Solution manual for

AN INTRODUCTION TO GAME THEORY

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Preface

This manual contains the solutions to all the exercises in my book *An Introduction to Game Theory* (Oxford University Press, 2004). The sources of the problems are given in the section entitled "Notes" at the end of each chapter of the book. Please alert me to errors.

I am grateful to the following people for pointing out errors in earlier versions of this manual: Akihito Asano, Joel Blit, Buddy Chang, Jingyu Chen, Douglas Dion, Xiaohua Fang, Satoshi Fukuda, Martin van Hees, Haifeng Huang, Jochen Konemann, Rebecca Morton, Carolyn Pitchik, William Sandholm, Ahmer Tarar, Levent Ulku, Georg Vanberg, Mingxiao Ye, and Zhanna Zhanabekova.

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1 Introduction

5.3 Altruistic preferences

Person 1 is indifferent between $(1,4)$ and $(3,0)$, and prefers both of these to $(2,1)$. The payoff function u defined by $u(x,y) = x + \frac{1}{2}y$, where x is person 1's income and y is person 2's, represents person 1's preferences. Any function that is an increasing function of u also represents her preferences. For example, the functions $k(x + \frac{1}{2}y)$ for any positive number k, and $(x + \frac{1}{2}y)^2$, do so.

6.1 Alternative representations of preferences

The function v represents the same preferences as does u (because $u(a) < u(b) < u(c)$ and $v(a) < v(b) < v(c)$), but the function w does not represent the same preferences, because $w(a) = w(b)$ while $u(a) < u(b)$.

2 Nash Equilibrium

16.1 Working on a joint project

The game in Figure 3.2 models this situation (as does any other game with the same players and actions in which the ordering of the payoffs is the same as the ordering in Figure 3.2).

	Work hard	*Goof off*
Work hard	$3,3$	$0,2$
Goof off	$2,0$	$1,1$

Figure 3.2 Working on a joint project (alternative version).

17.1 Games equivalent to the *Prisoner's Dilemma*

The game in the left panel differs from the *Prisoner's Dilemma* in both players' preferences. Player 1 prefers (Y, X) to (X, X) to (X, Y) to (Y, Y), for example, which differs from her preference in the *Prisoner's Dilemma*, which is (F, Q) to (Q, Q) to (F, F) to (Q, F), whether we let $X = F$ or $X = Q$.

The game in the right panel is equivalent to the *Prisoner's Dilemma*. If we let $X = Q$ and $Y = F$ then player 1 prefers (F, Q) to (Q, Q) to (F, F) to (Q, F) and player 2 prefers (Q, F) to (Q, Q) to (F, F) to (F, Q), as in the *Prisoner's Dilemma*.

18.1 Hermaphroditic fish

A strategic game that models the situation is shown in Figure 3.3.

	Either role	*Preferred role*
Either role	$\frac{1}{2}(H+L), \frac{1}{2}(H+L)$	L, H
Preferred role	H, L	S, S

Figure 3.3 A model of encounters between pairs of hermaphroditic fish whose preferred roles differ.

In order for this game to differ from the *Prisoner's Dilemma* only in the names of the players' actions, there must be a way to associate each action with an action in the *Prisoner's Dilemma* so that each player's preferences over the four outcomes are the same as they are in the *Prisoner's Dilemma*. Thus we need $L < S < \frac{1}{2}(H + L)$.

That is, the probability of a fish's encountering a potential partner must be large enough that $S > L$, but small enough that $S < \frac{1}{2}(H + L)$.

20.1 Games without conflict

Any two-player game in which each player has two actions and the players have the same preferences may be represented by a table of the form given in Figure 4.1, where a, b, c, and d are any numbers.

	L	R
T	a, a	b, b
B	c, c	d, d

Figure 4.1 A strategic game in which conflict is absent.

27.1 Variant of *Prisoner's Dilemma* with altruistic preferences

a. A game that model the situation is given in Figure 4.2.

	Quiet	*Fink*
Quiet	$4, 4$	$3, 3$
Fink	$3, 3$	$2, 2$

Figure 4.2 The payoffs in a variant of the *Prisoner's Dilemma* in which the players are altruistic.

This game is not the *Prisoner's Dilemma* because one (in fact both) of the players' preferences are not the same as they are in the *Prisoner's Dilemma*. Specifically, player 1 prefers $(Quiet, Quiet)$ to $(Fink, Quiet)$, while in the *Prisoner's Dilemma* she prefers $(Fink, Quiet)$ to $(Quiet, Quiet)$. (Alternatively, you may note that player 1 prefers $(Quiet, Fink)$ to $(Fink, Fink)$, while in the *Prisoner's Dilemma* she prefers $(Fink, Fink)$ to $(Quiet, Fink)$, or that player 2's preferences are similarly not the same as they are in the *Prisoner's Dilemma*.)

b. For an arbitrary value of α the payoffs are given in Figure 5.1. In order that the game be the *Prisoner's Dilemma* we need $3 > 2(1 + \alpha)$ (each player prefers *Fink* to *Quiet* when the other player chooses *Quiet*), $1 + \alpha > 3\alpha$ (each player prefers *Fink* to *Quiet* when the other player choose *Fink*), and $2(1 + \alpha) > 1 + \alpha$ (each player prefers $(Quiet, Quiet)$ to $(Fink, Fink)$). The last condition is satisfied for all nonnegative values of α. The first two conditions are both equivalent to $\alpha < \frac{1}{2}$. Thus the game is the *Prisoner's Dilemma* if and only if $\alpha < \frac{1}{2}$.

If $\alpha = \frac{1}{2}$ then all four outcomes $(Quiet, Quiet)$, $(Quiet, Fink)$, $(Fink, Quiet)$, and $(Fink, Fink)$ are Nash equilibria; if $\alpha > \frac{1}{2}$ then only $(Quiet, Quiet)$ is a Nash equilibrium.

	Quiet	*Fink*
Quiet	$2(1+\alpha), 2(1+\alpha)$	$3\alpha, 3$
Fink	$3, 3\alpha$	$1+\alpha, 1+\alpha$

Figure 5.1 The payoffs in a variant of the *Prisoner's Dilemma* in which the players are altruistic.

27.2 Selfish and altruistic social behavior

a. A game that model the situation is shown in Figure 5.2.

	Sit	*Stand*
Sit	$1, 1$	$2, 0$
Stand	$0, 2$	$0, 0$

Figure 5.2 Behavior on a bus when the players' preferences are selfish (Exercise 27.2*a*).

This game is not the *Prisoner's Dilemma*. If we identify *Sit* with *Quiet* and *Stand* with *Fink* then, for example, $(Stand, Sit)$ is worse for player 1 than (Sit, Sit), rather than better. If we identify *Sit* with *Fink* and *Stand* with *Quiet* then, for example, $(Stand, Stand)$ is worse for player 1 than (Sit, Sit), rather than better. The game has a unique Nash equilibrium, (Sit, Sit).

b. A game that models the situation is shown in Figure 5.3.

	Sit	*Stand*
Sit	$2, 2$	$0, 3$
Stand	$3, 0$	$1, 1$

Figure 5.3 Behavior on a bus when the players' preferences are altruistic (Exercise 27.2*b*).

This game is the *Prisoner's Dilemma*. Its unique Nash equilibrium is the action pair $(Stand, Stand)$.

c. Both people are more comfortable in the equilibrium that results when they act according to their selfish preferences.

30.1 Variants of the *Stag Hunt*

a. The equilibria of the game are the same as those of the original game: $(Stag, \ldots, Stag)$ and $(Hare, \ldots, Hare)$. Any player who deviates from the first profile obtains a hare rather than the fraction $1/n$ of the stag. Any player who deviates from the second profile obtains nothing, rather than a hare.

An action profile in which at least 1 and at most $m-1$ hunters pursue the stag is not a Nash equilibrium, because any one of them is better off catching a hare. An action profile in which at least m and at most $n-1$ hunters pursue the stag is not a Nash equilibrium, because any one of the remaining hunters is better off joining the pursuit of the stag (thereby earning herself the right to a share of the stag).

b. The set of Nash equilibria consists of the action profile $(Hare, \ldots, Hare)$ in which all hunters catch hares, together with any action profile in which exactly k hunters pursue the stag and the remaining hunters catch hares. Any player who deviates from the first profile obtains nothing, rather than a hare. A player who switches from the pursuit of the stag to catching a hare in the second type of profile is worse off, since she obtains a hare rather than the fraction $1/k$ of the stag; a player who switches from catching a hare to pursuing the stag is also worse off since she obtains the fraction $1/(k+1)$ of the stag rather than a hare, and $1/(k+1) < 1/k$.

No other action profile is a Nash equilibrium, by the following argument.

- If some hunters, but fewer than m, pursue the stag then each of them obtains nothing, and is better off catching a hare.
- If at least m and fewer than k hunters pursue the stag then each one that pursues a hare is better off switching to the pursuit of the stag.
- If more than k hunters pursue the stag then the fraction of the stag that each of them obtains is less than $1/k$, so each of them is better off catching a hare.

31.1 Extension of the *Stag Hunt*

Every profile $(e, \ldots, e)$, where e is an integer from 0 to K, is a Nash equilibrium. In the equilibrium $(e, \ldots, e)$, each player's payoff is e. The profile $(e, \ldots, e)$ is a Nash equilibrium since if player i chooses $e_i < e$ then her payoff is $2e_i - e_i = e_i < e$, and if she chooses $e_i > e$ then her payoff is $2e - e_i < e$.

Consider an action profile $(e_1, \ldots, e_n)$ in which not all effort levels are the same. Suppose that e_i is the minimum. Consider some player j whose effort level exceeds e_i. Her payoff is $2e_i - e_j < e_i$, while if she deviates to the effort level e_i her payoff is $2e_i - e_i = e_i$. Thus she can increase her payoff by deviating, so that $(e_1, \ldots, e_n)$ is not a Nash equilibrium.

(This game is studied experimentally by van Huyck, Battalio, and Beil (1990). See also Ochs (1995, 209–233).)

31.2 *Hawk–Dove*

A strategic game that models the situation is shown in Figure 7.1. The game has two Nash equilibria, $(Aggressive, Passive)$ and $(Passive, Aggressive)$.

	Aggressive	*Passive*
Aggressive	$0,0$	$3,1$
Passive	$1,3$	$2,2$

Figure 7.1 *Hawk–Dove.*

33.1 Contributing to a public good

The following game models the situation.

Players The n people.

Actions Each person's set of actions is $\{Contribute, Don't\ contribute\}$.

Preferences Each person's preferences are those given in the problem.

An action profile in which more than k people contribute is not a Nash equilibrium: any contributor can induce an outcome she prefers by deviating to not contributing.

An action profile in which k people contribute is a Nash equilibrium: if any contributor stops contributing then the good is not provided; if any noncontributor switches to contributing then she is worse off.

An action profile in which fewer than k people contribute is a Nash equilibrium only if no one contributes: if someone contributes, she can increase her payoff by switching to noncontribution.

In summary, the set of Nash equilibria is the set of action profiles in which k people contribute together with the action profile in which no one contributes.

34.1 Guessing two-thirds of the average

If all three players announce the same integer $k \geq 2$ then any one of them can deviate to $k - 1$ and obtain \$1 (since her number is now closer to $\frac{2}{3}$ of the average than the other two) rather than $\$\frac{1}{3}$. Thus no such action profile is a Nash equilibrium. If all three players announce 1, then no player can deviate and increase her payoff; thus $(1, 1, 1)$ is a Nash equilibrium.

Now consider an action profile in which not all three integers are the same; denote the highest by k^*.

- Suppose only one player names k^*; denote the other integers named by k_1 and k_2, with $k_1 \geq k_2$. The average of the three integers is $\frac{1}{3}(k^* + k_1 + k_2)$, so that $\frac{2}{3}$ of the average is $\frac{2}{9}(k^* + k_1 + k_2)$. If $k_1 \geq \frac{2}{9}(k^* + k_1 + k_2)$ then k^* is further from $\frac{2}{3}$ of the average than is k_1, and hence does not win. If $k_1 < \frac{2}{9}(k^* + k_1 + k_2)$ then the difference between k^* and $\frac{2}{3}$ of the average is $k^* - \frac{2}{9}(k^* + k_1 + k_2) = \frac{7}{9}k^* - \frac{2}{9}k_1 - \frac{2}{9}k_2$, while the difference between k_1 and $\frac{2}{3}$ of the average is $\frac{2}{9}(k^* + k_1 + k_2) - k_1 = \frac{2}{9}k^* - \frac{7}{9}k_1 + \frac{2}{9}k_2$. The difference between the former and the latter is $\frac{5}{9}k^* + \frac{5}{9}k_1 - \frac{4}{9}k_2 > 0$, so k_1 is closer to $\frac{2}{3}$ of the average than is k^*. Hence the player who names k^* does not win, and is better off naming k_2, in which case she obtains a share of the prize. Thus no such action profile is a Nash equilibrium.

- Suppose two players name k^*, and the third player names $k < k^*$. The average of the three integers is then $\frac{1}{3}(2k^* + k)$, so that $\frac{2}{3}$ of the average is $\frac{4}{9}k^* + \frac{2}{9}k$. We have $\frac{4}{9}k^* + \frac{2}{9}k < \frac{1}{2}(k^* + k)$ (since $\frac{4}{9} < \frac{1}{2}$ and $\frac{2}{9} < \frac{1}{2}$), so that the player who names k is the sole winner. Thus either of the other players can switch to naming k and obtain a share of the prize rather obtaining nothing. Thus no such action profile is a Nash equilibrium.

We conclude that there is only one Nash equilibrium of this game, in which all three players announce the number 1.

(This game is studied experimentally by Nagel (1995).)

34.2 Voter participation

a. For $k = m = 1$ the game is shown in Figure 8.1. It is the same, except for the names of the actions, as the *Prisoner's Dilemma*.

		B supporter	
		abstain	*vote*
A supporter	*abstain*	$1, 1$	$0, 2 - c$
	vote	$2 - c, 0$	$1 - c, 1 - c$

Figure 8.1 The game of voter participation in Exercise 34.2.

b. For $k = m$, denote the number of citizens voting for A by n_A and the number voting for B by n_B. The cases in which $n_A \leq n_B$ are symmetric with those in which $n_A \geq n_B$; I restrict attention to the latter.

$n_A = n_B = k$ (all citizens vote): A citizen who switches from voting to abstaining causes the candidate she supports to lose rather than tie, reducing her payoff from $1 - c$ to 0. Since $c < 1$, this situation is a Nash equilibrium.

$n_A = n_B < k$ (not all citizens vote; the candidates tie): A citizen who switches from abstaining to voting causes the candidate she supports to win rather than tie, increasing her payoff from 1 to $2 - c$. Thus this situation is not a Nash equilibrium.

$n_A = n_B + 1$ or $n_B = n_A + 1$ (a candidate wins by one vote): A supporter of the losing candidate who switches from abstaining to voting causes the candidate she supports to tie rather than lose, increasing her payoff from 0 to $1 - c$. Thus this situation is not a Nash equilibrium.

$n_A \geq n_B + 2$ or $n_B \geq n_A + 2$ (a candidate wins by two or more votes): A supporter of the winning candidate who switches from voting to abstaining does not affect the outcome, but saves the cost c, so such a situation is not a Nash equilibrium.

We conclude that the game has a unique Nash equilibrium, in which all citizens vote.

c. If $0 < k < m$ then a similar logic shows that there is no Nash equilibrium.

$n_A = n_B \leq k$: A supporter of B who switches from abstaining to voting causes B to win rather than tie, increasing her payoff from 1 to $2 - c$. Thus this situation is not a Nash equilibrium.

$n_A = n_B + 1$, or $n_B = n_A + 1$ and $n_A < k$: A supporter of the losing candidate who switches from abstaining to voting causes the candidates to tie, increasing her payoff from 0 to $1 - c$. Thus this situation is not a Nash equilibrium.

$n_A = k$ and $n_B = k + 1$: A supporter of the losing candidate (namely A) who switches from voting to abstaining does not affect the outcome but saves the cost c. Thus this situation is not a Nash equilibrium.

$n_A \geq n_B + 2$ or $n_B \geq n_A + 2$: A supporter of the winning candidate who switches from voting to abstaining does not affect the outcome but saves the cost c, so such a situation is not a Nash equilibrium.

If $k = 0 < m$ then the set of Nash equilibria of the game is the set of action profiles in which exactly one citizen votes for B.

34.3 Choosing a route

A strategic game that models this situation is:

Players The four people.

Actions The set of actions of each person is $\{X, Y\}$ (the route via X and the route via Y).

Preferences Each player's payoff is the negative of her travel time.

In every Nash equilibrium, two people take each route. (In any other case, a person taking the more popular route is better off switching to the other route.) For any such action profile, each person's travel time is either 29.9 or 30 minutes (depending on the route she takes). If a person taking the route via X switches to the route via Y her travel time becomes $22 + 12 = 34$ minutes; if a person taking the route via Y switches to the route via X her travel time becomes $12 + 21.8 = 33.8$ minutes. For any other allocation of people to routes, at least one person can decrease her travel time by switching routes. Thus the set of Nash equilibria is the set of action profiles in which two people take the route via X and two people take the route via Y.

Now consider the situation after the road from X to Y is built. There is no equilibrium in which the new road is not used, by the following argument. Because the only equilibrium before the new road is built has two people taking each route, the only possibility for an equilibrium in which no one uses the new road is for two people to take the route A–X–B and two to take A–Y–B, resulting in a total travel time for each person of either 29.9 or 30 minutes. However, if a person taking A–X–B switches to the new road at X and then takes Y–B her total travel time becomes $9 + 7 + 12 = 28$ minutes.

I claim that in any Nash equilibrium, one person takes A–X–B, two people take A–X–Y–B, and one person takes A–Y–B. For this assignment, each person's travel time is 32 minutes. No person can change her route and decrease her travel time, by the following argument.

- If the person taking A–X–B switches to A–X–Y–B, her travel time increases to $12 + 9 + 15 = 36$ minutes; if she switches to A–Y–B her travel time increases to $21 + 15 = 36$ minutes.

- If one of the people taking A–X–Y–B switches to A–X–B, her travel time increases to $12 + 20.9 = 32.9$ minutes; if she switches to A–Y–B her travel time increases to $21 + 12 = 33$ minutes.

- If the person taking A–Y–B switches to A–X–B, her travel time increases to $15 + 20.9 = 35.9$ minutes; if she switches to A–X–Y–B, her travel time increases to $15 + 9 + 12 = 36$ minutes.

For every other allocation of people to routes at least one person can switch routes and reduce her travel time. For example, if one person takes A–X–B, one person takes A–X–Y–B, and two people take A–Y–B, then the travel time of those taking A–Y–B is $21 + 12 = 33$ minutes; if one of them switches to A–X–B then her travel time falls to $12 + 20.9 = 32.9$ minutes. Or if one person takes A–Y–B, one person takes A–X–Y–B, and two people take A–X–B, then the travel time of those taking A–X–B is $12 + 20.9 = 32.9$ minutes; if one of them switches to A–X–Y–B then her travel time falls to $12 + 8 + 12 = 32$ minutes.

Thus in the equilibrium with the new road every person's travel time *increases*, from either 29.9 or 30 minutes to 32 minutes.

37.1 Finding Nash equilibria using best response functions

a. The *Prisoner's Dilemma* and *BoS* are shown in Figure 11.1; *Matching Pennies* and the two-player *Stag Hunt* are shown in Figure 11.2.

b. The best response functions are indicated in Figure 11.3. The Nash equilibria are (T, C), (M, L), and (B, R).

	Quiet	*Fink*
Quiet	$2\ ,2$	$0\ ,3^*$
Fink	$3^*,0$	$1^*,1^*$

Prisoner's Dilemma

	Bach	*Stravinsky*
Bach	$2^*,1^*$	$0\ ,0$
Stravinsky	$0\ ,0$	$1^*,2^*$

BoS

Figure 11.1 The best response functions in the *Prisoner's Dilemma* (left) and in *BoS* (right).

	Head	*Tail*
Head	$1^*,-1$	$-1\ ,\ \ 1^*$
Tail	$-1\ ,\ \ 1^*$	$1^*,-1$

Matching Pennies

	Stag	*Hare*
Stag	$2^*,2^*$	$0\ ,1$
Hare	$1\ ,0$	$1^*,1^*$

Stag Hunt

Figure 11.2 The best response functions in *Matching Pennies* (left) and the *Stag Hunt* (right).

38.1 Constructing best response functions

The analogue of Figure 38.2 in the book is given in Figure 12.1.

38.2 Dividing money

For each amount named by one of the players, the other player's best responses are given in the following table.

Other player's action	Sets of best responses
0	$\{10\}$
1	$\{9,10\}$
2	$\{8,9,10\}$
3	$\{7,8,9,10\}$
4	$\{6,7,8,9,10\}$
5	$\{5,6,7,8,9,10\}$
6	$\{5,6\}$
7	$\{6\}$
8	$\{7\}$
9	$\{8\}$
10	$\{9\}$

	L	*C*	*R*
T	$2\ ,2$	$1^*,3^*$	$0^*,1$
M	$3^*,1^*$	$0\ ,0$	$0^*,0$
B	$1\ ,0^*$	$0\ ,0^*$	$0^*,0^*$

Figure 11.3 The game in Exercise 37.1.

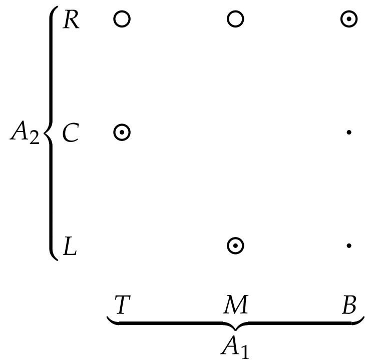


Figure 12.1 The players' best response functions for the game in Exercise 38.1b. Player 1's best responses are indicated by circles, and player 2's by dots. The action pairs for which there is both a circle and a dot are the Nash equilibria.

The best response functions are illustrated in Figure 12.2 (circles for player 1, dots for player 2). From this figure we see that the game has four Nash equilibria: $(5, 5)$, $(5, 6)$, $(6, 5)$, and $(6, 6)$.

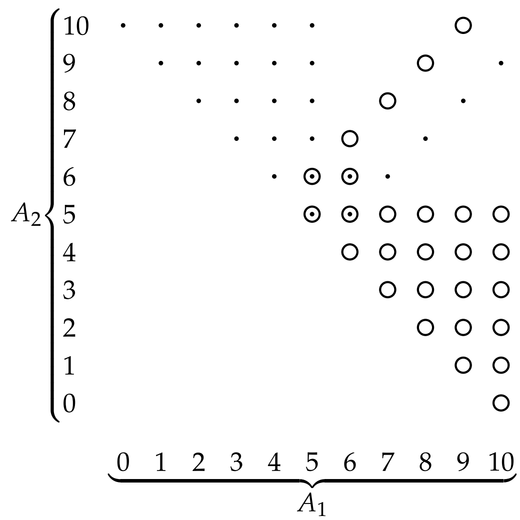


Figure 12.2 The players' best response functions for the game in Exercise 38.2.

41.1 Strict and nonstrict Nash equilibria

Only the Nash equilibrium (a_1^*, a_2^*) is strict. For each of the other equilibria, player 2's action a_2 satisfies $a_2^{***} \leq a_2 \leq a_2^{**}$, and for each such action player 1 has multiple best responses, so that her payoff is the same for a range of actions, only one of which is such that (a_1, a_2) is a Nash equilibrium.

41.2 Finding Nash equilibria using best response functions

First find the best response function of player 1. For any fixed value of a_2, player 1's payoff function $a_1(a_2 - a_1)$ is a quadratic in a_1. The coefficient of a_1^2 is negative and the function is zero at $a_1 = 0$ and at $a_1 = a_2$. Thus, using the symmetry of quadratic functions, $b_1(a_2) = \frac{1}{2}a_2$.

Now find the best response function of player 2. For any fixed value of a_1, player 2's payoff function $a_2(1 - a_1 - a_2)$ is a quadratic in a_2. The coefficient on a_2^2 is negative and the function is zero at $a_2 = 0$ and at $a_2 = 1 - a_1$. Thus if $a_1 \leq 1$ we have $b_2(a_1) = \frac{1}{2}(1 - a_1)$ and if $a_1 > 1$ we have $b_2(a_1) = 0$.

The best response functions are shown in Figure 13.1.

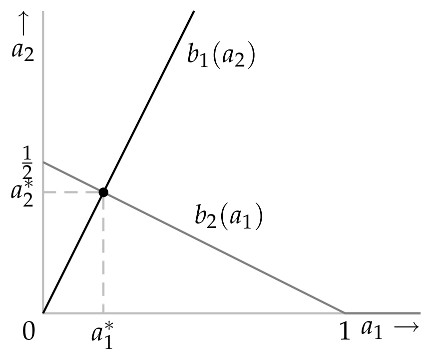


Figure 13.1 The best response functions for the game in Exercise 41.2.

A Nash equilibrium is a pair (a_1^*, a_2^*) such that $a_1^* = b_1(a_2^*)$ and $a_2^* = b_2(a_1^*)$. From the figure we see that there is a unique Nash equilibrium, with $a_1^* < 1$. Thus in this equilibrium $a_1^* = \frac{1}{2}a_2^*$ and $a_2^* = \frac{1}{2}(1 - a_1^*)$. Hence $a_1^* = \frac{1}{4}(1 - a_1^*)$, or $5a_1^* = 1$, or $a_1^* = \frac{1}{5}$. Hence $a_2^* = \frac{2}{5}$. Thus the game has a unique Nash equilibrium, $(\frac{1}{5}, \frac{2}{5})$.

42.1 A joint project

A strategic game that models this situation is:

Players The two people.

Actions The set of actions of each person i is the set of effort levels (the set of numbers x_i with $0 \leq x_i \leq 1$).

Preferences Person i's payoff to the action pair (x_1, x_2) is $\frac{1}{2}f(x_1, x_2) - c(x_i)$.

a. Assume that $f(x_1, x_2) = 3x_1x_2$ and $c(x_i) = x_i^2$. To find the Nash equilibria of the game, first find the players' best response functions. Player 1's best response to x_2 is the action x_1 that maximizes $\frac{3}{2}x_1x_2 - x_1^2$, or $x_1(\frac{3}{2}x_2 - x_1)$. This function is a quadratic that is zero when $x_1 = 0$ and when $x_1 = \frac{3}{2}x_2$. The coefficient of x_1^2 is negative, so the maximum of the function occurs at $x_1 = \frac{3}{4}x_2$. Thus player 1's best response function is

$$b_1(x_2) = \tfrac{3}{4}x_2.$$

Similarly, player 2's best response function is

$$b_2(x_1) = \tfrac{3}{4}x_1.$$

The best response functions are shown in Figure 14.1.

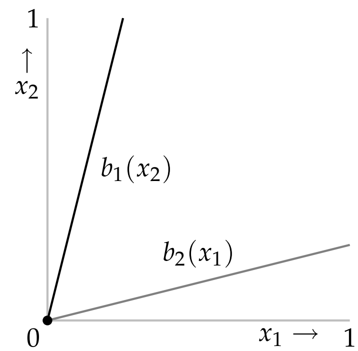


Figure 14.1 The best response functions for the game in Exercise 42.1a.

In a Nash equilibrium (x_1^*, x_2^*) we have $x_1^* = b_1(x_2^*)$ and $x_2^* = b_2(x_1^*)$, or $x_1^* = \frac{3}{4}x_2^*$ and $x_2^* = \frac{3}{4}x_1^*$. Substituting x_2^* in the first equation we obtain $x_1^* = \frac{9}{16}x_1^*$, so that $x_1^* = 0$. Thus $x_2^* = 0$.

We conclude that the game has a unique Nash equilibrium, $(x_1^*, x_2^*) = (0, 0)$. In this equilibrium, both players' payoffs are zero.

If each player i chooses $x_i = 1$ then the total output is 3, and each player's payoff is $\frac{3}{2} - 1 = \frac{1}{2}$, rather than 0 as in the Nash equilibrium.

b. When $f(x_1, x_2) = 4x_1x_2$ and $c(x_i) = x_i$, player 1's payoff function is

$$2x_1x_2 - x_1 = x_1(2x_2 - 1).$$

Thus if $x_2 < \frac{1}{2}$ her best response is $x_1 = 0$, if $x_2 = \frac{1}{2}$ then all values of x_1 are best responses, and if $x_2 > \frac{1}{2}$ her best response is $x_1 = 1$. That is, player 1's best response function is

$$b_1(x_2) = \begin{cases} 0 & \text{if } x_2 < \frac{1}{2} \\ \{x_1 : 0 \le x_1 \le 1\} & \text{if } x_2 = \frac{1}{2} \\ 1 & \text{if } x_2 > \frac{1}{2}. \end{cases}$$

Player 2's best response function is the same. (That is, $b_2(x) = b_1(x)$ for all x.) The best response functions are shown in Figure 15.1.

We see that the game has three Nash equilibria, $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$, and $(1, 1)$.

The players' payoffs at these equilibria are $(0, 0)$, $(0, 0)$, and $(1, 1)$. There is no pair of effort levels that yields both players payoffs higher than 1, but there are pairs of effort levels that yield both players payoffs higher than 0, for example $(1, 1)$, which yields the payoffs $(1, 1)$.

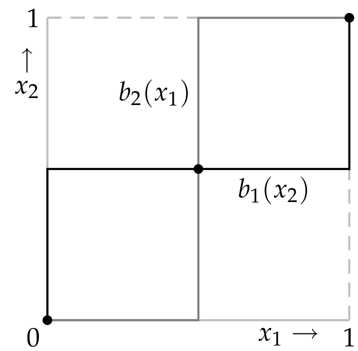


Figure 15.1 The best response functions for the game in Exercise 42.1b.

44.1 Contributing to a public good

The best response of player 1 to the contribution c_2 of player 2 is the value of c_1 that maximizes player 1's payoff $w + c_2 + (w - c_1)(c_1 + c_2)$. This function is a quadratic in c_1 (remember that $w + c_2$ is a constant). The coefficient of c_1^2 is negative, and the value of the function is equal to $w + c_2$ when $c_1 = w$ and when $c_1 = -c_2$. Thus the function attains a maximum at $c_1 = \frac{1}{2}(w - c_2)$. We conclude that player 1's best response function is

$$b_1(c_2) = \tfrac{1}{2}(w - c_2).$$

Player 2's best response function is similarly

$$b_2(c_1) = \tfrac{1}{2}(w - c_1).$$

A Nash equilibrium is a pair (c_1^*, c_2^*) such that $c_1^* = b_1(c_2^*)$ and $c_2^* = b_2(c_1^*)$, so that

$$c_1^* = \tfrac{1}{2}(w - c_2^*) = \tfrac{1}{2}(w - \tfrac{1}{2}(w - c_1^*)) = \tfrac{1}{4}w + \tfrac{1}{4}c_1^*$$

and hence $c_1^* = \frac{1}{3}w$. Substituting this value into player 2's best response function we get $c_2^* = \frac{1}{3}w$.

We conclude that the game has a unique Nash equilibrium $(c_1^*, c_2^*) = (\frac{1}{3}w, \frac{1}{3}w)$, in which each person contributes one third of her wealth to the public good.

In this equilibrium each player's payoff is $\frac{4}{3}w + \frac{4}{9}w^2$. If each player contributes $\frac{1}{2}w$ to the public good then her payoff is $\frac{3}{2}w + \frac{1}{2}w^2$, which exceeds $\frac{4}{3}w + \frac{4}{9}w^2$ for all w (since $\frac{3}{2} > \frac{4}{3}$ and $\frac{1}{2} > \frac{4}{9}$).

When there are n players the payoff function of player 1 is

$$\begin{aligned} w - c_1 + c_1 + c_2 + \cdots + c_n + (w - c_1)(c_1 + c_2 + \cdots + c_n) &= \\ w + c_2 + \cdots + c_n + (w - c_1)(c_1 + c_2 + \cdots + c_n)&. \end{aligned}$$

This function is a quadratic in c_1. The coefficient of c_1^2 is negative, and the value of the function is equal to $w + c_2 + \cdots + c_n$ when $c_1 = w$ and when $c_1 = -c_2 - c_3 - \cdots - c_n$. Thus the function attains a maximum at $c_1 = \frac{1}{2}(w - c_2 - c_3 - \cdots - c_n)$.

We conclude that player 1's best response function is

$$b_1(c_{-1}) = \tfrac{1}{2}(w - c_2 - c_3 - \cdots - c_n)$$

where c_{-1} is the list of the contributions of the players other than 1. Similarly, any player i's best response function is

$$b_i(c_{-i}) = \tfrac{1}{2}(w - (c_1 + c_2 + \cdots + c_n) + c_i).$$

A Nash equilibrium is an action profile $(c_1^*, \ldots, c_n^*)$ such that $c_i^* = b_i(c_{-i}^*)$ for all i. We can write the condition $c_1^* = b_1(c_{-1}^*)$ as

$$2c_1^* = w - c_2^* - c_3^* - \cdots - c_n^*,$$

or

$$w = 2c_1^* + c_2^* + c_3^* + \cdots + c_n^*.$$

Writing the other conditions $c_i^* = b_i(c_{-i}^*)$ similarly, we obtain the system of equations

$$\begin{aligned}
w &= 2c_1^* + c_2^* + c_3^* + \cdots + c_n^* \\
w &= c_1^* + 2c_2^* + c_3^* + \cdots + c_n^* \\
&\vdots \\
w &= c_1^* + c_2^* + c_3^* + \cdots + 2c_n^*
\end{aligned}$$

Subtracting the second equation from the first we conclude that $c_1^* = c_2^*$. Similarly subtracting each equation from the next we deduce that c_i^* is the same for all i. Denote the common value by c^*. From any of the equations we deduce that $w = (n+1)c^*$. Hence $c^* = w/(n+1)$.

In conclusion, when there are n players the game has a unique Nash equilibrium $(c_1^*, \ldots, c_n^*) = (w/(n+1), \ldots, w/(n+1))$. The total amount contributed in this equilibrium is $nw/(n+1)$, which increases as n increases, approaching w as n increases without bound.

Player 1's payoff in the equilibrium is $w + (n-1)w/(n+1) + (nw/(n+1))^2$. As n increases without bound, this payoff increases, approaching $2w + w^2$. If each player contributes $\frac{1}{2}w$ to the public good, each player's payoff is $w + \frac{1}{2}(n-1)w + n(w/2)^2$, which increases without bound as n increases without bound.

47.1 Strict equilibria and dominated actions

For player 1, T is weakly dominated by M, and strictly dominated by B. For player 2, no action is weakly or strictly dominated. The game has a unique Nash equilibrium, (M, L). This equilibrium is not strict. (When player 2 choose L, B yields player 1 the same payoff as does M.)

47.2 Nash equilibrium and weakly dominated actions

The only Nash equilibrium of the game in Figure 17.1 is (T, L). The action T is weakly dominated by M and the action L is weakly dominated by C. (There are of course many other games that satisfy the conditions.)

	L	C	R
T	$1,1$	$0,1$	$0,0$
M	$1,0$	$2,1$	$1,2$
B	$0,0$	$1,1$	$2,0$

Figure 17.1 A game with a unique Nash equilibrium, in which both players' equilibrium actions are weakly dominated. (The unique Nash equilibrium is (T, L).)

48.1 Voting

First consider an action profile in which the winner receives one more vote than the loser and at least one citizen who votes for the winner prefers the loser to the winner. Any citizen who votes for the winner and prefers the loser to the winner can, by switching her vote, cause her favorite candidate to win rather than lose. Thus no such action profile is a Nash equilibrium.

Next consider an action profile in which the winner receives one more vote than the loser and all citizens who vote for the winner prefer the winner to the loser. Because a majority of citizens prefer A to B, the winner in any such case must be A. No citizen who prefers A to B can induce a better outcome by changing her vote, since her favorite candidate wins. Now consider a citizen who prefers B to A. By assumption, every such citizen votes for B; a change in her vote has no effect on the outcome (A still wins). Thus every such action profile is a Nash equilibrium.

Finally consider an action profile in which the winner receives at least three more votes than the loser. In this case no change in any citizen's vote has any effect on the outcome. Thus every such profile is a Nash equilibrium.

In summary, the Nash equilibria are: any action profile in which A receives one more vote than B and all the citizens who vote for A prefer A to B, and any action profile in which the winner receives at least three more votes than the loser.

The only equilibrium in which no citizen uses a weakly dominated action is that in which every citizen votes for her favorite candidate.

49.1 Voting between three candidates

Fix some citizen, say i; suppose she prefers A to B to C. By the argument in the text, citizen i's voting for C is weakly dominated by her voting for A (and by her voting for B). Her voting for B is clearly not weakly dominated by her voting for C. I now argue that her voting for B is not weakly dominated by her voting for A. Suppose that the other citizens' votes are equally divided between B and C; no one votes for A. Then if citizen i votes for A the outcome is a tie between B and C, while if she votes for B the outcome is that B wins. Thus for this configuration of the other citizens' votes, citizen i is better off voting for B than she is voting for A. Thus her voting for B is not weakly dominated by her voting for A.

Now fix some citizen, say i, and consider the candidate she ranks in the middle, say candidate B. The action profile in which all citizens vote for B is a Nash equilibrium. (No citizen's changing her vote affects the outcome.) In this equilibrium, citizen i does not vote for her favorite candidate, but the action she takes is not weakly dominated. (Other Nash equilibria also satisfy the conditions in the exercise.)

49.2 Approval voting

First I argue that any action a_i of player i that includes a vote for i's least preferred candidate, say candidate k, is weakly dominated by the action a_i' that differs from a_i only in that candidate k does not receive a vote in a_i'. For any list a_{-i} of the other players' actions, the outcome of (a_i', a_{-i}) differs from that of (a_i, a_{-i}) only in that the total number of votes received by candidate k is one less in (a_i', a_{-i}) than it is in (a_i, a_{-i}). There are two possible implications for the winners of the election, depending on a_{-i}: either the set of winners is the same in (a_i, a_{-i}) as it is in (a_i', a_{-i}), or candidate k is a winner in (a_i, a_{-i}) but not in (a_i', a_{-i}). Because candidate k is player i's least preferred candidate, a_i' thus weakly dominates a_i.

I now argue that any action a_i of player i that excludes a vote for i's most preferred candidate, say candidate 1, is weakly dominated by the action a_i' that differs from a_i only in that candidate 1 receives a vote in a_i'. The argument is symmetric with the one in the previous paragraph. For any list a_{-i} of the other players' actions, the outcome of (a_i', a_{-i}) differs from that of (a_i, a_{-i}) only in that the total number of votes received by candidate 1 is one more in (a_i', a_{-i}) than it is in (a_i, a_{-i}). There are two possible implications for the winners of the election, depending on a_{-i}: either the set of winners is the same in (a_i, a_{-i}) as it is in (a_i', a_{-i}), or candidate 1 is a winner in (a_i', a_{-i}) but not in (a_i, a_{-i}). Because candidate 1 is player i's most preferred candidate, a_i' thus weakly dominates a_i.

Finally I argue that if citizen i prefers candidate 1 to candidate 2 to $\ldots$ to candidate k then the action a_i that consists of votes for candidates 1 and $k-1$ is not weakly dominated.

- The action a_i is not weakly dominated by any action that excludes votes for either candidate 1 or candidate $k-1$ (or both). Suppose a_i' excludes a vote for candidate 1. Then if the numbers of votes by the other citizens for candidates 1 and 2 are both equal to $m \geq 2$, and the total votes for all other candidates are at most $m-2$ (which is possible given that the number of citizens is at least three), then citizen i's taking the action a_i leads candidate 1 to win, while the action a_i' leads at best (from the point of view of citizen i) to a tie between candidates 1 and 2. Thus a_i' does not weakly dominate a_i. Similarly, suppose that a_i' excludes a vote for candidate $k-1$. Then if the numbers of votes by the other citizens for candidates $k-1$ and k are both equal to $m \geq 2$, while the total votes for all other candidates are at most $m-2$, then citizen i's taking the action a_i leads candidate $k-1$ to win, while the action a_i' leads at best (from the point of view of citizen i) to a tie between candidates $k-1$ and k.

- Now let a_i' be an action that includes votes for both candidate 1 and candidate $k-1$, and also for at least one other candidate, say candidate j. Suppose that the total votes by the other citizens for candidates 1 and j are both equal to $m \geq 2$, and the total votes for all other candidates are at most $m-2$. Then citizen i's taking the action a_i leads candidate 1 to win, while the action a_i' leads at best (from the point of view of citizen i) to a tie between candidates 1 and j. Thus a_i' does not weakly dominate a_i.

50.1 Other Nash equilibria of the game modeling collective decision-making

Denote by i the player whose favorite policy is the median favorite policy. The set of Nash equilibria includes every action profile in which (*i*) i's action is her favorite policy x_i^*, (*ii*) every player whose favorite policy is less than x_i^* names a policy equal to at most x_i^*, and (*iii*) every player whose favorite policy is greater than x_i^* names a policy equal to at least x_i^*.

To show this, first note that the outcome is x_i^*, so player i cannot induce a better outcome for herself by changing her action. Now, if a player whose favorite position is less than x_i^* changes her action to some $x < x_i^*$, the outcome does not change; if such a player changes her action to some $x > x_i^*$ then the outcome either remains the same (if some player whose favorite position exceeds x_i^* names x_i^*) or increases, so that the player is not better off. A similar argument applies to a player whose favorite position is greater than x_i^*.

The set of Nash equilibria also includes, for any positive integer $k \leq n$, every action profile in which k players name the median favorite policy x_i^*, at most $\frac{1}{2}(n-3)$ players name policies less than x_i^*, and at most $\frac{1}{2}(n-3)$ players name policies greater than x_i^*. (In these equilibria, the favorite policy of a player who names a policy less than x_i^* may be greater than x_i^*, and vice versa. The conditions on the numbers of players who name policies less than x_i^* and greater than x_i^* ensure that no such player can, by naming instead her favorite policy, move the median policy closer to her favorite policy.)

Any action profile in which all players name the same, arbitrary, policy is also a Nash equilibrium; the outcome is the common policy named.

More generally, any profile in which at least three players name the same, arbitrary, policy x, at most $(n-3)/2$ players name a policy less than x, and at most

$(n-3)/2$ players name a policy greater than x is a Nash equilibrium. (In both cases, no change in any player's action has any effect on the outcome.)

50.2 Another mechanism for collective decision-making

When the policy chosen is the mean of the announced policies, player i's announcing her favorite policy does not weakly dominate all her other actions. For example, if there are three players, the favorite policy of player 1 is 0.3, and the other players both announce the policy 0, then player 1 should announce the policy 0.9, which leads to the policy 0.3 ($= (0+0+0.9)/3$) being chosen, rather than 0.3, which leads to the policy 0.1.

51.2 Symmetric strategic games

The games in Exercise 31.2, Example 39.1, and Figure 47.2 (both games) are symmetric. The game in Exercise 41.2 is not symmetric. The game in Section 2.8.4 is symmetric if and only if $u_1 = u_2$.

52.2 Equilibrium for pairwise interactions in a single population

The Nash equilibria are (A, A), (A, C), and (C, A). Only the equilibrium (A, A) is relevant if the game is played between the members of a single population—this equilibrium is the only *symmetric* equilibrium.

3 Nash Equilibrium: Illustrations

58.1 Cournot's duopoly game with linear inverse demand and different unit costs

Following the analysis in the text, the best response function of firm 1 is

$$b_1(q_2) = \begin{cases} \frac{1}{2}(\alpha - c_1 - q_2) & \text{if } q_2 \leq \alpha - c_1 \\ 0 & \text{otherwise} \end{cases}$$

while that of firm 2 is

$$b_2(q_1) = \begin{cases} \frac{1}{2}(\alpha - c_2 - q_1) & \text{if } q_1 \leq \alpha - c_2 \\ 0 & \text{otherwise.} \end{cases}$$

To find the Nash equilibrium, first plot these two functions. Each function has the same general form as the best response function of either firm in the case studied in the text. However, the fact that $c_1 \neq c_2$ leads to two qualitatively different cases when we combine the two functions to find a Nash equilibrium. If c_1 and c_2 do not differ very much then the functions in the analogue of Figure 59.1 intersect at a pair of outputs that are both positive. If c_1 and c_2 differ a lot, however, the functions intersect at a pair of outputs in which $q_1 = 0$.

Precisely, if $c_1 \leq \frac{1}{2}(\alpha + c_2)$ then the downward-sloping parts of the best response functions intersect (as in Figure 59.1), and the game has a unique Nash equilibrium, given by the solution of the two equations

$$\begin{aligned} q_1 &= \tfrac{1}{2}(\alpha - c_1 - q_2) \\ q_2 &= \tfrac{1}{2}(\alpha - c_2 - q_1). \end{aligned}$$

This solution is

$$(q_1^*, q_2^*) = \left(\tfrac{1}{3}(\alpha - 2c_1 + c_2), \tfrac{1}{3}(\alpha - 2c_2 + c_1)\right).$$

If $c_1 > \frac{1}{2}(\alpha + c_2)$ then the downward-sloping part of firm 1's best response function lies below the downward-sloping part of firm 2's best response function (as in Figure 22.1), and the game has a unique Nash equilibrium, $(q_1^*, q_2^*) = (0, \frac{1}{2}(\alpha - c_2))$.

In summary, the game always has a unique Nash equilibrium, defined as follows:

$$\begin{cases} \left(\frac{1}{3}(\alpha - 2c_1 + c_2), \frac{1}{3}(\alpha - 2c_2 + c_1)\right) & \text{if } c_1 \leq \frac{1}{2}(\alpha + c_2) \\ \left(0, \frac{1}{2}(\alpha - c_2)\right) & \text{if } c_1 > \frac{1}{2}(\alpha + c_2). \end{cases}$$

The output of firm 2 exceeds that of firm 1 in every equilibrium.

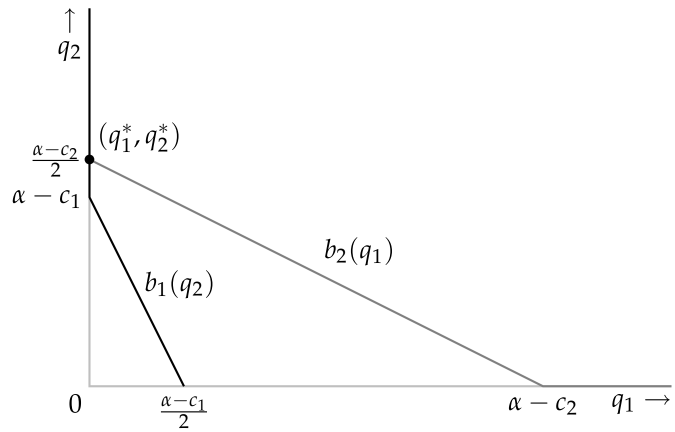


Figure 22.1 The best response functions in Cournot's duopoly game under the assumptions of Exercise 58.1 when $\alpha - c_1 < \frac{1}{2}(\alpha - c_2)$. The unique Nash equilibrium in this case is $(q_1^*, q_2^*) = (0, \frac{1}{2}(\alpha - c_2))$.

If c_2 decreases then firm 2's output increases and firm 1's output either falls, if $c_1 \leq \frac{1}{2}(\alpha + c_2)$, or remains equal to 0, if $c_1 > \frac{1}{2}(\alpha + c_2)$. The total output increases and the price falls.

59.1 Cournot's duopoly game with linear inverse demand and a quadratic cost function

Firm 1's profit is

$$\pi_1(q_1, q_2) = \begin{cases} q_1(\alpha - q_1 - q_2) - q_1^2 & \text{if } q_1 + q_2 \leq \alpha \\ -q_1^2 & \text{if } q_1 + q_2 > \alpha \end{cases}$$

or

$$\pi_1(q_1, q_2) = \begin{cases} q_1(\alpha - 2q_1 - q_2) & \text{if } q_1 + q_2 \leq \alpha \\ -q_1^2 & \text{if } q_1 + q_2 > \alpha. \end{cases}$$

When it is positive, this function is a quadratic in q_1 that is zero at $q_1 = 0$ and at $q_1 = (\alpha - q_2)/2$. Thus firm 1's best response function is

$$b_1(q_2) = \begin{cases} \frac{1}{4}(\alpha - q_2) & \text{if } q_2 \leq \alpha \\ 0 & \text{if } q_2 > \alpha. \end{cases}$$

Since the firms' cost functions are the same, firm 2's best response function is the same as firm 1's: $b_2(q) = b_1(q)$ for all q. The firms' best response functions are shown in Figure 23.1.

Solving the two equations $q_1^* = b_1(q_2^*)$ and $q_2^* = b_2(q_1^*)$ we find that there is a unique Nash equilibrium, in which the output of firm i ($i = 1, 2$) is $q_i^* = \frac{1}{5}\alpha$.

59.2 Cournot's duopoly game with linear inverse demand and a fixed cost

Firm i's payoff function is

$$\begin{cases} 0 & \text{if } q_i = 0 \\ q_i(P(q_1 + q_2) - c) - f & \text{if } q_i > 0. \end{cases}$$

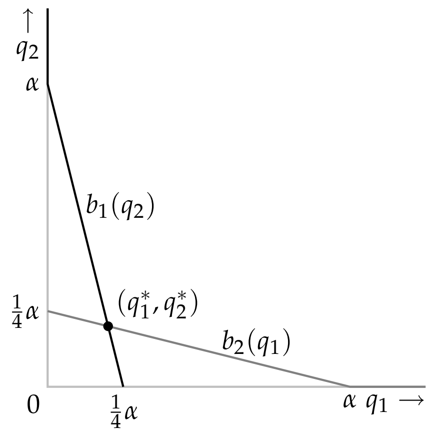


Figure 23.1 The best response functions in Cournot's duopoly game with linear inverse demand and a quadratic cost function, as in Exercise 59.1. The unique Nash equilibrium is $(q_1^*, q_2^*) = (\frac{1}{5}\alpha, \frac{1}{5}\alpha)$.

As before firm 1's best response to q_2 is $(\alpha - c - q_2)/2$ if firm 1's profit is nonnegative for this output; otherwise its best response is the output of zero. Firm 1's profit when it produces $(\alpha - c - q_2)/2$ and firm 2 produces q_2 is

$$\frac{\alpha - c - q_2}{2}\left(\alpha - c - \frac{\alpha - c - q_2}{2} - q_2\right) - f = \left(\frac{\alpha - c - q_2}{2}\right)^2 - f,$$

which is nonnegative if

$$\left(\frac{\alpha - c - q_2}{2}\right)^2 > f,$$

or if $q_2 \leq \alpha - c - 2\sqrt{f}$. Let $\overline{q} = \alpha - c - 2\sqrt{f}$. Then firm 1's best response function is

$$b_1(q_2) = \begin{cases} \frac{1}{2}(\alpha - c - q_2) & \text{if } q_2 < \overline{q} \\ \{0, \frac{1}{2}(\alpha - c - q_2)\} & \text{if } q_2 = \overline{q} \\ 0 & \text{if } q_2 > \overline{q}. \end{cases}$$

(If $q_2 = \overline{q}$ then firm 1's profit is zero whether it produces the output $\frac{1}{2}(\alpha - c - q_2)$ or the output 0; both outputs are optimal.)

Thus firm 1's best response function has a "jump": for outputs of firm 2 slightly less than $\overline{q}$ firm 1 wants to produce a positive output (and earn a small profit), while for outputs of firm 2 slightly greater than $\overline{q}$ it wants to produce an output of zero.

Firm 2's cost function is the same as firm 1's, so its best response function is the same.

Because of the jumps in the best response functions, there are four qualitatively different cases, depending on the value of f. If f is small enough that $\overline{q} > \frac{1}{2}(\alpha - c)$ (or, equivalently, $f < (\alpha - c)^2/16$) then the best response functions take the form given in Figure 24.1. In this case the existence of the fixed cost has no impact on the equilibrium, which remains $(q_1^*, q_2^*) = (\frac{1}{3}(\alpha - c), \frac{1}{3}(\alpha - c))$.

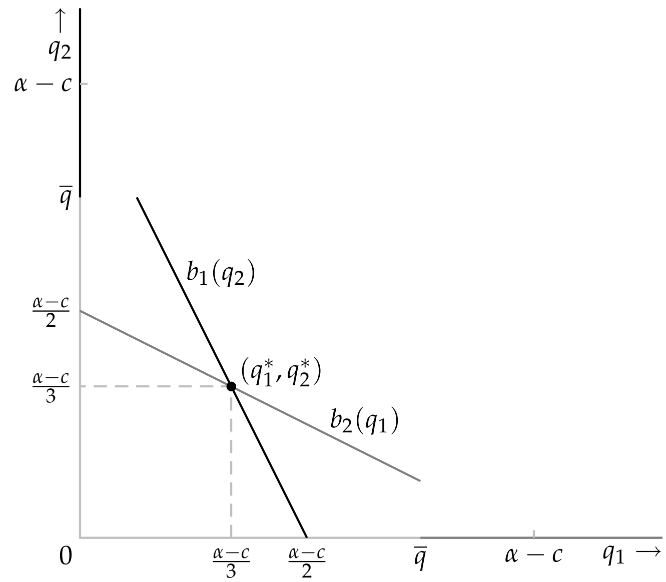


Figure 24.1 The best response functions in Cournot's duopoly game when the inverse demand function is $P(Q) = \alpha - Q$ (where this is positive) and the cost function of each firm is $f + cq$, with $f < (\alpha - c)^2/16$. The unique Nash equilibrium is $(q_1^*, q_2^*) = (\frac{1}{3}(\alpha - c), \frac{1}{3}(\alpha - c))$ (as in the case in which $f = 0$).

As f increases, the point at which the best response functions jump moves closer to the origin. Eventually $\overline{q}$ enters the range from $\frac{1}{3}(\alpha - c)$ to $\frac{1}{2}(\alpha - c)$ (which implies that $(\alpha - c)^2/16 < f < (\alpha - c)^2/9$), in which case the best response functions take the forms shown in the left panel of Figure 24.2. In this case there are three Nash equilibria: $(0, \frac{1}{2}(\alpha - c))$, $((\alpha - c)/3, (\alpha - c)/3)$, and $(\frac{1}{2}(\alpha - c), 0)$.

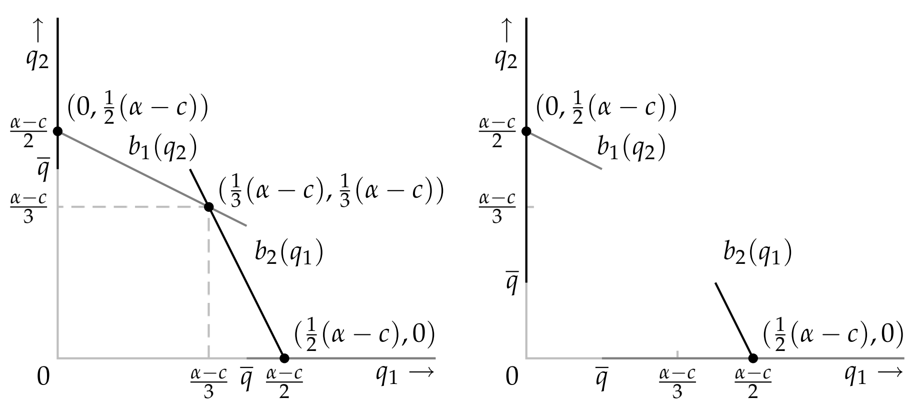


Figure 24.2 The best response functions in Cournot's duopoly game when the inverse demand function is $P(Q) = \alpha - Q$ (where this is positive) and the cost function of each firm is $f + cq$, with $(\alpha - c)^2/16 < f < (\alpha - c)^2/9$ (left panel) and $f > (\alpha - c)^2/9$ (right panel). In the first case the game has three Nash equilibria: $(0, \frac{1}{2}(\alpha - c))$, $(\frac{1}{3}(\alpha - c), \frac{1}{3}(\alpha - c))$, and $(\frac{1}{2}(\alpha - c), 0)$. In the second case it has two Nash equilibria: $(0, \frac{1}{2}(\alpha - c))$ and $(\frac{1}{2}(\alpha - c), 0)$.

As f increases further, there is a point at which $\overline{q}$ becomes less than $\frac{1}{3}(\alpha - c)$ but is still positive (implying that $(\alpha - c)^2/9 < f < (\alpha - c)^2/4$), so that the best response functions take the forms shown in the right panel of Figure 24.2. In this case there are two Nash equilibria: $(0, \frac{1}{2}(\alpha - c))$ and $(\frac{1}{2}(\alpha - c), 0)$.

Finally, if f is extremely large then a firm does not want to produce any output even if the other firm produces no output. This occurs when $f > (\alpha - c)^2/4$; the unique Nash equilibrium in this case is $(0, 0)$.

60.1 Variant of Cournot's duopoly game with market-share maximizing firms

Let firm 1 be the market-share maximizing firm. If $q_2 > \alpha - c$, there is no output of firm 1 for which its profit is nonnegative. Thus its best response to such an output of firm 2 is $q_1 = 0$. If $q_2 \leq \alpha - c$ then firm 1 wants to choose its output q_1 large enough that the price is c (and hence its profit is zero). Thus firm 1's best response to such a value of q_2 is $q_1 = \alpha - c - q_2$. We conclude that firm 1's best response function is

$$b_1(q_2) = \begin{cases} \alpha - c - q_2 & \text{if } q_2 \leq \alpha - c \\ 0 & \text{if } q_2 > \alpha - c. \end{cases}$$

Firm 2's best response function is the same as in Section 3.1.3, namely

$$b_2(q_1) = \begin{cases} (\alpha - c - q_2)/2 & \text{if } q_2 \leq \alpha - c \\ 0 & \text{if } q_2 > \alpha - c. \end{cases}$$

These best response functions are shown in Figure 26.1. The game has a unique Nash equilibrium, $(q_1^*, q_2^*) = (\alpha - c, 0)$, in which firm 2 does not operate. (The price is c, and firm 1's profit is zero.)

If both firms maximize their market shares, then the downward-sloping parts of their best response functions coincide in the analogue of Figure 26.1. Thus every pair (q_1, q_2) with $q_1 + q_2 = \alpha - c$ is a Nash equilibrium.

60.2 Nash equilibrium of Cournot's duopoly game and the collusive outcome

The firms' total profit is $(q_1 + q_2)(\alpha - c - q_1 - q_2)$, or $Q(\alpha - c - Q)$, where Q denotes total output. This function is a quadratic in Q that is zero when $Q = 0$ and when $Q = \alpha - c$, so that its maximizer is $Q^* = \frac{1}{2}(\alpha - c)$.

If each firm produces $\frac{1}{4}(\alpha - c)$ then its profit is $\frac{1}{8}(\alpha - c)^2$. This profit exceeds its Nash equilibrium profit of $\frac{1}{9}(\alpha - c)^2$.

If one firm produces $Q^*/2$, the other firm's best response is $b_i(Q^*/2) = \frac{1}{2}(\alpha - c - \frac{1}{4}(\alpha - c)) = \frac{3}{8}(\alpha - c)$. That is, if one firm produces $Q^*/2$, the other firm wants to produce *more* than $Q^*/2$.

