb{R}$. Suppose that there exists a number λ and an interior point x^* of S such that

$$f_i'(x^*) - \lambda g_i'(x^*) = 0 \text{ for } i = 1, \ldots, n.$$

Suppose further that $g(x^*) = c$.

Define the function $\mathscr{L} : S \to \mathbb{R}$ by

$$\mathscr{L}(x) = f(x) - \lambda(g(x) - c) \text{ for all } x \in S.$$

- If $\mathscr{L}$ is concave—in particular if f is concave and λg is convex—then x^* solves the problem $\max_{x \in S} f(x)$ subject to $g(x) = c$.
- If $\mathscr{L}$ is convex—in particular if f is convex and λg is concave—then x^* solves the problem $\min_{x \in S} f(x)$ subject to $g(x) = c$.

For proofs of generalizations of Propositions 16.14 and 16.15 (to problems with many constraints), see Sydsæter (1981, Theorems 5.20 and 5.21).

References

Acemoglu, Daron, and James A. Robinson (2000), “Why did the West extend the franchise? Democracy, inequality, and growth in historical perspective.” *Quarterly Journal of Economics*, 115, 1167–1199, 10.1162/003355300555042. [533]

Acemoglu, Daron, and James A. Robinson (2001), “A theory of political transitions.” *American Economic Review*, 91, 938–963, 10.1257/aer.91.4.938. [533]

Acemoglu, Daron, and James A. Robinson (2017), “Why did the West extend the franchise? A correction.”, https://voices.uchicago.edu/jamesrobinson/2018/06/26/why-did-the-west-extend-the-franchise-a-correction/. [Unpublished paper.] [533]

Ali, S. Nageeb, B. Douglas Bernheim, and Xiaochen Fan (2019), “Predictability and power in legislative bargaining.” *Review of Economic Studies*, 86, 500–525, 10.1093/restud/rdy013. [501]

Ali, S. Nageeb, Maximilian Mihm, and Lucas Siga (2025), “The political economy of zero-sum thinking.” *Econometrica*, 93, 41–70, 10.3982/ECTA22474. [235]

Anesi, Vincent, and Daniel J. Seidmann (2015), “Bargaining in standing committees with an endogenous default.” *Review of Economic Studies*, 82, 825–867, 10.1093/restud/rdv009. [500, 501]

Ansolabehere, Stephen, and James M. Snyder (2000), “Valence politics and equilibrium in spatial election models.” *Public Choice*, 103, 327–336. [288]

Aragonès, Enriqueta, and Thomas R. Palfrey (2002), “Mixed equilibrium in a Downsian model with a favored candidate.” *Journal of Economic Theory*, 103, 131–161, 10.1006/jeth.2001.2821. [288]

Arrow, Kenneth J. (1951), *Social choice and individual values*, first edition. Wiley, New York, 10.12987/9780300186987. [68]

Ashworth, Scott (2006), “Campaign finance and voter welfare with entrenched incumbents.” *American Political Science Review*, 100, 55–68, 10.1017/s0003055406062009. [437]

Aumann, Robert J., and Mordecai Kurz (1977), “Power and taxes.” *Econometrica*, 45, 1137–1161, 10.2307/1914063. [411]

Austen-Smith, David (1984), “Two-party competition with many constituencies.” *Mathematical Social Sciences*, 7, 177–198, 10.1016/0165-4896(84)90004-0. [288]

Austen-Smith, David, and Jeffrey S. Banks (1996), “Information aggregation, rationality, and the Condorcet Jury Theorem.” *American Political Science Review*, 90, 34–45, 10.2307/2082796. [234]

Austen-Smith, David, and Jeffrey S. Banks (1999), *Positive political theory I*. University of Michigan Press, Ann Arbor, 10.3998/mpub.14228. [xv, 69]

Austen-Smith, David, and Jeffrey S. Banks (2005), *Positive political theory II*. University of Michigan Press, Ann Arbor, 10.3998/mpub.105376. [xv, 93, 288, 323, 486, 501]

Banks, Jeffrey S. (1985), “Sophisticated voting outcomes and agenda control.” *Social Choice and Welfare*, 1, 295–306, 10.1007/bf00649265. [186]

Banks, Jeffrey S., and John Duggan (2000), “A bargaining model of collective choice.” *American Political Science Review*, 94, 73–88, 10.2307/2586381. [491, 492, 501]

Baron, David P. (1994), “Electoral competition with informed and uninformed voters.” *American Political Science Review*, 88, 33–47, 10.2307/2944880. [437]

Baron, David P. (1996), “A dynamic theory of collective goods programs.” *American Political Science Review*, 90, 316–330, 10.2307/2082887. [501]

Baron, David P., and John A. Ferejohn (1989), “Bargaining in legislatures.” *American Political Science Review*, 83, 1181–1206, 10.2307/1961664. [501]

Bell, David E. (1982), “Regret in decision making under uncertainty.” *Operations Research*, 30, 961–981, 10.1287/opre.30.5.961. [124]

Bernhardt, Dan, Peter Buisseret, and Sinem Hidir (2020), “The race to the base.” *American Economic Review*, 110, 922–942, 10.1257/aer.20181606. [286, 288]

Bernhardt, Dan, John Duggan, and Francesco Squintani (2007), “Electoral competition with privately-informed candidates.” *Games and Economic Behavior*, 58, 1–29, 10.1016/j.geb.2006.03.004. [288]

Bernhardt, Dan, John Duggan, and Francesco Squintani (2009), "Private polling in elections and voter welfare." *Journal of Economic Theory*, 144, 2021–2056, 10.1016/j.jet.2008.05.013. [265, 288]

Besley, Timothy, and Stephen Coate (1997), "An economic model of representative democracy." *Quarterly Journal of Economics*, 112, 85–114, 10.1162/003355397555136. [365]

Black, Duncan (1948a), "The decisions of a committee using a special majority." *Econometrica*, 16, 245–261, 10.2307/1907278. [186]

Black, Duncan (1948b), "On the rationale of group decision-making." *Journal of Political Economy*, 56, 23–34, 10.1086/256633. [93, 186]

Black, Duncan (1958), *The theory of committees and elections*. Cambridge University Press, Cambridge. [68, 186]

Blackorby, Charles, David Donaldson, and John A. Weymark (1984), "Social choice with interpersonal utility comparisons: a diagrammatic introduction." *International Economic Review*, 25, 327–356, 10.2307/2526200. [69]

Bleichrodt, Han, and Peter P. Wakker (2015), "Regret theory: a bold alternative to the alternatives." *Economic Journal*, 125, 493–532, 10.1111/ecoj.12200. [124]

