Variants The key point behind the results is that under unanimity rule a juror's vote makes a difference to the outcome only if every other juror votes for conviction. Consequently, a juror, when deciding how to vote, rationally assesses the defendant's probability of guilt under the assumption that every other juror votes for conviction. The fact that this implication of unanimity rule drives the results suggests that the Nash equilibria might be quite different if less than unanimity were required for conviction. The analysis of such rules is difficult, but indeed the Nash equilibria they generate differ significantly from the Nash equilibria under unanimity rule. In particular, the analogue of the mixed strategy Nash equilibria considered earlier generates a probability that an innocent defendant is convicted that approaches zero as the jury size increases, as Feddersen and Pesendorfer (1998) show.

The idea behind the equilibria of the model in the next exercise is related to the ideas in this section, though the model is different.

? EXERCISE 307.1 (Swing voter's curse) Whether candidate 1 or candidate 2 is elected depends on the votes of two citizens. The economy may be in one of two states, A and B. The citizens agree that candidate 1 is best if the state is A and candidate 2 is best if the state is B. Each citizen's preferences are represented by the expected value of a Bernoulli payoff function that assigns a payoff of 1 if the best candidate for the state wins (obtains more votes than the other candidate), a payoff of 0 if the other candidate wins, and payoff of $\frac{1}{2}$ if the candidates tie. Citizen 1 is informed of the state, whereas citizen 2 believes it is A with probability 0.9 and B with probability 0.1. Each citizen may either vote for candidate 1, vote for candidate 2, or not vote.

a. Formulate this situation as a Bayesian game. (Construct the table of payoffs for each state.)

b. Show that the game has exactly two pure Nash equilibria, in one of which citizen 2 does not vote and in the other of which she votes for 1.

c. Show that an action of one of the players in the second equilibrium is weakly dominated.

d. Why is "swing voter's curse" an appropriate name for the determinant of citizen 2's decision in the first equilibrium?

9.8 Appendix: auctions with an arbitrary distribution of valuations

9.8.1 *First-price sealed-bid auctions*

In this section I construct a symmetric equilibrium of a first-price sealed-bid auction for a distribution F of valuations that satisfies the assumptions in Section 9.6.2 and is differentiable on $(\underline{v}, \overline{v})$. (Unlike the remainder of the book, the section uses calculus.)

As before, denote the bid of type v of player i (i.e. player i when her valuation is v) by $\beta_i(v)$. In a symmetric equilibrium, every player uses the same bidding function: for some function β we have $\beta_i = \beta$ for every player i. A reasonable guess is that if such an equilibrium exists, β is increasing: the higher a player's valuation, the more she bids. Under the additional assumption that β is differentiable, I derive a condition that it must satisfy in any symmetric equilibrium. Exactly one function satisfies this condition, and this function is in fact increasing (as you are asked to show in an exercise).

Suppose that all $n-1$ players other than i bid according to the increasing differentiable function β. Then, given the assumptions on F, the probability of a tie is zero, and hence for any bid b, the expected payoff of player i when her valuation is v and she bids b is

$$(v-b)\Pr(\text{Highest bid is } b) = (v-b)\Pr(\text{All } n-1 \text{ other bids} \leq b). \qquad (308.1)$$

Now, a player bidding according to the function β bids at most b, for $\beta(\underline{v}) \leq b \leq \beta(\overline{v})$, if her valuation is at most $\beta^{-1}(b)$ (the inverse of β evaluated at b). Thus the probability that the bids of the $n-1$ other players are all at most b is the probability that the highest of $n-1$ randomly selected valuations—a random variable denoted $\mathbf{X}$ in Section 9.6.2—is at most $\beta^{-1}(b)$. Denoting the cumulative distribution function of $\mathbf{X}$ by H, the expected payoff in (308.1) is thus

$$(v-b)H(\beta^{-1}(b)) \text{ if } \beta(\underline{v}) \leq b \leq \beta(\overline{v})$$

(and 0 if $b < \beta(\underline{v})$, $v-b$ if $b > \beta(\overline{v})$).