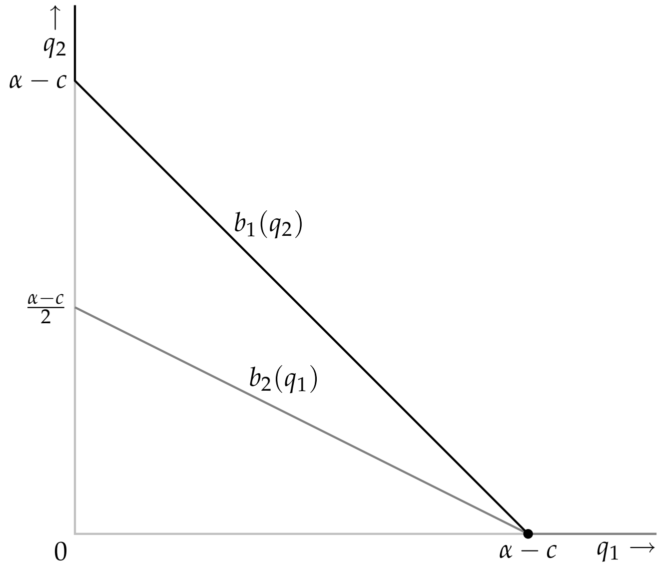


Figure 26.1 The best response functions in a variant of Cournot's duopoly game in which in which the inverse demand function is $P(Q) = \alpha - Q$ (where this is positive) and the cost function of each firm is cq, and firm 1 maximizes its market share, rather than its profit. The unique Nash equilibrium is $(q_1^*, q_2^*) = (\alpha - c, 0)$.

61.1 Cournot's game with many firms

Firm 1's payoff function is

$$\begin{cases} q_1(\alpha - c - q_1 - q_2 - \cdots - q_n) & \text{if } q_1 + q_2 + \cdots + q_n \leq \alpha \\ -cq_1 & \text{if } q_1 + q_2 + \cdots + q_n > \alpha. \end{cases}$$

As in the case of two firms, this function is a quadratic in q_1 where it is positive, and is zero when $q_1 = 0$ and when $q_1 = \alpha - c - q_2 - \cdots - q_n$. Thus firm 1's best response function is

$$b_1(q_{-1}) = \begin{cases} (\alpha - c - q_2 - \cdots - q_n)/2 & \text{if } q_2 + \cdots + q_n \leq \alpha - c \\ 0 & \text{if } q_2 + \cdots + q_n > \alpha - c. \end{cases}$$

(Recall that q_{-1} stands for the list of the outputs of all the firms except firm 1.)

The best response functions of every other firm is the same.

The conditions for $(q_1^*, \ldots, q_n^*)$ to be a Nash equilibrium are

$$\begin{aligned} q_1^* &= b_1(q_{-1}^*) \\ q_2^* &= b_2(q_{-2}^*) \\ &\vdots \\ q_n^* &= b_2(q_{-n}^*) \end{aligned}$$

or, in an equilibrium in which all the firms' outputs are positive,

$$q_1^* = \tfrac{1}{2}(\alpha - c - q_2^* - q_3^* - \cdots - q_n^*)$$

$$\begin{aligned} q_2^* &= \tfrac{1}{2}(\alpha - c - q_1^* - q_3^* - \cdots - q_n^*) \\ &\vdots \\ q_n^* &= \tfrac{1}{2}(\alpha - c - q_1^* - q_2^* - \cdots - q_{n-1}^*). \end{aligned}$$

We can write these equations as

$$\begin{aligned} 0 &= \alpha - c - 2q_1^* - q_2^* - \cdots - q_{n-1}^* - q_n^* \\ 0 &= \alpha - c - q_1^* - 2q_2^* - \cdots - q_{n-1}^* - q_n^* \\ &\vdots \\ 0 &= \alpha - c - q_1^* - q_2^* - \cdots - q_{n-1}^* - 2q_n^*. \end{aligned}$$

If we subtract the second equation from the first we obtain $0 = -q_1^* + q_2^*$, or $q_1^* = q_2^*$. Similarly subtracting the third equation from the second we conclude that $q_2^* = q_3^*$, and continuing with all pairs of equations we deduce that $q_1^* = q_2^* = \cdots = q_n^*$. Let the common value of the firms' outputs be q^*. Then each equation is $0 = \alpha - c - (n+1)q^*$, so that $q^* = (\alpha - c)/(n+1)$.

In summary, the game has a unique Nash equilibrium, in which the output of every firm i is $(\alpha - c)/(n+1)$.

The price at this equilibrium is $\alpha - n(\alpha - c)/(n+1)$, or $(\alpha + nc)/(n+1)$. As n increases this price decreases, approaching c as n increases without bound: $\alpha/(n+1)$ decreases to 0 and $nc/(n+1)$ decreases to c.

62.1 Nash equilibrium of Cournot's game with small firms

- If $P(Q^*) < \underline{p}$ then every firm producing a positive output makes a negative profit, and can increase its profit (to 0) by deviating and producing zero.

- If $P(Q^* + \underline{q}) > \underline{p}$, take a firm that is either producing no output, or an arbitrarily small output. (Such a firm exists, since demand is finite.) Such a firm earns a profit of either zero or arbitrarily close to zero. If it deviates and chooses the output $\underline{q}$ then total output changes to at most $Q^* + \underline{q}$, so that the price still exceeds $\underline{p}$ (since $P(Q^* + \underline{q}) > \underline{p}$). Hence the deviant makes a positive profit.

63.1 Interaction among resource-users

The game is given as follows.

Players The firms.

Actions Each firm's set of actions is the set of all nonnegative numbers (representing the amount of input it uses).

Preferences The payoff of each firm i is

$$\begin{cases} x_i(1-(x_1+\cdots+x_n)) & \text{if } x_1+\cdots+x_n \leq 1 \\ 0 & \text{if } x_1+\cdots+x_n > 1. \end{cases}$$

Up to the 6th printing of the book, this exercise asked the reader to find values of α and c such that the game is the same as the one in Exercise 61.1. If $c = 0$ is allowed, the answer is $c = 0$ and $\alpha = 1$. However, when $c = 0$ the example in Section 3.1.3 and the game in Exercise 61.1 have multiple equilibria; in the 7th printing and subsequently, Section 3.1.3 assumes $c > 0$.

The Nash equilibria of the game consist of the action profile $(x_1, \ldots, x_n) = (1/(n+1), \ldots, 1/(n+1))$ and any action profile $(x_1, \ldots, x_n)$ in which the sum of the actions of every set of $n-1$ players is at least 1.

In the first Nash equilibrium, each firm's output is $(1/(n+1))(1-n/(n+1)) = 1/(n+1)^2$; in the other equilibria, each firm's output is 0. If $x_i = 1/(2n)$ for $i = 1, \ldots, n$ then each firm's output is $1/(4n)$, which exceeds $1/(n+1)^2$ for $n \geq 2$. (We have $1/(4n) - 1/(n+1)^2 = (n-1)^2/(4n(n+1)^2) > 0$ for $n \geq 2$.)

67.1 Bertrand's duopoly game with constant unit cost

The pair (c, c) of prices remains a Nash equilibrium; the argument is the same as before. Further, as before, there is no other Nash equilibrium. The argument needs only very minor modification. For an arbitrary function D there may exist no monopoly price p^m; in this case, if $p_i > c$, $p_j > c$, $p_i \geq p_j$, and $D(p_j) = 0$ then firm i can increase its profit by reducing its price slightly below $\overline{p}$ (for example).

67.2 Bertrand's duopoly game with discrete prices

Yes, (c, c) is still a Nash equilibrium, by the same argument as before.

In addition, $(c+1, c+1)$ is a Nash equilibrium (where c is given in cents). In this equilibrium both firms' profits are positive. If either firm raises its price or lowers it to c, its profit becomes zero. If either firm lowers its price below c, its profit becomes negative.

No other pair of prices is a Nash equilibrium, by the following argument, similar to the argument in the text for the case in which a price can be any nonnegative number.

- If $p_i < c$ then the firm whose price is lowest (or either firm, if the prices are the same) can increase its profit (to zero) by raising its price to c.

- If $p_i = c$ and $p_j \geq c+1$ then firm i can increase its profit from zero to a positive amount by increasing its price to $c+1$.

- If $p_i > p_j \geq c+1$ then firm i can increase its profit (from zero) by lowering its price to $c+1$.

- If $p_i = p_j \geq c + 2$ and $p_j < \alpha$ then either firm can increase its profit by lowering its price by one cent. (If firm i does so, its profit changes from $\frac{1}{2}(p_i - c)(\alpha - p_i)$ to $(p_i - 1 - c)(\alpha - p_i + 1) = (p_i - 1 - c)(\alpha - p_i) + p_i - 1 - c$. We have $p_i - 1 - c \geq \frac{1}{2}(p_i - c)$ and $p_i - 1 - c > 0$, since $p_i \geq c + 2$.)

- If $p_i = p_j \geq c + 2$ and $p_j \geq \alpha$ then either firm can increase its profit by lowering its price to p^m.

68.1 Bertrand's oligopoly game

Consider a profile $(p_1, \ldots, p_n)$ of prices in which $p_i \geq c$ for all i and at least two prices are equal to c. Every firm's profit is zero. If any firm raises its price its profit remains zero. If a firm charging more than c lowers its price, but not below c, its profit also remains zero. If a firm lowers its price below c then its profit is negative. Thus any such profile is a Nash equilibrium.

To show that no other profile is a Nash equilibrium, we can argue as follows.

- If some price is less than c then the firm charging the lowest price can increase its profit (to zero) by increasing its price to c.

- If exactly one firm's price is equal to c then that firm can increase its profit by raising its price a little (keeping it less than the next highest price).

- If all firms' prices exceed c then the firm charging the highest price can increase its profit by lowering its price to some price between c and the lowest price being charged.

68.2 Bertrand's duopoly game with different unit costs

a. If all consumers buy from firm 1 when both firms charge the price c_2, then $(p_1, p_2) = (c_2, c_2)$ is a Nash equilibrium by the following argument. Firm 1's profit is positive, while firm 2's profit is zero (since it serves no customers).

- If firm 1 increases its price, its profit falls to zero.

- If firm 1 reduces its price, say to p, then its profit changes from $(c_2 - c_1)(\alpha - c_2)$ to $(p - c_1)(\alpha - p)$. Since c_2 is less than the maximizer of $(p - c_1)(\alpha - p)$, firm 1's profit falls.

- If firm 2 increases its price, its profit remains zero.

- If firm 2 decreases its price, its profit becomes negative (since its price is less than its unit cost).

Under this rule no other pair of prices is a Nash equilibrium, by the following argument.

- If $p_i < c_1$ for $i = 1, 2$ then the firm with the lower price (or either firm, if the prices are the same) can increase its profit (to zero) by raising its price above that of the other firm.
- If $p_1 > p_2 \geq c_2$ then firm 2 can increase its profit by raising its price a little.
- If $p_2 > p_1 \geq c_1$ then firm 1 can increase its profit by raising its price a little.
- If $p_2 \leq p_1$ and $p_2 < c_2$ then firm 2's profit is negative, so that it can increase its profit by raising its price.
- If $p_1 = p_2 > c_2$ then firm 2 can increase its profit by lowering its price a little.

b. Now suppose that the rule for splitting up the customers when the prices are equal specifies that firm 2 receives some customers when both prices are c_2. By the argument for part *a*, the only possible Nash equilibrium is $(p_1, p_2) = (c_2, c_2)$. (The argument in part *a* that every other pair of prices is not a Nash equilibrium does not use the fact that customers are split equally when $(p_1, p_2) = (c_2, c_2)$.) But if $(p_1, p_2) = (c_2, c_2)$ and firm 2 receives some customers, firm 1 can increase its profit by reducing its price a little and capturing the entire market.

69.1 Bertrand's duopoly game with fixed costs

At the pair of prices $(\overline{p}, \overline{p})$, both firms' profits are zero. (Firm 1 receives all the demand and obtains the profit $(\overline{p} - c)(\alpha - \overline{p}) - f = 0$, and firm 2 receives no demand.) This pair of prices is a Nash equilibrium by the following argument.

- If either firm raises its price its profit remains zero (it receives no customers).
- If either firm lowers its price then it receives all the demand and earns a negative profit (since f is less than the maximum of $(p - c)(\alpha - p)$).

No other pair of prices (p_1, p_2) is a Nash equilibrium, by the following argument.

- If $p_1 = p_2 < \overline{p}$ then firm 1's profit is negative; firm 1 can increase its profit by raising its price.
- If $p_1 = p_2 > \overline{p}$ then firm 2's profit is zero; firm 2 can obtain a positive profit by lowering its price a little.
- If $p_i < p_j$ and firm i's profit is positive then firm j can increase its profit from zero to almost the current level of i's profit by changing its price to be slightly less than p_i.
- If $p_i < p_j$ and firm i's profit is zero then firm i can earn a positive profit by raising its price a little.
- If $p_i < p_j$ and firm i's profit is negative then firm i can increase its profit to zero by raising its price above p_j.

74.1 Electoral competition with asymmetric voters' preferences

The unique Nash equilibrium remains (m, m); the direct argument is exactly the same as before. (The dividing line between the supporters of two candidates with different positions changes. If $x_i < x_j$, for example, the dividing line is $\frac{1}{3}x_i + \frac{2}{3}x_j$ rather than $\frac{1}{2}(x_i + x_j)$. The resulting change in the best response functions does not affect the Nash equilibrium.)

74.2 Electoral competition with three candidates

Note regarding statement of exercise up to third printing of book: The exercise requires the additional assumption that less than one-third of the citizens' favorite positions are equal to the median favorite position. (This assumption is satisfied, for example, if the density of favorite position is nonatomic (i.e. the distribution function of favorite positions is continuous).)

If a single candidate enters, then either remaining candidate can enter at the same position and tie for first place, which she regards as better than staying out of the race. Thus there is no Nash equilibrium in which a single candidate enters.

If more than one candidate enters and not all these candidates tie for first place, then at least one of them loses and would be better off staying out of the race. Thus in any Nash equilibrium, all the candidates who enter tie for first place.

If two candidates enter, then by the argument in the text for the case in which there are only two candidates, each takes the position m in an equilibrium. But then by the assumption that less than a third of the citizens' favorite positions are equal to m, the third candidate can enter, capture the votes of more than a third of the citizens, and hence win outright. Thus there is no Nash equilibrium in which two candidates enter.

If all three candidates enter and choose the same position, each candidate receives one third of the votes. If the common position is equal to m, then by the assumption about the dispersion of the citizens' favorite positions, any candidate can win outright by moving either slightly to the left or slightly to the right of m (by doing so she obtains more than a third of the votes). If the common position is different from m then any candidate can win outright (obtaining more than one-half of the votes) by moving to m. Thus there is no Nash equilibrium in which all three candidates enter and choose the same position.

Finally, suppose that all three candidates enter and do not all choose the same position. By the second argument, they all tie for first place. First suppose that their positions are all different, say $x < y < z$. If $x < m < y < z$ then the candidate at x can move to m and win outright and if $x < y < m < z$ then the candidate at z can move to m and win outright. If $y = m$ then either the candidate at x or the candidate at y can move close to m can win outright (by the assumption on the dispersion in the citizens' favorite positions). Now suppose that two candidates' positions are the same. If the common position is m, then the remaining candidate can move close to m and win outright (by the assumption on the dispersion in the citizens' favorite positions). If the common position differs from m, then the remaining candidate can move to m and win outright.

We conclude that the game has no Nash equilibrium.

75.1 U.S. presidential election

The game has a unique equilibrium, in which the both candidates choose the position m_1 (the median favorite position in the state with the most electoral college votes). The outcome is a tie.

The following argument shows that this pair of positions is a Nash equilibrium. If a candidate deviates to a position less than m_1, she loses in state 1 and wins in state 2, and thus loses overall. If a candidate deviates to a position greater than m_1, she loses in both states.

To see that there is no other Nash equilibrium, first consider a pair of positions for which candidate 1 loses in state 1, and hence loses overall. By deviating to m_1, she either wins in state 1, and hence wins overall, or, if candidate 2's position is m_1, ties in state 1, and ties overall. Thus her deviation induces an outcome she prefers. The same argument applies to candidate 2, so that in any equilibrium the candidates tie in state 1. Now, if the candidates' positions are either different, or the same and different from m_1, either candidate can win outright rather than tying for first place by moving to m_1. Thus there is a single equilibrium, in which both candidates' positions are m_1.

75.2 Electoral competition between candidates who care only about the winning position

First consider a pair (x_1, x_2) of positions for which either $x_1 < m$ and $x_2 < m$, or $x_1 > m$ and $x_2 > m$.

- If $x_1 \neq x_2$ and the winner's position is different from her favorite position then the winner can move slightly closer to her favorite position and still win.
- If $x_1 \neq x_2$ and the winner's position is equal to her favorite position then the other candidate can move to m, which is closer to her favorite position than the winner's position, and win.
- If $x_1 = x_2 < m$ then the candidate whose favorite position exceeds m can move to m and cause the winning position to be m rather than $x_1 = x_2$.
- If $x_1 = x_2 > m$ then the candidate whose favorite position is less than m can move to m and cause the winning position to be m rather than $x_1 = x_2$.

Now suppose the candidates' positions are on opposite sides of m: either $x_1 < m < x_2$, or $x_2 < m < x_1$.

- If each candidate's position is on the same side of m as her favorite position and one candidate wins outright, then the loser can win outright by moving to m, which she prefers to the position of the other candidate.

- If each candidate's position is on the same side of m as her favorite position and the candidates tie for first place, then by moving slightly closer to m either candidate can win. If her movement is small enough she prefers her new position to the previous compromise $\frac{1}{2}(x_1 + x_2)$ $(= m)$.

- If each candidate's position is on the opposite side of m to her favorite position then the winner, or either player in the case of a tie, can move to her favorite position and either win outright or cause the winning position to be the other candidate's position, in both cases improving the outcome from her point of view.

Now suppose that $x_1 = m$ and $x_2 < m$. If $x_1^* < m$ then candidate 1 is better off choosing a slightly smaller value of x_1 (in which case she still wins). If $x_1^* > m$ then candidate 1 is better off choosing a slightly larger value of x_1 (in which case she still wins). Thus (x_1, x_2) is not a Nash equilibrium. A similar argument applies to pairs (x_1, x_2) for which $x_1 = m$ and $x_2 > m$, and for which $x_1 \neq m$ and $x_2 = m$.

Finally, if $(x_1, x_2) = (m, m)$, then the candidates tie. If either candidate changes her position then she loses, and the winning position does not change. Thus this pair of positions is a Nash equilibrium.

76.1 Citizen-candidates

If $b \leq 2c$ then the game has a Nash equilibrium in which a single citizen, with favorite position m, stands as a candidate. In this equilibrium, the candidate's payoff is $b - c$ and the payoff of every other citizen i is $-|x_i - m|$, where x_i is i's favorite position. The argument is as follows.

- If the citizen who stands as a candidate withdraws she obtains the payoff $K < b - c$.

- If another citizen with the favorite position m stands, she obtains the payoff $\frac{1}{2}b - c \geq 0$ (given $b \leq 2c$), as opposed to the payoff of 0 if she does not stand.

- If a citizen with favorite position $x_i \neq m$ stands, she loses, and obtains the payoff $-|x_i - m| - c < -|x_i - m|$.

If two citizens with favorite position m become candidates, each candidate's payoff is $\frac{1}{2}b - c$. If one withdraws then she obtains the payoff of 0, so for equilibrium we require $b \geq 2c$. Now consider a citizen whose favorite position is close to m. If she enters she wins outright, obtaining the payoff $b - c$. Since $b \geq 2c$, this payoff is positive, and hence exceeds her payoff if she does not stand (which is negative, since the winner's position is then different from her favorite position).

Thus there is no equilibrium in which two citizens with favorite position m stand as candidates.

Now consider the possibility of an equilibrium in which two citizens with favorite positions different from m stand as candidates. For an equilibrium the candidates must tie, otherwise one loses, and can do better by withdrawing. Thus the positions, say x_1 and x_2, must satisfy $\frac{1}{2}(x_1 + x_2) = m$. If x_1 and x_2 are close enough to m then any other citizen loses if she becomes a candidate. Thus there are equilibria in which two citizens with positions symmetric about m, and sufficiently close to m, become candidates.

76.2 Electoral competition for more general preferences

a. If x^* is a Condorcet winner then for any $y \neq x^*$ a majority of voters prefer x^* to y, so y is not a Condorcet winner. Thus there is no more than one Condorcet winner.

b. Suppose that one of the remaining voters prefers y to z to x, and the other prefers z to x to y. For each position there is another position preferred by a majority of voters, so no position is a Condorcet winner.

c. Now suppose that x^* is a Condorcet winner. Then the strategic game described the exercise has a unique Nash equilibrium in which both candidates choose x^*. This pair of actions is a Nash equilibrium because if either candidate chooses a different position she loses. For any other pair of actions either one candidate loses, in which case that candidate can deviate to the position x^* and at least tie, or the candidates tie at a position different from x^*, in which case either of them can deviate to x^* and win.

   If there is no Condorcet winner then for every position there is another position preferred by a majority of voters. Thus for every pair of distinct positions the loser can deviate and win, and for every pair of identical positions either candidate can deviate and win. Thus there is no Nash equilibrium.

77.1 Competition in product characteristics

Suppose there are two firms. If the products are different, then either firm increases its market share by making its product more similar to that of its rival. Thus in every possible equilibrium the products are the same. But if $x_1 = x_2 \neq m$ then each firm's market share is 50%, while if it changes its product to be closer to m then its market share rises above 50%. Thus the only possible equilibrium is $(x_1, x_2) = (m, m)$. This pair of positions is an equilibrium, since each firm's market share is 50%, and if either firm changes its product its market share falls below 50%.

Now suppose there are three firms. If all firms' products are the same, each obtains one-third of the market. If $x_1 = x_2 = x_3 = m$ then any firm, by changing its product a little, can obtain close to one-half of the market. If $x_1 = x_2 = x_3 \neq m$

then any firm, by changing its product a little, can obtain more than one-half of the market. If the firms' products are not all the same, then at least one of the extreme products is different from the other two products, and the firm that produces it can increase its market share by making it more similar to the other products. Thus when there are three firms there is no Nash equilibrium.

80.1 Direct argument for Nash equilibria of *War of Attrition*

- If $t_1 = t_2$ then either player can increase her payoff by conceding slightly later (in which case she obtains the object for sure, rather than getting it with probability $\frac{1}{2}$).
- If $0 < t_i < t_j$ then player i can increase her payoff by conceding at 0.
- If $0 = t_i < t_j < v_i$ then player i can increase her payoff (from 0 to almost $v_i - t_j > 0$) by conceding slightly after t_j.

Thus there is no Nash equilibrium in which $t_1 = t_2$, $0 < t_i < t_j$, or $0 = t_i < t_j < v_i$ (for $i = 1$ and $j = 2$, or $i = 2$ and $j = 1$). The remaining possibility is that $0 = t_i < t_j$ and $t_j \geq v_i$ for $i = 1$ and $j = 2$, or $i = 2$ and $j = 1$. In this case player i's payoff is 0, while if she concedes later her payoff is negative; player j's payoff is v_j, her highest possible payoff in the game.

80.2 Variant of *War of Attrition*

The game is

Players The two parties to the dispute.

Actions Each player's set of actions is the set of possible concession times (nonnegative numbers).

Preferences Player i's preferences are represented by the payoff function

$$u_i(t_1, t_2) = \begin{cases} 0 & \text{if } t_i < t_j \\ \frac{1}{2}(v_i - t_i) & \text{if } t_i = t_j \\ v_i - t_j & \text{if } t_i > t_j. \end{cases}$$

where j is the other player.

Three representative cross-sections of player i's payoff function are shown in Figure 36.1.

From this figure we deduce that the best response function of player i is

$$B_i(t_j) = \begin{cases} \{t_i : t_i > t_j\} & \text{if } t_j < v_i \\ \{t_i : t_i \geq 0\} & \text{if } t_j = v_i \\ \{t_i : 0 \leq t_i < t_j\} & \text{if } t_j > v_i. \end{cases}$$

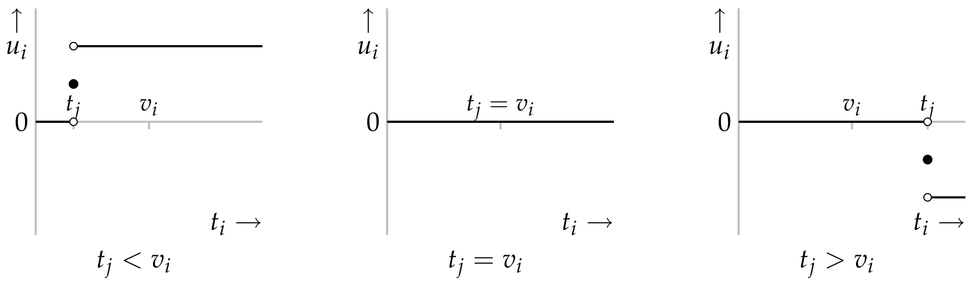


Figure 36.1 Three cross-sections of player i's payoff function in the variant of the *War of Attrition* in Exercise 80.2.

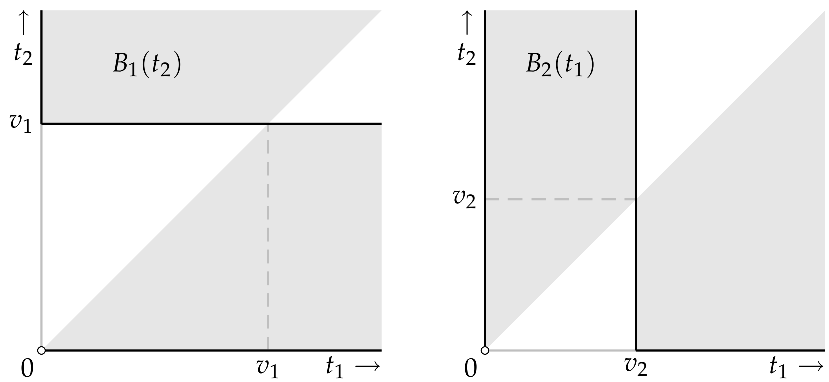


Figure 36.2 The players' best response functions in the variant of the *War of Attrition* in Exercise 80.2 for $v_1 > v_2$. Player 1's best response function is in the left panel; player 2's is in the right panel. (The sloping edges are excluded.)

The best response functions are shown in Figure 36.2 for a case in which $v_1 > v_2$.

Superimposing the two best response functions, we see that if $v_1 > v_2$ then the set of Nash equilibrium action pairs is the union of the shaded regions in Figure 37.1, namely the set of all pairs (t_1, t_2) such that either

$$t_1 \leq v_2 \text{ and } t_2 \geq v_1,$$

or

$$t_1 \geq v_2,\, t_1 > t_2, \text{ and } t_2 \leq v_1.$$

81.1 Timing product release

A strategic game that models this situation is:

Players The two firms

Actions The set of actions of each player is the set of possible release times, which we can take to be the set of numbers t for which $0 \leq t \leq T$.

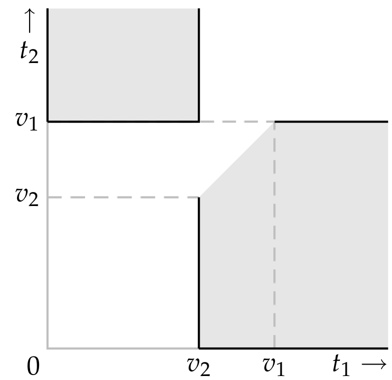


Figure 37.1 The set of Nash equilibria of the variant of the *War of Attrition* in Exercise 80.2 when $v_1 > v_2$.

Preferences Each firm's preferences are represented by its market share; the market share of firm i when it releases its product at time t_i and its rival releases its product at time t_j is

$$\begin{cases} h(t_i) & \text{if } t_i < t_j \\ \frac{1}{2} & \text{if } t_i = t_j \\ 1 - h(t_j) & \text{if } t_i > t_j. \end{cases}$$

Three representative cross-sections of firm i's payoff function are shown in Figure 37.2.

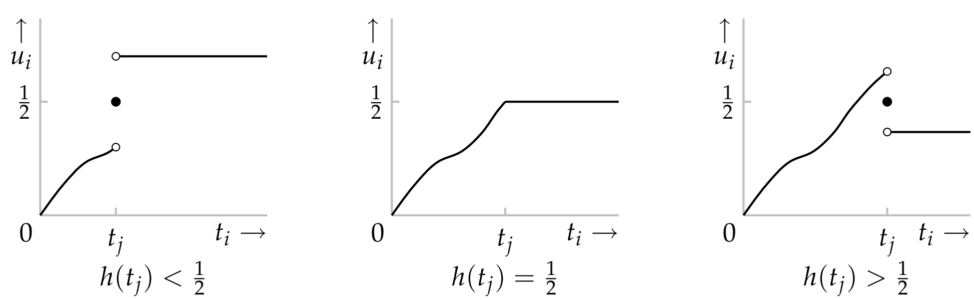


Figure 37.2 Three cross-sections of firm i's payoff function in the game in Exercise 81.1.

From the payoff function we see that if t_j is such that $h(t_j) < \frac{1}{2}$ then the set of firm i's best responses is the set of release times after t_j. If t_j is such that $h(t_j) = \frac{1}{2}$ then the set of firm i's best responses is the set of release times greater than or equal to t_j. If t_j is such that $h(t_j) > \frac{1}{2}$ then firm i wants to release its product just before t_j. Since there is no latest time before t_j, firm i has no *best* response in this case. (It has good responses, but none is optimal.) Denoting the time t for which $h(t) = \frac{1}{2}$ by t^*, the firms' best response functions are shown in Figure 38.1.

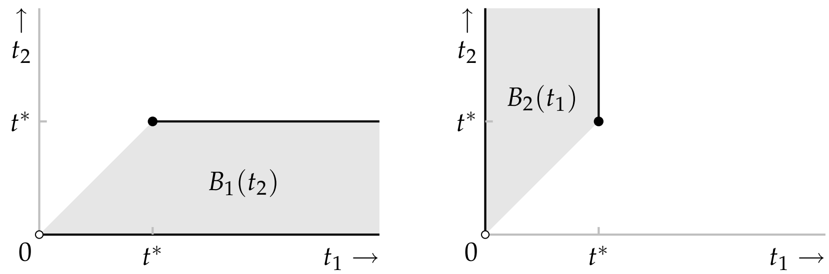


Figure 38.1 The firms' best response functions in the game in Exercise 81.1. Firm 1's best response function is in the left panel; firm 2's is in the right panel.

Combining the best response functions we see that the game has a unique Nash equilibrium, in which both firms release their products at the time t^* (where $h(t^*) = \frac{1}{2}$).

81.2 A fight

The game is defined as follows.

Players The two people.

Actions The set of actions of each player i is the set of amounts of the resource that player i can devote to fighting (the set of numbers y_i with $0 \leq y_i \leq 1$).

Preferences The preferences of player i are represented by the payoff function

$$u_i(y_1, y_2) = \begin{cases} f(y_1, y_2) & \text{if } y_i > y_j \\ \frac{1}{2} f(y_1, y_2) & \text{if } y_1 = y_2 \\ 0 & \text{if } y_i < y_j. \end{cases}$$

If $y_i < y_j$ then player j can increase her payoff by reducing y_j a little, keeping it greater than y_i (output increases, and she still wins). So no action profile in which $y_1 \neq y_2$ is a Nash equilibrium.

If $y_1 = y_2 < 1$ then either player i can increase her payoff by increasing y_i to slightly above y_j (output falls a little, but i's share of it increases from $\frac{1}{2}$ to 1). So no action profile in which $y_1 = y_2 < 1$ is a Nash equilibrium.

The only action profile that remains is $(y_1, y_2) = (1, 1)$. This profile is a Nash equilibrium: each player's payoff is 0, and remains 0 if she reduces the amount of the resource she devotes to fighting (given the other player's action).

85.1 Nash equilibrium of second-price sealed-bid auction

The set of Nash equilibria of a second-price sealed-bid auction in which player n wins the object is

$$\{(b_1, \ldots, b_n) : b_i \leq v_n \text{ for } i = 1, \ldots, n-1 \text{ and } b_n \geq v_1\}.$$

(One member of this set is $(0, \ldots, 0, v_1)$. The question asks only for one equilibrium; this answer describes all equilibria.)

Any member of the set is a Nash equilibrium by the following argument.

- Player n wins and obtains the payoff $v_n - \max_{1 \leq i \leq n-1} b_i \geq 0$; the payoff of every other player is 0.
- If any player $i = 1, \ldots, n-1$ changes her bid, either the outcome remains the same or she wins and pays v_1, which yields her a payoff of at most 0.
- If player n changes her bid, either the outcome remains the same or she loses and obtains the payoff 0.

No other action profile is a Nash equilibrium in which player n wins because if any player i with $1 \leq i \leq n-1$ bids more than v_n then the payoff of player n is negative if she wins, and if player n bids less than v_1 player 1 can deviate to a bid above player n's bid and obtain a positive payoff.

86.1 Second-price sealed-bid auction with two bidders

If player 2's bid b_2 is less than v_1 then any bid of b_2 or more is a best response of player 1 (she wins and pays the price b_2). If player 2's bid is equal to v_1 then every bid of player 1 yields her the payoff zero (either she wins and pays v_1, or she loses), so every bid is a best response. If player 2's bid b_2 exceeds v_1 then any bid of less than b_2 is a best response of player 1. (If she bids b_2 or more she wins, but pays the price $b_2 > v_1$, and hence obtains a negative payoff.) In summary, player 1's best response function is

$$B_1(b_2) = \begin{cases} \{b_1 : b_1 \geq b_2\} & \text{if } b_2 < v_1 \\ \{b_1 : b_1 \geq 0\} & \text{if } b_2 = v_1 \\ \{b_1 : 0 \leq b_1 < b_2\} & \text{if } b_2 > v_1. \end{cases}$$

By similar arguments, player 2's best response function is

$$B_2(b_1) = \begin{cases} \{b_2 : b_2 > b_1\} & \text{if } b_1 < v_2 \\ \{b_2 : b_2 \geq 0\} & \text{if } b_1 = v_2. \\ \{b_2 : 0 \leq b_2 \leq b_1\} & \text{if } b_1 > v_2. \end{cases}$$

These best response functions are shown in Figure 40.1.

Superimposing the best response functions, we see that the set of Nash equilibria is the shaded set in Figure 40.2, namely the set of pairs (b_1, b_2) such that either

$$b_1 \leq v_2 \text{ and } b_2 \geq v_1$$

or

$$b_1 \geq v_2,\ b_1 \geq b_2, \text{ and } b_2 \leq v_1.$$

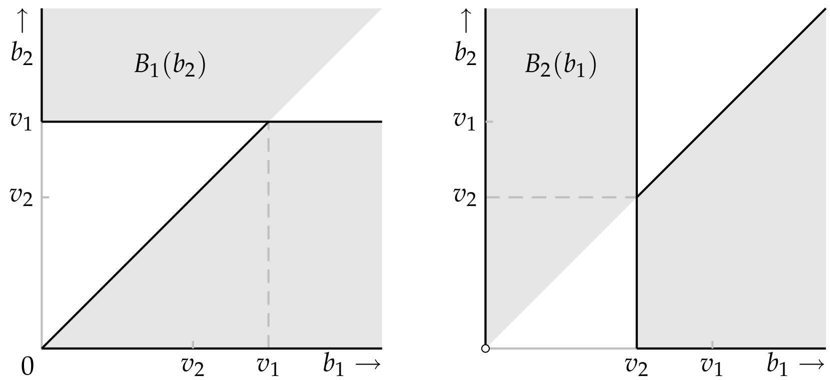


Figure 40.1 The players' best response functions in a two-player second-price sealed-bid auction (Exercise 86.1). Player 1's best response function is in the left panel; player 2's is in the right panel. (Only the edges marked by a black line are included.)

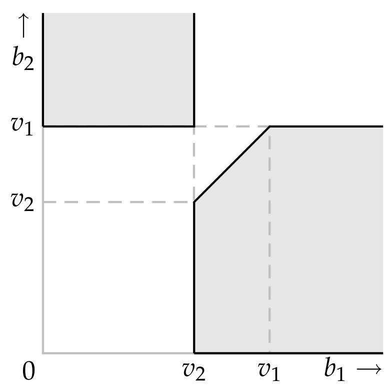


Figure 40.2 The set of Nash equilibria of a two-player second-price sealed-bid auction (Exercise 86.1).

87.1 Auctioning the right to choose

Denote the favorite action of player i by a_i^* for $i = 1, 2$. I claim that the bid $b_i^* = u_i(a_i^*) - u_i(a_j^*)$ for player i, where j is the other player, weakly dominates all of player i's other bids.

Suppose that $b_j < b_i^*$. Then the bid b_i^* yields player i the payoff $u_i(a_i^*) - b_j$, any other bid greater than b_j yields the same payoff, any bid less than b_j yields the payoff $u_i(a_j^*)$ (player j chooses the action in this case), and the bid b_j yields one or other of these payoffs (depending whether $i = 1$ or $i = 2$). We have $u_i(a_i^*) - b_j > u_i(a_j^*)$ because $b_j < b_i^* = u_i(a_i^*) - u_i(a_j^*)$, so the bid b_i^* yields a payoff at least as large as every other bid.

A symmetric argument shows that the bid b_i^* is optimal if $b_j > b_i^*$; the case $b_j = b_i^*$ is similar.

87.2 Nash equilibrium of first-price sealed-bid auction

The profile $(b_1, \ldots, b_n) = (v_2, v_2, v_3, \ldots, v_n)$ is a Nash equilibrium by the following argument.

- If player 1 raises her bid she still wins, but pays a higher price and hence obtains a lower payoff. If player 1 lowers her bid then she loses, and obtains the payoff of 0.
- If any other player changes her bid to any price at most equal to v_2 the outcome does not change. If she raises her bid above v_2 she wins, but obtains a negative payoff.

88.1 First-price sealed-bid auction

A profile of bids in which the two highest bids are not the same is not a Nash equilibrium because the player naming the highest bid can reduce her bid slightly, continue to win, and pay a lower price.

By the argument in the text, in any equilibrium player 1 wins the object. Thus she submits one of the highest bids.

If the highest bid is less than v_2, then player 2 can increase her bid to a value between the highest bid and v_2, win, and obtain a positive payoff. Thus in an equilibrium the highest bid is at least v_2.

If the highest bid exceeds v_1, player 1's payoff is negative, and she can increase this payoff by reducing her bid. Thus in an equilibrium the highest bid is at most v_1.

Finally, any profile $(b_1, \ldots, b_n)$ of bids that satisfies the conditions in the exercise is a Nash equilibrium by the following argument.

- If player 1 increases her bid she continues to win, and reduces her payoff. If player 1 decreases her bid she loses and obtains the payoff 0, which is at most her payoff at $(b_1, \ldots, b_n)$.
- If any other player increases her bid she either does not affect the outcome, or wins and obtains a negative payoff. If any other player decreases her bid she does not affect the outcome.

89.1 Third-price auction

a. The argument that a bid of v_i weakly dominates any lower bid is the same as for a second-price auction.

Now compare bids of $b_i > v_i$ and v_i. Suppose that one of the other players' bids is between v_i and b_i and all the remaining bids are less than v_i. If player i bids v_i she loses, and obtains the payoff of 0. If she bids b_i she wins, and pays the third highest bid, which is less than v_i. Thus she is better off bidding b_i than she is bidding v_i.

b. Each player's bidding her valuation is not a Nash equilibrium because player 2 can deviate and bid more than v_1 and obtain the object at the price v_3 instead of not obtaining the object.

c. Any action profile in which every player bids b, where $v_2 \leq b \leq v_1$ is a Nash equilibrium. (If player 1's raises her bid, the outcome does not change. If she lowers her bid, her payoff becomes zero, which is no higher than her payoff in the action profile. If any other player raises her bid then she wins and pays b, obtaining a nonpositive payoff; if any other player lowers her bid the outcome does not change.)

The set of all Nash equilibria is given as follows. (Note that the question asks only for one equilibrium, not all equilibria.) A profile of bids is a Nash equilibrium if and only if it satisfies one of the following two conditions.

- Player 1 wins, at least two players bid at least v_2, and the third highest bid is at least v_k, where k is the player who submits the second highest bid.
- Player k wins for some k with $2 \leq k \leq n-1$, at least two players bid at least v_1, the third highest bid is at most v_k, and the index of the player submitting the second highest bid is greater than k.

(The "second highest bid" is the winning bid among the bids that remain when the winning bid is removed, and the "third highest bid" is the winning bid when the winning bid and second highest bid are removed.)

90.1 All-pay auctions

Second-price all-pay auction with two bidders: The payoff function of bidder i is

$$u_i(b_1, b_2) = \begin{cases} -b_i & \text{if } b_i < b_j \\ v_i - b_j & \text{if } b_i > b_j, \end{cases}$$

with $u_1(b, b) = v_1 - b$ and $u_2(b, b) = -b$ for all b. This payoff function differs from that of player i in the *War of Attrition* only in the payoffs when the bids are equal. The set of Nash equilibria of the game is the same as that for the *War of Attrition*: the set of all pairs $(0, b_2)$ where $b_2 \geq v_1$ and $(b_1, 0)$ where $b_1 \geq v_2$. (The pair (b, b) of actions is not a Nash equilibrium for any value of b because player 2 can increase her payoff by either increasing her bid slightly or by reducing it to 0.)

First-price all-pay auction with two bidders: In any Nash equilibrium the two highest bids are equal, otherwise the player with the higher bid can increase her payoff by reducing her bid a little (keeping it larger than the other player's bid). But no profile of bids in which the two highest bids are equal is a Nash equilibrium, because the player with the higher index who submits this bid can increase her payoff by slightly increasing her bid, so that she wins rather than loses.

91.1 Multiunit auctions

Discriminatory auction To show that the action of bidding v_i and w_i is not dominant for player i, we need only find actions for the other players and alternative bids for player i such that player i's payoff is higher under the alternative bids than it is under the v_i and w_i, given the other players' actions. Suppose that each of the other players submits two bids of 0. Then if player i submits one bid between 0 and v_i and one bid between 0 and w_i she still wins two units, and pays less than when she bids v_i and w_i.

Uniform-price auction Suppose that some bidder other than i submits one bid between w_i and v_i and one bid of 0, and all the remaining bidders submit two bids of 0. Then bidder i wins one unit, and pays the price w_i. If she replaces her bid of w_i with a bid between 0 and w_i then she pays a lower price, and hence is better off.

Vickrey auction Suppose that player i bids v_i and w_i. Consider separately the cases in which the bids of the players other than i are such that player i wins 0, 1, and 2 units.

Player i wins 0 units: In this case the second highest of the other players' bids is at least v_i, so that if player i changes her bids so that she wins one or more units, for any unit she wins she pays at least v_i. Thus no change in her bids increases her payoff from its current value of 0 (and some changes lower her payoff).

Player i wins 1 unit: If player i raises her bid of v_i then she still wins one unit and the price remains the same. If she lowers this bid then either she still wins and pays the same price, or she does not win any units. If she raises her bid of w_i then either the outcome does not change, or she wins a second unit. In the latter case the price she pays is the previously-winning bid she beat, which is at least w_i, so that her payoff either remains zero or becomes negative.

Player i wins 2 units: Player i's raising either of her bids has no effect on the outcome; her lowering a bid either has no effect on the outcome or leads her to lose rather than to win, leading her to obtain the payoff of zero.

91.2 Waiting in line

The situation is modeled by a variant of a discriminatory multiunit auction in which 100 units are available, and each person attaches a positive value only to one unit and submits a bid for only one unit.

We can argue along the lines of Exercise 88.1.

- The first 100 people to arrive must do so at the same time. If not, at least one of them could arrive a little later and still be in the first 100.

- The first 100 people to arrive must be persons 1 through 100. Suppose, to the contrary, that one of these people is person i with $i \geq 101$, and person j with $j \leq 100$ is not in the group that arrives first. Then the common waiting time of the first 100 must be at most v_{101}, otherwise person i obtains a negative payoff. But then person j can deviate and arrive slightly earlier than the group of 100, and obtain a positive payoff.

- The common waiting time of the first 100 people must be at least v_{101}. If not, then person 101 could arrive slightly before the first 100 and obtain a positive payoff.

- The common waiting time of the first 100 people must be at most v_{100}. If not, then person 100 obtains a negative payoff, while by arriving later her payoff is zero.

- At least one person i with $i \geq 101$ arrives at the same time as the first 100 people. If not, then any person i with $i \leq 100$ can arrive slightly later and still be one of the first 100 to arrive.

This argument shows that in a Nash equilibrium persons 1 through 100 choose the same waiting time t^* with $v_{101} \leq t^* \leq v_{100}$, all the remaining people choose waiting times of at most t^*, and at least one of the remaining people chooses a waiting time equal to t^*. Any such action profile is a Nash equilibrium: any person i with $i \leq 100$ obtains a smaller payoff if she arrives earlier and a payoff of zero if she arrives later. Any person i with $i \geq 101$ obtains a negative payoff if she arrives before the first 100 people and a payoff of zero if she arrives at or after the first 100 people.

Thus the set of Nash equilibria is the set of action profiles $(t_1, \ldots, t_{200})$ in which $t_1 = \cdots = t_{100}$, this common waiting time, say t^*, satisfies $v_{101} \leq t^* \leq v_{100}$, $t_i \geq t^*$ for all $i \geq 101$, and $t_j = t^*$ for some $j \geq 101$.

When goods are rationed by line-ups in the world, people in general do not all arrive at the same time. The feature missing from the model that seems to explain the dispersion in arrival times is uncertainty on the part of each player about the other players' valuations.

91.3 Internet pricing

The situation may be modeled as a multiunit auction in which k units are available, and each player attaches a positive value to only one unit and submits a bid for only one unit. The k highest bids win, and each winner pays the $(k+1)$st highest bid.

By a variant of the argument for a second-price auction, in which "highest of the other players' bids" is replaced by "highest rejected bid", each player's action of bidding her value is weakly dominates all her other actions.

92.1 Lobbying as an auction

First-price auction In the action pair, each interest group's payoff is -100. Consider group A. If it raises the price it will pay for y, then the government still chooses y, and A is worse off. If it lowers the price it will pay for y, then the government chooses z and A's payoff remains -100. Now suppose it changes its bid from y to x and bids p. If $p < 103$, then the government chooses z and A's payoff remains -100. If $p \geq 103$, then the government chooses x and A's payoff is at most -103. Group A cannot increase its payoff by changing its bid from y to z, for similar reasons. A similar argument applies to group B's bid.

Menu auction In the action pair, each group's payoff is -3. Consider group A. If it changes its bids then either the outcome remains x and it pays at least 3, so that its payoff is at most -3, or the outcome becomes y and it pays at least 6, in which case its payoff is at most -3, or the outcome becomes z and it pays at least 0, in which case its payoff is at most -100. (Note that if it reduces its bids for both x and y then z is chosen.) Thus no change in its bids increases its payoff. Similar considerations apply to group B's bid.

97.2 Alternative standards of care under negligence with contributory negligence

First consider the case in which $X_1 = \hat{a}_1$ and $X_2 \leq \hat{a}_2$. The pair $(\hat{a}_1, \hat{a}_2)$ is a Nash equilibrium by the following argument.

If $a_2 = \hat{a}_2$ then the victim's level of care is sufficient (at least X_2), so that the injurer's payoff is given by (95.1) in the text. Thus the argument that the injurer's action $\hat{a}_1$ is a best response to $\hat{a}_2$ is exactly the same as the argument for the case $X_2 = \hat{a}_2$ in the text.

Since X_1 is the same as before, the victim's payoff is the same also, so that by the argument in the text the victim's best response to $\hat{a}_1$ is $\hat{a}_2$. Thus $(\hat{a}_1, \hat{a}_2)$ is a Nash equilibrium.

To show that $(\hat{a}_1, \hat{a}_2)$ is the only Nash equilibrium of the game, we study the players' best response functions. First consider the injurer's best response function. As in the text, we split the analysis into three cases.

$a_2 < X_2$: In this case the injurer does not have to pay any compensation, regardless of her level of care; her payoff is $-a_1$, so that her best response is $a_1 = 0$.

$a_2 = X_2$: In this case the injurer's best response is $\hat{a}_1$, as argued when showing that $(\hat{a}_1, \hat{a}_2)$ is a Nash equilibrium.

$a_2 > X_2$: In this case the injurer's best response is at most $\hat{a}_1$, since her payoff is equal to $-a_1$ for larger values of a_1.

Thus the injurer's best response takes a form like that shown in the left panel of Figure 46.1. (In fact, $b_1(a_2) = \hat{a}_1$ for $X_2 \leq a_2 \leq \hat{a}_2$, but the analysis depends only on the fact that $b_1(a_2) \leq \hat{a}_1$ for $a_2 > X_2$.)

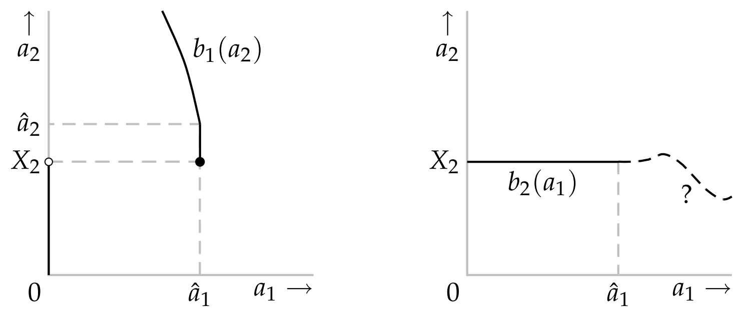


Figure 46.1 The players' best response functions under the rule of negligence with contributory negligence when $X_1 = \hat{a}_1$ and $X_2 = \hat{a}_2$. Left panel: the injurer's best response function b_1. Right panel: the victim's best response function b_2. (The position of the victim's best response function for $a_1 > \hat{a}_1$ is not significant, and is not determined in the solution.)

Now consider the victim's best response function. The victim's payoff function is

$$u_2(a_1, a_2) = \begin{cases} -a_2 & \text{if } a_1 < \hat{a}_1 \text{ and } a_2 \geq X_2 \\ -a_2 - L(a_1, a_2) & \text{if } a_1 \geq \hat{a}_1 \text{ or } a_2 < X_2. \end{cases}$$

As before, for $a_1 < \hat{a}_1$ we have $-a_2 - L(a_1, a_2) < -\hat{a}_2$ for all a_2, so that the victim's best response is X_2. As in the text, the nature of the victim's best responses to levels of care a_1 for which $a_1 > \hat{a}_1$ are not significant.

Combining the two best response functions we see that $(\hat{a}_1, \hat{a}_2)$ is the unique Nash equilibrium of the game.

Now consider the case in which $X_1 = M$ and $a_2 = \hat{a}_2$, where $M \geq \hat{a}_1$. The injurer's payoff is

$$u_1(a_1, a_2) = \begin{cases} -a_1 - L(a_1, a_2) & \text{if } a_1 < M \text{ and } a_2 \geq \hat{a}_2 \\ -a_1 & \text{if } a_1 \geq M \text{ or } a_2 < \hat{a}_2. \end{cases}$$

Now, the maximizer of $-a_1 - L(a_1, \hat{a}_2)$ is $\hat{a}_1$ (see the argument following (95.1) in the text), so that if M is large enough then the injurer's best response to $\hat{a}_2$ is $\hat{a}_1$. As before, if $a_2 < \hat{a}_2$ then the injurer's best response is 0, and if $a_2 > \hat{a}_2$ then the injurer's payoff decreases for $a_1 > M$, so that her best response is less than M. The injurer's best response function is shown in the left panel of Figure 47.1.

The victim's payoff is

$$u_2(a_1, a_2) = \begin{cases} -a_2 & \text{if } a_1 < M \text{ and } a_2 \geq \hat{a}_2 \\ -a_2 - L(a_1, a_2) & \text{if } a_1 \geq M \text{ or } a_2 < \hat{a}_2. \end{cases}$$

If $a_1 \leq \hat{a}_1$ then the victim's best response is $\hat{a}_2$ by the same argument as the one in the text. If a_1 is such that $\hat{a}_1 < a_1 < M$ then the victim's best response is at most $\hat{a}_2$ (since her payoff is decreasing for larger values of a_2). This information about

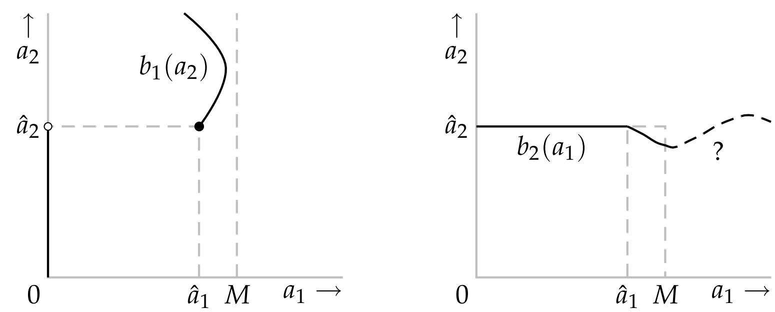


Figure 47.1 The players' best response functions under the rule of negligence with contributory negligence when $(X_1, X_2) = (M, \hat{a}_2)$, with $M \geq \hat{a}_1$. Left panel: the injurer's best response function b_1. Right panel: the victim's best response function b_2. (The position of the victim's best response function for $a_1 > M$ is not significant, and is not determined in the text.)

the victim's best response function is recorded in the right panel of Figure 47.1; it is sufficient to deduce that $(\hat{a}_1, \hat{a}_2)$ is the unique Nash equilibrium of the game.

97.3 Equilibrium under strict liability

In this case the injurer's payoff is $-a_1 - L(a_1, a_2)$ and the victim's is $-a_2$ for all (a_1, a_2). Thus the victim's optimal action is 0, regardless of the injurer's action. (The victim takes no care, given that, regardless of her level of care, the injurer is obliged to compensate her for any loss.) Thus in a Nash equilibrium the injurer chooses the level of care that maximizes $-a_1 - L(a_1, 0)$ and the victim chooses $a_2 = 0$.

If the function $-a_1 - L(a_1, 0)$ has a unique maximizer then the game has a unique Nash equilibrium; if there are multiple maximizers then the game has many Nash equilibria, though the players' payoffs are the same in all the equilibria. The relation between $\hat{a}_1$ and the equilibrium value of a_1 depends on the character of $L(a_1, a_2)$. If, for example, L decreases more sharply as a_1 increases when $a_2 = 0$ than when a_2 is positive, the equilibrium value of a_1 exceeds $\hat{a}_1$.

4 Mixed Strategy Equilibrium

101.1 Variant of *Matching Pennies*

The analysis is the same as for *Matching Pennies*. There is a unique steady state, in which each player chooses each action with probability $\frac{1}{2}$.

106.2 Extensions of *BoS* with vNM preferences

In the first case, when player 1 is indifferent between going to her less preferred concert in the company of player 2 and the lottery in which with probability $\frac{1}{2}$ she and player 2 go to different concerts and with probability $\frac{1}{2}$ they both go to her more preferred concert, the Bernoulli payoffs that represent her preferences satisfy the condition

$$u_1(S,S) = \tfrac{1}{2}u_1(S,B) + \tfrac{1}{2}u_1(B,B).$$

If we choose $u_1(S,B) = 0$ and $u_1(B,B) = 2$, then $u_1(S,S) = 1$. Similarly, for player 2 we can set $u_2(B,S) = 0$, $u_2(S,S) = 2$, and $u_2(B,B) = 1$. Thus the Bernoulli payoffs in the left panel of Figure 49.1 are consistent with the players' preferences.

In the second case, when player 1 is indifferent between going to her less preferred concert in the company of player 2 and the lottery in which with probability $\frac{3}{4}$ she and player 2 go to different concerts and with probability $\frac{1}{4}$ they both go to her more preferred concert, the Bernoulli payoffs that represent her preferences satisfy the condition

$$u_1(S,S) = \tfrac{3}{4}u_1(S,B) + \tfrac{1}{4}u_1(B,B).$$

If we choose $u_1(S,B) = 0$ and $u_1(B,B) = 2$ (as before), then $u_1(S,S) = \frac{1}{2}$. Similarly, for player 2 we can set $u_2(B,S) = 0$, $u_2(S,S) = 2$, and $u_2(B,B) = \frac{1}{2}$. Thus the Bernoulli payoffs in the right panel of Figure 49.1 are consistent with the players' preferences.

	Bach	*Stravinsky*
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

	Bach	*Stravinsky*
Bach	$2, \frac{1}{2}$	$0, 0$
Stravinsky	$0, 0$	$\frac{1}{2}, 2$

Figure 49.1 The Bernoulli payoffs for two extensions of *BoS*.

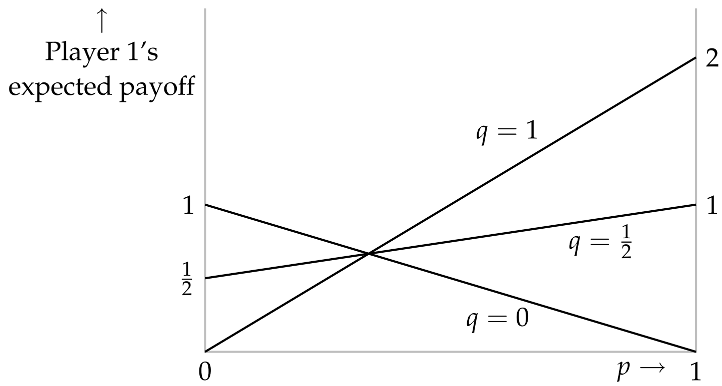


Figure 50.1 Player 1's expected payoff as a function of the probability p that she assigns to B in *BoS*, when the probability q that player 2 assigns to B is 0, $\frac{1}{2}$, and 1.

110.1 Expected payoffs

For *BoS*, player 1's expected payoff is shown in Figure 50.1.

For the game in the right panel of Figure 21.1 in the book, player 1's expected payoff is shown in Figure 50.2.

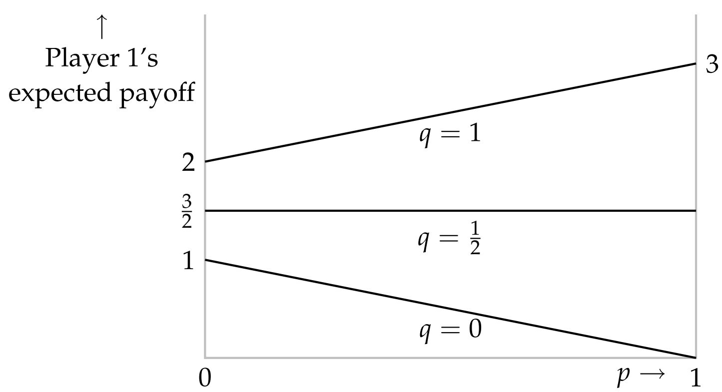


Figure 50.2 Player 1's expected payoff as a function of the probability p that she assigns to *Refrain* in the game in the right panel of Figure 21.1 in the book, when the probability q that player 2 assigns to *Refrain* is 0, $\frac{1}{2}$, and 1.