Börgers, Tilman (2004), "Costly voting." *American Economic Review*, 94, 57–66, 10.1257/000282804322970706. [124]

Bossert, Walter, and John A. Weymark (2004), "Utility in social choice." In *Handbook of utility theory* (Salvador Barberà, Peter J. Hammond, and Christian Seidl, eds.), volume 2, 1099–1177, Kluwer, Boston, 10.1007/978-1-4020-7964-1_7. [69]

Bouton, Laurent, Micael Castanheira, and Aniol Llorente-Saguer (2017), "Multicandidate elections: aggregate uncertainty in the laboratory." *Games and Economic Behavior*, 101, 132–150, 10.1016/j.geb.2015.10.003. [149]

Brennan, Geoffrey, and Alan Hamlin (1998), "Expressive voting and electoral equilibrium." *Public Choice*, 95, 149–175, 10.1023/A:1004936203144. [288]

Brett, Craig, and John A. Weymark (2017), "Voting over selfishly optimal nonlinear income tax schedules." *Games and Economic Behavior*, 101, 172–188, 10.1016/j.geb.2016.02.008. [410]

Brett, Craig, and John A. Weymark (2020), "Majority rule and selfishly optimal nonlinear income tax schedules with discrete skill levels." *Social Choice and Welfare*, 54, 337–362, 10.1007/s00355-019-01188-4. [410]

Brusco, Sandro, Marcin Dziubiński, and Jaideep Roy (2012), "The Hotelling–Downs model with runoff voting." *Games and Economic Behavior*, 74, 447–469, 10.1016/j.geb.2011.08.004. [365]

Calvert, Randall L. (1985), "Robustness of the multidimensional voting model: candidate motivations, uncertainty, and convergence." *American Journal of Political Science*, 29, 69–95, 10.2307/2111212. [288, 323]

Camion, Paul (1959), "Chemins et circuits hamiltoniens des graphes complets." *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences*, 249, 2151–2152. [186]

Campbell, Colin M. (1999), "Large electorates and decisive minorities." *Journal of Political Economy*, 107, 1199–1217, 10.1086/250094. [124]

Chamberlain, Gary, and Michael Rothschild (1981), "A note on the probability of casting a decisive vote." *Journal of Economic Theory*, 25, 152–162, 10.1016/0022-0531(81)90022-3. [124]

Cho, In-Koo, and David M. Kreps (1987), "Signaling games and stable equilibria." *Quarterly Journal of Economics*, 102, 179–221, 10.2307/1885060. [463]

Cho, Seok-Ju (2014), "Voting equilibria under proportional representation." *American Political Science Review*, 108, 281–296, 10.1017/s0003055414000136. [150]

Cho, Seok-ju, and John Duggan (2009), "Bargaining foundations of the median voter theorem." *Journal of Economic Theory*, 144, 851–868, 10.1016/j.jet.2008.06.011. [485, 501]

Cho, Seok-ju, and John Duggan (2015), "A folk theorem for the one-dimensional spatial bargaining model." *International Journal of Game Theory*, 44, 933–948, 10.1007/s00182-014-0460-8. [485, 501]

Coate, Stephen (2004), "Pareto-improving campaign finance policy." *American Economic Review*, 94, 628–655, 10.1257/0002828041464443. [437]

Coate, Stephen, and Michael Conlin (2004), "A group rule-utilitarian approach to voter turnout: theory and evidence." *American Economic Review*, 94, 1476–1504, 10.1257/0002828043052231. [199]

Condorcet, Marquis de (Marie Jean Antoine Nicolas de Caritat) (1785), *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix*. Imprimerie Royale, Paris, 10.1017/cbo9781139923972. [67]

Coughlin, Peter J. (1986), "Elections and income distribution." *Public Choice*, 50, 27–91, 10.1007/bf00124925. [410]

Dasgupta, Partha, and Eric S. Maskin (2008), "On the robustness of majority rule." *Journal of the European Economic Association*, 6, 949–973, 10.1162/JEEA.2008.6.5.949. [68]

d'Aspremont, Claude, and Louis Gevers (1977), "Equity and the informational basis of collective choice." *Review of Economic Studies*, 44, 199–209, 10.2307/2297061. [69]

Davis, Otto A., Morris H. DeGroot, and Melvin J. Hinich (1972), "Social preference orderings and majority rule." *Econometrica*, 40, 147–157, 10.2307/1909727. [68]

De Sinopoli, Francesco, and Giovanna Iannantuoni (2007), "A spatial model where proportional rule leads to two-party equilibria." *International Journal of Game Theory*, 35, 267–286, 10.1007/s00182-006-0056-z. [149]

Debreu, Gerard (1954), "Representation of a preference ordering by a numerical function." In *Decision processes* (Robert M. Thrall, Clyde H. Coombs, and Robert L. Davis, eds.), chapter 11, 159–165, Wiley, New York. [542]

Demange, Gabrielle (1982), "Single-peaked orders on a tree." *Mathematical Social Sciences*, 3, 389–396, 10.1016/0165-4896(82)90020-8. [68]

Dhillon, Amrita, and Ben Lockwood (2004), "When are plurality rule voting games dominance-solvable?" *Games and Economic Behavior*, 46, 55–75, 10.1016/s0899-8256(03)00050-2. [149]

Downs, Anthony (1957), *An economic theory of democracy*. Harper and Row, New York. [288]

Droop, Henry R. (1881), "On methods of electing representatives." *Journal of the Statistical Society of London*, 44, Part II, 141–196, 10.2307/2339223. [365]

Duggan, John (2013), "Uncovered sets." *Social Choice and Welfare*, 41, 489–535, 10.1007/s00355-012-0696-9. [187]

Duggan, John (2014), "A survey of equilibrium analysis in spatial models of elections." [Unpublished paper, University of Rochester.] [288, 323]

Duggan, John, and César Martinelli (2015), "The political economy of dynamic elections: a survey and some new results." [Unpublished paper.] [463]

Duggan, John, and César Martinelli (2017), "The political economy of dynamic elections: accountability, commitment, and responsiveness." *Journal of Economic Literature*, 55, 916–984, 10.1257/jel.20150927. [463]

Duggan, John, and César Martinelli (2020), "Electoral accountability and responsive democracy." *Economic Journal*, 130, 675–715, 10.1093/ej/uez070. [463]

Duggan, John, and Yoji Sekiya (2009), "Voting equilibria in multi-candidate elections." *Journal of Public Economic Theory*, 11, 875–889, 10.1111/j.1467-9779.2009.01433.x. [136, 149]

