Solutions to Practice Final

Q1 a) $\frac{\partial y^*}{\partial n} \cdot \frac{n}{y^*} = f'(k^*) \cdot \frac{\partial k^*}{\partial n} \frac{n}{f(k^*)} = \frac{f'(k^*)}{f(k^*)} \cdot k^* \left(\frac{\partial k^*}{\partial n} \cdot \frac{n}{k^*} \right)$

where k^* is defined as

$$sf(k^*) = (n+g+\delta) k^*$$

$$\Rightarrow \quad sf'(k^*) \cdot \frac{\partial k^*}{\partial n} = k^* + (n+g+\delta) \frac{\partial k^*}{\partial n}$$

$$\Rightarrow \quad \frac{\partial k^*}{\partial n} = \frac{k^*}{sf'(k^*) - (n+g+\delta)} < 0$$

$$\frac{\partial y^*}{\partial n} \cdot \frac{n}{y^*} = \underbrace{\frac{f'(k^*)}{f(k^*)} \cdot k^*} \left[\frac{k^*}{sf'(k^*) - (n+g+\delta)} \cdot \frac{n}{k^*} \right] = \alpha_K \left[\frac{n}{sf'(k^*) - (n+g+\delta)} \right]$$

$$= \alpha_K \cdot \frac{n}{\underbrace{\frac{(n+g+\delta)k^*}{f(k^*)}}_{=s} f'(k^*) - (n+g+\delta)} = \alpha_K \cdot \frac{n}{(n+g+\delta)(\alpha_K - 1)},$$

α_K is the elasticity of output per unit of effective ~~labour~~ capital

$$\alpha_K = \frac{f'(k^*)\,k^*}{f(k^*)}$$

b) $\frac{\partial y^*}{\partial n} \cdot \frac{n}{y^*} = -\frac{1}{2}$

Δn 2% $\rightarrow$ 1%

$\Delta\% n$ is -50% $\Rightarrow$ $\Delta\%$ in y^*

$-50\% * (-\frac{1}{2}) = 25\%$

$n = .02$, $g = .01$, $\delta = .01$

The time it takes for k & y to move $\frac{1}{2}$ way to the balanced growth path is given by $t^* = \frac{-\ln(.5)}{\lambda}$ where $\lambda = (1-\alpha_K)(n+g+\delta)$

$= \frac{\alpha_K}{\alpha_K - 1} \frac{n}{n+g+\delta}$

α_K does not change since production function doesn't change.

if $n = .01$, $n+g+\delta = .03$

$$-\frac{1}{2} = \frac{\alpha_K}{\alpha_K - 1} \cdot \frac{0.01}{0.03} \Rightarrow \frac{\alpha_K}{1-\alpha_K} = \frac{3}{2} \Rightarrow \alpha_K = \frac{3}{5}$$

$$\lambda = (1 - \tfrac{3}{5}) \cdot 0.03 = \frac{0.06}{5}$$

if $n = .02$, $n+g+\delta = .04$

$$-\frac{1}{2} = \frac{\alpha_K}{\alpha_K - 1} \cdot \frac{0.02}{0.04} \Rightarrow \frac{\alpha_K}{\alpha_K - 1} = -1 \Rightarrow \alpha_K = \frac{1}{2}$$

$$\lambda = (1 - \tfrac{1}{2}) \cdot 0.04 = \frac{0.04}{2}$$

① c) $\frac{\dot{K}}{K} - \frac{\dot{L}}{L} = 4\%$, $\frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} = 8\%$

$$R = \left(\frac{\dot{Y}}{Y} - \frac{\dot{L}}{L}\right) - \alpha_K \left(\frac{\dot{K}}{K} - \frac{\dot{L}}{L}\right)$$

$$= 8\% - \tfrac{3}{5} \times 4\% = 5.6\% \;/\; = 8\% - \tfrac{1}{2} \times 4\% = 6\%$$

