Economics 326: Advanced Economic Theory—Micro

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Answers to Midterm Examination

1. (a) The game is shown in Figure 1.
   (b) All four action pairs are Nash equilibria.
   (c) No action of either player is weakly dominated. (For each player, each action is equivalent to the other.)

	Retain	Give
Retain	$0, 0$	$x, 0$
Give	$0, x$	x, x

Figure 1. The game in Problem 1.

2. A pair (b_1, b_2) with $b_1 + b_2 > c$ is not a Nash equilibrium because either player can decrease her contribution slightly without affecting the provision of the good.

   A pair (b_1, b_2) in which $b_1 + b_2 = c$ is not a Nash equilibrium if $b_i > v_i$ because player i can then change her contribution from b_i to 0 and increase her payoff from $v_i - b_i$ to 0. Every other pair (b_1, b_2) with $b_1 + b_2 = c$ is a Nash equilibrium: for $i = 1$, 2, both an increase and a decrease in b_i make player i worse off (for an increase, the good is still provided and i's contribution is larger, and for a decrease the good is no longer provided).

   A pair (b_1, b_2) in which $b_1 + b_2 < c$ is not a Nash equilibrium if $v_1 - (c - b_2) > 0$ because player 1 can then change her contribution from b_1 to $c - b_2$ and thereby cause the good to be provided, increasing her payoff from 0 to $v_1 - (c - b_2)$. Similarly, such a pair is not a Nash equilibrium if $v_2 - (c - b_1) > 0$. A pair (b_1, b_2) in which $b_1 + b_2 < c$ is a Nash equilibrium if $v_1 - (c - b_2) \leq 0$ and $v_2 - (c - b_1) \leq 0$ because any change in b_i ($i = 1$, 2) that changes the outcome to that in which the good is provided involves a contribution for player i that exceeds v_i

   We conclude that the set of Nash equilibria consists of

   - the set of all pairs (b_1, b_2) such that $b_1 + b_2 = c$, $b_1 \leq v_1$, and $b_2 \leq v_2$

- the set of all pairs (b_1, b_2) such that $b_1 \leq c - v_2$ and $b_2 \leq c - v_1$.

These equilibria are illustrated in Figure 2 for a case in which $v_i < c$ for $i = 1, 2$.

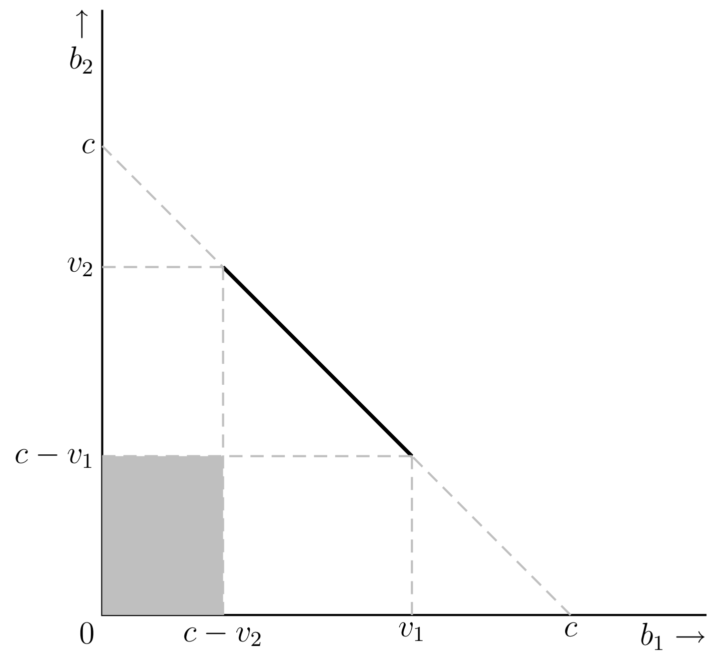

Figure 2. The set of Nash equilibria of the game in Problem 2 for a case which $v_i < c$ for $i = 1, 2$.