Duverger, Maurice (1951), *Les partis politiques*. Armand Colin, Paris, 10.3917/lsrel.duver.1981.01. [365]

Duverger, Maurice (1964), *Political parties*, third edition. Methuen, London. [365]

Eguia, Jon X. (2007), "Citizen candidates under uncertainty." *Social Choice and Welfare*, 29, 317–331, 10.1007/s00355-006-0207-y. [357]

Elkind, Edith, Martin Lackner, and Dominik Peters (2022), "Preference restrictions in computational social choice: a survey.", 10.48550/arXiv.2205.09092. [Unpublished paper, arXiv:2205.09092v1] [69]

Enelow, James M., and Melvin J. Hinich (1984), *The spatial theory of voting*. Cambridge University Press, Cambridge. [68]

Eraslan, Hülya (2002), "Uniqueness of stationary equilibrium payoffs in the Baron–Ferejohn model." *Journal of Economic Theory*, 103, 11–30, 10.1006/jeth.2001.2820. [492, 494, 501]

Evren, Özgür (2012), "Altruism and voting: a large-turnout result that does not rely on civic duty or cooperative behavior." *Journal of Economic Theory*, 147, 2124–2157, 10.1016/j.jet.2012.09.005. [124]

Farquharson, Robin (1969), *Theory of voting*. Yale University Press, New Haven. [For some corrections, see Richard G. Niemi, "An exegesis of Farquharson's *Theory of voting*", *Public Choice*, 40 (1983), 323-328.] [123, 186, 187]

Fearon, James D. (1999), "Electoral accountability and the control of politicians: selecting good types versus sanctioning poor performance." In *Democracy, Accountability, and Representation* (Adam Przeworski, Susan C. Stokes, and Bernard Manin, eds.), chapter 2, 55–97, Cambridge University Press, 10.1017/cbo9781139175104.003. [463]

Fearon, James D. (2011), "Self-enforcing democracy." *Quarterly Journal of Economics*, 126, 1661–1708, 10.1093/qje/qjr038. [533]

Feddersen, Timothy J., and Wolfgang Pesendorfer (1996), "The swing voter's curse." *American Economic Review*, 86, 408–424. [234]

Feddersen, Timothy J., and Wolfgang Pesendorfer (1998), "Convicting the innocent: the inferiority of unanimous jury verdicts under strategic voting." *American Political Science Review*, 92, 23–35, 10.2307/2585926. [232, 234]

Feddersen, Timothy J., and Alvaro Sandroni (2005), "The calculus of ethical voting." [Unpublished paper.] [196, 199]

Feddersen, Timothy J., and Alvaro Sandroni (2006a), "The calculus of ethical voting." *International Journal of Game Theory*, 35, 1–25, 10.1007/s00182-006-0047-0. [194, 195, 196, 199]

Feddersen, Timothy J., and Alvaro Sandroni (2006b), "A theory of participation in elections." *American Economic Review*, 96, 1271–1282, 10.1257/aer.96.4.1271. [199]

Feddersen, Timothy J., Itai Sened, and Stephen G. Wright (1990), "Rational voting and candidate entry under plurality rule." *American Journal of Political Science*, 34, 1005–1016, 10.2307/2111469. [149, 365]

Ferejohn, John A. (1986), "Incumbent performance and electoral control." *Public Choice*, 50, 5–25, 10.1007/bf00124924. [463]

Ferejohn, John A., and Morris P. Fiorina (1974), "The paradox of not voting: a decision theoretic analysis." *American Political Science Review*, 68, 525–536, 10.1017/s0003055400117368. [124]

Fishburn, Peter C. (1973), *The theory of social choice*. Princeton University Press, 10.1515/9781400868339. [33]

Fishburn, Peter C. (1974), "Social choice functions." *SIAM Review*, 16, 63–90, 10.1137/1016005. [69]

Fishburn, Peter C. (1977), "Condorcet social choice functions." *SIAM Journal on Applied Mathematics*, 33, 469–489, 10.1137/0133030. [187]

Forand, Jean Guillaume (2014), "Two-party competition with persistent policies." *Journal of Economic Theory*, 152, 64–91, 10.1016/j.jet.2014.03.014. [323]

Fudenberg, Drew, and Jean Tirole (1991), *Game theory*. MIT Press, Cambridge, Massachusetts. [557]

Gans, Joshua S., and Michael Smart (1996), "Majority voting with single-crossing preferences." *Journal of Public Economics*, 59, 219–237, 10.1016/0047-2727(95)01503-5. [68, 411]

Geanakoplos, John (2005), "Three brief proofs of Arrow's impossibility theorem." *Economic Theory*, 26, 211–215, 10.1007/s00199-004-0556-7. [68]

Gehrlein, William V., and Peter C. Fishburn (1976), "The probability of the paradox of voting: a computable solution." *Journal of Economic Theory*, 13, 14–25, 10.1016/0022-0531(76)90063-6. [21]

Gibbard, Allan (1973), "Manipulation of voting schemes: a general result." *Econometrica*, 41, 587–601, 10.2307/1914083. [93]

Goodman, Leo A. (1954), "On methods of aggregation." In *Decision processes* (Robert M. Thrall, Clyde H. Coombs, and Robert L. Davis, eds.), 39–48, Wiley, New York. [68]

Grossman, Gene M., and Elhanan Helpman (1996), "Electoral competition and special interest politics." *Review of Economic Studies*, 63, 265–286, 10.2307/2297852. [437]

Haan, Marco, and Bjørn Volkerink (2001), "A runoff system restores the principle of minimum differentiation." *European Journal of Political Economy*, 17, 157–162, 10.1016/s0176-2680(00)00033-1. [365]

Hägele, Günter, and Friedrich Pukelsheim (2001), "Llull's writings on electoral systems." *Studia Lulliana*, 41, 3–38. [68]

Hamlin, Alan, and Colin Jennings (2011), "Expressive political behaviour: foundations, scope and implications." *British Journal of Political Science*, 41, 645–670, 10.1017/s0007123411000020. [199]

Hamlin, Alan, and Colin Jennings (2019), "Expressive voting." In *Oxford Handbook of Public Choice* (Roger D. Congleton, ed.), volume 1, 333–350, Oxford University Press, 10.1093/oxfordhb/9780190469733.013.16. [199]

Hammond, Peter J. (1976), "Equity, Arrow's conditions, and Rawls' difference principle." *Econometrica*, 44, 793–804, 10.2307/1913445. [69]

