Lecture Notes for Prelims Course, 1972/3, at Cambridge.

1

## Notes on Notes etc.

1) The Lecture notes are intended to supplement ordinary reading and lectures. They are not remotely exhaustive. An asterisk denotes sections which are intended for the numerate only and can readily be skipped by everyone on first reading.

2) There are no good textbooks in English for what I want to do although there is an excellent one in French (English edition expected). It is

E. Malinvaud: Leçons de théorie microéconomique. Dunod. Paris 1969.

You will easily find in this book the part corresponding to various parts of the lectures.

Anyone seriously interested should, by the end of the sessions on micro-economics have read the essay entitled: "Allocation of Resources and the Price System" in T.C. Koopmans: Three Essays on The State of Economic Science, McGraw Hill, 1957.

K. Lancaster: Mathematical Economics, Macmillan N.Y. 1968 contains mistakes but is a useful reference.

E.A.G. Robinson: The Structure of Competitive Industry. C.U.P. is a must for the early part of the lectures.
If you want a critique of the whole approach to be followed read

J. Kornai: Anti-Equilibrium, North Holland 1971. You do not have to read it all to get the point.

E.H. Chamberlin: Monopolistic Competition and

J. Graaff: Theoretical Welfare Economics
serious students will want to have digested.

Further references will be given during lectures and again before I get to Macro-economics. But you should have a look also at

A.A. Walters: "Production and Cost Functions: An Econometric Survey". Econometrica, Vol. 31 No. 1-2 1963.

3) The lectures, (and notes), are intended to serve a small minority of those reading Economics. They are not required for examinations. Moreover there are many successful practical economists everywhere who are quite innocent of the matters to be discussed so that the lectures are not required to ensure comfort in later life: There are also many people who regard this kind of careful and abstract approach as not worthwhile and they are just as often respected academically as are those who hold the opposite view. So the lectures are not required for academic respectability.

If you are impatient with abstractions and/or find that mathematical reasoning does not come easily and fairly effortlessly, do not pursue these lectures; there are better things to do with your time. On the other hand if these objections do not apply and the outline given in the first lecture seems interesting enough to pursue, then if you are to get anything for this at all, you will have to do some work on your own. It is not my intention to utter only sentences which can be understood at first hearing. So if you do not like 'difficult lectures' then these are not for you.

# Principles of Economics I. (Frank Hahn)

## The Production Possibilities of the Firm

### 1) Goods

We shall distinguish goods by three properties: a) their characteristics b) the date at which they are available and c) the place at which they are available. Thus one distinguishes butter of a certain grade in Glasgow on the 22nd September 1972 from butter of the same grade in Glasgow on the 23rd September 1972 and ~~for~~ from butter of the same grade in Liverpool on the 23rd September 1972 etc. There are no great difficulties with b) or c) but a) may be hard in practice. This is particularily so with 'second-hand' goods and with services. Indeed there are interesting economic consequences from the actual difficulties of distinguishing between goods. For instance some people maintain that an important function of a University is to grade the 'goods' graduates represent. But one thing at a time and for the moment take goods as always well defined.

### 2) Activity of Firm

Production uses goods to make goods. One calls the goods used inputs and the goods produced outputs. A list on which are entered all the outputs and inputs of a firm is called an activity. This suggests that an activity is best thought of as a vector. It is convenient to regard inputs as negative and outputs as positive. If n goods are involved we then think of the vector in $E^n$ which is n-dimensional Euclidean space and each component of the vector is read off from use of the co-ordinates of that space. More formally,

Definition D.I.1. A vector $y \in E^n$ i.e. $y = (y_1 .. y_n)$ is called an activity of the firm if the components $y_i$ have the following interpretations

a) if $y_i > 0$ then $y_i$ is the amount of output of good i by the firm

b) if $y_i < 0$ then $y_i$ is the amount of input of good i by the firm.

### 3) Time*

It is useful to think of time as divided into discrete intervals and prudent at this stage to take the number of days as finite (else $E^n$ might have to be taken as infinite dimensional). Let us take our view point as today and write $y^o$ when we want to emphasise that we are looking at the firm's activity today (that is the day labelled zero; tomorrow gets the label 1 etc.).

Then note that $y^o$ contains goods dated today, tomorrow etc so it contains items which may not exist yet. One may for the moment say that $y^o$ includes current and planned inputs and outputs. This interpretation will be important later. For the moment I shall not be emphasising this aspect and so omit superscripts to y.

4) The Production Set of the Firm.

One is interested in the technological environment of the firm and it seems natural to describe it as the list of activities y it could choose - sometimes for vividness called the book of blue prints. One has here an important ambiguity: should we not distinguish between the book of blue-prints and that part of it which is already known to the firm? The answer is yes, we should so distinguish. Indeed reflection suggests that there may be activities, other than the ones so far described, which have inputs - say labour time, and outputs consisting of new pages of the book which have been learned. These are "knowledge producing activities" and they may be required even if every page of the book is already known to someone in the economy.

Again, it would be as unwise to pursue this difficult problem now as it would be to forget that it exists. At the moment we take the available set of activities y in the economy as identical with those known by the firm. This set we write as Y. Formally

Definition D.T.2. Y is called the production set of the firm if the members of this set, (written y), are the activities the firm could choose.

Example. a) Suppose there are three goods ($n = 3$) and that the production set of the firm is given by

$$Y = \{ y \mid y_1 \leqq \min(-a_2y_2, -a_3y_3),\ y_1 \geqq 0,\ y_i \leqq 0 \text{ for } i = 2,3,\ a_i > 0,\ i = 1,2 \}$$

[ The expression: $\min(x, z)$ says "take the smaller of the two members x and z" ].

Let us interprete this set. Consider any input pair $(y_2, y_3)$ say $(-3, -5)$. Let $a_2 = 1$, $a_3 = \frac{1}{2}$. So

$$(-a_2y_2, -a_3y_3) = (3, 2\tfrac{1}{2})$$

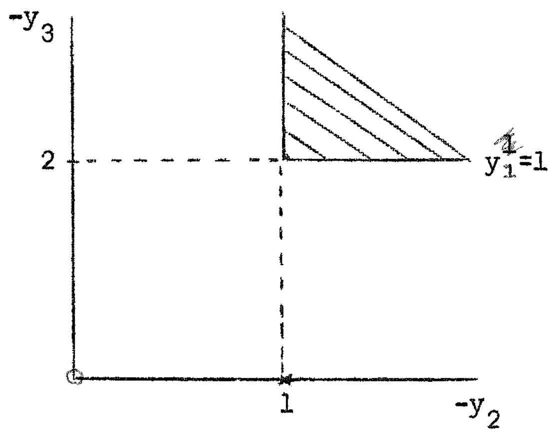
The smallest of these two numbers is $2\frac{1}{2}$ and so Y tells us that the firm can produce quantity of output $y_1$ bounded as follows:

$$0 \leqq y_1 \leqq 2\tfrac{1}{2}$$

Or alternatively suppose the firm wants to produce one unit of good 1 ($y_1 = 1$). What is the minimum quantity of input that it requires?: Since $a_2 = 1$ it must use at least one unit of good 2 as input. Since $a_3 = \frac{1}{2}$ it must use at least two units of good 3, then

$$\min (-a_2 y_2, -a_3 y_3) = \min (1 \times 1, \tfrac{1}{2} \times 2) = 1$$

Check that if it used any less of either input it could not produce one unit of good 1, while if it used more of one input and no more of the other it could still produce no more than one unit of output. In Fig. 1 I have drawn a curve showing all the combinations of inputs of goods 2 and 3 which just enable the firm to produce just one unit of output




Draw the curve for $y_1 = 2$, $y_1 = \frac{1}{2}$. Notice that from the definition of Y any combination of inputs given by a point in the shaded area also allows the production of one unit of good 1 (although it could produce more than that).

Example. b) Let

$$Y = \{ y \mid y_1 \leqq x_1 + x_2 + x_3,\ y_2 \leqq a_{11} x_1 + a_{12} x_2 + a_{13} x_3,\ y_3 \leqq a_{21} x_1 + a_{22} x_2 + a_{23} x_3, \text{ where}$$

2,3

$$y_1 \geqq 0,\ y_i \leqq 0\ i = 1,2\ \ x_i \geqq 0 \text{ all } i, \text{ and } a_{ij} < 0 \text{ all } i, j \}$$

(If you know about matrices you will see how Y can be written more succinctly). Let us consider all the ways of producing one unit of good 1.

(i) set $x_2 = x_3 = 0$. Then since we want $y_1 = 1$ we must set $x_1 \geqq 1$. Take $x_1 = 1$. Then we shall need at least $a_{11}$ of good 2 and $a_{21}$ of good 3 as inputs. These are here take as negative numbers but if we want to speak of positive quantities we simply say we need at least $-a_{11}$ of good 2 and $-a_{21}$ of good 3. Then setting $x_1 = x_3 = 0$ and the $x_1 = x_2 = 0$ and proceeding as before we obtain the points labelled $P^1$, $P^2$, $P^3$ in Fig. 2. Each one of these points $P^i$ shows the minimum amounts of the two inputs needed to produce one unit of output where $x_j = 0$ for $j \neq i$. I have also shown the shaded areas associated with each of these points which show that the firm could use more of the inputs than is indicated by $P^i$ to produce one unit of output.

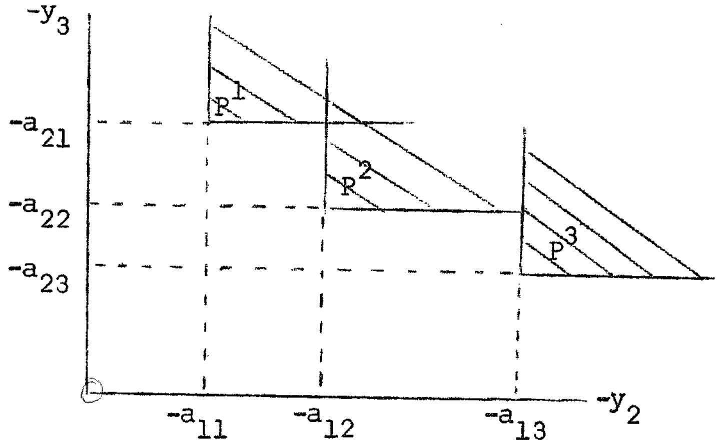



Fig. 2

(ii) Set $x_3 = 0$. Then any combination of non-negative $x_1$, $x_2$ such that $x_1 + x_2 = 1$ and $y_2 = a_{11}\,x_1 + a_{12}\,x_2$, $y_3 = a_{21}\,x_1 + a_{22}\,x_2$ allows the firm to produce one unit of output without using more of any one input than is strictly required.

In our diagram the co-ordinates of $P^1$ are multiplied by $x_1$ and added to the co-ordinates of $P^2$ multiplied by $x_2$. The resulting point is on the chord joining $P^1$ to $P^2$ (why?), when $x_1 > 0$, $x_2 > 0$. So each one of these points for different values of $x_1$ and $x_2$ shows a way of producing one unit of output without being able to use less of any one input, given $x_3 = 0$. By repeating this operation with $x_1 = 0$, $x_2 + x_3 = 1$ we obtain the following picture,

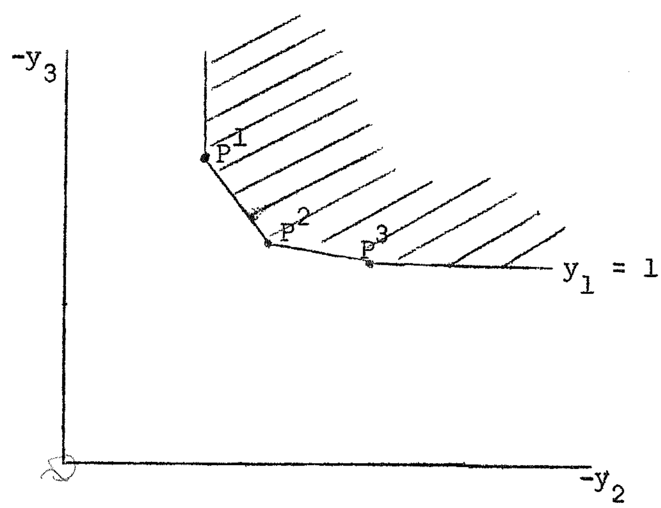



Fig. 3

giving us all the possible input combinations in Y which allow us to produce one unit of good 1. What is the picture for $y_1 = 2$?

Example. c)

$$\text{Let } Y = \{y \mid y_1 \leq (-y_2)^a(-y_3)^b,\ y_1 \geq 0,\ y_i \leq 0 \text{ for } i = 1,2$$

$$1 > a > 0,\ 1 > b > 0.\}$$

Let us write $z_2 = -y_2$, $z_3 = -y_3$ . Then if $z_2 = z_3 = 1$ one can at most produce one unit of good 1. To see whether the same amount can be produced by using less of any one input we differentiate

$$1 = y_1 = (z_2)^a(z_3)^b$$

totally at $z_2 = z_3 = 1$ and set equal to zero (why?). One has

$$0 = dy_1 = [a z_2^{a-1} z_3^{b}]\ dz_2 + [b z_2^{a} z_3^{b-1}] dz_2 =$$

$$= a \frac{y_1}{z_2} dz_2 + b \frac{y_1}{z_3}\ dz_3 = a\ dz_2 + b dz_3 \text{ when } y_1 = z_2 = z_3 = 1.$$

So $$-\frac{a}{b} = \frac{dz_3}{dz_2}$$

Verify that the by now familiar diagram looks as in Fig. 4

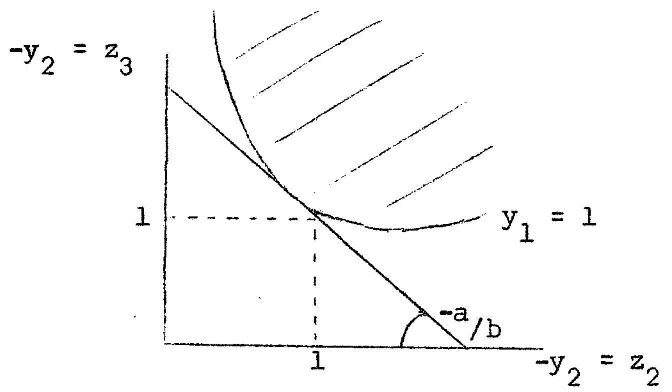



Fig. 4.

Example. d)

$$Y = \{y \mid y_1 \leq x_1,\ y_2 \leq x_2,\ y_3 \leq a_1 x_1 + b_1 x_2,\ y_4 \leq a_2 x_1 + b_2 x_2,\ y_i \leq 0,$$

$$i = 3,4 \quad a_i,\ b_i < 0 \text{ all } i,\ x_i \geq 0\ i = 1,2\ \}$$

Notice that the firm can produce two different goods. Let us ask: what combination of the two goods can the firm produce if it does not use more than one unit of either input?

Suppose the firm produced only good 1. Since $-y_3 \leq 1$, $-y_4 \leq 1$ we have that

$$y_1 = x_1 = \min\left(-\frac{1}{a_1}\ ,\ -\frac{1}{a_2}\right)$$

6

If the firm produced only good 2 the same reasoning gives

$$y_2 = x_2 = \min\left(-\frac{1}{b_1}, -\frac{1}{b_2}\right)$$

If the firm produces both goods one must have

$$= a_1x_1 - b_1x_2 \leqq 1 \qquad \text{(i)}$$

$$= a_2x_1 - b_2x_2 \leqq 1 \qquad \text{(ii)}$$

In Fig. 5 we plot (i) and (ii) as equations. All combinations of output which lie below or on both the lines representing the equations are possible. The heavy kinked curve shows the maximum of any one output which can be produced given the output of the other goods and the restriction; $y_2 \geqq 1$, $y_3 \geqq 1$.

$-y_3 \leq 1$ $-y_4 \leq 1$ ?

Plotted for

$a_1b_2 - a_2b_1 > 0$

Try other values.

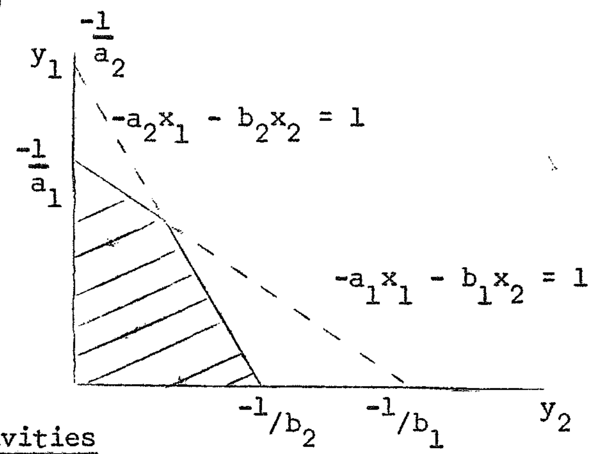



Fig. 5.

5) Efficient Activities

We shall now assume that Y always contains the null activity (that is, y with all components zero). Thus the firm always has the choice of not producing at all.

Let $\alpha$ be a scalar,

and if $0 \leqq \alpha \leqq 1$ abreviate to $\alpha \in [0, 1]$

and recall the following two simple properties of vectors:

a) If $y = (y_1 \ldots y_n)$, $\alpha y = (\alpha y_1 \ldots \alpha y_n)$

b) If $y = (y_1 \ldots y_n)$, $y' = (y_1' \ldots y_n')$ the $y + y' = (y_1 + y_1', y_2 + y_2' \ldots y_n + y_n')$.

[ Illustrate these two properties geometrically ]. I shall also use

c) $y = y'$ means $y_i = y_i'$ all $i$

$y > y'$ means $y_i$ not $< y_i'$ any $i$

$y >> y'$ means $y_i$ not $\leqq y_i'$ any $i$.

It is useful to begin with to seperate members of Y which we want to call efficient ~~for~~ from? the remainder. But by what do we want to characterise efficient activities? The natural procedure is to say that an activity is efficient if the technology does not permit any other activity by which either (i) one could produce more of any one good without producing less of another or using more of any input, or (ii) one could produce the same outputs but use less of some input without using more of another or (iii) both (i) and (ii). The reason why this is ~~an~~ a 'natural' definition is because in general one thinks of outputs as 'desirable' and of input as scarce. Formally

Definition D.I.3. An activity $y^* \varepsilon Y$ is called efficient if $y > y^*$ implies "y is not a member of Y". (Say $y^*$ is not dominated in Y).

Which points in Figs. 1, 3, 4 and 5 are efficient?

Suppose the firm wants to produce $y_1^* > 0$ units of good 1 and that there is some activity in Y which permits this. We want to characterise the efficient ways in which it can produce this amount of good 1. By D.I.3 that means searching for all the activities in Y which have $y_1^*$ as a component and are not dominated in Y. In example (a) there is only one such vector y with $y_1 = y_1^*$, $y_2 = \frac{1}{a_2} y_1^*$, $y_3 = \frac{1}{a_3} y_1^*$. In example (b) the set of efficient vectors for $y_1^*$ is given by y with

$$y_1^* = x_1 + x_2 + x_3 \qquad \text{all } x_i \geqq 0$$

$$y_2 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$

$$y_3 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3$$

In example (c) the efficient set for $y_1^*$ is y with $y_1 = y_1^*$ and any combination $y_1, y_2$ such that $\quad y_1^* = (-y_2)^a(-y_3)^b \quad$ etc.

It is seen that by varying the attainable value of $y_1$ we can in each case trace out a subset of Y which we may call the efficient set. Let $Y^*$ denote the subset of efficient activities in Y. Then in example (c) for instance:

$$Y^* = \{y \mid y_1 - (-y_2)^a(-y_3)^b = 0, \ y \ \varepsilon \ Y\} \qquad (1)$$

It will be clear that good one plays no special role in the definition of $Y^*$ and although I continue to single it out below you should restate all that has gone before and that follows by singling out some other good.

One can often find convenient ways of characterising the efficient subset of Y.

By the definition of efficient activities a necessary condition for $y^*$ to be efficient is that given any $(n - 1)$ components of $y^*$ the remaining component should be as large as possible in Y. (Notice (i) that inputs are taken as negative so that making an input component larger means making it less negative i.e. using less of it and (ii) that I am assuming that efficient points exist in Y.) So suppose we write

$$y(1) = (y_2 \dots y_n)$$

so that $y(1)$ is the activity y without the first component. Let the function $F(.)$ from $E^{n-1}$ to $E^1$ be defined by

$$F(\hat{y}(1)) = \max_{Y} (y_1 | (y_2 \dots y_n) = \hat{y}(1))$$

So that $F(\hat{y}(1))$ gives us the maximum value of $y_1$ of any activities Y which has $y(1) = \hat{y}(1)$. Then $\hat{y}_1 = F(\hat{y}(1))$ is necessary if the activity $\hat{y} = (\hat{y}_1, \hat{y}(1))$ is to be efficient in Y. I now introduce the following assumption.

Assumption A.I.1. If $y'(1) > y(1)$ then $F(y'(1)) < F(y(1))$.

Here is one interpretation of A.I.1: Suppose $y(1)$ represents a vector of inputs so that $y_1$ is quantity of output. The assumption says that using less of any input without using more of any other must reduce the maximum output attainable in Y.

Given A.I.1 one may characterise $Y^*$ by

$$Y^* = \{y \mid y_1 = F(y(1))\}$$

or more succinctly as the set of y such that

$$T(y) \equiv y_1 - F(y(1)) = 0 \tag{2}$$

where $T(y)$ is a function on $E^n$.

Compare (2) with the example in (1). There

$$F(y(1)) = ( - y_2 )^a ( - y_3 )^b \tag{3}$$

D.I.4. If the firm produces a single output, (e.g. good 1), then $F(y(1))$ is called the production function (p.f)

Notice that p.f. picks out efficient points in Y and that it may be a good deal more complicated than in (3).

Example I.4.(b) again

In that example as we have drawn the diagram one has for efficient production

$$x_3 = 0 \quad \text{for} \quad \frac{a_{21}}{a_{11}} \geqq \frac{y_3}{y_2} \geqq \frac{a_{22}}{a_{12}} \tag{4}$$

$$x_1 = 0 \quad \text{for} \quad \frac{a_{22}}{a_{12}} \geqq {}^{y_3}/y_2 \geqq \frac{a_{23}}{a_{13}} \tag{5}$$

So for the case (4) one solves

$$y_2 = a_{11}x_1 + a_{12}x_2$$

$$y_3 = a_{21}x_1 + a_{22}x_2$$

for $x_1$, $x_2$ in terms of $y_2$ and $y_3$ and in case (5) one solves

$$y_2 = a_{12}x_2 + a_{13}x_3$$

$$y_3 = a_{22}x_2 + a_{23}x_3$$

But $y_1 = x_1 + x_2 + x_3$ and so one finds

$$F(y(1)) = \begin{cases} \dfrac{(a_{22} - a_{21})y_2 + (a_{11} - a_{12})y_3}{a_{11}a_{22} - a_{12}a_{21}} & \text{when} \quad (4) \\[2ex] \dfrac{(a_{23} - a_{22})y_2 + (a_{12} - a_{13})y_3}{a_{12}a_{23} - a_{13}a_{22}} & \text{when} \quad (5) \end{cases}$$

Notice that A.I is not satisfied for

$$\frac{a_{23}}{a_{13}} > \frac{y_3}{y_2} > \frac{a_{21}}{a_{11}}$$

$\frac{y_3}{y_2} < \frac{a_{23}}{a_{13}}$

or $\frac{y_3}{y_2} > \frac{a_{21}}{a_{11}}$

(why?), so that for that range the production function is not defined.

I return to p.f. and (2) later. But before going on note

a) By what was said in I.4. the set of efficient activities in Y are as 'subjective' as Y itself; that is one thinks of the technological knowledge of the firm and not of what is, in a economy, technologically known by someone.

b) By I.1, goods include in their definition the date at which they are available.

6) Returns to Scale.

D.I.5. Constant Returns to Scale prevail if when $y \varepsilon Y$, $\alpha y \varepsilon Y$ all $\alpha \geqq 0$ (The production set is said to be a cone).

Check the examples and show that with the exception of (c0 they are all examples of (C.R) and that (c) is also, provided $a + b = 1$.

One is inclined to argue that C.R should be a property of every Y. For suppose $\alpha = 2$ so that in $\alpha y$ all inputs are duplicated, should it not be the case that all outputs are duplicated? But suppose the output is spherical storeage volume. Then the output will increase in more than proportion to the input used to make the spherical container. Even so C.R has given rise to some controversy on this matter and it must be admitted straight away that this is an instance where we may be in difficulties with the definition of goods. Two identical twins working seperately may not produce the same output, or may represent different goods, from their working together. Also you will notice that C.R implies that goods are finely divisible. For further discussion see A. Robinson: Structure of Competitive Industry.

We may with the aid of a definition clarify C.R.

D.I.6: Y is additive if when y and y' are in Y, $y + y' \varepsilon Y$.

Y is divisible if when $y \varepsilon Y$, $ky \varepsilon Y$ all $0 \leqq k \leqq 1$.

Theorem. I.1. If Y is divisible and additive then Y has C.R.

Proof. One wants to show that if $y \varepsilon Y$, $\alpha y \varepsilon Y$ all $\alpha > 0$. Let $k = \alpha - n$ where n is the largest integer not $> \alpha$. Then $ny = \overset{n}{\Sigma} y$ and by additivity is in Y. $ky \varepsilon Y$ by divisibility so by additivity again $ky + ny = \alpha y \varepsilon Y$.

Example I.4.(b)

This is the production cone for the example where the $y_1$ - axis goes "into" the paper.

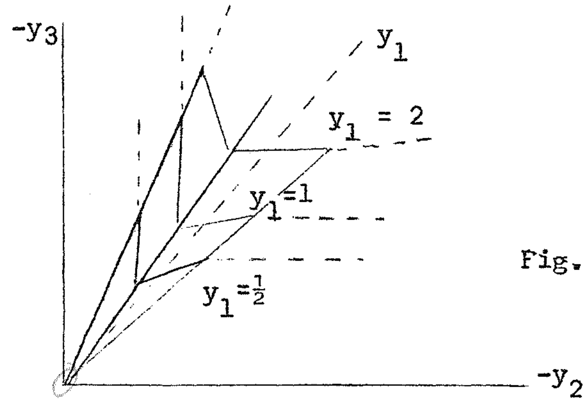



Fig. 6.

D.I.7. Diminishing Returns to Scale. (D.R) prevail in Y if when $y \varepsilon Y$, and $0 < \alpha < 1$, $\alpha y \varepsilon Y$ and $\alpha y$ is not efficient

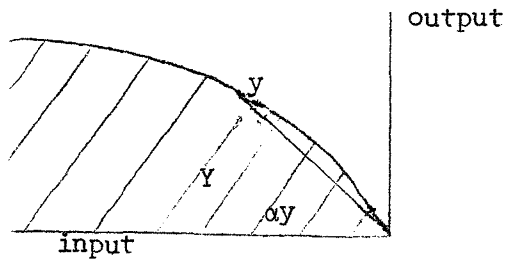



Fig. 7.