Q2. a) $\max_{\{N_t, M_{t+1}, C_t, L_t\}} E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$ s.t. $P_tC_t \le M_t - N_t + W_tL_t$

$M_{t+1} = M_t - N_t + W_tL_t - P_tC_t + R_tN_t + r_tK_t + \Pi_t^f + \Pi_t^{fi}$

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \{ \beta^t U(C_t, L_t) + \lambda_t^1 (M_t - N_t + W_tL_t - P_tC_t) + \lambda_t^2 (M_t - N_t + W_tL_t - P_tC_t + R_tN_t + r_tK_t + \Pi_t^f + \Pi_t^{fi} - M_{t+1}) \}$$

$$\frac{\partial \mathcal{L}}{\partial C_t} = E_t \{ \beta^t U_C(C_t, L_t) - P_t\lambda_t^1 - P_t\lambda_t^2 \} = 0 \qquad ①$$

$$\frac{\partial \mathcal{L}}{\partial L_t} = E_t \{ \beta^t U_L(C_t, L_t) + w_t\lambda_t^1 + w_t\lambda_t^2 \} = 0 \qquad ②$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = E_{t-1} \{ -\lambda_t^1 - \lambda_t^2 + R_t\lambda_t^2 \} = 0 \qquad ③$$

$$\frac{\partial \mathcal{L}}{\partial M_{t+1}} = E_t \{ -\lambda_t^2 + \lambda_{t+1}^1 + \lambda_{t+1}^2 \} = 0 \qquad ④$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t^1} = 0 \ \& \ \frac{\partial \mathcal{L}}{\partial \lambda_t^2} = 0$$

use ① & ② to get $(\lambda_t^1 + \lambda_t^2) = \dfrac{\beta^t U_C(C_t, L_t)}{P_t}$

and $\beta^t U_L(C_t, L_t) + w_t(\lambda_t^1 + \lambda_t^2) = 0$

so $-\beta^t U_L(C_t, L_t) = \dfrac{w_t \beta^t U_C(C_t, L_t)}{P_t}$

④ $\Rightarrow \lambda_t^2 = E_t(\lambda_{t+1}^1 + \lambda_{t+1}^2)$

So ③ can be written as $E_{t-1} \{ -(\lambda_t^1 + \lambda_t^2) + R_t E_t(\lambda_{t+1}^1 + \lambda_{t+1}^2) \} = 0$

or $E_{t-1} \left\{ \dfrac{\beta^t U_C(C_t, L_t)}{P_t} - R_t \beta^{t+1} \dfrac{U_C(C_{t+1}, L_{t+1})}{P_{t+1}} \right\} = 0$

& $M_{t+1} = R_tN_t + r_tK + \Pi_t^f + \Pi_t^{fi}$

& $P_tC_t = M_t - N_t - W_tL_t$

b) $\max\limits_{\{K_t, H_t\}} E_0 \sum\limits_{t=0}^{\infty} \beta^t \frac{U_{c,t+1}}{P_{t+1}} (P_t \theta_t K_t^{1-\alpha} H_t^{\alpha} - W_t H_t R_t - r_t K_t)$

this reduces to the period-by period problem

$\max\limits_{\{K_t, H_t\}} P_t \theta_t K_t^{1-\alpha} H_t^{\alpha} - W_t H_t R_t - r_t K_t$ capital goods : numeraire

FONC:

$$\frac{\partial \Pi_t^f}{\partial K_t} = (1-\alpha) P_t \theta_t K_t^{-\alpha} H_t^{\alpha} - r_t = 0 \quad \text{or } (1-\alpha) \frac{Y_t}{K_t} = \frac{r_t}{P_t}$$

$$\frac{\partial \Pi_t^f}{\partial H_t} = \alpha P_t \theta_t K_t^{1-\alpha} H_t^{\alpha-1} - W_t R_t = 0 \quad \text{or } \alpha \frac{Y_t}{H_t} = \frac{W_t R_t}{P_t} \quad (B)$$

c) A sequential market eq'bm is a set of prices $\{R_t, r_t, P_t, W_t\}$ and a set of quantities $\{N_t, M_{t+1}, H_t, C_t, Y_t, L_t, K_t, \Pi_t^f, \Pi_t^{fi}\}$ such that given the initial conditions and the processes $\{\varepsilon_t, X_t\}$