To find the Nash equilibria, you could alternatively find the players' best response functions. The best response function of player i is

$$B_i(b_j) = \begin{cases} \{b_i : b_i < c - b_j & \text{if } 0 \leq b_j < c - v_i \\ \{b_i : b_i \leq c - b_j & \text{if } b_j = c - v_i \\ c - b_j & \text{if } c - v_i < b_j < c \\ 0 & \text{if } b_j \geq c. \end{cases}$$

3. Firm 1's profit is

$$\pi_1(q_1, q_2) = \begin{cases} q_1(\alpha - q_1 - q_2) - q_1^2 & \text{if } q_1 + q_2 \leq \alpha \\ -q_1^2 & \text{if } q_1 + q_2 > \alpha \end{cases}$$

or

$$\pi_1(q_1, q_2) = \begin{cases} q_1(\alpha - 2q_1 - q_2) & \text{if } q_1 + q_2 \leq \alpha \\ -q_1^2 & \text{if } q_1 + q_2 > \alpha. \end{cases}$$

When it is positive, this function is a quadratic in q_1 that is zero at $q_1 = 0$ and at $q_1 = (\alpha - q_2)/2$. Thus firm 1's best response function is

$$b_1(q_2) = \begin{cases} \frac{1}{4}(\alpha - q_2) & \text{if } q_2 \leq \alpha \\ 0 & \text{if } q_2 > \alpha. \end{cases}$$

Since the firms' cost functions are the same, firm 2's best response function is the same as firm 1's: $b_2(q) = b_1(q)$ for all q. The firms' best response functions are shown in Figure 3.

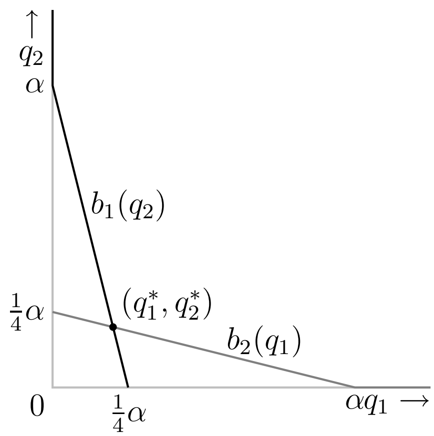

Figure 3. The best response functions in Cournot's duopoly game with linear inverse demand and a quadratic cost function, as in Problem 3. The unique Nash equilibrium is $(q_1^*, q_2^*) = (\frac{1}{5}\alpha, \frac{1}{5}\alpha)$.

Solving the two equations $q_1^* = b_1(q_2^*)$ and $q_2^* = b_2(q_1^*)$ we find that there is a unique Nash equilibrium, in which the output of firm i ($i = 1, 2$) is $q_i^* = \frac{1}{5}\alpha$.

4. The game has no Nash equilibrium in which both candidates enter, because at least one loses. It also has no Nash equilibrium in which no candidate enters, because either candidate can then enter and win. Thus in every Nash equilibrium, exactly one candidate enters. Denote by k_1 the position with the property that one third of the citizens' favorite positions is at most k_1 and by k_2 the position with the property that one third of the citizens' favorite positions is at least k_2. Then a strategy pair is a Nash equilibrium if and only if the position of the entrant is in the interval from k_1 to k_2 (otherwise the other candidate can enter and win).

5. (a) The game has no pure strategy Nash equilibrium. Consider an equilibrium in mixed strategies. Denote the probability player 1 assigns to A by p and the probability player 2 assigns to A by q.