The Fig. illustrates. In example I.4.(c), Y has D.R when $a + b < 1$. One of the 'explanations' often given for D.R is that higher output will be associated with increase organisational difficulties. But refer to lectures and Robinson.

D.I.8. Increasing Return to Scale. (I.R) prevail in Y if for every efficient activity $y \in Y$ and $0 < \alpha < 1$, $\alpha y \notin Y$.

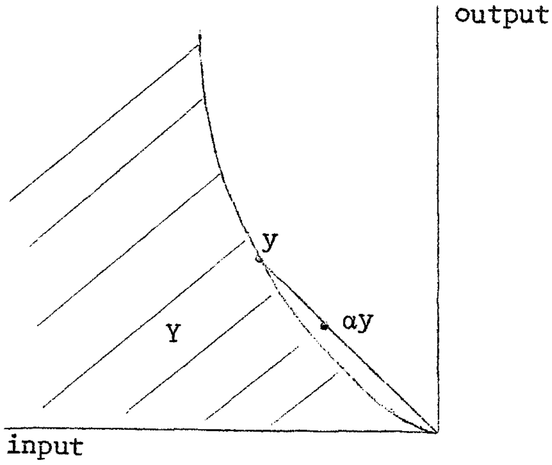



Fig. 8.

The container case is an example of I.R. But there are many less artifical ones and many of them turn of indivisibility of Y. Suppose that inputs only come in integer quantities then one might have a picture as in Fig. 9

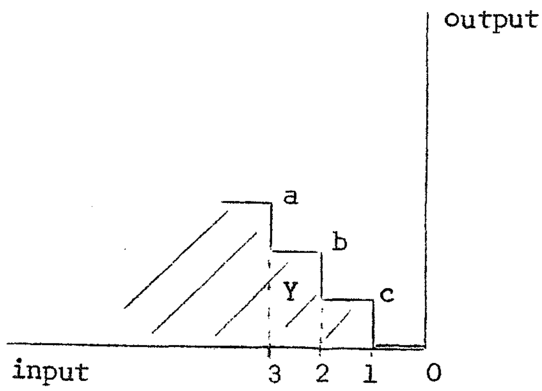



Fig. 9.

If one joined the points a, b, c, 0 one might have a picture like Fig. 7. Yet the 'inbetween' points are not attainable and there are I.R between Oc. Once again Robinson and lectures for further discussion. But note that as Fig. 9 hints, Y may be characterised by all three kinds of return.

It will be useful to have a mathematical way of distinguishing the Y which have C.R or D.R from those that have I.R

In Fig. 6 or 7 take any two members y and y' of Y. Join them by a straight line. Notice that every point on this line is a point in Y. But the equation of the straight line is given by

$$\alpha y + (1 - \alpha)y' \qquad 0 \leqq \alpha \leqq 1$$

and this leads to the following definition

<u>Definition D.I.9.</u> Any set, and in particular the production set Y, is said to be <u>convex</u> if when y and $y' \in Y$ then $\alpha y + (1 - \alpha)y' \in Y$ for any $\alpha$ with $0 \leqq \alpha \leqq 1$.

An example of a non-convex set with members s.

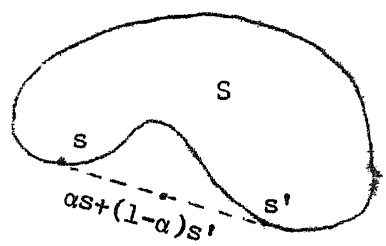



Fig. 10

<u>Theorem T.2.</u> If Y is convex it cannot have I.R.

<u>Proof.</u> Let $y \in Y$. We know that $\underline{0} \in Y$. Hence for all $0 < \alpha < 1$, convexity of Y implies: $\alpha y + (1 - \alpha)0 = \alpha y \in Y$ and so Y does not have I.R.

This simple result is rather important. Indeed we shall see that much of "traditional" economic theory depends crucially on the suposition that Y is indeed convex, which is not well born out by the facts.

Fig. 12 will help to elucidate the connection between the convexity of Y and the production function.

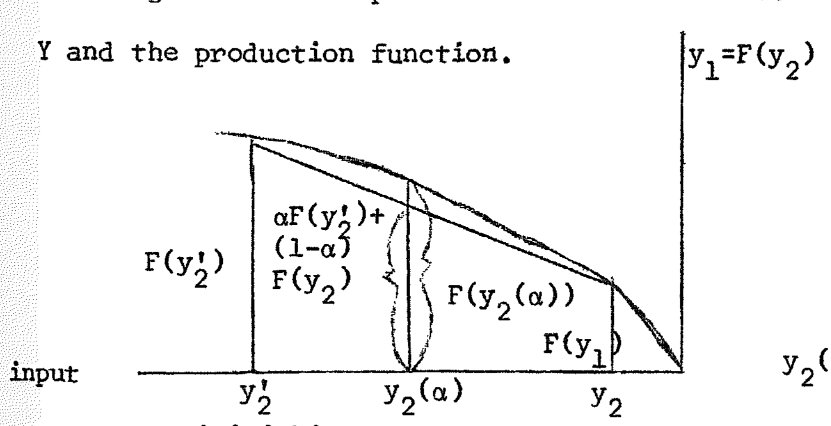



Fig. 12

$y_2(\alpha)=\alpha y_2'+(1-\alpha)y_2, \quad 0\leqq\alpha\leqq1$

So that for diminishing returns one has:

$$F(y_2(\alpha)) > \alpha F(y_2') + (1 - \alpha)\, f(y_2), \qquad \alpha \in [0, 1]$$

Generalising:

<u>D.I.11</u> Let $x \in E^n$ and $F(x) : E^n \to E^1$. Then

a) F(x) is said to be a <u>concave</u> function if for $x \neq x'$, $\alpha \in [0, 1]$

$$F(\alpha x + (1 - \alpha)x') \geqq \alpha F(x) + (1 - \alpha)F(x') \qquad (6)$$

and x <u>strictly concave function</u> if for $0 < \alpha < 1$ the inequality in (6) is strict

b) $F(x)$ is said to be a <u>convex function</u> if $-F(x)$ is concave.

Now one can prove what we have expected all along

<u>Theorem T.3.</u> If $Y$ is convex and a production function $F(y(1))$ can be defined on $Y$, then $F(y(1))$ is concave

<u>Proof.</u> Let $y$ and $y'$ be two efficient points in $F$ and write

$$y = (F(y(1), y(1)), \qquad y' = (F(y'(1), y'(1))$$

By convexity $y(\alpha) = \left[\, \alpha F(y(1)) + (1-\alpha) F(y'(1)),\ \alpha y(1) + (1-\alpha) y'(1) \,\right] \in Y.$

By the definition of $F$ then

$$F(\alpha y(1) + (1-\alpha)\, y'(1)) \geqq \alpha F(y(1)) + (1-\alpha)\, F(y'(1)).$$

# Principles of Economics II

## The Actions of Firms.

### 1) Prices:

Take it that there is a unit of account, say bancors in terms of which prices can be reckoned. For the moment this unit of account is all entry into a ledger. Then write $p_i$ as the number of units of bancors one unit of $i$ exchanges against. So $p_i$ is the price of good $i$ in terms of bancors. Let $p = (p_i \,..\, p_n)$ be the vector of prices and let us only concern ourselves with $p \in E_+^n$, i.e. those $p$ for which

$$p > 0.$$

From I.1 we know that $p_i$ can be the price of any object available at any place or time. For simplicity let us suppose that there is only one place at which all objects are available. Let the ~~suffix~~ subscript '1' denote a good, available "today"–i.e. at $t = 0$. Then if $p_1 > 0$ we can form new prices ?

$$q_i = {}^{p_i}/p_1 \; ; \quad q = (q_1 \ldots q) = \frac{1}{p_1}\, p$$

and $q_i$ will be the price of good $i$ in terms of good one. So $q_i$ is the number of units of good one we get for one unit of good $i$. When we chose to express prices in terms of a good which exists (rather than in terms of an abstract unit of account like bancors), we call this good the numéraire. Any good which has a positive price in terms of bancors can be chosen as numéraire.

Notice that $q$ gives the prices of all goods, i.e. including those available tomorrow or the day after etc. in terms of good one which is available today. So if $x_i$ represents a quantity of good $i$, the expression

$$\Sigma q_i x_i$$

is the value of a basket of goods: $x = (x_1 \,..\, x_n)$ in terms of a currently available good, (good one), and we think of it as a present value. That is it tells us how much of good one available today we could get for the basket $x$ which has goods available at all sorts of times. (E.g. if good one is pound notes available today and $p_1$ is the "bancor price" of pound notes, $\Sigma q_i x_i$ tells us how many present pound notes we can obtain for the bundle $x$ when prices are $q$). I return to some of these matters later.

2). The Simplest Market Environment.

We know that Y describes the set of activities y which are technologically feasible for the firm. We now want to describe the market feasible set of activities, by which I mean the terms at which the market allows the transformation of one good into another.

Suppose that the firm has no bancors in its accounts and can get no gifts when the story starts. Suppose further that p is given and fixed. Then the market allows it to chose all activities y which have the property

$$\Sigma p_i y_i \geqq 0 \qquad (1)$$

Or when $p_1 > 0$, we can write (1) as

$$\Sigma q_i y_i \geqq 0 \qquad (1')$$

Notice that once we know p, we know all the activities y which are market feasible for the firm. In (1') $q_i$ is as usual the number of units of good one which must be given up for, (or are received for), one unit of good i. Notice also that (1') is a present value and in words can be put as follows: the amount of good one the firm receives from the sales of output in any day must not be less than the amount of good one it spends in acquiring inputs at any date. (Recall that inputs are taken as negative quantities.)

In any event in this case p fully describes the market environment of the firm. When that is the case we shall say that the firm is a price taker. For the present I shall be investigating this simple case.

To understand why it is a simple case, consider a firm which produces one good ($y_1 > 0$) with one input ($y_2 < 0$). Suppose good one is cheese-cake in Cambridge today and good 2 is labour in Cambridge today. Suppose further that all today's cheese-cake in Cambridge is produced by this firm. Then it would surely be fanciful to suppose that the firm could exchange any amount of cheese-cake at a fixed price. Indeed, we know enough to suspect that as the firm tries to sell more cheese-cake, the bancor price of cheese-cake will have to be lowered. In any case we may have

$$p_1 = f(y_1)$$

to describe the dependance of $p_1$ on $y_1$. If the labour item ~~plays~~ is specialised {plays a specialised rôle} cheese-cake bakers one may also have

$$p_2 = g(y_2)$$

16



to describe the dependance of the price of labour on the amount of it used by the form. (1) now becomes

$$f(y_1)y_1 + g(y_2)\,y_2 \geqq 0$$

and to describe the market environment one needs to know the functions f and g.

But things may be even more complicated. Suppose there are just two cheese-cake firms in Cambridge today. The $p_1$ may depend on the total output of the two firms so that any firm will not only have to know the form of this dependance but also the output of the other firm. And so it goes on. It is plain that the assumption that firms are price-takers is extremely suspect.

If one thinks about these examples one will see that the price-taking assumption will probably serve best when there are "a very large number" of identical firms so that over the range which we are interested variations in the demand (supply) of any one good by any of the firms are negligible relatively to the total demand, (supply), of that good. This idea can be made precise. But I return to this whole question later. Here I simply assert that while price-taking is not likely to be a correct assumption it allows us to construct bench-marks which will prove useful.

Before I leave the assumption, for the moment, there is a more important difficulty I must draw to your attention. By our definition of goods, p will be a pretty large vector. It not only includes the price of bread today but also the price of bread tomorrow etc. etc. But in practice these prices are not known today. If there were markets today for all goods e.g. for bread tomorrow, this difficulty would not be decisive; but there are very few such future markets in the real world. In real life firms must form expectations of the terms at which goods available in the future can be exchanged; these terms are not given to them. Once again by shutting our eyes to this problem to begin with we shall get rather important insights into why it is in fact of great importance. It is as silly to think that the idealised, (and wrong), case of price taking can teach one no economics as it would be to suppose that the study of frictionless, (and so wrong), penduli can teach one no physics.

3). The Motives of the Firm.

The firm is an abstraction which as such of course can have no motives; people have motives. In life, not only are there often many people involved in production decisions but these people may not coincide with those who are going to receive the profits. So there are really great difficulties in getting to grip with this problem and later I shall return to discuss it.

At the moment let us think of a ruthless competitive struggle for survival which has been going on for a long time in a stationary environment. If certain relatively plausible assumptions are made one can show that the surviving firms will be those who have maximised profits. Of course I am continuing the assumption of section (2) above. In any case it is with reference to this Darwinian picture that the assumption which I am going to make has often been justified in the literature. So when I assume that firms maximise profits I want to be understood as saying that those firms have survived which happened to have followed the strategy. (Read the first essay of Friedman's Essays in Positive Economics, but do not be entirely persuaded by it. Also Samuelson's Nobel-lecture in Vol. 3 of his "Collected Scientific Papers" is interesting).

Let me now be formal about this assumption. Let $\pi$ be the profit of the firm. Then $\pi$ depends on p and y. In particular

$$\pi = \tilde{\pi}(p, y) = \Sigma p_i y_i$$

Why?

Assumption II.1. A price taking firm choses y from Y so that $\tilde{\pi}(p, y)$ is a maximum. Or equivalently if $y^*$ is chosen by the firm then $y^*$ solves

$$\max_{Y} \tilde{\pi}(p, y).$$

A.II.1 will now be employed for some time - but keep in mind that it needs to be scrutinised again.

4). Existence of a Maximum*

The problem of A.II.1 may not have a solution. For instance if $\lambda$ is a scalar and $S = \{\lambda \mid 0 \leq \lambda < 1\}$ then $\max_{S} \lambda$ does not exist. Why? Here is a Theorem, (not proved here, it is due to Weierstrass and all calculus texts have it), which settles the problem.

Theorem II.1*: Let x be a vector, $g(x)$ a continuous function on a given domain X of vectors x. (X is the set of x we can choose from). Then $g(x)$ has a maximum on X if X is closed and bounded.

a) A set is closed if it contains it's limit points

b) A set is bounded if there is a finite vector 'longer' than any member of X.

In what follows let Y have the property that $\tilde{\pi}(p, y)$ has a maximum on Y.

5) Profit-Maximising Conditions.

a) Necessary Conditions are those which are implied by the claim that $y^*$ solves the maximising problem.

b) Sufficient conditions are those which if they hold at $y^*$ imply that $y^*$ solves the maximising problem.

I shall discuss these conditions in turn. To be specific I take

$$Y = \{y \mid y_1 \leqq f(-y_2, -y_3),\ y_1 \geqq 0,\ y_i \leqq 0\ i = 2, 3,\ f \text{ strictly convave}\}$$

I shall also assume that f is differentiable.

6. A Simple Method.

Suppose that $y^*$ solves the problem

$$\max \tilde{\pi}(p, y) \equiv \max \left[ \Sigma p_i y_i \right] \qquad \text{(a)}$$

$$\text{subject to} \qquad y_1 \leqq f(-y_2, -y_3) \qquad \text{(b)} \qquad \text{I.}$$

$$y_1 \geqq 0,\ -y_i \geqq 0 \quad i = 2, 3 \qquad \text{(c)}$$

Theorem II.2. $y^*$ is efficient in Y if $p >> 0$

Proof. If not there is $y \in Y$, $y > y^*$ whence $py > py^*$ since $p >> 0$. So if profits are maximised the firm must be producing efficiently and

$$y_1^* = f(-y_2^*, -y_3^*) \qquad (2)$$

So

$$\tilde{\pi}(p, y^*) = p_1 f(-y_2^*, -y_3^*) + \sum_{i=2}^{3} p_i y_i^* \equiv \pi^* \qquad (3)$$

You may find it more convenient if I define the new vector $z = (z_1, z_2, z_3)$ as

$$z_1 = y_1, \qquad z_i = -y_i \quad (i = 2, 3)$$

and write (3) as

$$\tilde{\pi}(p, z^*) \equiv p_1 f(z_2^*, z_3^*) - \sum_{i=2}^{3} p_i z_i^* \equiv \pi^* \qquad (3')$$

19



Now let $dz_i = (z_i - z_i^*)$. We say if $z^*$ is our solution then for no $z \neq z^*$ which is possible should $\pi(p, z) > \pi^*$. Now $z_i = z_i^* + dz_i$ so for $z \neq z^*$ to be possible, I(c) tells us that we need

$$z_i^* + dz_i \geqq 0 \qquad i = 2, 3 \tag{4}$$

<u>Case 1.</u> <u>(Interior)</u> Let $z_i^* > 0$, $i = 2, 3$. Then for $dz = (dz_2, dz_3)$ <u>small enough</u>, $dz$ and $-dz$ are possible and

$$\pi - \pi^* \approx \frac{\partial \pi^*}{\partial z_2} dz_2 + \frac{\partial \pi^*}{\partial z_3} dz_3 = \sum_i (p_1 f_i^* - p_i) dz_i \tag{5}$$

<u>Definition II.1.</u> $f_i^* = \dfrac{\partial f(z_2^*, z_3^*)}{\partial z_i}$ is called the <u>marginal product</u> of input i.

<u>Warning</u> (a) I have assumed that f is differentiable. When this is not true e.g. example I.4.(b), we must modify (5) and marginal products <u>simpliciter</u> do not exist. simply?

(b) More nonsense has been written on this concept than on most things. Don't make up your minds until you see what can and cannot be proved.

If (5) > 0 plainly $z^*$ cannot be maximising. If (5) is negative then it will be > 0 for $-dz$ and since that is feasible $z^*$ cannot be maximising. Hence if $z^*$ is maximising

$$(5) = 0 \text{ for all small } dz \text{ when } z_i^* > 0 \quad i = 2, 3 \tag{6}$$

Setting $dz_2$ and $dz_3$ in turn equal to zero in (5) = 0 gives

$$p_1 f_i^* - p_i = 0 \qquad i = 2, 3, \tag{6'}$$

In words: <u>If some of input i is used at maximum profit then the value of its marginal product must equal its price.</u>

<u>Case 2 (Corner)</u> Suppose $z_2^* = 0$. Then from (4) only variations in profit obtainable with $dz_2 > 0$ need concern us. But then even if $p_1 f_i^* - p_i < 0$ we cannot improve profits because $dz_2 < 0$ is impossible.

Combining the two cases we have

<u>Theorem II.3.</u> $y^*$ (or $z^*$) must be such that

$$p_1 f_i^* - p_i \leqq 0, \; (p_1 f_i^* - p_i) z_i^* = 0, \quad i = 2, 3 \tag{7}$$

<u>Warning</u> Do <u>not</u> proceed until you can explain (7) to someone without mathematics.

Let us look at some implications of (7)

a) Conditions (7) are necessary if $z^*$ solves the problem

$$\min \sum_{i=2}^{3} p_i z_i$$

$$\text{subject to} \quad z_1^* = f(z_2, z_3) \qquad \text{II}$$

$$z_i^* \geqq 0 \qquad i = 2, 3$$

That is the cost of producing $z_1^*$ is as low as it can be. This should be obvious but I shall be pedantic and show it. Suppose $z_2^*$, $z_3^*$ solves the problem II. and take $z_i^* > 0$. Let $dz_i$ be a "small" deviation satisfying

$$f_2^* \, dz_2 + f_3^* \, dz_3 = dz_1^* = 0 \tag{8}$$

Such a deviation must not enable the cost of producing $y_1^*$ to fall below $\Sigma p_i z_i^*$. But when (7) holds $p_i = p_1 f_1^*$ (recall $z^* >> 0$) and so

$$\Sigma p_i \, dz_i = p_1 \, \Sigma f_i^* \, dz_i = 0 \text{ by (8)}$$

and there is no $dz_i$ satisfying (8) which reduces the cost of producing the given output.

b) Let $C(p_2, p_3, z_1^*) = \Sigma p_i z_i^*$ i.e. the lowest cost of producing the profit maximising output. Note that this cost depends on the parameters of problem II: input prices and output. Let

$$C_{z_1}^* = \frac{\partial C(p_2, p_3, z_1^*)}{\partial z_1}$$

<u>Definition II.2.</u> The change in the minimum cost of producing a given output for an infinitesimal change in the latter is called <u>marginal cost</u>

Now if $z_2^* > 0$, $z_3^* > 0$

$$dz_1^* = f_2^* dz_2 + f_3^* \, dz_3 \equiv \text{by (7)}$$

$$= (\Sigma p_i \, dz_i) \frac{1}{p_1}$$

So

$$p_1 = \Sigma \, p_i \frac{dz_i}{dz_1} = C_{z_1}^* \tag{9}$$

But if $z_1^* = 0$, $z_2^* = z_3^* = 0$ (see Notes I) and $dz_1 > 0$ is the only possibility. Since now $f_i^* \leqq p_i/p_1$ we obtain

Corollary II.3. $y^*$ (or $z^*$) must be such that

$$p_1 \leqq C^*_{z_1}, \quad (p_1 - C^*_{z_1})\, z^*_1 = 0 \tag{10}$$

In words: if profit maximising output is positive price = marginal cost and price can only be less than marginal cost if nothing is produced.

c) Lastly note that when $z^*_i > 0$ $i = 2, 3$, (7) can be written as

$$p_2/f^*_2 = p_3/f^*_3 = p_1 = C^*_{z_1}$$

and also
$$f^*_3/f^*_2 = p_3/p_2 \tag{11}$$

Definition II.3 $f^*_3/f^*_2$ is called the marginal rate of substitution of input two for input three.

Explain the terminology and condition (11) without mathematics.

6) Sufficiency

I start with an elementary mathematical result which will be proved in Mr. Heal's lectures on Convexity.

Theorem T.II.4.: Let $x \in E^n$ be a vector, $g(x)$ a concave function on $E^n$. Then for all $x, x^* \in E^n$

$$g(x) - g(x^*) \leqq \Sigma g^*_i (x_i - x^*_i)$$

where $g^*_i = \dfrac{\partial g(x^*)}{\partial x_i}$

Here is a one dimensional intuitive demonstration

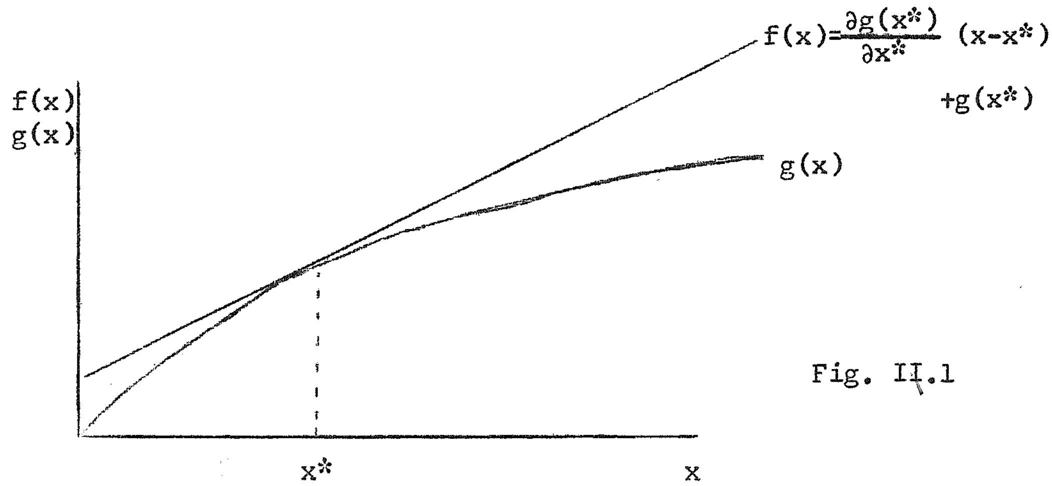



Fig. II.1

[ Here I take $x \in E^1_+$ i.e. the set of non-negative values of x. It is obvious that $f(x) \geq g(x)$ all $x \in E^1_+$. ]

Now let us ask whether if (7) holds at y, (z), that then y must be profit maximising. We easily prove

Theorem II.5. If f(.) is concave, (Y is convex), then if (7) holds at $z^*$ then $z^*$ solves problem I.

Proof. Suppose not. Let $\pi^*$ be profit at $z^*$, $\pi$ profit at any $z \neq z^*$. z satisfies I(b) and I(c). Then

$$\pi - \pi^* = [p_1(f(z_2, z_3) - f(z_2^*, z_3^*)) - p_2(z_2 - z_2^*) - p_3(z_3 - z_3^*)]$$

By (7) which holds at $z^*$

$$p_i(z_i - z_i^*) \geqq p_1 f_i^*(z_i - z_i^*) \quad i = 2, 3 \tag{12}$$

why? So

$$\pi - \pi^* \leqq p_1 \{[f(z_2, z_3) - f(z_2^*, z_3^*)] - [f_2^*(z_2 - z_2^*) + f_3^*(z_3 - z_3^*)]\}$$

$$\leqq 0 \text{ by Th. II.4}$$

So $\pi - \pi^* \leqq 0$ contradicting that $z^*$ does not maximise profits.

Now notice the following Corrollary. Suppose (12) holds with equality ($z_i^* > 0$, i = 2, 3): Suppose f(.) were convex for all

$$z_i = z_i^* \pm \varepsilon, \qquad \varepsilon > 0 \text{ and small}$$

and (7) holds at $z^*$. Then $z^*$ could not be a maximum. Prove that for yourself. One has

Corollary II.5: If $z^*$ solves problem I then (7) must hold and f must be concave in the neighbourhood of $z^*$.

It is clear that (7) can only qualify as a sufficient condition if f( ) is concave everywhere. Indeed if f( ) were convex everywhere i.e. if there were increasing returns everywhere in Y, the profit maximising theory of a price taker would be meaningless. Why? I return to this later.

Sometimes one assumes

$$f(\ ) \text{ concave if and only if } z_2 \geqq a > 0,\ z_3 \geqq b > 0$$

Then if (7) holds for $z_2^* \geqq a$, $z_3^* \geqq b$, (7) will be sufficient for a maximum. Prove that.