111.1 Examples of best responses

For *BoS*: for $q = 0$ player 1's unique best response is $p = 0$ and for $q = \frac{1}{2}$ and $q = 1$ her unique best response is $p = 1$. For the game in the right panel of Figure 21.1: for $q = 0$ player 1's unique best response is $p = 0$, for $q = \frac{1}{2}$ her set of best responses is the set of all her mixed strategies (all values of p), and for $q = 1$ her unique best response is $p = 1$.

114.1 Mixed strategy equilibrium of *Hawk–Dove*

Denote by u_i a payoff function whose expected value represents player i's preferences. The conditions in the problem imply that for player 1 we have

$$u_1(Passive, Passive) = \tfrac{1}{2}u_1(Aggressive, Aggressive) + \tfrac{1}{2}u_1(Aggressive, Passive)$$

and

$$u_1(Passive, Aggressive) = \tfrac{2}{3}u_1(Aggressive, Aggressive) + \tfrac{1}{3}u_1(Passive, Passive).$$

Given $u_1(Aggressive, Aggressive) = 0$ and $u_1(Passive, Aggressive = 1$, we have

$$u_1(Passive, Passive) = \tfrac{1}{2}u_1(Aggressive, Passive)$$

and

$$1 = \tfrac{1}{3}u_1(Passive, Passive),$$

so that

$$u_1(Passive, Passive) = 3 \text{ and } u_1(Aggressive, Passive) = 6.$$

Similarly,

$$u_2(Passive, Passive) = 3 \text{ and } u_2(Passive, Aggressive) = 6.$$

Thus the game is given in the left panel of Figure 51.1. The players' best response functions are shown in the right panel. The game has three mixed strategy Nash equilibria: $((0,1),(1,0))$, $((\tfrac{3}{4},\tfrac{1}{4}),(\tfrac{3}{4},\tfrac{1}{4}))$, and $((1,0),(0,1))$.

	Aggressive	*Passive*
Aggressive	0, 0	6, 1
Passive	1, 6	3, 3

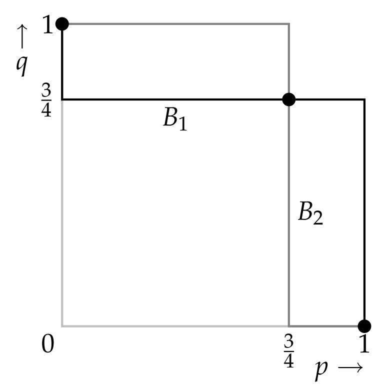


Figure 51.1 An extension of *Hawk–Dove* (left panel) and the players' best response functions when randomization is allowed in this game (right panel). The probability that player 1 assigns to *Aggressive* is p and the probability that player 2 assigns to *Aggressive* is q. The disks indicate the Nash equilibria (two pure, one mixed).

114.2 Games with mixed strategy equilibria

The best response functions for the left game are shown in the left panel of Figure 52.1. We see that the game has a unique mixed strategy Nash equilibrium $((\frac{1}{4}, \frac{3}{4}), (\frac{2}{3}, \frac{1}{3}))$. The best response functions for the right game are shown in the right panel of Figure 52.1. We see that the mixed strategy Nash equilibria are $((0,1),(1,0))$ and any $((p, 1-p),(0,1))$ with $\frac{1}{2} \leq p \leq 1$.

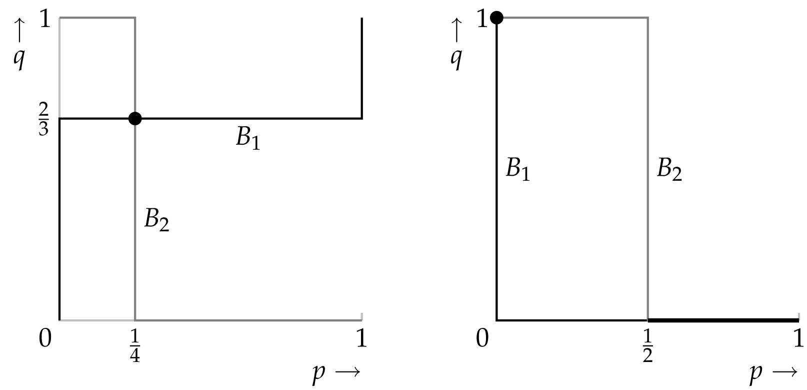


Figure 52.1 The players' best response functions in the left game (left panel) and right game (right panel) in Exercise 114.2. The probability that player 1 assigns to T is p and the probability that player 2 assigns to L is q. The disks and the heavy line indicate Nash equilibria.

114.3 A coordination game

The best response functions are shown in Figure 53.1. From the figure we see that the game has three mixed strategy Nash equilibria, $((1,0),(1,0))$ (the pure strategy equilibrium (*No effort*, *No effort*)), $((0,1),(0,1))$ (the pure strategy equilibrium (*Effort*, *Effort*)), and $((1-c,c),(1-c,c))$.

An increase in c has no effect on the pure strategy equilibria, and *increases* the probability that each player chooses to exert effort in the mixed strategy equilibrium (because this probability is precisely c).

The pure Nash equilibria are not affected by the cost of effort because a change in c has no effect on the players' rankings of the four outcomes. An increase in c reduces a player's payoff to the action *Effort*, given the other player's mixed strategy; the probability the other player assigns to *Effort* must increase in order to keep the player indifferent between *No effort* and *Effort*, as required in an equilibrium.

114.4 Swimming with sharks

As argued in the question, if you swim today, your expected payoff is $-\pi c + 2(1 - \pi)$, regardless of your friend's action. If you do not swim today and your friend

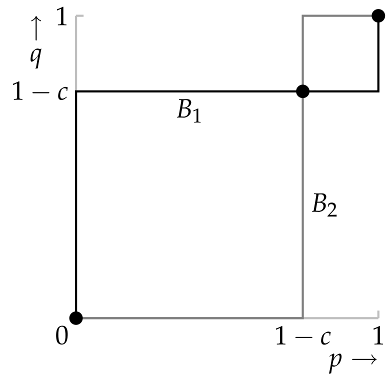


Figure 53.1 The players' best response functions in the coordination game in Exercise 115.1. The probability that player 1 assigns to *No effort* is p and the probability that player 2 assigns to *No effort* is q. The disks indicate the Nash equilibria (two pure, one mixed).

does, then with probability π your friend is attacked and you do not swim tomorrow, and with probability $1-\pi$ your friend is not attacked and you do swim tomorrow. Thus your expected payoff in this case is $\pi \cdot 0 + (1-\pi)\cdot 1 = 1-\pi$. If neither of you swims today then your expected payoff is $\max\{-\pi c + 1 - \pi, 0\}$, as argued in the problem. Hence player 1's payoffs in the game are given in Figure 53.2. (Player 2's payoffs are symmetric with player 1's.)

	Swim today	*Wait*
Swim today	$-\pi c + 2(1-\pi)$	$-\pi c + 2(1-\pi)$
Wait	$1-\pi$	$\max\{-\pi c + 1 - \pi, 0\}$

Figure 53.2 Player 1's payoffs in the (symmetric) game of swimming with sharks.

To find the mixed strategy Nash equilibria, first note that if $-\pi c + 1 - \pi > 0$, or $c < (1-\pi)/\pi$, then *Swim today* is the unique best response to both *Swim today* and *Wait*. Thus in this case there is a unique mixed strategy Nash equilibrium, in which both players choose *Swim today*.

At the other extreme, if $-\pi c + 2(1-\pi) < 0$, or $c > 2(1-\pi)/\pi$, then *Wait* is the unique best response to both *Swim today* and *Wait*. Thus in this case there is a unique mixed strategy Nash equilibrium, in which neither of you swims today, and consequently neither of you swims tomorrow.

In the intermediate case in which $0 < -\pi c + 2(1-\pi) < 1-\pi$, or $(1-\pi)/\pi < c < 2(1-\pi)/\pi$, the unique best response to *Swim today* is *Wait* and the unique best response to *Wait* is *Swim today*. Thus (*Swim today*, *Wait*) and (*Wait*, *Swim today*) are both mixed strategy Nash equilibria. In this case the game has also a mixed strategy Nash equilibrium in which the probability that each player assigns to each action is positive.

Denote by q the probability that player 2 chooses *Swim today*. Then player 1's expected payoff to *Swim today* is $-\pi c + 2(1-\pi)$ and her expected payoff to *Wait* is $q(1-\pi)$. (Because $-\pi c + 2(1-\pi) < 1-\pi$, we have $-\pi c + 1 - \pi < 0$, so that each player's payoff if both players *Wait* is 0.) Thus player 1's expected payoffs to her two actions are equal if and only if

$$-\pi c + 2(1-\pi) = q(1-\pi),$$

or $q = [-\pi c + 2(1-\pi)]/(1-\pi)$. The same calculation implies that player 2's expected payoffs to her two actions are equal if and only if the probability that player 1 assigns to *Swim today* is $[-\pi c + 2(1-\pi)]/(1-\pi) = 2 - \pi c/(1-\pi)$.

We conclude that if $(1-\pi)/\pi < c < 2(1-\pi)/\pi$ then the game has three mixed strategy Nash equilibria: $(\textit{Swim today}, \textit{Wait})$, $(\textit{Wait}, \textit{Swim today})$, and the mixed strategy pair in which each person swims today with probability $2 - \pi c/(1-\pi)$.

If $c = (1-\pi)/\pi$ the payoffs simplify to those given in the left panel of Figure 54.1. The set of mixed strategy Nash equilibria in this case is the set of all mixed strategy pairs $((p, 1-p), (q, 1-q))$ for which either $p = 1$ or $q = 1$. If $c = 2(1-\pi)/\pi$ the payoffs simplify to those given in the right panel of Figure 54.1. The set of mixed strategy Nash equilibria in this case is the set of all mixed strategy pairs $((p, 1-p), (q, 1-q))$ for which either $p = 0$ or $q = 0$.

	Swim	*Wait*
Swim	$1-\pi, 1-\pi$	$1-\pi, 1-\pi$
Wait	$1-\pi, 1-\pi$	$0, 0$

	Swim	*Wait*
Swim	$0, 0$	$0, 1-\pi$
Wait	$1-\pi, 0$	$0, 0$

Figure 54.1 The game if Figure 53.2 for $c = (1-\pi)/\pi$ (left panel) and $c = 2(1-\pi)/\pi$ (right panel).

If you were alone your expected payoff to swimming on the first day would be $-\pi c + 2(1-\pi)$; your expected payoff to staying out of the water on the first day and acting optimally on the second day would be $\max\{-\pi c + 1 - \pi, 0\}$. Thus if $-\pi c + 2(1-\pi) > 0$, or $c < 2(1-\pi)/\pi$, you would swim on the first day (and stay out of the water on the second day if you get attacked on the first day), and if $c > 2(1-\pi)/\pi$ you would stay out of the water on both days.

In the presence of your friend, you swim on the first day if $c < (1-\pi)/\pi$ and stay out of the water if $c > 2(1-\pi)/\pi$. Thus for $c < (1-\pi)/\pi$ or $c > 2(1-\pi)/\pi$ each person acts in the same way as she would if she were alone.

If $(1-\pi)/\pi < c < 2(1-\pi)/\pi$, then the game has an equilibrium in which you swim on the first day (and your friend does not), one in which you do not (but your friend does), and one in which you both swim with probability $2 - \pi c/(1-\pi)$ (which decreases from 1 to 0 as c increases from $(1-\pi)/\pi$ to $2(1-\pi)/\pi$).

Thus for $(1-\pi)/\pi < c < 2(1-\pi)/\pi$ the presence of your friend either decreases the probability of your swimming on the first day or has no effect on this probability.

117.2 Choosing numbers

a. To show that the pair of mixed strategies in the question is a mixed strategy equilibrium, it suffices to verify the conditions in Proposition 116.2. Thus, given that each player's strategy specifies a positive probability for every action, it suffices to show that each action of each player yields the same expected payoff. Player 1's expected payoff to each pure strategy is $1/K$, because with probability $1/K$ player 2 chooses the same number, and with probability $1 - 1/K$ player 2 chooses a different number. Similarly, player 2's expected payoff to each pure strategy is $-1/K$, because with probability $1/K$ player 1 chooses the same number, and with probability $1 - 1/K$ player 2 chooses a different number. Thus the pair of strategies is a mixed strategy Nash equilibrium.

b. Let (p^*, q^*) be a mixed strategy equilibrium, where p_j^* and q_j^* are the probabilities assigned by players 1 and 2 to the integer j, for $j = 1, \ldots, K$.

Step 1 If $p_k^* > 0$ then $q_k^* \geq q_j^*$ for all j.

Proof. Player 1's expected payoff to any action j is q_j^* (her payoff is 1 if player 2 chooses j and is otherwise 0). So by Proposition 116.2, if $p_k^* > 0$ then $q_k^* \geq q_j^*$ for all j.

Step 2 If $p_k^* > 0$ then $q_k^* > 0$.

Proof. From Step 1, we must have $q_j^* > 0$ for some j (not every q_j^* can be 0!), so $q_k^* > 0$.

Step 3 If $p_k^* > 0$ then $p_k^* \leq 1/K$.

Proof. By Step 2, $q_k^* > 0$. Thus by Proposition 116.2, player 2's expected payoff to the action k is at least her expected payoff to any other action. Her expected payoff to the action j is $-p_j^*$, so $-p_k^* \geq -p_j^*$ for any j, or $p_k^* \leq p_j^*$ for any j. Thus we have $p_k^* \leq 1/K$ (p_j^* cannot exceed $1/K$ for all j!).

Step 4 $p_j^* = 1/K$ for all j.

Proof. The sum of the p_j^*'s equals 1, so the result follows from Step 3.

Step 5 $q_j^* = 1/K$ for all j.

Proof. By Step 4, every $p_j^* > 0$, so the result follows from Step 1 and the fact that the sum of q_j^*'s equals 1.

118.1 Silverman's game

The game has no pure strategy Nash equilibrium in which the players' integers are the same because either player can increase her payoff from 0 to 1 by naming the next higher integer. It has no Nash equilibrium in which the players' integers are different because the losing player (the player whose payoff is -1) can increase her payoff to 1 by changing her integer to be one more than the other player's integer. Thus the game has no pure strategy Nash equilibrium.

To show that the pair of mixed strategies in the question is a mixed strategy equilibrium, consider player i's expected payoff to each of her actions when player j uses the mixed strategy in the question:

1: $\frac{1}{3} \cdot 0 + \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 1 = 0$.

2: $\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot (-1) = 0$.

3 or 4: $\frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot (-1) = -\frac{1}{3}$.

5: $\frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 = 0$.

6–14: $\frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 1 = -\frac{1}{3}$.

15 or more: $\frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot (-1) = -1$.

Given these payoffs, player i's expected payoff to any mixed strategy that assigns positive probability to an action other than 1, 2, and 5 is negative, and no mixed strategy of player i yields a positive expected payoff. Thus the pair of strategies is a mixed strategy Nash equilibrium.

118.2 Voter participation

I verify that the conditions in Proposition 116.2 are satisfied.

First consider a supporter of candidate A. If she votes then candidate A ties if all $k-1$ of her comrades vote, an event with probability p^{k-1}, and otherwise candidate A loses. Thus her expected payoff is

$$p^{k-1} - c.$$

If she abstains, then candidate A surely loses, so her payoff is 0. Thus in an equilibrium in which $0 < p < 1$ the first condition in Proposition 116.2 implies that $p^{k-1} = c$, or

$$p = c^{1/(k-1)}.$$

Given $k \geq 2$ and $0 < c < 1$, we have $0 < p < 1$.

Now consider a supporter of candidate B who votes. With probability p^k all of the supporters of candidate A vote, in which case the election is a tie; with probability $1 - p^k$ at least one of the supporters of candidate A does not vote, in which case candidate B wins. Thus the expected payoff of a supporter of candidate B who votes is

$$p^k + 2(1 - p^k) - c.$$

If the supporter of candidate B switches to abstaining, then

- candidate B loses if all supporters of candidate A vote, an event with probability p^k
- candidate B ties if exactly $k-1$ supporters of candidate A vote, an event with probability $kp^{k-1}(1-p)$
- candidate B wins if fewer than $k-1$ supporters of candidate A vote, an event with probability $1 - p^k - kp^{k-1}(1-p)$.

Thus a supporter of candidate B who switches from voting to abstaining obtains an expected payoff of

$$kp^{k-1}(1-p) + 2(1 - p^k - kp^{k-1}(1-p)) = 2 - (2-k)p^k - kp^{k-1}.$$

Hence in order for it to be optimal for such a citizen to vote (i.e. in order for the second condition in Proposition 116.2 to be satisfied), we need

$$p^k + 2(1-p^k) - c \geq 2 - (2-k)p^k - kp^{k-1},$$

or

$$kp^{k-1}(1-p) + p^k \geq c.$$

Finally, consider a supporter of candidate B who abstains. With probability p^k all the supporters of candidate A vote, in which case the candidates tie; with probability $1 - p^k$ at least one of the supporters of candidate A does not vote, in which case candidate B wins. Thus the expected payoff of a supporter of candidate B who abstains is

$$p^k + 2(1-p^k).$$

If this citizen instead votes, candidate B surely wins (she gets $k+1$ votes, while candidate A gets at most k). Thus the citizen's expected payoff is

$$2 - c.$$

Hence in order for the citizen to wish to abstain, we need

$$p^k + 2(1-p^k) \geq 2 - c$$

or

$$c \geq p^k.$$

In summary, for equilibrium we need $p = c^{1/(k-1)}$ and

$$p^k \leq c \leq kp^{k-1}(1-p) + p^k.$$

Given $p = c^{1/(k-1)}$, $c = p^{k-1}$, so that the two inequalities are satisfied. Thus $p = c^{1/(k-1)}$ defines an equilibrium.

As c increases, the probability p, and hence the expected number of voters, *increases*.

118.3 Defending territory

(The solution to this problem, which corrects an error in Shubik (1982, 226), is due to Nick Vriend.) The game is shown in Figure 58.1, where each action (x, y) gives the number x of divisions allocated to the first pass and the number y allocated to the second pass.

		General B		
		$(2,0)$	$(1,1)$	$(0,2)$
General A	$(3,0)$	$1,-1$	$-1,\ 1$	$-1,\ 1$
	$(2,1)$	$1,-1$	$1,-1$	$-1,\ 1$
	$(1,2)$	$-1,\ 1$	$1,-1$	$1,-1$
	$(0,3)$	$-1,\ 1$	$-1,\ 1$	$1,-1$

Figure 58.1 The game in Exercise 118.3.

Denote a mixed strategy of A by (p_1, p_2, p_3, p_4) and a mixed strategy of B by (q_1, q_2, q_3).

First I argue that in every equilibrium $q_2 = 0$. If $q_2 > 0$ then A's expected payoff to $(3,0)$ is less than her expected payoff to $(2,1)$, and her expected payoff to $(0,3)$ is less than her expected payoff to $(1,2)$, so that $p_1 = p_4 = 0$. But then B's expected payoff to at least one of her actions $(2,0)$ and $(0,2)$ exceeds her expected payoff to $(1,1)$, contradicting $q_2 > 0$.

Now I argue that in every equilibrium $q_1 = q_3 = \frac{1}{2}$. Given $q_2 = 0$ we have $q_3 = 1 - q_1$, and A's payoffs are $2q_1 - 1$ to $(3,0)$ and to $(2,1)$, and $1 - 2q_1$ to $(1,2)$ and $(0,3)$. Thus if $q_1 < \frac{1}{2}$ then in any equilibrium we have $p_1 = p_2 = 0$. Then B's action $(2,0)$ yields her a higher payoff than does $(0,2)$, so that in any equilibrium $q_1 = 1$. But then A's actions $(3,0)$ and $(2,1)$ both yield higher payoffs than do $(1,2)$ and $(0,3)$, contradicting $p_1 = p_2 = 0$. Similarly, $q_1 > \frac{1}{2}$ is inconsistent with equilibrium. Hence in any equilibrium $q_1 = q_3 = \frac{1}{2}$.

Now, given $q_1 = q_3 = \frac{1}{2}$, A's payoffs to her four actions are all equal. Thus $((p_1, p_2, p_3, p_4), (q_1, q_2, q_3))$ is a Nash equilibrium if and only if B's payoff to $(2,0)$ is the same as her payoff to $(0,2)$, and this payoff is at least her payoff to $(1,1)$. The first condition is $-p_1 - p_2 + p_3 + p_4 = p_1 + p_2 - p_3 - p_4$, or $p_1 + p_2 = p_3 + p_4 = \frac{1}{2}$. Thus B's payoff to $(2,0)$ and to $(0,2)$ is zero, and the second condition is $p_1 - p_2 - p_3 + p_4 \leq 0$, or $p_1 + p_4 \leq \frac{1}{2}$ (using $p_1 + p_2 + p_3 + p_4 = 1$).

We conclude that the set of mixed strategy Nash equilibria of the game is the set of strategy pairs $((p_1, \frac{1}{2} - p_1, \frac{1}{2} - p_4, p_4), (\frac{1}{2}, 0, \frac{1}{2}))$ with $p_1 + p_4 \leq \frac{1}{2}$.

In this equilibrium General A splits her resources between the two passes with probability at least $\frac{1}{2}$ ($p_2 + p_3 = \frac{1}{2} - p_1 + \frac{1}{2} - p_4 = 1 - (p_1 + p_4) \geq \frac{1}{2}$) while General B concentrates all of her resources in one or other of the passes (with equal probability).

120.2 Strictly dominating mixed strategies

Denote the probability that player 1 assigns to T by p and the probability she assigns to M by r (so that the probability she assigns to B is $1 - p - r$). A mixed strategy of player 1 strictly dominates T if and only if

$$p + 4r > 1 \quad \text{and} \quad p + 3(1 - p - r) > 1,$$

or if and only if $1 - 4r < p < 1 - \frac{3}{2}r$. For example, the mixed strategies $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ and $(0, \frac{1}{3}, \frac{2}{3})$ both strictly dominate T.

120.3 Strict domination for mixed strategies

(*a*) True. Suppose that the mixed strategy α_i' assigns positive probability to the action a_i', which is strictly dominated by the action a_i. Then $u_i(a_i, a_{-i}) > u_i(a_i', a_{-i})$ for all a_{-i}. Let α_i be the mixed strategy that differs from α_i' only in the weight that α_i' assigns to a_i' is transferred to a_i. That is, α_i is defined by $\alpha_i(a_i') = 0$, $\alpha_i(a_i) = \alpha_i'(a_i') + \alpha_i'(a_i)$, and $\alpha_i(b_i) = \alpha_i'(b_i)$ for every other action b_i. Then α_i strictly dominates α_i': for any a_{-i} we have $U(\alpha_i, a_{-i}) - U(\alpha_i', a_{-i}) = \alpha_i'(a_i')(u(a_i, a_{-i}) - u_i(a_i', a_{-i})) > 0$.

(*b*) False. Consider a variant of the game in Figure 120.1 in the text in which player 1's payoffs to (T, L) and to (T, R) are both $\frac{5}{2}$ instead of 1. Then player 1's mixed strategy that assigns probability $\frac{1}{2}$ to M and probability $\frac{1}{2}$ to B is strictly dominated by T, even though neither M nor B is strictly dominated.

121.2 Eliminating dominated actions when finding equilibria

Player 2's action L is strictly dominated by the mixed strategy that assigns probability $\frac{1}{4}$ to M and probability $\frac{3}{4}$ to R (for example), so that we can ignore the action L. The players' best response functions in the reduced game in which player 2's actions are M and R are shown in Figure 59.1. We see that the game has a single mixed strategy Nash equilibrium, namely $((\frac{2}{3}, \frac{1}{3}), (0, \frac{1}{2}, \frac{1}{2}))$.

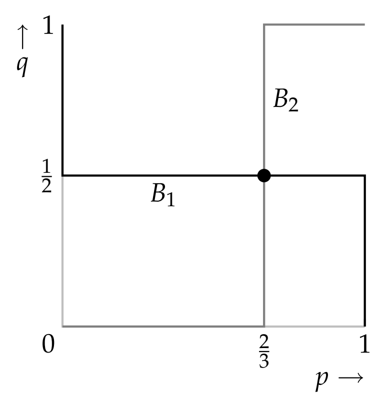


Figure 59.1 The players' best response functions in the game in Figure 122.1 after player 2's action L has been eliminated. The probability assigned by player 1 to T is p and the probability assigned by player 2 to M is q. The best response function of player 1 is black and that of player 2 is gray. The disk indicates the unique Nash equilibrium.

127.1 Equilibrium in the expert diagnosis game

When $E = rE' + (1-r)I'$ the consumer is indifferent between her two actions when $p = 0$, so that her best response function has a vertical segment at $p = 0$. Referring to Figure 126.1 in the text, we see that the set of mixed strategy Nash equilibria correspond to $p = 0$ and $\pi/\pi' \le q \le 1$.

127.2 Incompetent experts

The payoffs are given in Figure 60.1. (The actions are the same as those in the game in which every expert is fully competent.)

	A	R
H	$\pi, -rE - (1-r)[sI + (1-s)E]$	$(1-r)s\pi, -rE' - (1-r)[sI + (1-s)I']$
D	$r\pi + (1-r)[s\pi' + (1-s)\pi], -E$	$0, -rE' - (1-r)I'$

Figure 60.1 A game between a consumer with a problem and a not-fully-competent expert.

The players' best response functions are shown in Figure 60.2. The consumer's best response function depends on the value of s. The left panel of the figure shows a case in which s is large and the right panel shows the case in which s is small.

We see that when s is large the game has a unique mixed strategy Nash equilibrium, in which the probability the expert's strategy assigns to H is

$$p^* = \frac{E - [rE' + (1-r)I']}{(1-r)s(E - I')}$$

and the probability the consumer's strategy assigns to A is

$$q^* = \frac{\pi}{\pi'}.$$

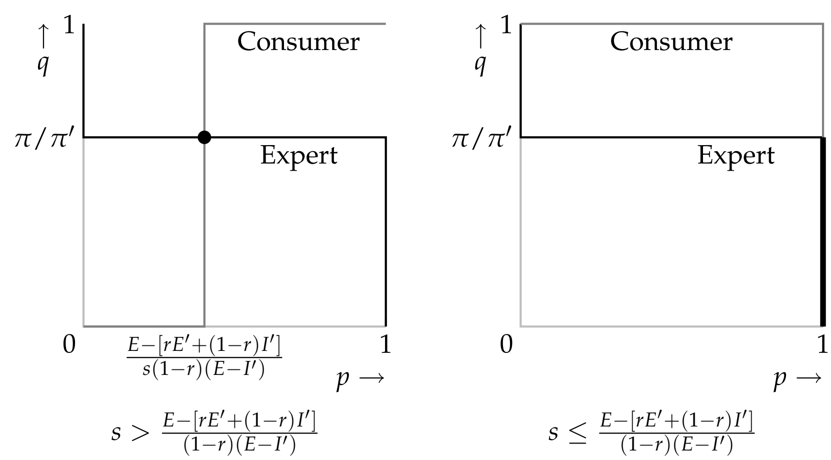


Figure 60.2 The players' best response functions in the game in Exercise 127.2. The probability assigned by the expert to H is p and the probability assigned by the consumer to A is q.

We see that q^* is independent of s. That is, the degree of competence has no effect on consumer behavior: consumers do not become more, or less, wary. The fraction of experts who are honest is a decreasing function of s, so that greater incompetence (smaller s) leads to a *higher* fraction of honest experts: incompetence breeds honesty! The intuition is that when experts become less competent, the potential gain from ignoring their advice increases (since $I' < E$), so that they need to be more honest to attract business.

When s is small the game has a continuum of mixed strategy Nash equilibria. In all equilibria the expert is always honest; the probability that the consumer accepts her advice ranges from 0 to π/π'. The value of s has no effect on these equilibria.

128.1 Choosing a seller

The game is given in Figure 61.1.

		Buyer 2: Seller 1	Buyer 2: Seller 2
Buyer 1	Seller 1	$\frac{1}{2}(1-p_1), \frac{1}{2}(1-p_1)$	$1-p_1, 1-p_2$
	Seller 2	$1-p_2, 1-p_1$	$\frac{1}{2}(1-p_2), \frac{1}{2}(1-p_2)$

Figure 61.1 The game in Exercise 128.1.

The character of its equilibria depend on the value of (p_1, p_2). If $p_1 = p_2 = 1$ every pair $((\pi_1, 1-\pi_1), ((\pi_2, 1-\pi_2))$ is a mixed strategy equilibrium (where π_i is the probability of buyer i's choosing seller 1) is a equilibrium. Now suppose that at least one price is less than 1.

- If $\frac{1}{2}(1-p_2) > 1-p_1$ (i.e. $p_2 < 2p_1 - 1$), each buyer's action of approaching seller 2 strictly dominates her action of approaching seller 1. Thus the game has a unique mixed strategy equilibrium, in which both buyers use a pure strategy: each approaches seller 2.

- If $\frac{1}{2}(1-p_2) = 1-p_1$ (i.e. $p_2 = 2p_1 - 1$), every mixed strategy is a best response of a buyer to the other buyer's approaching seller 2, and the pure strategy of approaching seller 2 is the unique best response to the other buyer's using any other strategy. Thus $((\pi_1, 1-\pi_1), ((\pi_2, 1-\pi_2))$ is a mixed strategy equilibrium if and only if either $\pi_1 = 0$ or $\pi_2 = 0$.

- If $\frac{1}{2}(1-p_1) > 1-p_2$ (i.e. $p_2 > \frac{1}{2}(1+p_1)$), each buyer's action of approaching seller 1 strictly dominates her action of approaching seller 2. Thus the game has a unique mixed strategy equilibrium, in which both buyers use a pure strategy: each approaches seller 1.

- If $\frac{1}{2}(1-p_1) = 1-p_2$ (i.e. $p_2 = \frac{1}{2}(1+p_1)$), every mixed strategy is a best response of a buyer to the other buyer's strategy of approaching seller 1, and the pure strategy of approaching seller 1 is the unique best response to any other strategy of the other buyer. Thus $((\pi_1, 1-\pi_1), ((\pi_2, 1-\pi_2))$ is a mixed strategy equilibrium if and only if either $\pi_1 = 1$ or $\pi_2 = 1$.

- For the case $2p_1 - 1 < p_2 < \frac{1}{2}(1+p_1)$, a buyer's expected payoff to choosing each seller is the same when

$$\tfrac{1}{2}(1-p_1)\pi + (1-p_1)(1-\pi) = (1-p_2)\pi + \tfrac{1}{2}(1-p_2)(1-\pi),$$

where π is the probability that the other buyer chooses seller 1, or when

$$\pi = \frac{1-2p_1+p_2}{2-p_1-p_2}.$$

The players' best response functions are shown in Figure 62.1. We see that the game has three mixed strategy equilibria: two pure equilibria in which the buyers approach different sellers, and one mixed strategy equilibrium in which each buyer approaches seller 1 with probability $(1-2p_1+p_2)/(2-p_1-p_2)$.

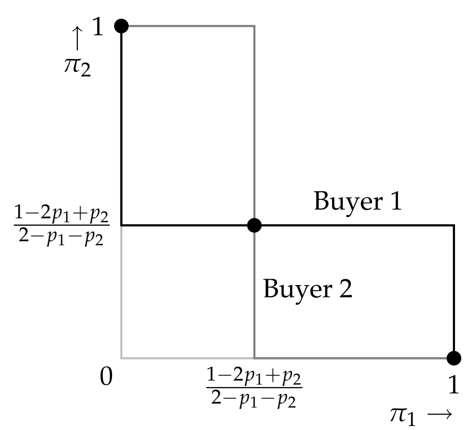


Figure 62.1 The players' best response functions in the game in Exercise 128.1. The probability with which buyer i approaches seller 1 is π_i.

The three main cases are illustrated in Figure 63.1. If the prices are relatively close, there are two pure strategy equilibria, in which the buyers choose different sellers, and a symmetric mixed strategy equilibrium in which both buyers approach seller 1 with the same probability. If seller 2's price is high relative to seller 1's, there is a unique equilibrium, in which both buyers approach seller 1. If seller 1's price is high relative to seller 2's, there is a unique equilibrium, in which both buyers approach seller 2.

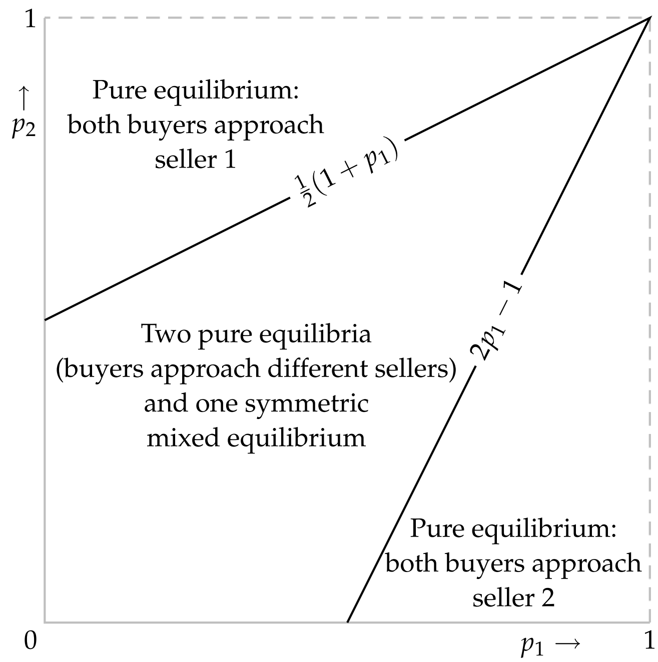


Figure 63.1 Equilibria of the game in Exercise 128.1 as a function of the sellers' prices.

130.2 Approaching cars

The game has three Nash equilibria: $(Stop, Continue)$, $(Continue, Stop)$, and a mixed strategy equilibrium in which each player chooses *Stop* with probability

$$\frac{1-\epsilon}{2-\epsilon}.$$

Only the mixed strategy equilibrium is symmetric; the expected payoff of each player in this equilibrium is $2(1-\epsilon)/(2-\epsilon)$.

The modified game also has a unique symmetric equilibrium. In this equilibrium each player chooses *Stop* with probability

$$\frac{1-\epsilon+\delta}{2-\epsilon}$$

if $\delta \leq 1$ and chooses *Stop* with probability 1 if $\delta \geq 1$. The expected payoff of each player in this equilibrium is $(2(1-\epsilon)+\epsilon\delta)/(2-\epsilon)$ if $\delta \leq 1$ and 1 if $\delta \geq 1$, both of which are larger than her payoff in the original game (given $\delta > 0$).

After reeducation, each driver's payoffs to stopping stay the same, while those to continuing fall. Thus if the behavioral norm (the probability of stopping) were to remain the same, every driver would find it beneficial to stop. Equilibrium is restored only if enough drivers switch to *Stop*, raising everyone's expected payoff. (Each player's expected payoff in a mixed strategy Nash equilibrium is her expected payoff to choosing *Stop*, which is $p + (1-\epsilon)(1-p)$, where p is the probability of a player's choosing *Stop*.)

130.3 Bargaining

The game is given in Figure 64.1.

	0	2	4	6	8	10
0	$5,5$	$4,6$	$3,7$	$2,8$	$1,9$	$0,10$
2	$6,4$	$5,5$	$4,6$	$3,7$	$2,8$	$0,0$
4	$7,3$	$6,4$	$5,5$	$4,6$	$0,0$	$0,0$
6	$8,2$	$7,3$	$6,4$	$0,0$	$0,0$	$0,0$
8	$9,1$	$8,2$	$0,0$	$0,0$	$0,0$	$0,0$
10	$10,0$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$

Figure 64.1 A bargaining game.

By inspection it has a single symmetric pure strategy Nash equilibrium, $(10,10)$.

Now consider situations in which the common mixed strategy assigns positive probability to two actions. Suppose that player 2 assigns positive probability only to 0 and 2. Then player 1's payoff to her action 4 exceeds her payoff to either 0 or 2. Thus there is no symmetric equilibrium in which the actions assigned positive probability are 0 and 2. By a similar argument we can rule out equilibria in which the actions assigned positive probability are any pair except 2 and 8, or 4 and 6.

If the actions to which player 2 assigns positive probability are 2 and 8 then player 1's expected payoffs to 2 and 8 are the same if the probability player 2 assigns to 2 is $\frac{2}{5}$ (and the probability she assigns to 8 is $\frac{3}{5}$). Given these probabilities, player 1's expected payoff to her actions 2 and 8 is $\frac{16}{5}$, and her expected payoff to every other action is less than $\frac{16}{5}$. Thus the pair of mixed strategies in which every player assigns probability $\frac{2}{5}$ to 2 and $\frac{3}{5}$ to 8 is a symmetric mixed strategy Nash equilibrium.

Similarly, the game has a symmetric mixed strategy equilibrium (α^*, α^*) in which α^* assigns probability $\frac{4}{5}$ to the demand of 4 and probability $\frac{1}{5}$ to the demand of 6.

In summary, the game has three symmetric mixed strategy Nash equilibria in which each player's strategy assigns positive probability to at most two actions: one in which probability 1 is assigned to 10, one in which probability $\frac{2}{5}$ is assigned to 2 and probability $\frac{3}{5}$ is assigned to 8, and one in which probability $\frac{4}{5}$ is assigned to 4 and probability $\frac{1}{5}$ is assigned to 6.

132.2 Reporting a crime when the witnesses are heterogeneous

Denote by p_i the probability with which each witness with cost c_i reports the crime, for $i = 1, 2$. For each witness with cost c_1 to report with positive probability less than one, we need

$$v - c_1 = v \cdot \Pr\{\text{at least one other person calls}\}$$

$$= v\left(1-(1-p_1)^{n_1-1}(1-p_2)^{n_2}\right),$$

or

$$c_1 = v(1-p_1)^{n_1-1}(1-p_2)^{n_2}. \tag{65.1}$$

Similarly, for each witness with cost c_2 to report with positive probability less than one, we need

$$\begin{aligned} v - c_2 &= v \cdot \Pr\{\text{at least one other person calls}\} \\ &= v\left(1-(1-p_1)^{n_1}(1-p_2)^{n_2-1}\right), \end{aligned}$$

or

$$c_2 = v(1-p_1)^{n_1}(1-p_2)^{n_2-1}. \tag{65.2}$$

Dividing (65.1) by (65.2) we obtain

$$1 - p_2 = c_1(1-p_1)/c_2.$$

Substituting this expression for $1 - p_2$ into (65.1) we get

$$p_1 = 1 - \left(\frac{c_1}{v}\cdot\left(\frac{c_2}{c_1}\right)^{n_2}\right)^{1/(n-1)}.$$

Similarly,

$$p_2 = 1 - \left(\frac{c_2}{v}\cdot\left(\frac{c_1}{c_2}\right)^{n_1}\right)^{1/(n-1)}.$$

For these two numbers to be probabilities, we need each of them to be nonnegative and at most one, which requires

$$\left(\frac{c_2^{n_2}}{v}\right)^{1/(n_2-1)} < c_1 < \left(v c_2^{n_1-1}\right)^{1/n_1}.$$

132.3 Contributing to a public good

In a mixed strategy equilibrium each player obtains the same expected payoff whether or not she contributes. A player's contribution makes a difference to the outcome only if exactly $k-1$ of the other players contribute. Thus the difference between the expected benefit of contributing and that of not contributing is

$$vQ_{n-1,k-1}(p) - c,$$

which must be 0 in a mixed strategy equilibrium.

For $v = 1$, $n = 4$, $k = 2$, and $c = \frac{3}{8}$ this equilibrium condition is

$$Q_{3,1}(p) = \tfrac{3}{8}.$$

Now, $Q_{3,1}(p) = 3p(1-p)^2$, so an equilibrium value of p satisfies

$$3p(1-p)^2 = \tfrac{3}{8},$$

or

$$p^3 - 2p^2 + p - \tfrac{1}{8} = 0,$$

or

$$(p - \tfrac{1}{2})(p^2 - \tfrac{3}{2}p + \tfrac{1}{4}) = 0.$$

Thus $p = \frac{1}{2}$ or $p = \frac{3}{4} - \frac{1}{2}\sqrt{\frac{5}{4}} \approx 0.19$. (The other root of the quadratic is greater than one, and thus not meaningful as a solution of the problem.)

We conclude that the game has two symmetric mixed strategy Nash equilibria: one in which the common probability is $\frac{1}{2}$ and one in which this probability is $\frac{3}{4} - \frac{1}{2}\sqrt{\frac{5}{4}}$.

136.1 Best response dynamics in Cournot's duopoly game

The best response functions of both firms are the same, so if the firms' outputs are initially the same, they are the same in every period: $q_1^t = q_2^t$ for every t. For each period t, we thus have

$$q_i^t = \tfrac{1}{2}(\alpha - c - q_i^t).$$

Given that $q_i^1 = 0$ for $i = 1, 2$, solving this first-order difference equation we have

$$q_i^t = \tfrac{1}{3}(\alpha - c)[1 - (-\tfrac{1}{2})^{t-1}]$$

for each period t. When t is large, q_i^t is close to $\frac{1}{3}(\alpha - c)$, a firm's equilibrium output.

In the first few periods, these outputs are 0, $\frac{1}{2}(\alpha - c)$, $\frac{1}{4}(\alpha - c)$, $\frac{3}{8}(\alpha - c)$, $\frac{5}{16}(\alpha - c)$.

136.2 Best response dynamics in Bertrand's duopoly game

If $p_i > c + 1$ then firm j has a unique best response, equal to the lesser of $p_i - 1$ and the monopoly price. Thus if both prices initially exceed $c + 1$ then for every period t in which at least one price exceeds $c + 1$ the maximal price in period $t + 1$ is (*i*) less than the maximal price in period t and (*ii*) at least $c + 1$. Thus the process converges to the Nash equilibrium $(c + 1, c + 1)$.

If $p_i = c$ then all prices $p_j \geq c$ are best responses. Thus if the pair of prices is initially (c, c), many subsequent sequences of prices are consistent with best response dynamics. We can divide the sequences into three cases.

- Both prices are equal to c in every subsequent period.
- In some period both prices are at least $c + 1$, in which case eventually the Nash equilibrium $(c + 1, c + 1)$ is reached (by the analysis for the first part of the exercise).

- In every period one of the prices is equal to c, while the other price is greater than c; the identity of the firm charging c changes from period to period. The pairs of prices eventually alternate between $(c, c+1)$ and $(c+1, c)$ (neither of which are Nash equilibria).

139.1 Finding all mixed strategy equilibria of two-player games

Left game:

- There is no equilibrium in which each player's mixed strategy assigns positive probability to a single action (i.e. there is no pure equilibrium).
- Consider the possibility of an equilibrium in which one player assigns probability 1 to a single action while the other player assigns positive probability to both her actions. For neither action of player 1 is player 2's payoff the same for both her actions, and for neither action of player 2 is player 1's payoff the same for both her actions, so there is no mixed strategy equilibrium of this type.
- Consider the possibility of a mixed strategy equilibrium in which each player assigns positive probability to both her actions. Denote by p the probability player 1 assigns to T and by q the probability player 2 assigns to L. For player 1's expected payoff to her two actions to be the same we need

  $$6q = 3q + 6(1-q),$$

  or $q = \frac{2}{3}$. For player 2's expected payoff to her two actions to be the same we need

  $$2(1-p) = 6p,$$

  or $p = \frac{1}{4}$. We conclude that the game has a unique mixed strategy equilibrium, $((\frac{1}{4}, \frac{3}{4}), (\frac{2}{3}, \frac{1}{3}))$.

Right game:

- By inspection, (T, R) and (B, L) are the pure strategy equilibria.
- Consider the possibility of a mixed strategy equilibrium in which one player assigns probability 1 to a single action while the other player assigns positive probability to both her actions.
  - $\{T\}$ for player 1, $\{L, R\}$ for player 2: no equilibrium, because player 2's payoffs to (T, L) and (T, R) are not the same.
  - $\{B\}$ for player 1, $\{L, R\}$ for player 2: no equilibrium, because player 2's payoffs to (B, L) and (B, R) are not the same.
  - $\{T, B\}$ for player 1, $\{L\}$ for player 2: no equilibrium, because player 1's payoffs to (T, L) and (B, L) are not the same.

- $\{T, B\}$ for player 1, $\{R\}$ for player 2: player 1's payoffs to (T, R) and (B, R) are the same, so there is an equilibrium in which player 1 uses T with probability p if player 2's expected payoff to R, which is $2p + 1 - p$, is at least her expected payoff to L, which is $p + 2(1 - p)$. That is, the game has equilibria in which player 1's mixed strategy is $(p, 1 - p)$, with $p \geq \frac{1}{2}$, and player 2 uses R with probability 1.

- Consider the possibility of an equilibrium in which both players assign positive probability to both their actions. Denote by q the probability that player 2 assigns to L. For player 1's expected payoffs to T and B to be the same we need $0 = 2q$, or $q = 0$, so there is no equilibrium in which both players assign positive probability to both their actions.

In summary, the mixed strategy equilibria of the game are $((0, 1), (1, 0))$ (i.e. the pure equilibrium (B, L)) and $((p, 1 - p), (0, 1))$ for $\frac{1}{2} \leq p \leq 1$ (of which one equilibrium is the pure equilibrium (T, R)).

141.1 Finding all mixed strategy equilibria of a two-player game

By inspection, (T, R) and (B, L) are pure strategy equilibria.

Now consider the possibility of an equilibrium in which player 1's strategy is pure while player 2's strategy assigns positive probability to two or more actions.

- If player 1's strategy is T then player 2's payoffs to M and R are the same, and her payoff to L is less, so an equilibrium in which player 2 randomizes between M and R is possible. In order that T be optimal we need $1 - q \geq q$, or $q \leq \frac{1}{2}$, where q is the probability player 2's strategy assigns to M. Thus every mixed strategy pair $((1, 0), (0, q, 1 - q))$ in which $q \leq \frac{1}{2}$ is a mixed strategy equilibrium.

- If player 1's strategy is B then player 2's payoffs to L and R are the same, and her payoff to M is less, so an equilibrium in which player 2 randomizes between L and R is possible. In order that B be optimal we need $2q + 1 - q \leq 3q$, or $q \geq \frac{1}{2}$, where q is the probability player 2's strategy assigns to L. Thus every mixed strategy pair $((0, 1), (q, 0, 1 - q))$ in which $q \geq \frac{1}{2}$ is a mixed strategy equilibrium.

Now consider the possibility of an equilibrium in which player 2's strategy is pure while player 1's strategy assigns positive probability to both her actions. For each action of player 2, player 1's two actions yield her different payoffs, so there is no equilibrium of this sort.

Next consider the possibility of an equilibrium in which both player 1's and player 2's strategies assign positive probability to two actions. Denote by p the probability player 1's strategy assigns to T. There are three possibilities for the pair of player 2's actions that have positive probability.

L and *M*: For an equilibrium we need player 2's expected payoff to L to be equal to her expected payoff to M and at least her expected payoff to R. That is, we need
$$2 = 3p + 1 - p \geq 3p + 2(1-p).$$
The inequality implies that $p = 1$, so that player 1's strategy assigns probability zero to B. Thus there is no equilibrium of this type.

L and *R*: For an equilibrium we need player 2's expected payoff to L to be equal to her expected payoff to R and at least her expected payoff to M. That is, we need
$$2 = 3p + 2(1-p) \geq 3p + 1 - p.$$
The equation implies that $p = 0$, so there is no equilibrium of this type.

M and *R*: For an equilibrium we need player 2's expected payoff to M to be equal to her expected payoff to R and at least her expected payoff to L. That is, we need
$$3p + 1 - p = 3p + 2(1-p) \geq 2.$$
The equation implies that $p = 1$, so there is no equilibrium of this type.

The final possibility is that there is an equilibrium in which player 1's strategy assigns positive probability to both her actions and player 2's strategy assigns positive probability to all three of her actions. Let p be the probability player 1's strategy assigns to T. Then for player 2's expected payoffs to her three actions to be equal we need
$$2 = 3p + 1 - p = 3p + 2(1-p).$$
For the first equality we need $p = \frac{1}{2}$, violating the second equality. That is, there is no value of p for which player 2's expected payoffs to her three actions are equal, and thus no equilibrium in which she chooses each action with positive probability.

We conclude that the mixed strategy equilibria of the game are the strategy pairs of the forms $((1,0),(0,q,1-q))$ for $0 \leq q \leq \frac{1}{2}$ ($q = 0$ is the pure equilibrium (T,R)) and $((0,1),(q,0,1-q))$ for $\frac{1}{2} \leq q \leq 1$ ($q = 1$ is the pure equilibrium (B,L)).

141.2 Rock, paper, scissors

The game is shown in Figure 69.1.

	Rock	*Paper*	*Scissors*
Rock	$0, \; 0$	$-1, \; 1$	$1, -1$
Paper	$1, -1$	$0, \; 0$	$-1, \; 1$
Scissors	$-1, \; 1$	$1, -1$	$0, \; 0$

Figure 69.1 *Rock, paper, scissors*

By inspection the game has no pure strategy equilibrium, and no mixed strategy equilibrium in which one player's strategy is pure and the other's is strictly mixed.

In the remaining possibilities both players use at least two actions with positive probability. Suppose that player 1's mixed strategy assigns positive probability to *Rock* and to *Paper*. Then player 2's expected payoff to *Paper* exceeds her expected payoff to *Rock*, so in any such equilibrium player 2 must assign positive probability only to *Paper* and *Scissors*. Player 1's expected payoffs to *Rock* and *Paper* are equal only if player 2 assigns probability $\frac{2}{3}$ to *Paper* and probability $\frac{1}{3}$ to *Scissors*. But then player 1's expected payoff to *Scissors* exceeds her expected payoffs to *Rock* and *Paper*. So there is no mixed strategy equilibrium in which player 1 assigns positive probability only to *Rock* and to *Paper*.

Given the symmetry of the game, the same argument implies that there is no equilibrium in which player 1 assigns positive probability to only two actions, nor any equilibrium in which player 2 assigns positive probability to only two actions.

The remaining possibility is that each player assigns positive probability to all three of her actions. Denote the probabilities player 1 assigns to her three actions by (p_1, p_2, p_3) and the probabilities player 2 assigns to her three actions by (q_1, q_2, q_3). Player 1's actions all yield her the same expected payoff if and only if there is a value of c for which

$$\begin{aligned} -q_2 + q_3 &= c \\ q_1 \qquad - q_3 &= c \\ -q_1 + q_2 \qquad &= c. \end{aligned}$$

Adding the three equations we deduce $c = 0$, and hence $q_1 = q_2 = q_3 = \frac{1}{3}$. A similar calculation for player 2 yields $p_1 = p_2 = p_3 = \frac{1}{3}$.

In conclusion, the game has a unique mixed strategy equilibrium, in which each player uses the strategy $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Each player's equilibrium payoff is 0.

In the modified game in which player 1 is prohibited from using the action *Scissors*, player 2's action *Rock* is strictly dominated. The remaining game has a unique mixed strategy equilibrium, in which player 1 chooses *Rock* with probability $\frac{1}{3}$ and *Paper* with probability $\frac{2}{3}$, and player 2 chooses *Paper* with probability $\frac{2}{3}$ and *Scissors* with probability $\frac{1}{3}$. The equilibrium payoff of player 1 is $-\frac{1}{3}$ and that of player 2 is $\frac{1}{3}$.

141.3 Election campaigns

A strategic game that models the situation is shown in Figure 71.1, where action k means devote resources to locality k.

By inspection the game has no pure strategy equilibrium and no equilibrium in which one player's strategy is pure and the other is strictly mixed. (For each action of each player, the other player has a single best action.)

		Party B		
		1	2	3
	1	$0,0$	$a_1,-a_1$	$a_1,-a_1$
Party A	2	$a_2,-a_2$	$0,0$	$a_2,-a_2$
	3	$a_3,-a_3$	$a_3,-a_3$	$0,0$

Figure 71.1 The game in Exercise 141.3.

Now consider the possibility of an equilibrium in which party A assigns positive probability to exactly two actions. There are three possible pairs of actions. Throughout the argument I denote the probability party A's strategy assigns to her action i by p_i, and the probability party B's strategy assigns to her action i by q_i.

1 and 2: Party B's action 3 is strictly dominated by her mixed strategy that assigns probability $\frac{1}{2}$ to each of her actions 1 and 2, so that we can eliminate it from consideration. For party A's actions 1 and 2 to yield the same expected payoff we need $q_2 a_1 = q_1 a_2$, or, given $q_2 = 1 - q_1$, $q_1 = a_1/(a_1 + a_2)$. For party B's actions 1 and 2 to yield the same expected payoff we similarly need $p_1 = a_2/(a_1 + a_2)$. Finally, for party A's expected payoff to her action 3 to be no more than her expected payoff to her other two actions, we need

$$a_3 \leq \frac{a_1 a_2}{a_1 + a_2}.$$

We conclude that if $a_3 \leq a_1 a_2/(a_1 + a_2)$ (or equivalently $a_1 a_3 + a_2 a_3 \leq a_1 a_2$) then the game has a mixed strategy equilibrium

$$\left(\left(\frac{a_2}{a_1 + a_2}, \frac{a_1}{a_1 + a_2}, 0\right), \left(\frac{a_1}{a_1 + a_2}, \frac{a_2}{a_1 + a_2}, 0\right)\right). \tag{71.1}$$

1 and 3: Party B's action 2 is strictly dominated her mixed strategy that assigns probability $\frac{1}{2}$ to each of her actions 1 and 3, so that we can eliminate it from consideration. But then party A's action 2 strictly dominates her action 3, so there is no equilibrium in which she assigns positive probability to action 3. Thus there is no equilibrium of this type.

2 and 3: For similar reasons, there is no equilibrium of this type.

The remaining possibility is that there is an equilibrium in which each player assigns positive probability to all three of her actions. In order that party A's actions yield the same expected payoff we need

$$a_1(q_2 + q_3) = a_2(q_1 + q_3) = a_3(q_1 + q_2),$$

or, using $q_1 + q_2 + q_3 = 1$,

$$q_1 = \frac{a_1 a_2 + a_1 a_3 - a_2 a_3}{a_1 a_2 + a_1 a_3 + a_2 a_3}, \quad q_2 = \frac{a_1 a_2 - a_1 a_3 + a_2 a_3}{a_1 a_2 + a_1 a_3 + a_2 a_3}, \quad q_3 = \frac{-a_1 a_2 + a_1 a_3 + a_2 a_3}{a_1 a_2 + a_1 a_3 + a_2 a_3}. \tag{71.2}$$

For these three numbers to be positive we need

$$a_1a_2 + a_1a_3 - a_2a_3 > 0, \quad a_1a_2 - a_1a_3 + a_2a_3 > 0, \quad -a_1a_2 + a_1a_3 + a_2a_3 > 0.$$

Since $a_1 > a_2 > a_3$, these inequalities are satisfied if and only if $a_1a_3 + a_2a_3 > a_1a_2$.

Similarly, in order that party B's actions yield the same expected payoff we need

$$p_1 = \frac{a_2a_3}{a_1a_2 + a_1a_3 + a_2a_3}, \quad p_2 = \frac{a_1a_3}{a_1a_2 + a_1a_3 + a_2a_3}, \quad p_3 = \frac{a_1a_2}{a_1a_2 + a_1a_3 + a_2a_3}. \tag{72.1}$$

These three numbers are positive, given $a_i > 0$ for all i.

Thus if $a_1a_3 + a_2a_3 > a_1a_2$ there is an equilibrium in which player 1's mixed strategy is (p_1, p_2, p_3) and player 2's mixed strategy is (q_1, q_2, q_3).

In summary,

- if $(a_1 + a_2)a_3 \leq a_1a_2$ then the game has a unique mixed strategy equilibrium given by (71.1)
- if $(a_1 + a_2)a_3 > a_1a_2$ then the game has a unique mixed strategy equilibrium given by (71.2) and (72.1).

That is, if the first two localities are sufficiently more valuable than the third then both parties concentrate all their efforts on these two localities, while otherwise they both randomize between all three localities.

142.1 A three-player game

By inspection the game has two pure strategy equilibria, namely (A, A, A) and (B, B, B).

Now consider the possibility of an equilibrium in which one or more of the players' strategies is pure, and at least one is strictly mixed. If player 1 uses the action A and player 2 uses a strictly mixed strategy then player 3's uniquely best action is A, in which case player 2's uniquely best action is A. Thus there is no equilibrium in which player 1 uses the action A and at least one of the other players randomizes. By similar arguments, there is no equilibrium in which player 1 uses the action B and at least one of the other players randomizes, or indeed any equilibrium in which some player's strategy is pure while some other player's strategy is mixed.

The remaining possibility is that there is an equilibrium in which each player's strategy assigns positive probability to each of her actions. Denote the probabilities that players 1, 2, and 3 assign to A by p, q, and r respectively. In order that player 1's expected payoffs to her two actions be the same we need

$$qr = 4(1 - q)(1 - r).$$

Similarly, for player 2's and player 3's expected payoffs to their two actions to be the same we need

$$pr = 4(1 - p)(1 - r) \quad \text{and} \quad pq = 4(1 - p)(1 - q).$$

The unique solution of these three equations is $p = q = r = \frac{2}{3}$ (isolate r in the second equation and q in the third equation, and substitute into the first equation).

We conclude that the game has three mixed strategy equilibria: $((1,0),(1,0),(1,0))$ (i.e. the pure strategy equilibrium (A,A,A)), $((0,1),(0,1),(0,1))$ (i.e. the pure strategy equilibrium (B,B,B)), and $((\frac{2}{3},\frac{1}{3}),(\frac{2}{3},\frac{1}{3}),(\frac{2}{3},\frac{1}{3}))$.

146.1 All-pay auction with many bidders

Denote the common mixed strategy by F. Look for an equilibrium in which the largest value of z for which $F(z) = 0$ is 0 and the smallest value of z for which $F(z) = 1$ is $z = K$.

A player who bids a_i wins if and only if the other $n-1$ players all bid less than she does, an event with probability $(F(a_i))^{n-1}$. Thus, given that the probability that she ties for the highest bid is zero, her expected payoff is

$$(K - a_i)(F(a_i))^{n-1} + (-a_i)(1 - (F(a_i))^{n-1}).$$

Given the form of F, for an equilibrium this expected payoff must be constant for all values of a_i with $0 \le a_i \le K$. That is, for some value of c we have

$$K(F(a_i))^{n-1} - a_i = c \text{ for all } 0 \le a_i \le K.$$

For $F(0) = 0$ we need $c = 0$, so that $F(a_i) = (a_i/K)^{1/(n-1)}$ is the only candidate for an equilibrium strategy.