Harsanyi, John C. (1977a), "Morality and the theory of rational behavior." *Social Research*, 44, 623–656. [199]

Harsanyi, John C. (1977b), "Rule utilitarianism and decision theory." *Erkenntnis*, 11, 25–53, 10.1007/BF00169843. [199]

Harsanyi, John C. (1980), "Rule utilitarianism, rights, obligations and the theory of rational behavior." *Theory and Decision*, 12, 115–133, 10.1007/BF00154357. [199]

Herrera, Helios, David K. Levine, and César Martinelli (2008), "Policy platforms, campaign spending and voter participation." *Journal of Public Economics*, 92, 501–513, 10.1016/j.jpubeco.2007.11.001. [437]

Hinich, Melvin J., John O. Ledyard, and Peter C. Ordeshook (1972), "Nonvoting and the existence of equilibrium under majority rule." *Journal of Economic Theory*, 4, 144–153, 10.1016/0022-0531(72)90145-7. [288]

Horan, Sean (2021), "Agendas in legislative decision-making." *Theoretical Economics*, 16, 235–274, 10.3982/TE2618. [186]

Horan, Sean, Martin J. Osborne, and M. Remzi Sanver (2019), "Positively responsive collective choice rules and majority rule: a generalization of May's theorem to many alternatives." *International Economic Review*, 60, 1489–1504, 10.1111/iere.12394. [67, 68]

Hotelling, Harold (1929), "Stability in competition." *Economic Journal*, 39, 41–57, 10.2307/2224214. [256, 288]

Humphreys, Macartan, and Michael Laver (2010), "Spatial models, cognitive metrics, and majority rule equilibria." *British Journal of Political Science*, 40, 11–30, 10.1017/S0007123409990263. [68]

Imai, Haruo, and Hannu Salonen (2012), "Bargaining and rent seeking." Discussion Paper 80, Aboa Centre for Economics, Turku, Finland. [495]

Indriðason, Indriði H. (2011), "Proportional representation, majoritarian legislatures, and coalitional voting." *American Journal of Political Science*, 55, 954–970, 10.1111/j.1540-5907.2011.00530.x. [150]

Itsumi, Yoshitaka (1974), "Distributional effects of linear income tax schedules." *Review of Economic Studies*, 41, 371–381, 10.2307/2296756. [411]

Kartik, Navin, Richard Van Weelden, and Stephane Wolton (2017), “Electoral ambiguity and political representation.” *American Journal of Political Science*, 61, 958–970, 10.1111/ajps.12310. [315, 323]

Kawamori, Tomohiko (2004), *Legislative bargaining and parties’ patience*. Master’s thesis, Graduate School of Economics, University of Tokyo. [494, 501]

Kawamori, Tomohiko (2005), “Players’ patience and equilibrium payoffs in the Baron-Ferejohn model.” *Economics Bulletin*, 3, 1–5. [493, 494, 495, 501]

Kim, Jaehoon, and Mark Fey (2007), “The swing voter’s curse with adversarial preferences.” *Journal of Economic Theory*, 135, 236–252, 10.1016/j.jet.2006.04.001. [235]

Kleiner, Andreas, and Benny Moldovanu (2017), “Content-based agendas and qualified majorities in sequential voting.” *American Economic Review*, 107, 1477–1506, 10.1257/aer.20160277. [185, 187]

Krasa, Stefan, and Mattias K. Polborn (2009), “Is mandatory voting better than voluntary voting?” *Games and Economic Behavior*, 66, 275–291, 10.1016/j.geb.2008.05.004. [124]

Laffond, Gilbert, and Jean-François Laslier (1991), “Slater’s winners of a tournament may not be in the Banks set.” *Social Choice and Welfare*, 8, 365–369, 10.1007/bf00183047. [178]

Ledyard, John O. (1981), “The paradox of voting and candidate competition: a general equilibrium analysis.” In *Essays in contemporary fields of economics* (George Horwich and James P. Quirk, eds.), 54–80, Purdue University Press, West Lafayette, Indiana. [123, 124]

Ledyard, John O. (1984), “The pure theory of large two-candidate elections.” *Public Choice*, 44, 7–41, 10.1007/bf00124816. [273, 288]

Lindbeck, Assar, and Jörgen W. Weibull (1987), “Balanced-budget redistribution as the outcome of political competition.” *Public Choice*, 52, 273–297, 10.1007/bf00116710. [410]

Loomes, Graham, and Robert Sugden (1982), “Regret theory: an alternative theory of rational choice under uncertainty.” *Economic Journal*, 92, 805–824, 10.2307/2232669. [124]

Martinelli, César, and Akihiko Matsui (2002), “Policy reversals and electoral competition with privately informed parties.” *Journal of Public Economic Theory*, 4, 39–61, 10.1111/1467-9779.00087. [323]

Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green (1995), *Microeconomic theory*. Oxford University Press, New York. [542, 544]

Maskin, Eric S., and Jean Tirole (2004), "The politician and the judge: accountability in government." *American Economic Review*, 94, 1034–1054, 10.1257/0002828042002606. [463]

May, Kenneth O. (1952), "A set of independent necessary and sufficient conditions for simple majority decision." *Econometrica*, 20, 680–684, 10.2307/1907651. [67]

McKelvey, Richard D., and Richard G. Niemi (1978), "A multistage game representation of sophisticated voting for binary procedures." *Journal of Economic Theory*, 18, 1–22, 10.1016/0022-0531(78)90039-x. [186]

McKelvey, Richard D., and Richard E. Wendell (1976), "Voting equilibria in multidimensional choice spaces." *Mathematics of Operations Research*, 1, 144–158, 10.1287/moor.1.2.144. [68]

McMurray, Joseph C. (2013), "Aggregating information by voting: the wisdom of the experts versus the wisdom of the masses." *Review of Economic Studies*, 80, 277–312, 10.1093/restud/rds026. [234]

Meirowitz, Adam, and Kenneth W. Shotts (2009), "Pivots versus signals in elections." *Journal of Economic Theory*, 144, 744–771, 10.1016/j.jet.2008.08.008. [288]

Miller, Nicholas R. (1977), "Graph-theoretical approaches to the theory of voting." *American Journal of Political Science*, 21, 769–803, 10.2307/2110736. [186, 187]

Mirrlees, James A. (1971), "An exploration in the theory of optimum income taxation." *Review of Economic Studies*, 38, 175–208, 10.2307/2296779. [390]

Moulin, Hervé (1980), "On strategy-proofness and single peakedness." *Public Choice*, 35, 437–455, 10.1007/bf00128122. [93]