One can get a little more insight into sufficiency by commonsense. In Fig. II.2 I have plotted $p_1 f_2^*$ as a function of $z_2$ where $z_3$ is held constant at $z_3^*$

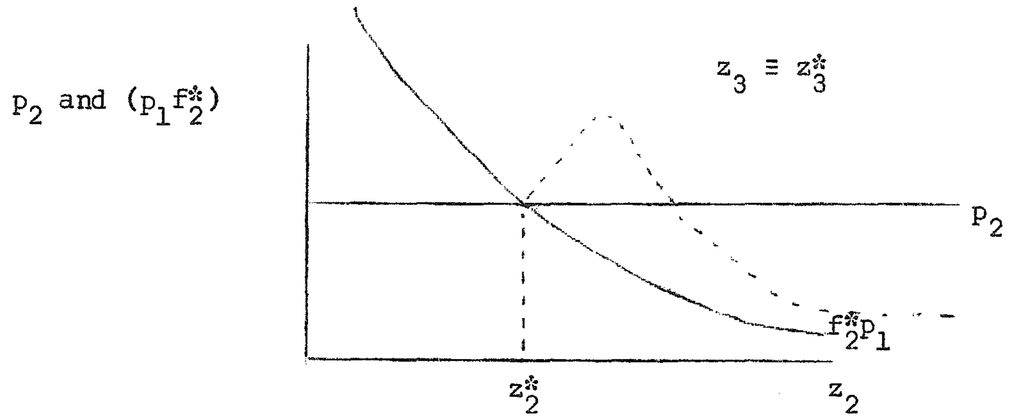



Fig. II.2

If the curve $f_2^* p_1$ looked like the broken curve, $(z_2^*, z_3^*)$ could not be a point of maximum profit. Why? So one has at once that sufficiency requires

$$f_{ii}^* = \frac{\partial^2 f(z_2^*, z_3^*)}{(\partial z_2^*)^2} \leqq 0$$

i.e. the marginal product of every input must be non-increasing as we have more of it.

This can be generalised by the use of Th. II.4. Take x and g(x) as defined there. By Taylor's theorem, for $||x - x^*||$ "small" enough

$$g(x) - g(x^*) = \Sigma g_i^*(x_i - x_i^*) + \frac{1}{2!} \sum_j \sum_i g_{ij}^* (x_i - x_i^*)(x_j - x_j^*) + 0\ (\varepsilon)$$

$$\left(g_{ij}^* = \frac{\partial^2 g(x^*)}{\partial x_i \partial x_j}\right)$$

So using this and Th. II.4 on has

<u>Corollary II.4</u>. If g(x) is a concave function on $E^n$ then at all $x^* \varepsilon E^n$

$$\sum_i \sum_j g_{ij}^*(x_i - x_i^*)(x_j - x_j^*) \leqq 0 \qquad (13)$$

(If you don't know Taylor's expansion yet ignore this until you do).

It is now clear that f( ) concave, gives $f_{ii}^* \leqq 0$. (set $z_j = z_j^*$ for $j \neq 2$ in (13).

7) <u>Profit and Cost Functions</u>

I have shown some of the main features of maximising profits in a simple context. The next section is more general. But first I want to discuss the positive content, i.e. the falsifiable propositions, which are implied by the theory. For this profit functions (and cost functions) are very useful.

Definition II.4. $\pi(p)$ is called the profit function where

$$\pi(p) = \sup \{ \Sigma p_i y_i \mid y \in Y \}.$$

(Do not worry about the 'highbrow' sup. The assumptions which I use allow us to take sup $\equiv$ max).

Definition II.5. $C(p, y_1)$ is called the cost function where

$$C(p, y_1) = \inf \{-(p_2 y_2 + p_3 y_3) \mid (y_1, y_2 y_3) \in Y \}.$$

(Do not worry about "inf". Interpret as min.)}

So $\pi(p)$ gives the profits of a price taking firm maximising at p and $C(p, y_1)$ gives the costs of this firm when it is producing $y_1$.

Theorem II.6 For all $k > 0$: $k\,\pi(p) = \pi(kp)$

Proof. Left to you.

Remark* Th. II.6 says: "$\pi(p)$ is homogeneous of degree one in p"

A function $g(x)$, $x \in E^n$ is homogeneous of degree r in x when

$$k^r g(x) = g(kx), \qquad k > 0.$$

Th. II.6 becomes interesting when we see what it predicts about the actions of the firm.

Let us define

D. II.6: The set $y(p)$ is called the supply set, (correspondence) of the firm where

$$y(p) = \{(y_1, y_2, y_3) \mid \Sigma p_i y_i \geqq \Sigma p_i y_i' \text{ all } (y_1', y_2', y_3') \in Y \}.$$

Remark. (a) A member of $y(p)$ is an activity in Y which maximises profits at p over y. The terminology 'supply set' is a little confusing since $-y_2$ for instance is the demand for input z. But think about it and do not let it confuse you.

(b) If for all p, $y(p)$ has only one member we shall call it the supply function.

With this out of the way we can prove

Corrollary II.6.: For all p and $k > 0$

$$y(p) = y(kp)$$

[Since these may be equalities between sets this means

$$y(p) \subset y(kp) \text{ and } y(kp) \subset y(p).$$

i.e. every $y \in y(p)$ is also a member of $y(kp)$ and vice versa ].

Proof. Suppose not, so that there is a vector y

$$y \in y(p),\ y \notin y(kp)$$

Then by the definitions there is $y' \in y(kp)$ with

$$\Sigma\, p_i k\ y_i \ <\ \Sigma p_i\ k\ y'_i$$

Dividing both sides by k shows $y \notin y(p)$, a contradiction.

Remarks (a) The prediction is: if all prices are different in the same proportion the firm will not find any activity which is preferable to the preferred activities at the old price.

(b) If y(p) is a function we make the stronger statement that the firm will chose to do the same thing at p and kp.

(c) If y(p) is a function, set $k = \frac{1}{p_1}$, $p_1 > 0$ and note

$$y(p) = y(kp) = y(q) \qquad q_i = {}^{p_i}/p_1$$

and notice that the supply of the firm depends only in relative prices (see section 1). (y(p) is homogeneous of degree zero in p)

Let us see whether we can state more precisely the circumstances in which y(p) is a function, i.e. contains only one member. Here is the relevent theorem.

Theorem II.7. If Y is strictly convex (f(.) strictly concave), y(p) is a function.

Proof. Suppose y(p) has two distinct members, y and y'. Then by the definition

$$\Sigma\, p_i y_i \ = \ \Sigma\, p_i y'_i$$

By the definition of strict convexity to two things are true:

a) $y(\alpha) \ = \ \alpha y + (1 - \alpha)y' \in Y \qquad\qquad \alpha \in [0, 1]$

b) There exists $y'' > y(\alpha)$ with $y'' \in Y$

But then since $\Sigma\, p_i y_i(\alpha) = \Sigma p_i y_i = \Sigma\, p_i y'_i$, $\Sigma p_i y_i(\alpha) > \Sigma p_i y_i$ etc

Contradicting y and $y' \in y(p)$.

So diminishing returns ensure that y(p) is a function. Since the example we are working with has f(.) strictly concave we may indeed treat y(p) as a function.

Next let us see how $y(p)$ and $\pi(p)$ are related. This very useful relationship is summed up in

<u>Theorem II.8</u>. Let $\pi_i(p) \equiv \frac{\partial \pi(p)}{\partial p_i}$. Then

$$\pi_i(p) = y_i(p) \qquad i = 1,\ldots,3 \tag{14}$$

<u>Proof</u>. Notice that $\pi(p) = \max \tilde{\pi}(p, y) = \tilde{\pi}(p, y(p))$ and that by the necessary conditions of a maximum

$$\frac{\partial \tilde{\pi}(p, y(p))}{\partial y_i} = 0 \qquad i = 1,\ldots, 3 \tag{15}$$

But

$$\frac{\partial \pi(p)}{\partial p_i} = \pi_i(p) = \frac{\partial \tilde{\pi}(p, y(p))}{\partial p_i} + \Sigma \frac{\partial \tilde{\pi}(p, y(p))}{\partial y_i} \frac{dy_i}{dp_i} = \text{by (15)}$$

$$= \frac{\partial \tilde{\pi}(p, y(p))}{\partial p_i} = y_i(p)$$

This as we shall see is a useful result for it implies

$$\frac{\partial^2 \pi(p)}{(\partial p_i)^2} = \pi_{ii}(p) = \partial y_i / \partial p_i$$

and we know something about the sign of $\pi_{ii}$.

<u>Remarks</u>* Mathematicians may wonder whether we are justified in assuming $\pi$ to be twice differentiable. The Theorem which follows can be shown to imply that indeed we are justified "almost everywhere".

<u>Theorem II.9</u>. $\pi(p)$ is a <u>convex</u> function of $p$.

<u>Proof</u>. Let $p \neq p'$; $p(\alpha) = \alpha p + (1 - \alpha)p'$. $\qquad \alpha \in [0, 1]$

we want to prove

$$\pi(p(\alpha)) \leqq \alpha\, \pi(p) + (1 - \alpha)\, \pi(p')$$

But

$$\begin{aligned} \pi(p) &\equiv \Sigma p_i y_i(p) \geqq \Sigma p_i y_i(p(\alpha)) \quad &(a) \\ \pi(p') &\equiv \Sigma p_i' y_i(p') \geqq \Sigma p_i' y_i(p(\alpha)) \quad &(b) \end{aligned} \tag{16}$$

Why? Add $\alpha$ 16(a) to $(1 - \alpha)$ 16(b):

$$\alpha\, \pi(p) + (1 - \alpha)\, \pi(p') \geqq \Sigma p_i(\alpha)\, y_i(p(\alpha) \equiv \pi(p(\alpha)).$$

Corrollary II.9.

$$\sum_i (p_i' - p_i)(y_i'(p') - y_i(p)) \geqq 0 \tag{17}$$

and

$$\sum_i \sum_j \pi_{ij}(p)(p_i' - p_i)(p_j' - p_j) \geqq 0 \tag{18}$$

Proof. In 16(a) set $\alpha = 0$ so $p(\alpha) = p'$, in 16(b) set $\alpha = 1$ so $p(\alpha) = p$. Verify (and do not proceed until you have done so) that the inequalities are still valid. Subtract the right hand side of each inequality from both sides and add the two resulting inequalities to obtain (17).

By taking $p'$ close enough to $p$ one has

$$y_i'(p') - y_i(p) = \sum_j \pi_{ij}(p_j' - p_j) \quad \text{by (15)}$$

which proves (18). Compare (18) with Cor. II.4.

Corrollary II.9' (a) $\pi_{11}(p) \geqq 0$ so $\dfrac{\partial y_1(p)}{\partial p_1} \geqq 0$ and the profit maximising output is a non-decreasing function of it's price.

(b) $\pi_{ii}(p) \geqq 0$ so $\dfrac{\partial y_i(p)}{\partial p_i} \geqq 0$ $i = 2, 3$. Since $y_i \leqq 0$, the amount of input used is a non-increasing function of it's price.

What is the economics of all this? In Fig. II.3 I have drawn $\pi(p)$ as a function of $p_1$ when $p_2$ and $p_3$ are fixed

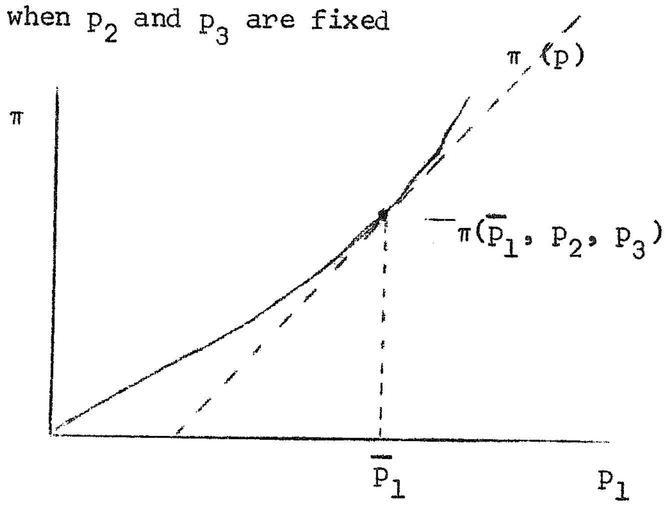



Fig. II.3

Take $\pi(\bar{p}_1, p_2, p_3)$. It's slope $\pi_1(\bar{p}_1, p_2, p_3) = y_1(\bar{p}_1, p_2, p_3)$. When the price is taken at $p_1 > \bar{p}_1$ the firm has the choice of staying where it was at $\bar{p}_1$. If it did that $\pi(p_1, p_2, p_3) - \pi(\bar{p}_1, p_2, p_3) = y_1(\bar{p}_1, p_2, p_3)(p_1 - \bar{p}_1) =$

$$= \pi_1(\bar{p}_1, p_2, p_3)(p_1 - \bar{p}_1)$$

So the profit curve would be given by the broken line i.e. the tangent to $\pi(\bar{p}_1, p_2, p_3)$. If the firm can do better than that profits will increase by more (fall by less), than indicated by the broken line. In any case $\pi(p)$ is a convex function. Do not proceed until you can explain this result to a non-mathematical economist of modest intelligence.

As a matter of fact Cor. II.9' can be strengthened when f(.) is strictly concave and y(p) is a function.

Let $q_i = {}^{p_i}/p$ and write (7) as

$$q_i = f_i(z_2(q), z_3(q)) \quad i=2, 3 \tag{19}$$

$z_i(q) \equiv -y_i(q)$. Consider a small change in q of dq. Then since (19) must hold at q + dq we get

$$dq_i = \sum_{j=z} f_{ij}\, dz_j \qquad i = 2, 3 \tag{20}$$

where $f_{ij} = \dfrac{\partial f_i(z\ (q), z_3(q))}{\partial z_j}$

It is plain $dz_2 = dz_3 = 0$ is not a solution of (20). So if in 16(a) or (b), $p(\alpha)$ is not proportional to p or p' i.e. if $p(\alpha)$ represents different relative prices we can assert:

$$y(p) \neq y(p(\alpha))$$

But then if the inequality in 16(a) is not strict then by the argument of the proof of Th. II.7 we have a contradiction since Y is strictly convex. So the inequalities are strict. One has

<u>Corrollary II.9''</u> (a) If Y strictly convex and $p \neq kp'$, $k > 0$ then

$$\sum_{ij} \pi_{ij}(p)(p_i' - p_i)(p_j' - p_j) > 0$$

(b) The profit maximising <u>output</u> is an <u>increasing</u> function and the profit maximising input a <u>decreasing</u> function of it's price.

Notice now that these are <u>predictions</u> and they can be falsified. In particular one might be tempted to argue that whenever the output of a firm is larger the price of the output is larger. This would be falsified: think of a car firm. The trouble may be that there are increasing returns. But there may be other troubles: we take Y as given. But it may change not only through time, (we have agreed to ignore <u>that</u> for the moment), but with p. Why? Because as I have argued Y is not as objective as all that and at different p the firm may have cause to consult the existing book of blue prints more closely. On the other hand there may be many firms, (say dairy farmers), for whom our predictions will work very well.

I now turn to Cost functions. I leave it to you to prove

Th. II.10. Cost functions are concave in p.

Proof. Hint: use Th. II.9 and notice definition of convex functions and of $C(p, y_1)$.

Th. II.11. When f(.) is strictly concave marginal costs are increasing

Proof: We know that: $p_1 = \cancel{C_{y_1}}\ C_{y_1}$

$$\text{and} \quad \frac{\partial}{\partial p_1} p_1 = 1 = \cancel{C_{y_1 y_1}} \left(C_{y_1 y_1}\right) \frac{\partial y_1(p)}{\partial p_1}$$

But by Cor. II.9'': $\dfrac{\partial y_1(p)}{\partial p_1} > 0$ so

$$C_{y_1 y_1} > 0.$$

I shall be using Cost functions a good deal later in these lectures. Here for completeness I want to lay out a number of fairly simple results.

Definition II.7. $c(p, y_1) = \dfrac{1}{y_1} C(p, y_1)$ is called the average cost function.

Theorem II.12. If Y has constant returns to scale

$$kC(p, y_1) = C(p, ky_1) \qquad k > 0$$

so setting $k = \dfrac{1}{y_1}$ $\qquad c(p, y_1) = c(p, 1) = c(p)$

i.e. average cost depends only on p and not on $y_1$

Proof. By C.R: $y \in Y$ implies $ky \in Y$, $k > 0$. Let $k = \dfrac{1}{y_1}$ and let $y_i/y_1 = -a_{1i}$, $i = 2, 3$. Then

$$(1; -a_{12}, -a_{13}) \in Y$$

Suppose $a_{1i}$ $(i = 2, 3)$ has been chose to minimise the cost of producing one unit of output. Then $\Sigma\, a_{1i} y_1$ must minimise the cost of $y_1$ units. (Why?).

Lastly let us consider a matter which, as we shall see, will be of special interest to us.

Let

$$c(p, y_1) = \min_Y \frac{1}{y_1} \Sigma p_i z_i = \min_Y \tilde{C}(p_2, p_3, y_1, z_2, z_3) \frac{1}{y_1}$$

and let $z_2(p) = z_2^*$, $z_3(p) = z_3^*$. Then as usual

$$\frac{\partial \tilde{C}(p_2, p_3, y_1, z_2^*, z_3^*)}{\partial z_i} = 0 \qquad i = 2, 3$$

Now let us suppose that for some reason or another the firm must use $z_3^*$ of input three, or better that the firm must maximise profits under the additional constraint

$$z_3^* = z_3 \tag{21}$$

Refer to lectures for an economic basis to this problem). Let

$$\tilde{c}(p,\ z_3^*,\ y_1)$$

be minimum average cost when (21) is a constraint. Then we notice

a) At p (when $z_3^*$ would be chosen so (21) is not a constraint)

$$\tilde{c}(p,\ z_3^*,\ y_1) \ = \ c(p,\ y_1^*)$$

b) At $p' \neq kp$,

$$\tilde{c}(p',\ z_3^*\ y_1) \geqq c(p',\ y_1^*) \qquad \text{Why?}$$

c) At p

$$\frac{\partial c(p,\ y_1^*)}{\partial y_1} = \frac{\partial \tilde{c}(p,\ z_3^*,\ y_1^*)}{\partial y_1} \tag{22}$$

This result follows from

$$\frac{\partial c(p,\ y_1^*)}{\partial y_1} = -\frac{1}{{y_1^*}^2}\ \tilde{C}(p_2,\ p_3, y_1^*,\ z_2^*,\ z_3^*) + \frac{\tilde{C}y_1}{y_1^*}\left[\frac{1}{y_1^*}\ \Sigma \frac{\partial \tilde{C}(p_2, p_3, y_1^*, z_2^*, z_3^*)}{\partial z_i}\ \frac{dz_i}{dy_1}\right]$$

and the last term is zero whence variations in $z^*$ play no role in evaluating this derivative. The picture is in Fig. II.4

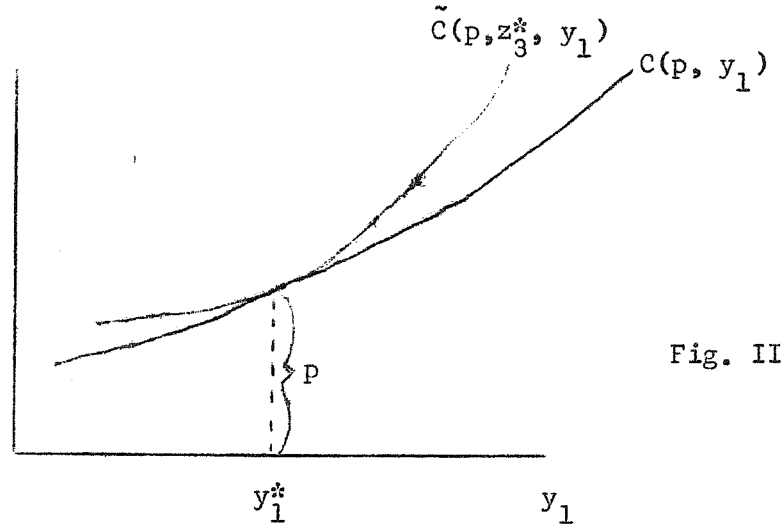



Fig. II.4

8) <u>An example.</u>

Let $f(z_2,\ z_3) = z_2^a\ z_3^b$; $a + b = 1 - c,\ a > 0,\ b > 0,\ c > 0.$

Then by (7) with $p >> 0$

$$\begin{aligned} p_1\ f_2^* &= p_1 \frac{y_1}{z_2} a = p_2 \\ p_1\ f_3^* &= p_1 \frac{y_1}{z_3} b = p_3 \end{aligned} \tag{23}$$

Solve these for $z_2$, $z_3$ in terms of $y_1$ and parameters and substitute in $y_1 = z_2^a\ z_3^b$ to get

$$y_1 = ({}^{p_1}/{p_2})^a \left(\frac{p_1}{p_2}\right)^b y_1^{a+b}\ a^a\ b^b \tag{24}$$

Or solving for $y_1$

$$y_1 = ({}^{p_1/}{}_{p_2})^{\frac{a}{c}} \, (\frac{p_1}{p_2})^{\frac{b}{c}} \, a^{\frac{a}{c}} \, b^{\frac{b}{c}} \tag{25}$$

And (25) is our equation for $y_1(p)$. Find $y_i(p)$ $i = 2, 3$.

Also

$$\pi(p) = p_1 y_1(p) - \Sigma p_i z_i(p) = p_1(25) - (a + b)\, p_1 \,(25) =$$

$$= p_1 \,(25)\, c \tag{26}$$

And that is the profit function. Check $\pi(p)$ etc. Find the Cost function.

9). <u>A Fundamental Theorem.</u>

Let $x \in E^n$ be a vector, $g_i(x)$, $i = 0,\ldots,m$, $(m + 1)$ functions on $E^n$. We are given the problem

$$\left.\begin{array}{lll} & \max g_o(x) & \quad (a) \\ \text{subject to} & g_i(x) \geqq 0 \quad i = 1,\ldots,m & \quad (b) \end{array}\right\} \quad \text{III}$$

Suppose $\mu = (\mu_1,\ldots,\mu_m)$ and m-vector and

$$U = \{ \mu \mid \mu \geqq 0 \}$$

and consider the following related problem: Find $\mu \in U$, $x$ satisfying III(b) so that the function $v(x, \mu)$ given by

$$v(x, \mu) = g_o(x) + \sum_i \mu_i g_i(x)$$

is <u>maximised</u> with respect to $x$ and <u>minimised</u> with respect to $\mu$. Formally

Find $x^*$ with $g_i(x^*) \geqq 0$ $i = 1,\ldots,m$ and $\mu^* \in U$ so that all $x$ with $g_i(x) \geqq 0$, $v(x, \mu) \leqq v(x^*, \mu^*) \leqq v(x^*, \mu)$, all $\mu \in U$ IV

We show that the $x^*$ solving problem IV solves problem III.

From the right hand inequality of IV

$$g_o(x^*) + \Sigma\mu_i^* g_i(x^*) \leqq g_o(x^*) + \Sigma\mu_i g_i(x^*) \text{ all } \mu \in U.$$

Suppose $\Sigma\mu_i^* g_i(x^*) > 0$. Then $\mu = k\mu^* \in U$ with $k < 1$ and $\Sigma\mu_i g_i(x^*) < \Sigma\mu_i^* g_i(x^*)$, contradicting the required inequality. But by definition $\Sigma\mu_i^* g_i(x^*) \geqq 0$ whence

$$\Sigma\, \mu_i^* g_i(x^*) = 0 \tag{27}$$

But by III(b) and $\mu^* \in U$, (27) gives

$$g_i(x^*) \geqq 0,\; \mu_i g_i(x^*) = 0 \text{ all } i = 1 \,..\, m \tag{28}$$

Now look at the left hand inequality of IV . By (27)

$$g_o(x) + \Sigma \mu_i^* g_i(x) \leqq g_o(x^*)$$

But $\Sigma \mu_i^* g_i(x) \geqq 0$ and so

$$g_o(x) \leqq g_o(x^*) \text{ all } x \text{ with } g_i(x) \geqq 0 \quad i = 1,\ldots, m$$

We have proved the following result: If $x^*$ solves problem IV then it solves problem III. We have not proved the reverse namely that if $x^*$ solves III that there exists $\mu^*$ so that $(x^*, \mu^*)$ solve IV. For that we need more assumptions and more mathematics. In the theorem which follows I make the assumptions but I will not prove the necessary part.

Theorem (Kuhn-Tucker). Let $x \varepsilon E^n$, $g_i(x)$, $i = 0,\ldots, m$ be $(m + 1)$ concave functions from $E^n$ to the reals. Let

$$G = \{ x \mid g_i(x) \geqq 0 \quad i = 1,\ldots, m \}$$

and assume that G has an interior point (i.e. there is $x' \varepsilon G$; $g_i(x') > 0$ $i = 1,\ldots, m.$) Then a necessary and sufficient condition for $x^*$ to solve the problem $\max g_o(x)$ with $x \varepsilon G$ is that there exist $\mu^* \varepsilon E_+^m$, (i.e. $\mu^* > 0$), such that $x^*$, $\mu^*$ solves

$$\max_{x \varepsilon G} \min_{\mu \varepsilon E_+^m} [ g_o(x) + \Sigma \mu_i g_i(x) ].$$

The full proof, serious mathematical economists will get at some stage of their lectures. At the moment, provided you have checked on concavity, use it like a cooking-recipe.

Here is how one does it. Take problem I. Then let $z = -y$ and

$$g_1(z) = f(z_2, z_3) - z_1 \geqq 0$$

$$g_i(z) = z_{i-1} \geqq 0 \quad i = 2, 3, 4.$$

Check that they are concave. Also

$$g_o(z) = p_1 z_1 - p_2 z_2 - p_3 z_3$$

By the theorem we know that there is $\mu^* = (\mu_1^*, \mu_2^*, \mu_3^*, \mu_4^*)$ such that $x^*$ must maximise

$$v(z, \mu^*) \equiv g_o(z) + \Sigma \mu_i^* g_i(z) \tag{29}$$

since we have assumed differentiability and we now have an unconstrained problem we must at the solution $z^*$ have

(i) $$\frac{\partial v(z^*, \mu^*)}{\partial z_i} = 0 \qquad i = 1,\ldots, 3 \tag{30}$$

This gives

$$p_1 - \mu_1^* + \mu_2^* = 0 \qquad (a)$$

or (30')

$$-p_i + \mu_1^* f_2^* + \mu_{i+1}^* = 0 \qquad (b) \qquad i = 2, 3$$

(ii) If $z_1^* = 0$ then one may have $\mu_2^* > 0$ by (28) whence

$$p_1 - \mu_1^* \leqq 0, \ (p_1 - \mu_1^*) z_1^* = 0 \qquad (31)$$

(iii) If $z_1^* = 0$ certainly $z_i^* = 0$ $i = 2, 3$ and $\mu_{i+1}^* > 0$ is possible. On the other hand since $p_i > 0$ our assumption ensure $z_i^* > 0$ implies $z_1^* > 0$. So using (31), (30')(b) gives

$$(p_1 f_i^* - p_i) \leqq 0, \ (p_1 f_i^* - p_i) z_i^* = 0 \qquad i = 2, 3 \qquad (32)$$

which is (7)

(iv) It is plain that one has $\mu_1^*$ = marginal cost when $z_1^* > 0$. But there is an instructive way examining the measuring of $\mu_1^*$ (and $\mu_i^*$ generally).

