Given name:____________________ **Family name:**____________________

Student number:____________________ **Signature:**__________________________

UNIVERSITY OF TORONTO
Faculty of Arts and Science

PLEASE HAND IN

APRIL/MAY EXAMINATIONS 2006

ECO326H1F Section L0101 (Advanced Economic Theory—Micro)
Instructor: Li, Hao and Martin J. Osborne

Duration: 3 hours

No aids allowed

This examination paper consists of **8** pages and **6** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Please check the box corresponding to the section in which you are registered:

- ☐ **Section L0101 (Osborne): answer questions 1–5.**
- ☐ **Section L0201 (Li): answer questions 1–4 and 6.**

TO OBTAIN CREDIT, YOU MUST GIVE ARGUMENTS TO SUPPORT YOUR ANSWERS. You may use the last 3 pages of the exam for rough work.

For graders' use:

	Score
1 (20)	
2 (20)	
3 (20)	
Subtotal	

	Score
4 (20)	
5 (20)	
6 (20)	
Subtotal	

Total (100)	

1. Two people have to select one of three alternatives, A, B, and C. Their preferences between the alternatives may differ; neither player is indifferent between any two alternatives. The following method is used to select an alternative: each person submits a *ranking* of the alternatives and the alternative for which the *sum* of the submitted ranks is smallest is selected. Each person may submit any ranking she wishes as long as it is strict—does not contain any ties. (For example, it is possible to rank A first (1), B second (2), and C third (3).) If two or more alternatives are tied for the smallest total ranking, the tied alternative whose name is closest to the start of the alphabet is selected. (For example, if player 1 submits the ranking ABC and player 2 submits the ranking BAC then the total rankings of A and B are equal and A is selected.)

   (a) [5] In a model of this situation as a strategic game, what is the set of actions of each player?

   Solution: Each player's set of actions consists of the six possible rankings of the three alternatives, ABC, ACB, BAC, BCA, CAB, and CBA.

   (b) [5] Player 1 prefers A to B to C. Is any action of player 1 weakly dominated?

   Circle one: Yes No

   Reason (required for credit!):

   Solution: Yes: if fact, all rankings except ABC and ACB are weakly dominated by ABC. For example, player 1's switching from CAB to ABC either does not affect the outcome or causes it to change from C to either A or B, depending on player 2's action. (In addition, CAB is weakly dominated by ACB, CBA is weakly dominated by BAC and ACB, and BCA is weakly dominated by BAC.)
   The outcome as a function of the actions of the players is given as follows:

	ABC	ACB	BAC	BCA	CAB	CBA
ABC	A	A	A	B	A	A
ACB	A	A	A	A	A	C
BAC	A	A	B	B	A	B
BCA	B	A	B	B	C	B
CAB	A	A	A	C	C	C
CBA	A	C	B	B	C	C

   (c) [5] Are there any preferences for the players such that the action pair in which each player submits a ranking equal to her true preferences is not a Nash equilibrium?

   Circle one: Yes No

   Reason (required for credit!):

   Solution: Yes. Suppose player 1 prefers A to B to C and submits the ranking ABC. Suppose that player 2 prefers C to B to A. Then if she submits the ranking CBA the outcome is A, whereas if she submits the ranking BCA the outcome is B, which is better for player 2.

(d) [5] Find a Nash equilibrium of the game in the case that player 1's true ranking is ABC and player 2's true ranking is BAC.

Solution: The Nash equilibria are (ACB, ABC), (ACB, ACB), (CAB, ACB), (ACB, BAC), (CAB, BAC), (ACB, BCA), and (ACB, CAB). The outcome in each case is that A is chosen. (Note: the question asks you only to find one Nash equilibrium.)

2. Each of three people chooses a positive integer (1, 2, 3, ...). If all three people choose the *same* integer, each person's payoff is $\frac{1}{3}$. If all three choose different integers, the payoff of the player who chooses the *smallest* integer is 1 and the payoffs of the others are 0. If two people choose the same integer and the third person chooses a different integer, the third person's payoff is 1 and the others' payoffs are 0.

