Given name:________________ Family name:________________

Student number:________________ Signature:________________

UNIVERSITY OF TORONTO
Faculty of Arts and Science

PLEASE HAND IN

DECEMBER EXAMINATIONS 2006

ECO326H1F Sections L0101 and L5101 (Advanced Economic Theory—Micro)
Instructor: Martin J. Osborne and Ettore Damiano

Duration: 3 hours

No aids allowed

This examination paper consists of **10** pages and **7** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Please check the box corresponding to the section in which you are registered:

- ☐ **Section L0101 (Osborne): answer questions 1–6.**
- ☐ **Section L5101 (Damiano): answer questions 1–5 and 7.**

TO OBTAIN CREDIT, YOU MUST GIVE ARGUMENTS TO SUPPORT YOUR ANSWERS. You may use the last 3 pages of the exam for rough work.

For graders' use:

	Score
1 (17)	
2 (18)	
3 (17)	
Subtotal	

	Score
4 (10)	
5 (16)	
6 (16)	
7 (16)	
Subtotal	

Total (100)	

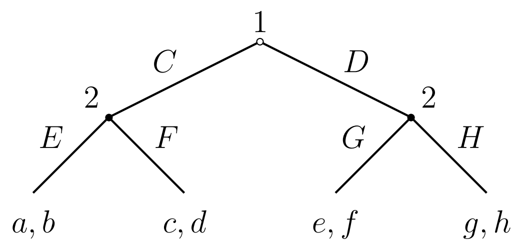
1. (a) [8] Either give an example of a strategic game with the following properties, or argue that no such game exists.

- The game has exactly one Nash equilibrium.
- If one action of one of the players is deleted, the resulting game has exactly one Nash equilibrium, and each player is *better off* in this equilibrium than in the equilibrium of the original game.

Solution: Consider the game below. This game has a unique Nash equilibrium, (B, R). If the action B of player 1 is deleted, the resulting game has a unique Nash equilibrium, (T, L), in which both players are better off than they are in (B, R).

	L	R
T	$3, 3$	$0, 2$
B	$4, 0$	$1, 1$

(b) [9] Consider an extensive game with the structure shown in the figure below. Assume that neither player is indifferent between any two terminal histories. Are there any values for the payoffs such that the game has a Nash equilibrium in which both players are better off than they are in any subgame perfect equilibrium?



Circle one: Yes No

Reason (required for credit!):

Solution: Because neither player is indifferent between any two actions, the game has a unique subgame perfect equilibrium. Assume, without loss of generality, that this subgame perfect equilibrium is (C, EG). Then $a > e$, $b > d$, and $f > h$. Because $b > d$, the game has no Nash equilibrium in which the outcome is (C, F) and because $f > h$, it has no Nash equilibrium in which the outcome is (D, H). Thus the only terminal history that can be the outcome of a Nash equilibrium other than (C, E) is (D, G). But given $a > e$, player 1 is worse off in this outcome than in the subgame perfect equilibrium. We conclude that there are no values for the payoffs such that the game has a Nash equilibrium in which both players are better off than they are in any subgame perfect equilibrium.

2. Consider the following two-player strategic game. Assume that the four numbers a_1, b_1, c_1, and d_1 are all different, and the four numbers a_2, b_2, c_2, d_2 are all different.

Further assume that $a_1 > c_1$.

	A	B
A	a_1, a_2	b_1, b_2
B	c_1, c_2	d_1, d_2

(a) [4] Find conditions on the payoffs for which the game has a unique mixed strategy Nash equilibrium and in this equilibrium each player assigns positive probability to each of her actions.

Solution: Given that $a_1 > c_1$, if $b_1 > d_1$ then A strictly dominates B. Thus for the game to have an equilibrium in which player 1 assigns positive probability to each of her actions we need $b_1 < d_1$. Further, if $a_2 > b_2$ then (A, A) is a pure strategy equilibrium. Thus we need $a_2 < b_2$ and hence $c_2 > d_2$.
In summary, the game has a unique mixed strategy Nash equilibrium, in which each player assigns positive probability to each action if and only if $b_1 < d_1$, $a_2 < b_2$, and $c_2 > d_2$.