Let us notice that we have taken what is produced of good one as equal to sales. But let us now distinguish sales written $\hat{z}_1$, from production $z_1$. Let us write $\hat{z} = (\hat{z}_1, z_2, z_3)$ and modify

$$g_1(\hat{z}) = f(z_2, z_3) + a - \hat{z}_1 \geqq 0$$

where a may be interpreted as the amount of good 1 the firm has before production starts. Above we took a = 0 and when that is done the problem is as before. Also

$$g_2(\hat{z}) = \hat{z}_1 - a$$

Now consider $\max_{\hat{z}} v(\hat{z}, \mu^*)$. This maximum value, i.e. the maximum profits attainable will depend on the parameters p and a so we write

$$V(p, a) = \max_{\hat{z}} v(\hat{z}, \mu^*) = v(\hat{z}^*, \mu^*, a)$$

Consider

$$\frac{\partial V(p, a)}{\partial a} = \Sigma \frac{\partial v}{\partial \hat{z}_i^*} \frac{dz_i}{da} + \Sigma \frac{\partial v}{\partial \mu_i^*} \frac{d\mu_i^*}{da} + \frac{\partial v}{\partial a} \qquad (33)$$

By (30) all the terms under the first summation sign are zero. Next by (28) $\frac{\partial v}{\partial \mu_i^*} = 0$ certainly if $g_i(\hat{z}^*) = 0$. If $g_i(\hat{z}^*) > 0$ then $\mu_i^* = 0$ and remain zero for small perturbations. Hence the terms under the second summation sign are zero. But you calculate when $z_1^* = \hat{z}_1 - a > 0$

$$\frac{\partial v}{\partial a} = \mu_1^*$$

and so from (33) $\mu_1^*$ measures the increase in the maximum profit made possible by a small increase in a. If I offered the firm a little of good 1 at it's maximum profit position then $\mu_1^*$ would be the maximum price it would be willing to pay for it. One calls $\mu_1^*$ the shadow price of good one - "shadow" because it is not a quoted price. Since $z_1^* > 0$ the firm will not be willing to buy a little more of good one at a price is excess of marginal cost. Notice that $\hat{z}_1 - a = 0$ may allow $\mu_2^* > 0$ as

$$\frac{\partial v}{\partial a} = \mu_1^* - \mu_2^* \leqq \mu_1^* .$$

Warning, The Method of this section is important but it takes some time to understand. It will come with practice. Try the following problem: Take $p >> 0$ and

$$Y = \{ y \mid y_1 \leqq (-y_2)^a(-y_3{}^b),\ y_1 \geqq 0,\ y_i \leqq 0 \quad i = 2, 3,\ a + b = 1 - c$$

$$c > 0,\ a > 0,\ b > 0 \}$$

Profits are to be maximised on Y and the following further constraints:

$$y_i + k_i \geqq 0 \qquad i = 2, 3, \qquad k_i > 0$$

Interpret the $\mu_i^*$'s of your problem. Do not say: "I will not do it I already understand it all" for your are almost certainly wrong.

# Principles of Economics III

## The Firm (Continued)

### 1) Tidying Up

In this section I want to take up a number of lose ends of II, before returning to the main story.

(a) Constant Returns to Scale. For much of II I assumed that there were diminishing returns to scale (pf strictly concave). This, amongst other things, had the advantage that (i) we could deal with supply functions (see II .7 ) and (ii) that it was not difficult to suppose that a maximum existed.

Now if Y has constant returns to scale and $\tilde{\pi}(p, y) > 0$ some $y \in Y$, then it is easy to see that a maximum profit choice will not exist unless we do something about it. For $\Sigma p_i y_i > 0$ and constant returns give $\Sigma p_i y_i < \Sigma p_i y_i k$, all $k > 1$ and $ky \in Y$ all $k > 1$, by assumption and so profits can be indefinitely increased. On the other hand if a maximum profit choice exists then

$$\pi(p) = 0 \qquad \text{III.1}$$

This follows from the fact that $\pi(p) > 0$ we have already seen to be impossible and $\pi(p) < 0$ is also impossible since it can be improved upon by chosing $y = 0 \in Y$. We thus have the following.

Th. III.1. If Y is a cone (C.R.S.) then either at $p > 0$ no profit maximising choice exists or $\pi(p) = 0$.

Th. III.2. If Y is a cone and $\pi(p) = 0$ then $y(p)$ is a convex set and except for $y(p) = 0$, $y(p)$ is not a function.

Proof. Let $y(p) \neq 0$, $\Sigma p_i y_i(p) = 0$. Since C.R.S., $\Sigma p_i y'_i = 0$, $y' = ky(p)$ and $y' \in Y$. Convexity is obvious.

Now you know that unbounded y is silly for two reasons. Reason one is that as a firm choses larger and larger y we will not be able to continue to suppose that it is a price taker. Reason two is that some of the inputs, say labour-services or land are available in finite quantities in any economy.

So what we do is this: we cook the story by arbitrarily chosing a large number $K > 0$ and assuming that the firm in fact faces a fictional production set $Y^*$ where

$$Y^* = \{ y \mid y \in Y, \| y \| \leq K \}$$

The argument is that if one considers an economy with these fictional production sets one will not go far wrong since the world will see to it that prices are in fact established such that no-one could make unbounded profits. I return to this much later in these lectures.

In Fig. III.1 I illustrate the argument by a simple example.

Let $Y = \{ y \mid y_1 \leqq -ay_2,\ a > 0,\ y_1 \geqq 0,\ y_2 \leqq 0 \}$.

or $Y^* = \{ y \mid y \in Y \text{ and } [y_1^2 + y_2^2]^{\frac{1}{2}} \leqq K \}$

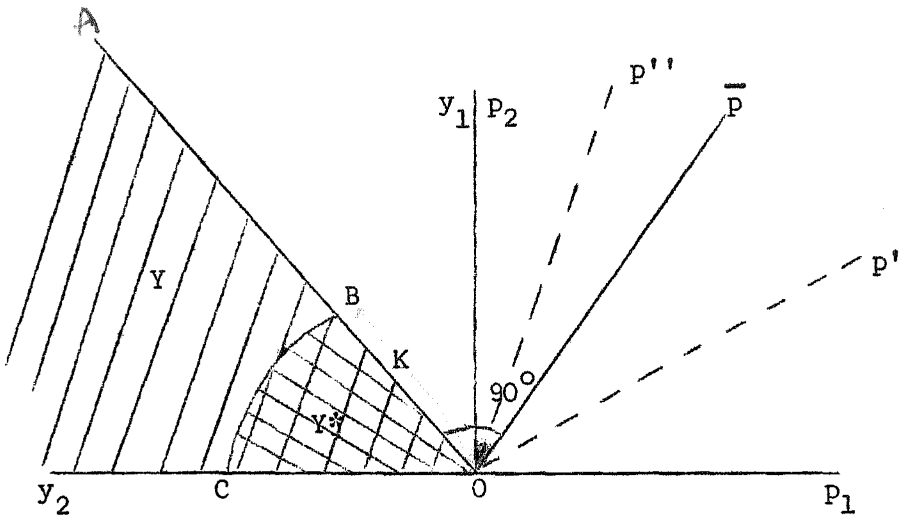



Fig. III.1

(i) Look first at the $(y_1, y_2)$ quadrant. The production set is every point on or below the line OA. Only points on the line are efficient (why?) The length K of the line is by a well known Greek Theorem equal to $y_1^2 + y_2^2$. Rotating the line towards the $y_2$ axis traces out part of a circle and any point on this circle is a vector of length K. Hence Y* consists of the closed? ? cloud set OBC.

(ii) In the right hand quadrant I measure prices, i.e. $p_1$ and $p_2$ on each of the axis. Consider the vector $O\bar{p}$ at right angles to OB. It is known (see Heal's lectures) that the product of two vectors which are at right angles to each other is zero. I.e. $OB \times O\bar{p} = \bar{p}_1 y_1 + \bar{p}_2 y_2 = 0$. It is obvious that the length of either of these vectors is of no consequence. (I.e. the result does not depend on $p_1, p_2$ but on the ratio of these prices. This sounds I hope familiar; see II. Also $kOB \times O\bar{p} = 0$ for $k < 1$). So at the relative prices implied by $\bar{p}$ the producer's maximum profit will be the same which ever point on OB (including the end point), he choses. Hence $y(\bar{p})$ is a correspondence.

But Op'' makes an acute angle with OB so OB x Op'' > 0 and the producer will chose the point B - a corner solution on Y*. Why? Also OB x Op' < 0 and the producer will close the point O.

So let us write $y_1(p) = y_1(\frac{p_1}{p_2}) = y_1(q)$ and plot it in Fig. III.2

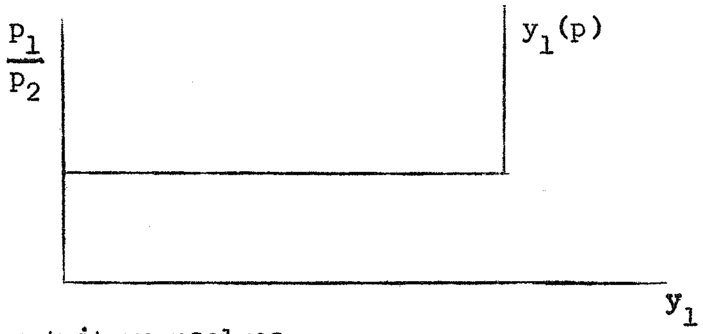



Fig. III.2

you can interpret it yourselves.

Lastly recall that the average cost function under constant returns to scale can be written as c(p), $p = p_1 \dots p_n$. (See II ). So far a single product firm $(y_1 \geqq 0)$, $(y_i \leqq 0 \quad i > 1)$, we can put what we have learned quite generally:

$y_1(p)$ is a set for all p such that $p_1 = c(p)$

$y_1(p)$ is zero for all p such that $p_1 < c(p)$

$y_1(p)$ is at its max in the fictional set for $p_1 > c(p)$

(b) <u>Marginal Products again.</u>

In II we noted that if f( ) is not differentiable then one will not have an obvious measuring to marginal product.

Consider again the production function of I.4(b) given on page 9 of I. Let us see what we can say about $\frac{\partial F}{\partial y_2}$. Here we go:

$$\frac{\partial F}{\partial y_2} = \frac{a_{22} - a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \equiv A \text{ say when } \frac{a_{21}}{a_{11}} > \frac{y_3}{y_2} > \frac{a_{22}}{a_{12}} \qquad \text{III.1}$$

$$\frac{\partial F}{\partial y_2} = \frac{a_{23} - a_{22}}{a_{12}a_{23} - a_{13}a_{22}} \equiv B \text{ say, when } \frac{a_{22}}{a_{12}} > {}^{y_3}/_{y_2} > \frac{a_{23}}{a_{13}} \qquad \text{III.2}$$

So in these ranges $\frac{\partial F}{\partial y_2}$ is well defined. But suppose

$${}^{y_3}/_{y_2} = \frac{a_{22}}{a_{12}}$$

38



Then using a little more of input two bring us in to range III.1. So we have to distinguish between <u>left</u> and <u>right</u> hand derivatives i.e.

$$\left(\frac{\partial F}{\partial y_2}\right)^+ = B, \qquad \left(\frac{\partial F}{\partial y_2}\right)^- = A.$$

So what about our maximising conditions?

$A < p_2/p_1 < B$ ?

Suppose $\underline{p_2 < p_1}$ B. The plainly profits can be increased by having more of input two so at an equlibrium

$$p_1 \left(\frac{\partial F}{\partial y_2}\right)^+ \leqq p_2 \qquad \text{III.3}$$

analagous !!!

By an <u>unslogons</u> argument

$$p_1 \left(\frac{\partial F}{\partial y_2}\right)^- \geqq p_2 \qquad \text{III.4}$$

And so the equilibrium condition is

$$p_1 \left(\frac{\partial F}{\partial y_2}\right)^- \geqq p_2 \geqq p_1 \left(\frac{\partial F}{\partial y_2}\right)^+ \qquad \text{III.5}$$

which has a perfectly good commonsense interpretation, hasn't it?

Now, condition III.5 is not~~es~~ as powerful as are the usual ones. After all, if for instance, having a little more of input two does not allow you to produce any more at all, (because say inputs two and three must be used in fiscal proportions), then the right hand side of III.5 is zero and that does not tell us very much.

There are economists who believe that the world looks more like III.5 with very wide limits than it does like II.8. But although this is an empirical matter much that is said on this matter is muddled with quite different problems such as whether there is any meaningful way in which inputs, or a subset of inputs, can be aggregated. There is also a "short and long run" distinction to be considered. Since I come to these matters later I postpone a full discussion until then.

But notice that care must be taken with these arguments. Consider a double-decker bus manned by a driver and a conductor. It is possible to dispense with the conductor (and have the driver collect the money), or to have two conductors one upstairs and one downstairs. The bounds of III.5 may not be too far apart. But you do need a driver and without him output is zero. But now consider the firm which is deciding for the first time how to produce and it can chose not only it's labour inputs but also whether to have a single decker or double decker bus. Then the story will be different and having one driver fewer (in the plans)will still allow positive output provided the equilibrium number of drivers exceeds one. So you see at once that the problem will be different for a firm which is already committed than for one which is not. I shall return to this.

2) Back to Square One

In this section I want to discuss problems which arise from our treatment of Y. In the next section I consider problems which arise from our assumption of price taking and absence of increasing Returns.

(a) The Future

I started this story by saying that Y is at any time rather subjective - it gives the production possibilities known to the firm and not the production possibilities known in general. When technical knowledge is stationary we would expect diffusion of knowledge and sooner or later we might expect the actual and known production possibilities to coincide. But technical knowledge is not stationary and so we better be explicit and write $Y_f$ as the production set known to firm f, and for good measure let us put in a t to indicate that it is the production set of f as viewed from t.

But now we have a problem. Recall that $Y_f(t)$ will refer to goods at different dates and in particular at dates after t. Certainly any technique known to f at t will presumably also be known to it at t + 1, (the firm has a memory), and so

$$Y_f(t) \overset{?}{\subseteq} Y_f(t + 1)$$

But if the firm learns something new about production (because either something new has been invented or because it becomes acquainted with some existing knowledge), it will not be true that $Y_f(t) = Y_f(t + 1)$. But the firm does not know at t what it will learn with certainty, for if it did it would have nothing to learn. So the plot thickens. When that happens the worse thing to do is to relax precision in concepts and to go off into after-dinner conversation.

Consider, for concreteness the following examples

Example A. There are only two time periods (the argument could proceed equally well if there were T such periods). For concreteness also I now label a good by two indices $(i, t)$. Here $i$ gives the physical characteristic of the good, $t$ its date. Suppose $i = 1, 2$, $t = 1, 2$, so when $y(1) \varepsilon Y_f(1)$ one has

$$y(1) = (y_{11}(1), y_{21}(1), y_{12}(1), y_{22}(1))$$

and this is a plan known by f to be feasible at 1. But now suppose that

$$y(1) \varepsilon Y_f(1) \Rightarrow (y_{11}(1), y_{21}(1), 0, 0) \text{ and } (0, 0, y_{12}(1), y_{22}(1)) \varepsilon Y_f(1)$$

and $Y_f(1)$ is additive. In commonsense terms this means that the plan can be split into two independant plans for $t = 1$ and $t = 2$. But then the firm which is uncertain about $Y_f(2)$ will do best to make no plans for $t = 2$ at all and simply wait and see what the future brings. That is the firm will maximise

$$p_1y_{11}(1) + p_2y_{21}(1) \text{ on } Y_f(1)$$

and then $$p_1' y_{22}(2) + p_2'y_{22}(2) \text{ on } Y_f(2)$$

So: if the present value of profits is maximised by maximising the profits of each period separately i.e. if profits in period t are independant of actions in period t - 1, then ignorance about future technology would cause no new problems. Here is the reason why this avenue of excape is unlikely to be very wide.

(b) Durable Inputs. Suppose that in our example input 2 is durable in the sense that when it is used in period one some of it is left over for period two. Formally, we now have a new good, units of "second hand" or used input 2. This can be thought of as an output of the firm in period one. If there were a market for this good with prices known in period one nothing new would be added to our analysis. For in maximising first period profits the receipts from selling used units of input 2 would be included. When period two arrives the firm still has the choice of whether to sell or keep it.

But markets in second hand goods are very thin and especially is that true for inputs, which in my case may not be divisible. In general firms probably do not even attempt to calculate the price of such second hand inputs and act as if they had no market at all. But then the firm must certainly consider that its first period plan will ~~have~~ leave it with second hand inputs which may not be well ~~omitted~~ suited to the best plans available when period two technology is known.

So in real situations a firm will be uncertain (a) about future technology, and (b) future prices. In general this uncertainty will matter because decisions taken today will have consequences for tomorrow and the day after. There are various formal theories available to discuss the actions of the firm. For instance one of these says that the firm considers all possible future situations, attaches probabilities to their occurence and maximises the actuarial, (expected), present value of profits. This yields answers but is open to the crucial objection that it requires enormous computational effort on the part of the firm. Alternative approaches note that when rational decisions are difficult social routines, "rules of thumb" grow up. Various of such rules have been suggested but e.g. Kaldor 1962 ? none of them are very convincing. Lastly there are theories which compromise between these two extremes. If the truth be told: we do not really know the answer.

The question, as usual then arises, how badly wrong we shall be by abstracting from some of these complications. Consider (a) that the stock of knowledge is large relatively to new accretions and that we need not think of production sets as changing drastically or unforseeably all the time; (b) while there is some fluctuation in most prices they too are not usually very violent or terribly unpredictable. So one might certainly argue that a theory which abstracts from the difficulties which we have been considering may not do terribly badly in 'normal' times (though it is bound to be wrong). On the other hand in times of crises and turbulence, (perhaps the U.K. in 1972?) a mechanical use of the abstracted theory is likely to lead us badly astray. (There will be more discussion in lectures and I return to these matters much later).

(c) The Past

~~(a) Example B: The Past~~

Let us consider a slightly different example to bring out the next important point.

Example B. Let $y(1)$ and $y(2)$ be output in period one and period two (they are scalars) and $z_1(1) \geqq 0$ and $z_1(2) \geqq 0$ input of type one in period one and two. Assume that it takes one period to obtain input of type two which is not durable. So when the story starts the firm cannot use more of that input than it has available as a result of past decisions. So let us write $b_2(t)$ as the amount of input two bought in period $(t)$ but which will not be available until $t + 1$. Let $z_2(t) \geqq 0$ be the amount of input two used in production at $t$. Then the firm will be constrained by:

$$\begin{aligned} z_2(1) &\leqq b_2(0) \qquad \text{(a)} \\ z_2(2) &\leqq b_2(1) \qquad \text{(b)} \end{aligned} \qquad \text{III.6}$$

The firms' profits are, when superscript e denotes "expected and discounted"

$$p\, y(1) + p^e\, y(2) - q_1 z_1(1) - q_1^e\, z_1(2) - q_2 b_2(1) \qquad \text{III.7}$$

where $p$ = price of output, $q_i$ = price of input $i$. Notice that at $t = 1$ the firm can do nothing about what it bought at $t = 0$,: "bygones are forever bygones". Now suppose

$$y(t) = f(z_1(t), z_2(t)), \quad t = 1, 2 \qquad \text{III.8}$$

is the production function. Notice that it is assumed to have the property discussed in example A.

It is now straightforward to apply the usual method to the problem:

max III.7 subject to III.6 and III.8

I leave details to you. But notice that we maximise

$$\pi + \sum_{t=1} \lambda(t) \left[b_2(t-1) - z_2(t)\right] + \sum_{t=1} \mu(t) \left[f(z_1(t), z_2(t)) - y(t)\right] +$$

$$\sum_i \sum_t \xi_i(t)\, z_i(t) \cancel{+} \quad ?? \qquad \text{III.9}$$

where $\pi$ = III.7 and I have not bothered to put in the constraints $z_i \geqq 0$ etc.

Notice that even if prices are not expected to change, output in the two periods may be different. In fact if prices had been constant for a long time in the past and are now different at $t = 1$, it will in general take the firm two periods to adjust to the new situation. Here the past

43



is important in understanding the present. For the past (decisions) will constrain what the firm can do currently. This leads to the following definition which I leave to you to generalise beyond the confines of this example.

Definition III.1. The action taken by the firm in period one is called a short period action if $\lambda(1) \neq q_2(1)$. Otherwise call it a long period action.

The idea is I hope obvious in light of what we learned in II. If say $\lambda(1) > q_2(1)$, the price the firm would be willing to pay for an extra unit of input two in period one exceeds the market price. But the firm is stuck with the decisions which it took in the past. The distinction between short and long period actions is of great practical importance; I discuss it more fully in lectures.

Example C.

Transport services are provided by buses. Once you have chosen the type of bus you know how many drivers you need. If it takes time to change the type of bus then for some time you are stuck with these proportions whatever relative prices of bus services and drivers. But you say: in due course I will switch to double deckers. That is what you are doing now is short period.

It is the case that the analytical, conceptual and empirical problems posed by placing an economic agent in real time are (a) great (b) good to work on. But do not despair of sense. The exercise we do when we abstract from these matters often help us to say what is not true without telling us what is true. Also some progress has been made which serious students will learn at a later stage. On the other hand enough has been said to show how silly it would be to use say the tricks of lecture II mechanically. Don't do it!

(d) Price Taking and Increasing Returns

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We know that to assume price taking and increasing returns at optimum output levels ~~are~~ is inconsistent. It is plain that we observe cases where returns are still increasing at the point chosen by the firm. So we must often abandon price taking as an assumption.

There are a number of reasons why price-taking is a wrong description of some markets

(i) The Firm is large simply by virtue of increasing returns

(ii) The goods produced by any one firm are not usually identical to those produced by any other. Recall that goods are distinguished by location which in itself tends to differentiate the output of one firm from that of another. But firms also have an incentive to differentiate their products from those of other firms precisely because it enables them to escape the price taking straight jacket. What all this means is that the outputs of two firms will not be perfect substitutes for consumers.

(iii) firms may have natural monopolies: e.g. oil well or legal monopolies e.g. telephones. There are other reasons; (see lectures).

From our point of view it is important to subdivide the problem into two different cases

a) The Atomistic Case: here the firm is small enough to assume correctly that it's own actions will not influence the actions of other firms.

b) The Oligopoly Case: occurs when the firm in taking an action can only calculate it's consequences if it knows the reactions of other firms.

I discuss the two cases in turn.

a) In general we may write

$$p_i = p_i(y) \qquad i = 1 \ldots n \qquad \text{III.7b}$$

This says that the price the firm can charge (if $y_i > 0$) or must pay (if $y_i < 0$) depends on it's activity choice $y$. So if $y_i > 0$ III.7 tells us something about the property of the demand for good i by customers and if $y_i < 0$ it tells us something about the property of the supply of good i to the firm.

Let
$$p_{ij} = \frac{\partial p_i(y)}{\partial y_j}$$

$$\sigma_{ij} = p_{ij} \frac{y_j}{p_i} \equiv \frac{\partial \log p_i}{\partial \log y_j}$$

Consider $p_{ij}$ when $y_i > 0$. It tells us by how much the price of good i will have to be changed if the firm wishes to sell a little more (or less) of it. It is usual to take $p_{ii} < 0$. But as a number it is not very informative.

Why? So we prefer to work with $\sigma_{ij}$. [When $y_i < 0$, $p_{ii}$ tells us the change in the price of the $i^{th}$ input the firm has to pay if it wants a little more (or less) of it. Again it is usual to take $p_{ii} < 0$ ($\sigma_{ii} > 0$) i.e. the firm has to pay more per unit of input i if it wants more of it ]. But then we can define

$$e_{ii} = (\sigma_{ii})^{-1}$$

and call $e_{ii}$ the <u>own elasticity of demand</u> when $y_i > 0$ and the <u>own elasticity of supply</u> when $y_i < 0$. Why?

Suppose a firm produces two brands of soap. We would not be suprised if one brand is "i" and the other "j" if $\sigma_{ij} \neq 0$. Then with

$$e_{ij} = (\sigma_{ij})^{-1}$$

we call $e_{ij}$ the <u>cross-elasticity of demand</u>. In the present example we would expect $e_{ij} > 0$. Why?

In general we should expect that many $e_{ij} = 0$. For instance we do not expect that a change in the price of the labour <u>used by the firm</u>, will have a perceptible effect on the demand for that firm, output.

Profits are now given by

$$\pi = \Sigma p_i(y)\, y_i \qquad \text{III.8b}$$

Suppose first that:

$$p_{ij} = 0 \text{ for } j \neq i \text{ all } i.$$

Then

$$\frac{\partial \pi}{\partial y_i} = p_i(y) + p_{ii}\, y_i = p_i(y)\,(1 + \sigma_{ii}) \qquad \text{III.9b}$$

Notice: if $\sigma_{ii} = 0$ (then $e_{ii} = -\infty$), III.9b reduces to the price taking case.

<u>Definition III.2.</u> III.9b is called the <u>marginal revenue</u> of output i ($y_i > 0$) or the <u>marginal cost</u> of input i ($y_i < 0$).

<u>Th. III.3.</u> When a firm faces a downward sloping demand curve for its output then price of output > marginal revenue of output. If it faces an upward sloping supply curve of an input then price of input < marginal cost of input.

<u>Proof.</u> Obvious.

46

Now go back to the example worked in section (5) of II and verify that a <u>necessary</u> condition for the maximisation of III.8 under present assumptions yield the following modification of Theorem II.3:

<u>Theorem III.4.</u> If $p_{ij} = 0 \quad i \neq j$ all $i$ and $y^*$ solves the problem

$$\max \text{III.8}(y) \text{ subject to } y_1 \leqq f(-y_2, -y_3),\ y_1 \geqq 0,\ y_i \leqq 0 \quad i = 2, 3$$

then we must have

$$p_1(1 + \sigma_{11})\, f_i^* - p_i(1 + \sigma_{ii}) \leqq 0,\ [p_1(1 + \sigma_{11}) f_i^* - p_i(1 + \sigma_{ii})] z_i^* = 0$$

$$\cancel{z_i^* = 0},\quad i = 2, 3.$$

<u>Example 2.1.</u> Suppose a firm employs men and women to manufacture baskets. Let $y_b \geqq 0$ be the output of baskets, $y_m \leqq 0$ the input of men, $y_w \leqq 0$ the input of women. The women have very few alternative job opportunities and so we assume that $e_{mm} > e_{ww}$. Also $e_{mw} = e_{wm} = 0$. Show that even if

$$y_b = f(-(y_m + y_w))$$

women will be paid a lower wage than men when both are employed.