Moulin, Hervé (1986), "Choosing from a tournament." *Social Choice and Welfare*, 3, 271–291, 10.1007/bf00292732. [186, 187]

Moulin, Hervé (1988), *Axioms of cooperative decision making*. Cambridge University Press, Cambridge, 10.1017/ccol0521360552. [xv, 69, 93, 409]

Mueller, Dennis C. (2003), *Public choice III*. Cambridge University Press, Cambridge, 10.1017/cbo9780511813771. [xv]

Myatt, David P. (2007), "On the theory of strategic voting." *Review of Economic Studies*, 74, 255–281, 10.1111/j.1467-937x.2007.00421.x. [149]

Myatt, David P. (2015), "A theory of voter turnout." [Unpublished paper.] [124]

Nash, John F. (1950), "The bargaining problem." *Econometrica*, 18, 155–162, 10.2307/1907266. [24, 68]

Osborne, Martin J. (1993), "Candidate positioning and entry in a political competition." *Games and Economic Behavior*, 5, 133–151, 10.1006/game.1993.1007. [365]

Osborne, Martin J. (2004), *An introduction to game theory*. Oxford University Press, New York. [264, 271, 539, 546, 550, 552, 554]

Osborne, Martin J. (2024), "Comment on a passage in 'Why did the West extend the franchise? Democracy, inequality, and growth in historical perspective' by Daron Acemoglu and James A. Robinson." Working Paper 778, Department of Economics, University of Toronto. [533]

Osborne, Martin J., Jeffrey S. Rosenthal, and Matthew A. Turner (2000), "Meetings with costly participation." *American Economic Review*, 90, 927–943, 10.1257/aer.90.4.927. [93]

Osborne, Martin J., and Ariel Rubinstein (1994), *A course in game theory*. MIT Press, Cambridge, Massachusetts. [93, 233, 409, 539, 546, 547, 549, 552, 554, 556, 559, 564, 565]

Osborne, Martin J., and Ariel Rubinstein (2003), "Sampling equilibrium, with an application to strategic voting." *Games and Economic Behavior*, 45, 434–441, 10.1016/s0899-8256(03)00147-7. [150]

Osborne, Martin J., and Ariel Rubinstein (2023), *Models in microeconomic theory*, second edition. Open Book Publishers, Cambridge, 10.11647/OBP.0361. [68, 93, 539, 544, 554, 557]

Osborne, Martin J., and Al Slivinski (1996), "A model of political competition with citizen-candidates." *Quarterly Journal of Economics*, 111, 65–96, 10.2307/2946658. [365]

Osborne, Martin J., and Rabee Tourky (2008), "Party formation in single-issue politics." *Journal of the European Economic Association*, 6, 974–1005, 10.1162/jeea.2008.6.5.974. [93]

Osborne, Martin J., and Matthew A. Turner (2010), "Cost benefit analyses versus referenda." *Journal of Political Economy*, 118, 156–187, 10.1086/650305. [234]

Palfrey, Thomas R. (1989), "A mathematical proof of Duverger's law." In *Models of strategic choice in politics* (Peter C. Ordeshook, ed.), 69–91, University of Michigan Press, Ann Arbor. [146, 149]

Palfrey, Thomas R., and Howard Rosenthal (1983), "A strategic calculus of voting." *Public Choice*, 41, 7–53, 10.1007/bf00124048. [123]

Palfrey, Thomas R., and Howard Rosenthal (1985), "Voter participation and strategic uncertainty." *American Political Science Review*, 79, 62–78, 10.2307/1956119. [123, 124]

Prat, Andrea (2006), "Rational votes and political advertising." In *The Oxford Handbook of Political Economy* (Barry R. Weingast and Donald A. Wittman, eds.), chapter 3, 50–63, Oxford University Press, Oxford, 10.1093/oxfordhb/9780199548477.003.0003. [437]

Puppe, Clemens (2018), "The single-peaked domain revisited: a simple global characterization." *Journal of Economic Theory*, 176, 55–80, 10.1016/j.jet.2018.03.003. [69]

Rae, Douglas, and Michael Taylor (1971), "Decision rules and policy outcomes." *British Journal of Political Science*, 1, 71–90, 10.1017/s0007123400008954. [68]

Rasch, Bjørn Erik (2000), "Parliamentary floor voting procedures and agenda setting in Europe." *Legislative Studies Quarterly*, 25, 3–23. [186]

Riker, William H. (1982), "The two-party system and Duverger's Law: an essay on the history of political science." *American Political Science Review*, 76, 753–766, 10.2307/1962968. [365]

Roberts, Kevin W. S. (1977), "Voting over income tax schedules." *Journal of Public Economics*, 8, 329–340, 10.1016/0047-2727(77)90005-6. [68, 411]

Rockafellar, R. Tyrrell (1970), *Convex analysis*. Princeton University Press, 10.1515/9781400873173. [566]

Röell, Ailsa A. (2012), "Voting over nonlinear income tax schedules." [Unpublished paper (earlier version 1996).] [410]

Roemer, John E. (1994), "A theory of policy differentiation in single issue electoral politics." *Social Choice and Welfare*, 11, 355–380, 10.1007/bf00183303. [323]

Roemer, John E. (1997), "Political-economic equilibrium when parties represent constituents: the unidimensional case." *Social Choice and Welfare*, 14, 479–502, 10.1007/s003550050082. [323]

Romer, Thomas (1975), "Individual welfare, majority voting, and the properties of a linear income tax." *Journal of Public Economics*, 4, 163–185, 10.1016/0047-2727(75)90016-x. [411]

Rosenthal, Jeffrey S. (2020), "Maximum binomial probabilities and game theory voter models." *Advances and Applications in Statistics*, 64, 75–85, 10.17654/as064010075. [116]

Rothstein, Paul (1990), "Order restricted preferences and majority rule." *Social Choice and Welfare*, 7, 331–342, 10.1007/bf01376281. [68, 69, 411]

Rothstein, Paul (1991), "Representative voter theorems." *Public Choice*, 72, 193–212, 10.1007/bf00123744. [411]

Royden, Halsey L. (1968), *Real analysis*, second edition. Macmillan, New York. [544]

Rubinstein, Ariel (1991), "Comments on the interpretation of game theory." *Econometrica*, 59, 909–924, 10.2307/2938166. [486]

Saporiti, Alejandro (2009), "Strategy-proofness and single-crossing." *Theoretical Economics*, 4, 127–163, https://econtheory.org/ojs/index.php/te/article/view/20090127. [93]

