ECONOMICS TRIPOS. Part II

FRIDAY 30 MAY 1975. 1.30 TO 4.45

PAPER 5. MATHEMATICAL ECONOMICS

Candidates are advised to spend the first fifteen minutes of the examination reading through the questions. Candidates should attempt **at least three** *questions. They can gain extra credit for answers to questions with an asterisk, and by answering more than three complete questions. Relatively little credit will be given for answers to parts of questions. If there is not time to complete a question candidates should explain how they would have proceeded.*

1 There are n competitive boats operating in a fishing ground, where n is variable and large enough to be taken as continuous. Each has a total cost function

$$C(x) = A + wl(S, x)$$

$$\frac{\partial l}{\partial S} < 0, \quad \frac{\partial l}{\partial x} > 0, \quad \frac{\partial^2 l}{\partial x^2} > 0,$$

p69

where A is a fixed cost, w is the constant wage rate per hour, $l(S, x)$ is the input of labour in man-hours, x is weight of fish caught and S is the stock of fish in the fishing ground. The stock of fish varies according to the relation: $\dot{S} = f(S) - nx$, where f is a strictly concave differentiable function which achieves a positive maximum.

The price of fish (per unit weight), p, is set in world markets and is constant. Derive the conditions for long-run equilibrium (when the stock of fish is constant) without government intervention, and show that efficient production in the long run can be achieved by an *ad valorem* tax on output of

$$nw \frac{\partial l}{\partial S} \Big/ p \frac{df}{dS}.$$

[*Note:* You may assume that each boat takes no account of the effect of the size of its catch on either $\dot{S}$ or S.]

2 State precise conditions for the existence of a unique 'ideal' cost of living index, independent of income, for an individual. Show that when it exists it lies between the Laspeyre and Paashe indices. Discuss how the theory might be reformulated to allow the index to take account of new commodities.

[TURN OVER

3 There are three levels of income in an economy and an individual in level i at time $t-1$ has a probability p_{ij} of being in level j in period t, where the p_{ij} are given by the following matrix

$$\begin{array}{ll|ccc} & & \multicolumn{3}{c}{\text{Income in } t} \\ & & £5{,}000 & £2{,}000 & £1{,}000 \\ \hline \text{Income in } t-1 & £5{,}000 & \frac{2}{3} & \frac{1}{3} & 0 \\ & £2{,}000 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ & £1{,}000 & 0 & \frac{1}{3} & \frac{2}{3} \end{array}$$

Find the steady-state distribution of income (assuming a large population). Suppose at $t = 0$ everyone in the population had income of £1,000 p.a. How long would it take for the proportion of the population at each level to be within 10 per cent of its steady-state level?

Comment on the assumptions of the model.

4 The choice of new machines available at date t is described by the production function

$$y(t) = e^{gt}f(k(t)e^{-gt}),$$

where y is output per man employable on the machine, and k is the investment per man employable. Once installed the machine retains its initial operating characteristics and it is impossible to increase output per man employed on the machine. For such a model economy provide a description of a competitive balanced growth equilibrium, and find an expression for the share of profits in G.N.P. Explain how different expectations will influence the equilibrium rate of profit and distribution of income.

5* (a) Define the profit function of a competitive firm. Prove that it is convex, homogeneous of degree one in prices and that its partial differential coefficient with respect to the jth price measures the profit maximising input (output) of goods.

(b) Let

$$\pi_i(p_1, p_2, p_0 e^{-g_i t}), \quad i = 1, 2, \quad g_i > 0$$

be the profit functions of two firms which make up the economy; p_0 is the price of labour. What is being assumed about technological change? Assuming perfect competition and constant returns to scale calculate the rate of change in wages, $\frac{dp_0}{dt}\frac{1}{p_0}$, which would keep some index of commodity prices unchanged. Thence comment on the view that to prevent rising goods prices wages should rise at the same rate as 'productivity is rising'.

1775

6 Consumers derive utility from the consumption of the characteristics contained in the goods they buy. There are r characteristics and m goods ($m \geqslant r$). The consumption technology is described by the matrix T, where t_{ij} is the amount of characteristic i in one unit of good j. Goods are perfectly divisible and their prices are p.

(i) Formulate and comment on the linear programme which determines the shadow prices of a fixed vector of characteristics.

(ii) If $r = 2$, $m = 3$, $U(\mathbf{z}) = \frac{4}{5}\log(z_1 - 1) + \frac{1}{5}\log(z_2 - 3)$, $\mathbf{p}' = (1, \frac{4}{5}, 1)$, $T = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$, derive and sketch the Engel curve for each good for incomes above 1.

7* In a neo-classical growth model with one good the fully employed population is growing at the rate $n > 0$ and

$$y = f(k), \quad f'(k) > 0, \quad f''(k) < 0$$

is the 'well behaved' production function where y = output per worker, k = capital per worker. The population is divided into two groups. Group one receives all its income from the ownership of capital and group two (the workers) receives income from wages and from the capital it owns. The two groups have propensities to save s_1 and s_2 with $s_1 > s_2$. Let k_i = the amount of capital owned by group i per worker so that $k_1 + k_2 = k$. Write down the differential equations governing the path of k_1 and k_2. Discuss the stationary solutions. In the co-ordinates k and k_1 plot the curves $\dot{k} = 0$ and $\dot{k}_1 = 0$ and consider the asymptotic properties of the model. Say what you think of the economics of the construction.