<u>Example 2.2.</u> In the above example assume that the men and women join to form a Trade Union. This sets a common wage below which neither men nor women will work. Show that the Union can set a wage which will improve the lot of both men and women.

Notice that Th.III.4 shows that inputs will be paid less than the value of their marginal product whenever the demand and/or the supply of input is less than perfectly elastic.

So far, so good. If we drop the assumption of zero cross elasticities we get a more complicated result but we know how to get it - do we not? The interesting question is not that elaboration but the reminder that we are <u>no longer wanting</u> to assume that f is concave and so we must pay attention to <u>sufficient</u> conditions

We showed in II that increasing returns and price taking leads to contradictions. What is the present situation? We may write for our example

$$\pi(y) = p_1(y)\, f(-y_2, -y_3) + p_2(y)\, y_2 + p_3(y)\, y_3$$

and we can easily prove

<u>Theorem III.5.</u> If $\pi(y)$ is a concave function then the conditions of Th.III.4 are necessary and sufficient to solve $\max_Y \pi(y)$

<u>Proof</u> I consider only the case of an interior solution; the corner case is left to you.

We need only prove sufficiency i.e.: if conditions of III.4 hold at $y^*$ then profits are maximised. We may write these conditions as

$$\pi_i(y^*) = 0 \quad i = 1,\ldots,3 \text{ where } \pi_i(y^*) = \frac{\partial\pi(y^*)}{\partial y_i}$$

Let $y \in Y$, $y \neq y^*$. By Th.II.4 and $\pi(.)$ concave:

$$\pi(y) - \pi(y^*) \leqq \Sigma\, \pi_i(y^*)\,(y_i - y_i^*) = 0 \qquad \text{III.9c}$$

so that for all such y, $\pi(y) \leqq \pi(y^*)$ as was to be proved.

<u>Remarks.</u> a) Let us write

$$\pi(y) = p_1(y)\, y_1 - C(p_2(y), p_3(y), y_1)$$

where C(.) is the minimum cost function. Notice that this function now depends on inputs because prices do. But check that

$$\frac{\partial C}{\partial p_i} = y_i \qquad i = 2, 3$$

as before. Assume that all cross elasticities are zero. The marginal cost of output now depends (i) on the production function and (ii) on how the price of inputs varies when there is a change in the amount of them which is used. Let $\frac{\partial C}{\partial y_1}$ represent marginal cost at <u>constant</u> input prices, and $\frac{\partial^2 C}{\partial y_1^2}$ the rate at which this marginal cost is changing. Then if the firm is producing under increasing returns one can have $\frac{\partial^2 C}{\partial y_1^2} < 0$. On the other hand if $\frac{dC}{dy_1}$ is the marginal cost when input prices may vary, then $\frac{d^2C}{dy_1^2}$ may be > 0 inspite of increasing returns. Why? Because while there is some economising in inputs per unit of output the marginal cost of inputs is rising. If you can differentiate you can calculate these derivatives for yourselves and see precisely how the behaviour of marginal cost depends on the production function and on the supply conditions of inputs.

If one writes $R(y) \equiv p_1(y)y_1$ for total revenue, $R_1 = \frac{\partial R(y)}{\partial y_1}$,

$$R_{11} = \frac{\partial^2 R(y)}{\partial y_1^2} \quad \text{then}$$

$$\pi_1(y^*) = R_1(y^*) - \frac{dC}{dy_1} = 0$$

$$\text{and} \qquad \pi_{11}(y^*) = R_{11}(y^*) - \frac{d^2C}{dy_1^2} \qquad \text{III.10}$$

One sees at once that at a maximum, $\pi_{11}(y^*) \not> 0$. To assume $\pi(y)$ concave allows one to say that not only are profits no higher than $\pi(y^*)$ in the vicinity of $y^*$ but they are no higher anywhere. Notice that applying a Taylor expansion to III.9c yields $\pi_{ii}(y^*) \leqq 0$ all i. What III.10 says is: if marginal cost are falling at the maximum profit point they must be falling less rapidly than marginal revenue is falling. (See lectures for diagram).

(b) It will be clear that we can now no longer express the profit maximising choices as functions of (or more generally, as depending on), prices. For prices are now no longer parameters describing the market environment of the firm. In fact that environment is described by the functions $p_i(y)$. This is an important point to be borne in mind. When a firm is a price taker we can talk of the firm having a supply function, (correspondence), depending only on the prices; when the firm is not a price taker this is in general impossible. I return to this later in these lectures.

Once the price taking assumption is dropped the way is open to other modifications. Here is one.

Example 2.3. Suppose the firm is a price taker in the markets for inputs (goods 2 and 3) but not in the market for output (good 1). Let a be the amount spent by the firm on advertising its output and assume

$$p_1 = p_1(y_1, a), \quad \frac{\partial p_1}{\partial a} > 0 \qquad a \geqq 0$$

(Interpret this). Profits are now:

$$p_1(y_1, a)\, y_1 + \sum_{i=2}^{3} p_i y_i - a$$

49



The firm maximises profits subject to the usual constraints.

Let

$$R(y_1, a) = p_1(y_1, a)\, y_1, \quad R_{y_1} = \frac{\partial R}{\partial y_1}, \quad R_a = \frac{\partial R}{\partial a} \quad \text{etc.}$$

$$C(p_2, p_3, y_1) \;=\; \text{mainimum cost of producing } y_1.$$

Then at a maximum one has

$$\pi(y_1^*, a^*, p_2, p_3) \;=\; R(y_1^*, a) - C(p_2, p_3, y_1^*) - a^*$$

and

$$\left.\begin{array}{ll} R^*_{y_1} - C^*_{y_1} \leqq 0, \; (R_{y_1} - C_{y_1})\, g_1^* = 0 & \text{(a)} \\ R^*_a - 1 \leqq 0, \; (R_a - 1)\, a^* = 0 & \text{(b)} \end{array}\right\} \qquad \text{III.11}$$

Notice that the firm must now chose y and a and we may think of the latter choice as how most profitably to sell a given output.

If we assume $\pi(.)$ to be strictly concave at the optimum we have as usual for $(y_1, a)$ "close" to $(y_1^*, a^*)$:

$$\pi(y_1, a, p_2, p_3) - \pi(y_1^*, a_1^*, p_2, p_3) < (R^*_{y_1} - C^*_{y_1})(y_1 - y_1^*) + (R^*_a - 1)(a - a^*) = 0 \qquad \text{III.12}$$

Let us see what use we can make of III.12.

First notice that for all $(y_1, a)$ close enough to $(y_1^*\; a^*)$ one has

$$\pi(y_1, a, p_2, p_3) - \pi(y_1^*, a_1^*, p_2, p_3) = \frac{1}{2!}\left[ (R^*_{y_1y_1} - C^*_{y_1y_1})(y_1 - y_1^*)^2 + R^*_{y_1a}(y_1 - y_1^*)(a - a^*) + R^*_{aa}(a - a^*)^2 \right] \text{III.}$$

III 12b

? ver?

where we have used $R_{y_1} = R_{ay_1}$. By III.12 we have III.13b < 0.

With this out of the way let us examine the effect on the firm's action of imposing a tax on advertising. To keep things simply assume

$$R_{y_1a} \;=\; C_{y_1y_1} \;=\; 0$$

(is that a good assumption?). If the firm pays ta in tax for every amount a spent on advertising we now have

$$\pi(y_1, a, p_2, p_3, t) \;=\; R(y_1, a) - C(p_2, p_3, y_1) - (1 + t)\, a$$

To start with we take t = 0 and assume $a^* > 0$ so III.11(b) is

$$R^*_a = 1 + t \qquad \text{III.13}$$

50



Now let t be made positive. Then taking $y_1^* > 0$, in the new position the firm choses III.11(a) and III.12 must again hold. So

$$\text{from III.12} \quad R^*_{aa}\frac{da}{dt} + R^*_{ay_1}\frac{dy_1}{dt} = R^*_{aa}\frac{da}{dt} = 1 \qquad \text{(a)}$$

III.14

$$\text{from III.13} \quad R^*_{y_1a}\frac{da}{dt} + R^*_{y_1y_1}\frac{dy_1}{dt} = R^*_{y_1y_1}\frac{dy_1}{dt} = 0 \qquad \text{(b)}$$

Since we take $R_{y_1y_1} \neq 0$ one has $\frac{dy_1}{dt} = 0$. But from III.13 ?

$$R^*_{aa} < 0$$

$$\text{and so} \qquad \frac{da}{dt} < 0$$

So the imposition of the tax will leave output unchanged and reduce advertising expenditure. But $\frac{\partial p_1}{\partial a} > 0$ so the tax on advertising <u>will reduce the price of output</u> under present assumptions.

But now take $R_{y_1a} > 0$ while $C_{y_1y_1} = 0$ as before. Multiply the l.h.s. + r.h.s. of III.11(a) by $\frac{da}{dt}$ and the l.h.s. and r.h.s. of III.14(b) by $\frac{dy_1}{dt}$ and add:

$$R^*_{aa}\left(\frac{da}{dt}\right)^2 + 2R^*_{ay_1}\frac{da}{dt}\frac{dy_1}{dt} + R^*_{y_1y_1}\left(\frac{dy_1}{dt}\right)^2 = \frac{da}{dt}$$

By III.13 then once again $\frac{da}{dt} < 0$ i.e. advertising expenditure will be reduced. On the other hand since we must have $R^*_{y_1y_1} < 0$ (why?), III.14 (b) now gives $\frac{dy_1}{dt} < 0$ so that output is also reduced and we can not in general say what will happen to the equilibrium price of output when a tax is imposed on advertising.

But it is not surprising that our predictions will depend on the parameters. What one wants to do in these cases is to improve one's understanding of what the crucial parameters are. A little thought will tell us that our prediction is likely to depend not only on the efficiency of advertising in allowing a higher price to be charged for a given output (i.e. on $\frac{\partial p_1}{\partial a}$) but also on whether advertising affects the elasticity of demand. Suppose for instance that

$$p_1(y_1, a) = y_1^{-\alpha} a^{\beta}, \quad 0 < \alpha < 1, \ 1 > \beta > 0$$

$$\text{So} \qquad \log p_1 = -\alpha \log y_1 + \beta \log a$$

51



and the elasticity of demand = $-\ ^{1}/\alpha$, a constant which is independant of a. Now check that $R_{y_1} > 0$, $R_{y_1 y_1} < 0$, $R_a > 0$, $R_{aa} < 0$. Next we know (see III.9), that 96 (i.e. p11)

p 8 or p 11 ?!! or p 13 ?!

$$R_{y_1} = p_1 (1 - \alpha)$$

So III.14(b) becomes

$$(1 - \alpha) \left[ \frac{\partial p_1}{\partial a} \frac{da}{dt} + \frac{\partial p_1}{\partial y_1} \frac{dy_1}{dt} \right] = (1 - \alpha) \frac{dp_1}{dt} = 0 \qquad \text{III.15}$$

So that in this case the tax on advertising has no effect on price of output at all. Next note that if $C_{y_1 y_1} > 0$, (instead of $C_{y_1 y_1} = 0$), the r.h.s. of III.15 is not zero but

$$C_{y_1 y_1} \frac{dy_1}{dt}$$

But it is easily checked that one still has $\frac{dy_1}{dt} < 0$ whence we can now predict that certainly the price of output will be reduced by the tax. Lastly if $C_{y_1 y_1} < 0$ that conclusion will be reversed. Now think about the economics of these examples. Notice that increasing returns to scale is one of the main pleas made by business etc. in favour of advertising. But before reaching conclusions remember that we are looking at one firm in isolation. Try some other examples e.g. the effect of a tax (or subsidy) on labour.

Notice how the operational content of the theory comes from sufficient conditions as in Th.III.5.

Theorem III.6. Suppose that a) $\pi^* = 0$ and b) $\frac{\partial p_1}{\partial y_1} < 0$. Then if $C(y_1)$ is the minimum average cost of producing $y_1$ (I have supressed price arguments), the

$$\frac{\partial C(y_1^*)}{\partial y_1} < 0$$

Proof. At a maximum

$$p_1 (1 + \sigma_{11}) = \frac{\partial C}{\partial y_1}$$

By assumption $\sigma_{11} < 0$ and $p_1 = c_1(y_1^*)$ since $\pi^* = 0$. So $c(y_1^*)(1 + \sigma_{11}) = \frac{\partial C}{\partial y_1}$

and

$$c(y_1^*) > \frac{\partial C}{\partial y_1}$$

52

But if average cost > marginal cost then average costs are falling.

Draw the diagram for this case. Notice that it is of interest because in the old literature [ e.g. J. Robinson: Economics of Imperfect Competition ] it was argued that entry of new firms into the production of similar commodities will eliminate all profit. This is a somewhat confusing story with a large literature. It will be discussed in lectures. But read the appropriate section of Chamberlin: Theory of Monopolistic Competition.

b) The Oligopoly Case

I start with two observations (i) we do not really know the answers and (ii) I can only sketch some of the main considerations. (These problems will be treated much more fully in lectures for Part II).

The problem, or at least one part of it, is best illustrated in a simple Duopoly example. We think now of two producers a and b producing an identical output. Let $x_a$ be the amount produced by a and let $x_b$ be the amount produced by b and let p be the price (a scalar) at which output is sold. Notice that it is assumed that both producers must charge the same price. It is assumed that

$$p = p\,(x_a + x_b) \text{ or } p = p(x) \text{ when } x = x_a + x_b.$$

Let both producers have the some minimum cost function $C(x_i)$, $i = a, b$.

Because there are only two producers each one knows that if he causes a change in price (by changing output), the other producer may be induced to change his output also. But he does not know how the other producer will change it.

let us call

$$v_a(x) = \frac{dx_b}{dx_a}$$

$$v_b(x) = \frac{dx_a}{dx_b}$$

where $v_a(.)$ is the change in b's output a expects his own change to cause and $v_b(.)$ is the change in a's output b expects his own change to cause. Frish called $v_a$, a's conjectural variation and similarily for $v_b$.

At a's maximum profit we have:

$$\left.\begin{aligned} &p(x) + p'x_a + p'v_a(x)\,x_a = C'(x_a) \qquad (a) \\ \text{and for b}\quad &p(x) + p'x_b + p'v_b(x)x_b = C'(x_b) \qquad (b) \end{aligned}\right\} \qquad \text{III.16}$$

Notice that the equation (a) involves $x_b$ and the equation (b) involves $x_a$, so that we can only discover what each firm will close to do if we already know what the other firm is doing.

To be more precise: given any $x_b$, III.16(a) will determine the optimum choice $x_a$ of a, given his conjectural variation. We write

$$x_a = R_a(x_b)$$

$$x_b = R_b(x_a)$$

where $R_a(x_b)$ is a's profit maximising choice of output given $x_b$. In the literature $R_a(.)$ and $R_b(.)$ are called <u>reaction functions</u>.

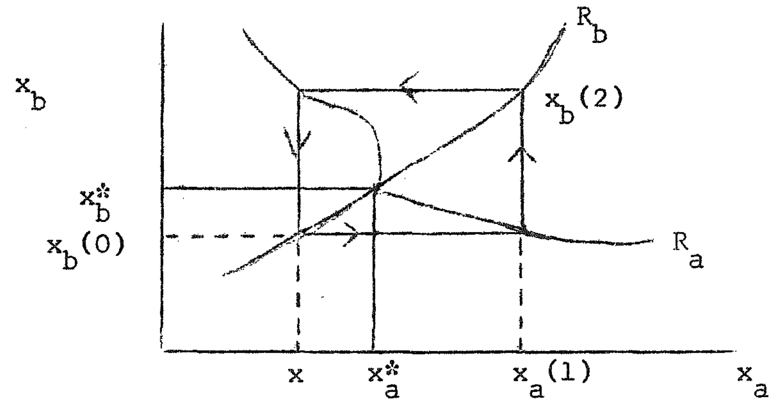



Fig. III.3

The fig. is self-explanatory although the slope which I have given the two reaction functions is plausible (is it?) but by no means the only possible one. $(x_a^*, x_b^*)$ is the solution to the equations in III.16.

Cournot who was one of the first to study this problem set $v_a(.) = v_b(.) \equiv 0$. This has always struck me as very dubious since it seems to remove all the most interesting features of the problem. But specify a cost and demand function and try out some Cournot examples.

In the fig. I have also drawn a rectangle and arrows. I am indicating a difficulty. Suppose we tell the following story. Firm b has chosen $x_b(0)$. In the next period firm a will ~~the~~ chose $x_a(1)$. Then in the period after that b will chose $x_b(2)$. etc. You see that in this sequential story the two producers will keep going round the mulberry bush.

That of course is due to (i) the story I have decided to tell and (ii) to the slopes I have given to the curves. As a matter of fact in the Cournot-case mild assumptions would lead the story to the intersection point. But we can notice two things: (i) there is no good reason why in general the intersection should be reached and (ii) one finds it hard to believe that the reaction curves would not be shifting during such a process. Why should they shift? Because during the process each firm learns something about the behaviour of the other and so its conjectural variations will change.

The plot thickens but will be thicker yet. For suppose firm a never changes it's conjectural variations and always behaves according to $R_a(x_b)$. Then firm (b) by varying it's output can discover what this reaction function is and almost certainly do better than it would do if it stuck to $R_b(x_a)$.

Let us be precise. Suppose firm (b) discovers a's reaction function $R_a(x_b)$. Then equation III.16(b) becomes

$$p(x_b + R_a(x_b) + p'[1 + R_a'(x_b)] x_b = C'(x_b)$$

which is an equation in $x_b$ alone. There is no reason why the solution to that equation should be the $x_b^*$ of Fig. III.3. Moreover firm (a) may now make much lower profits.

So we have now found that firms must in general take care not to let their rivals discover too acurately how they will react. It is here that we make contact with the theory of games. It is plain that the way in which we have formulated the problem is unsatisfactory. But I must confess at once that while the game theoretic formulation is an improvement we do not believe that it has solved the problem.

I cannot even scratch the surface of game theory but you might enjoy this example.

Example 2.4. Suppose (I keep things simple), that firm a must produce either $x_a$ or $x_a'$ and firm b must produce either $x_b$ or $x_b'$. In the table below the first entry is profit for firm (a) the second profit for firm b. For instance if firm a choses $x_a$ and firm b choses $x_b'$ then a gets a profit of 1 and b a profit of 2

| | $x_b$ | $x_b'$ |
|---|---|---|
| $x_a$ | (2, 1) | (1, 2) |
| $x_a'$ | (1, 2) | (2, 1) |

Suppose a choses $x_a$. Then whatever b choses it can make certain of a profit of 1. Similarily if a choses $x_a'$ it can make certain of a profit 1. Notice that if b discovers a's choice it will always be able to ensure a profit of 2.

But now assume that firm a choses $x_a$ with probability q and $x_a'$ with probability 1 - q. In the same way let p be the probability of b chosing $x_b$ and (1 - p) the probability of it chosing $x_b'$.

Then $2p + 1 - p$ is the expected gain to a of chosing $x_a$

$p + 2(1 - p)$ is the expected gain to a of chosing $x_a'$

So since it choses $x_a$ and $x_a'$ with probability q, 1 - q, the expected gain, $E(\pi)$ of this behaviour is given by

$$E(\pi) = q(2p + (1 - p)) + (1 - q)(p + 2(1-p))$$

$$= 2 - q + p(2q - 1) \qquad \text{III.17}$$

But now notice the following nice property. If firm a choses $q = \frac{1}{2}$ then it can make sure of an expected profit of $1\frac{1}{2}$ whatever p is chosen by b. If we think of this "game" as repeated many times and if we think that the firm is always out to maximise the "sure" expected gain then it will behave in this fashion.

The example is sufficient to show (a) that there is some fascinating work to be done here and (b) that there is a great deal of work to be done. For it is clear that the simple game example can also be criticised.

But we have not discussed the most serious problem yet. Here is a result which I leave it to you to prove.

Theorem III.7. There always exists a distribution of total profits between two duopolists which ensures that each has at least as much profit when they join forces and act as a monopolist as they would have had when they do not join forces.

This therefore is of considerable importance. It points to centrifugal forces which are bound to be at work - there will always be a tendency for duopolists (oligopolists) to join forces to form a monopoly. What are the counter-forces?: evidently the distribution of the swag. To assert that there exists a distribution is not the same thing as saying that if two duopolists joined up the actual distribution would have each one at least as satisfied as before. For more discussion see lectures. The point that there will be tendencies towards "joint profit maximisation" may help to explain a good deal of observed behaviour in this field.

Enough has been said to sketch the enormous complexity of these problems. More will be said in lectures. I conclude with one last remark.

If we think of duopoly or oligopoly we find very soon that the hypothesis that every firm is rationally seeking its self interest leads at best to enormously complicated calculations. Remember that when there are say 4 producers, sub groups of them can form joint-profit maximising coalitions. But if we find the calculations complex so do the participants. When rational behaviour becomes very complicated we seek social conventions and rules of thumb to help us out. One such rule in this context is price leadership. That is, it becomes established that one firm sets the price which all others follow. Moreover should one firm attempt to lower its price below that of the leader all other firms will do so as well (and so possibly not make it worth while to break the discipline of price leadership). If a firm raises its price, the others stay put. This results in each firm facing a kinked demand curve, the kink occuring at the price set by the leader. Why? Draw the curve. This part of the story will be more fully discussed in lectures.

# Principles of Economics IV

## The Economy of Producers.

### (1) Total Production Sets etc.

I shall now once again ignore for a time the complications discussed in III although I shall return to some of them. The prime aim at the moment is to see what can be said of the economy as a whole without bringing in the consumer.

Recall that $y_f$ is an activity of firm f and $Y_f$ the set of activities which are technologically feasible

Definition IV.1. The aggregate production activity vector y of the economy is defined by

$$y = \sum_f y_f, \quad y_f \in Y_f \text{ each } f$$

and the aggregate production set Y, by

$$Y = \{ y \mid y = \Sigma y_f \text{ with } y_f \in Y_f \text{ each } f \}.$$

Notice that since inputs in $y_f$ are taken as negative and outputs as positive, the components of y are the net inputs and net outputs of the economy.

Theorem IV.1. If $Y_f$ is convex each f then Y is convex

Proof: Obvious.

I shall make the following assumptions

Assumption IV.1. a) Y is bounded and closed and convex.

b) $0 \in Y_f$ each f (possibility of inaction)

c) $Y \cap R^n_{++} = \emptyset$ (no net output without net input)

d) If $y \in Y$ and $y' < y$ then $y' \in Y$ (Free disposal)

Of these, (a) is most restrictive not only because convexity is not a very appealing assumption, but because boundedness seems silly. But you will be happy to learn at a more advanced stage that this is not something we need to postulate (we can postulate something more appealing and just as good) and that it is done here to make life simple. In (c) one takes

$$R^n_{++} = \text{the set of strictly positive } n - \text{vectors.}$$

The Free Disposal assumption means what it says and if you have ever dealt with dustmen you will know that it is dubious. But again one can go a long way without it (by allowing some negative prices in subsequent analysis) and it is here to keep things nice and simple.

One certainly now can prove the following

Theorem IV.2. Let $y_f(p)$ be the activity choice by f at prices p, (see II), and $y(p) = \sum_f y_f(p)$. Then

a) $py(p)$ solves: $\max(py)$ on Y

b) $py(p) \geqq 0$ all $p > 0$

Proof

a) Suppose not. Then there is $y' \varepsilon Y$, $py' > py(p)$.

By the definitions then: $p \, \Sigma y'_f > p \, \Sigma y_f(p)$, $y'_f \varepsilon Y_f$ each f

so for some f one has

$$py'_f > py_f(p)$$

a contradiction of the definition of $y_p(p)$

b) By A IV.1 (b) $py_f(p) \geqq 0$ all f.

So that the economy of independant producers maximises the total profits of all producers over the total production possibility set. One can now prove the following basic result.

Theorem IV.3: For $p >> 0$: $y' > y(p) \Rightarrow y' \notin Y$

Proof. If not then $py' > py(p)$, $y' \varepsilon Y$, a contradiction of Th. IV.2 (a).

So what this says is that, (when all prices are strictly positive), the economy cannot attain an activity of higher net output of any good without having less net output of some other good or some more net inputs, than it has at the profit maximising activity. So in that sense profit maximisation entails production efficiency. Since efficiency turns out to be interesting let us devote a section to studying it further.

(2) Production Efficiency.

Definition IV.2. An activity $y^* \varepsilon Y$ is said to be production efficient if:

$$Y \cap \{ y \mid y > y^* \} = \emptyset$$

I leave the intepretation of D.IV.2 to you, but it will be convenient to have a notation for the set of activities which are <u>production superior</u> to y* and so I write

$$S(y^*) = \{ y \mid y > y^* \}$$

Notice that $S(y^*)$ is an open <u>convex</u> set. Why?

I now appeal to the theorem of separating hyperplanes (see Mathematical notes), which says that if there are two convex disjoint sets there is a vector $q \neq 0$ and a scalar $C > 0$ such that

$$qy \geqq C \qquad \text{all } y \in S(y^*) \qquad (1)$$

$$qy \leqq C \qquad \text{all } y \in Y \qquad (2)$$

<u>Lemma IV.1</u>. $q > 0$. Clearly $y >> 0$ and "large enough" is in $S(y^*)$. If some $q_i < 0$ one can make $y_i > 0$ and arbitrarily large without leaving $S(y^*)$ and contradicting (1). So $q_i \geqq 0$ all i. But $q \neq 0$.

So since $q > 0$ we can interpret it as a vector of prices, and we have straight away a theorem which used to excite writers on Socialist economics in the past and is now exciting some Russians.

<u>Theorem IV.4.</u> If y* is production efficient then there exists a price vector q which if it ruled would cause profit maximising firms to choose $y_f(q)$ such that

$$y^* = \sum_f y_f(q)$$

<u>Proof</u>. Take the price vector to be q of (1) and (2). Notice that in $S(y^*)$ there are points arbitrarily close to y* and for all of these (1) holds. So by straight forward limiting argument

$$qy^* \geqq C$$

But $y^* \in Y$ and so by (2) $\quad qy^* \leqq C$

which gives: $\quad qy^* = C.$

But then (2) now says

$$qy \leqq qy^* \quad \text{all } y \in Y$$

and by Th. IV.2) the present theorem is proved.

I will discuss this further in lectures, here is a simple example.

<u>Example IV.1</u>

Suppose y with $y_1 > 0$, $y_2 > 0$, $y_3 < 0$ is possible.

