Given name:____________________ **Family name:**____________________

Student number:____________________ **Signature:**____________________

UNIVERSITY OF TORONTO
Faculty of Arts and Science

ECO 326 H Section L0101 (Advanced Economic Theory—Micro)
Instructor: Martin J. Osborne

MIDTERM EXAMINATION
October 2005

Duration: 1 hour 50 minutes

No aids allowed

This examination paper consists of **14** pages and **5** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Answer all questions.

TO OBTAIN CREDIT, YOU MUST GIVE ARGUMENTS TO SUPPORT YOUR ANSWERS.

For graders' use:

	Score
1 (15)	
2 (20)	
3 (20)	
4 (20)	
5 (25)	
Total (100)	

1. [15] Each of $n \geq 3$ people announces an integer from the set $\{1, \ldots, K\}$. The person whose integer is closest to the average of the announced integers wins \$1. If there is a tie for the integer closest to the average, \$1 is split equally between the people whose integer is closest to the average. Find all the pure strategy equilibria of the strategic game that models this situation. (Be sure to argue that you have found all equilibria.)

   Solution: An action profile $(a_1, \ldots, a_n)$ is a Nash equilibrium if and only if $a_1 = a_2 = \cdots = a_n$. (That is, the game has K Nash equilibria, $(1, \ldots, 1)$, ..., $(K, \ldots, K)$.)

   Any such action profile is a Nash equilibrium because any deviation leads the deviating player to lose rather than share the dollar.

   In any other action profile at least one player loses and any such player can deviate to the integer closest to the average of the other players' actions and obtain at least a share of the dollar. Thus no other action profile is a Nash equilibrium.

2. [20] Two people can choose how much to contribute to the provision of a public good. If person 1 contributes c_1 and person 2 contributes c_2 then the amount of the public good provided is $c_1 + c_2$ and person i's payoff (for $i = 1, 2$) is

$$v_i\sqrt{c_1 + c_2} - c_i,$$

where v_1 and v_2 are constants with $v_1 \neq v_2$. Each person can choose any nonnegative number for her contribution.

Find the Nash equilibria of the strategic game that models this situation. (The character of the equilibria depend on the values of v_1 and v_2.)

Solution: The best response of player i to c_j is the value of c_i that maximizes $v_i\sqrt{c_1 + c_2} - c_i$. This function is strictly concave, so that if its maximizer is positive, this maximizer is the solution of the first-order condition

$$\tfrac{1}{2}v_i(c_1 + c_2)^{-1/2} - 1 = 0.$$

The solution is $c_i = \frac{1}{4}(v_i)^2 - c_j$, where $j = 2$ if $i = 1$, and $j = 1$ if $i = 2$. This solution is positive if $c_j < \frac{1}{4}(v_i)^2$. If $c_j \geq \frac{1}{4}(v_i)^2$ then i's payoff is decreasing in c_i, so that i's best response is 0.

In summary, player i's best response to c_j is

$$b_i(c_j) = \begin{cases} 0 & \text{if } c_j \geq \frac{1}{4}(v_i)^2 \\ \frac{1}{4}(v_i)^2 - c_j & \text{if } c_j < \frac{1}{4}(v_i)^2 \end{cases}$$

For $v_1 > v_2$, the best response functions are equal to those given in Figure 44.1 in the book.

We deduce that for any $v_1 \neq v_2$ the game has a unique Nash equilibrium:

$$\begin{cases} (\frac{1}{4}(v_1)^2, 0) & \text{if } v_1 > v_2 \\ (0, \frac{1}{4}(v_2)^2) & \text{if } v_1 > v_2. \end{cases}$$

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3. [20] Consider a variant of Hotelling's model of electoral competition in which there are *four* candidates. Assume that each candidate has to choose a position—no candidate has the option of staying out of the competition—and that if there is a tie for first place, then all the tied candidates have an equal probability of winning.

   Suppose that the voters' favorite positions are uniformly distributed between 0 and 1. That is, for any number x with $0 \leq x \leq 1$, the fraction of the population of voters whose favorite position is less than x is x. Find a Nash equilibrium of the game in this case. [Note: you are not asked to find all equilibria, but you do need to argue carefully that the action profile you find is an equilibrium.]

Solution: In one Nash equilibrium, two candidates choose $\frac{1}{4}$ and two candidates choose $\frac{3}{4}$. The outcome is that all four candidates tie. In this action profile, each candidate obtains $\frac{1}{4}$ of the votes; thus each candidate's probability of winning is $\frac{1}{4}$.

The action profile is a Nash equilibrium because if any candidate chooses a different position, she loses.

Consider, for example, a candidate whose position is $\frac{1}{4}$. (The arguments for a candidate at $\frac{3}{4}$ is symmetric.) Denote this candidate by i.

- If i deviates to a position less than $\frac{1}{4}$, she obtain less than $\frac{1}{4}$ of the votes and the other candidate at $\frac{1}{4}$ obtains more than $\frac{1}{4}$, so that i loses.
- If i deviates to a position between $\frac{1}{4}$ and $\frac{3}{4}$, she obtains $\frac{1}{4}$ of the vote, whereas the other candidate at $\frac{1}{4}$ obtains more than $\frac{1}{4}$, so that i loses.
- If i deviates to $\frac{3}{4}$, she obtains $\frac{1}{6}$ of the votes, whereas the other candidate at $\frac{1}{4}$ obtains $\frac{1}{2}$ of the votes, so that i loses.
- If i deviates to a position greater than $\frac{3}{4}$, she obtains less than $\frac{1}{4}$ of the votes whereas the other candidate at $\frac{1}{4}$ obtains $\frac{1}{2}$ of the votes, so that i loses.