(b) Suppose that a_1 decreases slightly, but remains larger than c_1. All other payoffs remain the same. How does this change affect

i. [4] the equilibrium probability player 1 assigns to A?

Solution: This probability remains the same, because player 2's payoffs do not change.

ii. [4] the equilibrium probability player 2 assigns to A?

Solution: This probability is given by

$$\frac{d_1 - b_1}{a_1 - c_1 + d_1 - b_1}$$

Thus this probability increases when a_1 decreases.

iii. [6] the equilibrium payoff of player 1?

Solution: Player 1's equilibrium payoff is equal to the payoff to each of her actions A and B. In particular, it is equal to the payoff to her action B. Now if $c_1 > d_1$ then the change in player 2's mixed strategy puts more weight on a higher payoff, so that player 1's equilibrium payoff increases; if $c_1 < d_1$ then player 1's equilibrium payoff decreases.

3. Two players use the following variant of the ultimatum game to split \$$c$. Player 1 starts by naming an amount \$$p$, which can be any nonnegative number. Then player 2 either takes \$$p$ from player 1, or gives \$$p$ to player 1. If player 2 chooses to take \$$p$ from player 1, the two players proceed to divide the \$$c$ by playing an ultimatum game in which player 1 is the proposer. If instead player 2 decides to give \$$p$ to player 1, the two player proceed to divide the \$$c$ by playing an ultimatum game in which player 2 is the proposer. Both players care about the total amount of money they have at the end of the game. (Suppose, for example, that player 1 names \$$p$, player 2 takes this amount, and player 1 then offers y to player 2. In this case, if player 2 accepts y then player 1's payoff is $c - y - p$ and player 2's payoff is $y + p$, and if player 2 rejects y then player 1's payoff is $-p$ and player 2's payoff is p.)

(a) [5] Model this scenario as an extensive game with perfect information. (A diagram is sufficient.)

Solution: See Figure 1.

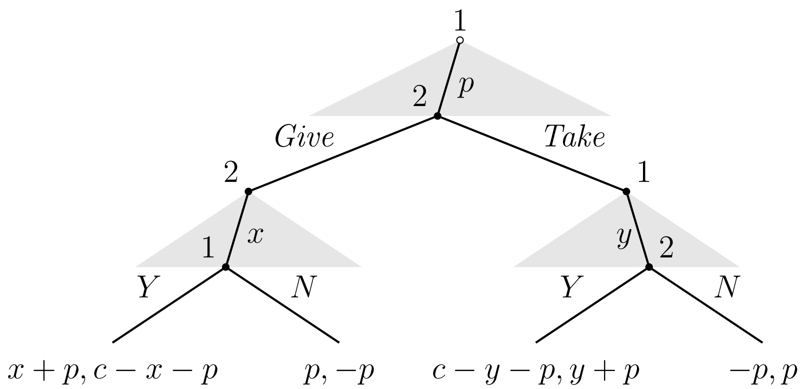


Figure 1. The game in Problem 3.

(b) [12] Find all subgame perfect equilibria of the game.

Solution: The subgame following any history in which player 2 chooses *Take* is an ultimatum game in which player 1 is the proposer. Thus in the unique subgame perfect equilibrium of this subgame, player 1 offers the division $(c, 0)$ and player 2 accepts all proposals. Similarly, the subgame following any history in which player 2 chooses *Give* is an ultimatum game in which player 2 is the proposer, so that in the unique subgame perfect equilibrium of this subgame player 2 proposes $(0, c)$ and player 1 accepts all proposals.

Now consider a history after which player 1 names some amount p. The payoff of player 2 is p if she takes p from player 1 and $c - p$ if she chooses to give p to player 1. Thus in any subgame perfect equilibrium player 2 takes p if $p > \frac{1}{2}c$ and gives p to player 1 if $p < \frac{1}{2}c$. If $p = \frac{1}{2}c$, she is indifferent between giving p and taking p.

Finally consider player 1's choice at the start of the game. Her payoff is $c - p$ for any $p > \frac{1}{2}c$, p for any $p < \frac{1}{2}c$, and $\frac{1}{2}c$ for $p = \frac{1}{2}c$. Thus in any subgame perfect equilibrium player 1 names $\frac{1}{2}c$ at the start of the game, player 2 chooses *Take* if $p > \frac{1}{2}c$ and *Give* if $p < \frac{1}{2}c$, player 1 proposes $(c, 0)$ if she is the proposer and accepts any proposal when player 2 is the proposer, and player 2 proposes $(0, c)$ when she is the proposer and accepts any offer proposed by player 1.