60



Consider the set of such $y \varepsilon Y$ with $y_3 \geqq -1$, and that it is as in Fig. IV.1

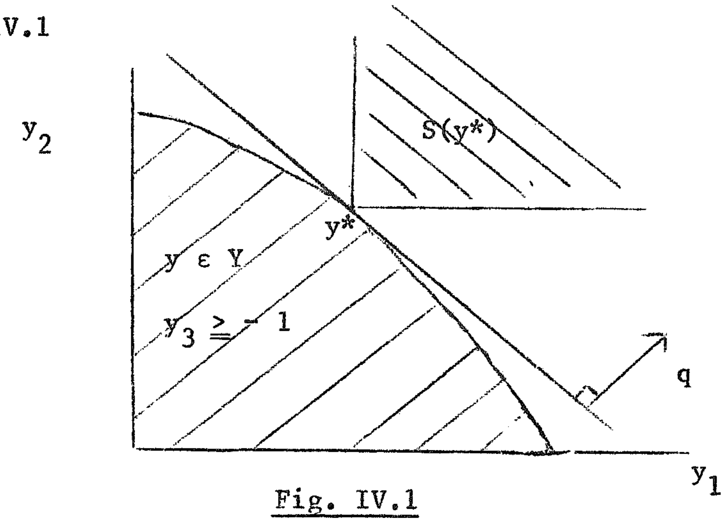



Fig. IV.1

It explains itself, does it not?

(3) Equilibrium: The Special case of Constant Returns.

Given p we know y(p), that is, the set of activities which may result from the profit maximising activities of all firms. But how are we to determine p? In general we should agree that we need more information to do that: we need to know the demand for net outputs and the supply of net inputs by households. If these depended on p as well then we would be inclined to say that we can determine the equilibrium p by using the further conditions that demand should equal supply in each market.

But as you know economists form the Classics to Mr. Sraffa have wanted to determine the equilibrium p from considerations of production alone (the demand side is regarded as somehow 'less real' or 'psychic' and perhaps as subversive). So I now enquire into the conditions on $Y_f$ which will allow us to carry out this programme. It is a striking fact that the theory which makes Ricardo right was provided by Samuelson ("The Non-Substitution Theorem".)

We introduce the following special postulates:

Assumption IV.2. For each f

a) $Y_f$ is a cone (constant returns to scale)

b) If $y_f \varepsilon Y_f$, $y_{fi} \leqq 0$ all $i \neq f$ (only one produced output, $y_{ff}$, or no joint production)

c) There is an input, say $i = 0$, and only one such input such that it is not an output of any firm.

d) $y_f \in Y_f,\ y_{ff} > 0 \rightarrow y_{f0} < 0$ all f (The input "0" is required by all firms with non-zero output.)

I discuss A.IV.2 in lectures. But note that (b) and (d) are at least as restrictive as is the postulate of constant returns.

I have introduced an input "0" which for concreteness we may call "labour". The vector $y_f$ is (n + 1) dimensional:

$$y_f = (y_{f0}, y_{f1}, \ldots, y_{ff}, y_{ff+1}, \ldots, y_{fn})$$

and we now let there be n firms or sectors i.e.

$$f = 1, \ldots, n.$$

So there are as many firms as there are produced goods.

We know the following facts (III) about an economy in which all goods are produced under constant returns:

a) The minimum unit cost function of each firm depends only on p. Write

$$C_f = C_f(p) \qquad\qquad p \in R_+^{n+1}$$

b) $C_f(.)$ is homogeneous of degree one in p and concave in p.

c) No firm can be in equilibrium at a positive profit and no firm will produce at a negative profit. So if the economy is in equilibrium we have:

$$\left.\begin{array}{l} p_f \leq C_f(p) \quad \text{all } f \\ \left[ p_f - C_f(p) \right] y_{ff} = 0 \quad \text{all } f \end{array}\right\} \qquad (3)$$

The relations (3) follow from A.IV.3(b) since each firm produces only one good.

d) We know that

$$\frac{\partial C_f(p)}{\partial p_i} \equiv C_{fi}(p) = -\frac{y_{fi}}{y_{ff}} \equiv a_{fi} \quad \text{say, all } f \text{ and } i$$

since that was demonstrated in Theorem II.8. Which you all know!

Of course, $a_{fi}$ is the amount of input i per unit of output f used when prices are p.

e) We all know Euler's Theorm on homogeneous functions of the first degree:

If $x \in R^n$, $f(x\,k) = k\,f(x)$, $k > 0$ then

$$f(x) = \Sigma f_i(x)x, \quad f_i(x) = \frac{\partial f(x)}{\partial x_i}$$

So (3) can be written:

$$\left.\begin{array}{lll} p_f \leqq \sum\limits_{i=0} a_{fi}\, p_i & \text{all } f & \quad\text{a)} \\ p_f = \sum\limits_{i=0} a_{fi}\, p_i & \text{for } y_{ff} > 0 & \quad\text{b)} \end{array}\right\} (3^*)$$

(Notice that each $a_{fi}$ depends on p).

I shall now introduce

<u>Assumption IV.3</u>: There is a $\bar{y}_f \in Y_f$ each f and so $\bar{y} \in Y$ such that

$$\sum_{i=1} \bar{y}_{if} > 0 \quad \text{all } f$$

$$- \Sigma \bar{y}_{i0} \leqq 1$$

(i.e. By using no more than one unit of labour it is possible to have a positive net output of every produced good).

Suppose then that we are interested in an equilibrium of the economy in which the net output of every produced good is positive. We know that this is technologically feasible. One must therefore have

$$\sum_{f=1} a_{fi}\, y_{ff} < y_{ff} \quad i = 1,\ldots,n \quad y_{ff} > 0 \text{ all } f \tag{4}$$

$$- \sum_f a_{f0}\, y_{ff} \leqq 1.$$

Do not proceed until you understand.

So if $x = (y_{11},\ldots,y_{nn})$, $A = [a_{ij}]$, an n x n matrix we, have from (4)

$$x[I - A] \gg 0 \tag{5}$$

has a solution $x \gg 0$, where I is the n-dimensional unit matrix.

Now notice that in the matrix I - A the diagonal elements are the only positive elements. This allows us to prove

Lemma IV.1. If $x [I - A] \gg 0$ has a solution $x \gg 0$ then (a) $[I - A]$ is non singular and (b) $[I - A]^{-1}$ has no negative elements.

Proof. (i) Suppose first that $z [I - A] \geqq 0$ has a solution with some component of z negative. Let

$$z_k/x_k \leqq z_i/x_i \quad \text{all } i \text{ (by assumption } x \gg 0).$$

So certainly $z_k/x_k < 0$. Then if

$$w = z - (z_k/x_k) x \rightarrow w \geqq 0$$

and

$$w [I - A] = z [I - A] - (z_k/x_k) x [I - A] \gg 0 \qquad (6)$$

why? But in the vector w the $k^{th}$ component $w_k = 0$, so the $k^{th}$ element in the vector $w [I - A]$ is given by

$$-\sum_{i \neq k} a_{ik} w_i \leqq 0$$

what is $\geq$?

which contradicts (6). Hence $z [I - A] \geqq 0 \rightarrow z \geqq 0$.

(ii) But both $z [I - A] \geqq 0$ and $(-z) [I - A] \geqq 0$ imply $z \geqq 0$ and $-z \geqq 0$ by (i) whence $z [I - A] = 0$ implies $z = 0$, whence $(I - A)$ is non-singular

(iii) For any vector $b > 0$, $z [I - A] = b$ has the solution

$$z = b [I - A]^{-1}, \quad z \geqq 0 \text{ by (i)}$$

it follows that $[I - A]^{-1} \geqq 0$.

OK. we are now ready for the first stage. It is claimed that the economy has an equilibrium with strictly positive net outputs of all produced goods. Then we know that a necessary condition for that to be true is that there is $p^* > 0$ with

$$p^*_f = C_f(p^*) \qquad f = 1,\dots,n$$

or using 3*(b):

$$p^*_f - \sum_{i=1} a^*_{fi} p^*_i = a^*_{f0} p^*_0 \qquad f = 1,\dots,n \qquad (7)$$

where

$$a^*_{fi} = a_{fi}(p^*).$$

So if $a^*_o = (a^*_{1o}, \dots, a^*_{no})$ etc. one has in obvious notation

$$[I - A^*]\hat{p}^* = a^*_o p^*_o$$

$\hat{p}^* = (p^*_1, \dots, p^*_n)$. But our Lemma applies to $[I - A^*]$, why?

and so

$$\hat{p}^* = [I - A^*]^{-1} a^*_o p^*_o \geq 0 \tag{8}$$

But then by the Lemma if $p^*_o = 0$ the $\hat{p}^* = 0$ which is impossible.

Hence $p^*_o > 0$. By A.IV.2(d) $a_o >> 0$, where $\hat{p}^* >> 0$. So if

$$(p^*_1, \dots, p^*_n)\ 1/p^*_o$$

is the price vector in terms of labour we have

$$(^1/p^*_o)\ \hat{p}^* = [I - A^*]^{-1} a^*_o \tag{9*}$$

For the moment I am assuming that an equilibrium with (7) is possible. Later I sketch a proof. At the moment I am interested in a more striking result.

Suppose that (9*) gives us the equilibrium (relative) prices when a given strictly positive basket of net outputs is specified. Now specify another basket. Will relative prices be different?

Suppose that it is claimed that with the new basket, the price vector q* solves

$$q^*_f = C_f(q^*) \quad \text{all } f.$$

Let

$$b^*_{fi} = \frac{\partial C_f(q^*)}{\partial q_i}, \quad b^*_o = (b^*_{1o}, \dots, b^*_{fo})$$ etc so that by the same argument as before

$$(^1/q^*_o)\ \hat{q}^* = [I - B^*]^{-1} b^*_o \tag{10}$$

Now surely setting $q^*_o = p^*_o = 1$

$$0 = [I - A^*]\hat{p}^* - a^*_o \geq [I - B^*]\hat{p}^* - b^*_o \tag{11}$$

65



for all that says is that if profits are maximised by firms choosing the techniques $(A^*, a^*_o)$ when prices are $\hat{p}^*$ then these profits cannot be less than they would have had if they had chosen $(B^*, b^*_o)$ which by assumption they could have done. From (11), remembering $[I - B^*]^{-1} \geq 0$, one has

$$\hat{q}^* = [I - B^*]^{-1} b^*_o \leqq \hat{p}^* \tag{12}$$

But by the same argument

$$0 = [I - B^*] \hat{q}^* - b^*_o \geqq [I - A^*] \hat{q}^* - a^*_o$$

or

$$\hat{p}^* = [I - A^*]^{-1} a^*_o \leqq \hat{q}^* \tag{13}$$

Combining (12, 13) gives

$$\hat{p}^* = \hat{q}^* \quad \text{when } p^*_o = q^*_o = 1$$

So relative prices are the same for both baskets. Here is the theorem.

Theorem IV.5: If A.IV.2; 3 then if the economy is in equilibrium at one positive net output vector at relative prices $\hat{p}^*$ then these prices must be the equilibrium prices for all attainable positive net output vectors. In particular, the equilibrium relative prices are independent of either the composition or scale of demand.

Remark: I have in the last sentence gone slightly beyond what has been proved: for I have been dealing with a strictly positive net output vector. Because, however, Y is taken as closed there is a simple limiting argument to show that the result applies to all equilibria with semi-positive net output vectors.

In lectures this result will be further discussed. Here I want to draw your attention to an obvious point.

It is easy to prove that under the conditions of Lemma IV.1

$$[I - A^*]^{-1} = [I + A^* + A^{*2} + \ldots\ldots]$$

Let us then consider (9*) with $p^*_o = 1$

$$\hat{p}^* = [I + A^* + A^{*2} + \ldots\ldots] a^*_o$$

Looking at the first row one has

$$\hat{p}^*_1 = a^*_{o1} + \Sigma a^*_{1j} a^*_{oj} + \sum_k a^*_{ok} \sum_j a_{1j} a_{jk} + \ldots \tag{14}$$

The first term on r.h.s. of (14) is the direct labour cost per unit of producing good 1. But one unit of that good requires $a^*_{1j}$ of good j which in turn has a labour cost of $a^*_{1j}.a^*_{oj}$ and so on, so that the second term is the first stage indirect cost of labour of producing one unit of good one. But to produce $a_{1j}$ units of good j requires $a_{jk}\ a_{1j}$ units of good k which requires $a_{ok}\ a_{jk}\ a_{1j}$ expenditure on labour and so the meaning of the third term is obvious. As we proceed the terms become smaller (why?). So we have

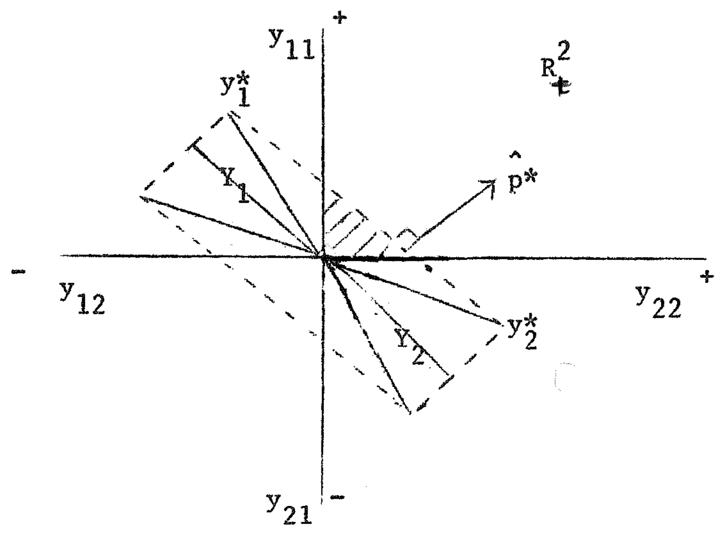
Corollary IV.5. Under the assumptions of Th.IV.5 the price of any good in terms of labour is equal to the sum of the direct and indirect amounts of labour needed to produce one unit of it.

So we have here a labour theory of prices. Notice that our choice of calling the non-produced input 'labour' was rather arbitrary. In the lectures I discuss the case for calling it land.

(4) Efficiency once again.

So far I have assumed that an equilibrium exists which is quite proper in elementary lectures. But it turns out that we can pursue this problem further and learn some economics rather than mathematics.

In Fig. IV.2 I have drawn the production curves of two producers $(Y_1, Y_2)$. I assume that $y_1 \in Y_1$ produces good 1 $y_{11} \geq 0$ uses good 2, $y_{12} \leq 0$ and labour $y_{10} \leq 0$. Also $y_2 \in Y_2$ has $y_{20} \leq 0$, $y_{21} \leq 0$, $y_{22} \geq 0$. I have drawn a projection (since I cannot draw easily in three dimensions) and have shown $y_f$ with $y_{fo} \equiv -1$, $f = 1, 2$, i.e. the feasible activities if each firm only uses one unit of labour.



I here also drawn $\overline{Y}$ which is obtained by taking all points in the diagram which can be written

$$\alpha y_1 + (1 - \alpha)\, y_2 \geqq y \qquad 1 \geqq \alpha \geqq 0,\ y_1 \in Y_1,\ y_2 \in Y_2$$

for if I do that I get all the combinations of output which are possible and do not use more than one unit of labour. I have drawn the intersection of $\overline{Y}$ with the non-negative orthant $R^2_+$ by shading. The frontier of ks intersection you notice is a straight line. It is the economy's transformation curve. But then unless the price vector $\hat{p}^*$ is as shown for any point on the transformation curve some firm would make a positive profit or a loss. For since $\hat{p}^*$ is normal (orthogonal) to that curve, one has for any two points which use one unit of labour

$$p_1^* (y_1 - \overline{y}_1) + p_2^* (y_2 - \overline{y}_2) = 0. \quad ?$$

I shall discuss this further in lectures. I shall also illustrate the force of A.IV.2 by means of Fig. IV.2. But here is how we can go about a general way of stating the insight.

<u>Theorem IV.6</u>. Under A.IV.2, A.IV.3 for any $y^* \in Y$ which is production efficient there exists $p^*$ independant of $y^*$ such that

$$\max_{Y_f} p^* y_f = p^* y_f^* = 0 \qquad \text{all } f$$

<u>Proof</u>. (i) We assume that there is one unit of labour and define

$$F = \{ y \mid y = \sum_f y_f,\ y_f \in Y_f \text{ all } f,\ -\Sigma y_{fo} \leqq 1 \}$$

which gives the set of activities y feasible for the economy if it uses no more than one unit of labour. One has of course:

F is closed and convex

Also let

$$H(y^*) = \{ y \mid y > y^* \}$$

so that

H(y*) is open and convex

Then if $y^* >> 0$ is Pareto efficient in F one has

$$F \cap H(y^*) = \emptyset.$$

One can therefore find the hyperplane separating $H(y^*)$ and $F$. That is, there is $p \in R_+^{n+1}$ such that

$$py \geqq py^* = 0 \quad \text{all } y \in H(y^*) \tag{15}$$

$$py \leqq py^* = 0 \quad \text{all } y \in F \tag{16}$$

Let $(A^*, a_o^*)$ be the activity matrix which generates $y^*$. Certainly

$$py^* = [I - A^*]\hat{p} - a_o^* p_o = 0 \tag{17}$$

and of course by (16) these activities are profit maximising.

Let $y^{**} >> 0$, $y^{**} \neq y^*$, be another activity in $F$ which is production efficient and let it be generated by the matrix $(B^*, b_o^*)$. Since by (16)

$$py^{**} \leqq 0$$

one has

$$[I - B^*]\hat{p} - b_o p_o \leqq [I - A^*]\hat{p} - a_o p_o = 0 \tag{18}$$

which is (11) so as before

$$[I - B^*]^{-1} b_o^* \leqq \hat{p}\left(\frac{1}{p_o}\right)$$

But if $y^{**}$ is production efficient there is $q \in R_+^{n+1}$ such that

$$qy \leqq qy^{**} = 0 \quad \text{all } y \in F$$

So proceeding as before we get (13) and deduce

$$\hat{q}\frac{1}{q_o} = \hat{p}\frac{1}{p_o}$$

So we have proved, as before, that the relative prices which lead <u>any</u> production efficient $y$ to be profit maximising are independent of the choice of production efficient activity. Nice?!

## (5) <u>Difficulties</u>

The most obvious difficulty is time. Recall that we agreed long ago to treat goods at different dates as different. But then labour at different dates is different and the postulate of one non-produced input will be hard to sustain. Very closely allied to those difficulties is the problem of durable inputs. Because if these exist the output of a firm consists not only of the good which it produces but also of the used durable input transferred from one moment to the next.

This is a 2nd year course so we cannot devote time to a careful exploration of these problems and I shall only discuss the simplest cases. In particular I shall not now discuss the case of durable inputs.

Suppose that $y_f$ still has the properties of A.IV.2 but that it has a temporal structure of the following kind: inputs at date t give rise to output at date t + 1. So if we now are explicit about time structure we write

$$y_f \in Y_f : y_f = (y_{fo}(t),\ldots,y_{ff-1}(t),\ y_{ff}(t+1),\ y_{ff+1}(t),\ldots,y_{fn}(t))$$

Assumption IV.4. Labour is paid at the end of the production period.

This assumption is not at all required but is so usual in the literature that I make it for completeness sake.

When the firm has to pay for inputs one period before it can sell the resulting output it must charge itself the rate of interest or profit foregone. I write this rate as r and

$$R = 1 + r$$

What we are now looking for is a set of relative prices - say prices in terms of labour-which we write as $\hat{p}$, (with $p_o = 1$), which remains independent of t and at r causes producers to make the same profit maximising choices.

Let us therefore now write the minimum cost function:

$$C_f = C_f\,(R\,\hat{p},\ 1)$$

which gives the minimum cost in terms of labour of producing one unit of output at $(\hat{p}, R)$. As before

$$C_{fi} = \frac{\partial C_f}{\partial (Rp_i)} = a_{fi}$$

and $a_{fi}$ depends on $(\hat{p}, R)$. For an equilibrium with positive net output for each firm one must have

$$p_f = C_f(R\,\hat{p},\ 1) \qquad f = 1,\ldots,n \tag{19}$$

Using homogeneity this becomes

$$p_f = \Sigma\, a_{fi}\, R\, p_i + a_{fo} \qquad f = 1,\ldots,n \tag{20}$$

Or in obvious matrix notation

$$[I - RA]\hat{p} = a_o \tag{21}$$

In (21) A depends on $(\hat{p}, R)$. Also (21) gives us n equations in (n + 1) unknown: $(\hat{p}, R)$. Therefore the theorem which we have discussed before is modified to:

<u>Th.IV.7</u>: For any R such that $[I - RA]^{-1}$ exists and is non-negative for some $\hat{p}$, the equilibrium prices are independant of the composition of demand.

Notice that I am taking a short cut in stating the conditions of the theorem. If you look back at Lemma IV.1 and the assumption which I have used you will see that what we are saying is that we can only allow R which is not so large as to make impossible the assumption that there is a strictly positive net output vector which is now defined by:

net output vector = output vector - Rx input vector.

Otherwise the story is unchanged and I leave it to you to prove the theorem. Notice that

$$[I - RA]^{-1} = [I + RA + R^2A^2 + \dots]$$

So that the labour theory of prices gets modified. How? Well obviously by making indirect labour input bear an interest charge. Work it out!

Now let me briefly deal with nonsense. It is plain that (19), (or 21), leaves us one degree of freedom. We can for instance arbitrarily fix one price in terms of labour. Or we might fix the "real wage" by demanding that for some weight $\alpha_i > 0$:

$$\Sigma\, \alpha_i\, p_i^{-1} = \text{constant.} \tag{22}$$

Or we could fix R. All of these procedures will "close the system". But of course what we fix should have some justification. For instance a Malthusian might have an argument for (22) might he not? Professor Joan Robinson - Japanese etc. evidence notwithstanding-believes that all savings is done by capitalists and that they have animal spirits. By assuming that investment = saving in equilibrium she gets an equation for r.

(Of course one should elaborate here a little). But you could also have a Pigovian story that the supply of labour depends on the <u>real wage</u>, (i.e. on $\hat{p}^{-1}$), and that the demand for labour = supply. Or you could have a story that the saving people do depends on $(\hat{p}, R)$ and again that investment depends on $(\hat{p}, R)$ and whoops you have another equation. You could even determine R by Keynesian liquidity preference. But the point is this: it is nonsense to say that in general R is "determined" <u>before</u> $\hat{p}$ is or indeed vice versa. There is <u>nothing</u> in the logic of the story to falsify the general theorem that $(\hat{p}, R)$ are determined <u>simultaneously</u> by (19) and market clearing equation.

Of course in Cambridge this is a bone of contention. Think about it and see whether the argument has a flaw!

Lastly let us see how different values of R go with different equilibrium $\hat{p}$.

Differentiate (19) with respect to R to get

$$\frac{\partial p_f}{\partial R} = \Sigma a_{fi} \left(p_i + R\frac{\partial \hat{p}_i}{\partial R}\right) \quad f = 1,\dots,n \tag{23}$$

Or in matrix notation

$$[I - RA] \frac{d\hat{p}}{dR} = A\hat{p} \tag{24}$$

But $A\hat{p} > 0$ and

$$[I - RA]^{-1} > 0$$

whence

$$\frac{d\hat{p}}{dR} > 0 \tag{25}$$

So if now we write the "real wage" as w with

$$w = \Sigma \alpha_i p_i^{-1}$$

one has

$$\frac{\partial w}{\partial R} < 0 \tag{26}$$

which is the "factor price-frontier" proposition i.e. real wages are lower the higher the rate of profit.

<u>Question</u>: Does that mean that the <u>share of labour</u> in the value of output is lower the higher R??

Now you are ready if you want to, to read and understand all about double-switching etc. etc. I shall briefly mention it in lectures. But here is a theorem for the fastidious.

Theorem IV.8. If $Y_f$ has a differentiable frontier for each f then with every equilibrium $(\hat{p}, R)$ there is associated one and only one technique $(A, a_o)$.

Proof. Left to you.

# Principles of Economics V

## The Choices of Households.

1) Introduction: In this lecture I consider the actions of a household "that knows what it wants and knows how to get it". In a precise sense this will be a theory of rational actions. Many people object that households in fact are not rational. A good many of these objectors do not know the theory. But they may be right. One of the objects of the theory will thus have to be to generate propositions which can in principle be shown to be false. This we shall do. But in learning the theory clear your minds of the fetters of what you regard as "common sense". The common sense of anyone is usually a vulgarisation of an old-fashioned theory or plain prejudice. Also notice that the theory is beautiful.

2) The Consumption space $X_h$

A household's action is a vector $x_h$ the components of which (e.g. $x_{hi}$) are the amounts of various goods consumed. The set $X_h$ is the set of all $x_h$ which are physically possible for the household h. For instance, it cannot consume more than twenty four hours of leisure per day, or if it consumes only one hour of leisure per day it will have to consume at least n calories, etc.

I shall take it that there are n goods and make the following assumption

A.V.1: $X_h \subset R^n_+$ and $X_h$ is closed and convex.

Notice that A.V.1. postulates (a) that goods are consumed in non-negative quantities and (b) that goods are finely divisible. For if not, $x_h \in X_h$ and $x'_h \in X_h$ would not allow $\alpha x_h + (1 - \alpha)\, x'_h \in X_h$ for all $0 \leq \alpha \leq 1$.

In modern work A.V.1 can be greatly relaxed. For instance, we have results on consumption spaces where goods can only be consumed in integral quantities. Moreover, it is possible to develop the theory for much more general spaces which contain objects other than goods. All this will have to be ignored.

3) <u>The ordering of $X_h$.</u>

"The household knows what it wants" is interpreted as meaning that the household has a <u>complete preordering of $X_h$</u>. More specifically, for $x_h$ and $x'_h \in X_h$ read

$x_h \; R \; x'_h$ as "$x'_h$ is not preferred to $x_h$"

sometimes I shall also use

$x_h \; R \; x'_h$ <u>and</u> $x'_h \; R x_h$ read $x_h \; I \; x'_h$ "$x_h$ and $x'_h$ are indifferent"

$x_h \; R \; x'_h$ and <u>not</u> $x'_h \; R \; x_h$, read $x_h \; P \; x'_h$ "$x_h$ is preferred to $x'_h$",

I shall make the following assumptions:

Assumption V.2. a) <u>Connexity</u>: For all $x_h$, $x'_h \in X_h$, <u>either</u> $x_h \; R \; x'_h$ <u>or</u> $x'_h \; R \; x_h$ <u>or</u> both.

b) <u>Transitivity</u> for $x''_h$, $x'_h$ and $x_h \in X_h$

$x''_h \; R \; x'_h$ and $x'_h \; R \; x_h$ <u>implies</u> $x''_h \; R \; x_h$.

c) <u>Continuity</u>. For any $x^o_h \in X_h$ the sets:

$W^+(x^o_h) = \{ x_h \mid x_h \; R \; x^o_h, \; x_h \in X_h \}$ and

$W^-(x^o_h) = \{ x_h \mid x^o_h \; R \; x_h, \; x_h \in X_h \}$

are both closed.

d) <u>Non-Satiation</u>. There is <u>no</u> $x^o_h \in X_h$ such that $x^o_h \; R \; x_h$ all $x_h \in X_h$.