I now claim that in a symmetric equilibrium in which every player bids according to β, we have $\beta(v) \leq v$ if $v > \underline{v}$ and $\beta(\underline{v}) = \underline{v}$. If $v > \underline{v}$ and $\beta(v) > v$, then a player with valuation v wins with positive probability (players with valuations less than v bid less than $\beta(v)$, because β is increasing) and obtains a negative payoff if she does so. She obtains a payoff of zero by bidding v, so for equilibrium we need $\beta(v) \leq v$ whenever $v > \underline{v}$. Given that β satisfies this condition, if $\beta(\underline{v}) > \underline{v}$ then a player with valuation $\underline{v}$ wins with positive probability, and obtains a negative payoff if she does so. Thus $\beta(\underline{v}) \leq \underline{v}$. If $\beta(\underline{v}) < \underline{v}$, then players with valuations slightly greater than $\underline{v}$ also bid less than $\underline{v}$ (because β is continuous), so that a player with valuation $\underline{v}$ who increases her bid slightly wins with positive probability and obtains a positive payoff if she does so, rather than obtaining the payoff of zero. We conclude that $\beta(\underline{v}) = \underline{v}$.

Now, the expected payoff of a player of type v when every other player uses the bidding function β is differentiable on $(\underline{v}, \beta(\overline{v}))$ (given that β is increasing and differentiable, and $\beta(\underline{v}) = \underline{v}$) and, if $v > \underline{v}$, is increasing at $\underline{v}$. Thus the derivative of this expected payoff with respect to b is zero at any best response less than $\beta(\overline{v})$:

$$-H(\beta^{-1}(b)) + \frac{(v-b)H'(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))} = 0. \qquad (308.2)$$

(The derivative of β^{-1} at the point b is $1/\beta'(\beta^{-1}(b))$.)

In a symmetric equilibrium in which every player bids according to β, the best response of type v of any given player to the other players' strategies is $\beta(v)$. Because β is increasing, we have $\beta(v) < \beta(\overline{v})$ for $v < \overline{v}$, so $\beta(v)$ must satisfy (308.2) whenever $\underline{v} < v < \overline{v}$. If $b = \beta(v)$, then $\beta^{-1}(b) = v$, so that substituting $b = \beta(v)$ into (308.2) and multiplying by $\beta'(v)$ yields

$$\beta'(v)H(v) + \beta(v)H'(v) = vH'(v) \text{ for } \underline{v} < v < \overline{v}.$$

The left-hand side of this differential equation is the derivative with respect to v of $\beta(v)H(v)$. Thus for some constant C we have

$$\beta(v)H(v) = \int_{\underline{v}}^{v} xH'(x)\,dx + C \text{ for } \underline{v} < v < \overline{v}.$$

The function β is bounded, so considering the limit as v approaches $\underline{v}$, we deduce that $C = 0$.

We conclude that if the game has a symmetric Nash equilibrium in which each player's bidding function is increasing and differentiable on $(\underline{v}, \overline{v})$, then this function is defined by

$$\beta^*(v) = \frac{\int_{\underline{v}}^{v} xH'(x)\,dx}{H(v)} \text{ for } \underline{v} < v \leq \overline{v}$$

and $\beta^*(\underline{v}) = \underline{v}$. Now, the function H is the cumulative distribution function of $\mathbf{X}$, the highest of $n-1$ independently drawn valuations. Thus $\beta^*(v)$ is the expected value of $\mathbf{X}$ conditional on its being less than v: $\beta^*(v) = \mathrm{E}[\mathbf{X} \mid \mathbf{X} < v]$, as claimed in Section 9.6.2.

We may alternatively express the numerator in the expression for $\beta^*(v)$ as $vH(v) - \int_{\underline{v}}^{v} H(x)dx$ (using integration by parts), so that given $H(v) = (F(v))^{n-1}$ (the probability that $n-1$ valuations are at most v), we have

$$\beta^*(v) = v - \frac{\int_{\underline{v}}^{v} H(x)\,dx}{H(v)} = v - \frac{\int_{\underline{v}}^{v} (F(x))^{n-1}\,dx}{(F(v))^{n-1}} \text{ for } \underline{v} < v \leq \overline{v}. \qquad (309.1)$$

In particular, $\beta^*(v) < v$ for $\underline{v} < v \leq \overline{v}$.

?? EXERCISE 309.2 (Property of the bidding function in a first-price auction) Show that the bidding function β^* defined in (309.1) is increasing.

? EXERCISE 309.3 (Example of Nash equilibrium in a first-price auction) Verify that for the distribution F uniform from 0 to 1 the bidding function defined by (309.1) is $(1 - 1/n)v$.

9.8.2 Revenue equivalence of auctions

I argued in the text that the expected price paid by the winner of a first-price auction is the same as the expected price paid by the winner of a second-price auction. A much more general result may be established.

Suppose that n risk-neutral bidders are involved in a sealed-bid auction in which the price is an arbitrary function of the bids (not necessarily the highest, or