Satterthwaite, Mark A. (1975), "Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions." *Journal of Economic Theory*, 10, 187–217, 10.1016/0022-0531(75)90050-2. [93]

Schuessler, Alexander A. (2000a), "Expressive voting." *Rationality and Society*, 12, 87–119, 10.1177/104346300012001005. [197, 199]

Schuessler, Alexander A. (2000b), *A logic of expressive choice*. Princeton University Press, 10.1515/9780691222417. [199]

Schultz, Christian (1996), "Polarization and inefficient policies." *Review of Economic Studies*, 63, 331–344, 10.2307/2297855. [323]

Schwartz, Thomas (2008), "Parliamentary procedure: principal forms and political effects." *Public Choice*, 136, 353–377, 10.1007/s11127-008-9300-3. [186]

Sen, Amartya K. (1970), *Collective choice and social welfare*. Holden-Day, San Francisco. [68]

Shepsle, Kenneth A., and Barry R. Weingast (1984), "Uncovered sets and sophisticated voting outcomes with implications for agenda institutions." *American Journal of Political Science*, 28, 49–74, 10.2307/2110787. [186, 187]

Smart, D. R. (1974), *Fixed point theorems*. Cambridge University Press. [545]

Sydsæter, Knut (1981), *Topics in mathematical analysis for economists*. Academic Press, London. [566, 567]

Taylor, Curtis R., and Huseyin Yildirim (2010), "A unified analysis of rational voting with private values and group-specific costs." *Games and Economic Behavior*, 70, 457–471, 10.1016/j.geb.2010.02.008. [124]

von Neumann, John, and Oskar Morgenstern (1944), *Theory of games and economic behavior*, first edition. Princeton University Press, Princeton. [543]

Ward, Benjamin (1961), "Majority rule and allocation." *Journal of Conflict Resolution*, 5, 379–389, 10.1177/002200276100500405. [186]

Wendell, Richard E., and Stuart J. Thorson (1974), "Generalizations of social decisions under majority rule." *Econometrica*, 42, 893–912. [44, 68]

Wittman, Donald (1977), "Candidates with policy preferences: a dynamic model." *Journal of Economic Theory*, 14, 180–189, 10.1016/0022-0531(77)90091-6. [323]

Wittman, Donald (1983), "Candidate motivation: a synthesis of alternative theories." *American Political Science Review*, 77, 142–157, 10.2307/1956016. [323]

Index

agenda formation, 179–180
amendment agenda, 171 (5.8)
  Banks set, 176 (5.8)
  sophisticated sequence, 174 (5.10)
  sophisticated voting, 174 (5.7), 180 (5.9)
anonymous collective choice rule, 12 (1.9)
anonymous social welfare ordering, 61 (1.35)
Arrow's impossibility theorem, 53 (1.9)
assessment (extensive game), 561 (16.36)
asymmetric information voting game, 206 (7.1)
  equilibrium, 209 (7.3)

balanced collection of minimal majorities, 487 (14.5)
Banks set, 172 (5.9)
  amendment agenda, 176 (5.8)
  and top cycle set, 172 (5.6)
bargaining game with voting, 470 (14.1)
  distributive, 478 (14.3)
    equilibrium, 479 (14.3)
    stationary equilibrium, 487 (14.5)
  equilibrium, 473 (14.1), 484 (14.4)
  recurrent, 496–500
  rotating recognition, 496 (14.2)
  stationary strategy, 486 (14.4)
  undominated voting, 474 (14.2)
    equilibrium, 476 (14.2)
Bayesian extensive game, 558 (16.32)
Bayesian game, 550 (16.23)
  Nash equilibrium, 551 (16.25)
  strategy, 550 (16.24)
behavioral strategy, 561 (16.34)
belief system (extensive game), 561 (16.35)
binary agenda, 159 (5.2)
  amendment, 171 (5.8)
  Banks set, 172 (5.9)
  convex, 183 (5.14)
  optimistically sincere strategy, 181 (5.13)
  partitional, 181 (5.12)
  strict Condorcet winner, 162 (5.1)
  successive, 169 (5.6)
  top cycle set, 163 (5.4)
  uncovered set, 178 (5.11)
binary relation, 540 (16.1)
Borda aggregation, 50 (1.6)
Borda rule, 9 (1.7)
Brouwer's fixed point theorem, 545 (16.3)

citizen-candidates
  equilibrium, 354 (10.5)
    agglomerated, 358 (10.6)
    multi-candidate, 363 (10.7), 363 (10.8)
    one-candidate, 355 (10.5)
    two-candidate, 360 (10.7)
  sincere voting, 352 (10.3)
  strategic voting, 353 (10.4)
city-block preferences, 40 (1.24)
  Condorcet winner, 42 (1.6)
coalitional game with transferable payoff, 563 (16.38)
  core, 563 (16.39)
  Shapley value, 564 (16.41)
    characterization, 564 (16.10), 565 (16.11)
  value, 563 (16.40)
coalitional redistribution game, 401 (11.9)
  dissatisfaction-minimization, 408 (11.4)
  Harsanyi coalitional form, 404 (11.11), 404 (11.2)
  Shapley value, 406 (11.3)
  threat game, 402 (11.10)
    equilibrium, 403 (11.1)
collective choice function, 81 (2.2)
  strategy-proof, 81 (2.3), 82 (2.1)
  unanimous, 87 (2.5)
collective choice problem, 6 (1.3)
  odd-strict, 158 (5.1)
  single-crossing preferences, 35 (1.23)
  single-peaked preferences, 30 (1.20)
collective choice rule, 8 (1.4)

- anonymous, 12 (1.9)
- Borda rule, 9 (1.7)
- dictatorial, 12 (1.3), 87 (2.3)
- Gibbard-Satterthwaite theorem, 87 (2.3)
- majority rule, 14 (1.11)
- Nash independent, 24 (1.16)
- neutral, 13 (1.10)
- nonnegatively responsive, 19
- plurality rule, 9 (1.6)
- positively responsive, 17 (1.13), 21 (1.15)
- scoring rule, 10

collective choice with costly reporting, 85 (2.4)
complete binary relation, 540 (16.2)
Condorcet aggregation, 50 (1.7)
Condorcet cycle, 21 (1.5)
Condorcet domain, strict, 25 (1.17)
Condorcet winner, 20 (1.14)
- city-block preferences, 42 (1.6)
- Euclidean preferences, 47 (1.8)
- max preferences, 45 (1.7)