These assumptions will be further discussed in lectures.

It may be worthwhile commenting here on (c). What it implies is this: Let $x^1_h \; P \; x^2_h$. Then let

$$x_h(\alpha) = \alpha \, x^1_h + (1 - \alpha) \, x^2_h \qquad 0 \leq \alpha \leq 1$$

Then for $\alpha$ "close to unity" $x_h(\alpha) \; P \; x^2_h$ and for $\alpha$ "close to zero", $x^1_h \; P \; x_h(\alpha)$. In other words, bundles "close to $x^1_h$" will be preferred to $x^2_h$ and bundles close to $x^2_h$, are inferior to $x^1_h$.

4) The Numerical Representation of an Ordering: $U_h(x_h)$.

Let us ask: does there exist a continuous function $U_h(x_h)$

$$U_h(x_h): \; X_h \to R^1$$

($R^1$ = the real line), such that

$$U_h(x_h) \geqq U_h(x_h') \text{ iff } x_h \; R \; x_h' \qquad (1)$$

$$U_h(x_h) > U_h(x_h') \text{ iff } x_h \; P \; x_h' \qquad (2)$$

If such a function exists, call it a utility function.

Suppose that a utility function exists. Let

$$V_h = V_h(U_h(x_h)), \; V_h' > 0 \text{ everywhere.}$$

Then $V_h$ will also be a utility function. Why? One has

Theorem V.1: A utility function satisfying (1), (2) is defined only up to a monotone transformation (it is ordinal).

Thus $U_h(.)$ measures nothing: it represents an ordering of $X_h$. In particular there is no intrinsic meaning to the partial derivatives of $U_h(x_h)$ (if they exist, they are called marginal utilities) since these partials are not invariant under monotone transformation of $U_h(.)$.

A utility function need not exist if the ordering does not satisfy A.V.2.

Example. Let $X_h \subset R^2_+$. Let h have a lexicographical ordering:

$$x_h \; P \; x_h'$$

iff $x_h \neq x_h'$ and $x_{h1} > x_{h1}'$ or if $x_{h1} = x_{h1}'$, $x_{h2} > x_{h2}'$. Notice that for any two distinct points in $X_h$ one must be preferred to the other so that no two distinct points are indifferent. But then if a utility function exists, every point in the plane $X_h$ must get a different number i.e. be assigned a different point on the real line. But that cannot be done; there are not enough numbers to do that.

So if we are to have a utility function we must be able to "economise in numbers" and this requires that the ordering be continuous. For consider

$$W^+(x_h^o) \cap W^-(x_h^o)$$

Since $x_h^o$ is a limit point of both sets and both sets are closed the intersection is not empty since it must contain $x_h^o$. Let $x_h$ be any other l.p. of $W^+(x_h^o)$. Either $x_h \in W^-(x_h^o)$ or $x_h \notin W^-(x_h^o)$. If $x_h \in W^-(x_h^o)$ then $x_h^o \; R \; x_h$ and since $x_h \in W^+(x_h^o)$ one has $x_h \; R \; x_h^o$ and so $x_h \; I \; x_h^o$. Moreover $x_h$ is a limit point of $W^-(x_h^o)$ since every small neighbourhood of $x_h$ contains points in $W^+(x_h^o)$. If $x_h \notin W^-(x_h^o)$ then by definition $x_h \; P \; x_h^o$. But then $x_h$ cannot be a limit point of $W^+(x_h^o)$. Why? So indeed $x_h \in W^+(x_h^o) \cap W^-(x_h^o)$.

So the intersection of the two sets gives us a set of points of $X_h$ which are indifferent to $x_h^o$ and to each other.

The rest is now fairly easy but requires one to perform some little manipulation with the rationals and I omit all that. Heuristically the picture looks like this:

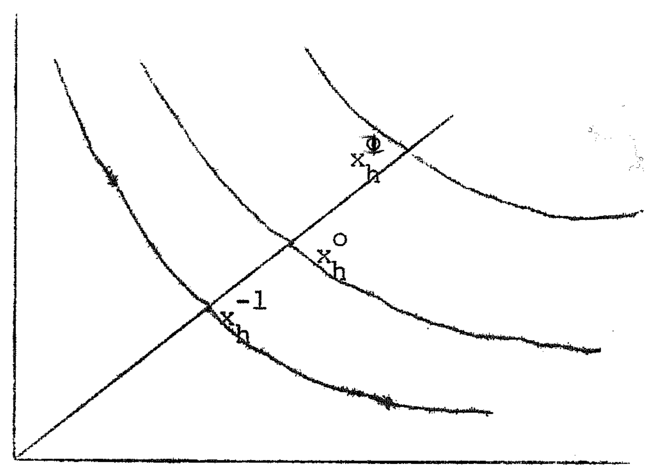



Fig. V.1.

chosen $x_h^{-1}, x_h^o, x_h^1$ ?

In Fig. V.1 I have choose $x_h^-$, $x_h^o$, $x_h^1$ to lie on a line though the origin. I have drawn in $W^+(x_h^{-1}) \cap W^-(x_h^{-1})$, $W^+(x_h^o) \cap W^-(x_h^o)$ etc. Also $x_h^o >> x_h^{-1}$, $x_h^1 >> x_h^o$ etc. and I have made the mild assumption $x_h^o \; P \; x_h^{-1}$, $x_h^1 \; P \; x_h^o$. Then we can plainly label the curves by the numbers we read off on the straight line (I here ignore the problem of 'irrationals'). These numbers can serve as a utility function.

I hope this heuristic account suffices for an understanding of

<u>Theorem V.2</u>. If A.V.2 and $X_h \subset R_+^n$ then there exists a continuous real valued function $U_h(x_h) : X_h \to R^1$ which satisfies (1) and (2).

Later I shall discuss special forms of utility functions and indicate certain assumptions which allow us to deduce functions which are defined up to a linear transformation.

5) Convexity.

The assumption which I shall be using is that the ordering by "R" of $X_h$ is convex.

Definition D.V.1. (a) The ordering is convex if $x_h$ P $x'_h$ then

$$x_h(\alpha) \text{ P } x'_h \quad \text{all} \quad 0 \leqq \alpha < 1$$

$$x_h(\alpha) = \alpha x'_h + (1 - \alpha) x_h.$$

(b) The ordering is strictly convex if $x_h$ R $x'_h$ then

$$x_h(\alpha) \text{ P } x'_h \quad \text{all} \quad 0 \leqq \alpha < 1.$$

Convexity does not exclude flat segments of indifference surfaces but strict convexity does so. Why?

To assume convexity of the ordering is neither harmless nor does it have a special appeal. I may prefer a sausage and a glass of milk to a sausage and a glass of whisky but greatly dislike the convex combination of the two bundles. Or one may prefer living in Cambridge to living in London but hate all points "in between". (This is stretching things a bit, is it not?). So convexity will depend a good deal on what commodity space is chosen and, of course, on divisibility. The troubles that arise from non-convex preferences need not be serious although they can be. But here I stick to convexity.

Now convex preferences will be reflected in the utility function. First a definition

Definition D.V.2. A function $f(x) : R^n \to R^1$ is quasi-concave if for any $x^o \in R^n$ the set $\{ x \mid f(x) \geqq f(x^o) \}$ is convex. The function is strictly quasi-concave if the above set is strictly convex.

If then f(x) is a utility function for all x in the above set one has $x \text{ R } x^o$. Certainly you can now prove for yourselves

Theorem V.3. A quasi-concave utility function represents a convex ordering; a strictly quasi-concave utility function represents a strictly convex ordering.

Remark*: A concave function is quasi-concave but the reverse need not be the case.

6) The set of Budget-Feasible Choices: $B_h(p)$

One supposes that each household has an endowment of goods $\bar{x}_h \in R^n_+$. Notice that (a) the endowment includes leisure as one item (b) that some endowments can be zero and (c) that we do not know the wealth of h until we know the price vector $p \in R^n_+$.

In addition each h is entitled to a fraction $d_{hf}$ of the profits $p.y_f$ of firm f. One has

$$0 \leqq d_{hf} \leqq 1, \quad \sum_h d_{hf} = 1$$

The constraint on profit shares assumes that all firms are owned by households although it is not postulated that all households hold entitlements to the profits of any or of all firms.

In the notation just introduced we now have

Definition D.V.3. The wealth $w_h$ of h is defined as

$$w_h(p) = p \cdot \bar{x}_h + \sum_f d_{hf} (p \cdot y_f)$$

Notice that $w_h$ depends on p.

This then leads to the

Definition D.V.4. a) The set of budget-feasible choices at p open to h is $B_h(p)$,

$$B_h(p) = \{ x_h \mid px_h \leqq w_h(p) \}.$$

b) The set of feasible choices at p: $B^*_h(p)$ is

$$B_h(p) \cap X_h.$$

I shall also use the technical assumption:

Assumption V.3. There exists $\bar{\bar{x}}_h \in X_h$ such that $\bar{\bar{x}}_h \leqq x_h$ and $\bar{\bar{x}}_{hi} < \bar{x}_{hi}$ if $\bar{x}_{hi} > 0$.

This assumption I shall use later.

7) <u>The demand $x_h(p, w_h)$</u>

In what follows I shall only consider $p >> 0$. To a general equilibrium theorist, this is bad - for a second year course it is sensible. When some $p_i = 0$, technical problems arise but nothing of what follows is invalidated.

Notice that $B^*_h(p)$ is a <u>bounded</u> and <u>closed</u> and <u>convex</u> set for $p >> 0$. Why? Therefore the problem

$$\max \ U_h(x_h) \quad \text{for } x_h \in B^*_h(p) \tag{3}$$

has a solution since $U_h(.)$ is a continuous function on a compact set. The solution is found in the usual way and first order **conditions** are

$$\left( \frac{\partial U_h}{\partial x_{hk}} - \lambda p_k \right) x_{hk} = 0, \ \frac{\partial U}{\partial x_{hk}} - \lambda p_k \leqq 0 \text{ all } k, \ \lambda = \text{Lagrangean} \tag{3*}$$

<u>Definition. D.V.5</u>. The demand of household h at $p >> 0$ and $w_h(p)$ is the set of $x_h$ which solve (3). This set is written:

$$x_h\ (p,\ w_h\ (p))$$

and is often called the <u>demand correspondence</u>.

The following theorem will enable us, by making the appropriate assumption, to deal with the simplest case.

<u>Theorem V.4.</u> If the ordering of h is strictly convex then $x_h\ (p,\ w_h(p))$ contains only one element for each $p_j$; it is a, (vector valued), <u>demand function</u>.

<u>Proof</u>. Suppose not, i.e., $x_h \neq x'_h$ are both elements of $x_h(p, w_h(p))$. Then it must be that $x_h \ R\ x'_h$ and $x'_h \ R\ x_h$ since $x_h \in B_h(p)$, $x'_h \in B_h(p)$. By strict convexity and in the usual notation

$$x_h\ (\alpha)\ P\ x_h,\ x_h(\alpha)\ P\ x'_h$$

and one verifies $\quad x_h(\alpha) \in B^*(p)$.

But then neither $x_h$ nor $x'_h$ can solve (3) contrary to assumptions.

From now onwards I shall assume a strictly convex ordering - this is no way limits the essential generality of what follows.

Also it will be useful to have the following

<u>Theorem V.5</u>: $x_h(p, w_h(p))$ is a frontier point of $B_h^*(p)$.

<u>Proof</u>. Suppose not i.e. $px_h(p, w_h(p)) < w_h(p)$

By non-satiation there exists $x_h \in X_h$ so that

$$x_h \, P \, x_h(p, w_h(p))$$

By convexity $\quad x_h(\alpha) \, P \, x_h(p, w_h(p)) \qquad 0 \leq \alpha < 1$

But also for some $\alpha$ in the range

$$px_h(\alpha) \leqq w_h(p)$$

and $x_h(\alpha) \in B^*(p)$ (since $X_h$ is convex). But the $x_h(p, w_h(p))$ cannot be the solution to (3), contrary to definition.

<u>Theorem V.6</u>. $x_h(p, w_h(p)) = x_h(kp, \cancel{w_h(p)}\, w(kp))$, $k > 0$ (demand function is homogeneous of degree zero in p).

<u>Proof</u>: Obvious.

Lastly I prove the following result for those who have not spent their few pence on Arrow-Hahn. <u>It should be skipped</u> by the not so mathematical.

<u>Theorem V.7*</u>: $x_h(p, w_h(p))$ is continuous at all $p >> 0$.

<u>Proof</u>. (i) Let $p^v >> 0$ all v a sequence with $p^v \to p^o$. Let

$$x_h^v = x_h(p^v, w_h(p^v))$$

We want to prove that $x_h^v \to x_h^o$. If not then there is $\varepsilon > 0$ and

$$|| x_h^v - x_h^o || \geqq \varepsilon \quad \text{all v.}$$

(ii) Define:

$$\beta^v = \frac{\varepsilon}{|| x_h^v - x_h^o ||} \quad \text{and note } 1 \geqq \beta_h^v \geqq 0$$

$$x_h^{1v} = \beta^v x_h^v + (1 - \beta^v) x_h^o$$

$X_h$ is convex so $x_h^{1v} \in X_h$ all v.

(iii) If it is the case that $U_h(x_h^o) > U_h(x_h^{1v})$ then

$$U_h(x_h^{1v}) \geqq U_h(x_h^v)$$

For if not a convex combination of two points would be inferior to each of them which is impossible when $U_h(.)$ is quasi concave.

(iv) Now

$$|| x_h^{1v} - x_h^o || = \beta^v || x_h^v - x_h^o || = \varepsilon$$

so $x^{1v}$ is bounded and has a limit point $x_h' \varepsilon X_h$ since $X_h$ is closed.

But

$$p^v x_h^v \leqq w_h(p^v), \text{ all } v$$

$$\text{so} \quad p^v x^{1v} \leqq \beta^v w_h(p^v) + (1 - \beta^v) p^v x_h^o =$$

$$p^v x_h^o + \beta^v (w_h(p^v) - p^v x_h^o) \qquad (5)$$

Since $p^v x_h^o \to w_h(p^o)$, $w_h(p^v) \to w_h(p^o)$ and $\beta^v$ bounded one has

$$p^v x_h^{1v} \to p^o x_h' \leqq w_h(p^o) \qquad (6)$$

Since $x_h' \neq x_h^o$ by assumption and $x_h' \varepsilon B_h^*(p^o)$ one has

$$U_h (x_h^o) > U_h (x_h')$$

So for v large enough

$$U_h (x_h^o) > U_h (x_h^{1v}) \qquad (7)$$

and so by (iii) $\quad U_h (x_h^{1v}) \geqq U_h (x_h^v)$

(v) Let $\bar{\bar{x}}_h$ be the vector defined in A.V.3.

$$x_h (\lambda) = \lambda \bar{\bar{x}}_h + (1 - \lambda) x_h^o \qquad 0<\lambda\leqq 1.$$

Then by continuity for some $\lambda > 0$ one has from (7)

$$U_h(x_h^{1v}) < U_h (x_h (\lambda))$$

and so as before

$$U_h (x_h^v) \leqq U_h (x_h(\lambda)) \qquad (8)$$

But by the definition of $x_h^v$ (8) implies

$$p^v x_h (\lambda) \geqq w_h (p^v)$$

Take limits to get

$$p^o \lambda \bar{\bar{x}}_h + p^o(1 - \lambda)x_h^o \geqq w_h(p^o)$$

$$\text{or} \quad p^o \lambda \bar{\bar{x}}_h + (1 - \lambda) w_h(p^o) \geqq w_h(p^o)$$

$$\text{or} \quad p^o \bar{\bar{x}}_h \geqq w_h(p^o)$$

But if $w_h(p^o) > 0$ this is impossible by the definition of $\bar{\bar{x}}_h$. Hence $x_h^v \to x_h^o$.

(8) Expenditure Functions.

Let us ask what is the dual problem to (3). I claim that it is the following. If $\overline{U}_h$ is the utility attained when (3) is solved then solving

$$\min px_h \quad \text{on} \quad \{ x_h \mid U_h (x_h) \geqq \overline{U}_h \}$$

will "almost always" give the same answer as solving problem 3 does. I shall write

$$e (p, \overline{U}_h) = \min \{p \, x_h \mid U_h (x_h) \geqq \overline{U}_h \}$$

that is, as the minimum expenditure required at p to guarantee utility of $\overline{U}_h$. I now prove my assertion.

Theorem V.8. a) If $x^*_h$ solves $\max U_h(x_h)$ on $B^*_h$ (p) then $x^*_h$ solves $\min px_h$ subject to $x_h \; R \; x^*_h$.

b) If $x^*_h$ solves $\min px_h$ subject to $x_h \; R \; x^o_h$ then it also solves $\max U_h(x_h)$ subject to $px_h \leqq px^*_h$ provided there exists $\bar{\bar{x}}_h < px^*_h$.

Proof (a) If not then for some $x_h \; R \; x^*_h$

$$px_h < px^*_h$$

By non-satiation there exists $x'_h \; P \; x^*_h$. Let

$$x_h(\alpha) = \alpha \, x_h + (1 - \alpha) \, x'_h.$$

For $\alpha$ close enough to unity

$$p \, x_h(\alpha) \leqq p \, x^*_h$$

But since $x^*_h \in B^*(p)$ so does $x_h(\alpha)$ and by convexity

$$x_h(\alpha) \; P \; x^*_h$$

contradicting the definition of $x^*_h$.

(b) Define

$$x_h(\alpha) = (1 - \alpha) \, x_h + \alpha \, \bar{\bar{x}}_h \quad \text{any } x_h \text{ such that } px_h \leqq px^*_h$$

So $\qquad p \, x_h(\alpha) < p \, x^*_h \qquad 0 < \alpha \leqq 1$

Suppose $x_h(\alpha) \; R \; x^o_h$. Then by the definition of $x^*_h$

$$p \, x_h(\alpha) \geqq p \, x^*_h$$

a contradiction, and so

$$x^o_h \; R \; x_h \, (\alpha)$$

But the set

$$W\ (x_h^o) \quad = \quad \{\ x_h \mid x_h^o\ R\ x_h)$$

is closed and so $x_h(1) = x_h$ belongs to it and

$$x_h^o\ R\ x_h$$

But we know that $\qquad x_h^*\ R\ x_h^o$

So by transitivity

$$x_h^*\ R\ x_h \text{ all } x_h \text{ such } p\ x_h \leqq p\ x_h^*$$

which was to be proved.

In what follows I shall take it that the conditions of the Theorem hold.

We now have a rather powerful tool and one which is quite similar to the cost functions of production theory. In particular

Theorem V.9. a) $e(kp, U_h) = ke(p, U_h), \quad k > 0$

b) $e\ (p, U_h)$ is a concave function of p.

Proof: Left to you (see T.II.10)

The expenditure function has the envelope property. I prove this important result now but compare to (T.II.8)

Let

$$e_k\ (p, \bar{U}_h) \quad = \quad \frac{\partial e(p, \bar{U}_h)}{\partial p_k}$$

$$e_{kj}\ (p, \bar{U}_h) \quad = \quad \frac{\partial^2 e(p, \bar{U}_h)}{\partial p_k \partial p_j}$$

Then

Theorem V.10

$$e_k(p, \bar{U}_h) \quad = \quad x_{hk}\ (p, \bar{U}_h) \qquad k \quad = \quad 1, \ldots, n$$

where $x_{hk}\ (p, \bar{U}_k)$ is the $k^{th}$ component of the utility maximising demand vector when maximum utility is $\bar{U}_h$ and prices are p.

Proof. Let $h \in R^n$ and consider $p + h$

Then

$$e(p + h, \overline{U}_h) \leqq (p + h)\, x_h\, (p, \overline{U}_h) \;=\; p\, x_h(p, \overline{U}_h) + h\, x_h\, (p, \overline{U}_h) \;=$$

$$e(p, \overline{U}_h) + h\, x_h(p, \overline{U}_h) \qquad (9)$$

$$e(p, \overline{U}_h) \leqq p\, x_h(p + h, \overline{U}_h) \;=\; (p + h)\, x_h\, (p + h, \overline{U}_h) - h\, x_h\, (p + h, \overline{U}_h) \;=$$

$$e(p + h, \overline{U}_h) - h\, x_h(p + h, \overline{U}_h) \qquad (10)$$

From (9)

$$e(p + h, \overline{U}_h) - e(p, \overline{U}_h) - h\, x_h(p, \overline{U}_h) \leqq 0 \qquad (9^*)$$

From (10)

$$e(p + h, \overline{U}_h) - e(p, \overline{U}_h) - h\, x_h(p, \overline{U}_h) \geqq h\, [\, x_h(p + h, \overline{U}_h)$$

$$- x_h(p, \overline{U}_h)\, ] \qquad (10^*)$$

Divide both sides of 10* by $|| \; h \; ||$ and let $h \to 0$. Then since $x_h$ is continuous the r.h.s. of 10* approaches zero i.e.

$$\lim_{h \to 0} \frac{[\, e(p + h, \overline{U}_h) - e(p, \overline{U}_h) - h\, x_h(p, \overline{U}_h)\, ]}{|h|} \geqq 0 \quad (11)$$

Dividing (9*) by $|h|$ letting $h \to 0$ and combining with (11) now gives

$$\lim_{h \to 0} \frac{[\, e(p + h, \overline{U}_h) - e(p, \overline{U}_h) - hx_h(p, \overline{U}_h)\, ]}{|h|} = 0 \qquad (12)$$

So e(.) is differentiable for any h. Let

$$h \;=\; (0\;0\;1\;0\;0)$$

where (1) is in the $k^{th}$ place. Then from (12)

$$e_k\, (p, \overline{U}_h) \;=\; x_{hk}(p, \overline{U}_h)$$

as was to be proved.

This result will be further discussed in lectures.

9) Compensated Demand functions and Slutsky equation.

T.V.10 provides us with a number of easy but famous results. Here they are

Let
$$x_{hkj}(p, \overline{U}_h) = \frac{\partial x_{hk}(p, \overline{U}_h)}{\partial p_j}$$

and let $\sum_{n-1}(.)$ denote summation over (n-1) indices which include the index k. Then

**Theorem V.11**

(a) $x_{hkk}(p, \overline{U}_h) < 0$

(b) $x_{hkj}(p, \overline{U}_h) = x_{hjk}(p, \overline{U}_h)$

(c) $\sum_{j=1}^{n} x_{hkj}(p, \overline{U}_h)\, p_j = 0$

See over for statement in words!

(d) $\sum_{n-1}\sum_{n-1} x_{hkj}(p, \overline{U}_h)\, p_j p_k < 0$

Proof. For notational ease I shall now simply omit the arguments of functions but you must remember that all partial and second partial derivatives are taken at $(p, \overline{U}_h)$

a) By T.V.10

$$e_{kj} = x_{hkj}$$

as can be seen by differentiating both sides of $e_k = x_{hk}$. But e is strictly concave and so $e_{kk} < 0$. (See Corollory II.4)

b) Since e is a continuous twice differentiable function

$$e_{kj} = e_{jk}$$

or
$$x_{hkj} = x_{hjk}$$

This is the Slutsky symmetry result.

c) Since e is homogeneous of degree one in p, $e_k$ is homogeneous of degree zero in p. (Why?) By Euler's Theorem on homogeneous functions one has

$$\sum e_{kj}\, p_j = \sum x_{hkj}\, p_j = 0$$

d) This again follows from strict concavity of e. I here spell it out.

Let $h = (0, h_2, \ldots, h_n)$. We know, do we not, that for a strictly concave function $f(x)$, $f(x + h) - f(x) < \nabla f(x)\, h$, where $\nabla f(x)$ is the gradient vector of the first order partial differential coefficients of $f(x)$. So in our case

$$e(p + h, \bar{U}_h) - e(p, \bar{U}_h) < \sum_k e_k(p, \bar{U}_h)\, h_k \qquad (13)$$

But by a Taylor expansion of $e(.)$ about $(p, \bar{U}_h)$ of $p$ up to the second order, one has

$$e(p + h, \bar{U}_h) - e(p, \bar{U}_h) = \sum_k e_k(p, \bar{U}_h)\, h_k + \frac{1}{2!} \sum_j \sum_k e_{kj}(p, \bar{U}_h) h_k h_j + O(\varepsilon)$$

Using (13) then gives

$$\sum_j \sum_k e_{kj}\, (p, \bar{U}_h)\, h_k\, h_j < 0 \qquad (14)$$

Since $h_1 = 0$, and the other values of $h_i$ can be anything e.g. $h_i = p_i$ one gets (d) when $x_{hkj}$ is substituted for $e_{kj}$ in (14).

Now notice that, in all the above results, the households' utility has remained fixed at $\bar{U}_h$. Hence we have been investigating the effect of price changes on demand when somehow utility is kept constant. In fact we have been studying the compensated demand curve. We have shown that under our present postulates it must slope downwards. In fact here is a restatement of T.V.11 in words

a) A compensated change in its own price must reduce the demand for a good.

b) A compensated change in the price of good j must have the same effect on the demand for good k as does a compensated change in the price of good k on the demand for good j

c) If all prices change in the same proportion demand remains unchanged, i.e. demand is homogeneous of degree zero in p

d) Let $\Sigma\, p_k\, x_{hk}$ be the expenditure on a group of goods where the number of goods is less than n. Then a compensated equi-proportionate change in the prices of the goods in that group will reduce $\Sigma\, p_k\, x_{hk}$. This will be further discussed in lectures and in section 11.

These results are nice and empirically useful. Because of their importance here is an alternative simple proof which is only sketched, the rest is left to you.

Consider $p' \neq p$ and let $x_h = x_h(p, \bar{U}_h)$, $x'_h = x_h(p', \bar{U}_h)$. So that the choices $x_h$ and $x'_h$ in the two price situation give the same utility. By T.V.8 one has when preference strictly convex

$$p\, x_h < p\, x'_h$$

and

$$p'\, x'_h < p'\, x_h$$

From which

$$(p' - p)(x'_h - x_h) < 0 \tag{15}$$

Setting $p'_i = p_i$ all $i \neq k$ you at once get the discrete version of (a). Taking a Taylor expression up to the second order one gets (d). Symmetry cannot be established in this way.

So far we have found the behaviour of compensated demand - what about the behaviour of demand?