4. Consider the following variant of the bargaining game of alternating offers. The size of the pie is \$1 and both players discount future payoffs with the discount factor $0 < \delta < 1$. As in the original bargaining game of alternating offers, first player 1 makes an offer, which player 2 can accept or reject. But then if player 2 rejects player 1's offer, *player 1* makes *another* offer, which player 2 can again accept or reject. If player 2 rejects this second offer of player 1, she makes a counteroffer, which player 1 can accept or reject.

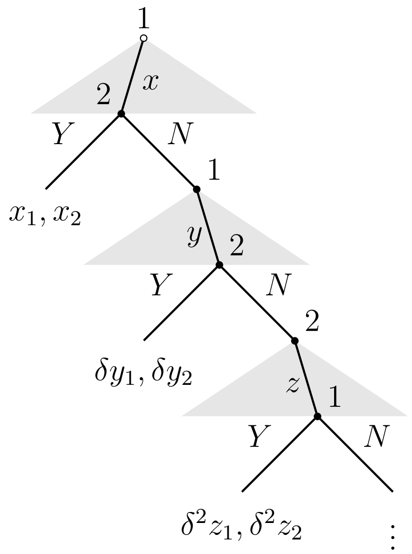


Figure 2. The first three periods of the variant of the bargaining game of alternating offers in Problem 4.

If player 1 rejects the counteroffer, then play continues as if we were at the start of the game: player 1 gets the opportunity to make offers for two periods before player 2 can make a counteroffer. The first three periods of the game are shown in Figure 2.

A strategy profile is a subgame perfect equilibrium of this game if and only if it satisfies the one deviation property. Find the subgame perfect equilibrium (equilibria?) of the game.

(a) [5] Player 1 always proposes

$$\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$$

and accepts any offer $(y, 1-y)$ in which $y \geq \delta/(1+\delta)$, and player 2 always proposes

$$\left(\frac{\delta}{1+\delta}, \frac{1}{1+\delta}\right)$$

and accepts any offer $(x, 1-x)$ in which $1-x \geq \delta/(1+\delta)$.

Solution: No. Consider a history that ends with player 1's making a "first round" offer $(x, 1-x)$. By accepting the offer, player 2 receives a payoff of $1-x$. If player 2 rejects this offer, while all future actions remain as specified by the candidate equilibrium strategy pair, in the next round player 1 will make the offer

$$\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right),$$

which player 2 will accept. This yields player 2 the payoff $\delta^2/(1+\delta)$. Since $\delta^2/(1+\delta) < \delta/(1+\delta)$, player 2 can increase her payoff by accepting rather then rejecting any "first round" offer $(x, 1-x)$ such that $\delta^2/(1+\delta) < 1-x < \delta/(1+\delta)$. Thus the strategy pair is not a subgame perfect equilibrium.

(b) [5] Refer to a proposal of player 1 at the start of the game or after player 1 has rejected an offer of player 2 as a "first round" offer, and to a proposal of player 1 in a period following a rejection by player 2 of a first round offer as a "second round" offer.

The strategy pair to consider is:

- Player 1's first round offer is always
$$\left(\frac{1-\delta^2}{1-\delta^3}, \frac{\delta^2(1-\delta)}{1-\delta^3}\right),$$
her second round offer is always
$$\left(1-\frac{\delta(1-\delta)}{1-\delta^3}, \frac{\delta(1-\delta)}{1-\delta^3}\right),$$
and she accepts every offer $(y, 1-y)$ of player 2 in which $y \geq \frac{\delta(1-\delta^2)}{1-\delta^3}$.
- Player 2 always proposes
$$\left(\frac{\delta(1-\delta^2)}{1-\delta^3}, \frac{1-\delta}{1-\delta^3}\right),$$
accepts any first round offer $(x, 1-x)$ of player 1 for which $1-x \geq \frac{\delta^2(1-\delta)}{1-\delta^3}$, and accepts any second round offer $(z, 1-z)$ of player 1 for which $1-z \geq \frac{\delta(1-\delta)}{1-\delta^3}$.