The function F is a cumulative probability distribution on the interval from 0 to K because $F(0) = 0$, $F(K) = 1$, and F is increasing. Thus F is indeed an equilibrium strategy.

We conclude that the game has a mixed strategy Nash equilibrium in which each player randomizes over all her actions according to the probability distribution $F(a_i) = (a_i/K)^{1/(n-1)}$; each player's equilibrium expected payoff is 0.

Each player's mean bid is K/n.

146.2 Bertrand's duopoly game

Denote the common mixed strategy by F. If firm 1 charges p it earns a profit only if the price charged by firm 2 exceeds p, an event with probability $1 - F(p)$. Thus firm 1's expected profit is

$$(1 - F(p))(p - c)D(p).$$

This profit is constant, equal to B, over some range of prices, if $F(p) = 1 - B/((p - c)D(p))$ over this range of prices. Because $(p - c)D(p)$ increases without bound as p increases without bound, for any value of B the number $F(p)$ approaches 1 as p increases without bound. Further, for any $B > 0$, there exists some $\underline{p} > c$ such that $(\underline{p} - c)D(\underline{p}) = B$, so that $F(\underline{p}) = 0$. Finally, because $(p - c)D(p)$ is an increasing function, so is F. Thus F is a cumulative probability distribution function.

We conclude that for any $\underline{p} > c$, the game has a mixed strategy equilibrium in which each firm's mixed strategy is given by

$$F(p) = \begin{cases} 0 & \text{if } p < \underline{p} \\ 1 - \dfrac{(\underline{p} - c)D(\underline{p})}{(p - c)D(p)} & \text{if } p \geq \underline{p}. \end{cases}$$

147.2 Preferences over lotteries

The first piece of information about the decision-maker's preferences among lotteries is consistent with her preferences being represented by the expected value of a payoff function: set $u(a_1) = 0$, $u(a_2)$ equal to any number between $\frac{1}{2}$ and $\frac{1}{4}$, and $u(a_3) = 1$.

The second piece of information about the decision-maker's preferences is not consistent with these preferences being represented by the expected value of a payoff function, by the following argument. For consistency with the information about the decision-maker's preferences among the four lotteries, we need

$$0.4u(a_1) + 0.6u(a_3) > 0.5u(a_2) \; + \; 0.5u(a_3) > \\ 0.3u(a_1) + 0.2u(a_2) \; + \; 0.5u(a_3) > 0.45u(a_1) + 0.55u(a_3).$$

The first inequality implies $u(a_2) < 0.8u(a_1) + 0.2u(a_3)$ and the last inequality implies $u(a_2) > 0.75u(a_1) + 0.25u(a_3)$. Because $u(a_1) < u(a_3)$, we have $0.75u(a_1) + 0.25u(a_3) > 0.8u(a_1) + 0.2u(a_3)$, so that the two inequalities are incompatible.

149.2 Normalized vNM payoff functions

Let $\bar{a}$ be the best outcome according to her preferences and let $\underline{a}$ be the worse outcome. Let $\eta = -u(\underline{a})/(u(\bar{a}) - u(\underline{a}))$ and $\theta = 1/(u(\bar{a}) - u(\underline{a})) > 0$. Lemma 148.1 implies that the function v defined by $v(x) = \eta + \theta u(x)$ represents the same preferences as does u; we have $v(\underline{a}) = 0$ and $v(\bar{a}) = 1$.

150.1 Games equivalent to the *Prisoner's Dilemma*

The left-hand game is not equivalent, by the following argument. Using either player's payoffs, for equivalence we need η and $\theta > 0$ such that

$$0 = \eta + \theta \cdot 0, 2 = \eta + \theta \cdot 1, 3 = \eta + \theta \cdot 2, \text{and } 4 = \eta + \theta \cdot 3.$$

From the first equation we have $\eta = 0$ and hence from the second we have $\theta = 2$. But these values do not satisfy the last two equations. (Alternatively, note that in the game in the left panel of Figure 107.1, player 1 is indifferent between (D, D) and the lottery in which (C, D) occurs with probability $\frac{1}{2}$ and (D, C) occurs with probability $\frac{1}{2}$, while in the left-hand game in Figure 150.2 she is not.)

The right-hand game is equivalent, by the following argument. For the equivalence of player 1's payoffs, we need η and $\theta > 0$ such that

$$0 = \eta + \theta \cdot 0, 3 = \eta + \theta \cdot 1, 6 = \eta + \theta \cdot 2, \text{and } 9 = \eta + \theta \cdot 3.$$

The first two equations yield $\eta = 0$ and $\theta = 3$; these values satisfy the second two equations. A similar argument for player 2's payoffs yields $\eta = -4$ and $\theta = 2$.

5 Extensive Games with Perfect Information: Theory

156.2 Examples of extensive games with perfect information

a. The game is given in Figure 77.2.

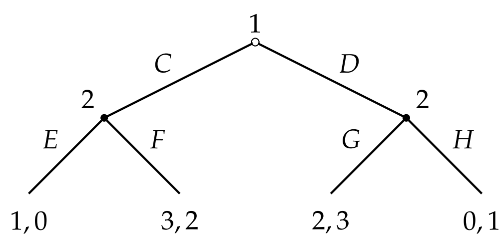


Figure 77.2 The game in Exercise 156.2*a*.

b. The game is specified as follows.

Players 1 and 2.

Terminal histories (C, E, G), (C, E, H), (C, F), D.

Player function $P(\varnothing) = 1$, $P(C) = 2$, $P(C, E) = 1$.

Preferences Player 1 prefers (C, F) to D to (C, E, G) to (C, E, H); player 2 prefers (C, E, G) to (C, F) to (C, E, H), and is indifferent between this outcome and D.

c. The game is specified as follows.

Players Rosa, Ernesto, and Karl.

Terminal histories All sequences (R, X, Y) and (E, X, Y) where X and Y are either B or H.

Player function $P(\varnothing) =$ Karl, $P(R) =$ Rosa, $P(E) =$ Ernesto, and $P(R, X) =$ Ernesto and $P(E, X) =$ Rosa for $X = B$ and $X = H$.

Preferences For any value of X, Karl and Ernesto prefer (X, H, H) to (X, B, B) to (X, B, H) and (X, H, B), between which they are both indifferent. For any value of X, Rosa prefers (X, B, B) to (X, H, H) to (X, B, H) and (X, H, B), between which she is indifferent.

The game in shown in Figure 78.1, where the order of the payoffs is Karl, Rosa, Ernesto.

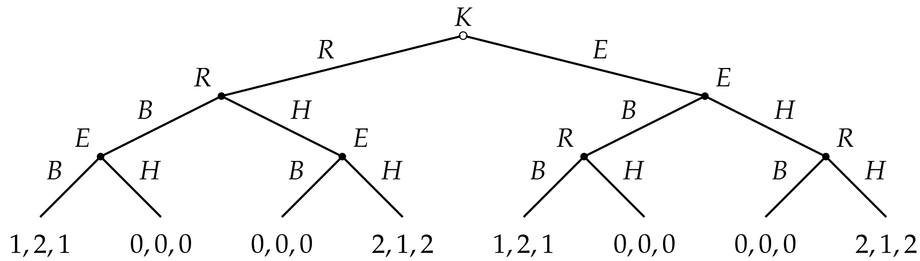


Figure 78.1 The game in Exercise 156.2*c*.

161.1 Strategies in extensive games

In the entry game, the challenger moves only at the start of the game, where it has two actions, *In* and *Out*. Thus it has two strategies, *In* and *Out*. The incumbent moves only after the history *In*, when it has two actions, *Acquiesce* and *Fight*. Thus it also has two strategies, *Acquiesce* and *Fight*.

In the game in Exercise 156.2c, Rosa moves after the histories R (Karl chooses her to move first), (E, B) (Karl chooses Ernesto to move first, and Ernesto chooses B), and (E, H) (Karl chooses Ernesto to move first, and Ernesto chooses H). In each case Rosa has two actions, B and H. Thus she has eight strategies. Each strategy takes the form (x, y, z), where each of x, y, and z are either B or H; the strategy (x, y, z) means that she chooses x after the history R, y after the history (E, B), and z after the history (E, H).

163.1 Nash equilibria of extensive games

The strategic form of the game in Exercise 156.2*a* is given in Figure 78.2.

	EG	*EH*	*FG*	*FH*
C	1, 0	1, 0	3, 2	3, 2
D	2, 3	0, 1	2, 3	0, 1

Figure 78.2 The strategic form of the game in Exercise 156.2*a*.

The Nash equilibria of the game are (C, FG), (C, FH), and (D, EG).
The strategic form of the game in Figure 160.1 is given in Figure 78.3.

	E	*F*
CG	1, 2	3, 1
CH	0, 0	3, 1
DG	2, 0	2, 0
DH	2, 0	2, 0

Figure 78.3 The strategic form of the game in Figure 160.1.

The Nash equilibria of the game are (CH, F), (DG, E), and (DH, E).

163.2 Voting by alternating veto

The following extensive game models the situation.

Players The two people.

Terminal histories $(\cancel{X}, \cancel{Y})$, $(\cancel{X}, \cancel{Z})$, $(\cancel{Y}, \cancel{X})$, $(\cancel{Y}, \cancel{Z})$, $(\cancel{Z}, \cancel{X})$, and $(\cancel{Z}, \cancel{Y})$ (where $\cancel{A}$ means veto A).

Player function $P(\varnothing) = 1$ and $P(\cancel{X}) = P(\cancel{Y}) = P(\cancel{Z}) = 2$.

Preferences Person 1's preferences are represented by the payoff function u_1 for which $u_1(\cancel{Y}, \cancel{Z}) = u_1(\cancel{Z}, \cancel{Y}) = 2$ (both of these terminal histories result in X's being chosen), $u_1(\cancel{X}, \cancel{Z}) = u_1(\cancel{Z}, \cancel{X}) = 1$, and $u_1(\cancel{X}, \cancel{Y}) = u_1(\cancel{Y}, \cancel{X}) = 0$. Person 2's preferences are represented by the payoff function u_2 for which $u_2(\cancel{X}, \cancel{Y}) = u_2(\cancel{Y}, \cancel{X}) = 2$, $u_2(\cancel{X}, \cancel{Z}) = u_2(\cancel{Z}, \cancel{X}) = 1$, and $u_2(\cancel{Y}, \cancel{Z}) = u_2(\cancel{Z}, \cancel{Y}) = 0$.

This game is shown in Figure 79.1.

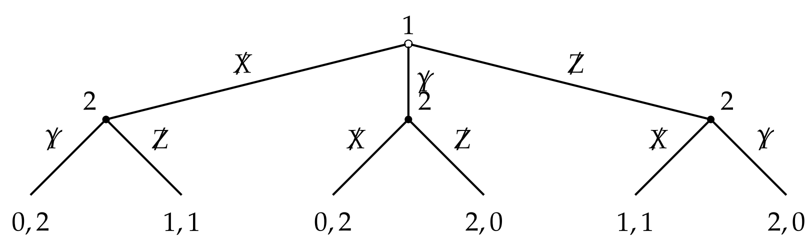


Figure 79.1 An extensive game that models the alternate strikeoff method of selecting an arbitrator, as specified in Exercise 163.2.

The strategic form of the game is given in Figure 79.2 (where $\cancel{A}\cancel{B}\cancel{C}$ is person 2's strategy in which it vetoes A if person 1 vetoes X, B if person 1 vetoes Y, and C if person 1 vetoes Z). Its Nash equilibria are $(\cancel{Z}, \cancel{Y}\cancel{X}\cancel{X})$ and $(\cancel{Z}, \cancel{Z}\cancel{X}\cancel{X})$.

	$\cancel{Y}\cancel{X}\cancel{X}$	$\cancel{Y}\cancel{X}\cancel{Y}$	$\cancel{Y}\cancel{Z}\cancel{X}$	$\cancel{Y}\cancel{Z}\cancel{Y}$	$\cancel{Z}\cancel{X}\cancel{X}$	$\cancel{Z}\cancel{X}\cancel{Y}$	$\cancel{Z}\cancel{Z}\cancel{X}$	$\cancel{Z}\cancel{Z}\cancel{Y}$
$\cancel{X}$	0, 2	0, 2	0, 2	0, 2	1, 1	1, 1	1, 1	1, 1
$\cancel{Y}$	0, 2	0, 2	2, 0	2, 0	0, 2	0, 2	2, 0	2, 0
$\cancel{Z}$	1, 1	2, 0	1, 1	2, 0	1, 1	2, 0	1, 1	2, 0

Figure 79.2 The strategic form of the game in Figure 79.1.

164.2 Subgames

The subgames of the game in Exercise 156.2*c* are the whole game and the six games in Figure 80.1.

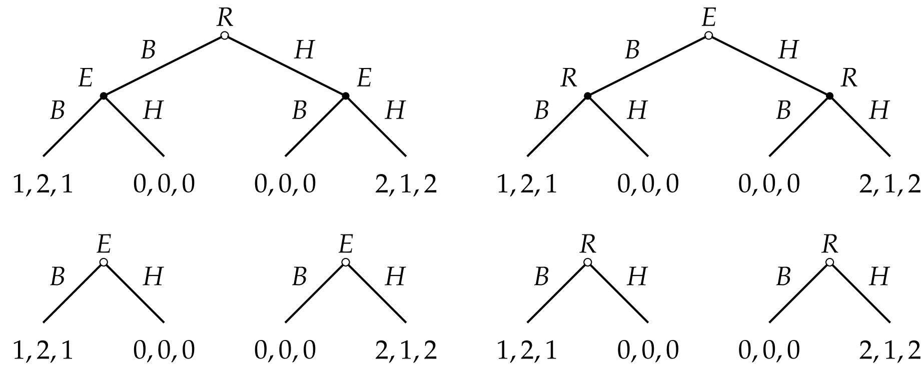


Figure 80.1 The proper subgames of the game in Exercise 156.2*c*.

168.1 Checking for subgame perfect equilibria

The Nash equilibria (CH, F) and (DH, E) are not subgame perfect equilibria: in the subgame following the history (C, E), player 1's strategies CH and DH induce the strategy H, which is not optimal.

The Nash equilibrium (DG, E) is a subgame perfect equilibrium: (*a*) it is a Nash equilibrium, so player 1's strategy is optimal at the start of the game, given player 2's strategy, (*b*) in the subgame following the history C, player 2's strategy E induces the strategy E, which is optimal given player 1's strategy, and (*c*) in the subgame following the history (C, E), player 1's strategy DG induces the strategy G, which is optimal.

173.2 Finding subgame perfect equilibria

The game in Exercise 156.2*a* has a unique subgame perfect equilibrium, (C, FG).

The game in Exercise 156.2*c* has a unique subgame perfect equilibrium in which Karl's strategy is E, Rosa's strategy is to choose B after the history R, B after the history (E, B), and H after the history (E, H), and Ernesto's strategy is to chooses B after the history (R, B), H after the history (R, H), and H after the history E. (The outcome is that Karl chooses Ernesto to move first, he chooses H, and then Rosa chooses H.)

The game in Figure 173.1 has six subgame perfect equilibria: (C, EG), (D, EG), (C, EH), (D, FG), (C, FH), (D, FH).

173.3 Voting by alternating veto

The game has a unique subgame perfect equilibrium $(\cancel{Z}, \cancel{Y}\cancel{X}\cancel{X})$. The outcome is that action Y is taken.

Thus the Nash equilibrium $(\not{Z}, \not{Z}\not{X}\not{X})$ (see Exercise 163.2) is not a subgame perfect equilibrium. However, this equilibrium generates the same outcome as the unique subgame perfect equilibrium.

If player 2 prefers Y to X to Z then in the unique subgame perfect equilibrium of the game in which player 1 moves first the outcome is that X is chosen, while in the unique subgame perfect equilibrium of the game in which player 2 moves first the outcome is that Y is chosen. (For all other strict preferences of player 2 (i.e. preferences in which player 2 is not indifferent between any pair of policies) the outcome of the subgame perfect equilibria of the two games are the same.)

173.4 Burning a bridge

An extensive game that models the situation has the same structure as the entry game in Figure 156.1 in the book. The challenger is army 1, the incumbent army 2. The action *In* corresponds to attacking; *Acquiesce* corresponds to retreating. The game has a single subgame perfect equilibrium, in which army 1 attacks, and army 2 retreats.

If army 2 burns the bridge, the game has a single subgame perfect equilibrium in which army 1 does not attack.

174.1 Sharing heterogeneous objects

Let $n = 2$ and $k = 3$, and call the objects a, b, and c. Suppose that the values person 1 attaches to the objects are 3, 2, and 1 respectively, while the values player 2 attaches are 1, 3, 2. If player 1 chooses a on the first round, then in any subgame perfect equilibrium player 2 chooses b, leaving player 1 with c on the second round. If instead player 1 chooses b on the first round, in any subgame perfect equilibrium player 2 chooses c, leaving player 1 with a on the second round. Thus in every subgame perfect equilibrium player 1 chooses b on the first round (though she values a more highly.)

Now I argue that for any preferences of the players, $G(2,3)$ has a subgame perfect equilibrium of the type described in the exercise. For any object chosen by player 1 in round 1, in any subgame perfect equilibrium player 2 chooses her favorite among the two objects remaining in round 2. Thus player 2 never obtains the object she least prefers; in any subgame perfect equilibrium, player 1 obtains that object. Player 1 can ensure she obtains her more preferred object of the two remaining by choosing that object on the first round. That is, there is a subgame perfect equilibrium in which on the first round player 1 chooses her more preferred object out of the set of objects excluding the object player 2 least prefers, and on the last round she obtains x_3. In this equilibrium, player 2 obtains the object less preferred by player 1 out of the set of objects excluding the object player 2 least prefers. That is, player 2 obtains x_2. (Depending on the players' preferences, the game also may have a subgame perfect equilibrium in which player 1 chooses x_3 on the first round.)

174.2 An entry game with a financially-constrained firm

a. Consider the last period, after any history. If the incumbent chooses to fight, the challenger's best action is to exit, in which case both firms obtain the profit zero. If the incumbent chooses to cooperate, the challenger's best action is to stay in, in which case both firms obtain the profit $C > 0$. Thus the incumbent's best action at the start of the period is to cooperate.

Now consider period $T - 1$. Regardless of the outcome in this period, the incumbent will cooperate in the last period, and the challenger will stay in (as we have just argued). Thus each player's action in the period affects its payoff only because it affects its profit in the period. Thus by the same argument as for the last period, in period $T - 1$ the incumbent optimally cooperates, and the challenger optimally stays in if the incumbent cooperates. If, in period $T - 1$, the incumbent fights, then the challenger also optimally stays in, because in the last period it obtains $C > F$.

Working back to the start of the game, using the same argument in each period, we conclude that in every period before the last the incumbent cooperates and the challenger stays in regardless of the incumbent's action. Given $C > f$, the challenger optimally enters at the start of the game.

That is, the game has a unique subgame perfect equilibrium, in which

- the challenger enters at the start of the game, exits in the last period if the challenger fights in that period, and stays in after every other history after which it moves
- the incumbent cooperates after every history after which it moves.

The incumbent's payoff in this equilibrium is TC and the challenger's payoff is $TC - f$.

b. First consider the incumbent's action after the history in which the challenger enters, the incumbent fights in the first $T - 2$ periods, and in each of these periods the challenger stays in. Denote this history h_{T-2}. If the incumbent fights after h_{T-2}, the challenger exits (it has no alternative), and the incumbent's total profit in the last two periods is M. If the incumbent cooperates after h_{T-2} then by the argument for the game in part *a*, the challenger stays in, and in the last period the incumbent also cooperates and the challenger stays in. Thus the incumbent's payoff in the last two periods if it cooperates after the history h_{T-2} is $2C$. Because $M > 2C$, we conclude that the incumbent fights after the history h_{T-2}.

Now consider the incumbent's action after the history in which the challenger enters, the incumbent fights in the first $T - 3$ periods, and in each period the challenger stays in. Denote this history h_{T-3}. If the incumbent fights after h_{T-3}, we know, by the previous paragraph, that if the challenger stays in then the incumbent will fight in the next period, driving the challenger out. Thus the challenger will obtain an additional profit of $-F$ if it stays in and 0 if it exits. Consequently the challenger exits if the incumbent fights after h_{T-3}, making a fight by the incumbent optimal (it yields the incumbent the additional profit $2M$).

Working back to the first period we conclude that the incumbent fights and the challenger exits. Thus the challenger's optimal action at the start of the game is to stay out.

In summary, the game has a unique subgame perfect equilibrium, in which

- the challenger stays out at the start of the game, exits after any history in which the incumbent fought in every period, exits in the last period if the incumbent fights in that period, and stays in after every other history.
- the incumbent fights after the challenger enters and after any history in which it has fought in every period, and cooperates after every other history.

The incumbent's payoff in this equilibrium is TM and the challenger's payoff is 0.

176.1 Dollar auction

The game is shown in Figure 84.1. It has four subgame perfect equilibria. In all the equilibria player 2 passes after player 1 bids \$2. After other histories the actions in the equilibria are as follows.

- Player 1 bids \$3 after the history $(\$1, \$2)$, player 2 passes after the history \$1, and player 1 bids \$1 at the start of the game.
- Player 1 passes after the history $(\$1, \$2)$, player 2 passes after the history \$1, and player 1 bids \$1 at the start of the game.
- Player 1 passes after the history $(\$1, \$2)$, player 2 bids \$2 after the history \$1, and player 1 passes at the start of the game.
- Player 1 passes after the history $(\$1, \$2)$, player 2 bids \$2 after the history \$1, and player 1 bids \$2 at the start of the game.

There are three subgame perfect equilibrium outcomes: player 1 passes at the start of the game (player 2 gets the object without making any payment), player 1 bids \$1 and then player 2 passes (player 1 gets the object for \$1), and player 1 bids \$2 and then player 2 passes (player 1 gets the object for \$2).

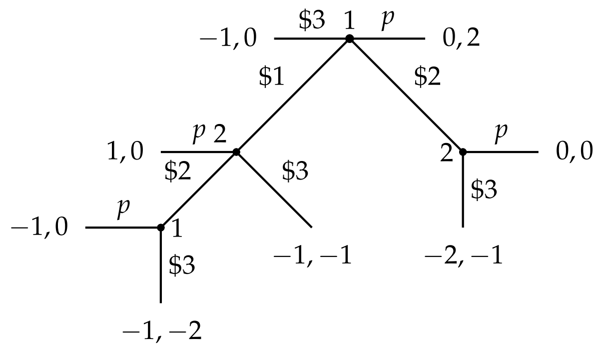


Figure 84.1 The extensive form of the dollar auction for $w = 3$ and $v = 2$. A pass is denoted p.

177.1 Firm–union bargaining

a. The following extensive game models the situation.

Players The firm and the union.

Terminal histories All sequences of the form (w, Y, L) and (w, N) for nonnegative numbers w and L (where w is a wage, Y means accept, N means reject, and L is the number of workers hired).

Player function $P(\varnothing)$ is the union, and, for any nonnegative number w, $P(w)$ and $P(w, Y)$ are the firm.

Preferences The firm's preferences are represented by its profit, and the union's preferences are represented by the value of wL (which is zero after any history (w, N)).

b. First consider the subgame following a history (w, Y), in which the firm accepts the wage demand w. In a subgame perfect equilibrium, the firm chooses L to maximize its profit, given w. For $L \leq 50$ this profit is $L(100 - L) - wL$, or $L(100 - w - L)$. This function is a quadratic in L that is zero when $L = 0$ and when $L = 100 - w$ and reaches a maximum in between. Thus the value of L that maximizes the firm's profit is $\frac{1}{2}(100 - w)$ if $w \leq 100$, and 0 if $w > 100$.

Given the firm's optimal action in such a subgame, consider the subgame following a history w, in which the firm has to decide whether to accept or reject w. For any w the firm's profit, given its subsequent optimal choice of L, is nonnegative; if $w < 100$ this profit is positive, while if $w \geq 100$ it is 0. Thus in a subgame perfect equilibrium, the firm accepts any demand $w < 100$ and either accepts or rejects any demand $w \geq 100$.

Finally consider the union's choice at the beginning of the game. If it chooses $w < 100$ then the firm accepts and chooses $L = (100 - w)/2$, yielding the union a payoff of $w(100 - w)/2$. If it chooses $w > 100$ then the firm either accepts and chooses $L = 0$ or rejects; in both cases the union's payoff is 0.

Thus the best value of w for the union is the number that maximizes $w(100 - w)/2$. This function is a quadratic that is zero when $w = 0$ and when $w = 100$ and reaches a maximum in between; thus its maximizer is $w = 50$.

In summary, in a subgame perfect equilibrium the union's strategy is $w = 50$, and the firm's strategy accepts any demand $w < 100$ and chooses $L = (100 - w)/2$, and either rejects a demand $w \geq 100$ or accepts such a demand and chooses $L = 0$. The outcome of any equilibrium is that the union demands $w = 50$ and the firm chooses $L = 25$.

c. Yes. In any subgame perfect equilibrium the union's payoff is $(50)(25) = 1250$ and the firm's payoff is $(25)(75) - (50)(25) = 625$. Thus both parties are better off at the outcome (w, L) than they are in the unique subgame perfect equilibrium if and only if $L \leq 50$ and

$$\begin{aligned} wL &> 1250 \\ L(100 - L) - wL &> 625 \end{aligned}$$

or $L \geq 50$ and

$$\begin{aligned} wL &> 1250 \\ 2500 - wL &> 625. \end{aligned}$$

These conditions are satisfied for a nonempty set of pairs (w, L). For example, if $L = 50$ the conditions are satisfied by $25 < w < 37.5$; if $L = 100$ they are satisfied by $12.5 < w < 18.75$.

d. There are many Nash equilibria in which the firm "threatens" to reject high wage demands. In one such Nash equilibrium the firm threatens to reject any positive wage demand. In this equilibrium the union's strategy is $w = 0$, and the firm's strategy rejects any demand $w > 0$, and accepts the demand $w = 0$ and chooses $L = 50$. (The union's payoff is 0 no matter what demand it makes; given $w = 0$, the firm's optimal action is $L = 50$.)

177.2 The "rotten kid theorem"

The situation is modeled by the following extensive game.

Players The parents and the child.

Terminal histories The set of sequences (a, t), where a (an action of the child) and t (a transfer from the parents to the child) are numbers.

Player function $P(\varnothing)$ is the child, $P(a)$ is the parents for every value of a.

Preferences The child's preferences are represented by the payoff function $c(a) + t$ and the parents' preferences are represented by the payoff function $\min\{p(a) - t, c(a) + t\}$.

To find the subgame perfect equilibria of this game, first consider the parents' optimal actions in the subgames of length 1. Consider the subgame following the choice of a by the child. We have $p(a) > c(a)$ (by assumption), so if the parents makes no transfer her payoff is $c(a)$. If she transfers \$1 to the child then her payoff increases to $c(a) + 1$. As she increases the transfer her payoff increases until $p(a) - t = c(a) + t$; that is, until $t = \frac{1}{2}(p(a) - c(a))$. (If she increases the transfer any more, she has less money than her child.) Thus the parents' optimal action in the subgame following the choice of a by the child is $t = \frac{1}{2}(p(a) - c(a))$.

Now consider the whole game. Given the parents' optimal action in each subgame, a child who chooses a receives the payoff $c(a) + \frac{1}{2}(p(a) - c(a)) = \frac{1}{2}(p(a) + c(a))$. Thus in a subgame perfect equilibrium the child chooses the action that maximizes $p(a) + c(a)$, the sum of her own private income and her parents' income.

177.3 Comparing simultaneous and sequential games

a. Denote by (a_1^*, a_2^*) a Nash equilibrium of the strategic game in which player 1's payoff is maximal in the set of Nash equilibria. Because (a_1^*, a_2^*) is a Nash equilibrium, a_2^* is a best response to a_1^*. By assumption, it is the only best response to a_1^*. Thus if player 1 chooses a_1^* in the extensive game, player 2 must choose a_2^* in any subgame perfect equilibrium of the extensive game. That is, by choosing a_1^*, player 1 is assured of a payoff of at least $u_1(a_1^*, a_2^*)$. Thus in any subgame perfect equilibrium player 1's payoff must be at least $u_1(a_1^*, a_2^*)$.

b. Suppose that $A_1 = \{T, B\}$, $A_2 = \{L, R\}$, and the payoffs are those given in Figure 86.1. The strategic game has a unique Nash equilibrium, (T, L), in which player 2's payoff is 1. The extensive game has a unique subgame perfect equilibrium, (B, LR) (where the first component of player 2's strategy is her action after the history T and the second component is her action after the history B). In this subgame perfect equilibrium player 2's payoff is 2.

	L	R
T	$1, 1$	$3, 0$
B	$0, 0$	$2, 2$

Figure 86.1 The payoffs for the example in Exercise 177.3*b*.

c. Suppose that $A_1 = \{T, B\}$, $A_2 = \{L, R\}$, and the payoffs are those given in Figure 87.1. The strategic game has a unique Nash equilibrium, (T, L), in which player 2's payoff is 2. A subgame perfect equilibrium of the extensive game is (B, RL) (where the first component of player 2's strategy is her action after the history T and the second component is her action after the history B). In this subgame perfect equilibrium player 1's payoff is 1. (If you read Chapter 4, you can find the mixed strategy Nash equilibria of the strategic game; in all these equilibria, as in the pure strategy Nash equilibrium, player 1's expected payoff exceeds 1.)

	L	R
T	$2,2$	$0,2$
B	$1,1$	$3,0$

Figure 87.1 The payoffs for the example in Exercise 177.3*c*.

179.1 Subgame perfect equilibria of ticktacktoe

Player 2 puts her O in the center. If she does so, each player has a strategy that guarantees at least a draw in the subgame. Player 1 guarantees at least a draw by next marking one of the two squares adjacent to her first X and then subsequently completing a line of X's, if possible, or, if not possible, blocking a line of O's, if necessary, or, if not necessary, moving arbitrarily. Player 2 guarantees at least a draw as follows.

- If player 1's second X is adjacent to her first X or is in a corner not diagonally opposite player 1's first X, player 2 should, on each move, either complete a line of O's, if possible, or, if not possible, block a line of X's, if necessary, or, if not necessary, move arbitrarily.
- If player 1's second X is in some other square then player 2 should, on her second move, mark one of the corners not diagonally opposite player 1's first X, and then, on each move, either complete a line of O's, if possible, or, if not possible, block a line of X's, if necessary, or, if not necessary, move arbitrarily.

For each of player 2's other opening moves, player 1 has a strategy in the subgame that wins, as follows.

- Suppose player 2 marks the corner diagonally opposite player 1's first X. If player 1 next marks another corner, player 2 must next mark the square between player 1's two X's; by marking the remaining corner, player 1 wins on her next move.
- Suppose player 2 marks one of the other corners. If player 1 next marks the corner diagonally opposite her first X, player 2 must mark the center, then player 1 must mark the remaining corner, leading her to win on her next move.
- Suppose player 2 marks one of the two squares adjacent to player 1's X. If player 1 next marks the center, player 2 must mark the corner opposite player 1's first X, in which case player 1 can mark the other square adjacent to her first X, leading her to win on her next move.

- Suppose player 2 marks one of the other squares, other than the center. If player 1 next marks the center, player 2 must mark the corner opposite player 1's first X, in which case player 1 can mark the corner that blocks a row of O's, leading her to win on her next move.

179.2 Toetacktick

The following strategy leads to either a draw or a win for player 1: mark the central square initially, and on each subsequent move mark the square symmetrically opposite the one just marked by the second player.

179.3 Three Men's Morris, or Mill

Number the squares 1 through 9, starting at the top left, working across each row. The following strategy of player 1 guarantees she wins, so that the subgame perfect equilibrium outcome is that she wins. First player 1 chooses the central square (5).

- Suppose player 2 then chooses a corner; take it to be square 1. Then player 1 chooses square 6. Now player 2 must choose square 4 to avoid defeat; player 1 must choose square 7 to avoid defeat; and then player 2 must choose square 3 to avoid defeat (otherwise player 1 can move from square 6 to square 3 on her next turn). If player 1 now moves from square 6 to square 9, then whatever player 2 does she can subsequently move her counter from square 5 to square 8 and win.

- Suppose player 2 then chooses a noncorner; take it to be square 2. Then player 1 chooses square 7. Now player 2 must choose square 3 to avoid defeat; player 1 must choose square 1 to avoid defeat; and then player 2 must choose square 4 to avoid defeat (otherwise player 1 can move from square 5 to square 4 on her next turn). If player 1 now moves from square 7 to square 8, then whatever player 2 does she can subsequently move from square 8 to square 9 and win.

6 Extensive Games with Perfect Information: Illustrations

183.1 Nash equilibria of the ultimatum game

For *every* amount x there are Nash equilibria in which person 1 offers x. For example, for any value of x there is a Nash equilibrium in which person 1's strategy is to offer x and person 2's strategy is to accept x and any offer more favorable, and reject every other offer. (Given person 2's strategy, person 1 can do no better than offer x. Given person 1's strategy, person 2 should accept x; whether person 2 accepts or rejects any other offer makes no difference to her payoff, so that rejecting all less favorable offers is, in particular, optimal.)

183.2 Subgame perfect equilibria of the ultimatum game with indivisible units

In this case each player has finitely many actions, and for both possible subgame perfect equilibrium strategies of player 2 there is an optimal strategy for player 1.

If player 2 accepts all offers then player 1's best strategy is to offer 0, as before.

If player 2 accepts all offers except 0 then player 1's best strategy is to offer one cent (which player 2 accepts).

Thus the game has two subgame perfect equilibria: one in which player 1 offers 0 and player 2 accepts all offers, and one in which player 1 offers one cent and player 2 accepts all offers except 0.

183.3 Dictator game and impunity game

Dictator game Person 2 has no choice; person 1 optimally chooses the offer 0.

Impunity game The analysis of the subgames of length one is the same as it is in the ultimatum game. That is, in any subgame perfect equilibrium person 2 either accepts all offers, or accepts all positive offers and rejects 0. Now consider the whole game. Regardless of person 2's behavior in the subgames, person 1's best action is to offer 0.

Thus the game has two subgame perfect equilibria. In both equilibria person 1 offers 0. In one equilibrium person 2 accepts all offers, and in the other equilibrium she accepts all positive offers and rejects 0. The outcome of the first equilibrium is that person 1 offers 0, which person 2 accepts; the outcome of the second equilibrium is that person 1 offers 0, which person 2 rejects. In both equilibria person 1's payoff is c and person 2's payoff is 0.

183.4 Variants of ultimatum game and impunity game with equity-conscious players

Ultimatum game First consider the optimal response of person 2 to each possible offer. If person 2 accepts an offer x her payoff is $x - \beta_2|(1-x) - x|$, while if she rejects an offer her payoff is 0. Thus she accepts an offer x if $x - \beta_2|(1-x) - x| > 0$, or

$$x - \beta_2|1 - 2x| > 0, \tag{90.1}$$

rejects an offer x if $x - \beta_2|1-2x| < 0$, and is indifferent between accepting and rejecting if $x - \beta_2|1-2x| = 0$.

Which values of x satisfy (90.1)? Because of the absolute value in the expression, we can conveniently consider the cases $x \leq \frac{1}{2}$ and $x > \frac{1}{2}$ separately.

- For $x \leq \frac{1}{2}$ the condition is $x - \beta_2(1-2x) > 0$, or $x > \beta_2/(1+2\beta_2)$.
- For $x \geq \frac{1}{2}$ the condition is $x + \beta_2(1-2x) > 0$, or $x(1-2\beta_2) + \beta_2 > 0$. The values of x that satisfy this inequality depend on whether β_2 is greater than or less than $\frac{1}{2}$.

  $\beta_2 \leq \frac{1}{2}$: All values of x satisfy the inequality.

  $\beta_2 > \frac{1}{2}$: The inequality is $x < \beta_2/(2\beta_2 - 1)$ (the right-hand side of which is less than 1 only if $\beta_2 > 1$).

In summary, person 2 accepts any offer x with $\beta_2/(1+2\beta_2) < x < \beta_2/(2\beta_2 - 1)$, may accept or reject the offers $\beta_2/(1+2\beta_2)$ and $\beta_2/(2\beta_2-1)$, and rejects any offer x with $x < \beta_2/(1+2\beta_2)$ or $x > \beta_2/(2\beta_2-1)$. The shaded region of Figure 90.1 shows, for each value of β_2, the set of offers that person 2 accepts. Note, in particular, that, for every value of β_2, person 2 accepts the offer $\frac{1}{2}$.

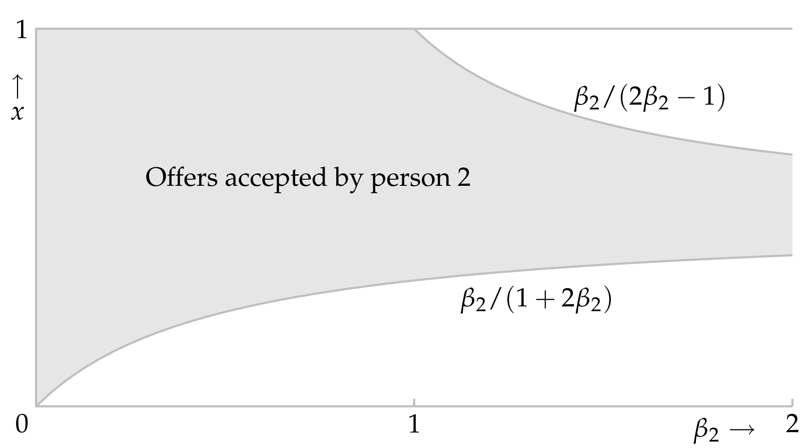


Figure 90.1 The set of offers x that person 2 accepts for each value of $\beta_2 \leq 2$ in the variant of the ultimatum game with equity-conscious players studied in Exercise 183.4.

Now consider person 1's decision. Her payoff is 0 if her offer is rejected and $1 - x - \beta_1|(1-x) - x| = 1 - x - \beta_1|1-2x|$ if it is accepted. We can conveniently separate the analysis into three cases.

$\beta_1 < \frac{1}{2}$: Person 1's payoff when her offer x is accepted is positive for $0 \leq x < 1$ and is decreasing in x. Thus person 1's optimal offer is the smallest one that person 2 accepts. If person 2's strategy rejects the offer $\beta_2/(1+2\beta_2)$, then as in the analysis of the original game when person 2's strategy rejects 0, person 1 has no optimal response. Thus in any subgame perfect equilibrium person 2 accepts $\beta_2/(1+2\beta_2)$, and person 1 offers this amount.

$\beta_1 = \frac{1}{2}$: Person 1's payoff to an offer that is accepted is positive and constant from $x = 0$ to $x = \frac{1}{2}$, then decreasing. Thus if person 2 accepts the offer $\beta_2/(1+2\beta_2)$ then every offer x with $\beta_2/(1+2\beta_2) \leq x \leq \frac{1}{2}$ is optimal, while if person 2 rejects the offer $\beta_2/(1+2\beta_2)$ then every offer x with $\beta_2/(1+2\beta_2) < x \leq \frac{1}{2}$ is optimal.

$\beta_1 > \frac{1}{2}$: Person 1's payoff to an offer that is accepted is increasing up to $x = \frac{1}{2}$ and then decreasing, and is positive at $x = \frac{1}{2}$, so that her optimal offer is $\frac{1}{2}$ (which person 2 accepts).

We conclude that the set of subgame perfect equilibria depends on the values of β_1 and β_2, as follows.

$\beta_1 < \frac{1}{2}$: the set of subgame perfect equilibria is the set of all strategy pairs for which

- person 1 offers $\beta_2/(1+2\beta_2)$
- person 2 accepts all offers x with $\beta_2/(1+2\beta_2) \leq x < \beta_2/(2\beta_2 - 1)$, rejects all offers x with $x < \beta_2/(1+2\beta_2)$ or $x > \beta_2/(2\beta_2 - 1)$, and either accepts or rejects the offer $\beta_2/(2\beta_2 - 1)$.

$\beta_1 = \frac{1}{2}$: the set of subgame perfect equilibria is the set of all strategy pairs for which

- person 1's offer x satisfies $\beta_2/(1+2\beta_2) \leq x \leq \frac{1}{2}$
- person 2 accepts all offers x with $\beta_2/(1+2\beta_2) < x < \beta_2/(2\beta_2 - 1)$, rejects all offers x with $x < \beta_2/(1+2\beta_2)$ or $x > \beta_2/(2\beta_2 - 1)$, either accepts or rejects the offer $\beta_2/(2\beta_2 - 1)$, and either accepts or rejects the offer $\beta_2/(1+2\beta_2)$ unless person 1 makes this offer, in which case person 2 definitely accepts it.

$\beta_1 > \frac{1}{2}$: the set of subgame perfect equilibria is the set of all strategy pairs for which

- person 1 offers $\frac{1}{2}$

- person 2 accepts all offers x with $\beta_2/(1+2\beta_2) < x < \beta_2/(2\beta_2-1)$, rejects all offers x with $x < \beta_2/(1+2\beta_2)$ or $x > \beta_2/(2\beta_2-1)$, and either accepts or rejects the offer $\beta_2/(2\beta_2-1)$ and the offer $\beta_2/(1+2\beta_2)$.

The subgame perfect equilibrium outcomes are:

$\beta_1 < \frac{1}{2}$: person 1 offers $\beta_2/(1+2\beta_2)$, which person 2 accepts

$\beta_1 = \frac{1}{2}$: person 1 makes an offer x that satisfies $\beta_2/(1+2\beta_2) \leq x \leq \frac{1}{2}$, and person 2 accepts this offer

$\beta_1 > \frac{1}{2}$: person 1 offers $\frac{1}{2}$, which person 2 accepts.

In particular, in all cases the offer made by person 1 in equilibrium is accepted by person 2.

Impunity game First consider the optimal response of person 2 to each possible offer. If person 2 accepts an offer x her payoff is $x - \beta_2|(1-x) - x|$, while if she rejects an offer her payoff is $-\beta_2(1-x)$. Thus she accepts an offer x if $x - \beta_2|(1-x) - x| > -\beta_2(1-x)$, or

$$x(1-\beta_2) + \beta_2(1 - |1-2x|) > 0, \qquad (92.1)$$

rejects an offer x if $x(1-\beta_2) + \beta_2(1-|1-2x|) < 0$, and is indifferent between accepting and rejecting if $x(1-\beta_2) + \beta_2(1-|1-2x|) = 0$.

As before, we can conveniently consider the cases $x \leq \frac{1}{2}$ and $x > \frac{1}{2}$ separately.

- For $x \leq \frac{1}{2}$ the condition is $x(1+\beta_2) > 0$, or $x > 0$.
- For $x \geq \frac{1}{2}$ the condition is $x(1-3\beta_2) + 2\beta_2 > 0$, which is satisfied by all values of x if $\beta_2 \leq \frac{1}{3}$, and for all x with $x < 2\beta_2/(3\beta_2-1)$ if $\beta_2 > \frac{1}{3}$.

In summary, person 2 accepts any offer x with $0 < x < 2\beta_2/(3\beta_2-1)$, may accept or reject the offers 0 and $2\beta_2/(3\beta_2-1)$, and rejects any offer x with $x > 2\beta_2/(3\beta_2-1)$.

Now consider person 1. If she offers x, her payoff is

$$\begin{cases} 1 - x - \beta_1|1-2x| & \text{if person 1 accepts } x \\ 1 - x - \beta_1(1-x) & \text{if person 1 rejects } x. \end{cases}$$

If $\beta_1 < \frac{1}{2}$ then in both cases person 1's payoff is decreasing in x; for $x = 0$ the payoffs are equal. Thus, given person 2's optimal strategy, in any subgame perfect equilibrium person 1's optimal offer is 0, which person 2 may accept or reject.

If $\beta_1 = \frac{1}{2}$ then person 1's payoff when person 2 accepts x is constant from 0 to $\frac{1}{2}$, then decreases. Her payoff when person 2 rejects x is decreasing in x, and the two payoffs are equal when $x = 0$. Thus the optimal offers of person 1 are 0, which person 2 may accept or reject, and any x with $0 < x \leq \frac{1}{2}$, which person 2 accepts.

If $\beta_1 > \frac{1}{2}$ then person 1's highest payoff is obtained when $x = \frac{1}{2}$, which person 2 accepts. Thus $x = \frac{1}{2}$ is her optimal offer.

In summary, in all subgame perfect equilibria the strategy of person 2 accepts all offers x with $0 < x < 2\beta_2/(3\beta_2 - 1)$, rejects all offers x with $x > 2\beta_2/(3\beta_2 - 1)$, and either accepts or rejects the offer 0 and the offer $2\beta_2/(3\beta_2 - 1)$. Person 1's offer depends on the value of β_1 and β_2, as follows.

$\beta_1 < \frac{1}{2}$: person 1 offers 0

$\beta_1 = \frac{1}{2}$: person 1's offer x satisfies $0 \leq x \leq \frac{1}{2}$

$\beta_1 > \frac{1}{2}$: person 1 offers $x = \frac{1}{2}$.

The subgame perfect equilibrium outcomes are:

$\beta_1 < \frac{1}{2}$: person 1 offers 0, which person 2 may accept or reject

$\beta_1 = \frac{1}{2}$: person 1 either offers 0, which person 2 either accepts or rejects, or makes an offer x that satisfies $0 < x \leq \frac{1}{2}$, which person 2 accepts

$\beta_1 > \frac{1}{2}$: person 1 offers $\frac{1}{2}$, which person 2 accepts.

In particular, if $\beta_1 \leq \frac{1}{2}$ there are equilibria in which person 1 offers 0, and person 2 rejects this offer.

Comparison of subgame perfect equilibria of ultimatum and impunity games
The equilibrium outcomes of the two games are the same unless $0 < \beta_1 \leq \frac{1}{2}$, or $\beta_1 = 0$ and $\beta_2 > 0$, in which case person 1's offer in the ultimatum game is higher than her offer in the impunity game.

185.1 Bargaining over two indivisible objects

An extensive game that models the situation is shown in Figure 93.1, where the action $(x, 2 - x)$ of player 1 means that she keeps x objects and offers $2 - x$ objects to player 2.

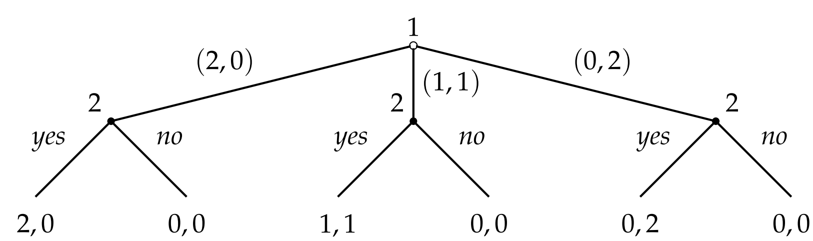


Figure 93.1 An extensive game that models the procedure described in Exercise 185.1 for allocating two identical indivisible objects between two people.

Denote a strategy of player 2 by a triple abc, where a is the action (y or n, for *yes* or *no*) taken after the offer $(2, 0)$, b is the action taken after the offer $(1, 1)$, and c is the action taken after the offer $(0, 2)$.

The subgame perfect equilibria of the game are $((2, 0), yyy)$ (resulting in the division $(2, 0)$), and $((1, 1), nyy)$ (resulting in the division $(1, 1)$).

The strategic form of the game is given in Figure 94.1. Its Nash equilibria are $((2,0), yyy)$, $((2,0), yyn)$, $((2,0), yny)$, $((2,0), ynn)$, $((2,0), nny)$, $((1,1), nyy)$, $((1,1), nyn)$, $((0,2), nny)$, and $((2,0), nnn)$. The first four equilibria result in the division $(2,0)$, the next two result in the division $(1,1)$, and the last two result in the divisions $(0,2)$ and $(0,0)$ respectively.

	yyy	yyn	yny	ynn	nyy	nyn	nny	nnn
$(2,0)$	$2,0$	$2,0$	$2,0$	$2,0$	$0,0$	$0,0$	$0,0$	$0,0$
$(1,1)$	$1,1$	$1,1$	$0,0$	$0,0$	$1,1$	$1,1$	$0,0$	$0,0$
$(0,2)$	$0,2$	$0,0$	$0,2$	$0,0$	$0,2$	$0,0$	$0,2$	$0,0$

Figure 94.1 The strategic form of the game in Figure 93.1

The outcomes $(0,2)$ and $(0,0)$ are generated by Nash equilibria but not by any subgame perfect equilibria.

185.2 Dividing a cake fairly

a. If player 1 divides the cake unequally then player 2 chooses the larger piece. Thus in any subgame perfect equilibrium player 1 divides the cake into two pieces of equal size.

b. In a subgame perfect equilibrium player 2 chooses P_2 over P_1, so she likes P_2 at least as much as P_1.

To show that in fact player 2 is indifferent between P_1 and P_2, suppose to the contrary that she prefers P_2 to P_1. I argue that in this case player 1 can slightly increase the size of P_1 in such a way that player 2 still prefers the now-slightly-smaller P_2. Precisely, by the continuity of player 2's preferences, there is a subset P of P_2, not equal to P_2, that player 2 prefers to its complement $C \setminus P$ (the remainder of the cake). Thus if player 1 makes the division $(C \setminus P, P)$, player 2 chooses P. The piece P_1 is a subset of $C \setminus P$ not equal to $C \setminus P$, so player 1 prefers $C \setminus P$ to P_1. Thus player 1 is better off making the division $(C \setminus P, P)$ than she is making the division (P_1, P_2), contradicting the fact that (P_1, P_2) is a subgame perfect equilibrium division. We conclude that in any subgame perfect equilibrium player 2 is indifferent between the two pieces into which player 1 divides the cake.

I now argue that player 1 likes P_1 as least as much as P_2. Suppose, to the contrary, that she prefers P_2 to P_1. Then by the continuity assumption there is a subset P of P_2 that she prefers to $C \setminus P$. Player 2 is indifferent between P_1 and P_2, so player 2 prefers $C \setminus P$ (which is larger than P_1) to P (which is smaller than P_2). Thus if player 1 makes the division $(P, C \setminus P)$ then player 2 chooses $C \setminus P$, leaving P for player 1, which player 1 prefers to $C \setminus P$, and hence to P_1 (which is smaller than $C \setminus P$). That is, player 1 has a deviation that leads to an outcome she prefers, contradicting the assumption that (P_1, P_2) is a subgame perfect equilibrium division. Hence player 1 likes P_1 at least as much as P_2 in any subgame perfect equilibrium.

To show that player 1 may strictly prefer P_1 to P_2, consider a cake that is perfectly homogeneous except for the presence of a single cherry. Assume that player 2 values a piece of the cherry in exactly the same way that she values a piece of the cake of the same size, while player 1 prefers a piece of the cherry to a piece of the cake of the same size. Then there is a subgame perfect equilibrium in which player 1 divides the cake equally, with one piece containing all of the cherry, and player 2 chooses the piece without the cherry. (In this equilibrium, as in all equilibria, player 2 is indifferent between the two pieces—but note that there is no subgame perfect equilibrium in which she chooses the piece with the cherry in it. A strategy pair in which she acts in this way is not an equilibrium, because player 1 can deviate and increase slightly the size of the cherryless piece of cake, inducing player 2 to choose that piece.)

186.1 Holdup game

The game is defined as follows.

Players Two people, person 1 and person 2.

Terminal histories The set of all sequences (low, x, Z), where x is a number with $0 \leq x \leq c_L$ (the amount of money that person 1 offers to person 2 when the pie is small), and $(high, x, Z)$, where x is a number with $0 \leq x \leq c_H$ (the amount of money that person 1 offers to person 2 when the pie is large) and Z is either Y ("yes, I accept") or N ("no, I reject").

Player function $P(\varnothing) = 2$, $P(low) = P(high) = 1$, and $P(low, x) = P(high, x) = 2$ for all x.

Preferences Person 1's preferences are represented by payoffs equal to the amounts of money she receives, equal to $c_L - x$ for any terminal history (low, x, Y) with $0 \leq x \leq c_L$, equal to $c_H - x$ for any terminal history $(high, x, Y)$ with $0 \leq x \leq c_H$, and equal to 0 for any terminal history (low, x, N) with $0 \leq x \leq c_L$ and for any terminal history $(high, x, N)$ with $0 \leq x \leq c_H$. Person 2's preferences are represented by payoffs equal to $x - L$ for the terminal history (low, x, Y), $x - H$ for the terminal history $(high, x, Y)$, $-L$ for the terminal history (low, x, N), and $-H$ for the terminal history $(high, x, N)$.

187.1 Agenda control

First consider the optimal strategies of the legislature. Denote the committee's proposal by y. If $y > y_0$ or $y < -y_0$ then the legislature optimally rejects y, while if $-y_0 < y < y_0$ it optimally accepts y; if $y = y_0$ or $y = -y_0$ then both acceptance and rejection are optimal. Thus the legislature has four optimal strategies, which differ in their response to the proposals y_0 and $-y_0$.

The optimal action of the committee, given the legislature's optimal strategies, depends on the relation between y_0 and y_c.

- If $y_0 > y_c$ or $y_0 < -y_c$ then for any optimal strategy of the legislature, the committee optimally proposes y_c, which the legislature accepts. Thus in a subgame perfect equilibrium the committee proposes y_c and the legislature uses one of its optimal strategies.
- If $0 \leq y_0 \leq y_c$ then for any optimal strategy of the legislature, the committee optimally proposes y_0, which the legislature may accept or reject. Thus in a subgame perfect equilibrium the committee proposes y_0 and the legislature uses one of its optimal strategies. (Note that the outcome is the same whether the legislature accepts the proposal or rejects it.)
- If $-y_c \leq y_0 < 0$ then if the legislature uses a strategy that accepts $-y_0$, the committee optimally proposes $-y_0$, while if the legislature uses a strategy that rejects $-y_0$, the committee has no optimal strategy. Thus in a subgame perfect equilibrium the committee proposes $-y_0$ and the legislature uses an optimal strategy that accepts $-y_0$.

The outcome y^* proposed by the committee in a subgame perfect equilibrium is shown as a function of the status quo y_0 in Figure 96.1. For $-y_c < y_0 < 0$, an increase in the value of y_0 leads to a decrease in the value of the equilibrium outcome.

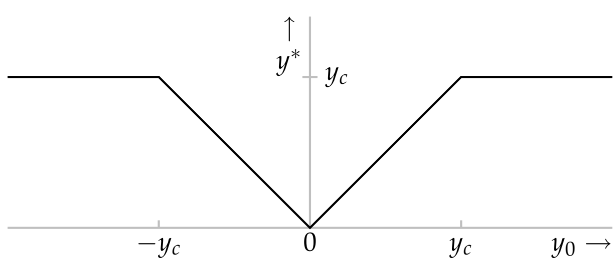


Figure 96.1 The outcome y^* proposed by the committee in a subgame perfect equilibrium of the game in Exercise 187.1, as a function of the status quo y_0.

189.1 Stackelberg's duopoly game with quadratic costs

From Exercise 59.1, the best response function of firm 2 is the function b_2 defined by

$$b_2(q_1) = \begin{cases} \frac{1}{4}(\alpha - q_1) & \text{if } q_1 \leq \alpha \\ 0 & \text{if } q_1 > \alpha. \end{cases}$$

Firm 1's subgame perfect equilibrium strategy is the value of q_1 that maximizes $q_1(\alpha - q_1 - b_2(q_1)) - q_1^2$, or $q_1(\alpha - q_1 - \frac{1}{4}(\alpha - q_1)) - q_1^2$, or $\frac{1}{4}q_1(3\alpha - 7q_1)$. The maximizer is $q_1 = \frac{3}{14}\alpha$.

We conclude that the game has a unique subgame perfect equilibrium, in which firm 1's strategy is the output $\frac{3}{14}\alpha$ and firm 2's strategy is its best response function b_2.

The outcome of the subgame perfect equilibrium is that firm 1 produces $q_1^* = \frac{3}{14}\alpha$ units of output and firm 2 produces $q_2^* = b_2(\frac{3}{14}\alpha) = \frac{11}{56}\alpha$ units. In a Nash equilibrium of Cournot's (simultaneous-move) game each firm produces $\frac{1}{5}\alpha$ (see Exercise 59.1). Thus firm 1 produces more in the subgame perfect equilibrium of the sequential game than it does in the Nash equilibrium of Cournot's game, and firm 2 produces less.

191.1 Stackelberg's duopoly game with fixed costs

We have $f < (\alpha - c)^2/16$ ($f = 4$; $(\alpha - c)^2/16 = 9$), so the best response function of firm 2 takes the form shown in Figure 24.1 (in the solution to Exercise 59.2). To determine the subgame perfect equilibrium we need to compare firm 1's profit when it produces $\overline{q} = 8$ units of output, so that firm 2 produces 0, with its profit when it produces the output that maximizes its profit on the positive part of firm 2's best response function.

If firm 1 produces 8 units of output and firm 2 produces 0, firm 1's profit is $8(12 - 8) = 32$. Firm 1's best output on the positive part of firm 2's best response function is $\frac{1}{2}(\alpha - c) = 6$. If it produces this output then firm 2 produces $\frac{1}{2}(\alpha - c - q_1) = \frac{1}{2}(12 - 6) = 3$, and firm 1's profit is $6(12 - 9) = 18$. Thus firm 1's profit is higher when it produces enough to induce firm 2 to produce zero. We conclude that the game has a unique subgame perfect equilibrium, in which firm 1's strategy is to produce 8 units, and firm 2's strategy is to produce $\frac{1}{2}(\alpha - c - q_1) = \frac{1}{2}(12 - q_1)$ units if firm 1 produces $q_1 < 8$ and 0 if firm 1 produces $q_1 \geq 8$ units.

192.1 Sequential variant of Bertrand's duopoly game

a. *Players* The two firms.

Terminal histories The set of all sequences (p_1, p_2) of prices (where each p_i is a nonnegative number).

Player function $P(\varnothing) = 1$ and $P(p_1) = 2$ for all p_1.

Preferences The payoff of each firm i to the terminal history (p_1, p_2) is its profit

$$\begin{cases} (p_i - c)D(p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c)D(p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j, \end{cases}$$

where j is the other firm.

b. A strategy of firm 1 is a price (e.g. the price c). A strategy of firm 2 is a function that associates a price with every price chosen by firm 1 (e.g. $s_2(p_1) = p_1 - 1$, the strategy in which firm 2 always charges 1 cent less than firm 1).

c. First consider firm 2's best responses to each price p_1 chosen by firm 1.

- If $p_1 < c$, any price greater than p_1 is a best response for firm 2.
- If $p_1 = c$, any price at least equal to c is a best response for firm 2.
- If $p_1 = c + 1$, firm 2's unique best response is to set the same price.
- If $p_1 > c + 1$, firm 2's unique best response is to set the price $\min\{p^m, p_1 - 1\}$ (where p^m is the monopoly price).

Now consider the optimal action of firm 1. Given firm 2's best responses,

- if $p_1 < c$, firm 1's profit is positive
- if $p_1 = c$, firm 1's profit is zero
- if $p_1 = c + 1$, firm 1's profit is positive
- if $p_1 > c + 1$, firm 1's profit is zero.

