Given name:____________________ **Family name:**____________________

Student number:____________________ **Signature:**____________________

UNIVERSITY OF TORONTO
Faculty of Arts and Science

ECO 326H1S Section L0101 (Advanced Economic Theory—Micro)
Instructor: Martin J. Osborne

MIDTERM EXAMINATION
October 2008

Duration: 1 hour 50 minutes

No aids allowed

This examination paper consists of **5** pages and **5** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Answer all questions.

TO OBTAIN CREDIT, YOU MUST GIVE ARGUMENTS TO SUPPORT YOUR ANSWERS.

1. Consider the following strategic game with vNM preferences. Throughout the question, consider mixed strategies as well as pure strategies.

	X	Y	Z
T	$1, 2$	$4, 0$	$2, 1$
M	$4, 0$	$1, 2$	$2, 1$
B	$2, 4$	$2, 4$	$1, 1$

(a) [4] Find the actions of each player, if any, that are strictly dominated.

Solution: The action B of player 1 is strictly dominated by her mixed strategy that assigns probability $\frac{1}{2}$ to T and probability $\frac{1}{2}$ to M. No other action of either player is strictly dominated.

(b) [4] Find the actions of each player, if any, that are weakly dominated but not strictly dominated.

Solution: The action Z of player 2 is weakly dominated by her mixed strategy that assigns probability $\frac{1}{2}$ to X and probability $\frac{1}{2}$ to Y.

(c) [12] Find all the Nash equilibria of the game.

Solution: Because player 1's action B is strictly dominated, the Nash equilibria of the game are the same as the Nash equilibria of the game

	X	Y	Z
T	$1, 2$	$4, 0$	$2, 1$
M	$4, 0$	$1, 2$	$2, 1$

This game has no pure strategy Nash equilibrium and no equilibrium in which player 1 uses a pure strategy.

Suppose that player 1 chooses T with probability p. Then player 2's expected payoffs are $2p$ to X, $2(1-p)$ to Y, and 1 to Z. Thus player 2's best response function is

- Y if $p < \frac{1}{2}$
- all mixed strategies if $p = \frac{1}{2}$
- X if $p > \frac{1}{2}$.

The game has no Nash equilibrium in which player 2 uses the pure strategies X or Y, so in any equilibrium $p = \frac{1}{2}$. For player 1's expected payoffs to her actions to be equal we need

$$q + 4r + 2(1 - q - r) = 4q + r + 2(1 - q - r),$$

where q is the probability player 2 assigns to X and r is the probability she assigns to Y. Thus we need $q = r$.

We conclude that the set of Nash equilibria of the game is the set of mixed strategy pairs $((\frac{1}{2}, \frac{1}{2}, 0), (q, q, 1 - 2q))$ where $0 \leq q \leq \frac{1}{2}$.

2. [20] Two people compete for a prize. Each person i chooses her effort level e_i, a nonnegative number. Person 1 is more skilled than person 2. Precisely, her probability of winning when the pair of efforts is (e_1, e_2) is

$$\frac{\theta e_1}{\theta e_1 + e_2}$$

and person 2's probability of winning is

$$\frac{e_2}{\theta e_1 + e_2},$$

where $\theta > 1$. Each person values the prize at V. Person 1's payoff is

$$\frac{\theta e_1}{\theta e_1 + e_2} V - e_1$$

and person 2's payoff is

$$\frac{e_2}{\theta e_1 + e_2} V - e_2.$$

Does the strategic game that models this situation have a Nash equilibrium in which $e_1 = e_2$? If so, find all such equilibria; if not, argue why no such equilibrium exists.

Solution: Find person 1's best response function: the first derivative of person 1's payoff function is

$$\frac{\theta e_2}{(\theta e_1 + e_2)^2} V - 1$$

and the second derivative is

$$-\frac{2\theta^2 e_2}{(\theta e_1 + e_2)^3} V,$$

which is nonpositive. Thus the payoff function is concave, and hence person 1's best response to e_2 satisfies

$$\frac{\theta e_2}{(\theta e_1 + e_2)^2} V = 1.$$

Similarly, player 2's best response to e_1 satisfies

$$\frac{\theta e_1}{(\theta e_1 + e_2)^2} V = 1.$$

Thus in an equilibrium in which $e_1 = e_2 = e$, we need

$$\frac{\theta}{(1+\theta)^2 e} V = 1,$$

or

$$e = \frac{\theta}{(1+\theta)^2} V.$$

3. [20] Consider an example of Cournot's model of oligopoly in which there are two firms. As for the main example in the book, the inverse demand function is linear where it is positive: $P(Q) = \alpha - Q$ for $Q \leq \alpha$, and $P(Q) = 0$ for $Q > \alpha$. The cost function of each firm is linear, but unlike the example in the book, the unit costs of the firms differ: firm 1's cost function is $C_1(q_1) = c_1 q_1$ and firm 2's cost function is $C_2(q_2) = c_2 q_2$, where $c_1 < \alpha$ and $c_2 < \alpha$.

   Find the Nash equilibrium (equilibria?) of Cournot's game for this example *under the assumption that* $c_2 < 2c_1 - \alpha$. (You do **not** need to consider any other case.)