Solution: Yes. Consider the move at the start of each of the six distinct subgames.

- Consider a history that ends with player 1 making a first round offer $(x, 1-x)$. By accepting the offer, player 2 receives a payoff of $1-x$. If player 2 rejects the offer, while all future actions remain as specified by the candidate equilibrium strategy pair, in the next round player 1 makes the offer $(1-\delta(1-\delta)/(1-\delta^3), \delta(1-\delta)/(1-\delta^3))$, which player 2 accepts, obtaining the payoff $\delta^2(1-\delta)/(1-\delta^3)$. Thus player 2 optimally accepts $(x, 1-x)$ if $1-x \geq \delta^2(1-\delta)/1-\delta^3$ and rejects it otherwise, as her strategy calls for her to do.
- Consider a history that ends with player 1 making a second round offer $(z, 1-z)$. By accepting the offer, player 2 receives a payoff of $1-z$. If player 2 rejects the offer, in the next round she makes the offer $(\delta(1-\delta^2)/(1-\delta^3), (1-\delta)/(1-\delta^3)$, which player 1 accepts, yielding player 2 the payoff $\delta(1-\delta)/(1-\delta^3)$. Thus player 2 optimally accepts $(z, 1-z)$ if $1-z \geq \delta(1-\delta)/(1-\delta^3)$ and rejects it otherwise, as her strategy calls for her to do.
- Consider a history that ends with player 2 making an offer $(y, 1-y)$. By accepting the offer, player 1 receives a payoff of y. If player 1 rejects the offer, in the next round she makes the offer $((1-\delta^2)/(1-\delta^3), \delta^2(1-\delta)/(1-\delta^3))$, which player 2 accepts, yielding player 1 the payoff $\delta(1-\delta^2)/(1-\delta^3)$. Thus player 1 optimally accepts $(y, 1-y)$ if $y \geq \delta(1-\delta^2)/(1-\delta^3)$ and rejects it otherwise, as her strategy calls for her to do.

- Consider a history that ends with an offer of player 2 being rejected. Any first round offer $(x, 1-x)$ of player 1 with $1-x > \delta^2(1-\delta)/(1-\delta^3)$ is accepted by player 2 and yields player 1 a lower payoff than the equilibrium offer. Any offer $(x, 1-x)$ with $1-x < \delta^2(1-\delta)/(1-\delta^3)$ is rejected and is followed by a second round offer $(1-\delta(1-\delta)/(1-\delta^3), \delta(1-\delta)/(1-\delta^3))$ of player 1 that is accepted, yielding player 1 the payoff $\delta - \delta^2(1-\delta)/(1-\delta^3)$, which is less than $(1-\delta^2)/(1-\delta^3)$.
- If player 1 makes a second round offer $(z, 1-z)$ with $1-z < \delta(1-\delta)/(1-\delta^3)$, player 2 rejects it, yielding player 1 the payoff $\delta^2(1-\delta^2)/(1-\delta^3)$, which is less than $1-\delta(1-\delta)/(1-\delta^3)$.
- If player 2 makes an offer $(y, 1-y)$ with $y < \delta(1-\delta^2)/(1-\delta^3)$, player 2 rejects it, yielding player 2 the payoff $\delta^3(1-\delta)/(1-\delta^3)$, which is less than $(1-\delta)/(1-\delta^3)$.

The strategy pair satisfies the one deviation property, and hence is a subgame perfect equilibrium.

5. Two people have to choose the type of computer operating system to use. There are two options, W and M. Each person is one of two possible types: type w, who has a preference for W, all other things equal, and type m, who has a preference for M. Each person's payoff depends mainly on the operating systems chosen by the *other* person, but depends partly on her own preference. Specifically, a person who chooses Y obtains the payoff

$$0.1U + 0.9V,$$

where U is 1 if the operating system Y matches the person's type (W for w and M for m) and 0 otherwise, and V is 1 if the operating system Y matches the *other* person's *action* and 0 otherwise.

Each person knows her own type but not the type of any other person. She believes that the type of any other given person is W with probability α and M with probability $1-\alpha$ (where $0 \leq \alpha \leq 1$).