Thus the only price p_1 for which there is a best response of firm 2 that leads to a positive profit for firm 1 is $c + 1$.

We conclude that in every subgame perfect equilibrium firm 1's strategy is $p_1 = c + 1$, and firm 2's strategy assigns to each price chosen by firm 1 one of its best responses, so that firm 2's strategy takes the form

$$s_2(p_1) = \begin{cases} k(p_1) & \text{if } p_1 < c \\ k' & \text{if } p_1 = c \\ c + 1 & \text{if } p_1 = c + 1 \\ \min\{p^m, p_1 - 1\} & \text{if } p_1 > c + 1 \end{cases}$$

where $k(p_1) > p_1$ for all p_1 and $k' \geq c$.

The outcome of every subgame perfect equilibrium is that both firms choose the price $c + 1$.

196.1 Three interest groups buying votes

a. Consider the possibility of a subgame perfect equilibrium in which bill X passes. In any such equilibrium, groups Y and Z make no payments. But now given that Y makes no payments and that $V_X = V_Z$, group Z can match X's payments to the two legislators to whom X's payments are smallest, spend at most 200, and gain the passage of bill Z. Thus there is no subgame perfect equilibrium in which bill X passes. Similarly there is no subgame perfect equilibrium in which bill Y passes. Thus in every subgame perfect equilibrium bill Z passes.

b. By making payments of more than 50 to each legislator, group X ensures that neither group Y nor group Z can profitably buy the passage of its favorite bill. (In any subgame perfect equilibrium, group X's payments to each legislator are exactly 50.) Thus in every subgame perfect equilibrium the outcome is that bill X is passed.

c. For any payments of group X that sum to at most 300, group Y can make payments that are (*i*) at least as high to at least two legislators and (*ii*) high enough that group Z cannot profitably buy off more than one legislator. Specifically, consider the subgame following group X's action. Denote group X's payment to legislator i by x_i, and assume that $x_1 \leq x_2 \leq x_3$. If group Y pays $\max\{51, x_1 + 1\}$ to legislator 1, $\max\{51, x_2 + 1\}$ to legislator 2, and 51 to legislator 3, then group Z cannot profitably buy off more than one legislator. Hence in a subgame perfect equilibrium of the subgame group Z makes no payments; group Y buys off legislators 1 and 2, and bill Y is passed.

196.2 Interest groups buying votes under supermajority rule

a. However group X allocates payments summing to 700, group Y can buy off five legislators for at most 500. Thus in any subgame perfect equilibrium neither group makes any payment, and bill Y is passed.

b. If group X pays each legislator 80 then group Y is indifferent between buying off five legislators, in which case bill Y is passed, and in making no payments, in which case bill X is passed. If group Y makes no payments then X is selected, and group X is better off than it is if it makes no payments. There is no subgame perfect equilibrium in which group Y buys off five legislators, because if it were to do so group X could pay each legislator slightly more than 80 to ensure the passage of bill X. Thus in every subgame perfect equilibrium group X pays each legislator 80, group Y makes no payments, and bill X is passed.

c. If only a simple majority is required to pass a bill, in case *a* the outcome under majority rule is the same as it is when five votes are required.

In case *b*, group X needs to pay each legislator 100 in order to prevent group Y from winning. If it does so, its total payments are less than V_X, so doing so is optimal. Thus in this case the payment to each legislator is *higher* under majority rule.

196.3 Sequential positioning by two political candidates

The following extensive game models the situation.

Players The candidates.

Terminal histories The set of all sequences $(x_1, \ldots, x_n)$, where x_i is a position of candidate i (a number) for $i = 1, \ldots, n$.

Player function $P(\varnothing) = 1$, $P(x_1) = 2$ for all x_1, $P(x_1, x_2) = 3$ for all $(x_1, x_2), \ldots,$ $P(x_1, \ldots, x_{n-1}) = n$ for all $(x_1, \ldots, x_{n-1})$.

Preferences Each candidate's preferences are represented by a payoff function that assigns n to every terminal history in which she wins outright, k to every terminal history in which she ties for first place with $n-k$ other candidates, for $1 \leq k \leq n-1$, and 0 to every terminal history in which she loses, where positions attract votes as in Hotelling's model of electoral competition (Section 3.3).

This game has a finite horizon, so we may use backward induction to find its subgame perfect equilibria. Suppose there are two candidates. First consider candidate 2's best response to each strategy of candidate 1. Suppose candidate 1's strategy is m. Then candidate 2 loses if she chooses any position different from m and ties with candidate 1 if she chooses m. Thus candidate 2's best response to m is m. Now suppose candidate 1's strategy is $x_1 \neq m$. Then candidate 2 wins if she chooses any position between x_1 and $2m - x_1$; thus every such position is a best response.

Given candidate 2's best responses, the best strategy for candidate 1 is m, leading to a tie. (Every other strategy of candidate 1 leads her to lose.)

We conclude that in every subgame perfect equilibrium candidate 1's strategy is m; candidate 2's strategy chooses m after the history m and some position between x_1 and $2m - x_1$ after any other history x_1.

196.4 Sequential positioning by three political candidates

The following extensive game models the situation.

Players The candidates.

Terminal histories The set of all sequences $(x_1, \ldots, x_n)$, where x_i is either *Out* or a position of candidate i (a number) for $i = 1, \ldots, n$.

Player function $P(\varnothing) = 1$, $P(x_1) = 2$ for all x_1, $P(x_1, x_2) = 3$ for all $(x_1, x_2), \ldots,$ $P(x_1, \ldots, x_{n-1}) = n$ for all $(x_1, \ldots, x_{n-1})$.

Preferences Each candidate's preferences are represented by a payoff function that assigns n to every terminal history in which she wins, k to every terminal history in which she ties for first place with $n-k$ other candidates, for $1 \leq k \leq n-1$, 0 to every terminal history in which she stays out, and -1 to every terminal history in which she loses, where positions attract votes as in Hotelling's model of electoral competition (Section 3.3).

When there are two candidates the analysis of the subgame perfect equilibria is similar to that in the previous exercise. In every subgame perfect equilibrium candidate 1's strategy is m; candidate 2's strategy chooses m after the history m, some position between x_1 and $2m - x_1$ after the history x_1 for any position x_1, and any position after the history *Out*.

Now consider the case of three candidates when the voters' favorite positions are distributed uniformly from 0 to 1. I claim that every subgame perfect equilibrium results in the first candidate's entering at $\frac{1}{2}$, the second candidate's staying out, and the third candidate's entering at $\frac{1}{2}$.

To show this, first consider the best response of candidate 3 to each possible pair of actions of candidates 1 and 2. Figure 101.1 illustrates these optimal actions in every case that candidate 1 enters. (If candidate 1 does not enter then the subgame is exactly the two-candidate game.)

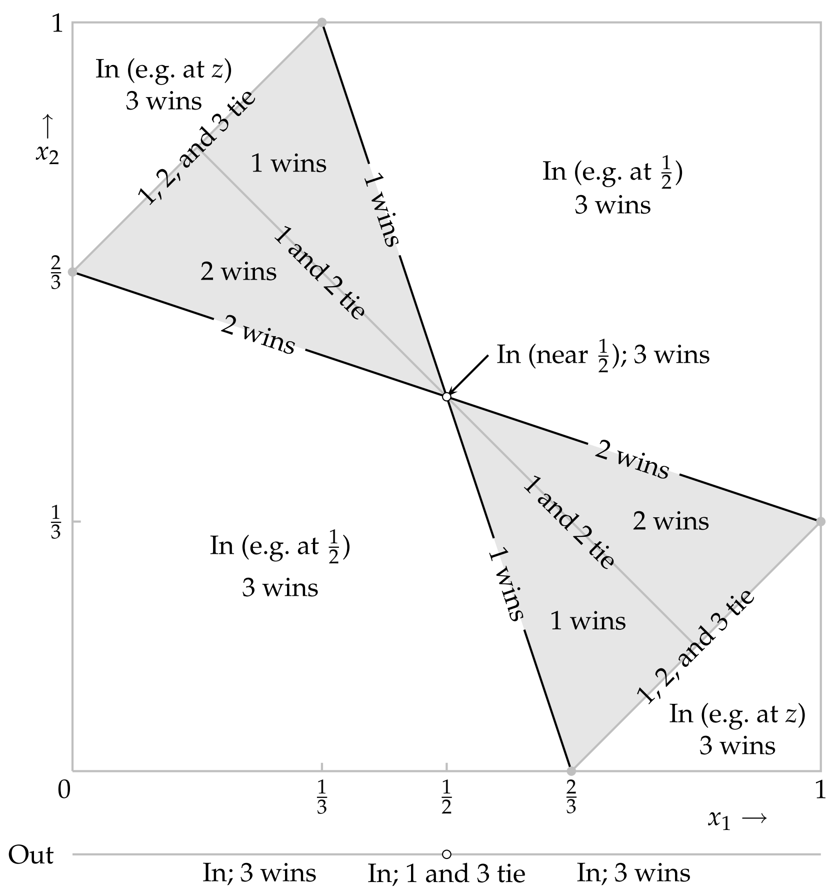


Figure 101.1 The outcome of a best response of candidate 3 to each pair of actions by candidates 1 and 2. The best response for any point in the gray shaded area (including the black boundaries of this area, but excluding the other boundaries) is *Out*. The outcome at each of the four small disks at the outer corners of the shaded area is that all three candidates tie. The value of z is $1 - \frac{1}{2}(x_1 + x_2)$.

Now consider the optimal action of candidate 2, given x_1 and the outcome of candidate 3's best response, as given in Figure 101.1. In the figure, take a value of x_1 and look at the outcomes as x_2 varies; find the value of x_2 that induces the best outcome for candidate 2. For example, for $x_1 = 0$ the only value of x_2 for which candidate 2 does not lose is $\frac{2}{3}$, at which point she ties with the other two candidates. Thus when candidate 1's strategy is $x_1 = 0$, candidate 2's best action, given candidate 3's best response, is $x_2 = \frac{2}{3}$, which leads to a three-way tie. We find that the outcome of the optimal value of x_2, for each value of x_1, is given as follows.

$$\begin{cases} \text{1, 2, and 3 tie } (x_2 = \frac{2}{3}) & \text{if } x_1 = 0 \\ \text{2 wins} & \text{if } 0 < x_1 < \frac{1}{2} \\ \text{1 and 3 tie (2 stays out)} & \text{if } x_1 = \frac{1}{2} \\ \text{2 wins} & \text{if } \frac{1}{2} < x_1 < 1 \\ \text{1, 2, and 3 tie } (x_2 = \frac{1}{3}) & \text{if } x_1 = 1. \end{cases}$$

Finally, consider candidate 1's best strategy, given the responses of candidates 2 and 3. If she stays out then candidates 2 and 3 enter at m and tie. If she enters then the best position at which to do so is $x_1 = \frac{1}{2}$, where she ties with candidate 3. (For every other position she either loses or ties with both of the other candidates.)

We conclude that in every subgame perfect equilibrium the outcome is that candidate 1 enters at $\frac{1}{2}$, candidate 2 stays out, and candidate 3 enters at $\frac{1}{2}$. (There are many subgame perfect equilibria, because after many histories candidate 3's optimal action is not unique.)

(The case in which there are many potential candidates, is discussed on the page `http://www.economics.utoronto.ca/osborne/research/CONJECT.HTM`.)

198.1 The race $G_1(2,2)$

The consequences of player 1's actions at the start of the game are as follows.

Take two steps: Player 1 wins.

Take one step: Go to the game $G_2(1,2)$, in which player 2 initially takes two steps and wins.

Do not move: If player 2 does not move, the game ends. If she takes one step we go to the game $G_1(2,1)$, in which player 1 takes two steps and wins. If she takes two steps, she wins. Thus in a subgame perfect equilibrium player 2 takes two steps, and wins.

We conclude that in a subgame perfect equilibrium of $G_1(2,2)$ player 1 initially takes two steps, and wins.

201.1 A race in which the players' valuations of the prize differ

By the arguments in the text for the case in which both players' valuations of the prize are between 6 and 7, the subgame perfect equilibrium outcomes of all games in which $k_1 \leq 2$ or $k_2 \leq 3$ are the same as they are when both players' valuations of the prize are between 6 and 7. If $k_2 \geq 5$ then player 1 is the winner in all subgame perfect equilibria, because even if player 2 reaches the finish line after taking one step at a time, her payoff is negative.

The games $G_i(3,4)$, $G_i(4,4)$, $G_i(5,4)$, and $G_i(6,4)$ remain. If, in the games $G_2(3,4)$ and $G_2(4,4)$, player 2 takes a single step then play moves to a game that player 1 wins. Thus player 2 is better off not moving; the subgame perfect equilibrium outcome is that player 1 takes one step at a time, and wins. In the game $G_i(5,4)$, the player who moves first can, by taking a single step, reach a game in which she wins regardless of the identity of the first-mover. Thus in this game the winner is the first-mover. Finally, in the game $G_1(6,4)$ it is not worth player 1's while taking two steps, to reach a game in which she wins, because her payoff would ultimately be negative. And if she takes one step, play moves to a game in which player 2 is the first-mover, and wins. Thus in this game player 2 wins. Figure 103.1 shows the subgame perfect equilibrium outcomes.

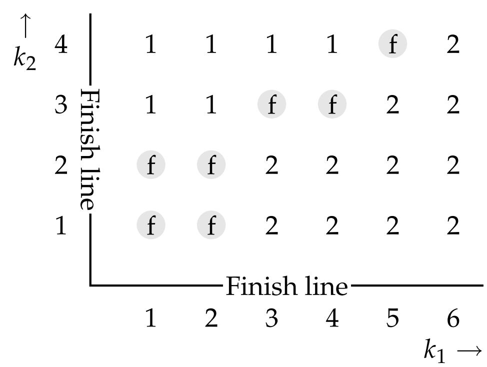


Figure 103.1 The subgame perfect equilibrium outcomes for the race in Exercise 201.1. Player 1 moves to the left, and player 2 moves down. The labels on the values of (k_1, k_2) indicate the subgame perfect equilibrium outcomes, as in the text.

201.2 Removing stones

For $n = 1$ the game has a unique subgame perfect equilibrium, in which player 1 takes one stone. The outcome is that player 1 wins.

For $n = 2$ the game has a unique subgame perfect equilibrium in which

- player 1 takes two stones
- after a history in which player 1 takes one stone, player 2 takes one stone.

The outcome is that player 1 wins.

For $n = 3$, the subgame following the history in which player 1 takes one stone is the game for $n = 2$ in which player 2 is the first mover, so player 2 wins. The subgame following the history in which player 1 takes two stones is the game for $n = 1$ in which player 2 is the first mover, so player 2 wins. Thus there is a subgame perfect equilibrium in which player 1 takes one stone initially, and one in which she takes two stones initially. In both subgame perfect equilibria player 2 wins.

For $n = 4$, the subgame following the history in which player 1 takes one stone is the game for $n = 3$ in which player 2 is the first-mover, so player 1 wins. The subgame following the history in which player 1 takes two stones is the game for $n = 2$ in which player 2 is the first-mover, so player 2 wins. Thus in every subgame perfect equilibrium player 1 takes one stone initially, and wins.

Continuing this argument for larger values of n, we see that if n is a multiple of 3 then in every subgame perfect equilibrium player 2 wins, while if n is not a multiple of 3 then in every subgame perfect equilibrium player 1 wins. We can prove this claim by induction on n. The claim is correct for $n = 1$, 2, and 3, by the arguments above. Now suppose it is correct for all integers through $n - 1$. I will argue that it is correct for n.

First suppose that n is divisible by 3. The subgames following player 1's removal of one or two stones are the games for $n - 1$ and $n - 2$ in which player 2 is the first-mover. Neither $n - 1$ nor $n - 2$ is divisible by 3, so by hypothesis player 2 is the winner in every subgame perfect equilibrium of both of these subgames. Thus player 2 is the winner in every subgame perfect equilibrium of the whole game.

Now suppose that n is not divisible by 3. As before, the subgames following player 1's removal of one or two stones are the games for $n - 1$ and $n - 2$ in which player 2 is the first-mover. Either $n - 1$ or $n - 2$ is divisible by 3, so in one of these subgames player 1 is the winner in every subgame perfect equilibrium. Thus player 1 is the winner in every subgame perfect equilibrium of the whole game.

202.1 Hungry lions

Denote by $G(n)$ the game in which there are n lions.

The game $G(1)$ has a unique subgame perfect equilibrium, in which the single lion eats the prey.

Consider the game $G(2)$. If lion 1 does not eat, it remains hungry. If it eats, we reach a subgame identical to $G(1)$, which we know has a unique subgame perfect equilibrium, in which lion 2 eats lion 1. Thus $G(2)$ has a unique subgame perfect equilibrium, in which lion 1 does not eat the prey.

In $G(3)$, lion 1's eating the prey leads to $G(2)$, in which we have just concluded that the first mover (lion 2) does not eat the prey (lion 1). Thus $G(3)$ has a unique subgame perfect equilibrium, in which lion 1 eats the prey.

For an arbitrary value of n, lion 1's eating the prey in $G(n)$ leads to $G(n - 1)$. If $G(n - 1)$ has a unique subgame perfect equilibrium, in which the prey is eaten, then $G(n)$ has a unique subgame perfect equilibrium, in which the prey is not eaten; if $G(n - 1)$ has a unique subgame perfect equilibrium, in which the prey is not eaten, then $G(n)$ has a unique subgame perfect equilibrium, in which the prey is eaten. Given that $G(1)$ has a unique subgame perfect equilibrium, in which the prey is eaten, we conclude that if n is odd then $G(n)$ has a unique subgame perfect equilibrium, in which lion 1 eats the prey, and if n is even it has a unique subgame perfect equilibrium, in which lion 1 does not eat the prey.

203.1 A race with a liquidity constraint

In the absence of the constraint, player 1 initially takes one step. Suppose she does so in the game with the constraint. Consider player 2's options after player 1's move.

Player 2 takes two steps: Because of the liquidity constraint, player 1 can take at most one step. If she takes one step, player 2's optimal action is to take one step, and win. Thus player 1's best action is not to move; player 2's payoff exceeds 1 (her steps cost 5, and the prize is worth more than 6).

Player 2 moves one step: Again because of the liquidity constraint, player 1 can take at most one step. If she takes one step, player 2 can take two steps and win, obtaining a payoff of more than 1 (as in the previous case).

Player 2 does not move: Player 1, as before, can take one step on each turn, and win; player 2's payoff is 0.

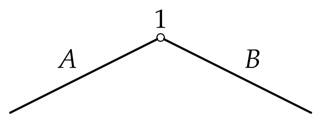
We conclude that after player 1 moves one step, player 2 should take either one or two steps, and ultimately win; player 1's payoff is -1. A better option for player 1 is not to move, in which case player 2 can move one step at a time, and win; player 1's payoff is zero.

Thus the subgame perfect equilibrium outcome is that player 1 does not move, and player 2 takes one step at a time and wins.

7 Extensive Games with Perfect Information: Extensions and Discussion

210.2 Extensive game with simultaneous moves

The game is shown in Figure 107.2.



	C	D
C	$4,2$	$0,0$
D	$0,0$	$2,4$

	E	F
E	$3,1$	$0,0$
F	$0,0$	$1,3$

Figure 107.2 The game in Exercise 210.2.

The subgame following player 1's choice of A has two Nash equilibria, (C,C) and (D,D); the subgame following player 1's choice of B also has two Nash equilibria, (E,E) and (F,F). If the equilibrium reached after player 1 chooses A is (C,C), then regardless of the equilibrium reached after she chooses (E,E), she chooses A at the beginning of the game. If the equilibrium reached after player 1 chooses A is (D,D) and the equilibrium reached after she chooses B is (F,F), she chooses A at the beginning of the game. If the equilibrium reached after player 1 chooses A is (D,D) and the equilibrium reached after she chooses B is (E,E), she chooses B at the beginning of the game.

Thus the game has four subgame perfect equilibria: (ACE, CE), (ACF, CF), (ADF, DF), and (BDE, DE) (where the first component of player 1's strategy is her choice at the start of the game, the second component is her action after she chooses A, and the third component is her action after she chooses B, and the first component of player 2's strategy is her action after player 1 chooses A at the start of the game and the second component is her action after player 1 chooses B at the start of the game).

In the first two equilibria the outcome is that player 1 chooses A and then both players choose C, in the third equilibrium the outcome is that player 1 chooses A and then both players choose D, and in the last equilibrium the outcome is that player 1 chooses B and then both players choose E.

210.3 Two-period *Prisoner's Dilemma*

The extensive game is specified as follows.

Players The two people.

Terminal histories The set of pairs $((W, X), (Y, Z))$, where each component is either Q or F.

Player function $P(\varnothing) = \{1, 2\}$ and $P(W, X) = \{1, 2\}$ for any pair (W, X) in which both W and X are either Q or F.

Actions The set $A_i(\varnothing)$ of player i's actions at the initial history is $\{Q, F\}$, for $i = 1, 2$; the set $A_i(W, X)$ of player i's actions after any history (W, X) in which both W and X are either Q or F is $\{Q, F\}$, for $i = 1, 2$.

Preferences Each player's preferences are represented by the payoffs described in the problem.

Consider the subgame following some history (W, X) (where W and X are both either Q or F). In this subgame each player chooses either Q or F, and her payoff to each resulting terminal history is the sum of her payoff to (W, X) in the *Prisoner's Dilemma* given in Figure 15.1 and her payoff to the pair of actions chosen in the subgame, again as in the *Prisoner's Dilemma*. Thus the subgame differs from the *Prisoner's Dilemma* given in Figure 15.1 only in that every payoff to a given player is increased by her payoff to the pair of actions (W, X). Thus the subgame has a unique Nash equilibrium, in which both players choose F.

Now consider the whole game. Regardless of the actions chosen at the start of the game, the outcome in the second period is (F, F). Thus the payoffs to the pairs of actions chosen in the first period are the payoffs in the *Prisoner's Dilemma* plus the payoff to (F, F). We conclude that the game has a unique subgame perfect equilibrium, in which each player chooses F after every history.

211.1 Timing claims on an investment

The following extensive game models the situation.

Players The two people.

Terminal histories The sequences of the form $((N, N), (N, N), \ldots, (N, N), x_t)$, where $1 \leq t \leq T$, x_t is (C, C), (C, N), or (N, C) if $t \leq T - 1$ and (C, C), (C, N), (N, C), or (N, N) if $t = T$, C means "claim", and N means "do not claim".

Player function The set of players assigned to every nonterminal history is $\{1, 2\}$ (the two people).

Actions The set of actions of each player after every nonterminal history is $\{C, N\}$.

Preferences Each player's preferences are represented by a payoff equal to the amount of money she obtains.

The consequences of the players' actions in period T are given in Figure 109.1. We see that the subgame starting in period T has a unique Nash equilibrium, (C, C), in which each player's payoff is T.

	C	N
C	T, T	$2T, 0$
N	$0, 2T$	T, T

Figure 109.1 The consequences of the players' actions in period T of the game in Exercise 211.1.

Thus if $T = 1$ the game has a unique subgame perfect equilibrium, in which both players claim.

Now suppose that $T \geq 2$, and consider period $T - 1$. The consequences of the players' actions in this period, given the equilibrium in the subgame starting in period T, are shown in Figure 109.2. (The entry in the bottom right box, (T, T), is the pair of equilibrium payoffs in the subgame in period T.) If $T > 2$ then $2(T - 1) > T$, so that the subgame starting in period $T - 1$ has a unique subgame perfect equilibrium, (C, C), in which each player's payoff is $T - 1$. If $T = 2$ then the whole game has two subgame perfect equilibria, in one of which both players claim in both periods, and another in which neither claims in period 1 and both claim in period 2.

	C	N
C	$T-1, T-1$	$2(T-1), 0$
N	$0, 2(T-1)$	T, T

Figure 109.2 The consequences of the players' actions in period $T - 1$ of the game in Exercise 211.1, given the equilibrium actions in period T.

For $T > 2$, working back to period 1 we see that the game has two subgame perfect equilibria: one in which each player claims in every period, and one in which neither player claims in period 1 but both players claim in every subsequent period.

211.2 A market game

The following extensive game models the situation.

Players The seller and m buyers.

Terminal histories The set of sequences of the form $((p_1, \ldots, p_m), j)$, where each p_i is a price (nonnegative number) and j is either 0 or one of the sellers (an integer from 1 to m), with the interpretation that p_i is the offer of buyer i, $j = 0$ means that the seller accepts no offer, and $j \geq 1$ means that the seller accepts buyer j's offer.

Player function $P(\varnothing)$ is the set of buyers and $P(p_1, \ldots, p_m)$ is the seller for every history $(p_1, \ldots, p_m)$.

Actions The set $A_i(\varnothing)$ of actions of buyer i at the start of the game is the set of prices (nonnegative numbers). The set $A_s(p_1, \ldots, p_m)$ of actions of the seller after the buyers have made offers is the set of integers from 0 to m.

Preferences Each player's preferences are represented by the payoffs given in the question.

To find the subgame perfect equilibria of the game, first consider the subgame following a history $(p_1, \ldots, p_m)$ of offers. The seller's best action is to accept the highest price, or one of the highest prices in the case of a tie.

I claim that a strategy profile is a subgame perfect equilibrium of the whole game if and only if the seller's strategy is the one just described, and among the buyers' strategies $(p_1, \ldots, p_m)$, every offer p_i is at most v and at least two offers are equal to v.

Such a strategy profile is a subgame perfect equilibrium by the following argument. If the buyer with whom the seller trades raises her offer then her payoff becomes negative, while if she lowers her offer she no longer trades and her payoff remains zero. If any other buyer raises her offer then either she still does not trade, or she trades at a price greater than v and hence receives a negative payoff.

No other profile of actions for the buyers at the start of the game is part of a subgame perfect equilibrium by the following argument.

- If some offer exceeds v then the buyer who submits the highest offer can induce a better outcome by reducing her offer to a value below v, so that either the seller does not trade with her, or, if the seller does trade with her, she trades at a lower price.

- If all offers are at most v and only one is equal to v, the buyer who offers v can increase her payoff by reducing her offer a little.

- If all offers are less than v then one of the buyers whose offer is not accepted can increase her offer to some value between the winning offer and v, induce the seller to trade with her, and obtain a positive payoff.

In any equilibrium the buyer who trades with the seller does so at the price v. Thus her payoff is zero. The other buyers do not trade, and hence also obtain the payoff of zero.

212.1 Price competition

The following game models the situation.

Players The two sellers and the two buyers.

Terminal histories All sequences $((p_1, p_2), (x_1, x_2))$ where p_i (for $i = 1, 2$) is the price posted by seller i and x_i (for $i = 1, 2$) is the seller chosen by buyer i (either seller 1 or seller 2).

Player function $P(\varnothing)$ is the set consisting of the two sellers; $P(p_1, p_2)$ for any pair (p_1, p_2) of prices is the set consisting of the two buyers.

Actions The set of actions of each seller at the start of the game is the set of prices (nonnegative numbers), and the set of actions of each buyer after any history (p_1, p_2) is the set consisting of seller 1 and seller 2.

Preferences Each seller's preferences on lotteries over the terminal histories are represented by the expected value of a Bernoulli payoff function that assigns the payoff p to a sale at the price p. Each buyers' preferences on lotteries over the terminal histories are represented by the expected value of a Bernoulli payoff function that assigns the payoff $1 - p$ to a purchase at the price p. The payoff of a player who does not trade is 0.

In any subgame perfect equilibrium, the buyers' strategies in the subgame following any history (p_1, p_2) must be a Nash equilibrium of the game in Exercise 128.1. This game has a unique Nash equilibrium unless $\frac{1}{2}(1 + p_1) \leq p_2 \leq 2p_1 - 1$. If $\frac{1}{2}(1 + p_1) < p_2 < 2p_1 - 1$ the game has three Nash equilibria, two pure and one mixed.

I claim that for any price $p \geq \frac{1}{2}$ the extensive game in this exercise has a subgame perfect equilibrium in which if $\frac{1}{2}(1 + p_1) < p_2 < 2p_1 - 1$ then if either $p_1 \leq p$ or $p_2 \leq p$, the equilibrium in the subgame is the pure Nash equilibrium in which buyer 1 approaches seller 1 and buyer 2 approaches seller 2, while if $p_1 > p$ and $p_2 > p$, the equilibrium in the subgame is the mixed strategy equilibrium.

Precisely, I claim that for any $p \geq \frac{1}{2}$ the following strategy pair is a subgame perfect equilibrium of the game.

Sellers' strategies Each seller announces the price p.

Buyers' strategies

- After a history (p_1, p_2) in which $2p_1 - 1 < p_2 < \frac{1}{2}(1 + p_1)$ and either $p_1 \leq p$ or $p_2 \leq p$ (or both), buyer 1 approaches seller 1 and buyer 2 approaches seller 2.
- After a history (p_1, p_2) in which $2p_1 - 1 < p_2 < \frac{1}{2}(1 + p_1)$, $p_1 > p$, and $p_2 > p$, each buyer approaches seller 1 with probability $(1 - 2p_1 + p_2)/(2 - p_1 - p_2)$.
- After a history (p_1, p_2) in which $p_2 \leq 2p_1 - 1$, both buyers approach seller 2.
- After a history (p_1, p_2) in which $p_2 \geq \frac{1}{2}(1 + p_1)$, both buyers approach seller 1.

By Exercise 128.1, the buyers' strategy pair is a Nash equilibrium in every subgame. The sellers' payoffs in the pure equilibrium in which one buyer approaches each seller are (p_1, p_2); their payoffs in the pure equilibrium in which both buyers approach seller 1 is $(p_1, 0)$; and their payoffs in the pure equilibrium in which both buyers approach seller 1 is $(0, p_2)$. Their payoffs in the mixed strategy equilibrium are more difficult to calculate. They are $(\pi_1^*(p_1, p_2), \pi_2^*(p_1, p_2)) = ((1-(1-\pi)^2)p_1, (1-\pi^2)p_2)$, where $\pi = (1-2p_1+p_2)/(2-p_1-p_2)$. After some algebra we obtain

$$(\pi_1^*(p_1, p_2), \pi_2^*(p_1, p_2)) = \left(\frac{3p_1(1-p_2)(1-2p_1+p_2)}{(2-p_1-p_2)^2}, \frac{3p_2(1-p_1)(1-2p_2+p_1)}{(2-p_1-p_2)^2}\right).$$

These equilibrium payoffs are illustrated in Figure 112.1.

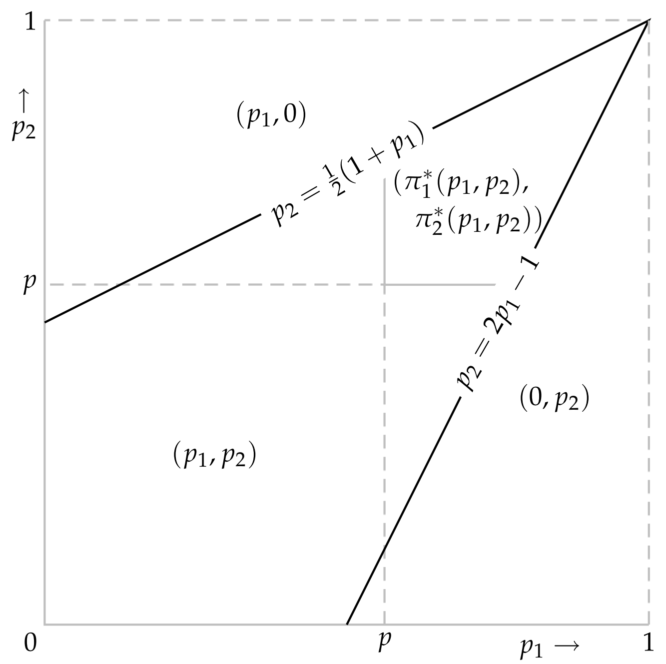


Figure 112.1 The sellers' payoffs in the game in Exercise 212.1 as a function of their prices, given the buyers' equilibrium strategies.

Now consider the sellers' choices of prices. Given that $p_2 = p \geq \frac{1}{2}$ and the buyers' strategies are those defined above, seller 1's payoff when she sets the price p_1 is

$$\begin{cases} p_1 & \text{if } p_1 \leq p \\ \pi_1^*(p_1, p) & \text{if } p < p_1 \leq \frac{1}{2}(1+p) \\ 0 & \text{if } p > \frac{1}{2}(1+p). \end{cases}$$

By the claim in the question (verified at the end of this solution), $\pi_1^*(p_1, p_2)$ is decreasing in p_1 for $p_1 \geq p_2$, so that seller 1's best response to p is p. An analogous argument shows that seller 2's best response to p is p.

We conclude that the strategy pair defined above is a subgame perfect equilibrium.

The verification of the last claim of the question (not required as part of an answer) follows. We have

$$\pi_1^*(p_1, p_2) = \frac{3p_1(1-p_2)(1-2p_1+p_2)}{(2-p_1-p_2)^2}.$$

The derivative of this function with respect to p_1 is

$$\frac{3(1-p_2)\left[(2-p_1-p_2)^2(1-2p_1+p_2-2p_1)+2(2-p_1-p_2)p_1(1-2p_1+p_2)\right]}{(2-p_1-p_2)^4}$$

or

$$\frac{3(1-p_2)(2-p_1-p_2)\left[(2-p_1-p_2)(1-4p_1+p_2)+2p_1(1-2p_1+p_2)\right]}{(2-p_1-p_2)^4}.$$

This expression is negative if

$$(2-p_1-p_2)(1-4p_1+p_2)+2p_1(1-2p_1+p_2) < 0,$$

or

$$p_1 > \frac{(2-p_2)(1+p_2)}{7-5p_2}.$$

The right-hand side is less than p_2 if

$$(2p_2-1)(p_2-1) < 0,$$

which is true if $\frac{1}{2} < p_2 < 1$, so that seller 1's equilibrium payoff is decreasing in p_1 whenever $p_1 > p_2 > \frac{1}{2}$.

214.1 Bertrand's duopoly game with entry

The unique Nash equilibrium of the subgame that follows the challenger's entry is (c, c), as we found in Section 3.2.2. The challenger's profit is $-f < 0$ in this equilibrium. By choosing to stay out the challenger obtains the profit of 0, so in any subgame perfect equilibrium the challenger stays out. After the history in which the challenger stays out, the incumbent chooses its price p_1 to maximize its profit $(p_1 - c)(\alpha - p_1)$.

Thus for any value of $f > 0$ the whole game has a unique subgame perfect equilibrium, in which the strategies are:

Challenger

- at the start of the game: stay out
- after the history in which the challenger enters: choose the price c

Incumbent

- after the history in which the challenger enters: choose the price c
- after the history in which the challenger stays out: choose the price p_1 that maximizes $(p_1 - c)(\alpha - p_1)$.

216.1 Electoral competition with strategic voters

Consider a strategy profile in which each candidate chooses the median m of the citizens' favorite positions and the citizens' strategies are defined as follows.

- After a history in which every candidate chooses m, each citizen i votes for candidate j, where j is the smallest integer greater than or equal to in/q. (That is, the citizens split their votes equally among the n candidates. If there are 3 candidates and 15 citizens, for example, citizens 1 through 5 vote for candidate 1, citizens 6 through 10 vote for candidate 2, and citizens 11 through 15 vote for candidate 3.)

- After a history in which all candidates enter and every candidate except j chooses m, each citizen votes for candidate j if her favorite position is closer to j's position than it is to m, and for some candidate ℓ whose position is m otherwise. (All citizens who do not vote for j vote for the *same* candidate ℓ.)

- After any other history, the citizens' action profile is any Nash equilibrium of the voting subgame in which no citizen's action is weakly dominated.

Every such strategy profile induces the outcome in which all candidates enter and choose the median of the citizens' favorite positions, and tie for first place. After every history of one of the first two types, every citizen votes for one of the candidates who is closest to her favorite position, so no citizen's strategy is weakly dominated. After a history of the third type, no citizen's strategy is weakly dominated by construction.

Every such strategy profile is a subgame perfect equilibrium by the following argument.

In each voting subgame the citizens' strategy profile is a Nash equilibrium:

- after the history in which the candidates' positions are the same, equal to m, no citizen's vote affects the outcome

- after a history in which all candidates enter and every candidate but j chooses m, a change in any citizen's vote either has no effect on the outcome or makes it worse for her

- after any other history the citizens' strategy profile is a Nash equilibrium by construction.

Now consider the candidates' choices at the start of the game. If any candidate deviates by choosing a position different from that of the other candidates, she loses, rather than tying for first place. If any candidate deviates by staying out of the race, the outcome is worse for her than adhering to the equilibrium, and tying for first place. Thus each candidate's strategy is optimal given the other players' strategies.

[The claim that every voting subgame has a (pure) Nash equilibrium in which no citizen's action is weakly dominated, which you are not asked to prove, may be demonstrated as follows. Given the candidates' positions, choose the candidate, say j, ranked last by the smallest number of citizens. Suppose that all citizens except those who rank j last vote for j; distribute the votes of the citizens who rank j last as equally as possible among the other candidates. Each citizen's action is not weakly dominated (no citizen votes for the candidate she ranks last) and, given $q \geq 2n$, no change in any citizen's vote affects the outcome, so that the list of citizens' actions is a Nash equilibrium of the voting subgame.]

217.1 Electoral competition with strategic voters

I first argue that in any equilibrium each candidate that enters is in the set of winners. If some candidate that enters is not a winner, she can increase her payoff by deviating to *Out*.

Now consider the voting subgame in which there are more than two candidates and not all candidates' positions are the same. Suppose that the citizens' votes are equally divided among the candidates. I argue that this list of citizens' strategies is not a Nash equilibrium of the voting subgame.

For either the citizen whose favorite position is 0 or the citizen whose favorite position is 1 (or both), at least two candidates' positions are better than the position of the candidate furthest from the citizen's favorite position. Denote a citizen for whom this condition holds by i. (The claim that citizen i exists is immediate if the candidates occupy at least three distinct positions, or they occupy two distinct positions and at least two candidates occupy each position. If the candidates occupy only two positions and one position is occupied by a single candidate, then take the citizen whose favorite position is 0 if the lone candidate's position exceeds the other candidates' position; otherwise take the citizen whose favorite position is 1.)

Now, given that each candidate obtains the same number of votes, if citizen i switches her vote to one of the candidates whose position is better for her than that of the candidate whose position is furthest from her favorite position, then this candidate wins outright. (If citizen i originally votes for one of these superior candidates, she can switch her vote to the other superior candidate; if she originally votes for neither of the superior candidates, she can switch her vote to either one of them.) Citizen i's payoff increases when she thus switches her vote, so that the list of citizens' strategies is not a Nash equilibrium of the voting subgame.

We conclude that in every Nash equilibrium of every voting subgame in which there are more than two candidates and not all candidates' positions are the same at least one candidate loses. Because no candidate loses in a subgame perfect equilibrium (by the first argument in the proof), in any subgame perfect equilibrium either only two candidates enter, or all candidates' positions are the same.

If only two candidates enter, then by the argument in the text for the case $n = 2$, each candidate's position is m (the median of the citizens' favorite positions).

Now suppose that more than two candidates enter, and their common position is not equal to m. If a candidate deviates to m then in the resulting voting subgame only two positions are occupied, so that for every citizen, any strategy that is not weakly dominated votes for a candidate at the position closest to her favorite position. Thus a candidate who deviates to m wins outright. We conclude that in any subgame perfect equilibrium in which more than two candidates enter, they all choose the position m.

220.1 Top cycle set

a. The top cycle set is the set $\{x, y, z\}$ of all three alternatives because x beats y beats z beats x.

b. The top cycle set is the set $\{w, x, y, z\}$ of all four alternatives. As in the previous case, x beats y beats z beats x; also y beats w.

221.1 Designing agendas

We have: x beats y beats z beats x; x, y, and z all beat v; v beats w; and w does not beat any alternative. Thus the top cycle set is $\{x, y, z\}$.

An agenda that yields x is shown in Figure 116.1. A similar agenda, with y and x interchanged, yields y, and one with x and z interchanged yields z.

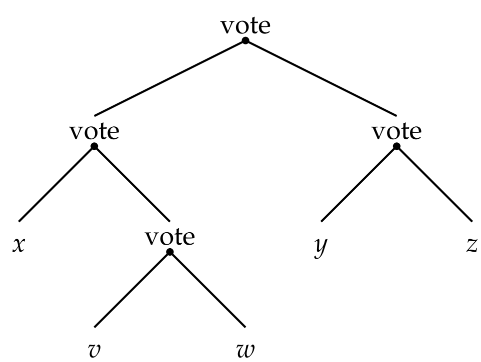


Figure 116.1 A binary agenda for which the alternative x is the outcome of sophisticated voting for the committee in Exercise 221.1.

No binary agenda yields w because for every other alternative a, a majority of committee members prefer a to w. No binary agenda yields v because the only alternative that v beats is w, which itself is beaten by every other alternative.

221.2 An agenda that yields an undesirable outcome

An agenda for which the outcome of sophisticated voting is z is given in Figure 117.1.

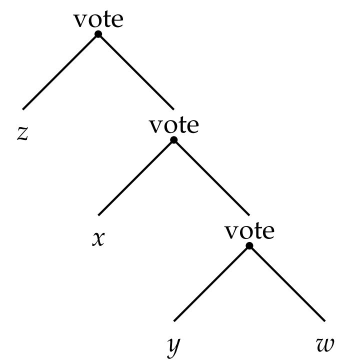


Figure 117.1 A binary agenda for which the alternative z is the outcome of sophisticated voting for the committee in Exercise 221.2.

224.1 Exit from a declining industry

Period t_1 is the largest value of t for which $P_t(k_1) \geq c$, or $60 - t \geq 10$. Thus $t_1 = 50$. Similarly, $t_2 = 70$.

If both firms are active in period t_1, then firm 2's profit in this period is $-ck_2 = -10(20) = -200$. (Note that the price is zero, because $k_1 + k_2 > 50$.) Its profit in any period t in which it is alone in the market is $(100 - t - c - k_2)k_2 = (70 - t)(20)$. Thus its profit from period $t_1 + 1$ through period t_2 is

$$(19 + 18 + \ldots + 1)(20) = 3800.$$

Hence firm 2's loss in period t_1 when both firms are active is (much) less than the sum of its profits in periods $t_1 + 1$ through t_2 when it alone is active.

225.1 Effect of borrowing constraint in declining industry

Period t_0 is the largest value of t for which $P_t(k_1 + k_2) \geq c$, or $100 - t - 60 \geq 10$, or $t \leq 30$. Thus $t_0 = 30$. From Exercise 224.1 we have $t_1 = 50$ and $t_2 = 70$.

Suppose that firm 2 stays in the market for k periods after t_0, then exits in period $t_0 + k + 1$. Firm 1's total profit from period $t_0 + 1$ on if it stays until period t_1 is

$$\begin{aligned}(P_{t_0+1}(k_1 + k_2) - c)k_1 + \ldots + (P_{t_0+k}(k_1 + k_2) &- c)k_1 + \\ (P_{t_0+k+1}(k_1) &- c)k_1 + \ldots + (P_{t_1}(k_1) - c)k_1,\end{aligned}$$

or

$$\begin{aligned}40[(100 - 30 - 1 - 60 - 10) + \ldots &+ (100 - 30 - k - 60 - 10) + \\ (100 - 30 - k - 1 - 40 &- 10) + \ldots + (100 - 50 - 40 - 10)],\end{aligned}$$

or

$$40[-1 - \ldots - k + (19 - k) + \ldots + 0],$$

or

$$40[-\tfrac{1}{2}k(k+1)+\tfrac{1}{2}(19-k)(20-k)]$$

(using the fact that the sum of the first n positive integers is $\frac{1}{2}n(n+1)$), or

$$20(380-40k).$$

In order that this profit be nonpositive we need $40k \geq 380$, or $k \geq 9.5$. Thus firm 2 needs to survive until at least period 40 (30 + 10) in order to make firm 1's exit in period $t_0 + 1$ optimal.

Firm 2's total loss from period 31 through period 40 when both firms are in the market is

$$(P_{31}(k_1+k_2)-c)k_2+\ldots+(P_{40}(k_1+k_2)-c)k_2,$$

or

$$20[(100-31-60-10)+\ldots+(100-40-60-10)],$$

or

$$20(-1+\ldots+-10),$$

or 1100.

Thus firm 2 needs to be able to bear a debt of at least 1100 in order for there to be a subgame perfect equilibrium in which firm 1 exits in period $t_0 + 1$.

227.1 Variant of ultimatum game with equity-conscious players

The game is defined as follows.

Players The two people.

Terminal histories The set of sequences (x, β_2, Z), where x is a number with $0 \leq x \leq c$ (the amount of money that person 1 offers to person 2), β_2 is 0 or 1 (the value of β_2 selected by chance), and Z is either Y ("yes, I accept") or N ("no, I reject").

Player function $P(\varnothing) = 1$, $P(x) = c$ for all x, and $P(x, \beta_2) = 2$ for all x and all β_2.

Chance probabilities For every history x, chance chooses 0 with probability p and 1 with probability $1 - p$.

Preferences Each person's preferences are represented by the expected value of a payoff equal to the amount of money she receives. For any terminal history (x, β_2, Y) person 1 receives $c - x$ and person 2 receives x; for any terminal history (x, β_2, N) each person receives 0.

Given the result from Exercise 183.4 stated in the question, an offer x of player 1 is accepted with probability either 0 or p if $x = 0$, is accepted with probability p if $0 < x < \frac{1}{3}$, is accepted with probability either p or 1 if $x = \frac{1}{3}$, and is accepted with probability 1 if $x > \frac{1}{3}$. By an argument like that for the original ultimatum game, in any equilibrium in which player 1 makes an offer of 0, player 2 certainly accepts the offer if $\beta_2 = 0$, and in any equilibrium in which player 1 makes an offer of $\frac{1}{3}$, player 2 certainly accepts the offer if $\beta_2 = 1$. Thus player 1's expected payoff to making the offer x is

$$\begin{cases} p(1-x) & \text{if } 0 \leq x < \frac{1}{3} \\ 1-x & \text{if } \frac{1}{3} \leq x < 1. \end{cases}$$

The maximizer of this function is $x = \frac{1}{3}$ if $p < \frac{2}{3}$ and $x = 0$ if $p > \frac{2}{3}$; if $p = \frac{2}{3}$ then both offers are optimal. (If you do not see that the maximizer takes this form, plot the expected payoff as a function of x.)

We conclude that if $p \neq \frac{2}{3}$, the subgame perfect equilibria of the game are given as follows.

$p < \frac{2}{3}$ Player 1 offers $\frac{1}{3}$. After a history in which $\beta_2 = 0$, player 2 accepts an offer x with $x > 0$ and either accepts or rejects the offer 0. After a history in which $\beta_2 = 1$, player 2 accepts an offer x with $x \geq \frac{1}{3}$ and rejects an offer x with $x < \frac{1}{3}$.

$p > \frac{2}{3}$ Player 1 offers 0. After a history in which $\beta_2 = 0$, player 2 accepts all offers. After a history in which $\beta_2 = 1$, player 2 accepts an offer x with $x > \frac{1}{3}$, rejects an offer x with $x < \frac{1}{3}$, and either accepts or rejects the offer $\frac{1}{3}$.

If $p = \frac{2}{3}$, both these strategy pairs are subgame perfect equilibria.

We see that if $p > \frac{2}{3}$ then in a subgame perfect equilibrium player 1's offers are rejected by every player 2 with for whom $\beta_2 = 1$ (that is, with probability $1 - p$).

227.2 Firm–union bargaining

First consider the firm's responses to the union's demand of x. If the surplus is H then the firm rejects $x > H$, accepts $x < H$, and either accepts or rejects $x = H$. Similarly, if the surplus is L then the firm rejects $x > L$, accepts $x < L$, and either accepts or rejects $x = L$.

First suppose that the firm accepts $x = H$ when the surplus is H and $x = L$ when the surplus is L. Then the union's expected payoff is

$$\begin{cases} x & \text{if } x \leq L \\ px & \text{if } L < x \leq H \\ 0 & \text{if } H < x. \end{cases}$$

Thus the union's best demand is L if $L > pH$, and H if $L < pH$; if $L = pH$ the union is indifferent between these two demands. (If you do not see that the union's best demand takes this form, plot its expected payoff as a function of x.)

By the same argument as in the ultimatum game, if the firm rejects $x = H$ when the surplus is H then the union has no optimal demand if $L < pH$, and if the firm rejects $x = L$ when the surplus is L then the union has no optimal demand if $L > pH$.

We conclude that if $p < L/H$ the game has two subgame perfect equilibria, in which the union demands L and the firm accepts a demand of x when the surplus is L if and only if $x \leq L$, accepts a demand of x when the surplus is H if $x < H$, rejects a demand of x when the surplus is H and $x > H$, and either accepts or rejects a demand of H when the surplus is H. In these equilibria the probability of a strike is 0.

Similarly, if $p > L/H$ the game has two subgame perfect equilibria, in which the union demands H and the firm accepts a demand of x when the surplus is H if and only if $x \leq H$, accepts a demand of x when the surplus is L if $x < L$, rejects a demand of x when the surplus is L and $x > L$, and either accepts or rejects a demand of L when the surplus is L. In these equilibria the probability of a strike is $1 - p$ (the probability the surplus is L).

Finally, if $p = L/H$ all of these strategy pairs are subgame perfect equilibria.

227.3 Sequential duel

The following game models the situation.

Players The two people.

Terminal histories All sequences of the form $(X_1, X_2, \ldots, X_k, S, H)$, where each X_i is either N ("don't shoot") or (S, M) ("shoot", "miss"), and H means "hit", together with the infinite sequence $(S, M, S, M, S, M, \ldots)$.

Player function $P(h) = 1$ for any history h that ends in M or N and in which the total number of S's and N's is even, $P(h) = 2$ for any history h that ends in M or N and in which the total number of S's and N's is odd, and $P(h) = c$ for any history h that ends in S.

Chance probabilities Whenever chance moves after a move of player 1 it chooses H with probability p_1 and M with probability $1 - p_1$; whenever it moves after a move of player 2 it chooses H with probability p_2 and M with probability $1 - p_2$;

Preferences Each player's preferences are represented by the expected value of a Bernoulli payoff function that assigns 1 to any history in which she survives and 0 to any history in which she is killed.

First consider the strategy pair in which neither player ever shoots. The outcome of this strategy pair is that both players survive. No outcome is better for either player, so in particular neither player has a strategy that leads to a better outcome for her in any subgame, given the other player's strategy.

Now consider the strategy pair in which each player always shoots. Suppose that player 2 always shoots. Consider any subgame. Suppose that player 1 switches to not shooting after some histories. If none of the histories after which she changes her action is reached when player 2 follows her strategy, the change in player 1's strategy has no effect on her payoff in the subgame. But if some of these histories are reached with positive probability when player 2 follows her strategy, then player 1's probability of being killed increases. That is, any change in player 1's strategy either has no effect on her payoff in the subgame or decreases that payoff. A symmetric argument applies to player 2. Thus the strategy pair in which each player always shoots is a subgame perfect equilibrium.

(If you are not convinced that player 1's probability of being killed increases when she switches from shooting to not shooting after some histories that are reached if player 2 follows her strategy, consider a subgame that starts with an action of player 1. If player 1 always shoots in this subgame, the probability she is killed (in the subgame) is

$$(1 - p_1)[p_2 + (1 - p_2)(1 - p_1)[p_2 + (1 - p_2)(1 - p_1)[p_2 + \cdots$$

If player 1 deviates to not shooting after some histories, the expression for the probability she is killed differs only in that some of the terms $1 - p_1$ are replaced by 1. Every such substitution increases the value of the expression, and thus reduces player 1's payoff.)

227.4 Sequential truel

The games are shown in Figure 122.1. (The action marked "0" is that of shooting into the air, which is available only in the second version of the game.)

First consider the game in which a player must shoot at another player.

To find the subgame perfect equilibria of this game, first consider the subgame Γ' in Figure 122.1. Whomever player C aims at, if she misses then she survives in the company of both A and B. If she aims at B and hits her, then she survives in the company of A; if she aims at A and hits her then she survives in the company of B. Thus C aims at B if $p_A < p_B$ and at A if $p_A > p_B$.

Now consider the subgame Γ. Whomever B aims at, the outcome is the same if she misses (because Γ' has a unique subgame perfect equilibrium). If B aims at A and hits her, then she survives with probability $1 - p_C$; if she aims at C and hits her, then she survives with probability 1. Thus (given $p_C > 0$), the subgame Γ thus has a unique subgame perfect equilibrium, in which B aims at C.

Finally, consider the whole game. Whomever A aims at, the outcome is the same if she misses (because Γ has a unique subgame perfect equilibrium). If she aims at B and hits her, then she survives with probability $1 - p_C$; if she aims at C and hits her, then she survives with probability $1 - p_B$. Thus A aims at C if $p_B < p_C$ and at B if $p_B > p_C$.

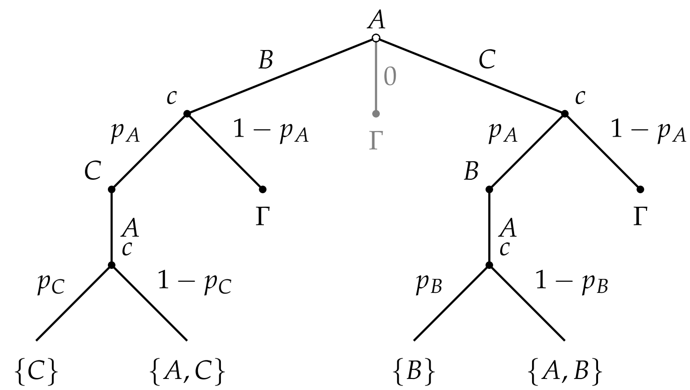
In summary, the game in which no player has the option of shooting into the air has the following unique subgame perfect equilibrium.

- At the start of the game, A aims at C if $p_B < p_C$ and at B if $p_B > p_C$.
- After a history in which A misses, B aims at C.

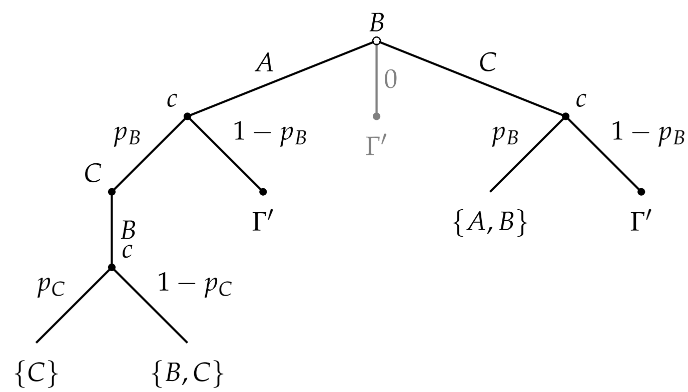
- After a history in which both A and B miss, C aims at B if $p_A < p_B$ and at A if $p_A > p_B$.

Player A aims the player who is her more dangerous opponent; she is better off if she eliminates this opponent than if she eliminates her weaker opponent.

Player C's survival probability is $(1-p_A)(1-p_B) = 1-p_A-p_B(1-p_A)$ if



where the game Γ is



and the game Γ' is

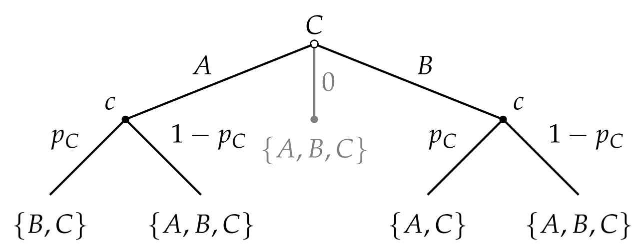


Figure 122.1 The games in Exercise 227.4. Only the actions indicated by black lines are available when players do not have the option of shooting into the air (the action "0"). The labels beside the actions of chance are the probabilities with which the actions are chosen; in each case the left action is "hit" and the right action is "miss".

$p_C > p_B$, and $1 - p_B(1 - p_A)$ if $p_C < p_B$. Thus she is better off if $p_C < p_B$ than if $p_C > p_B$.

Now consider the game in which each player has the option of shooting into the air. In the subgame Γ', player C's best action is to aim at B (given $p_A < p_B$). (If she shoots into the air then the set of survivors is $\{A, B, C\}$; if she aims at B she has some chance of eliminating her.)

In the subgame Γ we know that if B shoots, her target should be C. If she does so her probability of survival is $1 - (1 - p_B)p_C$. If she shoots into the air her probability of survival is $1 - p_C$. The former exceeds the latter, so in the subgame Γ player B aims at C.

Finally, given the equilibrium actions in the subgames, at the start of the game we know that if A fires she aims at C if $p_B < p_C$ and at B if $p_B > p_C$. Given $p_A < p_B$, her shooting into the air results in her certain survival, while her aiming at B or C results in her surviving with probability less than 1. Thus she shoots into the air.

We conclude that if $p_A < p_B$ then the game in which each player has the option of shooting into the air has a unique subgame perfect equilibrium, which differs from the subgame perfect equilibrium in which this option is absent only in that A shoots into the air at the beginning of the game.

Player A fires into the air because when she does so B and C fight between themselves; if she shoots at one of them she may eliminate her from the game, giving the remaining player an incentive to shoot at her.

228.1 Cohesion in legislatures

Let the initial governing coalition consist of legislators 1 and 2. The US game is defined as follows.

Players The three legislators.

Terminal histories All sequences

$$(i, (x_1, x_2, x_3), (A, B, C), j, (y_1, y_2, y_3), (A', B', C')),$$

where i and j are members of the governing coalition (possibly with $i = j$), (x_1, x_2, x_3) and (y_1, y_2, y_3) are partitions of one unit of payoff ($x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 1$, $x_i \geq 0$, and $y_i \geq 0$ for $i = 1, 2, 3$), and A, B, C, A', B', and C' are either *yes* (vote for bill) or *no* (vote against bill).

Player function

- $P(\varnothing) = c$ (chance)
- $P(i) = i$
- $P(i, (x_1, x_2, x_3)) = \{1, 2, 3\}$
- $P(i, (x_1, x_2, x_3), (A, B, C)) = c$

- $P(i, (x_1, x_2, x_3), (A, B, C), j) = j$
- $P(i, (x_1, x_2, x_3), (A, B, C), j, (y_1, y_2, y_3)) = \{1, 2, 3\}$.

Chance probabilities Chance assigns probability $\frac{1}{2}$ to 1 and probability $\frac{1}{2}$ to 2 whenever it moves.