(One interpretation is that an integer is a way of dressing. Everyone wants to dress differently from everyone else, and, in situations where people dress differently, everyone wants to be as "cool" as possible (cool = small integer).)

(a) [4] Find all the pure strategy Nash equilibria of the strategic game that models this situation.

Solution: An action profile is a Nash equilibrium if and only if either (i) exactly one player chooses 1 or (ii) exactly two players choose 1.

(b) [16] Does the game have a ***symmetric*** mixed strategy Nash equilibrium in which each player assigns positive probability only to 1 and 2? If so, find such an equilibrium. If not, argue why no such equilibrium exists.

Circle one: Yes No

Reason (required for credit!):

Solution: No, the game does not have such an equilibrium. Suppose that two of the players assign probability p to 1 and probability $1 - p$ to 2. Then if the remaining player chooses 2 her payoff is 1 with probability p^2, $\frac{1}{3}$ with probability $(1 - p)^2$, and 0 with the remaining probability. If she chooses 3 then her payoff is 1 with probability $p^2 + (1 - p)^2$ and 0 with the remaining probability. Thus her expected payoff to 3 exceeds her expected payoff to 2, so that any best response to the other players' strategies assigns probability 0 to 2.

(There *is* a value of p (namely $\frac{1}{2}$) such that the payoffs of a player to the actions 1 and 2 are equal when each of the other players chooses 1 with probability p and 2 with probability $1-p$, but for this value of p (and in fact for any value of p) the player's expected payoff to the action 3 exceeds her expected payoff to the action 2.)

3. A union is the sole supplier of labor to two firms that engage in quantity competition. For each firm, labor is the only input in production; assume that the quantity produced by firm i, $i = 1, 2$, is simply its employment level l_i. There are no other production

costs. The inverse demand function (the market-clearing price), $P(Q)$, is given by $\alpha - Q$ if $Q \leq \alpha$, and 0 if $Q > \alpha$, where α is a positive parameter and $Q = l_1 + l_2$ is the total output of the two firms. The union demands a single wage rate, w, that applies to both firms. Each firm i, $i = 1, 2$, observes the demand and then chooses its employment level l_i, which is a nonnegative number; the firms choose their employment levels simultaneously. The payoff to the union is the total wage bill, $(l_1 + l_2)w$. The payoff to firm i is its profit, $(P(l_1 + l_2) - w)l_i$.

(a) [4] Specify the strategic situation as an extensive game with perfect information and simultaneous moves.

Solution: The game is specified as follows:

Players $\{\text{Union}, \text{Firm 1}, \text{Firm 2}\}$

Terminal histories The set of sequences of the form $(w, (l_1, l_2))$, where w, l_1, and l_2 are nonnegative numbers.

Player function $P(\varnothing) = \text{Union}$ and $P(w) = \{\text{Firm 1}, \text{Firm 2}\}$ for every nonnegative number w.

Actions The set of actions available to the union at the start of the game is the set of nonnegative numbers and the set of actions available to each firm after any history w is also the set of nonnegative numbers.

Preferences The preferences of the union are represented by the payoff function $(l_1 + l_2)w$ and the preferences of each firm i are represented by its profit $(P(l_1 + l_2) - w)l_i$.

(b) [8] Use backward induction to find the unique subgame perfect equilibrium of the game.

Solution: First consider the subgame following the history in which the union chooses w. If $w \geq \alpha$ then the subgame has a unique Nash equilibrium, in which $(l_1, l_2) = (0, 0)$. If $w < \alpha$ then firm 1's best response to l_2 is $\frac{1}{2}(\alpha - l_2 - w)$ and firm 2's best response to l_1 is $\frac{1}{2}(\alpha - l_1 - w)$, so that the subgame has a unique Nash equilibrium, in which $(l_1, l_2) = (\frac{1}{3}(\alpha - w), \frac{1}{3}(\alpha - w))$.
Now consider the union's action at the start of the game. Given the equilibrium in the subgame following w, the union's payoff is $\frac{2}{3}w(\alpha - w)$ if $w \leq \alpha$ and 0 if $w > \alpha$. The value of w that maximizes this payoff is $\frac{1}{2}\alpha$.
We conclude that the game has a unique subgame perfect equilibrium, in which the strategy of the union is $w = \frac{1}{2}\alpha$ and the strategy of each firm i is to choose $l_i = \frac{1}{3}(\alpha - w)$ after the history w.