(a) [5] In the Bayesian game that models this situation, what is the set of states and what is the signal function of each person?

Solution: The set of states is $\{ww, wm, mw, mm\}$. The signal function of person 1 associates w with the states ww and wm and m with the states mw and mm. The signal function of player 2 associates w with the states ww and mw and m with the states wm and mm.

(b) [11] Find the Nash equilibria of the Bayesian game in which the action chosen by type w of player 1 is the same as the action chosen by type w of player 2 and the action chosen by type m of player 1 is the same as the action chosen by type m of player 2. (The character of the set of such equilibria depends on the value of α.)

Solution: We need to consider four strategy pairs, (WW, WW), (WM, WM), (MW, MW), and (MM, MM).
(WW, WW): The payoff of a player of type w is 1. If she switches to M, her payoff decreases to 0. The payoff of a player of type m is 0.9. If she switches to M, her payoff decreases to 0.1. Thus the strategy pair is a Nash equilibrium for all values of α.
(WM, WM): The payoff of a player of type w is $0.1 + 0.9\alpha$. If she switches to M, her payoff changes to $0.9(1-\alpha)$. Thus for a deviation not to be profitable we need $0.1 + 0.9\alpha \geq 0.9(1-\alpha)$, or $\alpha \geq \frac{4}{9}$. The payoff of a player of type m is $0.1 + 0.9(1-\alpha)$. If she switches to W, her payoff changes to 0.9α. Thus for a deviation not to be profitable we need $0.1 + 0.9(1-\alpha) \geq 0.9\alpha$, or $\alpha \leq \frac{5}{9}$. Thus the strategy pair is a Nash equilibrium if and only if $\frac{4}{9} \leq \alpha \leq \frac{5}{9}$.
(MW, MW): The payoff of a player of type w is $0.9(1-\alpha)$. If she switches to W, her payoff changes to $0.1 + 0.9\alpha$. Thus for a deviation not to be profitable we need $0.9(1-\alpha) \geq 0.1 + 0.9\alpha$, or $\alpha \leq \frac{4}{9}$. The payoff of a player of type m is 0.9α. If she switches to M, her payoff changes to $0.1 + 0.9(1-\alpha)$. Thus for a deviation not to be profitable we need $0.9\alpha \geq 0.1 + 0.9(1-\alpha)$, or $\alpha \geq \frac{5}{9}$. Thus the strategy pair is not a Nash equilibrium.
(MM, MM): The payoff of a player of type w is 0.9. If she switches to W, her payoff decreases to 0.1. The payoff of a player of type m is 1. If she switches to W, her payoff decreases to 0. Thus the strategy pair is a Nash equilibrium for all values of α.

For students in Section L0101 (Osborne) only.
DO NOT ANSWER THIS QUESTION IF YOU ARE IN SECTION L5101 (Damiano)!

6. [16] Consider Bertrand's duopoly game when the demand function is given by $D(p) = \alpha - p$ if $p \leq \alpha$, and $D(p) = 0$ for $p > \alpha$, and the cost function of each firm i is given by $C_i(q_i) = f + cq_i$ for $q_i > 0$, and $C_i(0) = 0$, where f is positive and less than the maximum of $(p - c)(\alpha - p)$ with respect to p. Assume that firm 1 gets all the demand when both firms charge the same price. Either find a Nash equilibrium of the game or show that no Nash equilibrium exists.

Solution: Denote by $\overline{p}$ the price p that satisfies $(p - c)(\alpha - p) = f$ and is less than the maximizer of $(p - c)(\alpha - p)$. (See Figure 69.1 in the book.)

I claim that the pair of prices $(\overline{p}, \overline{p})$ is a Nash equilibrium. At this pair of prices, both firms' profits are zero. (Firm 1 receives all the demand and obtains the profit $(\overline{p} - c)(\alpha - \overline{p}) - f = 0$, and firm 2 receives no demand.) This pair of prices is a Nash equilibrium by the following argument.