The game has other Nash equilibria, in which not all candidates win with positive probability. For example, any action profile in which one candidate is at x_1, one is at x_2, one is at $1 - x_2$, and one is at $1 - x_1$, with $x_1 < x_2$, $\frac{1}{4} < x_2 \leq \frac{1}{3}$, and $x_1 \leq 1 - 3x_2$ (which imply that $\frac{1}{4} < x_2 \leq \frac{1}{3}$) is a Nash equilibrium. In any such equilibrium, candidates 1 and 4 lose and candidates 2 and 3 tie for first place. Neither candidate 1 nor candidate 4 can deviate and win with positive probability, and neither candidate 2 nor candidate 3 can deviate and win outright.

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4. A third-price auction with perfect information is a variant of a second-price auction with perfect information in which the price paid by the winner (the player who submits the highest bid) is the third highest of the bids submitted. [That is, $n \geq 3$ players simultaneously submit bids for a single indivisible object. Player i's valuation of the object is v_i, where $v_1 > v_2 > \cdots > v_n$. The highest bid wins; in the event of a tie, the player whose index is smallest wins. (E.g. if players 1 and 2 tie for the highest bid, player 1 wins.)]

   Denote by G the strategic game that models this situation.

   (a) [10] Either find a (pure strategy) Nash equilibrium of the game in which the winner is player 1 and the price is less than v_2 (the second-highest valuation) or show that the game has no such equilibrium.

   Solution: Any action profile $(b_1, \ldots, b_n)$ with the following properties is such an equilibrium:

   - the winning bid is b_1
   - the second-highest bid is at least v_2 and is not submitted by player 2
   - the third-highest bid is less than v_2 and at least v_j, where j is the player who submits the second-highest bid.

   In such an action profile, player 1 wins and pays less than v_2. Denote by p^* the price player 1 pays. If the player who submits the second-highest bid changes her bid then either the outcome does not change or, if her bid exceeds b_1, she wins and pays the price p^*, which is at least her valuation (by the third condition). If any other player deviates either the outcome does not change or, if the deviant's bid exceeds b_1, the deviant wins and pays a price equal to the original second-highest bid, which is at least v_2 and hence at least equal to the deviant's valuation.
   (Note that you are asked only to find *one* equilibrium. An example of an action profile that satisfies the conditions is $(b_1, \ldots, b_n) = (v_1, v_n, v_3, v_2, v_n, \ldots, v_n)$.)

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(b) [10] Either find a (pure strategy) Nash equilibrium of the game in which the winner is player n (who has the lowest valuation) or show that the game has no such equilibrium.

Solution: Consider an action profile in which the winner is player n. Player n's bid b_n must be the highest, and the third-highest bid must be at most v_n, otherwise player n's payoff is negative so that she can do better by bidding 0. But now consider a deviation by the player submitting the second-highest bid. If she bids more than b_n then she wins and the price she pays is at most v_n, so her payoff increases. Hence no such action profile is a Nash equilibrium.

5. (a) [10] Find the range of values of a, b, c, d, and e, if any, for which the mixed strategy pair $((0, \frac{1}{3}, \frac{2}{3}), (\frac{1}{4}, 0, \frac{3}{4}))$ is a mixed strategy Nash equilibrium of the following strategic game.

Player 2

		L	C	R
	T	a, b	$2, 2$	$1, 3$
Player 1	M	$6, c$	$3, d$	$0, 3$
	B	$0, 1$	$4, 3$	$e, 0$

Solution: For the strategy pair to be a mixed strategy Nash equilibrium we need player 1's expected payoffs to M and B to be equal, and to be at least her expected payoff to T, given player 2's mixed strategy. Her expected payoffs to M is $\frac{3}{2}$, given player 2's mixed strategy, so we need

$$\tfrac{3}{4}e = \tfrac{3}{2} \geq \tfrac{1}{4}a + \tfrac{3}{4}$$

or$e = 2$ and $a \leq 3$. We need also player 2's expected payoffs to L and R to be equal, and to be at least her expected payoff to C, given player 1's mixed strategy. These conditions are equivalent to

$$\tfrac{1}{3}c + \tfrac{2}{3} = \tfrac{1}{3} \cdot 3 \geq \tfrac{1}{3}d + \tfrac{2}{3} \cdot 3$$

or $c = 1$ and $d \leq -3$.

In summary, the strategy pair is a mixed strategy Nash equilibrium if and only if $a \leq 3$, $c = 1$, $d \leq -3$, and $e = 2$; b may take any value.

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(b) [15] Find all the Nash equilibria, in both pure and mixed strategies, of the following game. For each equilibrium that you find, give both the strategies and the payoffs.

	L	M	R
T	$3, 0$	$1, 2$	$3, 1$
B	$3, 2$	$2, 3$	$1, 3$

Solution: L is strictly dominated for player 2. The players' best response functions in the game in which L is eliminated are shown in Figure 1. From these best response functions we see that the set of Nash equilibria is the set of mixed strategy pairs $((0, 1), (0, q, 1 - q))$ such that $q \geq \frac{2}{3}$.

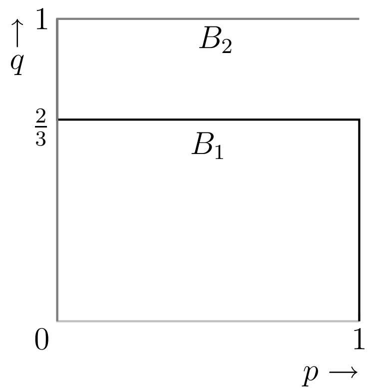

Figure 1. The players' best response functions in the game in Question 5b after the action L of player 2 has been eliminated.

You may use the next three pages for rough work.

For rough work (will not be graded)

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End of examination
Total pages: 14
Total marks: 100