Plainly when, at $(p, w_h(p))$, the household has maximised its utility one has

$$e(p, U_h) = w_h(p) \tag{16}$$

Now as long as (16) continues to hold when some price changes, $U_h$ constant, the household can stay on its compensated demand curve since the $w_h$ changes by just enough to enable h to continue to attain the same utility at minimum cost. So we can write

$$x_{hk}(p, U_h) = x_{hk}(p, e(p, U_h) - w_h(p)) \tag{17}$$

Let

$$e(p, U_h) - w_h(p) = y_h \tag{18}$$

In equilibrium $y_h = 0$, but when say the $j^{th}$ price is changed we can calculate $\frac{dy_h}{dp_j}$ as the compensation (positive or negative) the household needs in order that he should have just enough to maintain his old utility at minimum cost.

$\frac{\partial y_h}{\partial p_j}$ jointly: $y_h = y_h(p, U_h)$

Differentiating both sides of (17) partially with respect to $p_j$ one now has

$$x_{hkj}(p, U_h) = \frac{\partial x_{hk}(p, y_h)}{\partial p_j} + \frac{\partial x_{hk}}{\partial y_h} \frac{dy_h}{dp_j} . \qquad (19)$$

From (18) :

$$\frac{dy_h}{dp_j} = e_j - \frac{\partial w_h(p)}{\partial p_j}$$

But by Th.V.11.(a): $e_j = x_{hj}$

If we suppose $d_{hf} = 0$ all f, then

$$\frac{\partial w_h(p)}{\partial p_j} = \frac{\partial}{\partial p_j}(p\, \bar{x}_h) = \bar{x}_{hj}$$

So substituting in (19) and rearranging we obtain the famous Slutsky equation:

$\frac{\partial x_{hk}(p, U_h)}{\partial p_j}$

See p. 13 : [illegible]

$$\frac{\partial x_{hk}(p, y_h)}{\partial p_j} = x_{hkj}(p, U_h) + (\bar{x}_{hj} - x_{hj}) \frac{\partial x_{hk}}{\partial y_h} \qquad (20)$$

The left hand side of (20) is the change in demand for good k when the price of good j varies. The first term on the r.h.s. of (20) is the compensated change in the demand for k when $p_j$ varies. I.e. that change in demand which would occur if h stayed at his old utility level. It is called the substitution effect. The last term on the r.h.s. of (20) is called the income effect.

Notice that

$$(\bar{x}_{hj} - x_{hj}) = \frac{\partial}{\partial p_j}(w_h(p) - e(p, U_h))$$

and so measures the rise (fall) of wealth above (below) the minimum cost of staying at the old utility level. E.g. if that expression is positive the individual after the change of $p_j$ will have more wealth than he needs to stay at $U_h$ and so he can reach higher $U_h$. Then $\frac{\partial x_{hk}}{\partial y_h}$ measures the rate at which h changes his demand for k when p is constant but the gap between wealth and minimum expenditure to stay at $U_h$, changes. Let us call it the marginal propensity to consume good k and write

$$x_{hky} \equiv \frac{\partial x_{hk}}{\partial y_h} .$$

Here is the familiar picture

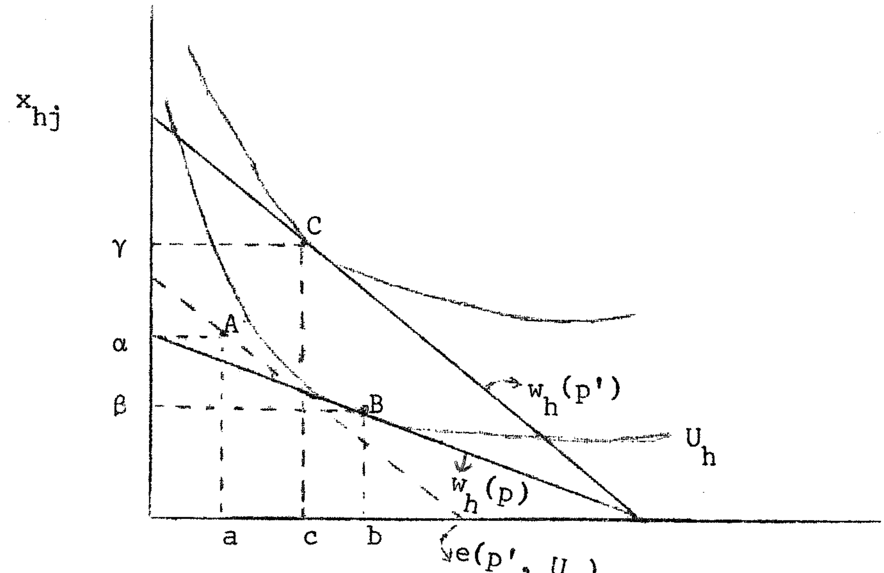



Fig. V.1

We start at B where $w_h(p) = e(p, U_h)$. $p'_i = p_i$ all $i \neq j$, $p'_j < p_j$. So $w_h(p')$ is as drawn, ~~as~~ is $e(p', U_h)$. At $w_h(p') > e(p', U_h)$. One reads off

$$- ab = x_{hkj}(p, U_h), \; ac = x_{hky}(\bar{x}_{hj} - x_{hj})$$

$$\frac{\partial x_{hk}}{\partial p_j} = ac - ab = -cb$$

$$\alpha\beta = x_{hjj}(p, U_h), \; \alpha\gamma = (\bar{x}_{hj} - x_{hj})$$

$$\frac{\partial x_{hj}}{\partial p_j} = \alpha\beta + \alpha\gamma.$$

All this will be discussed in lectures. Here to finish the section are some defintions

Definition D.V.6

a) Good j and k are substitutes (complements) according as $x_{hkj}(p, U_h) > 0$ (resp. $x_{hkj}(p, U_h) < 0$). Recall symmetry.

b) Goods j is a gross-substitute (gross-complement) for good k according as $\dfrac{\partial x_{hk}(p, y_h)}{\partial p_j} > 0$ (resp. $\dfrac{\partial x_{hk}(p, y_h)}{\partial p_j} < 0$). (i.e. income effects are included and symmetry is lost so that j may be a gross substitute for k and k a gross complement of j).

c) Good k is normal if $x_{hky} > 0$ and inferior if $x_{hky} < 0$.

Example. Prove that for the utility function

$$\log U_h = \alpha \log x_{h1} + (1 - \alpha) \log x_{h2}, \qquad 0 < \alpha < 1$$

both goods are normal and weak? gross substitutes for each other.

We now have deduced a good many implications for the demand of a household from the postulate that it knows what it wants and how to get it. Certainly we could, for instance, by cross-section studies over households which we take as similar or by keeping records of a given households' expenditure (as was done in Wisconsin), hope to estimate income and substitution effects separately. Read

A. Brown & A. Deaton: "Models of Consumer Behaviour: A Survey." E.J. Dec. 1972 for a survey.

But we can use the theory also for purposes of welfare economics and we can specialise the theory to make it both empirically more tractable and perhaps also more realistic. I shall now take a brief and very incomplete look at both of these matters.

Let us investigate this further. To do so I shall introduce an important new concept.

Definition D.V.7. The indirect utility function $V_h(p, y_h)$ is defined by

$$V_h(p, y_h) = U_h(x_h(p, y_h))$$

or equivalently:

$$V_h(p, y_h) = \max \{ U_h(x_h) \mid x_h \in B_h^*(p) \}$$

Think about this! $V_h(.)$ gives the maximum utility attainable at $(p, y_h)$. [ I have implicitly taken $y_h = 0$, else I should have to write $B_h^*(p, y_h)$ ].

Now notice:

$$V_{hk} \equiv \frac{\partial V_h}{\partial p_k} = \Sigma \frac{\partial U_{hj}}{\partial x_{hj}} \frac{dx_{hj}}{dp_k} = \text{by } (3^*) = \lambda \Sigma p_j \frac{\partial x_{hj}}{\partial p_k} \qquad (22)$$

Why? Because if $x_{hj} = 0$ and $\frac{\partial U_h}{\partial x_{hj}} - \lambda p_j < 0$ the $x_{hj}$ will remain zero for small changes in $p_k$ whence we simply leave it out of the summation in (22).

But

$$p(x_h - \bar{x}_h) = 0 \equiv y_h$$

So differentiating with respect to $p_k$:

$$\Sigma \, p_j \frac{\partial x_{hj}}{\partial p_k} = (\bar{x}_{hk} - x_{hk})$$

and so

$$V_{hk} \equiv \frac{\partial V_h}{\partial p_k} = (\bar{x}_{hk} - x_{hk})\,\lambda \qquad (23)$$

It is important to understand and know this result. Also one has

$$V_{hy} \equiv \frac{\partial V_h}{\partial y_h} = \Sigma \frac{\partial U_{hj}}{\partial x_{hj}} \frac{\partial x_{hj}}{\partial y_h} = \text{by } (3^*) = \lambda \cdot \Sigma \, p_j \frac{\partial x_{hj}}{\partial y_h} = \lambda \qquad (24)$$

So the Lagrangean in (3*) measures the increase in <u>maximum</u> utility from a little more wealth. It is thus <u>not</u> invariant under a monotone transformation of the utility function.

10) <u>Welfare Theory: Examples</u>.

<u>Example (1) Consumer Surplus</u>

If the household does not exchange at all and it receives no profit then its utility is $U_h(\bar{x}_h) = \bar{U}_h$ say. Associated with $\bar{U}_h$ is the minimum expenditure $e(p, \bar{U}_h)$. Then it may appear reasonable at first sight to call

$$w(p) - e(p, \bar{U}_h) \qquad (24b)$$

a measure of the gain the household in fact makes from exchange. For if (24b) > 0 plainly the household will exchange and we could tax (24b) away from the household and still allow it to reach $\bar{U}_h$. The difference between the wealth the household has and the wealth it would need, to have the "no-exchange" utility is one measure of what Marshall called <u>consumers' surplus</u>.

How do people gain from exchange? They do so from the fact that only on the <u>marginal transaction</u> is the marginal utility of what is gained equal to the marginal utility of what is given up. On the <u>intra-marginal transactions</u> the former exceeds the latter. Hence a kind of rent is earned. This will be further discussed in lectures but any one who does not understand this is not an economist!

But evidently (21) is trying to measure a "utility gain" by a value sum and there may not in general be a unique measure of consumer surplus. Let $U_h$ be the utility actually attained after exchange and $e(p, U_h)$ its minimum cost. Suppose that for one $\bar{p}$

$$e(\bar{p}, \bar{U}_h) = w(\bar{p})$$

Then

$$e(\bar{p}, U_h) - e(\bar{p}, \bar{U}_h)$$

could be different from (21) yet also a measure of consumer surplus. Why? So in general the measure will depend on p. If you have this under your belt let us return once more to consumer's surplus.

I am now interested in measuring the consumers' surplus on a single good, say k, given the prices and consumption of all other goods.

Let

$$z_{hk} = x_{hk} - \bar{x}_{hk},$$

the net demand for good k at p. I take $z_{hk} > 0$ at p, i.e. the guy is buying it at p. Now by (22)

$$V_{hk}\, dp_k = -\lambda\, z_{hk}\, dp_k$$

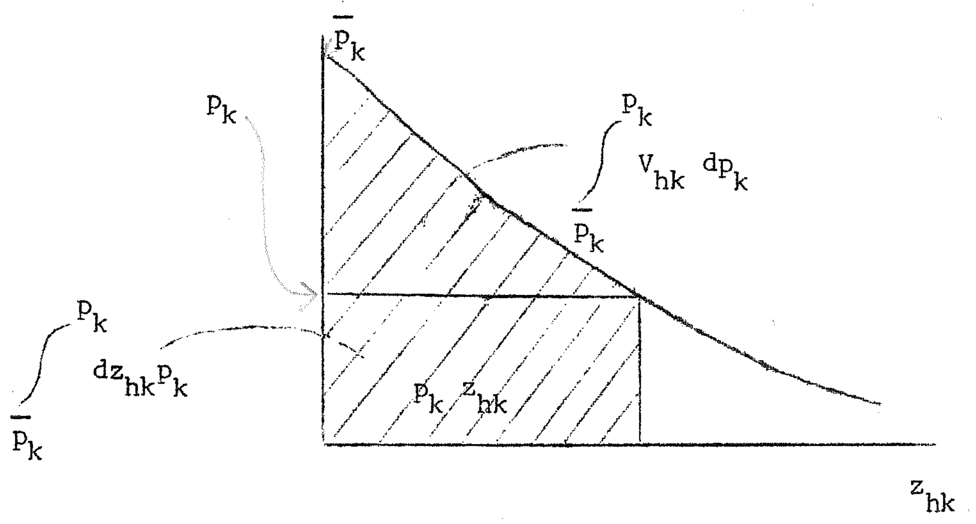
Suppose that when $p_k = \bar{p}_k$, other prices constant, $z_{hk} = 0$. Then

$$\int_{\bar{p}_k}^{p_k} V_{hk}\, dp_k = -\lambda \int_{\bar{p}_k}^{p_k} z_{hk}\, dp_k \qquad (25)$$

provided that λ is independant of $p_k$. Now the l.h.s. of (25) measures the total utility gained from facing the price $p_k$ rather than the price $\bar{p}_k$. We may chose $\lambda = 1$ since it will be constant anyway and integrate the r.h.s. of (25) by parts:

$$\int_{\bar{p}_k}^{p_k} V_{hk}\, dp_k = \int_{\bar{p}_k}^{p_k} dz_{hk}\, p_k - p_k\, z_{hk}$$

So geometrically one has




So the well known Marshallian triangle emerges if $\lambda$ can be taken to be independent of $p_k$.

But of course in general $\lambda$ is not constant and very often other prices are not constant either so that the greatest care must be taken in trying to arrive at actual numerical estimates.

<u>Example (2). Index Numbers.</u>

This is a large and fascinating topic and I can do no more than scratch the surface. A good reference is: F. Fisher and K. Shell: The Economic Theory of Price Indices, (Academic Press).

Let a superscript 1 denote "today" and a superscript 2, "tomorrow". One has $p^1$, $p^2$, $U_h^1$, $U_h^2$ etc.

We ask: At $p^2$ how much would we have to tax (positive or negative) h in order that he is indifferent between facing the budget constraint of 2 with this tax and the budget constraint he faced at 1?. That is, we could calculate

$$e(p^1, U_h^1) - e(p^2, U_h^1) \tag{26}$$

An equivalent formulation is to solve for $y_h^2/y_h^1$, where $y_h^1$ is given, from

$$V(p^1, y_h^1) = V(p^2, y_h^2) \tag{27}$$

Notice that (26) measures the gain (loss) in "consumer surplus" from a change in price for $p^2$ to $p^1$. Also the reference point, if you like "the weights", is situation (1). If we could solve (26) or (27) then giving (26) to h in "2" would leave him indifferent between the two situations. Plainly (26) therefore has a claim to be a measure of the change in "the cost of living".

But in principle we could reverse the question and treat 2 as our reference point. We would then get

$$e(p^2, U_h^2) \;-\; e(p^1, U_h^2) \qquad (28)$$

and we are asking: how much income would we have had to give to h in 1 so that he should be indifferent between the resulting budget constraint of 1 and that of 2? Unfortunately except in special cases, there is no reason for (26) and (28) to coincide.

Before I proceed notice (a) that the ambiguity of the "true cost of living index" i.e. whether (26) or (28) should be used, is identical to that which arises if one tries to measure consumers' surplus. (b) Both (26) and (28) assume two things: tastes are constant and the quality and number of goods is constant as between (1) and (2). This in general may be a bad assumption. The vulgar thing is to conclude that this difficulty must vitiate all attempts at a satisfactory index. Read Fisher-Shell to see what can be done although you should be able to prove some of their? these results more elegantly than they do.

Let me now re-formulate (26) as

$$e(p^2, U_h^1) \;/\; e(p^1, U_h^1) \qquad (26^*)$$

as one possible index. But we do not observe "indifference". So suppose we asked instead, which is what of course is often done in practice: how much income would we have to assure to h in 2 so that he could buy the goods he bought in 1? So we have the Laspeyres' price index

$$p^2x_h^1 / p^1x_h^1 \qquad (29)$$

But $p^1x_h^1 \;=\; e(p^1, U_h^1)$ and so

$$p^2x_h^1 \;\geq\; e(p^2, U_h^1) \qquad (30)$$

(29) follows from the fact that the minimum cost of attaining $U_h^1$ at prices $p^2$ cannot be greater than the cost of the bundle known to yield $U_h^1$ at prices $p^2$. The inequality may be strict for the obvious reason that there will now be substitution possibilities. Do <u>not</u> proceed until you understand. Draw a diagram in two dimensions.

So we have

$$\text{Laspeyeres} \geqq e(p^2, U_h^1)/e(p^1, U_h^1) .$$

Rewriting (28) as

$$e(p^2, U_h^2) / e(p^1, U_h^2) \tag{28*}$$

We consider the <u>Paashe Index</u>

$$p^2x_h^2/p^1x_h^2 \tag{31}$$

But $p^2x_h^2 = e(p^2, U_h^2)$ and by the same argument as before

$$p^1x_h^2 \geqq e(p^1, U_h^2) \tag{32}$$

whence

$$\text{Paashe} \leqq e(p^2, U_h^2)/e(p^1, U_h^2) \tag{33}$$

<u>Now if</u> (26*) = (28*) we would have

$$\text{Laspeyeres} \geqq (26^*) = (28^*) \geqq \text{Paashe} \tag{34}$$

and then, since Laspeyere and Paashe are observable one could argue that there is a good case for saying that the "true" index is some sort of mean of these two. In fact it can be shown that in the very special case of (26*) = (28*) the geometric mean of Laspeyeres and Paashe does best.

When will (26*) = (28*)? Start with a definition:

<u>Definition D.V.8</u>: A function f(x) is <u>homothetic</u> if

$$f(kx) = e(k)\, f(x). \quad k > 0.$$

Suppose then that $U_h(x_h)$ is homothetic. Then we can solve:

$$U_h^1 = U_h(x_h^1) = U_h(k\ x_h^2) = e(k)\ U_h\ (x_h^2) = e(k)\ U_h^2 \tag{35}$$

Right. Next notice that

$$e(p,\ U_h\ (k\ x_h)) = ke(p,\ U_h\ (x_h)) \tag{36}$$

Why? Because for a homothetic function

$$\frac{\partial f(k\ x)}{\partial x_i} \Big/ \frac{\partial f(k\ x)}{\partial x_j} = \frac{\partial f(x)}{\partial x_i} \Big/ \frac{\partial f(x)}{\partial x_j}$$

So when we are dealing with utility functions, the marginal rates of substitution between any two goods remain the same when all goods consumed are multiplied by k. But if prices are constant, (36) then follows. Make sure you understand!

But then using (35) or (36) we have

$$e(p^2,\ U_h^1)\ /\ e(p^1,\ U_h^1) = k\ e(p^2,\ U_h^2)\ /\ {}^{k}/e(p^1,\ U_h^2) = (28*)$$

and we have proved

<u>Theorem V.12</u> When $U_h(x_h)$ is homothetic (34) holds

(11) <u>Separability</u>.

Once again I shall deal sketchily with an interesting and much studied problem. Most of the recent ? decent work here is due to W.M. Gorman and I follow his procedure.

It seems reasonable or at least acceptable to suppose that a household engages in two - or more-stage budgeting. First it divides goods into categories such as "food" "clothes", "holidays" etc. and divides it's income between these categories. Then it proceeds to choose "within" each category.

Let $w_r$ be the wealth allocated to category r. Then one would expect

$$x_{hi} = x_{hi}(p^r,\ w_r) \quad i\ \varepsilon\ r \tag{37}$$

That is, the demand for good i in category r depends only on the price vector $p^r$ of goods in that category and on the wealth allocated to category r. Of course

$$\Sigma w_r = w$$

Let us say when demand functions can be written as in (37), that there is <u>two stage budgeting</u>.

Next let us say that the <u>utility function is separable</u> if

$$U_h(x_h) = U_h(v_1(x_h^1),\ldots,v_r(x_h^r),\ldots,v_R(x_h^R)) \tag{38}$$

One may think here of $v_r(.)$ as the 'sub-utility' or 'felicity' of bundle r. That is, we have an ordering over bundle r which has a numerical representation and then we have an ordering over the composites $v_r(.)$ which is represented by $U_h(\quad)$. One has the following

<u>Theorem (Gorman)</u>. If U is quasi-concave and continuous then it is separable iff two stage budgeting is possible.

Necessity is obvious. For if (38) then plainly the household must at its maximum utility have solved the problem

$$\max v_r(x_h^r) \quad \text{s.t} \quad p^r x_h^r \leqq w_r$$

and that must give a solution of the form (37). Sufficiency is harder and is not given here.

Let us see why separability is of great interest. To do that adopt the following notational convention

$$x^*_{hij} = \frac{\partial \hat{x}_{hi}(p, U_h)}{\partial p_j}$$

So that $x^*_{hij}$ is the substitution effect of the price change.

Now for (37)

$$x^*_{hij} = (x_{hiw})\, w^*_{rj} \quad \text{for } i \in r,\ j \in s \tag{39}$$

NOTE: $\frac{\partial x^*_{hi}}{\partial p_j} = 0$.

where $x_{hiw} = \frac{\partial x_{hi}}{\partial w_r}$ and $w^*_{rj}$ is the compensating change in the wealth allocated to group r when $p_j$ changes and utility is to remain constant. Notice by assumption $x_{hi}$ does <u>not</u> depend directly on $p_j$ when $i \in r$ and $j \in s$. But from symmetry

$$(x_{hiw})\, w^*_{rj} = (x_{hjw})\, w^*_{si} \tag{40}$$

98



or

$$w^*_{rj}/x_{hjw} = w^*_{si}/x_{hiw} \tag{41}$$

Now the l.h.s. of (41) does not depend on i and the right hand side does not depend on j so we can write

$$(41) = \lambda_{rs} \quad \text{say}$$

Whence

$$w^*_{rj} = \lambda_{rs} \; x_{hjw} \tag{42}$$

and substituting in (39)

$$x^*_{hij} = \lambda_{rs} \; x_{hjw} \, x_{hiw} \qquad i \in r, \; j \in s \tag{43}$$

which is very nice, is it not, since we can now in empirical work estimate substitution effects from income effects like $x_{hjw}$ and $x_{hiw}$. Since we need income effects anyway this saves a lot of work.

What is the meaning of $\lambda_{rs}$? To examine that multiply both sides of (42) by $p_j$, and add over $j \in s$. Recall that

$$p^s \, x^s_h = \Sigma \, p^s_j \, x^s_{hj} = w_s \quad \Rightarrow \quad \Sigma p^s_j \frac{\partial x^s_{hj}}{\partial w_s} = 1$$

so that we get

$$\sum_{j \in s} p_j \, w^*_{rj} = \lambda_{rs} \sum_{j \in s} p_j \, x_{hjw} \;\; (= \lambda_{rs}) \tag{44}$$

But the l.h.s. of (44) is the increase in the expenditure on goods in group r when the prices in group s change in the same proportion. It is like the expression in Theorem V.11.(d). (p85) So if we aggregate all goods in group r at constant prices $\sum_{i \in r} p^r_i \, x_{hi}$ and call this a new good $X_r$ and do the same for goods in group s and call the aggregate $X_s$ and say that the price of $X_s$ changes when all prices in group s change in the same proportion, the $\lambda_{rs}$ is the substitution term between $X_r$ and $X_s$. So, of course,

$$\lambda_{rs} = \lambda_{sr}$$

So we need only estimate in-between-group substitution terms and income terms in order to estimate demand.

Lastly, notice that

$$w^*_{ri} = \sum_{j\varepsilon r} p^r_j \, x^*_{hji} + x^r_{hi} \quad \text{for} \quad i \; \varepsilon \; r$$

So $$\sum_{i\varepsilon r} p^r_i \, w^*_{ri} - \Sigma \, p^r_i \, x^r_{hi} = \sum_{i\varepsilon r} p^r_i \, w^*_{ri} - w_r = \sum_{i\varepsilon r} \sum_{j\varepsilon r} p^r_j \, x^*_{hji} = \lambda_{rr} \qquad (45)$$

since the r.h.s. of (45) measures the substitution effect on the aggregate $X_r$ of an equi-proportionate change in the prices of the goods comprising the aggregate. So then as in T.V11, (c) we get

$$\sum_s \lambda_{rs} \sum_j p_j \, w^*_{rj} - w_r = 0$$

since $w_r$ is homogeneous of degree one in p.

This is only a very small part of the available theory. Notice that as we specialise our assumptions we get sharper predictions.

(12) Revealed Preference

This section is in the nature of an appendix. Serious study can be postponed until the third year.

Dear Professor Joan Robinson and many others have a dislike of a theory which starts with assuming that households have preferences. How is it to be tested? The answer is by observing the households in certain situations.

Suppose we use the notation of section (10) and find, by looking, that

$$p^2 \, x^2_h \geqq p^2 \, x^1_h \qquad (46)$$

Then we say: h could at $p^2$ have bought $x^1_h$, i.e. $x^1_h \; \varepsilon \; B^*(p^2)$, but did not. So if h does have decent strictly convex preferences it must be that

$$x^2_h \, P \, x^1_h$$

But then we must not observe

$$p^1 \, x^1_h \geqq p^1 \, x^2_h \qquad (47)$$

for by the same argument, (47) implies

$$x^1_h \, P \, x^2_h$$

and that would be horrid. So if (46) and not (47) we say the household satisfies the Weak Axiom of Revealed Preference. (W.R.P.)

In principle it would appear that if we choose enough suitable pairs ($p^1$, $p^2$) and incomes, & ~~that~~ if W.R.P. is satisfied for all of them, we could experimentally determine what the preferences of h are. But that is too hasty. For even though W.A.R. holds the household may choose intransitively. So we need it never to reveal intransitive choices, however long the chain. Formally, let

$$\left.\begin{array}{ll} p^i x_h^i \geqq p^i x_h^{i-1} & i = T \dots 1 \\ p^{i-1} x_h^{i-1} < p^{i-1} x_h^i & \end{array}\right\} \quad (48)$$

Then by W.A.R.

$$x_h^T \; P \; x_h^{T-1}, \; x_h^{T-1} \; P \; x_h^{T-2} \; \dots \; x_h^1 \; P \; x_h^o$$

For transitivity one wants

$$x_h^T \; P \; x_h^o \qquad (49)$$

That means that given a choice like (48) no experimental known choice between $x_h^r$ and $x_h^o$ should contradict (49). If this is the case the household is said to obey the Strong Axiom of Revealed Preference.

Lastly, to get a proper utility function the household must reveal certain continuity properties. But that is getting beyond where we reasonably want to get to.

The point is really this: It is extremely unlikely that any actual experiment can fully reveal preferences. What is true is this: we can use evidence to say that so far we have not observed anything which falsifies the assumption that the evidence is generated by the choices of rational households.