Given name:____________________ Family name:____________________

Student number:____________________ Signature:___________________________

UNIVERSITY OF TORONTO
Faculty of Arts and Science

PLEASE HAND IN

DECEMBER EXAMINATIONS 2005

ECO326H1F Section L0101 (Advanced Economic Theory—Micro)
Instructor: Martin J. Osborne

Duration: 3 hours

No aids allowed

This examination paper consists of **9** pages and **7** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Students in Section L0101 (Osborne): answer questions 1–6.
Students in Section L5101 (Damiano): answer questions 1–5 and 7.

TO OBTAIN CREDIT, YOU MUST GIVE ARGUMENTS TO SUPPORT YOUR ANSWERS. You may use the last 4 pages of the exam for rough work.

For graders' use:

	Score
1 (15)	
2 (17)	
3 (18)	
Subtotal	

	Score
4 (15)	
5 (20)	
6 (15)	
7 (15)	
Subtotal	

Total (100)	

1. Suppose that two firms that produce the same perfectly divisible good compete in a market. The total demand for the good when the price is p is $13-p$ (so that the price when the total amount sold is Q is $13-Q$). Each firm i can produce any amount $q_i \le 5$ at the cost q_i; neither firm can produce more than 5. (That is, the technology of each firm differs from the one considered in the book because each firm has limited capacity; each firm's unit cost is 1 up to its capacity.)

   (a) [8] Consider the strategic game in which the firms simultaneously choose outputs, and the price is $13-Q$ when the firms' total output is Q. Find the Nash equilibrium (equilibria?) of this game.

   Solution: In the absence of the capacity constraint, each firm produces 4 units in the unique Nash equilibrium. The capacity constraints cause the best response functions to be truncated at the output 5, which does not affect their point of intersection. Thus the unique Nash equilibrium of the game is $(4, 4)$, as when no capacity constraints exist.

   (b) [7] Consider the strategic game in which the firms simultaneously choose prices. Assume that if the prices are the same, the total demand is split equally between the firms, and if (a) the prices differ, (b) the total demand at the lower price exceeds 5, and (c) the higher price is less than 13, then the firm with the higher price faces positive demand. Is the pair of prices $(1, 1)$ a Nash equilibrium?

   Solution: No: for this pair of prices each firm's profit is 0, whereas if one of the firms raises its price a little, it obtains a positive profit.

2. Two bidders/players compete for a single object. The value of the object, v, is identical for the two bidders. The rules of the auction are as follows: The two players simultaneously submit bids for the object. Each bidder has only two possible bids, h and l, with $h > l$. The object is assigned to the highest bidder; ties are broken by a fair coin flip. Both bidders, regardless of whether they win or lose the auction, pay an amount of money equal to the *smallest* bid. Each bidder must pay, in addition, a fixed participation cost c. The payoff to a bidder is equal to the value of the object multiplied by the probability the bidder receives the object minus the price paid minus the participation cost.

   (a) [4] Model this scenario as a strategic game.

   Solution:

	h	l
h	$\frac{1}{2}v-h-c, \frac{1}{2}v-h-c$	$v-l-c, -l-c$
l	$-l-c, v-l-c$	$\frac{1}{2}v-l-c, \frac{1}{2}v-l-c$

   (b) [7] Assume $\frac{1}{2}v < h-l$. Find all Nash equilibria in pure and mixed strategies of the game.

Solution: The strategy profiles (h, l) and (l, h) are pure strategy Nash equilibria. Player i strictly prefers l to h when player j chooses h and strictly prefers h to l when j chooses l, so in any mixed strategy equilibrium both players choose each action with positive probability. Let p be the probability that j assigns to h. Then the expected payoff to player i from h is given by

$$p\left(\tfrac{1}{2}v - h - c\right) + (1-p)\,(v - l - c)$$

and the expected payoff to player i from l is given by

$$p\,(-l - c) + (1-p)\left(\tfrac{1}{2}v - l - c\right)$$

Equating these payoffs and solving for p yields $p = \frac{1}{2}v/(h-l)$. The strategy pair in which both players choose h with this probability is the only mixed strategy Nash equilibrium.

(c) [6] Now assume that each bidder has the option of not participating in the auction. When a bidder chooses to not participate her payoff is 0 regardless of the action taken by the other bidder. For each equilibrium you found in part (b), determine the values of c for which the equilibrium remains an equilibrium when the bidders can choose not to participate.

Solution: The pure strategy Nash equilibria in (b) are no longer equilibria because the player who loses the auction receives a negative payoff $(-l - c)$ and prefers not to participate and receive a payoff of 0.