Condorcet-plus domain, 26 (1.18)
consistency (extensive game), 562 (16.37)
convex binary agenda, 183 (5.14)
Copeland winner, 163 (5.5)
core (coalitional game), 563 (16.39)
costly voting, 103–118
coups, 516–517

dictatorial
- collective choice rule, 12 (1.3)
- preference aggregation function, 53 (1.30)

divided majority, 139–149
- Nash equilibrium, 144 (4.4)
- sampling equilibrium, 149 (4.7)
- uncertainty, 143 (4.2)

dynamic game of revolt, 507 (15.1)
- commitment to redistribution, 514–516
- coups, 516–517
- equilibrium, 510 (15.1), 513 (15.1)

electoral competition
- alienation, 256 (8.1)
- candidate of uncertain quality, 429 (12.3)
  - equilibrium, 431 (12.2)
  - interest group, 432 (12.4)
- citizen preferences over candidates, 275 (1.5)
- citizen-candidates
  - agglomerated equilibria, 358 (10.6)
  - equilibrium, 354 (10.5)
  - multi-candidate equilibria, 363 (10.7), 363 (10.8)
  - one-candidate equilibria, 355 (10.5)
  - sincere voting, 352 (10.3)
  - strategic voting, 353 (10.4)
  - two-candidate equilibria, 360 (10.7)
- costly voting, 267 (8.7), 271 (8.7)
- electing legislature, 282–287
- interest group, 422 (12.1), 434 (12.3)
- many office-motivated candidates
  - sequential, 345 (10.2), 346 (10.2), 349 (10.5)
  - sincere voting, 333–335
  - strategic voting, 336 (10.1), 339 (10.1)
- multi-period, 446
- Nash equilibrium interpretation, 287
- privately-informed candidates, 261 (8.6), 263 (8.6), 314–319
- rationalizable actions, 257 (8.2)
- regarding distribution, 379 (11.1)
- repeated, 319–322
- representative candidates, 300 (9.2)
- runoff, 335 (10.1)
- sequential, 250 (8.2), 251 (8.3)
- three office-motivated candidates
  - sequential, 347 (10.4)
- three policy-motivated candidates
  - sequential, 349 (10.4)
- two office-motivated candidates, 247 (8.1), 248 (8.1), 249 (8.1), 250 (8.2), 254 (8.4), 255 (8.4)
  - sequential, 346 (10.3)
- two policy-motivated candidates, 298 (9.1), 299 (9.1), 301 (9.2), 303 (9.3), 304 (9.3)
  - sequential, 347 (10.3)
- two-dimensional positions, 257–258
- two-period, 441 (13.1), 448 (13.2)
  - equilibrium, 444 (13.1), 448 (13.2)
  - well-informed candidates, 454 (13.3), 459 (13.3)
- uncertain median, 258 (8.5), 259 (8.5), 308 (9.5), 310 (9.4), 312 (9.5)
- uncertain partisanship, 277 (8.9), 278 (8.8), 280 (8.2)
- vote mobilization, 417 (12.2)
  - equilibrium, 422 (12.1)

endowed society, 400 (11.8)
ethical voting, 192–197
Euclidean preferences, 46 (1.26)
- Condorcet winner, 47 (1.8)

expected utility, 544 (16.2)

Definition Proposition Example Problem

expressive voting, 197–199
extensive game
  Bayesian, 558 (16.32)
  imperfect information, 559 (16.33)
    assessment, 561 (16.36)
    behavioral strategy, 561 (16.34)
    belief system, 561 (16.35)
    weak sequential equilibrium, 562 (16.37)
  perfect information, 552 (16.26), 553 (16.27)
    one-deviation property, 556 (16.31), 556 (16.7), 557 (16.9)
    strategy, 554 (16.28)
    subgame, 555 (16.29)
    subgame perfect equilibrium, 555 (16.30), 557 (16.8)

favorite alternatives, 8
fixed point theorem, 545 (16.3)

Gibbard-Satterthwaite theorem, 87 (2.3)

Hammond-equitable welfare ordering, 61 (1.37)

improvement of preference profile, 16 (1.12)
independence of irrelevant alternatives, 51 (1.28)
independence property (preferences over lotteries), 543 (16.13)
interpersonal comparisons, 58

lexicographic order, 541 (16.6)
leximin welfare ordering, 60 (1.34)
  characterization, 62 (1.11)
linear order, 541 (16.5)
lottery, 543 (16.11)

majority relation, 35 (1.22)
majority rule, 14 (1.11)
  characterization, 17 (1.1)
max preferences, 44 (1.25)
  Condorcet winner, 45 (1.7)
May's theorem, 17 (1.1)
  generalization, 25 (1.2)
mechanism, 90 (2.6)
median of distribution, 253
median of set, 31 (1.21)
median voter theorem
  electoral competition, 249 (8.2), 299 (9.1)
  single-crossing preferences, 37 (1.5)
  single-peaked preferences, 31 (1.4)
median-based collective choice game, 84 (2.4)
  single-peaked preferences, 84 (2.2)
mixed strategy equilibrium, 548 (16.22)
  existence, 549 (16.6)

Nash equilibrium
  Bayesian game, 551 (16.25)
  strategic game, 546 (16.17)
    existence, 546 (16.4)
Nash independence, 24 (1.16)
Nash welfare ordering, 59 (1.33)
  characterization, 67 (1.13)
neutral collective choice rule, 13 (1.10)
nonnegatively responsive collective choice rule, 19

odd-strict collective choice problem, 158 (5.1)
oligarchic society, 520 (15.2)
  equilibrium, 522 (15.3), 523 (15.2), 524 (15.4), 527 (15.3)
one-deviation property, 556 (16.31)
  subgame perfect equilibrium, 556 (16.7), 557 (16.9)
optimistically sincere strategy, 181 (5.13), 184 (5.10)
optimization
  equality constraint
    necessary conditions, 567 (16.14)
    sufficient conditions, 567 (16.15)
  unconstrained
    necessary conditions, 566 (16.12)
    sufficient conditions, 566 (16.13)

pandering to voters, 449–462
Pareto property, 52 (1.29)
partitional binary agenda, 181 (5.12)
permutation, 11 (1.8)
pessimistic sincerity, 185 (5.9)
pivotal vote, 141
plurality rule (collective choice), 9 (1.6)
plurality rule voting game, 130 (4.1)
  asymmetric information, 206 (7.1)
  divided majority, 139–149
  Nash equilibrium, 131 (4.1)
    existence, 134 (4.2)
  spatial, 138 (4.3)
  uncertain signal qualities, 220 (7.4)
  weak domination, 132 (4.1)
plurality rule with runoff, 22 (1.6)
policy game