Actions

- $A(\varnothing) = \{1, 2\}$
- $A(i) = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1, x_i \geq 0 \textit{ for all } i\}$ for $i = 1, 2$
- $A_k(i, (x_1, x_2, x_3)) = \{yes, no\}$ for all k, $i = 1, 2$, and all (x_1, x_2, x_3)
- $A(i, (x_1, x_2, x_3), (A, B, C)) = \{1, 2\}$ for all i, all (x_1, x_2, x_3), and all triples (A, B, C) in which A, B, and C are all either *yes* or *no*
- $A(i, (x_1, x_2, x_3), (A, B, C), j) = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1, x_i \geq 0 \textit{ for all } i\}$ for $i = 1, 2$, all (x_1, x_2, x_3), all triples (A, B, C) in which A, B, and C are all either *yes* or *no*, and $j = 1, 2$
- $A_k(i, (x_1, x_2, x_3), (A, B, C), j, (y_1, y_2, y_3)) = \{yes, no\}$ for all k, $i = 1, 2$, all (x_1, x_2, x_3), all triples (A, B, C) in which A, B, and C are all either *yes* or *no*, $j = 1, 2$, and all (y_1, y_2, y_3).

Preferences Each legislator i ranks the terminal histories according to the amount of money she receives: $x_i + y_i$ if both bills are passed, $x_i + d_i^2$ if only the first bill is passed, $d_i^1 + y_i$ if only the second bill is passed, and $d_i^1 + d_i^2$ if neither bill is passed.

We find a subgame perfect equilibrium as follows. Refer to d_i^t as legislator i's *reservation value* in period t. In the second period, denote by k the legislator whose reservation value is lower between the two who do not propose a bill. Each legislator i gets d_i^t if a bill does not pass, and hence, under the assumption that her vote is not weakly dominated, votes for a bill only if it gives her a payoff of at least d_i^t. The proposer needs one vote in addition to her own to pass a bill, and can obtain it most cheaply by proposing a bill that gives k the payoff d_k^2 and gives herself the remaining payoff $1 - d_k^2$ (which exceeds her reservation value, because all reservation values are less than $\frac{1}{2}$). Legislator k and the proposer vote for the bill, which thus passes. (Legislator k is indifferent between voting for or against the bill, but there is no subgame perfect equilibrium in which she votes against the bill, because if she uses such a strategy the proposer can increase her offer to k a little, leading k to strictly prefer voting for the bill.) The third player may vote for or against the bill (her vote has no effect on the outcome).

In the first period, the pattern of behavior is the same: the bill proposed gives the non-proposer with the lower reservation value that value.

In summary, in every subgame perfect equilibrium of the US game the strategy of each member i of the governing coalition has the following properties:

- after the move of chance in either period, propose the bill that gives the legislator with the smallest reservation value in that period her reservation value and gives i the remaining payoff
- after a bill is proposed in either period, vote for the bill if it assigns i a positive amount.

The equilibrium strategy of the other legislator j satisfies the condition:

- after a bill is proposed in either period, vote for the bill if it assigns j a positive amount.

(Each legislator's equilibrium strategy may either vote for or vote against a bill that gives her a payoff of zero.)

Thus in the US game there is no cohesion: the supporters of a bill may change from period to period, depending on the values of the reservation values.

The UK game is defined as follows.

Players The three legislators.

Terminal histories All sequences

$$(i, (x_1, x_2, x_3), (A, B, C), j, (y_1, y_2, y_3), (A', B', C')),$$

where i is a member of the governing coalition and j is any legislator, (x_1, x_2, x_3) and (y_1, y_2, y_3) are partitions of one unit of payoff ($x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 1$, $x_i \geq 0$, and $y_i \geq 0$ for $i = 1, 2, 3$), and A, B, C, A', B', and C' are either *yes* (vote for bill) or *no* (vote against bill).

Player function

- $P(\varnothing) = c$ (chance)
- $P(i) = i$
- $P(i, (x_1, x_2, x_3)) = \{1, 2, 3\}$
- $P(i, (x_1, x_2, x_3), (A, B, C)) = c$
- $P(i, (x_1, x_2, x_3), (A, B, C), j) = j$
- $P(i, (x_1, x_2, x_3), (A, B, C), j, (y_1, y_2, y_3)) = \{1, 2, 3\}$.

Chance probabilities Chance assigns probability $\frac{1}{2}$ to 1 and probability $\frac{1}{2}$ to 2 at the start of the game and after a history $(i, (x_1, x_2, x_3), (A, B, C))$ in which at least two of the votes A, B, and C are *yes*. Chance assigns probability $\frac{1}{3}$ to each legislator after a history $(i, (x_1, x_2, x_3), (A, B, C))$ in which at least two of the votes A, B, and C are *no*.

Actions

- $A(\varnothing) = \{1, 2\}$

- $A(i) = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1, x_i \geq 0 \textit{ for all } i\}$ for $i = 1, 2$
- $A_k(i, (x_1, x_2, x_3)) = \{\textit{yes}, \textit{no}\}$ for all k, $i = 1, 2$, and all (x_1, x_2, x_3)
- $A(i, (x_1, x_2, x_3), (A, B, C)) = \{1, 2\}$ for all i, all (x_1, x_2, x_3), and all triples (A, B, C) in which A, B, and C are all either *yes* or *no* and at least two are *yes*, and $A(i, (x_1, x_2, x_3), (A, B, C)) = \{1, 2, 3\}$ for all i, all (x_1, x_2, x_3), and all triples (A, B, C) in which A, B, and C are all either *yes* or *no* and at most one is *yes*
- $A(i, (x_1, x_2, x_3), (A, B, C), j) = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1, x_i \geq 0 \textit{ for all } i\}$ for $i = 1, 2$, all (x_1, x_2, x_3), all triples (A, B, C) in which A, B, and C are all either *yes* or *no*, and $j = 1, 2, 3$
- $A_k(i, (x_1, x_2, x_3), (A, B, C), j, (y_1, y_2, y_3)) = \{\textit{yes}, \textit{no}\}$ for all k, $i = 1, 2$, all (x_1, x_2, x_3), all triples (A, B, C) in which A, B, and C are all either *yes* or *no*, $j = 1, 2, 3$, and all (y_1, y_2, y_3).

Preferences Each legislator i ranks the terminal histories according to the amount of money she receives: $x_i + y_i$ if both bills are passed, x_i if only the first bill is passed, y_i if only the second bill is passed, and 0 if neither bill is passed.

To find the subgame perfect equilibria, start with the second period. The defeat of a bill leads each legislator to obtain the payoff of 0, so each legislator optimally votes for every bill (given that votes are restricted to be weakly undominated). Thus in any subgame perfect equilibrium the proposer's bill gives the proposer all the pie, and at least one of the other legislators votes for the bill. (As before, each of the other legislators is indifferent between voting for and voting against the bill, but there is no subgame perfect equilibrium in which the bill is voted down.)

In the first period, the same argument shows that the proposer's bill gives the proposer all the pie and that this bill passes. Further, in this period the other member of the governing coalition definitely votes for the bill. The reason is that if she does so, then her chance of being the proposer in the next period is $\frac{1}{2}$, so that her expected payoff is $\frac{1}{2}$. If she votes against, then the bill fails, so that she obtains a payoff of 0 in the first period and has a probability of $\frac{2}{3}$ of being in the governing coalition in the second period, so that her expected payoff is $\frac{1}{3}$. Thus she is better off voting for her comrade's bill than against it.

In summary, in every subgame perfect equilibrium of the UK game the strategy of each legislator i has the following properties:

- after the move of chance in either period, propose the bill that gives legislator i the payoff 1
- after a bill is proposed in the first period, vote for the bill if i is a member of the governing coalition.

Thus in the UK game the governing coalition is entirely cohesive.

230.1 Nash equilibria when players may make mistakes

The players' best response functions are indicated in Figure 127.1. We see that the game has two Nash equilibria, (A, A, A) and (B, A, A).

	A	B
A	$1^*, 1^*, 1^*$	$0, 0, 1^*$
B	$1^*, 1^*, 1^*$	$1^*, 0, 1^*$

A

	A	B
A	$0, 1^*, 0$	$1^*, 0, 0$
B	$1^*, 1^*, 0$	$0, 0, 0$

B

Figure 127.1 The player's best response functions in the game in Exercise 230.1.

The action A is not weakly dominated for any player. For player 1, A is better than B if players 2 and 3 both choose B; for players 2 and 3, A is better than B for all actions of the other players.

If players 2 and 3 choose A in the modified game, player 1's expected payoffs to A and B are

$$A\colon\ (1-p_2)(1-p_3) + p_1 p_2(1-p_3) + p_1(1-p_2)p_3 + (1-p_1)p_2 p_3$$

$$B\colon\ (1-p_2)(1-p_3) + (1-p_1)p_2(1-p_3) + (1-p_1)(1-p_2)p_3 + p_1 p_2 p_3.$$

The difference between the expected payoff to B and the expected payoff to A is

$$(1-2p_1)[p_2 + p_3 - 3p_2 p_3].$$

If $0 < p_i < \frac{1}{2}$ for $i = 1, 2, 3$, this difference is positive, so that (A, A, A) is not a Nash equilibrium of the modified game.

233.1 Nash equilibria of the chain-store game

Any terminal history in which the event in each period is either *Out* or (In, A) is the outcome of a Nash equilibrium. In any period in which challenger chooses *Out*, the strategy of the chain-store specifies that it choose F in the event that the challenger chooses *In*.

233.2 Subgame perfect equilibrium of the chain-store game

The outcome of the strategy pair is that the only the last 10 challengers enter, and the chain-store acquiesces to their entry. The payoff of each of the first 90 challengers is 1 and the payoff to the remaining 10 is 2. The chain-store's payoff is $90 \times 2 + 10 \times 1 = 190$.

No challenger can profitably deviate in any subgame (if one of the first 90 enters it is fought). However, I claim that the chain-store can increase its payoff by deviating after a history in which the first 89 challengers enter and are fought, and then challenger 90 enters. The chain-store's strategy calls for it to fight challenger 90 and then subsequently acquiesce to any entry, and the remaining challengers' strategies call for them to enter. But if instead the chain-store acquiesces to challenger 90, keeping the rest of its strategy the same, it increases its payoff by 1.

(Note that the chain-store cannot profitably deviate after a history in which fewer than 89 challengers enter and each of them is fought. Suppose, for example, that each of the first 88 challengers enters and is fought, and then challenger 89 enters. The chain-store's strategy calls for it to fight challenger 89, which induces challenger 90 to stay out; the remaining challengers enter, and the chain-store acquiesces. Its best deviation is to acquiesce to challenger 89's entry and that of all subsequent entrants, in which case all remaining challengers, including challenger 90, enter. The outcomes of the two strategies differ in periods 89 and 90. If the challenger sticks to its original strategy it obtains 0 in period 89 and 2 in period 90; if it deviates it obtains 1 in each period.)

234.1 Nash equilibria of the centipede game

Consider a strategy pair that results in an outcome in which player 1 stops the game in period $k \geq 2$. (That is, each player chooses C through period $k-1$ and the player who moves in period k chooses S.) Such a pair is not a Nash equilibrium because the player who moves in period $k-1$ can do better (in the whole game, not only the subgame) by choosing S rather than C, given the other player's strategy. Similarly the strategy pair in which each player always chooses C is not a Nash equilibrium. Thus in every Nash equilibrium player 1 chooses S at the start of the game.

8 Coalitional Games and the Core

241.2 *Stag Hunt*

The following coalitional game models the situation.

Players The hunters.

Actions The set of actions of every coalition with $k < n$ members is the set of all profiles $(x_1, \ldots, x_k)$ of nonnegative numbers in which each x_i is an amount of hare and $x_1 + \cdots + x_k = k$. The set of actions of the grand coalition is the union of the set of all profiles $(x_1, \ldots, x_k)$ of nonnegative numbers in which each x_i is an amount of hare and $x_1 + \cdots + x_k = n$, and the set of all profiles $(y_1, \ldots, y_n)$ of nonnegative numbers in which each y_i is an amount of a stag and $y_1 + \cdots + y_n = 1$.

Preferences The preferences of each player are represented by the payoff function $\alpha x + y$, where x is the amount of hares and y the amount of a stag she obtains.

A coalition with $k < n$ members catches k hares and thus can achieve any payoff distribution $(z_1, \ldots, z_k)$ among its members for which $z_1 + \cdots + z_k = \alpha k$. The grand coalition can either catch n hares or a stag. All its members prefer the fraction $1/n$ of a stag to a hare, so every payoff distribution it can achieve by catching hares is dominated by a payoff distribution it can achieve by catching a stag. Thus it can achieve any payoff distribution $(z_1, \ldots, z_n)$ for which $z_1 + \cdots + z_k = 1$.

We conclude that the game has transferable payoff, and the worth function is given by

$$v(S) = \begin{cases} \alpha k & \text{if } S \text{ contains } k < n \text{ members} \\ 1 & \text{if } S \text{ contains } n \text{ members.} \end{cases}$$

243.1 Cohesive games

Landowner–worker game: For any partition $\{S_1, \ldots, S_k\}$ of the players, $v(S_1) + \cdots + v(S_k)$ is at most $f(\ell)$, where ℓ is the number of members of the largest member of the partition, and hence at most $v(N)$.

Three-player majority game: For any partition $\{S_1, \ldots, S_k\}$ of the players we have $v(S_1) + \cdots + v(S_k) \leq 1 = v(N)$.

Stag Hunt: For any partition $\{S_1, \ldots, S_k\}$ of the players we have $v(S_1) + \cdots + v(S_k) = \alpha n < 1 = v(N)$.

Marriage market: The matching of the members of the grand coalition induced by any collection of actions of the coalitions in a partition can be achieved by some action of the grand coalition.

245.1 Three-player majority game

Let (x_1, x_2, x_3) be an action of the grand coalition. Every coalition consisting of two players can obtain one unit of output, so for (x_1, x_2, x_3) to be in the core we need

$$\begin{aligned} x_1 + x_2 &\geq 1 \\ x_1 + x_3 &\geq 1 \\ x_2 + x_3 &\geq 1 \\ x_1 + x_2 + x_3 &= 1. \end{aligned}$$

Adding the first three conditions we conclude that

$$2x_1 + 2x_2 + 2x_3 \geq 3,$$

or $x_1 + x_2 + x_3 \geq \frac{3}{2}$, contradicting the last condition. Thus no action of the grand coalition satisfies all the conditions, so that the core of the game is empty.

245.2 Variant of three-player majority game

A coalition can obtain one unit of output if and only if it contains player 1. (Note that players 2 and 3 together do not have a majority of the votes.) Thus for an action (x_1, x_2, x_3) of the grand coalition to be in the core we need

$$\begin{aligned} x_1 &\geq 1 \\ x_1 + x_2 &\geq 1 \\ x_1 + x_3 &\geq 1 \\ x_1 + x_2 + x_3 &= 1. \end{aligned}$$

The first and last conditions (and the restriction that amounts of output must be nonnegative) imply that $(x_1, x_2, x_3) = (1, 0, 0)$, which satisfies the other two conditions. Thus the core consists of the single action $(1, 0, 0)$ in which player 1 obtains all the output.

245.3 *Stag Hunt*

Let $(x_1, \ldots, x_n)$ be a distribution of payoff achievable by the grand coalition, so that

$$x_1 + \cdots + x_n = 1.$$

For $(x_1, \ldots, x_n)$ to be in the core we need $x_i \geq \alpha$ for every i, because the worth of a coalition containing a single player is α. I claim that if this condition is satisfied, no coalition of more than one player can improve upon $(x_1, \ldots, x_n)$, so that

$(x_1, \ldots, x_n)$ is in the core. The reason is that if $x_i \geq \alpha$ for each player i, then the total payoff under $(x_1, \ldots, x_n)$ of any coalition with $k < n$ members is at least αk, which is the worth of the coalition.

In summary, the core of the game is the set of divisions of the stag in which each player receives at least α.

The coalitional game assumes that any coalition may divide its spoils in any way it wishes, while the strategic game in Section 2.5 restricts a group to split its spoils equally. The solution concept of Nash equilibrium considers only deviations by single players, whereas the concept of the core considers deviations by groups of players, and assumes that any player will participate in a deviation that increases her payoff even if there are other deviations that yield her an even higher payoff.

245.4 Variant of *Stag Hunt*

The coalitional form of the game is given by

$$v(S) = \begin{cases} \alpha k & \text{if } S \text{ contains } k < m \text{ members} \\ 1 & \text{if } S \text{ contains at least } m \text{ members.} \end{cases}$$

The core is empty: for an allocation to be in the core, the total payoff of the members of each coalition of m players must be at least 1, which is incompatible with the requirement that the total payoff of all n players is 1. That is, the conflict over how to share the stag precludes a stable outcome. (Cf. the argument that the core of the three-player majority game is empty, in Exercise 245.1.)

246.1 Market with one owner and two heterogeneous buyers

By the arguments in Example 245.5, in any action in the core the owner does not keep the good, the buyer who obtains the good pays at most her valuation, and the other buyer makes no payment. Let a_N be an action of the grand coalition in which buyer 2 obtains the good and pays the owner p, and buyer 1 makes no payment. Then $p \leq v < 1$, so that the coalition consisting of the owner and buyer 1 can improve upon a_N: if the owner transfers the good to buyer 1 in exchange for $\frac{1}{2}(1 + p)$ units of money, both the owner and buyer 1 are better off than they are in a_N. Thus in any action in the core, buyer 1 obtains the good. The price she pays is at least v (otherwise the coalition consisting of the owner and buyer 2 can improve upon the action). No coalition can improve upon any action in which buyer 1 obtains the good and pays the owner at least v and at most 1 (and buyer 2 makes no payment), so the core consists of all such actions.

247.1 Vote trading

a. The core consists of the single action in which all three bills pass, yielding each legislator a payoff of 2. This action cannot be improved upon by any coalition because no single bill or pair of bills gives every member of any majority coalition a payoff of more than 2.

No other action is in the core, by the following argument.

- The action in which no bill passes (so that each legislator's payoff is 0) can be improved upon by the coalition of all three legislators, which by passing all three bills raises the payoff of each legislator to 2.
- The action in which only A passes can be improved upon by the coalition of legislators 2 and 3, who by passing bills A and B raise both of their payoffs.
- Similarly the action in which only B passes can be improved upon by the coalition of legislators 1 and 3, and the action in which only C passes can be improved upon by the coalition of legislators 1 and 2.
- The action in which bills A and B pass can be improved upon by the coalition of legislators 1 and 3, who by passing all three bills raise both their payoffs.
- Similarly the action in which bills A and C pass can be improved upon by the coalition of legislators 2 and 3, and the action in which bills B and C pass can be improved upon by the coalition of legislators 1 and 2.

b. The core consists of two actions: all three bills pass, and bills A and B pass. As in part *a*, the action in which all three bills pass cannot be improved upon by any coalition. The action in which bills A and B cannot be improved upon either: for no other set of bills are at least two legislators better off.

No other action is in the core, by the following argument.

- The action in which A passes can be improved upon by the coalition consisting of legislators 2 and 3, who can pass B instead.
- The action in which B passes can be improved upon by the coalition consisting of legislators 1 and 2, who can pass A and B instead.
- The action in which C passes can be improved upon by the coalition consisting of legislators 2 and 3, who can pass B instead.
- The action in which A and C pass can be improved upon by the coalition consisting of legislators 2 and 3, who can pass A and B instead.
- The action in which B and C pass can be improved upon by the coalition consisting of legislators 1 and 2, who can pass A and B instead.

c. The core is empty.

- The action in which no bill passes can be improved upon by the coalition consisting of legislators 1 and 2, who can pass A and B instead.

- The action in which any single bill passes can be improved upon by the coalition consisting of the two legislators whose payoffs are -1 if this bill passes; this coalition can do better by passing the other two bills.
- The action in which bills A and B pass can be improved upon by the coalition consisting of legislators 2 and 3, who can pass B instead.
- Similarly the action in which A and C pass can be improved upon by the coalition consisting of legislators 1 and 2, who can pass A instead, and the action in which B and C pass can be improved upon by the coalition consisting of legislators 1 and 2, who can pass B instead.
- The action in which all three bills pass can be improved upon by the coalition consisting of legislators 1 and 2, who can pass A and B instead.

248.1 Core of landowner–worker game

Let a_N be an action of the grand coalition in which the output received by each worker is at most $f(n) - f(n-1)$. No coalition consisting solely of workers can obtain any output, so no such coalition can improve upon a_N. Let S be a coalition of the landowner and $k-1$ workers. The total output received by the members of S in a_N is at least

$$f(n) - (n-k)(f(n) - f(n-1))$$

(because the total output is $f(n)$, and every *other* worker receives at most $f(n) - f(n-1)$). Now, the output that S can obtain is $f(k)$, so for S to improve upon a_N we need

$$f(k) > f(n) - (n-k)(f(n) - f(n-1)),$$

which contradicts the inequality given in the exercise.

249.1 Unionized workers in landowner–worker game

The following game models the situation.

Players The landowner and the workers.

Actions The set of actions of the grand coalition is the set of all allocations of the output $f(n)$. Every other coalition has a single action, which yields the output 0.

Preferences Each player's preferences are represented by the amount of output she obtains.

The core of this game consists of every allocation of the output $f(n)$ among the players. The grand coalition cannot improve upon any allocation x because for every other allocation x' there is at least one player whose payoff is lower in x' than it is in x. No other coalition can improve upon any allocation because no other coalition can obtain any output.

249.2 Landowner–worker game with increasing marginal products

We need to show that no coalition can improve upon the action a_N of the grand coalition in which every player receives the output $f(n)/n$. No coalition of workers can obtain any output, so we need to consider only coalitions containing the landowner. Consider a coalition consisting of the landowner and k workers, which can obtain $f(k+1)$ units of output by itself. Under a_N this coalition obtains the output $(k+1)f(n)/n$, and we have $f(k+1)/(k+1) < f(n)/n$ because $k < n$. Thus no coalition can improve upon a_N.

254.1 Range of prices in horse market

The equality of the number of owners who sell their horses and the number of nonowners who buy horses implies that the common trading price p^*

- is not less than σ_{k^*}, otherwise at most $k^* - 1$ owners' valuations would be less than p^* and at least k^* nonowners' valuations would be greater than p^*, so that the number of buyers would exceed the number of sellers
- is not less than β_{k^*+1}, otherwise at most k^* owners' valuations would be less than p^* and at least $k^* + 1$ nonowners' valuations would be greater than p^*, so that the number of buyers would exceed the number of sellers
- is not greater than β_{k^*}, otherwise at least k^* owners' valuations would be less than p^* and at most $k^* - 1$ nonowners' valuations would be greater than p^*, so that the number of sellers would exceed the number of buyers
- is not greater than σ_{k^*+1}, otherwise at least $k^* + 1$ owners' valuations would be less than p^* and at most k^* nonowners' valuations would be greater than p^*, so that the number of sellers would exceed the number of buyers.

That is, $p^* \geq \max\{\sigma_{k^*}, \beta_{k^*+1}\}$ and $p^* \leq \min\{\beta_{k^*}, \sigma_{k^*+1}\}$.

256.1 Horse trading game with single seller

The core consists of the set of actions of the grand coalition in which the owner sells her horse to the nonowner with the highest valuation (nonowner 1) at a price p^* for which $\max\{\beta_2, \sigma_1\} \leq p^* \leq \beta_1$. (The coalition consisting of the owner and nonwoner 2 can improve any action in which the price is less than β_2, the owner alone can improve upon any action in which the price is less than σ_1, and nonowner 1 alone can improve upon any action in which the price is greater than β_1.)

256.2 Horse trading game with large seller

In every action in the core, the owner sells one horse to buyer 1 and one horse to buyer 2. The prices at which the trades occur are not necessarily the same. The price p_1 paid by buyer 1 satisfies $\max\{\beta_3, \sigma_1\} \leq p_1 \leq \beta_1$ and the price p_2 paid by buyer 2 satisfies $\max\{\beta_3, \sigma_1\} \leq p_1 \leq \beta_2$.

258.1 House assignment with identical preferences

Because the players rank the houses in the same way, we can refer to the "best house", the "second best house", and so on. In any assignment in the core, the player who owns the best house is assigned this house (because she has the option of keeping it). Among the remaining players, the one who owns the second best house must be assigned this house (again, because she has the option of keeping it). Continuing to argue in the same way, we see that there is a single assignment in the core, in which every player is assigned the house she owns initially.

260.1 Emptiness of the strong core when preferences are not strict

Of the six possible assignments, $h_1h_2h_3$ (i.e. every player keeps the house she owns) and $h_3h_2h_1$ can both be improved upon by $\{1,2\}$ (and by $\{2,3\}$). All four of the other assignments are in the core.

None of the assignments in the core is in the strong core. The assignments $h_1h_3h_2$ and $h_3h_1h_2$ can both be weakly improved upon by $\{1,2\}$, and $h_2h_1h_3$ and $h_2h_3h_1$ can both be weakly improved upon by $\{2,3\}$.

261.1 Median voter theorem

Denote the median favorite position by m. If $x < m$ then every player whose favorite position is m or greater—a majority of the players—prefers m to x. Similarly, if $x > m$ then every player whose favorite position is m or less—a majority of the players—prefers m to x.

262.1 Cores of q-rule games

a. Denote the favorite policy of player i by x_i^* and number the players so that $x_1^* \leq \cdots \leq x_n^*$. The q-core is the set of all policies x for which

$$x_{n-q+1}^* \leq x \leq x_q^*.$$

Any such policy x is in the core because every coalition of q players contains at least one player whose favorite position is less than x and at least one player whose favorite position is greater than x, so that there is no position $y \neq x$ that all members of the coalition prefer to x.

Any policy $x < x_{n-q+1}^*$ is not in the core because the coalition of players $n-q+1$ through n can improve upon x: this coalition contains q players, all of whom prefer x_{n-q+1}^* to x. Similarly, no policy greater than x_q^* is in the core.

b. The core is the set of policies in the triangle defined by x_1^*, x_2^*, and x_3^*.

Every policy x in this set is in the core because for every other policy $y \neq x$ at least one player is worse off than she is at x.

No policy x outside the set is in the core because the policy $y \neq x$ closest to x in the set is preferred by all three players.

265.1 Deferred acceptance procedure with proposals by Y's

For the preferences given in Figure 263.1, the progress of the procedure when proposals are made by Y's is given in Figure 136.1. The matching produced is the same as that produced by the procedure when proposals are made by X's, namely (x_1, y_1), (x_2, y_2), x_3 (alone), and y_3 (alone).

	Stage 1		Stage 2		Stage 3	
y_1:	$\rightarrow x_1$					
y_2:	$\rightarrow x_2$					
y_3:	$\rightarrow x_1$	reject	$\rightarrow x_3$	reject	$\rightarrow x_2$	reject

Figure 136.1 The progress of the deferred acceptance procedure with proposals by Y's when the players' preferences are those given in Figure 263.1. Each row gives the proposals of one X.

266.1 Example of deferred acceptance procedure

For the preferences in Figure 266.1, the procedure when proposals are made by X's yields the matching (x_1, y_1), (x_2, y_2), (x_3, y_3); the procedure when proposals are made by Y's yields the matching (x_1, y_1), (x_2, y_3), (x_3, y_2).

In any matching in the core, x_1 and y_1 are matched, because each is the other's top-ranked partner. Thus the only two possible matchings are those generated by the two procedures. Player x_2 prefers y_2 to y_3 and player x_3 prefers y_3 to y_2, so the matching generated by the procedure when proposals are made by X's yields each X a better partner than does the matching generated by the procedure when proposals are made by Y's. Similarly, player y_2 prefers x_3 to x_2 and player y_3 prefers x_2 to x_3, so the matching generated by the procedure when proposals are made by Y's yields each Y a better partner than does the matching generated by the procedure when proposals are made by X's.

267.1 Strategic behavior under the deferred acceptance procedure

The matching produced by the deferred acceptance procedure with proposals by X's is (x_1, y_2), (x_2, y_3), (x_3, y_1). The matching produced by the deferred acceptance procedure with proposals by Y's is (x_1, y_1), (x_2, y_3), (x_3, y_2). Of the four other matchings, the coalition $\{x_3, y_2\}$ can improve upon (x_1, y_1), (x_2, y_2), (x_3, y_3)

and (x_1, y_2), (x_2, y_1), (x_3, y_3), and the coalition $\{x_1, y_1\}$ can improve upon (x_1, y_3), (x_2, y_1), (x_3, y_2) and (x_1, y_3), (x_2, y_2), (x_3, y_1). Thus the core consists of the two matchings produced by the deferred acceptance procedures.

If y_1 names the ranking (x_1, x_2, x_3) and every other player names her true ranking, the deferred acceptance procedure with proposals by X's yields the matching (x_1, y_1), (x_2, y_3), (x_3, y_2), as illustrated in Figure 137.1. Players y_1 and y_2 are matched with their favorite partners, so cannot profitably deviate by submitting any other ranking. Player y_3's ranking does not affect the outcome of the procedure. Thus, given that submitting her true ranking is a dominant strategy for every X, the game thus has a Nash equilibrium in which player y_1 submits the ranking (x_1, x_2, x_3) and every other player submits her true ranking.

	Stage 1		Stage 2		Stage 3		Stage 4
x_1:	$\rightarrow y_2$			reject	$\rightarrow y_1$		
x_2:	$\rightarrow y_1$					reject	$\rightarrow y_3$
x_3:	$\rightarrow y_1$	reject	$\rightarrow y_2$				

Figure 137.1 The progress of the deferred acceptance procedure with proposals by X's when the players' preferences differ from those in Exercise 267.1 only in that y_1's ranking is (x_1, x_2, x_3). Each row gives the proposals of one X.

267.2 Empty core in roommate problem

Notice that ℓ is at the bottom of each of the other players' preferences. Suppose that she is matched with i. Then j and k are matched, and $\{i, k\}$ can improve upon the matching. Similarly, if ℓ is matched with j then $\{i, j\}$ can improve upon the matching, and if ℓ is matched with k then $\{j, k\}$ can improve upon the matching. Thus the core is empty (ℓ has to be matched with *someone*!).

267.3 Spatial preferences in roommate problem

The core consists of the single matching μ^* defined as follows. First match the pair of players whose characteristics are closest. Then match the pair of players in the remaining set whose characteristics are closest. Continue until all players are matched.

Number the matches in the order they are made according to this procedure. If a coalition can improve upon μ^*, then a coalition consisting of two players can do so. Now, neither member of match k is better off being matched with a member of match ℓ for any $\ell > k$, so no two-player coalition can improve upon the matching. Thus μ^* is in the core.

For any other matching μ', at least one of the members of some match k defined by the procedure is matched with a different partner. If she is matched with a member of some match $\ell < k$ then the coalition consisting of the two members of match ℓ can improve μ'; if she is matched with a member of some match $\ell > k$ then the coalition consisting of the two member of match k can improve upon μ'. Thus no matching $\mu' \neq \mu^*$ is in the core.

9 Bayesian Games

276.1 Equilibria of a variant of *BoS* with imperfect information

If player 1 chooses S then type 1 of player 2 chooses S and type 2 chooses B. But if the two types of player 2 make these choices then player 1 is better off choosing B (which yields her an expected payoff of 1) than choosing S (which yields her an expected payoff of $\frac{1}{2}$). Thus there is no Nash equilibrium in which player 1 chooses S.

Now consider the mixed strategy Nash equilibria. If both types of player 2 use a pure strategy then player 1's two actions yield her different payoffs. Thus there is no equilibrium in which both types of player 2 use pure strategies and player 1 randomizes.

Now consider an equilibrium in which type 1 of player 2 randomizes. Denote by p the probability that player 1's mixed strategy assigns to B. In order for type 1 of player 2 to obtain the same expected payoff to B and S we need $p = \frac{2}{3}$. For this value of p the best action of type 2 of player 2 is S. Denote by q the probability that type 1 of player 2 assigns to B. Given these strategies for the two types of player 2, player 1's expected payoff if she chooses B is

$$\tfrac{1}{2} \cdot 2q = q$$

and her expected payoff if she chooses S is

$$\tfrac{1}{2} \cdot (1 - q) + \tfrac{1}{2} \cdot 1 = 1 - \tfrac{1}{2}q.$$

These expected payoffs are equal if and only if $q = \frac{2}{3}$. Thus the game has a mixed strategy equilibrium in which the mixed strategy of player 1 is $(\frac{2}{3}, \frac{1}{3})$, that of type 1 of player 2 is $(\frac{2}{3}, \frac{1}{3})$, and that of type 2 of player 2 is $(0, 1)$ (that is, type 2 of player 2 uses the pure strategy that assigns probability 1 to S).

Similarly the game has a mixed strategy equilibrium in which the strategy of player 1 is $(\frac{1}{3}, \frac{2}{3})$, that of type 1 of player 2 is $(0, 1)$, and that of type 2 of player 2 is $(\frac{2}{3}, \frac{1}{3})$.

For no mixed strategy of player 1 are both types of player 2 indifferent between their two actions, so there is no equilibrium in which both types randomize.

277.1 Expected payoffs in a variant of *BoS* with imperfect information

The expected payoffs are given in Figure 140.1.

	(B,B)	(B,S)	(S,B)	(S,S)
B	0	1	1	2
S	1	$\frac{1}{2}$	$\frac{1}{2}$	0

Type n_1 of player 1

	(B,B)	(B,S)	(S,B)	(S,S)
B	1	$\frac{2}{3}$	$\frac{1}{3}$	0
S	0	$\frac{2}{3}$	$\frac{4}{3}$	2

Type y_2 of player 2

	(B,B)	(B,S)	(S,B)	(S,S)
B	0	$\frac{1}{3}$	$\frac{2}{3}$	1
S	2	$\frac{4}{3}$	$\frac{2}{3}$	0

Type n_2 of player 2

Figure 140.1 The expected payoffs of type n_1 of player 1 and types y_2 and n_2 of player 2 in Example 276.2.

282.1 Fighting an opponent of unknown strength

The following Bayesian game models the situation.

Players The two people.

States The set of states is $\{strong, weak\}$.

Actions The set of actions of each player is $\{fight, yield\}$.

Signals Player 1 receives the same signal in each state, whereas player 2 receives different signals in the two states.

Beliefs The single type of player 1 assigns probability α to the state *strong* and probability $1-\alpha$ to the state *weak*. Each type of player 2 assigns probability 1 to the single state consistent with her signal.

Payoffs The players' Bernoulli payoffs are shown in Figure 140.2.

	F	Y
F	$-1,\ 1^*$	$1,\ 0$
Y	$0,\ 1^*$	$0,\ 0$

State: *strong*

	F	Y
F	$1,-1$	$1,\ 0^*$
Y	$0,\ 1^*$	$0,\ 0$

State: *weak*

Figure 140.2 The player's Bernoulli payoff functions in Exercise 282.1. The asterisks indicate the best responses of each type of player 2.

The best responses of each type of player 2 are indicated by asterisks in Figure 140.2. Thus if $\alpha < \frac{1}{2}$ then player 1's best action is *fight*, whereas if $\alpha > \frac{1}{2}$ her best action is *yield*. Thus for $\alpha < \frac{1}{2}$ the game has a unique Nash equilibrium, in which player 1 chooses *fight* and player 2 chooses *fight* if she is strong and *yield* if she is weak, and if $\alpha > \frac{1}{2}$ the game has a unique Nash equilibrium, in which player 1 chooses *yield* and player 2 chooses *fight* whether she is strong or weak.

282.2 An exchange game

The following Bayesian game models the situation.

Players The two individuals.

States The set of all pairs (s_1, s_2), where s_i is the number on player i's ticket (an integer from 1 to m).

Actions The set of actions of each player is $\{Exchange, Don't\ exchange\}$.

Signals The signal function of each player i is defined by $\tau_i(s_1, s_2) = s_i$ (each player observes her own ticket, but not that of the other player)

Beliefs Type s_i of player i assigns the probability $\Pr_j(s_j)$ to the state (s_1, s_2), where j is the other player and $\Pr_j(s_j)$ is the probability with which player j receives a ticket with the prize s_j on it.

Payoffs Player i's Bernoulli payoff function is given by $u_i((X,Y),\omega) = \omega_j$ if $X = Y = Exchange$ and $u_i((X,Y),\omega) = \omega_i$ otherwise.

Let M_i be the highest type of player i that chooses *Exchange*. If $M_i > 1$ then type 1 of player j optimally chooses *Exchange*: by exchanging her ticket, she cannot obtain a smaller prize, and may receive a bigger one. Thus if $M_i \geq M_j$ and $M_i > 1$, type M_i of player i optimally chooses *Don't exchange*, because the expected value of the prizes of the types of player j that choose *Exchange* is less than M_i. Thus in any possible Nash equilibrium $M_i = M_j = 1$: the only prizes that may be exchanged are the smallest.

282.3 Adverse selection

The game is defined as follows.

Players Firms A and T.

States The set of possible values of firm T (the integers from 0 to 100).

Actions Firm A's set of actions is its set of possible bids (nonnegative numbers), and firm T's set of actions is the set of possible cutoffs (nonnegative numbers) above which it will accept A's offer.

Signals Firm A receives the same signal in every state; firm T receives a different signal in every state.

Beliefs The single type of firm A assigns an equal probability to each state; each type of firm T assigns probability 1 to the single state consistent with its signal.

Payoff functions If firm A bids y, firm T's cutoff is at most y, and the state is x, then A's payoff is $\frac{3}{2}x - y$ and T's payoff is y. If firm A bids y, firm T's cutoff is greater than y, and the state is x, then A's payoff is 0 and T's payoff is x.

To find the Nash equilibria of this game, first consider the behavior of each type x of firm T. Type x is at least as well off accepting the offer y than it is rejecting it if and only if $y \geq x$. Thus any best response of type x to an offer y has a cutoff of at most y if $y > x$ and a cutoff of greater than y if $y < x$.

Now consider firm A. If it bids y then each type x of T with $x < y$ accepts its offer, and each type x of T with $x > y$ rejects the offer. Thus the expected value of the types that accept an offer $y \leq 100$ is $\frac{1}{2}q(y)$, where $q(y)$ is the largest integer at most equal to y, and the expected value of the types that accept an offer $y > 100$ is 50. If the offer y is accepted then A's payoff is $\frac{3}{2}x - y$, so that its expected payoff is $\frac{3}{2}(\frac{1}{2}q(y)) - y$ if $y \leq 100$ and $\frac{3}{2}(50) - y = 75 - y$ if $y > 100$. In both cases this expected payoff is negative. (In the first case it is approximately $\frac{1}{4}y$.) Thus firm A's optimal bid is 0!

We conclude that a strategy pair is a Nash equilibrium of the game if and only if firm A bids 0 and the cutoff for accepting an offer for each type x of firm T is greater than 0 if $x > 0$ and at least 0 if $x = 0$.

Even though firm A can increase firm T's value, it is not willing to make a positive bid in equilibrium because firm T's interest is in accepting only offers that exceed its value, so that the average type that accepts an offer has a value of only half the offer. As A decreases its offer, the value of the average firm that accepts the offer decreases: the selection of firms that accept the offer is adverse to A's interest.

284.1 Infection argument

In any Nash equilibrium, the action of player 1 when she receives the signal $\tau_1(\alpha)$ is R, because R strictly dominates L.

Now suppose that player 2's signal is $\tau_2(\alpha) = \tau_2(\beta)$. I claim that her best action is R, regardless of player 1's action in state β. If player 1 chooses L in state β then player 2's expected payoff to L is $\frac{3}{4} \cdot 0 + \frac{1}{4} \cdot 2 = \frac{1}{2}$, and her expected payoff to R is $\frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 0 = \frac{3}{4}$. If player 1 chooses R in state β then player 2's expected payoff to L is 0, and her expected payoff to R is 1. Thus in any Nash equilibrium player 2's action when her signal is $\tau_2(\alpha) = \tau_2(\beta)$ is R.

Now suppose that player 1's signal is $\tau_1(\beta) = \tau_1(\gamma)$. By the same argument as in the previous paragraph, player 1's best action is R, regardless of player 2's action in state γ. Thus in any Nash equilibrium player 1's action in this case is R.

Finally, given that player 1's action in state γ is R, player 2's best action in this state is also R.

287.1 Cournot's duopoly game with imperfect information

We have

$$b_1(q_L, q_H) = \begin{cases} \frac{1}{2}(\alpha - c - (\theta q_L + (1-\theta)q_H)) & \text{if } \theta q_L + (1-\theta)q_H \leq \alpha - c \\ 0 & \text{otherwise.} \end{cases}$$

The best response function of each type of player 2 is similar:

$$b_I(q_1) = \begin{cases} \frac{1}{2}(\alpha - c_I - q_1) & \text{if } q_1 \leq \alpha - c_I \\ 0 & \text{otherwise} \end{cases}$$

for $I = L, H$.

The three equations that define a Nash equilibrium are

$$q_1^* = b_1(q_L^*, q_H^*),\ q_L^* = b_L(q_1^*), \text{ and } q_H^* = b_H(q_1^*).$$

Solving these equations under the assumption that they have a solution in which all three outputs are positive, we obtain

$$\begin{aligned} q_1^* &= \tfrac{1}{3}(\alpha - 2c + \theta c_L + (1-\theta)c_H) \\ q_L^* &= \tfrac{1}{3}(\alpha - 2c_L + c) - \tfrac{1}{6}(1-\theta)(c_H - c_L) \\ q_H^* &= \tfrac{1}{3}(\alpha - 2c_H + c) + \tfrac{1}{6}\theta(c_H - c_L) \end{aligned}$$

If both firms know that the unit costs of the two firms are c_1 and c_2 then in a Nash equilibrium the output of firm i is $\frac{1}{3}(\alpha - 2c_i + c_j)$ (see Exercise 58.1). In the case of imperfect information considered here, firm 2's output is less than $\frac{1}{3}(\alpha - 2c_L + c)$ if its cost is c_L and is greater than $\frac{1}{3}(\alpha - 2c_H + c)$ if its cost is c_H. Intuitively, the reason is as follows. If firm 1 knew that firm 2's cost were high then it would produce a relatively large output; if it knew this cost were low then it would produce a relatively small output. Given that it does not know whether the cost is high or low it produces a moderate output, less than it would if it knew firm 2's cost were high. Thus if firm 2's cost is in fact high, firm 2 benefits from firm 1's lack of knowledge and optimally produces more than it would if firm 1 knew its cost.

288.1 Cournot's duopoly game with imperfect information

The best response $b_0(q_L, q_H)$ of type 0 of firm 1 is the solution of

$$\max_{q_0}[\theta(P(q_0 + q_L) - c)q_0 + (1-\theta)(P(q_0 + q_H) - c)q_0].$$

The best response $b_\ell(q_L, q_H)$ of type L of firm 1 is the solution of

$$\max_{q_\ell}(P(q_\ell + q_L) - c)q_\ell$$

and the best response $b_h(q_L, q_H)$ of type H of firm 1 is the solution of

$$\max_{q_h}(P(q_h + q_H) - c)q_h.$$

The best response $b_L(q_0, q_\ell, q_h)$ of type L of firm 2 is the solution of

$$\max_{q_L}[(1-\pi)(P(q_0 + q_L) - c_L)q_L + \pi(P(q_\ell + q_L) - c_L)q_L]$$

and the best response $b_H(q_0, q_\ell, q_h)$ of type H of firm 2 is the solution of

$$\max_{q_H}[(1-\pi)(P(q_0 + q_H) - c_H)q_H + \pi(P(q_h + q_H) - c_H)q_H].$$

A Nash equilibrium is a profile $(q_0^*, q_\ell^*, q_h^*, q_L^*, q_H^*)$ for which q_0^*, q_ℓ^*, and q_h^* are best responses to q_L^* and q_H^*, and q_L^* and q_H^* are best responses to q_0^*, q_ℓ^*, and q_h^*. When $P(Q) = \alpha - Q$ for $Q \leq \alpha$ and $P(Q) = 0$ for $Q > \alpha$ we find, after some exciting algebra, that

$$\begin{aligned}
q_0^* &= \frac{1}{3}\left(\alpha - 2c + c_H - \theta\left(c_H - c_L\right)\right) \\
q_\ell^* &= \frac{1}{3}\left(\alpha - 2c + c_L + \frac{(1-\theta)(1-\pi)(c_H - c_L)}{4-\pi}\right) \\
q_h^* &= \frac{1}{3}\left(\alpha - 2c + c_H - \frac{\theta(1-\pi)(c_H - c_L)}{4-\pi}\right) \\
q_L^* &= \frac{1}{3}\left(\alpha - 2c_L + c - \frac{2(1-\theta)(1-\pi)(c_H - c_L)}{4-\pi}\right) \\
q_H^* &= \frac{1}{3}\left(\alpha - 2c_H + c + \frac{2\theta(1-\pi)(c_H - c_L)}{4-\pi}\right).
\end{aligned}$$

When $\pi = 0$ we have

$$\begin{aligned}
q_0^* &= \frac{1}{3}\left(\alpha - 2c + c_H - \theta\left(c_H - c_L\right)\right) \\
q_\ell^* &= \frac{1}{3}\left(\alpha - 2c + c_L + \frac{(1-\theta)(c_H - c_L)}{4}\right) \\
q_h^* &= \frac{1}{3}\left(\alpha - 2c + c_H - \frac{\theta(c_H - c_L)}{4}\right) \\
q_L^* &= \frac{1}{3}\left(\alpha - 2c_L + c - \frac{(1-\theta)(c_H - c_L)}{2}\right) \\
q_H^* &= \frac{1}{3}\left(\alpha - 2c_H + c + \frac{\theta(c_H - c_L)}{2}\right),
\end{aligned}$$

so that q_0^* is equal to the equilibrium output of firm 1 in Exercise 287.1, and q_L^* and q_H^* are the same as the equilibrium outputs of the two types of firm 2 in that exercise.

When $\pi = 1$ we have

$$\begin{aligned}
q_0^* &= \frac{1}{3}\left(\alpha - 2c + c_H - \theta\left(c_H - c_L\right)\right) \\
q_\ell^* &= \frac{1}{3}\left(\alpha - 2c + c_L\right) \\
q_h^* &= \frac{1}{3}\left(\alpha - 2c + c_H\right) \\
q_L^* &= \frac{1}{3}\left(\alpha - 2c_L + c\right) \\
q_H^* &= \frac{1}{3}\left(\alpha - 2c_H + c\right),
\end{aligned}$$

so that q_ℓ^* and q_L^* are the same as the equilibrium outputs when there is perfect information and the costs are c and c_L (see Exercise 58.1), and q_h^* and q_H^* are the same as the equilibrium outputs when there is perfect information and the costs are c and c_H.

Now, for an arbitrary value of π we have

$$\begin{aligned}
q_L^* &= \frac{1}{3}\left(\alpha - 2c_L + c - \frac{2(1-\theta)(1-\pi)(c_H - c_L)}{4-\pi}\right) \\
q_H^* &= \frac{1}{3}\left(\alpha - 2c_H + c + \frac{2\theta(1-\pi)(c_H - c_L)}{4-\pi}\right).
\end{aligned}$$

To show that for $0 < \pi < 1$ the values of these variables lie between their values when $\pi = 0$ and when $\pi = 1$, we need to show that

$$0 \leq \frac{2(1-\theta)(1-\pi)(c_H - c_L)}{4-\pi} \leq \frac{(1-\theta)(c_L - c_H)}{2}$$

and

$$0 \leq \frac{2\theta(1-\pi)(c_H - c_L)}{4-\pi} \leq \frac{\theta(c_L - c_H)}{2}.$$

These inequalities follow from $c_H \geq c_L$, $\theta \geq 0$, and $0 \leq \pi \leq 1$.

290.1 Nash equilibria of game of contributing to a public good

Any type v_j of any player j with $v_j < c$ obtains a negative payoff if she contributes and 0 if she does not. Thus she optimally does not contribute.

Any type $v_i \geq c$ of player i obtains the payoff $v_i - c \geq 0$ if she contributes, and the payoff 0 if she does not, so she optimally contributes.

Any type $v_j \geq c$ of any player $j \neq i$ obtains the payoff $v_j - c$ if she contributes, and the payoff $(1 - F(c))v_j$ if she does not. (If she does not contribute, the probability that player i does so is $1 - F(c)$, the probability that player i's valuation is at least c.) Thus she optimally does not contribute if $(1 - F(c))v_j \geq v_j - c$, or $F(c) \leq c/v_j$. This condition must hold for all types of every player $j \neq i$, so we need $F(c) \leq c/\overline{v}$ for the strategy profile to be a Nash equilibrium.

291.1 Reporting a crime with an unknown number of witnesses

The following Bayesian game models the situation.

Players The two potential witnesses.

States 1 (player 1 is the only witness), 2 (player 2 is the only witness), and 12 (both players are witnesses).

Actions Each player's set of actions is $\{Call, Don't\ call\}$.

Signals Each player receives one of the signals *witness* or *not witness*. Player 1's signal function τ_1 satisfies $\tau_1(1) = \tau_1(12) = witness$ and $\tau_1(2) = not\ witness$; player 2's signal function τ_2 satisfies $\tau_2(2) = \tau_2(12) = witness$ and $\tau_2(1) = not\ witness$.

Beliefs For $i = 1, 2$, when player i receives the signal *witness* she assigns probability π to the state i and probability $1 - \pi$ to the state 12, and when she receives the signal *not witness* she assigns probability 1 to the state j (where j is the other player).

Preferences In state 12, each player's payoff to an action pair in which at least one player calls is $v - c$ if she calls and v if she does not call; her payoff to the action pair in which neither player calls is 0. In the other states, the payoff of the player who is the witness is $v - c$ if she calls and 0 if she does not; the payoff of the player who is not the witness is v if the witness calls and 0 if the witness does not call.

Note that the concept of a Bayesian game requires us to specify actions for each player independent of the state, so that in this game each player has actions even in the state in which she is not a witness. The payoffs of a player in a state in which she is not a witness reflect the fact that the action *Call* in that case has no effect on the outcome.

This game is shown in Figure 146.1.

Player 1: π (State 1), $1-\pi$ (State 12); Player 1: State 2. Player 2: State 1; Player 2: $1-\pi$ (State 12), π (State 2).

State 1

	C	N
C	$v-c, v$	$v-c, v$
N	$0, 0$	$0, 0$

State 12

	C	N
C	$v-c, v-c$	$v-c, v$
N	$v, v-c$	$0, 0$

State 2

	C	N
C	$v, v-c$	$0, 0$
N	$v, v-c$	$0, 0$

Figure 146.1 A Bayesian game that models the situation in Exercise 291.1. The action *Call* is denoted C, and the action *Don't call* is denoted N. In state 1, only player 1 is a witness, in state 2, only player 2 is a witness, and in state 12, both players are witnesses.

A player obtains the payoff $v - c$ if she chooses C and the payoff $(1 - \pi)v$ if she chooses N. Thus the game has a pure strategy Nash equilibrium in which each player chooses C in the state in which she is active if and only if $v - c \geq (1-\pi)v$, or $\pi \geq c/v$. (The action each player chooses in the state in which she is inactive is irrelevant.)

For a mixed strategy Nash equilibrium in which each player chooses C (if she is active) with probability p, where $0 < p < 1$, we need each player's expected payoffs to C and N to be the same, given that the other player chooses C with probability p. Thus we need $v - c = (1-\pi)pv$, or

$$p = \frac{v-c}{(1-\pi)v}.$$

If $\pi < c/v$, this number is less than 1, so that the game indeed has a mixed strategy Nash equilibrium in which each player calls with probability p.

When $\pi = 0$ we have $p = 1 - c/v$, as found in Section 4.8.

294.1 Weak domination in second-price sealed-bid action

Fix player i, and choose a bid for every type of every other player. Player i, who does not know the other players' types, is uncertain of the highest bid of the other players. Denote by $\overline{b}$ this highest bid. Consider a bid b_i of type v_i of player i for which $b_i < v_i$. The dependence of the payoff of type v_i of player i on $\overline{b}$ is shown in Figure 147.1.

		Highest of other players' bids			
		$\overline{b} < b_i$	$b_i = \overline{b}$ (m-way tie)	$b_i < \overline{b} < v_i$	$\overline{b} \geq v_i$
i's bid	$b_i < v_i$	$v_i - \overline{b}$	$(v_i - \overline{b})/m$	0	0
	v_i	$v_i - \overline{b}$	$v_i - \overline{b}$	$v_i - \overline{b}$	0

Figure 147.1 Player i's payoffs to her bids $b_i < v_i$ and v_i in a second-price sealed-bid auction as a function of the highest of the other player's bids, denoted $\overline{b}$.

Player i's expected payoffs to the bids b_i and v_i are weighted averages of the payoffs in the columns; each value of $\overline{b}$ gets the same weight when calculating the expected payoff to b_i as it does when calculating the expected payoff to v_i. The payoffs in the two rows are the same except when $b_i \leq \overline{b} < v_i$, in which case v_i yields a payoff higher than does b_i. Thus the expected payoff to v_i is at least as high as the expected payoff to b_i, and is greater than the expected payoff to b_i unless the other players' bids lead this range of values of $\overline{b}$ to get probability 0.

Now consider a bid b_i of type v_i of player i for which $b_i > v_i$. The dependence of the payoff of type v_i of player i on $\overline{b}$ is shown in Figure 148.1.

As before, player i's expected payoffs to the bids b_i and v_i are weighted averages of the payoffs in the columns; each value of $\overline{b}$ gets the same weight when calculating the expected payoff to v_i as it does when calculating the expected payoff to b_i. The payoffs in the two rows are the same except when $v_i < \overline{b} \leq b_i$, in

		Highest of other players' bids			
		$\overline{b} \le v_i$	$v_i < \overline{b} < b_i$	$b_i = \overline{b}$ (m-way tie)	$\overline{b} > b_i$
i's bid	v_i	$v_i - \overline{b}$	0	0	0
	$b_i > v_i$	$v_i - \overline{b}$	$v_i - \overline{b}$	$(v_i - \overline{b})/m$	0

Figure 148.1 Player i's payoffs to her bids v_i and $b_i > v_i$ in a second-price sealed-bid auction as a function of the highest of the other player's bids, denoted $\overline{b}$.

which case v_i yields a payoff higher than does b_i. (Note that $v_i - \overline{b} < 0$ for $\overline{b}$ in this range.) Thus the expected payoff to v_i is at least as high as the expected payoff to b_i, and is greater than the expected payoff to b_i unless the other players' bids lead this range of values of $\overline{b}$ to get probability 0.

We conclude that for type v_i of player i, every bid $b_i \neq v_i$ is weakly dominated by the bid v_i.

294.2 Nash equilibria of a second-price sealed-bid auction

For any player i, the game has a Nash equilibrium in which player i bids $\overline{v}$ (the highest possible valuation) regardless of her valuation and every other player bids $\underline{v}$ regardless of her valuation. The outcome is that player i wins and pays $\underline{v}$. Player i can do no better by bidding less; no other player can do better by bidding more, because unless she bids at least $\overline{v}$ she does not win, and if she makes such a bid her payoff is at best zero. (It is zero if her valuation is $\overline{v}$, negative otherwise.)

296.1 Auctions with risk-averse bidders

Consider player i. Suppose that the bid of each type v_j of player j is given by $\beta_j(v_j) = (1 - 1/[m(n-1)+1])v_j$. Then as far as player i is concerned, the bids of every other player are distributed uniformly between 0 and $1 - 1/[m(n-1)+1]$. Thus for $0 \le x \le 1 - 1/[m(n-1)+1]$, the probability that any given player's bid is less than x is $(1 + 1/[m(n-1)])x$ ($1 + 1/[m(n-1)]$ is the reciprocal of $1 - 1/[m(n-1)+1]$), and hence the probability that *all* the bids of the other $n-1$ players are less than x is $[(1 + 1/[m(n-1)])x]^{n-1}$. Consequently, if player i bids more than $1 - 1/[m(n-1)+1]$ then she surely wins, whereas if she bids $b_i \le 1 - 1/[m(n-1)+1]$ she wins with probability $[(1 + 1/[m(n-1)])b_i]^{n-1}$. Thus player i's payoff as a function of her bid b_i is

$$\begin{cases} (v_i - b_i)^{1/m}\left\{\left(1 + \dfrac{1}{m(n-1)}\right)b_i\right\}^{n-1} & \text{if } 0 \le b_i \le 1 - \dfrac{1}{m(n-1)+1} \\ (v_i - b_i)^{1/m} & \text{if } b_i > 1 - \dfrac{1}{m(n-1)+1}. \end{cases} \tag{148.1}$$

Now, the value of b_i that maximizes the function

$$(v_i - b_i)^{1/m} \left\{ \left(1 + \frac{1}{m(n-1)} \right) b_i \right\}^{n-1}$$

is the same as the value of b_i that maximizes the function

$$(v_i - b_i)^{1/m} (b_i)^{n-1},$$

which is $(n-1)v_i/(n-1+1/m)$ (by the mathematical fact stated in the exercise), or

$$\left(1 - \frac{1}{m(n-1)+1} \right) v_i.$$

We have

$$\left(1 - \frac{1}{m(n-1)+1} \right) v_i \le 1 - \frac{1}{m(n-1)+1}$$

(because $v_i \le 1$), and the function in (148.1) is decreasing in b_i for $b_i > 1 - 1/[m(n-1)+1]$, so $1 - 1/[m(n-1)+1]$ is the bid that maximizes player i's expected payoff, given that the bid of each type v_j of player j is $(1 - 1/[m(n-1)+1])v_j$.

We conclude that, as claimed, the game has a Nash equilibrium in which each type v_i of each player i bids $(1 - 1/[m(n-1)+1])v_i$.

In this equilibrium, the price paid by a bidder with valuation v who wins is $(1 - 1/[m(n-1)+1])v$ (the amount she bids). The expected price paid by a bidder in a second-price auction does not depend on the players' payoff functions. Thus this payoff is equal, by the revenue equivalence result, to the expected price paid by a bidder with valuation v who wins in a first-price auction in which each bidder is risk-neutral, namely $(1 - 1/n)v$. We have

$$\left(1 - \frac{1}{m(n-1)+1} \right) - \left(1 - \frac{1}{n} \right) = \frac{(m-1)(n-1)}{n(m(n-1)+1)},$$

which is positive because $m > 1$. Thus the expected price paid by a bidder with valuation v who wins is greater in a first-price auction than it is in a second-price auction. The probability that a bidder with any given valuation wins is the same in both auctions, so the auctioneer's expected revenue is greater in a first-price auction than it is in a second-price auction.

299.1 Asymmetric Nash equilibria of second-price sealed-bid common value auctions

Suppose that each type t_2 of player 2 bids $(1 + 1/\lambda)t_2$ and that type t_1 of player 1 bids b_1. Then by the calculations in the text, with $\alpha = 1$ and $\gamma = 1/\lambda$,

- a bid of b_1 by player 1 wins with probability $b_1/(1 + 1/\lambda)$
- the expected value of player 2's bid, *given that it is less than* b_1, is $\frac{1}{2}b_1$

- the expected value of signals that yield a bid of less than b_1 is $\frac{1}{2}b_1/(1+1/\lambda)$ (because of the uniformity of the distribution of t_2).