The mixed strategy equilibrium in (b) remains an equilibrium if the equilibrium expected payoff of each player is positive. This payoff is

$$p^*(-l - c) + (1 - p^*)(\tfrac{1}{2}v - l - c)$$

'where p^* is the equilibrium probability assigned to h by each player. This payoff is equal to

$$-l - c + \tfrac{1}{2}v(1 - p^*)$$

Thus the mixed strategy Nash equilibrium in (b) remains an equilibrium if and only if

$$c \le -l + \tfrac{1}{2}v(1 - p^*) = -l + \tfrac{1}{2}v\left(1 - \tfrac{1}{2}v/(h-l)\right).$$

3. Consider a variant of the "buying votes" model in which a supermajority is required to pass a bill.

[Reminder: In the "buying votes" model, one of two bills, X and Y, is to be passed by a legislature with k members. Interest group X values the passage of bill X at V_X and the passage of bill Y at 0, whereas interest group Y values the passage of bill X at 0 and the passage of bill Y at V_Y. First interest group X makes a payment to each legislator, then interest group Y does so. The payment made by a group to any legislator may be any nonnegative number. A legislator votes for bill X if she is paid more by group X than by group Y, and votes for bill Y if she is paid at least as much by group Y as by group X. (Note, in particular, that a legislator offered the same amount by each group votes for Y.) Each group's payoff is its value for the bill that is passed minus the payments it makes.]

In the variant considered in this question, a bill passes if and only if it receives at least k^* votes, where $k^* > \frac{1}{2}(k+1)$; if neither bill passes, a "default outcome" occurs. Both groups attach value 0 to the default outcome. Find the bill that is passed in any subgame perfect equilibrium when $k = 7$, $k^* = 5$, and

(a) [6] $V_X = V_Y = 700$

Solution: However group X allocates payments summing to 700, group Y can buy off five legislators for at most 500. Thus in any subgame perfect equilibrium neither group makes any payment, and bill Y passes.

(b) [6] $V_X = 750$, $V_Y = 400$.

Solution: If group X pays each legislator 80 then group Y is indifferent between buying off five legislators, in which case bill Y is passed, and in making no payments, in which case bill X is passed. If group Y makes no payments then X is selected, and group X is better off than it is if it makes no payments. There is no subgame perfect equilibrium in which group Y buys off five legislators, because if it were to do so group X could pay each legislator slightly more than 80 to ensure the passage of bill X. Thus in every subgame perfect equilibrium group X pays each legislator 80, group Y makes no payments, and bill X is passed.

(c) [6] For each of these cases, would the legislators be better off or worse off if a simple majority of votes were required to pass a bill?

Solution: If only a simple majority is required to pass a bill, in case *a* the outcome under majority rule is the same as it is when five votes are required. In case *b*, group X needs to pay each legislator 100 in order to prevent group Y from winning. If it does so, its total payments are less than V_X, so doing so is optimal. Thus in this case the payment to each legislator is *higher* under majority rule.

4. Consider the following variant of the bargaining game of alternating offers. The size of the pie is \$100. Neither player discounts future payoffs (i.e. both discount factors are equal to 1), but in any period that player 1 rejects an offer x she has to pay a penalty of $\$\frac{1}{2}x$ to a third party, and in any period that player 2 rejects an offer y she has to pay a penalty of $\$\frac{1}{2}y$ to a third party. Using the one deviation property, establish which, if any, of the following strategy profiles are subgame perfect equilibria of this game.

(a) [5] Player 1 always proposes $(50, 50)$ and accepts any offer $(x, 100 - x)$ in which $x \geq 50$, and player 2 always proposes $(50, 50)$ and accepts any offer $(100 - y, y)$ in which $y \geq 50$.

Solution: No. When rejecting an offer $(49, 51)$, player 1 receives a payoff of $50 - \frac{1}{2}49$, which is less than the payoff of 49 she receives if she accepts the offer. (The same is true for player 2.)

(b) [5] Player 1 always proposes $(60, 40)$ and accepts any offer $(x, 100 - x)$ in which $x \geq 40$, and player 2 always proposes $(40, 60)$ and accepts any offer $(100 - y, y)$ in which $y \geq 40$.