Definition Proposition Example Problem

candidate of uncertain quality, 429 (12.3)
equilibrium, 431 (12.2)
interest group, 432 (12.4)
equilibrium, 434 (12.3)
positive responsiveness
amendment agenda, 177 (5.6)
collective choice rule, 17 (1.13), 21 (1.15)
sophisticated voting, 168 (5.3)
preference aggregation function, 50 (1.27)
Arrow's impossibility theorem, 53 (1.9)
dictatorial, 53 (1.30)
independence of irrelevant alternatives, 51 (1.28)
Pareto property, 52 (1.29)
strict Condorcet winner, 57 (1.10)
preference intensities, 58
preference profile, 6 (1.2)
single-peaked, 30 (1.20)
single-peaked on tree, 34
strict, 6 (1.2)
preference relation, 540 (16.3)
continuous, 542 (16.9)
lotteries, 543 (16.12)
independence property, 543 (16.13)
payoff function representation, 542 (16.8), 542 (16.1)
single-peaked, 29 (1.19)
strict, 541 (16.7)
vNM, 544 (16.14)
preferences over lotteries
continuous, 543 (16.12)
expected utility, 544 (16.2)
independence, 543 (16.13)
representation, 544 (16.2)
profile, 545 (16.15)
proportional rule, 137 (4.5)

quasi-dominant strategies, 91

recurrent bargaining, 496–500
regret (two-alternative voting game), 121
repeated elections, 319–322
representative candidates (electoral competition), 300 (9.2), 304 (9.4)
revelation principle, 92
revolt
dynamic game, 507 (15.1)
equilibrium, 510 (15.1), 513 (15.1)
runoff, 22 (1.6), 335 (10.1)
sequential electoral competition, 350
sampling equilibrium, 149 (4.7)
scoring rule, 10
sequential rationality, 562 (16.37)
Shapley value, 564 (16.41)
axiomatic characterization, 564 (16.10), 565 (16.11)
coalitional redistribution game, 406 (11.3)
single-crossing preferences, 35 (1.23)
and single-peaked, 36 (1.13)
voting over transfer systems, 397 (11.2)
single-peaked function, 542 (16.10)
single-peaked preference profile, 30 (1.20)
single-peaked preference relation, 29 (1.19)
single-peaked preferences
and single-crossing, 36 (1.13)
on tree, 34
single-plateau preferences, 32
social welfare ordering, 59 (1.31)
additive invariance, 63 (1.38)
anonymous, 61 (1.35)
Hammond-equitable, 61 (1.37)
leximin, 60 (1.34)
multiplicative invariance, 66 (1.39)
Nash, 59 (1.33)
strong Pareto condition, 61 (1.36)
utilitarian, 59 (1.32)
society
collective choice, 6 (1.1)
endowed, 400 (11.8)
informed and uninformed citizens, 415 (12.1)
oligarchic, 520 (15.2)
voting over transfer systems, 386 (11.2), 395 (11.7)
sophisticated sequence
amendment agenda, 174 (5.10)
successive agenda, 169 (5.7)
sophisticated voting, 160 (5.3)
amendment agenda, 174 (5.7)
positive responsiveness, 168 (5.3)
strict Condorcet winner, 162 (5.1)
subgame perfect equilibrium, 161 (5.1)
successive agenda, 170 (5.4)
top cycle set, 166 (5.3)
strategic abstention, 203–225
strategic game, 545 (16.16), 548 (16.20)
Nash equilibrium, 546 (16.17)
existence, 546 (16.4)
payoff function, 546 (16.16)
strictly competitive, 547 (16.18)
Nash equilibrium, 547 (16.5)

vNM preferences, 548 (16.20)
mixed strategy, 548 (16.21)
mixed strategy equilibrium, 548 (16.22), 549 (16.6)
weak domination, 547 (16.19)
strict Condorcet domain, 25 (1.17)
strictly competitive strategic game, 547 (16.18)
Nash equilibrium, 547 (16.5)
strong Pareto condition, 61 (1.36)
successive agenda, 169 (5.6)
sophisticated sequence, 169 (5.7)
sophisticated voting, 170 (5.4)
top cycle set, 170 (5.5)

threshold strategy (two-alternative voting game), 108 (3.3)
top cycle set, 163 (5.4)
and Banks set, 172 (5.6)
Condorcet winner, 163 (5.2)
nonemptiness, 163 (5.2)
size, 164 (5.2)
sophisticated voting, 166 (5.3)
transfer plan, 389 (11.4)
selfishly-optimal, 389 (11.5)
transfer system, 387 (11.3)
linear, 392 (11.6)
transitive binary relation, 540 (16.2)
two-alternative voting game, 101 (3.1)
ethical individuals, 195 (6.2)
large population, 118
Nash equilibrium, 114 (3.3)
preferences with regret, 121
symmetric equilibrium, 109 (3.4), 111 (3.2)
threshold strategy, 108 (3.3)
voting probability, 116 (3.4)
weak domination, 102 (3.1)
two-period electoral competition
unobserved incumbent actions, 448 (13.2)
equilibrium, 448 (13.2)
unobserved incumbent preferences, 441 (13.1)
equilibrium, 444 (13.1)
well-informed candidates, 454 (13.3)
equilibrium, 459 (13.3)

unanimity rule voting game
asymmetric information, 227 (7.7)
equilibrium, 226 (7.2)
mixed strategy equilibrium, 231 (7.4)
Nash equilibrium, 229 (7.3)
uncertain signal qualities
voting game, 220 (7.4)
uncovered set, 178 (5.11)
amendment agenda, 180 (5.9)
utilitarian welfare ordering, 59 (1.32)
characterization, 64 (1.12)

vNM preference relation, 544 (16.14)
voting game
asymmetric information, 206 (7.1)
adversarial preferences, 219 (7.1)
diverse preferences, 233–234
equilibrium, 209 (7.3), 211 (7.1)
plurality rule, 130 (4.1)
proportional rule, 137 (4.5)
two-alternative, 101 (3.1)
uncertain costs, 106 (3.2)
voting game with uncertain signal qualities, 220 (7.4)
quality threshold equilibrium, 222 (7.6), 224 (7.2)
quality threshold strategy, 221 (7.5)
voting over transfer systems, 384–392
single-crossing preferences, 397 (11.2)

weak domination, 547 (16.19)
plurality rule voting game, 132 (4.1)
weak sequential equilibrium, 562 (16.37)

Definition Proposition Example Problem