(c) [8] Give a Nash equilibrium of the game in which the firms obtain greater payoffs than they do in the subgame perfect equilibrium.

Solution: In one such Nash equilibrium, the strategy of the firm is $w = 0$ and the strategy of each firm i is $l_i = 0$ if $w > 0$ and $l_i = \frac{1}{3}\alpha$ if $w = 0$. For this strategy profile, the union's payoff is 0; if it chooses any different value of w, its payoff remains 0. Further, for $w = 0$, the firms' actions constitute a Nash equilibrium of the subgame (given the calculation in part (b)).

The payoff of each firm i in this equilibrium is $\frac{1}{9}\alpha^2$. In the subgame perfect equilibrium, the payoff of each firm i is $\frac{1}{36}\alpha^2$.

4. A buyer and a seller each have private information about their own valuations of a single object that the seller may sell to the buyer. The buyer's valuation, denoted v_b, and the seller's valuation, denoted v_s, are independently drawn from a uniform distribution on the interval $[0, 1]$. The buyer names an offer price, p_b (a nonnegative number), and the seller simultaneously names an asking price, p_s (a nonnegative number). If $p_b < p_s$, there is no trade; the payoffs to both the buyer and the seller are 0. If $p_b \geq p_s$, trade occurs at the price $p = \frac{1}{2}(p_b + p_s)$; the payoff to the buyer is $v_b - p$ and the payoff to the seller is $p - v_s$.

   (a) [5] Specify the strategic situation as a Bayesian game.

   Solution: A Bayesian game that models the situation is defined as follow.

   Players The buyer and the seller.

   States The set of pairs (v_b, v_s), where $0 \leq v_b \leq 1$ and $0 \leq v_s \leq 1$.

   Actions The set of actions of each player is the set of possible prices, the set of nonnegative numbers.

   Signals The set of signals each player may receive is the set of numbers from 0 to 1. The buyer's signal function is defined by $\tau_b(v_b, v_s) = v_b$ and the seller's signal function is defined by $\tau_s(v_b, v_s) = v_s$.

   Beliefs After receiving the signal v_b, the buyer believes that the state takes the form (v_b, x) with the value of x uniformly distributed from 0 to 1. Similarly, after receiving the signal v_s, the seller believes that the state takes the form (x, v_s) with the value of x uniformly distributed from 0 to 1.

   Payoffs The buyer's payoff to $((p_b, p_s), (v_b, v_s))$ is $v_b - \frac{1}{2}(p_b + p_s)$ if $p_b \geq p_s$ and 0 otherwise; the seller's payoff is $\frac{1}{2}(p_b - p_s) - v_s$ if $p_b \geq p_s$ and 0 otherwise.

   (b) [15] Find all the values of x such that the game has a Nash equilibrium in which the buyer offers x if $v_b \geq x$ and 0 otherwise, and the seller asks x if $v_s \leq x$ and 1 otherwise.

   Solution: I argue that the strategy pair described in the question is a Nash equilibrium of the Bayesian game for all values of x with $0 \leq x \leq 1$.

   Fix such a value of $x < 1$. Let the strategy of the seller be the one given in the question. I argue that the action of each type of buyer given in the question is optimal. Consider type v_b of the buyer. Her expected payoff depends on her offer p_b as follows.

   - $p_b < x \Rightarrow$ expected payoff is 0.
   - $x \leq p_b < 1 \Rightarrow$ expected payoff is $\Pr\{v_s \leq x\}(v_b - \frac{1}{2}(p_b + x)) = x(v_b - \frac{1}{2}(p_b + x))$.
   - $p_b = 1 \Rightarrow$ expected payoff is $\Pr\{v_s \leq x\}(v_b - \frac{1}{2}(p_b + x)) + \Pr\{v_s > x\}(v_b - 1) = x(v_b - \frac{1}{2}(p_b + x)) + (1 - x)(v_b - 1)$.