8 In the following model of a world of two trading countries, indexed 1 and 2,

c_i is the constant average propensity to consume,

G_i is fixed government expenditure,

$\dfrac{dK_i}{dt}$ is investment in capital stock K_i,

Y_i is income.

Investment depends on income according to

$$\frac{dK_i}{dt} = \mu_i(v_i Y_i - K_i), \quad i = 1, 2.$$

Prices and exchange rates remain constant, and the average propensity to import by each country, m, is equal and constant. Prove that a sufficient condition for long run stability in the level of each country's income is that $\mu_i v_i + c_i < 1$, $i = 1, 2$, and find values for G_1, G_2 such that the countries are in long run balance of payments equilibrium with average exports equal to average imports.

[TURN OVER

9 In a two-good neoclassical growth model the output of the investment good industry I, is used to increase total capital stock, K, and is given by

$$I = F(K_i, L_i).$$

The output of the consumption good industry is C, where $C = G(K_c, L_c)$

$$K = K_i + K_c,$$

$$L = L_i + L_c = \text{total employment.}$$

Capital is malleable and everlasting, there is no technical progress, and both industries have ~~contrast~~ constant returns to scale and diminishing returns to each factor. Both factors are fully mobile. Describe the consumption–investment frontier and show that there exists a unique momentary competitive equilibrium if the savings–income ratio, s, is given and independent of the distribution of income. Will this be true if capital cannot be shifted from one sector to the other?

Give a qualitative account of the effect on the competitive rate of profit and the aggregate level of consumption of increasing s from 10 per cent to 20 per cent when

(i) the production functions in the two industries are identical,

(ii) they are different.

10 'It is not true that a competitive equilibrium does not exist if some production sets lack convexity and it is true that every competitive equilibrium is Pareto-efficient whether the production sets are convex or not.' Discuss.

1975

11* Consider the following discrete time trading process in an exchange economy: There are n goods (indexed i) whose price vector in period t is $\mathbf{p}(t)$. For a household h

$\mathbf{w}^h(t)$ is the vector of goods it owns,

$\mathbf{x}^h(t)$ is the vector of goods it would buy if it could trade freely at prices $\mathbf{p}(t)$,

$\mathbf{z}^h(t) = \mathbf{x}^h(t) - \mathbf{w}^h(t+1)$ is its unsatisfied excess demand at the end of period t, so that its net trade in period t is $\mathbf{w}^h(t+1) - \mathbf{w}^h(t)$.

$\mathbf{Z}(t) = \sum_h \mathbf{z}^h(t)$ is aggregate excess demand.

Trading satisfies four conditions:

(i) $\mathbf{p}(t)'\mathbf{w}^h(t+1) = \mathbf{p}(t)'\mathbf{w}^h(t)$ for all h, t.

(ii) For each i, either $\left.\begin{array}{l} z_i^h(t) \geqslant 0 \\ \text{or } z_i^h(t) \leqslant 0 \end{array}\right\}$ for all h.

(iii) $\sum_h \mathbf{w}^h(t) = \mathbf{w}$, constant for all t.

(iv) $p_i(t+1) = \max\,(p_i(t) + Z_i(t), 0)$ for all i and t.

(*a*) Comment on the economics of the trading conditions.

(*b*) Reformulate the model in continuous time assuming $\mathbf{w}^h(t)$ to be differentiable.

(*c*) Prove that in the continuous model the indirect utility function

$$V^h(\mathbf{p}(t), \mathbf{w}^h(t)) \equiv U^h(\mathbf{x}^h(t))$$

is a non-increasing function of time, and hence that the process converges to the set of Pareto-efficient equilibria.

12 Construct a model of life-cycle saving where $c^{-\alpha}(t)$, $\alpha > 0$, is the utility of consumption at t. What are the main predictions of the model?

[TURN OVER

13* There are two countries 1 and 2 in free trade with each other and no transport cost. Let y_{ij} $(i = 1, 2, j = 1, 2)$ be the output of good j in country i and x_{ij} $(i = 1, 2, j = 1, 2)$ the demand for good j in country i. Also

$$Y_i = \{(y_{i1}, y_{i2}) \mid g_i(y_{i1}, y_{i2}) \geqslant 0\}, \quad i = 1, 2$$

are the production sets of the two countries where $g_i(.)$ is strictly concave. There is perfect competition in each country so that if $\mathbf{p} = (p_1, p_2)$ the price vector

$$R_i(\mathbf{p}) = \max_{Y_i} \sum_j y_{ij} p_j, \quad i = 1, 2$$

is the income of country i. In each country the citizens all have the same utility function and receive the same fraction of income.

In an equilibrium of both countries, country 1 imports good 1 and exports good 2. The balance of payment B_1 of country 1 is given by

$$B_1 = -p_1[x_{11} - y_{11}] + p_2[x_{22} - y_{22}].$$

Prove that a sufficient condition for $\dfrac{\partial B_1}{\partial p_1} > 0$ at $B_1 = 0$ is that the sum of the marginal propensities to import exceed unity. Is the condition also necessary?

[*Hints:* (i) use the homogeneity of the demand and supply functions and the convexity of $R_i(p)$. (ii) You need the Slutsky equations.]

14 'Mathematics applied to economics has not yet produced anything of use or relevance.' Discuss this using examples.