Solution: Yes. Consider Player 1. Any offer $(100 - y, y)$ with $y < 40$ will be rejected by Player 2 and would lead to a nex round offer $(40, 60)$ from player 2 which is acepted by player 1. By offering $(60, 40)$, player 1 does strictly better. Since player 2 accepts the offer $(60, 40)$, player 1 would be unnecessarily generous to offer player 2 more than 40.
After rejecting any offer offer $(x, 100 - x)$ player one offers $(60, 40)$ which is accepted by player 2. Hence, Player 1's payoff is $60 - \frac{1}{2}x$ where $\frac{1}{2}x$ is the penalty Player 1 must pay. This payoff is larger than the payoff from accepting the offer $(x, 100 - x)$ if and only if $x < 40$. Therefore, player 1 cannot improve by accepting any offer with $x < 40$ nor by rejecting any offer with $x \geq 40$.

(c) [5] Player 1 always proposes $(100, 0)$ and accepts any offer $(x, 100 - x)$ in which $x \geq 100$, and player 2 always proposes $(100, 0)$ and accepts any offer $(100 - y, y)$ in which $y \geq 0$.

Solution: No. When rejecting an offer $(99, 1)$ player one receive a payoff of $100 - \frac{1}{2}99$ which is strictly smaller than the payoff of 99 from accepting the offer.

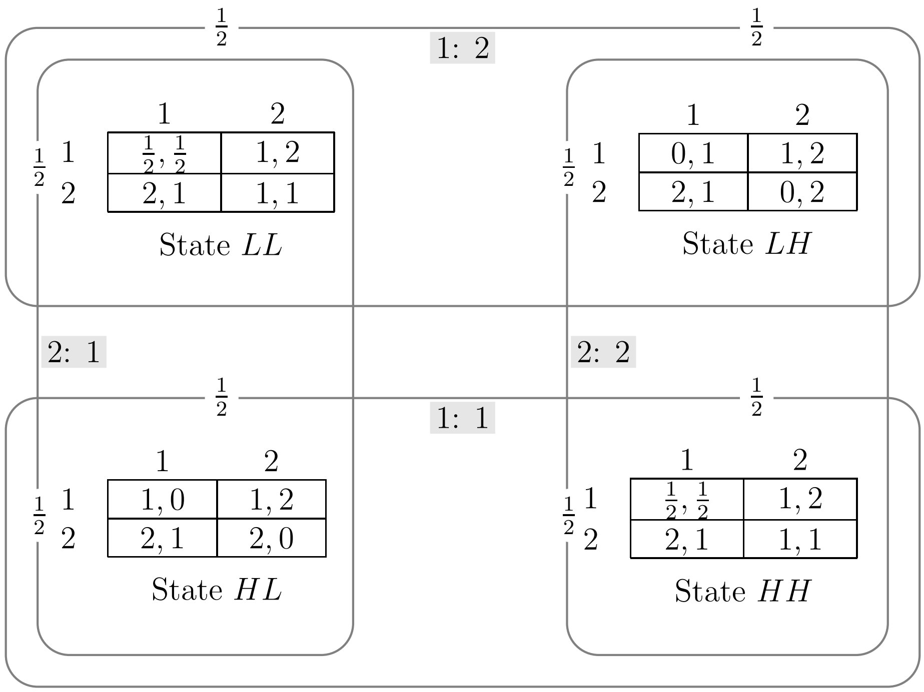
5. Two workers are deciding which of two firms to apply to for a job. The wage paid by firm 1 is 1 and the wage paid for firm 2 is 2. The productivity of each worker is either L or H, with $L < H$. Each worker can apply to only one firm and each firm hires exactly one worker. If both workers apply to the same firm, the firm chooses the one with the higher productivity or, if the applicants' productivities are the same, hires each with probability $\frac{1}{2}$. If the workers apply to different firms, each is hired by the firm to which she applies.

Each worker knows her own productivity, but not the productivity of the other worker. Each worker believes that the productivity of the other worker is L with probability $\frac{1}{2}$ and H with probability $\frac{1}{2}$. Each worker's payoff is the wage she receives, or 0 if she does not obtain a job.

(a) [8] Model this situation as a Bayesian game in which the players are the workers. (The firms have no decisions to make.) A diagram is sufficient.

Solution: The game is given in the following figure.

(b) [12] Find a Nash equilibrium of the Bayesian game in which each worker uses the same strategy and applies with positive probability to each firm if her type is L and with probability 1 to firm 2 if her type is H.