If $v_b > x$, this payoff is maximized when $p_b = x$. If $v_b \leq x$, it is maximized for any $p_b \leq x$. Thus the buyer's strategy specified in the question is optimal.
If $x = 1$, the buyer's payoff is 0 if she offers a price less than 1 or her valuation is 1 and she offers the price 1. Otherwise her payoff is negative. Thus in this case also the buyer's strategy specified in the question is optimal.
A symmetric argument shows that the action of each type of seller is optimal given the strategy of the buyer.

For students in Section L0101 (Osborne) only.
DO NOT ANSWER THIS QUESTION IF YOU ARE IN SECTION L0201 (Li)!

5. [20] Consider a variant of the bargaining game of alternating offers in which each player i loses c_i during each period of delay, rather than discounting her payoff. That is, if player i receives x units of the pie in period t, her payoff is $x - (t-1)c_i$. Assume that $c_1 < c_2$. Find a subgame perfect equilibrium of this game. *Be sure to specify the players' equilibrium strategies completely and to show that the pair you specify is a subgame perfect equilibrium.* (You may use the fact that a strategy pair is a subgame perfect equilibrium of the game if and only if it satisfies the one-deviation property.)

Solution: The following pair of strategies is a subgame perfect equilibrium (in fact, the only subgame perfect equilibrium):

- player 1 always proposes $(1, 0)$ and accepts a proposal (y_1, y_2) if and only if $y_1 \geq 1 - c_1$
- player 2 always proposes $(1 - c_1, c_1)$ and accepts all proposals.

To show that this strategy pair is a subgame perfect equilibrium, consider each of the four distinct subgames in turn.

Subgame starting with proposal by player 1 If player 1 follows her strategy she obtains all the pie, so she cannot profitably deviate.

Subgame starting with response by player 2 Denote by (x_1, x_2) the proposal to which player 2 is responding. Her strategy calls for her to accept the proposal, yielding her the payoff x_2. If she rejects the proposal, she proposes $(1 - c_1, c_1)$, which player 1 accepts, yielding her the payoff $c_1 - c_2$, which is negative.

Subgame starting with proposal by player 2 If player 2 follows her strategy she obtains the payoff c_1. If she offers player 1 more than $1 - c_1$, player 1 accepts, and player 2 is worse off. If she offers player 1 less than $1 - c_2$, player 1 rejects her offer and proposes $(1, 0)$, which she accepts, yielding her the payoff $-c_2$.

Subgame starting with response by player 1 Denote by (y_1, y_2) the proposal to which player 1 is responding.

- If $y_1 \geq 1 - c_1$ her strategy calls for her to accept the proposal, yielding her the payoff y_1. If instead she rejects the proposal, she proposes $(1, 0)$, which player 2 accepts, yielding her the payoff $1 - c_1$. Thus a deviation is not profitable.

- If $y_1 < 1 - c_1$ her strategy calls for her to reject the proposal, in which case she proposes $(1, 0)$, which player 2 accepts, yielding player 1 the payoff $1 - c_1$. If instead she accepts the proposal she obtains y_1. Thus a deviation is not profitable.

For students in Section L0201 (Li) only.
DO NOT ANSWER THIS QUESTION IF YOU ARE IN SECTION L0101 (Osborne)!

6. A chain store operates in two markets. In each market a single challenger must decide whether to enter the market. The chain store can be of two types, "weak" or "tough." Only the chain store knows its type; the two challengers initially believe that the chain store is tough with probability p. If any challenger enters, the chain store may acquiesce to its presence (choose A) or fight it (choose F). In each market k, $k = 1, 2$, there are three possible outcomes. If challenger k does not enter, its profit is 0, and the profit for both types of the chain store is 2. If challenger k enters and the chain store chooses A, the profit is 1 for the challenger, 0 for the weak type of the chain store, and -1 for the tough type. If challenger k enters and the chain store chooses F, the profit is -1 for both the challenger and the weak type of the chain store, and 0 for the tough type. The two challengers make their entry decisions sequentially, challenger 1 in market 1 first and then challenger 2 in market 2 after observing the outcome in market 1. Each challenger k cares only about its own profit. The chain store's payoff is the sum of the profits from the two markets. Note that it is a dominant action for the tough type of the chain store to choose F in each market.