Thus player 1's expected payoff if she bids b_1 is

$$(t_1 + \tfrac{1}{2}b_1/(1+1/\lambda) - \tfrac{1}{2}b_1) \cdot \frac{b_1}{1+1/\lambda},$$

or

$$\frac{\lambda}{2(1+\lambda)^2} \cdot (2(1+\lambda)t_1 - b_1)b_1.$$

This function is maximized at $b_1 = (1+\lambda)t_1$. That is, if each type t_2 of player 2 bids $(1+1/\lambda)t_2$, any type t_1 of player 1 optimally bids $(1+\lambda)t_1$. Symmetrically, if each type t_1 of player 1 bids $(1+\lambda)t_1$, any type t_2 of player 2 optimally bids $(1+1/\lambda)t_2$. Hence the game has the claimed Nash equilibrium.

299.2 First-price sealed-bid auction with common valuations

Suppose that each type t_2 of player 2 bids $\frac{1}{2}(\alpha+\gamma)t_2$ and type t_1 of player 1 bids b_1. To determine the expected payoff of type t_1 of player 1, we need to find the probability with which she wins, and the expected value of player 2's signal if player 1 wins. (The price she pays is her bid, b_1.)

Probability of player 1's winning: Given that player 2's bidding function is $\frac{1}{2}(\alpha+\gamma)t_2$, player 1's bid of b_1 wins only if $b_1 \geq \frac{1}{2}(\alpha+\gamma)t_2$, or if $t_2 \leq 2b_1/(\alpha+\gamma)$. Now, t_2 is distributed uniformly from 0 to 1, so the probability that it is at most $2b_1/(\alpha+\gamma)$ is $2b_1/(\alpha+\gamma)$. Thus a bid of b_1 by player 1 wins with probability $2b_1/(\alpha+\gamma)$.

Expected value of player 2's signal if player 1 wins: Player 2's bid, given her signal t_2, is $\frac{1}{2}(\alpha+\gamma)t_2$, so that the expected value of signals that yield a bid of less than b_1 is $b_1/(\alpha+\gamma)$ (because of the uniformity of the distribution of t_2).

Thus player 1's expected payoff if she bids b_1 is

$$2(\alpha t_1 + \gamma b_1/(\alpha+\gamma) - b_1) \cdot \frac{b_1}{\alpha+\gamma},$$

or

$$\frac{2\alpha}{(\alpha+\gamma)^2}((\alpha+\gamma)t_1 - b_1)b_1.$$

This function is maximized at $b_1 = \frac{1}{2}(\alpha+\gamma)t_1$. That is, if each type t_2 of player 2 bids $\frac{1}{2}(\alpha+\gamma)t_2$, any type t_1 of player 1 optimally bids $\frac{1}{2}(\alpha+\gamma)t_1$. Hence, as claimed, the game has a Nash equilibrium in which each type t_i of player i bids $\frac{1}{2}(\alpha+\gamma)t_i$.

306.1 Signal-independent equilibria in a model of a jury

If every juror votes for acquittal regardless of her signal then the action of any single juror has no effect on the outcome. Thus the strategy profile in which every juror votes for acquittal regardless of her signal is always a Nash equilibrium.

Now consider the possibility of a Nash equilibrium in which every juror votes for conviction regardless of her signal. Suppose that every juror other than juror 1 votes for conviction independently of her signal. Then juror 1's vote determines the outcome, exactly as in the case in which there is a single juror. Thus from the calculations in Section 9.7.2, type b of juror 1 optimally votes for conviction if and only if

$$z \leq \frac{(1-p)\pi}{(1-p)\pi + q(1-\pi)}$$

and type g of juror 1 optimally votes for conviction if and only if

$$z \leq \frac{p\pi}{p\pi + (1-q)(1-\pi)}.$$

The assumption that $p > 1 - q$ implies that the term on the right side of the second inequality is greater than the term on the right side of the first inequality, so that we conclude that there is a Nash equilibrium in which every juror votes for conviction regardless of her signal if and only if

$$z \leq \frac{(1-p)\pi}{(1-p)\pi + q(1-\pi)}.$$

307.1 Swing voter's curse

a. The Bayesian game is defined as follows.

Players Citizens 1 and 2.

States $\{A, B\}$.

Actions The set of actions of each player is $\{0, 1, 2\}$ (where 0 means do not vote).

Signals Citizen 1 receives different signals in states A and B, whereas citizen 2 receives the same signal in both states.

Beliefs Each type of citizen 1 assigns probability 1 to the single state consistent with her signal. The single type of citizen 2 assigns probability 0.9 to state A and probability 0.1 to state B.

Payoffs Both citizens' Bernoulli payoffs are 1 if either the state is A and candidate 1 receives the most votes or the state is B and candidate 2 receives the most votes; their payoffs are 0 if either the state is B and candidate 1 receives the most votes or the state is A and candidate 2 receives the most votes; and otherwise their payoffs are $\frac{1}{2}$. (These payoffs are shown in Figure 152.1.)

b. Type A of player 1's best action depends only on the action of player 2; it is to vote for 1 if player 2 votes for 2 or does not vote, and either to vote for 1 or not vote if player 2 votes for 1. Similarly, type B of player 1's best action is

	0	1	2
0	$\frac{1}{2}, \frac{1}{2}$	$1, 1$	$0, 0$
1	$1, 1$	$1, 1$	$\frac{1}{2}, \frac{1}{2}$
2	$0, 0$	$\frac{1}{2}, \frac{1}{2}$	$0, 0$

State A

	0	1	2
0	$\frac{1}{2}, \frac{1}{2}$	$0, 0$	$1, 1$
1	$0, 0$	$0, 0$	$\frac{1}{2}, \frac{1}{2}$
2	$1, 1$	$\frac{1}{2}, \frac{1}{2}$	$1, 1$

State B

Figure 152.1 The payoffs in the Bayesian game for Exercise 307.1.

to vote for 2 if player 2 votes for 1 or does not vote, and either to vote for 2 or not vote if player 2 votes for 2.

Player 2's best action is to vote for 1 if type A of player 1 either does not vote or votes for 2 (regardless of how type B of player 1 votes), not to vote if type A of player 1 votes for 1 and type B of player 1 either votes for 2 or does not vote, and either to vote for 1 or not to vote if both types of player 1 vote for 1.

Given the best responses of the two types of player 1, their only possible equilibrium actions are $(0, 0)$ (i.e. both do not vote), $(0, 2)$, $(1, 0)$, and $(1, 2)$. Checking player 2's best responses we see that the only equilibria are

- $(1, 2, 0)$ (player 1 votes for 1 in state A and for 2 in state B; player 2 does not vote).
- $(0, 2, 1)$ (player 1 does not vote in state A and votes for 2 in state B; player 2 votes for 1)

c. In the equilibrium $(0, 2, 1)$, type A of player 1's action is weakly dominated by the action of voting for 1: voting for 1 instead of not voting never makes her worse off, and makes her better off in the event that player 2 does not vote.

d. In the equilibrium $(1, 2, 0)$, player 2 does not vote because if she does then in the only case in which her vote affects the outcome (i.e. the only case in which she is a "swing voter"), it affects it adversely: if she votes for 1 then her vote makes no difference in state A, whereas it causes a tie instead of a win for candidate 2 in state B, and if she votes for 2, then her vote causes a tie instead of a win for candidate 1 in state A, and makes no difference in state B.

309.2 Property of the bidding function in a first-price sealed-bid auction

We have

$$\beta^{*\prime}(v) \;=\; 1 - \frac{(F(v))^{n-1}(F(v))^{n-1} - (n-1)(F(v))^{n-2}F'(v)\int_{\underline{v}}^{v}(F(x))^{n-1}\,dx}{(F(v))^{2n-2}}$$

$$\begin{aligned}
&= 1 - \frac{(F(v))^n - (n-1)F'(v)\int_{\underline{v}}^{v}(F(x))^{n-1}\,dx}{(F(v))^n} \\
&= \frac{(n-1)F'(v)\int_{\underline{v}}^{v}(F(x))^{n-1}\,dx}{(F(v))^n} \\
&> 0 \quad \text{if } v > \underline{v}
\end{aligned}$$

because $F'(v) > 0$ (F is increasing). (The first line uses the quotient rule for derivatives and the fact that the derivative of $\int^{v} f(x)dx$ with respect to v is $f(v)$ for any function f.)

[In the first printing of the book, this exercise asked also for a proof that a bidder with valuation $\underline{v}$ bids $\underline{v}$ and that a bidder with any other valuation bids less than her valuation. An argument for the latter is that for $v > \underline{v}$ the integral in (309.1) is positive; this argument is now included in the text. Regarding the former, the expression for $\beta^*(v)$ in (309.1) is not defined for $v = \underline{v}$ (the numerator and denominator of the quotient are both zero), though one can show that the limit of $\beta^*(v)$ as $v \to \underline{v}$ is $\underline{v}$: by L'Hôpital's rule, this limit is the ratio of the derivatives of the numerator and the denominator, namely

$$\frac{(F(v))^{n-1}}{(n-1)(F(v))^{n-2}F'(v)} = \frac{F(v)}{(n-1)F'(v)},$$

which is zero at $v = \underline{v}$ (because the numerator is zero and the denominator is positive).]

309.3 Example of Nash equilibrium in a first-price auction

From (309.1) we have

$$\begin{aligned}
\beta^*(v) &= v - \frac{\int_0^v x^{n-1}\,dx}{v^{n-1}} \\
&= v - \frac{\int_0^v x^{n-1}\,dx}{v^{n-1}} \\
&= v - v/n = (n-1)v/n.
\end{aligned}$$

310.2 Reserve prices in second-price sealed-bid auction

The argument that for each player a bid equal to her valuation weakly dominates all other bids is the same as the one in the absence of a reserve price.

Now consider the expected price at which the object is sold when the reserve price is r and each bidder submits her valuation. If both valuations are less than r, the object is not sold. If one valuation is less than r and the other is at least r, an event with probability $2r(1-r)$, the object is sold at the price r. If both valuations are at least r, the object is sold at a price equal to the smaller of the valuations. To deal with this last case, denote by $F(x)$ the probability that the smaller of the valuations is at most x. We have $F(x) = 1 - (1-x)^2$. (The probability that the smaller of the valuations is at least x is the probability that both valuations are at least x, which is $(1-x)^2$.) Thus the probability density of the smaller of the two valuations is $2(1-x)$ (the derivative of F). We conclude that the expected price at which the object is sold is

$$\begin{aligned} & r \cdot 2r(1-r) + \int_r^1 x \cdot 2(1-x)\,dx \\ &\qquad = 2r^2(1-r) + \tfrac{1}{3} - r^2 + \tfrac{2}{3}r^3 \\ &\qquad = \tfrac{1}{3} + r^2 - \tfrac{4}{3}r^3. \end{aligned}$$

This function is maximized at $r = \frac{1}{2}$. (Differentiate and set the derivative equal to zero. Note that $r = 0$ is a minimizer.)

Thus the expected price is maximized by a reserve price of $\frac{1}{2}$. For this reserve price the expected price is $\frac{5}{12}$, while for a reserve price of 0 it is $\frac{1}{3}$.

10 Extensive Games with Imperfect Information

316.1 Variant of card game

An extensive game that models the game is shown in Figure 155.1.

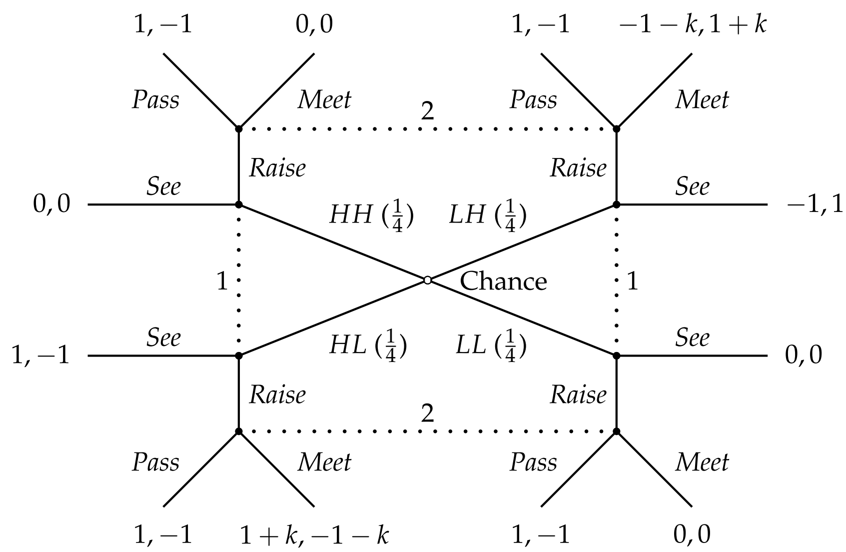


Figure 155.1 An extensive game that models the situation in Exercise 316.1.

318.2 Strategies in variants of card game and entry game

Card game: Each player has two information sets, and has two actions at each information set. Thus each player has four strategies: *SS*, *SR*, *RS*, and *RR* for player 1 (where *S* stands for *See* and *R* for *Raise*, the first letter of each strategy is player 1's action if her card is *High*, and the second letter if her action is her card is *Low*), and *PP*, *PM*, *MP*, and *MM* for player 2 (where *P* stands for *Pass* and *M* for *Meet*).
Entry game: The challenger has a single information set (the empty history) and has three actions after this history, so it has three strategies—*Ready*, *Unready*, and *Out*. The incumbent also has a single information set, at which two actions are available, so it has two strategies—*Acquiesce* and *Fight*.

319.3 Nash equilibrium of card game

The strategic form of the game is given in Figure 156.1. By arguments like those in Example 319.2, we find that the game has a unique Nash equilibrium, in which player 1 assigns probability $k/(k+2)$ to $(Raise, Raise)$ and probability $2/(k+2)$ to $(Raise, See)$, and player 2 assigns probability $k/(k+2)$ to *Pass* and probability $2/(k+2)$ to *Meet*. Thus in the equilibrium player 1 bluffs with probability $k/(k+2)$. As k increases, this probability increases, and the probability with which player 2 calls player 1's bluff (i.e. chooses *Meet*) decreases.

	Pass	*Meet*
Raise, *Raise*	$1, -1$	$0, 0$
Raise, *See*	$0, 0$	$\frac{1}{2}k, -\frac{1}{2}k$
See, *Raise*	$1, -1$	$-\frac{1}{2}k, \frac{1}{2}k$
See, *See*	$0, 0$	$0, 0$

Figure 156.1 The strategic form of the card game in Exercise 319.3.

320.1 Nash equilibria of variant of card game

The strategic form of the game is given in Figure 156.2, where R stands for *Raise*, S for *See*, P for *Pass*, and M for *Meet*, the first component of each player's strategy is her action if her card is *High*, and the second component of her strategy is her action if her card is *Low*.

	PP	PM	MP	MM
RR	$1, -1$	$\frac{3}{4}+\frac{1}{4}k, -\frac{3}{4}-\frac{1}{4}k$	$\frac{1}{4}-\frac{1}{4}k, -\frac{1}{4}+\frac{1}{4}k$	$0, 0$
RS	$\frac{1}{4}, -\frac{1}{4}$	$\frac{1}{4}+\frac{1}{4}k, -\frac{1}{4}-\frac{1}{4}k$	$0, 0$	$\frac{1}{4}k, -\frac{1}{4}k$
SR	$\frac{3}{4}, -\frac{3}{4}$	$\frac{1}{2}, -\frac{1}{2}$	$\frac{1}{4}-\frac{1}{4}k, -\frac{1}{4}+\frac{1}{4}k$	$-\frac{1}{4}k, \frac{1}{4}k$
SS	$0, 0$	$0, 0$	$0, 0$	$0, 0$

Figure 156.2 The strategic form of the game in Exercise 316.1.

First suppose that $0 < k < 1$. Player 1's strategy SS is strictly dominated by a mixed strategy that assigns probability $\frac{1}{2}$ to RR and to RS, so is not used with positive probability in any Nash equilibrium. In the reduced game, player 2's strategies PP and PM are strictly dominated by MM, and so can be eliminated from consideration.

I now argue that the resulting game has no Nash equilibrium in which player 1 assigns positive probability to SR. Player 1 is willing to assign positive probability to SR only if player 2 assigns probability 1 to MP, in which case player 1 wants to assign probability 0 to RS, making MM rather than MP optimal for player 2.

We are left with the game in Figure 157.1. This game has a unique Nash equilibrium, in which player 1 assigns probability k to RR and player 2 assigns probability k to MP.

	MP	MM
RR	$\frac{1}{4} - \frac{1}{4}k, -\frac{1}{4} + \frac{1}{4}k$	$0, 0$
RS	$0, 0$	$\frac{1}{4}k, -\frac{1}{4}k$

Figure 157.1 The strategies of the players in the game in Exercise 316.1 that may be assigned positive probability in a Nash equilibrium.

In conclusion, if $0 < k < 1$ the game has a unique Nash equilibrium, in which player 1 assigns probability k to RR and probability $1 - k$ to RS, and player 2 assigns probability k to MP and probability $1 - k$ to MM. In this equilibrium, player 1 bluffs (raises when her card is *Low*) with probability k, so that the larger is k, the more likely she is to bluff.

Now suppose that $k > 1$. By inspection the game has two pure strategy Nash equilibria in this case: (RS, MP) and (SS, MP). Any strategy pair in which player 1 assigns positive probability only to RS and SS and player 2 assigns probability 1 to MP is also a Nash equilibrium. In none of these equilibria does player 1 bluff.

The game has no other Nash equilibria, by the following argument.

- Player 1 assigns positive probability to SS only if player 2 assigns probability 1 to MP, to which player 1's best responses are RS and SS. Thus the only Nash equilibria in which player 1 assigns positive probability to SS are the equilibria in which player 1 assigns positive probability only to RS and SS and player 2 assigns probability 1 to MP.

- Now consider Nash equilibria in which player 1 assigns probability 0 to SS. In the game in which player 1's strategy SS is not used, player 2's strategies PP and PM are strictly dominated, and, when these strategies are eliminated, player 1's strategies RR and SR are strictly dominated. Thus the only Nash equilibrium in which player 1 assigns probability 0 to SS is the pure equilibrium (RS, MP).

331.1 Selten's horse

The strategic form of the game is shown in Figure 158.1. The players' best responses are indicated by asterisks. We see that the game has two pure strategy Nash equilibria, (D, c, L) and (C, c, R).

Consider the equilibrium (D, c, L). Player 2's action c is not sequentially rational (her action d yields her the payoff 4, given player 3's strategy), so there is no weak sequential equilibrium in which (D, c, L) is the strategy profile.

Now consider the equilibrium (C, c, R). The actions of players 1 and 2 are both optimal, given the other players' strategies. Player 3's information set is not

L

	c	d
C	$1\ ,1\ ,1^*$	$4^*,4^*,0$
D	$3^*,3^*,2^*$	$3\ ,3^*,2^*$

R

	c	d
C	$1^*,1^*,1^*$	$0^*,0\ ,1^*$
D	$0\ ,0^*,0$	$0^*,0^*,0$

Figure 158.1 The strategic form of the game in Exercise 331.1.

reached, so we are free to specify any belief there. If player 3 believes that the history is D with probability at most $\frac{1}{3}$, her action R is optimal. Thus the game has weak sequential equilibria in which the strategy profile is (C, c, R), and player 3's belief assigns probability of at most $\frac{1}{3}$ to D.

331.2 Weak sequential equilibrium and Nash equilibrium in subgames

Consider the assessment in which the Challenger's strategy is (Out, R), the Incumbent's strategy is F, and the Incumbent's belief assigns probability 1 to the history (In, U) at her information set. Each player's strategy is sequentially rational. The Incumbent's belief satisfies the condition of weak consistency because her information set is not reached when the Challenger follows her strategy. Thus the assessment is a weak sequential equilibrium.

The players' actions in the subgame following the history *In* do not constitute a Nash equilibrium of the subgame because the Incumbent's action F is not optimal when the Challenger chooses R. (The Incumbent's action F *is* optimal given her belief that the history is (In, U), as it is in the weak sequential equilibrium. In a Nash equilibrium she acts as if she has a belief that coincides with the Challenger's action in the subgame.)

335.1 Pooling and separating equilibria in a signaling game

Note: In the first three printings of the book, the exercise asks for conditions under which the game has *only* a separating equilibrium of a particular type and *only* a pooling equilibrium of a particular type. Finding these conditions is difficult; starting in the fourth printing, the question asks only for conditions under which the game has equilibria of these types. (The following answer gives only these latter conditions.)

Label the payoffs as in Figure 159.1.

In a weak sequential ("separating") equilibrium in which a strong challenger chooses *Ready* and a weak one chooses *Unready*, the incumbent's belief assigns probability 1 to the history $(Strong, Ready)$ at her top information set and probability 1 to the history $(Weak, Unready)$ at her bottom information set. Thus the incumbent chooses A at the top information set and F at the bottom one. Given these actions of the incumbent, the challenger's payoff decreases if she switches from R to U after the history *Strong*. For her payoff not to increase if she switches from U to R after the history *Weak* we need $a_1 \leq 3$. We conclude that the game

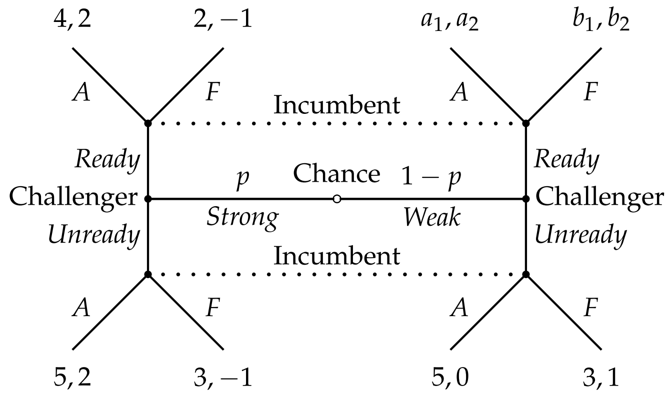


Figure 159.1 The game in Exercise 335.1.

has a weak sequential equilibrium in which the challenger chooses *Ready* after the history *Strong* and *Unready* after the history *Weak* if and only if $a_1 \leq 3$.

If an assessment in which both types of challenger choose U is a weak sequential equilibrium then at the incumbent's bottom information set she believes that the history is $(Strong, Unready)$ with probability p and $(Weak, Unready)$ with probability $1 - p$. Thus the incumbent's action at her bottom information set is F if $p < \frac{1}{4}$, A if $p > \frac{1}{4}$, and any mixture of A and F if $p = \frac{1}{4}$.

Now consider the incumbent's action at her top information set. In a weak sequential equilibrium in which the challenger chooses U after both the history *Strong* and the history *Weak*, the incumbent's belief at her top information set is not restricted, because this information set is not reached with positive probability. If $a_2 > b_2$ then A is the unique optimal action regardless of the incumbent's belief, whereas if $a_2 \leq b_2$ then F is optimal if the probability the incumbent assigns to $(Strong, Ready)$ is small enough.

Consider each case in turn.

$p < \frac{1}{4}$ For the challenger not to be able to profitably deviate after the history *Strong*, we need the incumbent to assign probability of at least $\frac{1}{2}$ to F at her top information set, which requires $a_2 \leq b_2$. Denote the probability that the incumbent assigns to A at her top information set by π. Then for the assessment to be a weak sequential equilibrium we need $\pi a_1 + (1 - \pi) b_1 \leq 3$. Thus for the game to have a weak sequential equilibrium in which both types of challenger choose U, we need $a_2 \leq b_2$ and $\pi a_1 + (1 - \pi) b_1 \leq 3$ for some $\pi \leq \frac{1}{2}$. (If $a_2 \leq b_2$ then the incumbent's belief at her top information set may be chosen to induce any value of π.)

Now, if $a_1 \geq b_1$ then the value of $\pi \leq \frac{1}{2}$ for which $\pi a_1 + (1 - \pi) b_1$ is minimal is $\pi = 0$, so that $\pi a_1 + (1 - \pi) b_1 \leq 3$ for some $\pi \leq \frac{1}{2}$ if and only if $b_1 \leq 3$. If $a_1 \leq b_1$ then the value of $\pi \leq \frac{1}{2}$ for which $\pi a_1 + (1 - \pi) b_1$ is minimal is $\pi = \frac{1}{2}$, so that $\pi a_1 + (1 - \pi) b_1 \leq 3$ for some $\pi \leq \frac{1}{2}$ if and only if $\frac{1}{2} a_1 + \frac{1}{2} b_1 \leq 3$.

We conclude that if $p < \frac{1}{4}$ then the game has a weak sequential equilibrium in which both types of challenger choose U if and only if $a_2 \leq b_2$ and either (a) $a_1 \geq b_1$ and $b_1 \leq 3$ or (b) $a_1 \leq b_1$ and $\frac{1}{2}(a_1 + b_1) \leq 3$.

$p > \frac{1}{4}$ In this case the challenger cannot profitably deviate after the history *Strong*, regardless of the incumbent's action at her top information set.

$a_2 > b_2$ The incumbent chooses A at her top information set regardless of her belief, so the challenger cannot profitably deviate after the history *Weak* if and only if $a_1 \leq 5$.

$a_2 \leq b_2$ In this case there are beliefs under which any mixture of A and F is optimal for the incumbent at her top information set. Thus the challenger cannot profitably deviate after the history *Weak* if and only if $\min\{a_1, b_1\} \leq 5$.

We conclude that if $p > \frac{1}{4}$, then the game has a weak sequential equilibrium in which both types of challenger choose U if and only if either (a) $a_2 > b_2$ and $a_1 \leq 5$, or (b) $a_2 \leq b_2$ and $\min\{a_1, b_1\} \leq 5$.

$p = \frac{1}{4}$ In this case both A and F (and any mixture of them) are optimal for the incumbent at bottom information set. The action A yields the challenger more than F does, so the game has a weak sequential equilibrium in which both types of challenger choose U if and only if the conditions for the case $p > \frac{1}{4}$ are satisfied.

335.2 Sir Philip Sydney game

Consider the strategy pair in which the offspring squawks if and only if it is hungry, and the parent gives it the food if and only if it squawks. The consistency condition requires the parent to believe that her offspring is hungry if and only if it squawks, so for the parent's strategy to be sequentially rational we need

$$1 + rV \geq S + r \quad \text{and} \quad S + r(1-t) \geq 1.$$

For the offspring's strategy to be sequentially rational we need

$$1 - t + rS \geq r \quad \text{and} \quad V + r \geq 1 - t + rS.$$

Combining these two conditions yields the requirements

$$\frac{1-S}{1-t} \leq r \leq \frac{1-S}{1-V} \quad \text{and} \quad \frac{1-V-t}{1-S} \leq r \leq \frac{1-t}{1-S}.$$

The condition $r < (1-V)/(1-S)$ is consistent with the left-hand side of the second inequality only if $t > 0$.

Now consider the strategy pair in which the offspring is quiet whether or not it is hungry, and the parent keeps the food whether or not the offspring squawks. The consistency condition requires the parent to believe that the offspring is hungry with probability p if it is quiet. The condition does not restrict the parent's belief if the offspring squawks; suppose that in this case the parent believes the offspring is not hungry. Then for the parent's strategy to be sequentially rational we need

$$p + (1-p)(1+rV) \geq S + r \quad \text{and} \quad 1 + rV(1-t) \geq S + r(1-t). \tag{161.1}$$

The parent's behavior does not depend on the offspring's action, so given that squawking is costly, the offspring's payoff when it is quiet is at least as high as its payoff when it squawks. Thus the strategy pair is a weak sequential equilibrium if the conditions in (161.1) are satisfied. These conditions are equivalent to

$$r \leq \frac{1-S}{1-(1-p)V} \quad \text{and} \quad r \leq \frac{1-S}{(1-t)(1-V)}.$$

If $r < (1-S)/(1-(1-p)V)$ then both these conditions are satisfied (because $1-(1-p)V > (1-t)(1-V)$ given $V < 1$).

340.1 Pooling equilibria of game in which expenditure signals quality

We know that in the second period the high-quality firm charges the price H and the low-quality firm charges any nonnegative price, and the consumer buys the good from a high-quality firm, does not buy the good from a low-quality firm that charges a positive price, and may or may not buy from a low-quality firm that charges a price of 0.

Consider an assessment in which each type of firm chooses (p^*, E^*) in the first period, the consumer believes the firm is high-quality with probability π if it observes (p^*, E^*) and low quality if it observes any other (price, expenditure) pair, and buys the good if and only if it observes (p^*, E^*).

The payoff of a high-quality firm under this assessment is $p^* + H - E^* - 2c_H$, that of a low-quality firm is $p^* - E^*$, and that of the consumer is $\pi(H - p^*) + (1 - \pi)(-p^*) = \pi H - p^*$.

This assessment is consistent—the only first-period action of the firm observed in equilibrium is (p^*, E^*), and after observing this pair the consumer believes, correctly, that the firm is high-quality with probability π.

Under what conditions is the assessment sequentially rational?

Firm If the firm chooses a (price, expenditure) pair different from (p^*, E^*) then the consumer does not buy the good, and the firm's profit is 0. Thus for the assessment to be an equilibrium we need $p^* + H - E^* - 2c_H \geq 0$ (for the high-quality firm) and $p^* - E^* \geq 0$ (for the low-quality firm).

Consumer If the consumer does not buy the good after observing (p^*, E^*) then its payoff is 0, so for the assessment to be an equilibrium we need $\pi H - p^* \geq 0$.

In summary, the assessment is a weak sequential equilibrium if and only if

$$\max\{E^*, E^* - H + 2c_H\} \leq p^* \leq \pi H.$$

342.1 Pooling equilibria of game in which education signals ability

Consider an assessment in which both types of worker choose the education level e^*. The consistency condition requires that a firm that observes e^* believe that the worker is type H with probability π and type L with probability $1 - \pi$. Thus the firms' equilibrium wage offers after observing e^* are both equal to $\pi H + (1 - \pi)L$, yielding a worker of ability K the payoff $\pi H + (1 - \pi)L - e^*/K$.

For the assessment to be a weak sequential equilibrium, neither type of worker must be able to increase her payoff by choosing a different value of e. The wage optimally offered by the firms to such a worker of course depends on the firms' beliefs. The belief that makes a profitable deviation by a worker least likely (and hence supports the widest range of equilibrium values of e^*) is that in which each firm believes that a worker who chooses $e \neq e^*$ has ability L. In response to this belief, each firm offers the wage L, yielding a worker of ability K the payoff $L - e/K$. If $e^* = 0$ then certainly neither type of worker can gain by deviating. If $e^* > 0$ then, given that the value of e that maximizes this payoff is 0, for equilibrium we need

$$\pi H + (1 - \pi)L - e^*/K \geq L \text{ for } K = L, H.$$

The value of the left-hand side is lower for $K = L$ than it is for $K = H$, so we need

$$\pi H + (1 - \pi)L - e^*/L \geq L,$$

or

$$e^* \leq \pi L(H - L).$$

In summary, for any $e^* \leq \pi L(H - L)$ the game has a pooling equilibrium in which both types of worker obtain the education level e^*.

Given $\pi \leq 1$, the education levels possible in a pooling equilibrium are all less than those possible in any separating equilibrium.

346.1 Comparing the receiver's expected payoff in two equilibria

The receiver's payoff as a function of the state t in each equilibrium is shown in Figure 163.1. The area above the black curve is smaller than the area above the gray curve: if you shift the black curve $\frac{1}{2}t_1$ to the left and move the section from 0 to $\frac{1}{2}t_1$ to the interval from $1 - \frac{1}{2}t_1$ to 1 then the area above the black curve is a subset of the area above the gray curve.

350.1 Variant of model with piecewise linear payoff functions

The equilibria of the variant are exactly the same as the equilibria of the original model.

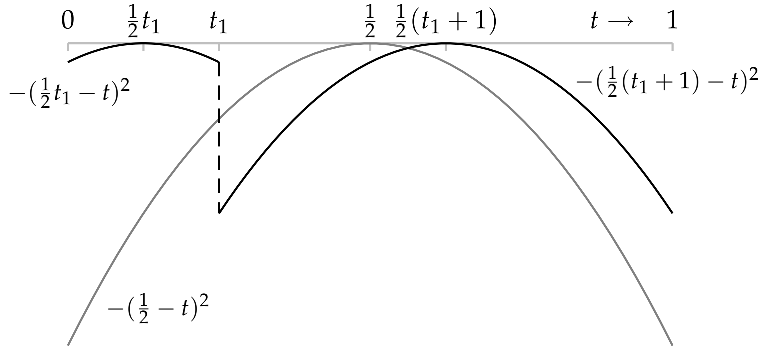


Figure 163.1 The gray curve gives the receiver's payoff in each state in the equilibrium in which no information is transferred. The black curve gives her payoff in each state in the two-report equilibrium.

350.2 Pooling equilibrium in a general model

Choose an arbitrary report r^*. I claim that the following assessment is a weak sequential equilibrium.

Sender's strategy Choose r^* in every state.

Receiver's belief For every report r, the distribution of the state is the same, equal to the initial distribution.

Receiver's strategy Choose the action y^* regardless of the sender's report.

This assessment is a weak sequential equilibrium by the following argument.

Sequential rationality of sender's strategy The sender's report has no effect on the outcome, so the choice of r^* in every state is optimal.

Consistency of receiver's belief The sender's report conveys no information about the state. If it is r^* then consistency requires that the receiver believe that the distribution of the state is equal to the initial distribution; if it is different from r^* then consistency imposes no restriction on the receiver's belief.

Sequential rationality of receiver's strategy By definition, the action y^* maximizes the receiver's expected payoff given her belief.

11 Strictly Competitive Games and Maxminimization

363.1 Maxminimizers in a bargaining game

If a player demands any amount x up to \$5 then her payoff is x regardless of the other player's action. If she demands \$6 then she may get as little as \$5 (if the other player demands \$5 or \$6). If she demands $x \geq \$7$ then she may get as little as $\$(11 - x)$ (if the other player demands $x - 1$). For each amount that a player demands, the smallest amount that you may get is given in Figure 165.2. We see that each player's maxminimizing pure strategies are \$5 and \$6 (for both of which the worst possible outcome is that the player receives \$5).

Amount demanded	0	1	2	3	4	5	6	7	8	9	10
Smallest amount obtained	0	1	2	3	4	5	5	4	3	2	1

Figure 165.2 The lowest payoffs that a player receives in the game in Exercise 38.2 for each of her possible actions, as the other player's action varies.

363.3 Finding a maxminimizer

The analog of Figure 364.1 in the text is Figure 165.3. From this figure we see that the maxminimizer for player 2 is the strategy that assigns probability $\frac{2}{5}$ to L. Player 2's maxminimized payoff is $-\frac{1}{5}$.

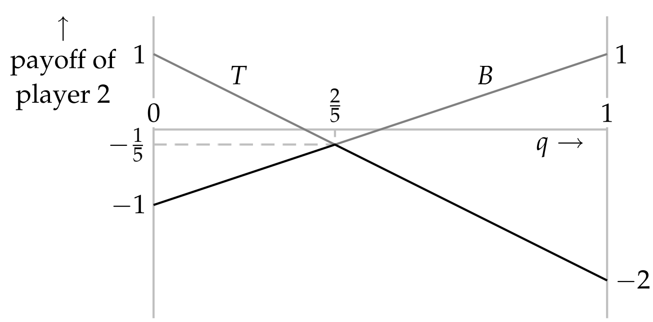


Figure 165.3 The expected payoff of player 2 in the game in Figure 363.1 for each of player 1's actions, as a function of the probability q that player 2 assigns to L.

364.2 Nash equilibrium payoffs and maxminimized payoffs

In the game in Figure 166.1 each player's maxminimized payoff is 1, while her payoff in the unique Nash equilibrium is 2.

	L	R
T	$2,2$	$1,0$
B	$0,1$	$0,0$

Figure 166.1 A game in which each player's Nash equilibrium payoff exceeds her maxminimized payoff.

365.1 Nash equilibrium payoffs and maxminimized payoffs

The game has a unique mixed strategy Nash equilibrium, in which player 1's strategy is $(\frac{1}{4}, \frac{3}{4})$ and player 2's strategy is $(\frac{2}{3}, \frac{1}{3})$. In this equilibrium player 1's payoff is 4.

Now consider the maxminimizer for player 1. Player 1's payoff as a function of the probability that she assigns to T is shown in Figure 166.2. We see that the maxminimizer for player 1 is $(\frac{1}{3}, \frac{2}{3})$, and this strategy guarantees player 1 a payoff of 4.

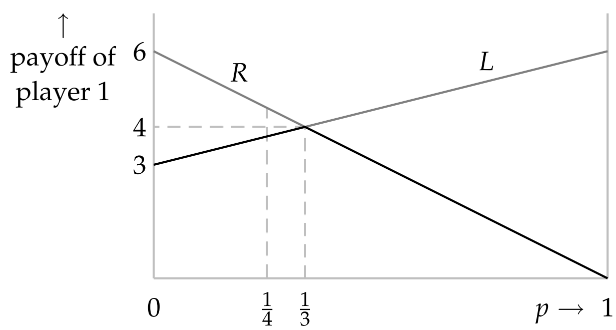


Figure 166.2 The expected payoff of player 1 in the game in Exercise 365.1 for each of player 2's actions, as a function of the probability p that player 1 assigns to T.

Player 1's payoffs, as a function of the probability q that player 2's strategy assigns to L, when she uses her Nash equilibrium strategy and her maxminimizer, are shown in Figure 167.1. Notice that her maxminimizer guarantees that she obtains her equilibrium payoff, while her equilibrium strategy does not.

366.2 Determining strictly competitiveness

Game in Exercise 365.1: Strictly competitive in pure strategies (because player 1's ranking of the four outcomes is the reverse of player 2's ranking). Not strictly

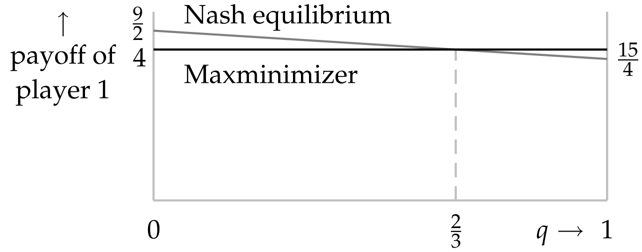


Figure 167.1 The expected payoff of player 1 in the game in Exercise 365.1 as a function of the probability q that player 2's mixed strategy assigns to L when player 1 uses her Nash equilibrium strategy (gray line) and her maxminimizer (black line).

competitive in mixed strategies (there exist no values of π and $\theta > 0$ such that $-u_1(a) = \pi + \theta u_2(a)$ for every outcome a; or, alternatively, player 1 is indifferent between (B, L) and the lottery that yields (T, L) with probability $\frac{1}{2}$ and (T, R) with probability $\frac{1}{2}$, whereas player 2 is not indifferent between these two outcomes).

Game in Figure 367.1: Strictly competitive both in pure and in mixed strategies. (Player 2's preferences are represented by the expected value of the Bernoulli payoff function $-u_1$ because $-u_1(a) = -\frac{1}{2} + \frac{1}{2}u_2(a)$ for every pure outcome a.)

369.2 Equilibrium payoffs in symmetric game

Let (α_1^*, α_2^*) be a mixed strategy Nash equilibrium of the game. Denote player 1's payoff in this equilibrium by v^*. By the symmetry of the game, (α_2^*, α_1^*) is also a mixed strategy Nash equilibrium; in this equilibrium player 2's payoff is v^*, so that player 1's payoff is $-v^*$. But by Corollary 369.1, the equilibrium payoff of each player in a strictly competitive game is unique, so $v^* = -v^*$, and hence $v^* = 0$.

370.2 Maxminimizing in BoS

Player 1's maxminimizer is $(\frac{1}{3}, \frac{2}{3})$ while player 2's is $(\frac{2}{3}, \frac{1}{3})$. Clearly neither pure equilibrium strategy of either player guarantees her equilibrium payoff. In the mixed strategy equilibrium, player 1's expected payoff is $\frac{2}{3}$. But if, for example, player 2 choose S instead of her equilibrium strategy, then player 1's expected payoff is $\frac{1}{3}$. Similarly for player 2.

372.1 Increasing payoffs and eliminating actions

a. Let U_i be player i's expected payoff function in the game G, let W_i be her expected payoff function in G', and let (α_1^*, α_2^*) be a mixed strategy Nash equilibrium of G'. We have $W_1(\alpha_1, \alpha_2) \geq U_1(\alpha_1, \alpha_2)$ for all every pair (α_1, α_2), so that $\max_{\alpha_1} W_1(\alpha_1, \alpha_2) \geq \max_{\alpha_1} U_1(\alpha_1, \alpha_2)$ for every strategy α_2, and hence

$$\min_{\alpha_2} \max_{\alpha_1} W_1(\alpha_1, \alpha_2) \geq \min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2).$$

Thus by part *a* of Proposition 368.1, player 1's payoff in every mixed strategy Nash equilibrium of the game G' is at least as high as her payoff in every mixed strategy Nash equilibrium of G.

b. By part *a* of Proposition 368.1, player 1's payoff in any equilibrium of G is $\min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2)$, where U_1 is player 1's expected payoff function. In G', for each value of α_2 the value of $\max_{\alpha_1} U_1(\alpha_1, \alpha_2)$ is no larger than it is in G, because player 1's set of strategies is smaller. Thus player 1's payoff in any equilibrium of G' is no larger than it is in any equilibrium of G.

c. In the unique equilibrium of the game on the left of Figure 168.1 player 1 receives a payoff of 3, while in the unique equilibrium of the game on the right she receives a payoff of 2. If she is prohibited from using her second action in this second game then she obtains an equilibrium payoff of 3, however.

3,3	1,1
1,0	0,1

3,3	1,1
4,0	2,1

Figure 168.1 The games for part *c* of Exercise 372.1.

372.2 Equilibrium in strictly competitive game

The claim is false. In the strictly competitive game in Figure 168.2 the action pair (T, L) is a Nash equilibrium, so that player 1's unique equilibrium payoff in the game is 0. But (B, R), which also yields player 1 a payoff of 0, is not a Nash equilibrium.

	L	R
T	$0, 0$	$1, -1$
B	$-1, 1$	$0, \ \ 0$

Figure 168.2 The game in Exercise 372.2.

372.3 *Morra*

a. In the strategic game there are two players, each of whom has four (relevant) actions, $S1G2$, $S1G3$, $S2G3$, and $S2G4$, where $SiGj$ denotes the strategy (Show i, Guess j). The payoffs in the game are shown in Figure 169.1. *b*. Let (p_1, p_2, p_3, p_4) be the probabilities that player 1 assigns to her four actions. In order that she obtain a payoff of at least 0 for each pure strategy of player 2, we need

$$\begin{aligned}
&-2p_2 + 3p_3 &&\geq 0\\
2p_1 & &-3p_4 &\geq 0\\
-3p_1 & &+4p_4 &\geq 0\\
&3p_2 - 4p_3 &&\geq 0.
\end{aligned}$$

	$S1G2$	$S1G3$	$S2G3$	$S2G4$
$S1G2$	$0, \ \ 0$	$2, -2$	$-3, \ \ 3$	$0, \ \ 0$
$S1G3$	$-2, \ \ 2$	$0, \ \ 0$	$0, \ \ 0$	$3, -3$
$S2G3$	$3, -3$	$0, \ \ 0$	$0, \ \ 0$	$-4, \ \ 4$
$S2G4$	$0, \ \ 0$	$-3, \ \ 3$	$4, -4$	$0, \ \ 0$

Figure 169.1 The game in Exercise 372.3.

The second and third inequalities imply that $p_1 \geq \frac{3}{2}p_4$ and $p_1 \leq \frac{4}{3}p_4$, so that $p_1 = p_4 = 0$, and hence $p_3 = 1 - p_2$. The first and fourth inequalities imply that $p_2 \leq \frac{3}{2}p_3$ and $p_2 \geq \frac{4}{3}p_3$, or $p_2 \leq \frac{3}{5}$ and $p_2 \geq \frac{4}{7}$. Thus any strategy $(0, p_2, 1 - p_2, 0)$ with $\frac{4}{7} \leq p_2 \leq \frac{3}{5}$ is a maxminimizer for player 1, and hence, given the symmetry of the game, also for player 2.

We conclude that the set of mixed strategy Nash equilibria of the game is the set of pairs of mixed strategies $((0, p_2, 1 - p_2, 0), (0, q_2, 1 - q_2, 0))$ with $\frac{4}{7} \leq p_2 \leq \frac{3}{5}$ and $\frac{4}{7} \leq q_2 \leq \frac{3}{5}$.

372.4 O'Neill's game

a. Denote the probability with which player 1 chooses each of her actions 1, 2, and 3, by p, and the probability with which player 2 chooses each of these actions by q. Then all four of player 1's actions yield the same expected payoff if and only if $4q - 1 = 1 - 6q$, or $q = \frac{1}{5}$, and similarly all four of player 2's actions yield the same expected payoff if and only if $p = \frac{1}{5}$. Thus $((\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}), (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}))$ is a Nash equilibrium of the game. The players' payoffs in this equilibrium are $(-\frac{1}{5}, \frac{1}{5})$.

b. Let (p_1, p_2, p_3, p_4) be an equilibrium strategy of player 1. In order that it guarantee her the payoff of $-\frac{1}{5}$, we need

$$\begin{aligned} -p_1 + p_2 + p_3 - p_4 &\geq -\tfrac{1}{5} \\ p_1 - p_2 + p_3 - p_4 &\geq -\tfrac{1}{5} \\ p_1 + p_2 - p_3 - p_4 &\geq -\tfrac{1}{5} \\ -p_1 - p_2 - p_3 + p_4 &\geq -\tfrac{1}{5}. \end{aligned}$$

Adding these four inequalities, we deduce that $p_4 \leq \frac{2}{5}$. Adding each pair of the first three inequalities, we deduce that $p_1 \leq \frac{1}{5}$, $p_2 \leq \frac{1}{5}$, and $p_3 \leq \frac{1}{5}$. We have $p_1 + p_2 + p_3 + p_4 = 1$, so we deduce that $(p_1, p_2, p_3, p_4) = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5})$. A similar analysis of the conditions for player 2's strategy to guarantee her the payoff of $\frac{1}{5}$ leads to the conclusion that $(q_1, q_2, q_3, q_4) = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5})$.

12 Rationalizability

379.2 Best responses to beliefs

Consider a two-player game in which player 1's payoffs are given in Figure 171.2. The action B of player 1 is a best response to the belief that assigns probability $\frac{1}{2}$ to both L and R, but is not a best response to any belief that assigns probability 1 to either action.

	L	R
T	3	0
M	0	3
B	2	2

Figure 171.2 The action B is a best response to a belief that assigns probability $\frac{1}{2}$ to L and to R, but is not a best response to any belief that assigns probability 1 to either L or R.

384.1 Mixed strategy equilibria of game in Figure 384.1

The game has no equilibrium in which player 2 assigns positive probability only to L and C, because if she does so then only M and B are possible best responses for player 1, but if player 1 assigns positive probability only to these actions then L is not optimal for player 2.

Similarly, the game has no equilibrium in which player 2 assigns positive probability only to C and R, because if she does so then only T and M are possible best responses for player 1, but if player 1 assigns positive probability only to these actions then R is not optimal for player 2.

Now assume that player 2 assigns positive probability only to L and R. There are no probabilities for L and R under which player 1 is indifferent between all three of her actions, so player 1 must assign positive probability to at most two actions. If these two actions are T and M then player 2 prefers L to R, while if the two actions are M and B then player 2 prefers R to L. The only possibility is thus that the two actions are T and B. In this case we need player 2 to assign probability $\frac{1}{2}$ to L and R (in order that player 1 be indifferent between T and B); but then M is better for player 1. Thus there is no equilibrium in which player 2 assigns positive probability only to L and R.

Finally, if player 2 assigns positive probability to all three of her actions then player 1's mixed strategy must be such that each of these three actions yields the same payoff. A calculation shows that there is no mixed strategy of player 1 with this property.

We conclude that the game has no mixed strategy equilibrium in which either player assigns positive probability to more than one action.

387.2 Finding rationalizable actions

I claim that the action R of player 2 is strictly dominated. Consider a mixed strategy of player 2 that assigns probability p to L and probability $1-p$ to C. Such a mixed strategy strictly dominates R if $p+4(1-p)>3$ and $8p+2(1-p)>3$, or if $\frac{1}{6}<p<\frac{1}{3}$. Now eliminate R from the game. In the reduced game, B is dominated by T. In the game obtained by eliminating B, L is dominated by C. Thus the only rationalizable action of player 1 is T and the only rationalizable action of player 2 is C.

387.3 *Morra*

The action $S1G2$ is a best response to a belief that assigns probability 1 to $S1G3$, the action $S1G3$ is a best response to the belief that assigns probability one to $S2G4$, the action $S2G3$ is a best response to the belief that assigns probability one to $S1G2$, and the action $S2G4$ is a best response to the belief that assigns probability one to $S2G3$. Thus no action of either player is strictly dominated, so that every action of each player is rationalizable. (In Definition 383.1 we can set Z_i equal to the set consisting of all the actions of player i, for $i=1,2$.)

387.4 Guessing two-thirds of the average

For any player, announcing K is a never-best response by the following argument.

- If the other two players announce (K,K) then announcing K yields the payoff of $\frac{1}{3}$ whereas announcing $K-1$ yields the payoff of 1.
- If the other two players make any other announcements then announcing K yields the payoff of 0 whereas announcing the smaller of the numbers announced by the other players yields at least $\frac{1}{3}$.

By Lemma 385.3, announcing K is thus strictly dominated.

Now eliminate the action K for each player. In the reduced game announcing $K-1$ is strictly dominated by the same argument. Thus we can eliminate $K-1$ for each player. Continuing similarly we can eliminate all announcements but 1. Thus each player's only rationalizable action is 1.

387.5 Hotelling's model of electoral competition

The positions 0 and ℓ are strictly dominated by the position m:

- if her opponent chooses m, a player who chooses m ties whereas a player who chooses 0 loses
- if her opponent chooses 0 or ℓ, a player who chooses m wins whereas a player who chooses 0 or ℓ either loses or ties
- if her opponent chooses any other position, a player who chooses m wins whereas a player who chooses 0 or ℓ loses.

In the game obtained by eliminating the two positions 0 and ℓ, the positions 1 and $\ell - 1$ are similarly strictly dominated. Continuing in the same way, we are left with the position m.

388.1 Contributing to a public good

Denote by $\bar{c}_{-i}$ the sum of the contributions of the players other than i.

a. We have $u_i(c_i, c_{-i}) = w_i + \bar{c}_{-i} + (w_i - c_i)(c_i + \bar{c}_{-i})$ and $u_i(\frac{1}{2}w_i, c_{-i}) = w_i + \bar{c}_{-i} + \frac{1}{2}w_i(\frac{1}{2}w_i + \bar{c}_{-i})$. Thus

$$u_i(\tfrac{1}{2}w_i, c_{-i}) - u_i(c_i, c_{-i}) = \tfrac{1}{2}w_i(\tfrac{1}{2}w_i + \bar{c}_{-i}) + (w_i - c_i)(c_i + \bar{c}_{-i}).$$

This function is a quadratic in c_i that is equal to $\frac{1}{2}w_i(\frac{1}{2}w_i + \bar{c}_{-i})$ when $c_i = -\bar{c}_{-i}$ and when $c_i = w_i$. The coefficient of c_i^2 is positive, so the function attains a minimum at $\frac{1}{2}(-\bar{c}_{-i} + w_i) < \frac{1}{2}w_i$ (given $c_j \geq 0$ for all j). The function is zero at $\frac{1}{2}w_i$, so for all values of $\bar{c}_{-i}$ it is positive whenever $c_i > \frac{1}{2}w_i$. Thus every contribution of more than $\frac{1}{2}w_i$ is strictly dominated by the contribution $\frac{1}{2}w_i$.

b. The best response of player i to $\bar{c}_{-i}$ is one of the feasible contribution levels close to

$$\max\{0, \tfrac{1}{2}(w - \bar{c}_{-i})\}$$

(see the solution to Exercise 44.1). Let $c \leq w/2$ and suppose that each of the other players contributes $\frac{1}{2}w - c$ (which is nonnegative). Then the other players' total contribution is $w - 2c$, so that player i's best response is to contribute c. That is, any contribution c of at most $w/2$ is a best response to the belief that assigns probability one to each of the other player's contributing $\frac{1}{2}w - c \leq \frac{1}{2}w$. Thus if we set $Z_i = [0, w/2]$ for all i in Definition 383.1 we see that any action of player i in $[0, w/2]$ is rationalizable. [Note: This argument does not show that actions outside $[0, w/2]$ are not rationalizable.]

c. Denote $w_1 = w_2 = w$. First eliminate contributions of more than $w_i/2$ by each player i.

In the reduced game the most that players 1 and 2 together contribute is w (because each contributes at most $w/2$). Now consider player 3. Her payoff function is

$$u_3(c_3, c_{-3}) = w_3 + \bar{c}_{-3} + (w_3 - c_3)(c_3 + \bar{c}_{-3}).$$

This function is a quadratic that is zero at $c_3 = -\bar{c}_{-3}$ and at w_3 and has a maximum at $c_3 = \frac{1}{2}(w_3 - \bar{c}_{-3})$. Because $c_{-3} \leq w$ and $w < \frac{1}{3}w_3$, this maximizer exceeds w. Thus player 3's payoff is increasing in her contribution for every remaining possible value of $\bar{c}_{-3}$. We conclude that in the reduced game every contribution of player 3 of less than w is strictly dominated by a contribution of w. Eliminate all such actions of player 3.

In the newly reduced game every contribution of player 3 is in the interval $[w, w_3/2]$. Now consider player 1. Her payoff is decreasing in her contribution if $c_1 > \frac{1}{2}(w - \bar{c}_{-1})$. We have $c_2 \geq 0$ and $c_3 \geq w$, so that $\bar{c}_{-1} \geq w$ and hence player 1's payoff is decreasing if $c_1 > 0$. Thus every action of player 1 is strictly dominated by a contribution of 0. The same analysis applies to player 2. Eliminate all such actions of player 1 and player 2.

Finally, in the game we now have, players 1 and 2 both contribute 0; it follows that all actions of player 3 are dominated except for a contribution of $w_3/2$, which is her best response to a total contribution of 0 by players 1 and 2.

We conclude that the unique action profile that survives iterated elimination of strictly dominated actions is $(0, 0, w_3/2)$.

388.2 Cournot's duopoly game

From Figure 58.1 we see that firm 1's payoff to any output greater than $\frac{1}{2}(\alpha - c)$ is less than its payoff to the output $\frac{1}{2}(\alpha - c)$ for any output q_2 of firm 2. Thus any output greater than $\frac{1}{2}(\alpha - c)$ is strictly dominated by the output $\frac{1}{2}(\alpha - c)$ for firm 1; the same argument applies to firm 2.

Now eliminate all outputs greater than $\frac{1}{2}(\alpha - c)$ for each firm. The maximizer of firm 1's payoff function for $q_2 = \frac{1}{2}(\alpha - c)$ is $\frac{1}{4}(\alpha - c)$, so from Figure 58.1 we see that firm 1's payoff to any output less than $\frac{1}{4}(\alpha - c)$ is less than its payoff to the output $\frac{1}{4}(\alpha - c)$ for any output $q_2 \leq \frac{1}{2}(\alpha - c)$ of firm 2. Thus any output less than $\frac{1}{4}(\alpha - c)$ is strictly dominated by the output $\frac{1}{4}(\alpha - c)$ for firm 1; the same argument applies to firm 2.

Now eliminate all outputs less than $\frac{1}{4}(\alpha - c)$ for each firm. Then by another similar argument, any output greater than $\frac{3}{8}(\alpha - c)$ is strictly dominated by $\frac{3}{8}(\alpha - c)$. Continuing in this way, we see from Figure 59.1 that in a finite number of rounds (given the finite number of possible outputs for each firm) we reach the Nash equilibrium output $\frac{1}{3}(\alpha - c)$.

391.1 Example of dominance-solvable game

The Nash equilibria of the game are (T, L), any $((0, 0, 1), (0, q, 1 - q))$ with $0 \leq q \leq 1$, and any $((0, p, 1 - p), (0, 0, 1))$ with $0 \leq p \leq 1$.

The game is dominance solvable, because T and L are the only weakly dominated actions, and when they are eliminated the only weakly dominated actions are M and C, leaving (B, R), with payoffs $(0, 0)$.

If T is eliminated, then L and C, no remaining action is weakly dominated; (M,R) and (B,R) both remain.

391.2 Dividing money

In the first round every action $a_i \leq 5$ of each player i is weakly dominated by 6. No other action is weakly dominated, because 100 is a strict best response to 0 and every other action $a_i \geq 6$ is a strict best response to $a_i + 1$. In the second round, 10 is weakly dominated by 6 for each player, and each other remaining action a_i of player i is a strict best response to $a_1 + 1$, so no other action is weakly dominated. Similarly, in the third round, 9 is weakly dominated by 6, and no other action is weakly dominated. In the fourth and fifth rounds 8 and 7 are eliminated, leaving the single action pair $(6,6)$, with payoffs $(5,5)$.

391.3 Voting

Suppose that more than two-thirds of the citizens rank candidate C below A and B. By the result of Exercise 49.1, each citizen's only weakly dominated action is a vote for her least preferred candidate. After eliminating this action for each citizen, every remaining action profile leads to a win by either A or B, because fewer than one-third of the citizens vote for C. Thus each citizen's voting for whomever of A and B she prefers weakly dominates her other remaining action, by the same argument as in the two-candidate game. We are left with the action profile in which every citizen votes for her favorite among the candidates A and B.

392.1 Bertrand's duopoly game

In the first round every price in excess of the monopoly price is weakly dominated by the monopoly price, and every price equal to at most c is weakly dominated by the price $c+1$. Every other price p is a strict best response to $p+1$, so no other price is weakly dominated. At each subsequent round the highest remaining price is weakly dominated by the next highest price. (Note that for any $p \geq c+1$ it is better to obtain all the demand at the price p than to obtain half of the demand at the price $p+1$.) The pair of prices that remains is $(c+1, c+1)$.

392.2 Strictly competitive extensive games with perfect information

Every finite extensive game with perfect information has a (pure strategy) subgame perfect equilibrium (Proposition 173.1). This equilibrium is a pure strategy Nash equilibrium of the strategic form of the game. Because the game has only two possible outcomes, one of the players prefers the Nash equilibrium outcome to the other possible outcome. By Proposition 368.1, this player's equilibrium strategy guarantees her equilibrium payoff, so this strategy weakly dominates all her nonequilibrium strategies. After all dominated strategies are eliminated, every remaining pair of strategies generates the same outcome.

13 Evolutionary Equilibrium

400.1 Evolutionary stability and weak domination

The ESS a^* does not necessarily weakly dominate every other action in the game. For example, in the game in Figure 395.1 of the text, X is an ESS but does not weakly dominate Y.

No action can weakly dominate an ESS. To see why, let a^* be an ESS and let b be another action. Because a^* is an ESS, (a^*, a^*) is a Nash equilibrium, so that $u(b, a^*) \leq u(a^*, a^*)$. Now, if $u(b, a^*) < u(a^*, a^*)$, certainly b does not weakly dominate a^*, so suppose that $u(b, a^*) = u(a^*, a^*)$. Then by the second condition for an ESS we have $u(b, b) < u(a^*, b)$. We conclude that b does not weakly dominate a^*.