Solution: Denote the probability with which is worker of type L applies to firm 1 by p. Then the expected payoff of a worker of type L who applies to firm 1 is

$$\tfrac{1}{2}(\tfrac{1}{2}p + 1 - p) + \tfrac{1}{2}(1)$$

and the expected payoff of such a worker who applies to firm 2 is

$$\tfrac{1}{2}(2p + 1 - p) + \tfrac{1}{2}(0).$$

For these two expected payoffs to be the same, we need

$$\tfrac{1}{2}(\tfrac{1}{2}p + 1 - p) + \tfrac{1}{2} = \tfrac{1}{2}(2p + 1 - p)$$

or

$$p = \tfrac{2}{3}.$$

Now consider a worker of type H. Her expected payoff if she applies to firm 1 is

$$\tfrac{1}{2}(1) + \tfrac{1}{2}(1) = 1$$

and her expected payoff if she applies to firm 2 is

$$\tfrac{1}{2}(2) + \tfrac{1}{2}(1) = \tfrac{3}{2}.$$

Thus such a worker prefers to apply to firm 2.

We conclude that the Bayesian game has a Nash equilibrium in which each worker applies to firm 1 with probability $\frac{2}{3}$ if her type is L and to firm 2 with probability 1 if her type is H.

For students in Section L0101 (Osborne) only.
DO NOT ANSWER THIS QUESTION IF YOU ARE IN SECTION L5101 (Damiano)!

6. An indivisible object is to be sold in an auction. There are two potential buyers, who bid *sequentially* (not simultaneously). Bidder 1 has valuation v_1 and bidder 2 has valuation v_2, where $v_1 > v_2+1$ and v_1 and v_2 are nonnegative integers. A bid can be any nonnegative *integer*. First bidder 1 announces a bid. Then bidder 2 either announces a higher bid, or quits; if she announces a higher bid, then bidder 1 either announces a higher bid or quits; and so on until a bidder quits. The bidder who remains (does not quit) obtains the object and pays the price she bid.

   (a) [10] Find a subgame perfect equilibrium of the extensive game with perfect information that models this situation. Specify a complete *strategy* for each player and show that the strategy pair is a subgame perfect equilibrium. To show that the strategy pair is a subgame perfect equilibrium, you may use the fact that a strategy pair in the game is a subgame perfect equilibrium if and only if it satisfies the one deviation property (even though the game does not have a finite horizon).

   Solution: Fix an integer z with $0 \le z \le v_2$. The game has a subgame perfect equilibrium (s_1, s_2) in which

$$s_1(h) = \begin{cases} z & \text{at start of game and if previous bid } < z \\ x+1 & \text{if previous bid } = x \text{ where } z \le x < v_1 \\ \text{quit} & \text{if previous bid } \ge v_1 \end{cases}$$

and

$$s_2(h) = \begin{cases} z & \text{if previous bid } < z \\ \text{quit} & \text{if previous bid } \ge z. \end{cases}$$

The outcome of such a subgame perfect equilibrium is that player 1 wins and pays the price z.

To show that this strategy pair is a subgame perfect equilibrium, I argue that it satisfies the one deviation property. First consider player 1's action after a history ending in a bid of x.

- If $x < z$ then she gets 0 if she quits, at most $v_1 - z$ if she bids between x and z, gets $v_1 - z$ if she bids z, and gets less than $v_1 - z$ if she bids more than z. Thus bidding z is optimal, given player 2's strategy and the rest of her own strategy.
- If $z \le x < v_1$ then she gets 0 if she quits, $v_1 - (x+1)$ if she bids $x+1$, and gets less than $v_1 - (x+1)$ if she bids more than $x+1$.
- If $x \ge v_1$ then she gets 0 if she quits and a negative payoff if she bids.

Now consider player 2's action after a history ending in a bid of x.

- If $x < z$ then she gets 0 whatever her action.
- If $x \ge z$ then quitting yields 0 whereas any bid yields a payoff of at most 0.

Another subgame perfect equilibrium is similar, but the initial bid of player 1 is $v_2 - 1$, and player 2 quits if the previous bid is v_2 or greater.

(b) [5] Consider the variant of the game in which player 2 (rather than player 1) submits the first bid. Does this extensive game have a *Nash* equilibrium in which player 2 obtains the object? Either specify such an equilibrium or argue that no such equilibrium exists.

Solution: The game has such a Nash equilibrium. Suppose that player 2's strategy is to bid 0 initially, and to bid $v_1 + 1$ after every other history. Then a best response of player 1 is to quit after every history. Further, if player 1 quits after every history, player 2's strategy of bidding 0 initially and $v_1 + 1$ after every other history is optimal. In this equilibrium, player 2 obtains the good and pays the price 0.