   Solution: By the same logic as for the example in the book, the best response function of firm 1 is

$$b_1(q_2) = \begin{cases} \frac{1}{2}(\alpha - c_1 - q_2) & \text{if } q_2 \leq \alpha - c_1 \\ 0 & \text{if } q_2 > \alpha - c_1 \end{cases}$$

   and the best response function of firm 2 is

$$b_2(q_1) = \begin{cases} \frac{1}{2}(\alpha - c_2 - q_1) & \text{if } q_1 \leq \alpha - c_2 \\ 0 & \text{if } q_1 > \alpha - c_2. \end{cases}$$

   These functions look like the ones in Figure 59.1 in the book, except that the y-intercept, $\frac{1}{2}(\alpha - c_2)$, of firm 2's best response function is above the point $\alpha - c_1$ above which $b_1(q_2)$ is zero. Thus the game has a unique Nash equilibrium, $(q_1^*, q_2^*) = (0, \frac{1}{2}(\alpha - c_2))$.

4. In an *average-price* sealed-bid auction, the player whose bid is highest wins and pays the *average* of the highest two bids. As in the models in the book, if there is a tie for the highest bid, the winner is the one with the lowest index among those submitting the highest bid. The remaining assumptions are also the same as for the models in the book; in particular, the valuations are ordered $v_1 > v_2 > \cdots > v_n$.

   (a) [5] Does such an auction have a Nash equilibrium in which the price is v_2?

   Solution: Yes: for example, $(v_2, v_2, v_3, v_4, \ldots, v_n)$. The price is v_2 and player 1 wins and gets a payoff of $v_1 - v_2$. If she raises her bid, she is still wins and reduces her payoff (the price increases); if she lowers her bid, she loses and gets a payoff of 0. If any other player raises her bid above v_2 she wins and gets a negative payoff; any other change in a bid of such a player has no effect on the outcome.

   (b) [5] Does such an auction have a Nash equilibrium in which the price is v_1?

   Solution: Yes: for example, $(v_1, v_1, v_3, v_4, \ldots, v_n)$. The argument is very similar to the one in the previous part.

   (c) [5] Does such an auction have a Nash equilibrium in which some player bids more than v_1?

   Solution: No: If two or more players bid more than v_1, the price is greater than v_1, and the winner is better off lowering her bid.
   If only one player bids more than v_1, she is better off reducing her bid, which reduces the price.

   (d) [5] Is a bid of more than v_i by player i weakly dominated in such an auction?

   Solution: Yes: any such bid is weakly dominated by a bid of v_i. Suppose i reduces her bid from $b_i > v_i$ to v_i. There are three possibilities:

   - In both cases she loses, in which case the change makes no difference.
   - In both cases she wins, in which case the price is lower when she bids v_i, and she is better off.
   - In the first case she wins and in the second case she loses. Given that she loses when she bids v_i, some other player bids at least v_i, so that her payoff when she bids b_i is negative. Thus she is better off bidding v_i.

5. Two players play the following game. Each chooses a number from the set $\{0, 1, \ldots, K\}$. These choices are independent. If the sum of the two numbers is odd, then player 2 pays player 1 that sum in dollars. If the sum is even, then player 1 pays player 2 the sum in dollars. Thus, for example, if player 1 chooses the number 3 and player 2 chooses the number 2, then player 2 pays 5 dollars to player 1. Assume that each player's von Neumann–Morgenstern payoff is equal to her own change in wealth and model the situation as a strategic game.

   (a) [3] Suppose that $K = 1$. Does the game have a pure strategy Nash equilibrium?

Solution: No: at $(0, 0)$ player 1 prefers 1, at $(1, 0)$ player 2 prefers 1, at $(1, 1)$ player 1 prefers 0, and at $(0, 1)$ player 2 prefers 0. The game is shown in Figure 1.

	0	1
0	$0, 0$	$1, -1$
1	$1, -1$	$-2, 2$

Figure 1. The game in Question 5(a).

(b) [7] Find all mixed strategy Nash equilibria when $K = 1$.

Solution: Denote by p the probability that player 1 assigns to 0 and by q the probability that player 2 assigns to 0. For player 1 to be indifferent between 0 and 1 we need

$$1 - q = q - 2(1 - q)$$

or $q = \frac{3}{4}$. For player 2 to be indifferent we need

$$-(1 - p) = -p + 2(1 - p)$$

or $p = \frac{3}{4}$. Thus the game has a unique mixed strategy Nash equilibrium, in which $p = \frac{3}{4}$ and $q = \frac{3}{4}$.

(c) [10] Now suppose that $K = 2$. Does this game have a mixed strategy Nash equilibrium in which both players mix only between actions 0 and 1?

Solution: Denote by p the probability that player 1 assigns to 0 and by q the probability that player 2 assigns to 1. For the players to be indifferent between 0 and 1 we need $p = \frac{3}{4}$ and $q = \frac{3}{4}$, as in the previous part. But then player 2's payoff to 0 and to 1 is $-\frac{1}{4}$, but her payoff to 2 is $\frac{3}{4}$, so the strategy pair is not a Nash equilibrium. Thus there is no mixed strategy Nash equilibrium in which each player mixes only between 0 and 1.

	0	1	2
0	$0, 0$	$1, -1$	$-2, 2$
1	$1, -1$	$-2, 2$	$3, -3$
2	$-2, 2$	$3, -3$	$-4, 4$

Figure 2. The game in Question 5(c).

End of examination
Total pages: 5
Total marks: 100