400.2 Example of evolutionarily stable actions

The payoff matrix of the game is given in Figure 177.1. The pure strategy symmetric Nash equilibria are $(1, 1)$, $(2, 2)$, and $(3, 3)$. The only pure evolutionarily stable strategy is 1, by the following argument. The action 1 is evolutionarily stable because $(1, 1)$ is a strict Nash equilibrium. The action 2 is not evolutionarily stable, because 1 is a best response to 2 and

$$u(1, 1) = 1 > 2\delta = u(2, 1).$$

The action 3 is not evolutionarily stable, because 2 is a best response to 3 and

$$u(2, 2) = 2 > 3\delta = u(3, 2).$$

In the case that each player has n actions, every pair (i, i) is a Nash equilibrium; only the action 1 is an ESS.

	1	2	3
1	$1, 1$	$2, 2\delta$	$3, 3\delta$
2	$2\delta, 2$	$2, 2$	$3, 3\delta$
3	$3\delta, 3$	$3\delta, 3$	$3, 3$

Figure 177.1 The game in Exercise 400.2.

402.1 Mixed strategy ESSs

By the first condition for an ESS, (α^*, α^*) is a Nash equilibrium. Thus a is a best response to α^* by Proposition 116.2. Now by the second condition for an ESS,

$U(\alpha, \alpha) < U(\alpha^*, \alpha)$, where α is the mixed strategy that assigns probability 1 to a, so that (a, a) is not a Nash equilibrium.

405.1 *Hawk–Dove–Retaliator*

First suppose that $v \geq c$. In this case the game has two pure symmetric Nash equilibria, (A, A) and (R, R). However, A is not an ESS, because R is a best response to A and $u(R, R) > u(A, R)$. The action pair (R, R) is a strict equilibrium, so R is an ESS. Now consider the possibility that the game has a mixed strategy equilibrium (α, α). If α assigns positive probability to either P or R (or both) then R yields a payoff higher than does P, so only A and R may be assigned positive probability in a mixed strategy equilibrium. But if a strategy α assigns positive probability to A and R and probability 0 to P, then R yields a payoff higher than does A against an opponent who uses α. Thus the game has no symmetric mixed strategy equilibrium in this case.

Now suppose that $v < c$. Then the only symmetric pure strategy equilibrium is (R, R). This equilibrium is strict, so that R is an ESS. Now consider the possibility that the game has a mixed strategy equilibrium (α, α). If α assigns probability 0 to A then R yields a payoff higher than does P against an opponent who uses α; if α assigns probability 0 to P then R yields a payoff higher than does A against an opponent who uses α. Thus in any mixed strategy equilibrium (α, α), the strategy α must assign positive probability to both A and P. If α assigns probability 0 to R then we need $\alpha = (v/c, 1 - v/c)$ (the calculation is the same as for *Hawk–Dove*). Because R yields a lower payoff against this strategy than do A and P, and the strategy is an ESS in *Hawk–Dove*, it is an ESS in the present game. The remaining possibility is that the game has a mixed strategy equilibrium (α, α) in which α assigns positive probability to all three actions. If so, then the expected payoff to this strategy is less than $\frac{1}{2}v$, because the pure strategy P yields an expected payoff less than $\frac{1}{2}v$ against any such strategy. But then $U(R, R) = \frac{1}{2}v > U(\alpha, R)$, violating the second condition in the definition of an ESS.

In summary:

- If $v \geq c$ then R is the unique ESS of the game.
- If $v < c$ then both R and the mixed strategy that assigns probability v/c to A and $1 - v/c$ to P are ESSs.

405.2 Variant of *BoS*

The action pair (C, C) is a strict Nash equilibrium, so C is an ESS.

The game has also a symmetric mixed strategy equilibrium in which each player's mixed strategy is $\alpha^* = (\frac{3}{4}, \frac{1}{4}, 0)$. Every mixed strategy $\beta = (p, 1 - p, 0)$ is a best response to α^*, so in order that α^* be an ESS we need

$$U(\beta, \beta) < U(\alpha^*, \beta).$$

We have $U(\beta,\beta) = 4p(1-p)$ and $U(\alpha^*,\beta) = \frac{9}{4}(1-p) + \frac{1}{4}p$, so the inequality is equivalent to

$$(p - \tfrac{3}{4})^2 > 0,$$

which is true for all $p \neq \frac{3}{4}$. Thus α^* is an ESS.

The only other symmetric mixed strategy equilibrium is one in which each player's strategy is $\alpha^{**} = (\frac{3}{7}, \frac{1}{7}, \frac{3}{7})$. This strategy is not an ESS, because $u(C,C) = 1$, whereas $u(\alpha^{**}, C) = \frac{3}{7} < 1$.

405.3 Bargaining

The game is given in Figure 64.1.

The pure strategy of demanding 10 is not an ESS because 2 is a best response to 10 and $u(2,2) > u(10,2)$.

Now let α be the mixed strategy that assigns probability $\frac{2}{5}$ to 2 and $\frac{3}{5}$ to 8. Each player's payoff at the strategy pair (α,α) is $\frac{16}{5}$. Thus the only actions a that are best responses to α are 2 and 8, so that the only mixed strategies that are best responses to α assign positive probability only to the actions 2 and 8. Let β be the mixed strategy that assigns probability p to 2 and probability $1-p$ to 8. We have

$$U(\beta,\beta) = 5p(2-p)$$

and

$$U(\alpha,\beta) = 6p + \tfrac{4}{5}.$$

We find that $U(\alpha,\beta) - U(\beta,\beta) = 5(p - \frac{2}{5})^2$, which is positive if $p \neq \frac{2}{5}$. Hence α is an ESS.

Finally let α be the mixed strategy that assigns probability $\frac{4}{5}$ to 4 and $\frac{1}{5}$ to 6. Each player's payoff at the strategy pair (α,α) is $\frac{24}{5}$. Thus the only actions a that are best responses to α are 4 and 6, so that the only mixed strategies that are best responses assign positive probability only to the actions 4 and 6. Let β be the mixed strategy that assigns probability p to 4 and probability $1-p$ to 6. We have

$$U(\beta,\beta) = 5p(2-p)$$

and

$$U(\alpha^*,\beta) = 2p + \tfrac{16}{5}.$$

We find that $U(\alpha,\beta) - U(\beta,\beta) = 5(p - \frac{4}{5})^2$, which is positive if $p \neq \frac{4}{5}$. Hence α^* is an ESS.

408.1 Equilibria of C and of G

First suppose that (α_1,α_2) is a mixed strategy Nash equilibrium of C. Then for all mixed strategies β_1 of player 1 and all mixed strategies β_2 of player 2 we have

$$U_1(\alpha_1,\alpha_2) \geq U_1(\beta_1,\alpha_2) \text{ and } U_2(\alpha_1,\alpha_2) \geq U_2(\alpha_1,\beta_2).$$

Thus

$$\begin{aligned} u((\alpha_1,\alpha_2),(\alpha_1,\alpha_2)) &= \tfrac{1}{2}U_1(\alpha_1,\alpha_2) + \tfrac{1}{2}U_2(\alpha_1,\alpha_2) \\ &\geq \tfrac{1}{2}U_1(\beta_1,\alpha_2) + \tfrac{1}{2}U_2(\alpha_1,\beta_2) \\ &= u((\beta_1,\beta_2),(\alpha_1,\alpha_2)), \end{aligned}$$

so that $((\alpha_1,\alpha_2),(\alpha_1,\alpha_2))$ is a Nash equilibrium of G. If (α_1,α_2) is a strict Nash equilibrium of C then the inequalities are strict, and $((\alpha_1,\alpha_2),(\alpha_1,\alpha_2))$ is a strict Nash equilibrium of G.

Now assume that $((\alpha_1,\alpha_2),(\alpha_1,\alpha_2))$ is a Nash equilibrium of G. Then

$$u((\alpha_1,\alpha_2),(\alpha_1,\alpha_2)) \geq u((\beta_1,\beta_2),(\alpha_1,\alpha_2)),$$

or

$$\tfrac{1}{2}U_1(\alpha_1,\alpha_2) + \tfrac{1}{2}U_2(\alpha_1,\alpha_2) \geq \tfrac{1}{2}U_1(\beta_1,\alpha_2) + \tfrac{1}{2}U_2(\alpha_1,\beta_2),$$

for all conditional strategies (β_1,β_2). Taking $\beta_2 = \alpha_2$ we see that α_1 is a best response to α_2 in C, and taking $\beta_1 = \alpha_1$ we see that α_2 is a best response to α_1 in C. Thus (α_1,α_2) is a Nash equilibrium of G.

409.2 Variant of BoS

The game has two evolutionarily stable conditional strategies, *LD* and *DL*, corresponding to the strict Nash equilibria of the contest game.

414.1 A coordination game between siblings

The game with payoff function v is shown in Figure 180.1. If $x < 2$ then (Y, Y) is a strict Nash equilibrium of the games, so Y is an evolutionarily stable action in the game between siblings. If $x > 2$ then the only Nash equilibrium of the game is (X, X), and this equilibrium is strict. Thus the range of values of x for which the only evolutionarily stable action is X is $x > 2$.

	X	Y
X	x, x	$\frac{1}{2}x, \frac{1}{2}$
Y	$\frac{1}{2}, \frac{1}{2}x$	$1, 1$

v

Figure 180.1 The game with payoff function v derived from the game in Exercise 414.1.

414.2 Assortative mating

Under assortative mating, all siblings take the same action, so the analysis is the same as that for asexual reproduction. (A difficulty with the assumption of assortative mating is that a rare mutant will have to go to great lengths to find a mate that is also a mutant.)

416.1 Darwin's theory of the sex ratio

A normal organism produces pn male offspring and $(1-p)n$ female offspring (ignoring the small probability that the partner of a normal organism is a mutant). Thus it has $pn \cdot ((1-p)/p)n + (1-p)n \cdot n = 2(1-p)n^2$ grandchildren.

A mutant has $\frac{1}{2}n$ male offspring and $\frac{1}{2}n$ female offspring, and hence $\frac{1}{2}n \cdot ((1-p)/p)n + \frac{1}{2}n \cdot n = \frac{1}{2}n^2/p$ grandchildren.

Thus the difference between the number of grandchildren produced by mutant and normal organisms is

$$\tfrac{1}{2}n^2/p - 2(1-p)n^2 = n^2\left(\frac{1}{2p}\right)(1-2p)^2,$$

which is positive if $p \neq \frac{1}{2}$. (The point is that if $p > \frac{1}{2}$ then the fraction of a mutant's offspring that are males is higher than the fraction of a normal organism's offspring that are males, and males each bear more offspring than females. Similarly, if $p < \frac{1}{2}$ then the fraction of a mutant's offspring that are females is higher than the fraction of a normal organism's offspring that are females, and females each bear more offspring than males.)

Thus any mutant with $p \neq \frac{1}{2}$ invades the population; only $p = \frac{1}{2}$ is evolutionarily stable.

14 Repeated Games: The *Prisoner's Dilemma*

423.1 Equivalence of payoff functions

Suppose that a person's preferences are represented by the discounted sum of payoffs with payoff function u and discount factor δ. Then if the two sequences of outcomes $(x^1, x^2, \ldots)$ and $(y^1, y^2, \ldots)$ are indifferent, we have

$$\sum_{t=0}^{\infty} \delta^{t-1} u(x^t) = \sum_{t=0}^{\infty} \delta^{t-1} u(y^t).$$

Now let $v(x) = \alpha + \beta u(x)$ for all x, with $\beta > 0$. Then

$$\sum_{t=0}^{\infty} \delta^{t-1} v(x^t) = \sum_{t=0}^{\infty} \delta^{t-1} [\alpha + \beta u(x^t)] = \sum_{t=0}^{\infty} \delta^{t-1} \alpha + \beta \sum_{t=0}^{\infty} \delta^{t-1} u(x^t)$$

and similarly

$$\sum_{t=0}^{\infty} \delta^{t-1} v(y^t) = \sum_{t=0}^{\infty} \delta^{t-1} [\alpha + \beta u(y^t)] = \sum_{t=0}^{\infty} \delta^{t-1} \alpha + \beta \sum_{t=0}^{\infty} \delta^{t-1} u(y^t),$$

so that

$$\sum_{t=0}^{\infty} \delta^{t-1} v(x^t) = \sum_{t=0}^{\infty} \delta^{t-1} v(y^t).$$

Thus the person's preferences are represented also by the discounted sum of payoffs with payoff function v and discount factor δ.

425.1 Subgame perfect equilibrium of finitely repeated *Prisoner's Dilemma*

Use backward induction. In the last period, the action C is strictly dominated for each player, so each player chooses D, regardless of history. Now consider period $T - 1$. Each player's action in this period affects only the outcome in this period—it has no effect on the outcome in period T, which is (D, D). Thus in choosing her action in period $T - 1$, a player considers only her payoff in that period. As in period T, her action D strictly dominates her action C, so that in any subgame perfect equilibrium she chooses D. A similar argument applies to all previous periods, leading to the conclusion that in every subgame perfect equilibrium each player chooses D in every period, regardless of history.

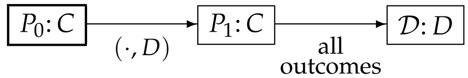


Figure 184.2 The strategy in Exercise 428.1a.

428.1 Strategies in an infinitely repeated *Prisoner's Dilemma*

a. The strategy is shown in Figure 184.2.

b. The strategy is shown in Figure 184.3.

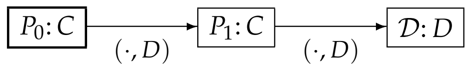


Figure 184.3 The strategy in Exercise 428.1b.

c. The strategy is shown in Figure 184.4.

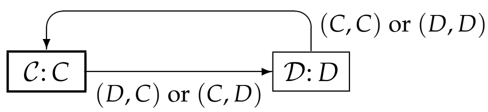


Figure 184.4 The strategy in Exercise 428.1c.

429.1 Grim trigger strategies in a general *Prisoner's Dilemma*

Suppose that player 1 uses the grim trigger strategy. If player 2 does too then her payoff is x in every period, so her discounted average payoff is x.

If player 2 chooses D in any period she obtains y in that period and 1 subsequently, so that her discounted average payoff is $(1-\delta)y+\delta$.

Thus the strategy pair in which both players use the grim trigger strategy is a Nash equilibrium if and only if $x \geq (1-\delta)y+\delta$, or

$$\delta \geq \frac{y-x}{y-1}.$$

430.1 Limited punishment strategies in an infinitely repeated *Prisoner's Dilemma*

Following the logic of Section 14.7.2, we find that the strategy pair in which each player uses k-period punishment is a Nash equilibrium if and only if

$$x(1-\delta^{k+1}) \geq y(1-\delta)+\delta(1-\delta^k),$$

or

$$(x-1)\delta^{k+1}+(1-y)\delta+y-x \leq 0.$$

431.1 *Tit-for-tat* in an infinitely repeated *Prisoner's Dilemma*

Following the logic of Section 14.7.3, the strategy pair in which each player uses *tit-for-tat* is a Nash equilibrium if and only if

$$x \geq \frac{y}{1+\delta} \quad \text{and} \quad x \geq y(1-\delta)+\delta,$$

or

$$\delta \geq \frac{y-x}{x} \quad \text{and} \quad \delta \geq \frac{y-x}{y-1}.$$

If $y \geq 2x$ then $(y-x)/x \geq 1$, so that the first inequality is not satisfied for any $\delta < 1$.

431.2 Nash equilibria of an infinitely repeated *Prisoner's Dilemma*

a. A player who adheres to the strategy obtains the discounted average payoff of 2. A player who deviates obtains the stream of payoffs $(3, 3, 1, 1, \ldots)$, with a discounted average of $(1-\delta)(3+3\delta)+\delta^2$. Thus for an equilibrium we require $(1-\delta)(3+3\delta)+\delta^2 \leq 2$, or $\delta \geq \frac{1}{2}\sqrt{2}$.

b. A player who adheres to the strategy obtains the payoff of 2 in every period. A player who chooses D in the first period and C in every subsequent period obtains the stream of payoffs $(3, 2, 2, \ldots)$. Thus for any value of δ a player can increase her payoff by deviating, so that the strategy pair is not a Nash equilibrium. Further, whatever the one-shot payoffs, a player can increase her payoff by deviating to D in a single period, so that for no payoffs is there any δ such that the strategy pair is a Nash equilibrium of the infinitely repeated game with discount factor δ.

c. A player who adheres to the strategy obtains the discounted average payoff of 2 (the outcome is (C, C) in every period). If player 2 deviates to D in every period then she induces the outcome to alternate between (C, D) and (D, D), yielding her a discounted average payoff of $(1-\delta)\cdot(3+3\delta^2+3\delta^4+\ldots)+(1-\delta)(\delta+\delta^3+\delta^5+\ldots) = (1-\delta)[3/(1-\delta^2)+\delta/(1-\delta^2)] = (3+\delta)/(1+\delta)$. For all $\delta < 1$ this payoff exceeds 2, so that the strategy pair is not a Nash equilibrium of the infinitely repeated game.

However, for different payoffs for the one-shot *Prisoner's Dilemma*, the strategy pair *is* a Nash equilibrium of the infinitely repeated game. The point is that the best deviation for player 2 leads to the sequence of outcomes that alternates between (C, D) and (D, D). If the average payoff of player 2 in these two outcomes is less than her payoff to the outcome (C, C) then the strategy pair is a Nash equilibrium for some values of δ. (For the payoffs in Figure 419.2 the average payoff of the two outcomes (C, D) and (D, D) is exactly equal to the payoff to (C, C).) Consider the general payoffs in Figure 186.1. The discounted average payoff of the sequence of outcomes that alternates

	C	D
C	x, x	$0, y$
D	$y, 0$	$1, 1$

Figure 186.1 A *Prisoner's Dilemma*.

between (C, D) and (D, D) is $(y + \delta)/(1 + \delta)$, while the discounted average of the constant sequence containing only (C, C) is x. Thus for the strategy pair to be a Nash equilibrium we need

$$\frac{y + \delta}{1 + \delta} \leq x,$$

or

$$\delta \geq \frac{y - x}{x - 1},$$

an inequality that is compatible with $\delta < 1$ if $x > \frac{1}{2}(y + 1)$—that is, if x exceeds the average of 1 and y.

433.1 Feasible payoff pairs in a *Prisoner's Dilemma*

The set of feasible payoff pairs is given in Figure 186.2.

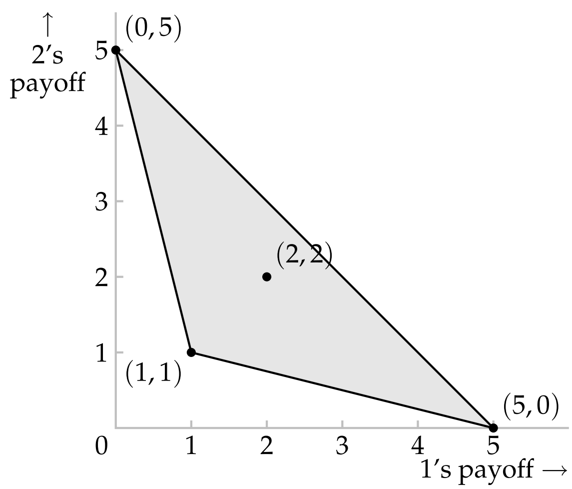


Figure 186.2 The set of feasible payoff pairs in the *Prisoner's Dilemma* with payoffs as in Figure 429.1 for $y = 5$ and $x = 2$.

439.1 Finitely repeated *Prisoner's Dilemma* with switching cost

a. Consider deviations by player 1, given that player 2 adheres to her strategy, in each subgame.

Subgame following initial history: If player 1 adheres to her strategy, her payoff is 3 in every period. If she deviates in the first period but otherwise follows her strategy, her payoff is 4 in the first period and 2 in every subsequent period. Given $T \geq 3$, player 1's deviation is not profitable.

Subgame following history ending in (C, C): If player 1 adheres to her strategy, her payoff is 3 in every period. If she deviates in the first period of the subgame but otherwise follows her strategy, her payoff is $4 - \epsilon$ in the first period of the subgame, and 2 in every subsequent period. Given $\epsilon > 1$, player 1's deviation is not profitable, even if it occurs in the last period of the game.

Subgame following history ending in (D, C) or (D, D): If player 1 adheres to her strategy, her payoff is 2 in every period. If she deviates in the first period of the subgame but otherwise follows her strategy, her payoff is $-\epsilon$ in the first period of the subgame, $2 - \epsilon$ in the next period, and 2 subsequently. Thus adhering to her strategy is optimal for player 1.

Subgame following history ending in (C, D): If player 1 adheres to her strategy, her payoff is $2 - \epsilon$ in the first period of the subgame, and 2 subsequently. If she deviates in the first period of the subgame but otherwise follows her strategy, her payoff is 0 in the first period of the subgame, $2 - \epsilon$ in the next period, and 2 subsequently. Given $\epsilon < 2$, player 1's deviation is not optimal even if it occurs in the last period of the game.

b. Given $\epsilon > 2$, a player does not gain from deviating from (C, C) in the next-to-last or last periods, even if she is not punished, and does not optimally punish a deviation by her opponent in the next-to-last period. Consider the strategy that chooses C at the start of the game and after any history that ends with (C, C), chooses D after any other history that has length at most $T - 2$, and chooses the action it chose in period $T - 1$ after any history of length $T - 1$ (where T is the length of the game). I claim that the strategy pair in which both players use this strategy is a subgame perfect equilibrium if $2 < \epsilon < 4$. Consider deviations by player 1, given that player 2 adheres to her strategy, in the subgames following the various possible histories.

Empty history: If player 1 adheres to her strategy, her payoff is 3 in every period. If she deviates in the first period but otherwise follows her strategy, her payoff is 4 in the first period and 2 in every subsequent period (her opponent switches to D). Given $T \geq 3$, player 1's deviation is not profitable.

History ending in (C, C), length $\leq T - 3$: If player 1 adheres to her strategy, her payoff is 3 in every period of the subgame. If she deviates in the first period of the subgame but otherwise follows her strategy, her payoff is $4 - \epsilon$ in the first period of the subgame, and 2 in every subsequent period (her opponent switches to D). Thus player 1's deviation is not profitable.

History ending in (C, C), length $\geq T - 2$: If player 1 adheres to her strategy, her payoff is 3 in each period of the subgame. If she deviates to D in the first period of the subgame, her payoff is $4 - \epsilon$ in that period, and 4 subsequently (her deviation is not punished). The length of the subgame is at most 2, so given $\epsilon > 2$, her deviation is not profitable.

History ending in (D, C) or (D, D): If player 1 adheres to her strategy, her payoff is 2 in every period. If she deviates in the first period of the subgame but otherwise follows her strategy, her payoff is $-\epsilon$ in the first period of the subgame, $2 - \epsilon$ in the next period, and 2 subsequently. Thus adhering to her strategy is optimal for player 1.

History ending in (C, D), length $\leq T - 3$: If player 1 adheres to her strategy, her payoff is $2 - \epsilon$ in the first period of the subgame (she switches to D), and 2 subsequently. If she deviates in the first period of the subgame but otherwise follows her strategy, her payoff is 0 in the first period of the subgame, $2 - \epsilon$ in the next period, and 2 subsequently, so the deviation is not profitable.

History ending in (C, D), length $T - 2$: If player 1 adheres to her strategy, her payoff is $2 - \epsilon$ in period $T - 1$ (she switches to D), and 2 in period T. If she deviates to C in the first period of the subgame but otherwise follows her strategy, she chooses C in period T, so her payoff is 0 in both period $T - 1$ and period T. Thus given $\epsilon < 4$ her deviation is not profitable.

History ending in (C, D), length $T - 1$: If player 1 adheres to her strategy, her payoff is 0 in period T (the outcome is (C, D)). If she deviates to D, her payoff is $2 - \epsilon$ in period T. Given $\epsilon > 2$, adhering to her strategy is thus optimal.

442.1 Deviations from grim trigger strategy

- If player 1 adheres to the strategy, she subsequently chooses D (because player 2 chose D in the first period). Player 2 chooses C in the first period of the subgame (player 1 chose C in the first period of the game), and then chooses D (because player 1 chooses D in the first period of the subgame). Thus the sequence of outcomes in the subgame is $((D, C), (D, D), (D, D), \ldots)$, yielding player 1 a discounted average payoff in the subgame of

$$(1-\delta)(3+\delta+\delta^2+\delta^3+\cdots) = (1-\delta)\left(3+\frac{\delta}{1-\delta}\right) = 3-2\delta.$$

- If player 1 refrains from punishing player 2 for her lapse, and simply chooses C in every subsequent period, then the outcome in period 2 and subsequently is (C, C), so that the sequence of outcomes in the subgame yields player 1 a discounted average payoff of 2.

If $\delta > \frac{1}{2}$ then $2 > 3 - 2\delta$, so that player 1 prefers to ignore player 2's deviation rather than to adhere to her strategy and punish player 2 by choosing D. (Note that the theory does not consider the possibility that player 1 takes player 2's play of D as a signal that she is using a strategy different from the grim trigger strategy.)

443.1 Delayed modified grim trigger strategies

Any deviation to C at the start of a subgame following a history of length $k - 1$ or less reduces a player's payoff and has no impact on the subsequent outcomes. No deviation in the first period of a later subgame is profitable if and only if $\delta \geq \frac{1}{2}$, by the argument in the text for the modified grim trigger strategy. Thus the strategy pair is a subgame perfect equilibrium if and only if $\delta \geq \frac{1}{2}$.

443.2 Different punishment lengths in subgame perfect equilibrium

Yes, an infinitely repeated *Prisoner's Dilemma* has such subgame perfect equilibria. As for the modified grim trigger strategy, each player's strategy has to switch to D not only if the other player chooses D but also if the player herself chooses D. The only subtlety is that the number of periods for which a player chooses D after a history in which not all the outcomes were (C, C) must depend on the identity of the player who first deviated. If, for example, player 1 punishes for two periods while player 2 punishes for three periods, then the outcome (C, D) induces player 1 to choose D for two periods (to punish player 2 for her deviation) whereas the outcome (D, C) induces her to choose D for three periods (while she is being punished by player 2). The strategy of each player in this case is shown in Figure 189.1. Viewed as a strategy of player 1, the top part of the figure entails punishment of player 2 and the bottom part entails player 1's reaction to her own deviation. Viewed as a strategy of player 2, the bottom part entails punishment of player 1 and the top part entails player 2's reaction to her own deviation.

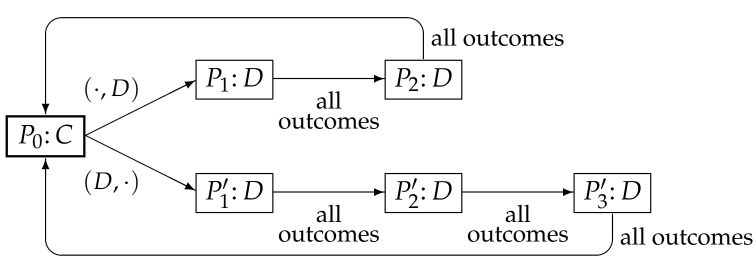


Figure 189.1 A strategy in an infinitely repeated *Prisoner's Dilemma* that punishes deviations for two periods and reacts to punishment by choosing D for three periods.

To find the values of δ for which the strategy pair in which each player uses the strategy in Figure 189.1 is a subgame perfect equilibrium, consider the result of each player's deviating at the start of a subgame.

First consider player 1. If she deviates when both players are in state P_0, she induces the outcome (D, C) followed by three periods of (D, D), and then (C, C) subsequently. This outcome path is worse for her than (C, C) in every period if and only if $\delta^3 - 2\delta + 1 \leq 0$, or if and only if δ is at least around 0.62 (as we found in Section 14.7.2). If she deviates when both players are in one of the other states then she is worse off in the period of her deviation and her deviation does not affect the subsequent outcomes. Thus player 1 cannot profitably deviate in the first period of any subgame if δ is at least around 0.62.

The same argument applies to player 2, except that a deviation when both players are in state P_0 induces (C, D) followed by three, rather than two periods of (D, D). This outcome path is worse for player 2 than (C, C) in every period if and only if $\delta^4 - 2\delta + 1 \leq 0$, or if and only if δ is at least around 0.55 (as we found in Section 14.7.2).

We conclude that the strategy pair in which each player uses the strategy in Figure 189.1 is a subgame perfect equilibrium if and only if $\delta^3 - 2\delta + 1 \leq 0$, or if and only if δ is at least around 0.62.

445.1 ***Tit-for-tat*** as a subgame perfect equilibrium

Suppose that player 2 adheres to *tit-for-tat*. Consider player 1's behavior in each subgame.

Whole game or subgame following history ending in (C, C) If player 1 adheres to *tit-for-tat* the outcome is (C, C) in every period, so that her discounted average payoff in the subgame is x. If she chooses D in the first period of the subgame, then adheres to *tit-for-tat*, the outcome alternates between (D, C) and (C, D), and her discounted average payoff is $y/(1+\delta)$. Thus we need $x \geq y/(1+\delta)$, or $\delta \geq (y-x)/x$, for a one-period deviation from *tit-for-tat* not to be profitable for player 1.

Subgame following history ending in (C, D) If player 1 adheres to *tit-for-tat* the outcome alternates between (D, C) and (C, D), so that her discounted average payoff is $y/(1+\delta)$. If she deviates to C in the first period of the subgame, then adheres to *tit-for-tat*, the outcome is (C, C) in every period, and her discounted average payoff is x. Thus we need $y/(1+\delta) \geq x$, or $\delta \leq (y-x)/x$, for a one-period deviation from *tit-for-tat* not to be profitable for player 1.

Subgame following history ending in (D, C) If player 1 adheres to *tit-for-tat* the outcome alternates between (C, D) and (D, C), so that her discounted average payoff is $\delta y/(1+\delta)$. If she deviates to D in the first period of the subgame, then adheres to *tit-for-tat*, the outcome is (D, D) in every period, and her discounted average payoff is 1. Thus we need $\delta y/(1+\delta) \geq 1$, or

$\delta \geq 1/(y-1)$, for a one-period deviation from *tit-for-tat* not to be profitable for player 1.

Subgame following history ending in (D, D) If player 1 adheres to *tit-for-tat* the outcome is (D, D) in every period, so that her discounted average payoff is 1. If she deviates to C in the first period of the subgame, then adheres to *tit-for-tat*, the outcome alternates between (C, D) and (D, C), and her discounted average payoff is $\delta y/(1+\delta)$. Thus we need $1 \geq \delta y/(1+\delta)$, or $\delta \leq 1/(y-1)$, for a one-period deviation from *tit-for-tat* not to be profitable for player 1.

The same arguments apply to deviations by player 2, so we conclude that (*tit-for-tat*, *tit-for-tat*) is a subgame perfect equilibrium if and only if $\delta = (y-x)/x$ and $\delta = 1/(y-1)$, or $y - x = 1$ and $\delta = 1/x$.

15 Repeated Games: General Results

452.3 Minmax payoffs of some games

a. If a player names $x < 5$ then her opponent can obtain more than 5 by naming $10 - x$. If she names 5 then the most her opponent can obtain is 5, by naming 5 or more. If she names 6 then the most her opponent can obtain is 5, by naming 5 or 6. If she names 7 or more her opponent can obtain 6 by naming 6. Thus each player's minmax payoff is 5.

b. Each firm's minmax payoff is at least 0, because a firm can ensure a payoff of zero by producing no output. By producing enough output to make the price zero, even if firm i produces nothing, the other firms can ensure that the price is zero whatever firm i does. Thus each firm's minmax payoff is exactly 0.

c. i. Whatever position one candidate chooses, the other can ensure, by locating at the same position, that the outcome is a tie. If one candidate's position is the voters' median favorite position, then the other candidate can do no better than a tie (by choosing the same position). Thus each candidate's minmax payoff is $\frac{1}{2}$.

ii. Two candidates can ensure that the third loses by choosing positions sufficiently close to each other on either side of the median. Thus each candidate's minmax payoff is 0.

454.2 Examples of application of Nash folk theorem

a. When the discount factor is close to 1, the approximate set of discounted average payoffs that can be obtained is shown in Figure 194.1.

b. The only possible pairs of payoffs in the game are $(1, 0)$ (candidate 1 wins outright), $(\frac{1}{2}, \frac{1}{2})$ (the candidates tie), and $(0, 1)$ (candidate 2 wins outright). The minmax payoff of each player is $\frac{1}{2}$ (see Exercise 452.3*c*), and the strategic game has a Nash equilibrium in which each candidate's payoff is $\frac{1}{2}$, so the set of discounted average payoff pairs to Nash equilibria of the infinitely repeated game consists of the single pair $(\frac{1}{2}, \frac{1}{2})$, regardless of the discount factor.

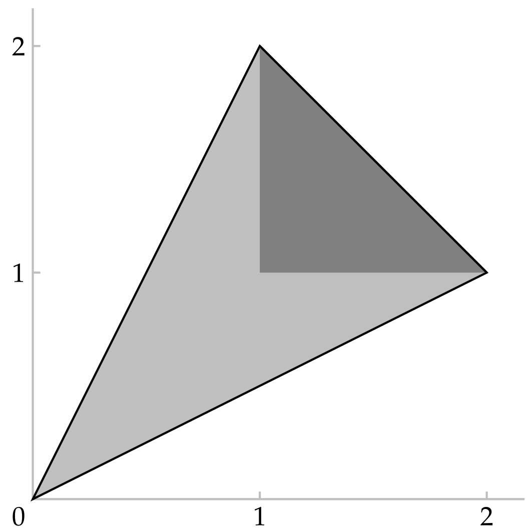


Figure 194.1 The light gray triangle is the set of feasible payoffs in *BoS*. The dark gray triangle is the approximate set of Nash equilibrium payoffs of the infinitely repeated game when the discount factor is close to 1.

454.3 Repeated Bertrand duopoly

a. Suppose that firm i uses the strategy s_i. If the other firm, j, uses s_j, then its discounted average payoff is

$$(1-\delta)\left(\tfrac{1}{2}\pi(p^m) + \tfrac{1}{2}\delta\pi(p^m) + \cdots\right) = \tfrac{1}{2}\pi(p^m).$$

If, on the other hand, firm j deviates to a price p then the closer this price is to p^m, the higher is j's profit, because the punishment does not depend on p. Thus by choosing p close enough to p^m the firm can obtain a profit as close as it wishes to $\pi(p^m)$ in the period of its deviation. Its profit during its punishment in the following k periods is zero. Once its punishment is complete, it can either revert to p^m or deviate once again. If it can profit from deviating initially then it can profit by deviating once its punishment is complete, so its maximal profit from deviating is

$$(1-\delta)\left(\pi(p^m) + \delta^{k+1}\pi(p^m) + \delta^{2k+2}\pi(p^m) + \cdots\right) = \frac{(1-\delta)\pi(p^m)}{1-\delta^{k+1}}.$$

Thus for (s_1, s_2) to be a Nash equilibrium we need

$$\frac{1-\delta}{1-\delta^{k+1}} \leq \tfrac{1}{2},$$

or

$$\delta^{k+1} - 2\delta + 1 \leq 0.$$

(This condition is the same as the one we found for a pair of k-period punishment strategies to be a Nash equilibrium in the *Prisoner's Dilemma* (Section 14.7.2).)

b. Suppose that firm i uses the strategy s_i. If the other firm does so then its discounted average payoff is $\frac{1}{2}\pi(p^m)$, as in part *a*. If the other firm deviates to some price p with $c < p < p^m$ in the first period, and maintains this price subsequently, then it obtains $\pi(p)$ in the first period and shares $\pi(p)$ in each subsequent period, so that its discounted average payoff is

$$(1-\delta)\left(\pi(p) + \tfrac{1}{2}\delta\pi(p) + \tfrac{1}{2}\delta^2\pi(p) + \cdots\right) = \tfrac{1}{2}(2-\delta)\pi(p).$$

If p is close to p^m then $\pi(p)$ is close to $\pi(p^m)$ (because π is continuous). In fact, for any $\delta < 1$ we have $2 - \delta > 1$, so that we can find $p < p^m$ such that $(2-\delta)\pi(p) > \pi(p^m)$. Hence the strategy pair is not a Nash equilibrium of the infinitely repeated game for any value of δ.

459.1 Costly price changing

a. Suppose firm 2 adheres to s. If firm 1 does so, its discounted average payoff is $\frac{1}{2}\pi(p^m)$. If it deviates, the best period in which to do so is the first period, because a deviation in this period incurs no cost. Any deviation yields the profit of 0 in every subsequent period, so the best deviations are prices just below p^m. Such a deviation yields a discounted average payoff close to $(1-\delta)\pi(p^m)$. Thus the strategy pair is a Nash equilibrium if and only if

$$\tfrac{1}{2}\pi(p^m) \geq (1-\delta)\pi(p^m),$$

or $\delta \geq \frac{1}{2}$.

b. The strategy pair is not a subgame perfect equilibrium. Consider a history in which firm 1 charged a price less than p^m in period 0 and firm 2 charged the price p^m. The strategy requires each firm to charge c in every subsequent period. But doing so is not a Nash equilibrium of the subgame: firm 2 obtains a profit of $-\epsilon$ if it does so, and is better off keeping its price the same as it was in the first period and obtaining a profit of at least zero.

459.2 Detection lags

a. The best deviations involve prices slightly less than p^*. Such a deviation by firm i yields a discounted average payoff close to

$$(1-\delta)\left(\pi(p^*) + \delta\pi(p^*) + \cdots + \delta^{k_i-1}\pi(p^*)\right) = (1-\delta^{k_i})\pi(p^*),$$

whereas compliance with the strategy yields the discounted average payoff $\frac{1}{2}\pi(p^*)$. Thus the strategy pair is a subgame perfect equilibrium for any value of p^* if $\delta^{k_1} \geq \frac{1}{2}$ and $\delta^{k_2} \geq \frac{1}{2}$, and is not a subgame perfect equilibrium for any value of p^* if $\delta^{k_1} < \frac{1}{2}$ or $\delta^{k_2} < \frac{1}{2}$. That is, the most profitable price for which the strategy pair is a subgame perfect equilibrium is p^m if $\delta^{k_1} \geq \frac{1}{2}$ and $\delta^{k_2} \geq \frac{1}{2}$ and is c if $\delta^{k_1} < \frac{1}{2}$ or $\delta^{k_2} < \frac{1}{2}$.

b. Denote by k_i^* the critical value of k_i found in part *a*. (That is, $\delta^{k_i^*} \geq \frac{1}{2}$ and $\delta^{k_i^*+1} < \frac{1}{2}$.)

If $k_i > k_i^*$ then no change in k_j affects the outcome of the price-setting subgame, so j's best action at the start of the game is θ, in which case i's best action is the same. Thus in one subgame perfect equilibrium both firms choose θ at the start of the game, and c regardless of history in the rest of the game.

If $k_i \leq k_i^*$ then j's best action is k_j^* if the cost of choosing k_j^* is at most $\frac{1}{2}\pi(p^m)$. Thus if the cost of choosing k_i^* is at most $\frac{1}{2}\pi(p^m)$ for each firm then the game has another subgame perfect equilibrium, in which each firm i chooses k_i^* at the start of the game and the strategy s_i in the price-setting subgame.

A promise by firm i to beat another firm's price is an inducement for consumers to inform firm i of deviations by other firms, and thus reduce its detection time. To this extent, such a promise tends to promote collusion.

459.3 Alternating moves

The strategy pair is a Nash equilibrium. If a player deviates she obtains 0 in the period of her deviation and the following period, and does not affect the following path.

The strategy pair is not a subgame perfect equilibrium. Consider a subgame following a history in which the last outcome is (Y, X) and it is player 1's turn to choose an action. If player 1 follows her strategy then in the first period of the subgame she chooses Y and obtains the payoff 0, and in the second period player 2 chooses Y, so that player 1's payoff is 1. If player 1 deviates from her strategy and chooses X in the first period of the subgame she obtains 2 in the first period and 0 in the second period (when player 2 returns to Y). In both cases player 1's payoff in every subsequent period is 1. Thus player 1 obtains the stream of payoffs 0, 1, 1, 1, … if she adheres to her strategy and 2, 0, 1, 1, … if she deviates. For any value of δ, she prefers the second stream.

16 Bargaining

468.1 Two-period bargaining with constant cost of delay

In the second period, player 1 accepts any proposal that gives a positive amount of the pie. Thus in any subgame perfect equilibrium player 2 proposes $(0, 1)$ in period 2, which player 1 accepts, obtaining the payoff $-c_1$.

Now consider the first period. Given the second period outcome of any subgame perfect equilibrium, player 2 accepts any proposal that gives her more than $1 - c_2$ and rejects any proposal that gives her less than $1 - c_2$. Thus in any subgame perfect equilibrium player 1 proposes $(c_2, 1 - c_2)$, which player 2 accepts.

In summary, the game has a unique subgame perfect equilibrium, in which

- player 1 proposes $(c_2, 1 - c_2)$ in period 1, and accepts all proposals in period 2
- player 2 accepts a proposal in period 1 if and only if it gives her at least $1 - c_2$, and proposes $(0, 1)$ in period 2 after any history.

The outcome of the equilibrium is that the proposal $(c_2, 1 - c_2)$ is made by player 1 and immediately accepted by player 2.

468.2 Three-period bargaining with constant cost of delay

The subgame following a rejection by player 2 in period 1 is a two-period game in which player 2 makes the first proposal. Thus by the result of Exercise 468.1, the subgame has a unique subgame perfect equilibrium, in which player 2 proposes $(1 - c_1, c_1)$, which player 1 immediately accepts.

Now consider the first period.

- If $c_1 \geq c_2$, player 2 rejects any offer of less than $c_1 - c_2$ (which she obtains if she rejects an offer), and accepts any offer of more than $c_1 - c_2$. Thus in an equilibrium player 1 offers her $c_1 - c_2$, which she accepts.
- If $c_1 < c_2$, player 2 accepts all offers, so that player 1 proposes $(1, 0)$, which player 2 accepts.

In summary, the game has a unique subgame perfect equilibrium, in which

- player 1 proposes $(1 - (c_1 - c_2), c_1 - c_2)$ if $c_1 \geq c_2$ and $(1, 0)$ otherwise in period 1, accepts any proposal that gives her at least $1 - c_1$ in period 2, and proposes $(1, 0)$ in period 3

- player 2 accepts any proposal that gives her at least $c_1 - c_2$ if $c_1 \geq c_2$ and accepts all proposals otherwise in period 1, proposes $(1 - c_1, c_1)$ in period 2, and accepts all proposals in period 3.

473.1 One-sided offers

The game has two distinct subgames—one in which the first move is a proposal by player 1 and one in which the first move is a response by player 2.

Subgame starting with proposal of player 1 If player 1 follows her strategy, she obtains x_1. If she offers player 2 more than $1 - x_1$, player 2 accepts the offer, making player 1 worse off. If she offers player 2 less than $1 - x_1$, player 2 rejects the offer, and player 1 obtains x_1 with one period of delay.

Subgame starting with response of player 2 Denote by (y_1, y_2) the proposal to which player 2 is responding. Her strategy calls for her to accept the proposal if and only if $y_2 \geq 1 - x_1$. If she rejects a proposal, she obtains x_2 with one period of delay, which is worth $\delta_2(1 - x_1)$ to her. Thus she should accept all proposals that give her at least $\delta_2(1 - x_1)$. Thus for her strategy to be optimal, given player 1's strategy, we need $\delta_2(1 - x_1) = 1 - x_1$, or $x_1 = 1$.

We conclude that the strategy pair is a subgame perfect equilibrium if and only if $x_1 = 1$. (The strategy pair in which $x_1 = 1$ is in fact the only subgame perfect equilibrium of the game.)

473.2 Alternating offer bargaining with constant cost of delay

First suppose that $c_1 < c_2$. A reasonable guess is that in a subgame perfect equilibrium in which player 1 always proposes $(1, 0)$, player 2 always accepts all proposals. Another reasonable guess is that the game has such an equilibrium in which player 2 always makes the same offer, say (z_1, z_2), and player 1 always uses the criterion "accept a proposal (x_1, x_2) if and only if $x_1 \geq z_1$" to respond to a proposal.

Is there a value of (z_1, z_2) such that this strategy pair is a subgame perfect equilibrium? A strategy pair is a subgame perfect equilibrium if and only if it satisfies the one-deviation property, so consider the conditions imposed on (z_1, z_2) by the one-deviation property. Examine each type of subgame in turn.

Subgame starting with proposal by player 1 If player 1 follows her strategy she obtains all the pie, so she cannot profitably deviate.

Subgame starting with response by player 2 Denote by (x_1, x_2) the proposal to which player 2 is responding. Her strategy calls for her to accept the proposal, yielding her the payoff x_2. If she rejects the proposal, she proposes (z_1, z_2), which player 1 accepts, yielding her the payoff $z_2 - c_2$. Thus for equilibrium we need $x_2 \geq z_2 - c_2$ for all x_2, which means that $0 \geq z_2 - c_2$, or $z_2 \leq c_2$.

Subgame starting with proposal by player 2 If player 2 follows her strategy she obtains the payoff z_2. If she offers player 1 more than z_1, player 1 accepts, and player 2 is worse off. If she offers player 1 less than z_1, player 1 rejects her offer and proposes $(1, 0)$, which she accepts, yielding her the payoff $-c_2$. Thus for equilibrium we need $z_2 \geq -c_2$.

Subgame starting with response by player 1 Denote by (y_1, y_2) the proposal to which player 1 is responding. If $y_1 \geq z_1$ her strategy calls for her to accept the proposal, yielding her the payoff y_1. If instead she rejects the proposal, she proposes $(1, 0)$, which player 2 accepts, yielding her the payoff $1 - c_1$. Thus for equilibrium we need $y_1 \geq 1 - c_1$ whenever $y_1 \geq z_1$, and hence $z_1 \geq 1 - c_1$. If $y_1 < z_1$ her strategy calls for her to reject the proposal, in which case she proposes $(1, 0)$, which player 2 accepts, yielding player 1 the payoff $1 - c_1$. If instead she accepts the proposal she obtains y_1. Thus for equilibrium we need $1 - c_1 \geq y_1$ whenever $y_1 < z_1$, and hence $1 - c_1 \geq z_1$.

From the analysis of the last subgame, we have $z_1 = 1 - c_1$, so that $z_2 = c_1$. Given $c_1 < c_2$, the one-deviation property is satisfied in every subgame, so that the strategy pair is a subgame perfect equilibrium.

In summary, the following pair of strategies is a subgame perfect equilibrium:

- player 1 always proposes $(1, 0)$ and accepts a proposal (y_1, y_2) if and only if $y_1 \geq 1 - c_1$
- player 2 always proposes $(1 - c_1, c_1)$ and accepts all proposals.

Now suppose that $c_1 = c_2 = c$. Let $c \leq z_1 \leq 1$ and consider the pair of strategies in which

- player 1 always proposes $(z_1, 1 - z_1)$ and accepts a proposal (y_1, y_2) if and only if $y_1 \geq z_1 - c$
- player 2 always proposes $(z_1 - c, 1 - z_1 + c)$ and accepts a proposal (x_1, x_2) if and only if $x_2 \geq 1 - z_1$.

I argue that this strategy pair is a subgame perfect equilibrium, by showing that it satisfies the one-deviation property. I consider each of the four distinct subgames in turn.

Subgame starting with proposal by player 1 If player 1 follows her strategy she obtains the payoff z_1. If she increases her offer to player 2, her offer is accepted and she is worse off. If she reduces her offer to player 2, her offer is rejected, and player 2 proposes $(z_1 - c, 1 - z_1 + c)$, which she accepts. Thus no deviation increases her payoff.

Subgame starting with response by player 2 Denote by (x_1, x_2) the proposal to which player 2 is responding. Her strategy calls for her to accept this proposal if and only if $x_2 \geq 1 - z_1$. If she rejects a proposal, she proposes

$(z_1 - c_1, 1 - z_1 + c)$, which player 1 accepts, yielding her the payoff $1 - z_1$. Thus no deviation increases her payoff.

Subgame starting with proposal by player 2 If player 2 follows her strategy she obtains the payoff $1 - z_1 + c$. If she offers player 1 more than $z_1 - c$, player 1 accepts, and player 2 is worse off. If she offers player 1 less than $z_1 - c$, player 1 rejects her offer and proposes $(z_1, 1 - z_1)$, which she accepts, yielding her the payoff $1 - z_1 - c$. Thus no deviation increases her payoff.

Subgame starting with response by player 1 Denote by (y_1, y_2) the proposal to which player 1 is responding. Her strategy calls for her to accept this proposal if and only if $y_1 \geq z_1 - c$. If she rejects a proposal, she proposes $(z_1, 1 - z_1)$, which player 2 accepts, yielding her the payoff $z_1 - c$. Thus no deviation increases her payoff.

479.1 One seller–two buyer game with random matching

The buyers are identical, and propose the same price, say b. Denote the price proposed by the seller by s. The condition for the seller to be indifferent between accepting and rejecting the proposal b of a buyer is

$$b = \tfrac{1}{2}\delta s + \tfrac{1}{2}\delta b,$$

as in the case that there is a single buyer. The condition for a buyer to be indifferent between accepting and rejecting the proposal s of the seller is

$$v - s = \tfrac{1}{4}\delta(v - s) + \tfrac{1}{4}\delta(v - b).$$

Solving these two equations we obtain

$$b = \frac{\delta(2-\delta)v}{4-3\delta} \quad \text{and} \quad s = \frac{(2-\delta)^2 v}{4-3\delta}.$$

When δ is close to 1, both of these prices are close to v.

479.2 One seller–two buyer game with choice of partner

First consider the seller.

Offers If she offers a lower price, it is accepted and she is worse off. If she offers a higher price, it is rejected. If her partner has valuation H, the partner counterproposes the price $\delta H/(1+\delta)$, which the seller accepts. If her partner has valuation L, the partner counterproposes either this price, which the seller accepts, or a lower price, which the seller rejects, leading the seller to choose the buyer with valuation H and propose $H/(1+\delta)$ in the next period, which the buyer accepts. Thus in no case is the seller better off proposing a price different from $H/(1+\delta)$.

Responses If she rejects an offer, she selects the buyer with valuation H, who accepts her proposed price of $H/(1+\delta)$. Thus she optimally rejects any price less than $\delta H/(1+\delta)$ and accepts any price greater than $\delta H/(1+\delta)$.

Choice of partner The buyer with valuation L neither proposes nor accepts any price greater than those proposed and accepted by the buyer with valuation H, so the seller optimally selects the buyer with valuation H.

Now consider the buyer with valuation H.

Offers If she proposes $\delta H/(1+\delta)$, the seller accepts it, and her payoff is $H - \delta H/(1+\delta)$. If she proposes a higher price, the seller accepts it, and she is worse off. If she proposes a lower price, the seller rejects it, continues bargaining with her, and proposes the price $H/(1+\delta)$, which she accepts, yielding her the payoff $\delta(H - H/(1+\delta) = \delta H - \delta H/(1+\delta)$, so that she is worse off than she is when she proposes the price $\delta H/(1+\delta)$.

Responses If she rejects an offer, she proposes the price $\delta H/(1+\delta)$, which the seller accepts, yielding her the payoff $\delta(H - \delta H/(1+\delta)) = \delta H/(1+\delta)$. If she accepts the price $H/(1+\delta)$ she obtains the same payoff ($H - H/(1+\delta) = \delta H/(1+\delta)$). Thus she optimally accepts any price of at most $H/(1+\delta)$.

Finally, consider the buyer with valuation L. First consider the case in which $L \geq \delta H/(1+\delta)$.

Offers If she proposes $\delta H/(1+\delta)$, the seller accepts it, and her payoff is $L - \delta H/(1+\delta) \geq 0$. If she proposes a higher price, the seller accepts it, and she is worse off. If she proposes a lower price, the seller rejects it and switches to the other buyer; her payoff is 0.

Responses If she rejects an offer, she proposes the price $\delta H/(1+\delta)$, which the seller accepts, yielding her the payoff $\delta(H - \delta H/(1+\delta)) = \delta H/(1+\delta)$. If she accepts the price $H/(1+\delta)$ she obtains the same payoff ($H - H/(1+\delta) = \delta H/(1+\delta)$). Thus she optimally accepts any price of at most $H/(1+\delta)$.

Now consider the case in which $L < \delta H/(1+\delta)$.

Offers If she proposes L, the seller rejects it and switches to the other buyer, who accepts her offer; thus her payoff is 0. If she proposes a higher price, her payoff is negative if the offer is accepted, so proposing L is optimal.

Responses If she accepts a price greater than L, her payoff is negative. If she rejects an offer, she proposes L, which the seller rejects; the seller switches to the other buyer, and her payoff is zero. Thus she optimally accepts any price of L or less.

480.1 One seller–two buyer game with public price announcements

First consider the seller.

Offers Her proposed price p^* is accepted by B_H, yielding her the payoff p^*. If she proposes a higher price, both buyers reject it, and propose the price L; the seller accepts B_H's proposal, and obtains the payoff $\delta L < p^*$.

Responses If she rejects both buyers' proposals, she proposes p^*, which B_H accepts, yielding her the payoff δp^*, so she should reject prices less than δp^* and accept prices higher than δp^*.

Now consider buyer B_H.

Offers Given $\delta H/(1+\delta) < L$, we have $\delta p^* = \delta^2 L + \delta(1-\delta)H < \delta^2 L + (1-\delta)(1+\delta)L = L$, so the seller accepts B_H's proposed price of L, yielding B_H the payoff $H - L$. If she proposes a lower price, then given the strategy of B_L to offer L, the seller trades with B_L, and B_H obtains the payoff of 0.

Responses If she rejects a price of at most L, buyer B_L subsequently accepts it, and her payoff is 0.

If she rejects a price from L to $\delta L + (1-\delta)H$, buyer B_L also rejects it, and both buyers propose the price L in the next period. The seller accepts B_H's proposal, yielding B_H the payoff $\delta(H-L)$. If B_H accepts such a price, her payoff is at least $H - \delta L - (1-\delta)H = \delta(H-L)$. Thus her decision to accept such a price is optimal.

If she rejects a higher price, she obtains $\delta(H-L)$, as in the previous case. If she accepts such a price her payoff is less than $\delta(H-L)$. Thus rejection is optimal.

Finally consider buyer B_L.

Offers Her proposed price of L is rejected by the seller (given that B_H proposes the same price), yielding her the payoff 0. If she proposes a higher price, the seller accepts it, and her payoff is negative.

Responses If she accepts a price greater than L (at the start of a subgame following a rejection of the price by B_H), her payoff is negative. If she rejects a lower price, her payoff is 0, whereas if she accepts such a price her payoff is positive.

486.1 Implications of PAR, SYM, and IIA

Consider the bargaining problem (U', d) in which U' is the triangle with corners at $(0,0)$, $(0,2)$, and $(2,0)$, and $d = (0,0)$. This problem is symmetric; the only agreement compatible with PAR and SYM is $(1,1)$. Now, U is a subset of U' that contains $(1,1)$, so the only agreement compatible with PAR, SYM, and IIA is $(1,1)$.

488.1 Bargaining solutions

a. Suppose that $d = (0,0)$ and U is the triangle with corners at $(0,0)$, $(0,1)$ and $(1,0)$. The solution assigns to this problem the point $(\frac{1}{2},\frac{1}{2})$. Now suppose that $d' = (0,0)$ and U' is the triangle with corners at $(0,0)$, $(0,1)$ and $(2,0)$. The solution assigns to this problem the point $(\frac{2}{3},\frac{2}{3})$. But $U' = \{(2u_1,u_2) : (u_1,u_2) \in U\}$ and $d' = (2d_1,d_2)$, so INV requires, given the bargaining solution of (U,d), that the bargaining solution of (U',d') be $(1,\frac{1}{2})$.

b. Suppose that $d = (0,0)$ and U is the triangle with corners at $(0,0)$, $(0,1)$ and $(1,0)$. The solution assigns to this problem the point $(\frac{1}{2},\frac{1}{2})$. Now suppose that $d' = (0,0)$ and U' is the quadrilateral with corners at $(0,0)$, $(0,\frac{1}{2})$, $(\frac{1}{2},\frac{1}{2})$, and $(1,0)$. The solution assigns to this problem the point $(\frac{2}{3},\frac{1}{3})$. But $d = d'$, $U' \subset U$, and the solution of (U,d) is in U', so IIA requires that the solution of (U',d) be the same as the solution of (U,d).

488.2 Wage bargaining

The Nash bargaining solution of (U,d) maximizes

$$(f(\ell^*) - \ell^* w)(\ell^* w + (L - \ell^*)w_0 - Lw_0),$$

or

$$(f(\ell^*) - \ell^* w)\ell^*(w - w_0).$$

This function is a quadratic in w that is equal to zero when $w = f(\ell^*)/\ell^*$ and when $w = w_0$. Thus the value of w that maximizes it is

$$\tfrac{1}{2}(f(\ell^*)/\ell^* + w_0),$$

the average of the average output of a worker and the "outside wage" w_0.

17 Appendix: Mathematics

497.1 Maximizer of quadratic function

We have $x(\alpha - x) = x(-x + \alpha) = x(ax + b) + c$ with $a = -1$, $b = \alpha$, and $c = 0$. Thus its maximizer is $-\frac{1}{2}b/a = \frac{1}{2}\alpha$.

499.3 Sums of sequences

In the first case set $r = \delta^2$ to transform the sum into $1 + r + r^2 + \cdots$, which is equal to $1/(1-r) = 1/(1-\delta^2)$.

In the second case split the sum into $(1 + \delta^2 + \delta^4 + \cdots) + (2\delta + 2\delta^3 + 2\delta^5 + \cdots)$; the first part is equal to $1/(1-\delta^2)$ and the second part is equal to $2\delta(1 + \delta^2 + \delta^4 + \cdots)$, or $2\delta/(1-\delta^2)$. Thus the complete sum is

$$\frac{1+2\delta}{1-\delta^2}.$$

504.2 Bayes' law

Your posterior probability of carrying X given that you test positive is

$$\frac{\Pr(\text{positive test}|X)\Pr(X)}{\Pr(\text{positive test}|X)\Pr(X) + \Pr(\text{positive test}|\neg X)\Pr(\neg X)}$$

where $\neg X$ means "not X". This probability is equal to $0.9p/(0.9p + 0.2(1-p)) = 0.9p/(0.2 + 0.7p)$, which is increasing in p (i.e. a smaller value of p gives a smaller value of the probability). If $p = 0.001$ then the probability is approximately 0.004. (That is, if 1 in 1,000 people carry the gene then if you test positive on a test that is 90% accurate for people who carry the gene and 80% accurate for people who do not carry the gene, then you should assign probability 0.004 to your carrying the gene.) If the test is 99% accurate in both cases then the posterior probability is $(0.99 \cdot 0.001)/[0.99 \cdot 0.001 + 0.01 \cdot 0.999] \approx 0.09$.

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