For an equilibrium we need player 1's expected payoffs to A and B to be the same, or

$$(1-q)v_A = qv_B + (1-q)\pi v_B,$$

which means that $q = 1 - v_B/[v_A + (1-\pi)v_B] = (v_A - \pi v_B)/(v_A + (1-\pi)v_B)$. Denote this probability by q^*. We need also player 2's expected payoffs to A and B to be the same, or

$$-(1-p)v_B = -pv_A - (1-p)\pi v_B,$$

which means that $p = 1 - v_A/[v_A + (1-\pi)v_B] = (1-\pi)v_B/(v_A + (1-\pi)v_B)$. Denote this probability by p^*. Thus the game has a unique Nash equilibrium $((p^*, 1-p^*), (q^*, 1-q^*))$.

(b) Player 1's expected payoff in the equilibrium of the game in part a is $(1-q)v_A$, where q is the equilibrium probability that player 2 chooses A, and is thus equal to $v_A v_B/[v_A + (1-\pi)v_B]$.

Thus if $h \le v_A v_B/[v_A + (1-\pi)v_B]$, a Nash equilibrium of the game is the mixed strategy equilibrium $((p^*, 1-p^*, 0), (q^*, 1-q^*))$.

If $h > v_A v_B/[v_A + (1-\pi)v_B]$, any strategy pair $((0,0,1),(q,1-q))$ with $1-h/v_A \le q \le (h-\pi v_B)/[(1-\pi)v_B]$ (including $((0,0,1),(q^*,1-q^*))$) is a Nash equilibrium. (If $h \ge v_A$ then $((0,0,1),(0,1))$ is also a Nash equilibrium.)

6. (a) Denote by p_i the probability with which each witness with cost c_i and value v_i reports the crime, for $i = 1, 2$. For each witness with cost c_1 to report with positive probability less than one, we need

$$\begin{aligned} v_1 - c_1 &= v_1 \cdot \Pr\{\text{at least one other person calls}\} \\ &= v_1 \left(1 - (1-p_1)(1-p_2)\right), \end{aligned}$$

or

$$c_1 = v_1(1-p_1)(1-p_2). \qquad (1)$$

Similarly, for each witness with cost c_2 and value v_2 to report with positive probability less than one, we need

$$\begin{aligned} v_2 - c_2 &= v_2 \cdot \Pr\{\text{at least one other person calls}\} \\ &= v_2 \left(1 - (1-p_1)^2\right), \end{aligned}$$

or

$$c_2 = v_2(1 - p_1)^2. \qquad (2)$$

From (2) we obtain

$$p_1 = 1 - (c_2/v_2)^{1/2}.$$

Substituting this value into (1) we obtain

$$c_1 = v_1(c_2/v_2)^{1/2}(1 - p_2),$$

so that

$$p_2 = 1 - (c_1/v_1)(v_2/c_2)^{1/2}.$$

For these values of p_1 and p_2 to be between 0 and 1 we need $c_2 < v_2$ and $(v_2/c_2)^{1/2} < v_1/c_1$.

In conclusion, if $c_2 < v_2$ and $(v_2/c_2)^{1/2} < v_1/c_1$ then the game has a Nash equilibrium in which each witness calls with positive probability less than 1. In this equilibrium, each witness with cost c_1 and value v_1 calls with probability $1-(c_2/v_2)^{1/2}$ and each witness with cost c_2 and value v_2 calls with probability $1-(c_1/v_1)(v_2/c_2)^{1/2}$.

(b) For an equilibrium of this type we need (1) to be satisfied with $p_2 = 0$, or $c_1 = v_1(1 - p_1)$, and $c_2 \geq v_2(1 - p_1)^2$. From the first condition we have $p_1 = 1 - c_1/v_1$ and thus the second condition is $c_2 \geq v_2(c_1/v_1)^2$. For $0 < p_1 < v_1$ we need $c_1 < v_1$.

In conclusion, if $c_1 < v_1$ and $(v_2/c_2)^{1/2} \geq v_1/c_1$ then the game has a Nash equilibrium in which each witness with cost c_1 and value v_1 calls with positive probability $1 - c_1/v_1$ and the witness with cost c_2 and value v_2 does not call.