For students in Section L5101 (Damiano) only.
DO NOT ANSWER THIS QUESTION IF YOU ARE IN SECTION L0101 (Osborne)!

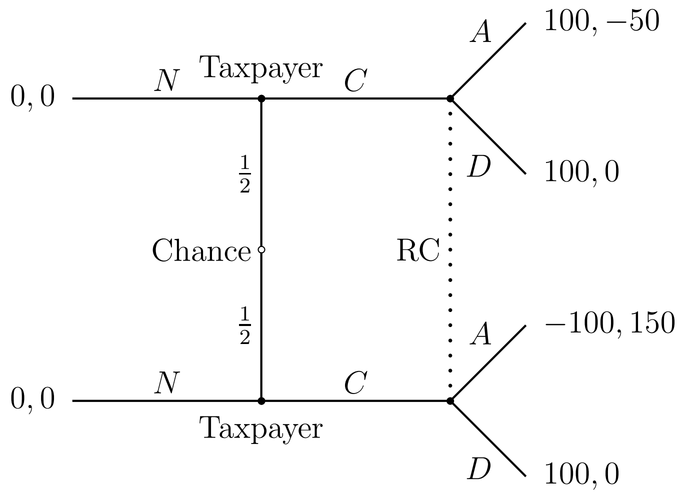
7. A taxpayer must decide whether or not to claim a \$100 deduction on her income tax. The taxpayer knows if she is eligible for the deduction whereas Revenue Canada knows only that there is a 50% chance the taxpayer is indeed eligible. If the taxpayer does not claim the deduction both the taxpayer and Revenue Canada receive a payoff of 0, regardless of whether the taxpayer was eligible for the deduction or not. If the taxpayer claims the deduction, Revenue Canada must decide whether to audit the taxpayer or reimburse the taxpayer without auditing her. If Revenue Canada decides not to audit the taxpayer the payoffs are \$100 to the taxpayer and 0 to Revenue Canada, irrespective of whether the taxpayer is eligible for the deduction or not. If Revenue Canada audits a taxpayer who legitimately claimed the deduction, the taxpayer still receives a payoff of \$100, while the payoff to Revenue Canada is −\$50, the cost of the audit. If Revenue Canada audits a taxpayer who was not eligible to take the deduction, the taxpayer's payoff is −\$100, while Revenue Canada's payoff is \$200 minus the cost of the audit (\$50).

(a) [5] Model this scenario as an extensive game with imperfect information.

Solution: See figure.

(b) [10] Find all weak sequential equilibria of the game.

Solution: First note for the eligible taxpayer, claiming the deduction strictly dominates not claiming it, so in any weak sequential equilibrium the eligible taxpayer claims. Next notice that there is no separating equilibrium. In any equilibrium in which the ineligible does not report, Revenue Canada believes that the taxpayer is eligible after observing a claim and hence does not audit the taxpayer. However, when Revenue Canada does not audit, the ineligible taxpayer prefers to claim the deduction.



Also there is no pooling equilibrium. In any equilibrium where the ineligible taxpayer claims the deduction, after observing a claim Revenue Canada believes that the claim is fraudulent with probability $\frac{1}{2}$, in which case it prefers auditing to not auditing. Hence in such an equilibrium, Revenue Canada always audits all claims. However, if Revenue Canada always audits, than the ineligible taxpayer prefers not to report the deduction.

The only candidate for an equilibrium is a semi-pooling equilibrium in which the ineligible taxpayer reports the claim with a probability strictly between 0 and 1. In any such equilibrium, the ineligible taxpayer must be indifferent between reporting and not reporting the deduction, which happens only when Revenue Canada audits with probability exactly $\frac{1}{2}$. Since Revenue Canada must randomize between auditing and not auditing, it must be indifferent between the two actions, which happens only when it believes that the claim is legitimate with probability $\frac{3}{4}$. In equilibrium, Revenue Canada's belief that the claim is legitimate must be equal to $\frac{1}{2}/(\frac{1}{2}+\frac{1}{2}p)$, where p is the equilibrium probability that the ineligible taxpayer claims the deduction. The probability p must be equal to $\frac{1}{3}$ in any weak sequential equilibrium. Hence there is exactly one weak sequential equilibrium; in the equilibrium, the eligible taxpayer always claims the deduction, the ineligible taxpayer makes the claim with probability $\frac{1}{3}$, Revenue Canada believes that the claim is legitimate with probability $\frac{3}{4}$ and audits the claim with probability $\frac{1}{2}$.

End of examination
Total pages: 9
Total marks: 100