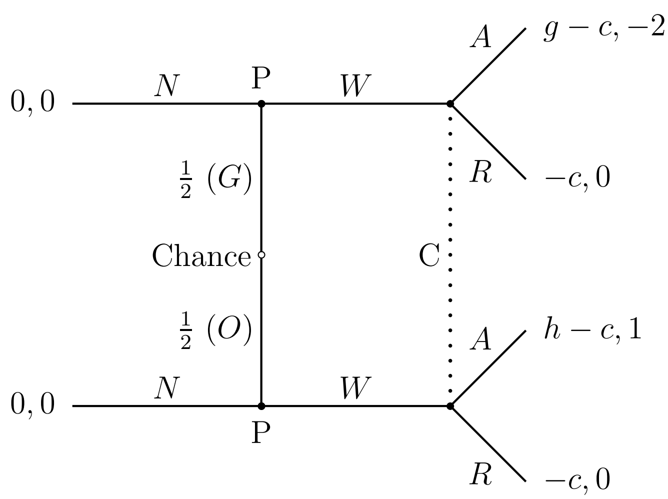
- If either firm raises its price its profit remains zero (it receives no customers).
- If either firm lowers its price then it receives all the demand and earns a negative profit (since f is less than the maximum of $(p - c)(\alpha - p)$).

For students in Section L5101 (Damiano) only.
DO NOT ANSWER THIS QUESTION IF YOU ARE IN SECTION L0101 (Osborne)!

7. A university professor must decide whether or not to write a letter of recommendation for a student who is applying for graduate school. The professor knows the student's ability while the graduate school admission committee only knows that there is a 50% chance the student is good and a 50% chance the student is outsdanding. If the professor does not write a letter of recommendation, the student will not be admitted to graduate school and the payoffs to both the professor and the admission committee are 0. If the professor writes the letter of recommendation, the admission committee must decide whether to accept or reject the student's application. The admission committee receives a payoff of 1 from admitting an outstanding student and a payoff of -2 from admitting a good student. The payoff from rejecting a student is 0, regardless of the student's quality. The professor incurs a cost of $c > 0$ from writing a letter of recommendation and receives a reward of $h > 0$ if an outstanding student is accepted, a reward of $g > 0$ if a good student is accepted, and no reward if the student is not accepted. The professor's payoff is the difference between the reward received and the cost of writing a letter.

   (a) [5] Model this scenario as an extensive game with imperfect information.

   Solution: See figure.



   (b) [11] Assume that $h > g > c$. Find all weak sequential equilibria in which the professor always writes a letter of recommendation if the student is outstanding.

Solution: (a) If P never writes a recommendation for a good student, then the admission committee must believe that the student is surely outstanding upon seeing a recommendation. Then, by sequential rationality, the admission committee must accept the student for sure. This is not an equilibrium though because, given the strategy of the admission committee, the professor is strictly better off by writing a recommendation for a good student.

(b) If P always write a recommendation letter for a good student, then, by belief consistency, the admission committee must believe that there is exactly a 50% chance that a recommended student is outstanding. Sequential rationality then requires that the committee rejects a recommended. This cannot be an equilibrium though because, given the strategy of the admission committee, the professor would be strictly better off not writing any recommendation.

(c) The only remaining candidate equilibrium is one in which the professor recommends a good student with probability strictly between 0 and 1. For this to be sequentially rational the payoff from writing a recommendation must coincide with 0. Since the expected payoff from writing a recommendation for a good student is given by $(g-c)q-c(1-q)$ where q is the equilibrium probability that the admission committee accepts a recommended applicant, $q=c/g$ in equilibrium. Since in equilibrium $0<q<1$ it must be that the admission committee is exactly indifferent between accepting and rejecting a recommended applicant. Hence it must be that $1\mu-2(1-\mu)=0$ where μ is the equilibrium belief that a recommended applicant is outstanding. In equilibrium we must have $\mu=2/3$. Since consistency requires that $\mu=\frac{1/2}{1/2+1/2p}$ we have that the equilibrium probability that the professor recommends a good applicant must be $p=1/2$.

The unique weak sequential equilibrium is the following: the professor always recommends an outstanding student, the professor recommends a good student with probability $p=1/2$, the admission committee, upon observing a recommended student believes that the student is outstanding with probability $\mu=2/3$ and accepts the student with probability $q=c/g$.

End of examination
Total pages: 10
Total marks: 100