   (a) [4] Show that in any weak sequential equilibrium, challenger 2 enters market 2 if it believes that the probability of the chain store being the tough type is less than $\frac{1}{2}$, and stays out if the probability is greater than $\frac{1}{2}$.

   Solution: If challenger 2 enters, it is a dominant action for the tough type to play F and for the weak type to play A. Given this, challenger 2's payoff if it enters the market is -1 if the chain store type is tough and 1 if the chain store type is weak. Challenger's payoff is 0 if it does not enter. Thus, challenger 2 should enter if the probability that the chain store type is tough is less than $\frac{1}{2}$ and stay out if the probability is greater than $\frac{1}{2}$.

   (b) [8] Find the unique weak sequential equilibrium outcome when $p > \frac{1}{2}$.

   Solution: In market 1, challenger 1 must believe that the probability that the chain store is tough is greater than $\frac{1}{2}$, and in market 2, regardless of what happens in market 1, challenger 2 must believe that the probability that the chain store is tough is greater than $\frac{1}{2}$. From (a), in any weak sequential equilibrium, both challengers will stay out.

   (c) [4] Show that when $p < \frac{1}{2}$, in any weak sequential equilibrium where challenger 1 enters with positive probability, the weak type of the chain store randomizes between A and F in market 1.

Solution: Suppose that in a weak sequential equilibrium the weak type plays A with probability 1 in market 1. Then, by Bayes' rule, if after observing F in market 1 challenger 2 must believe that the chain store's type is tough and thus stay out, while after observing A in market 1 challenger 2 must believe that the type is weak and thus enter. Given this, if it plays A in market 1 the weak type chain store's payoff is 0 from market 1 and 0 from market 2, while if it switches to F its payoff is -1 from market 1 and 2 from market 2, which has a greater sum, a contradiction. Now, suppose that in a weak sequential equilibrium the weak type plays F with probability 1 in market 1. By Bayes' rule, after observing F in market 1 challenger 2's belief that its type is tough remains p, which by assumption is less than $\frac{1}{2}$, and by (a) challenger 2 should enter. The weak type chain store will then optimally choose A, implying that its payoff from playing F in market 1 is -1 from market 1 and 0 from market 2. The sum of payoffs would be greater if the weak type switches to A in market 1 and plays A if challenger 2 enters in market 2, a contradiction.

(d) [4] Find a weak sequential equilibrium when $p < \frac{1}{4}$, in which challenger 1 enters with probability 1, the weak type of the chain store chooses F with probability $p/(1-p)$, and challenger 2 enters with probability $\frac{1}{2}$ after observing F in market 1.

Solution: The strategy profile is given in the question. By Bayes' rule, after observing F in market 1 challenger 2 believes that the chain store's type is tough with probability $p/[p + (1-p)p/(1-p)] = 1/2$, and after observing A in market 1 challenger 2 believes that the chain store's type is tough with probability 0. From (a), sequential rationality is satisfied for challenger 2. For challenger 1, if it stays out its payoff is 0, and if it enters its payoff is 1 with probability $(1-p)(1-2p)/(1-p) = 1-2p$ and -1 with probability 2p, with the expectation of $1-4p$, so sequential rationality is satisfied. Finally, for the weak type chain store, if it chooses A in market 1, its payoff is 0 from market 1 and 0 from market 2, while if it chooses F in market 1, its payoff is -1 in market 1, and 2 with probability $\frac{1}{2}$ and 0 with probability $\frac{1}{2}$, with the expectation of 0, so sequential rationality is satisfied.

End of examination
Total pages: 8
Total marks: 100