(1) The prices & quantities solves the HH's problem
(2) The prices & quantities slove the firm's problem
(3) The loan mkt clears $W_t H_t = N_t + X_t$
(4) labour mkt clears $H_t = L_t$
(5) capital mkt clears $K = K_t$
(6) resource constraint is satisfied $C_t = Y_t$

from (B)

d) $$R_t = \frac{\alpha P_t Y_t}{H_t W_t} = \frac{\alpha P_t C_t}{H_t W_t} = \frac{\alpha (M_t + X_t)}{(N_t + X_t)}$$

$$P_t C_t = M_t - N_t + W_t L_t = M_t - N_t + W_t H_t = M_t + \underbrace{(W_t H_t - N_t)}_{= X_t} = M_t + X_t$$

$$\frac{\partial R_t}{\partial X_t} = \frac{\alpha[(N_t + X_t) - (M_t + X_t)]}{(N_t + X_t)^2} = \frac{\alpha (N_t - M_t)}{(N_t + X_t)^2} < 0$$

N_t, use HH FOC: $U_c = \frac{1}{C_t - \frac{\varphi_0}{1+\varphi} L_t^{1+\varphi}}$, $U_L = \frac{1}{C_t - \frac{\varphi_0}{1+\varphi} L_t^{1+\varphi}} (-\varphi_0 L_t^{\varphi})$

The intertemporal trade-off is given by $\frac{\varphi_0 L_t^{\varphi}}{(C_t - \frac{\varphi_0}{1+\varphi} L_t^{1+\varphi})} = \frac{W_t}{P_t} \frac{1}{(C_t - \frac{\varphi_0}{1+\varphi} L_t^{1+\varphi})}$

$$\Rightarrow \varphi_0 L_t^{\varphi} = \frac{W_t}{P_t}$$

also $\alpha P_t Y_t = w_t H_t R_t = w_t L_t R_t$

$$\Rightarrow \alpha\left(\frac{Y_t}{L_t}\right)\frac{1}{R_t} = \frac{w_t}{P_t}$$

combing these we get

$$\varphi_0 L_t^{\varphi} = \alpha\left(\frac{Y_t}{L_t}\right)\left(\frac{1}{R_t}\right) = \alpha \theta_t K^{1-\alpha} L_t^{-1+\alpha} \cdot \frac{1}{R_t}$$

$$\Rightarrow L_t^{\varphi - \alpha + 1} = \left(\frac{\alpha}{\varphi_0}\right)\theta_t K^{1-\alpha}\frac{1}{R_t} \quad \Rightarrow \quad L_t = \left[\left(\frac{\alpha}{\varphi_0}\right)\theta_t K^{1-\alpha}\frac{1}{R_t}\right]^{\frac{1}{\varphi - \alpha + 1}}$$

$$\frac{\partial L_t}{\partial X_t} = \frac{1}{\varphi - \alpha + 1}\left[\frac{\alpha}{\varphi_0}\theta_t K^{1-\alpha}\frac{1}{R_t}\right]^{\frac{1}{\varphi - \alpha+1}-1} \cdot \frac{\alpha}{\varphi_0}\cdot \theta_t K^{1-\alpha}\left(-\frac{1}{R_t^2}\right)\cdot\frac{\partial R_t}{\partial X_t}$$

since $\frac{\partial R_t}{\partial X_t} < 0 \quad \Rightarrow \frac{\partial L_t}{\partial X_t} > 0$ $\quad \alpha \in (0,1) \quad 1-\alpha > 0$

break into cases. $\quad \varphi - \alpha + 1 > 0$ ✓✓ $\quad \leq 0 \qquad \varphi$? $\quad$ prove $\varphi > 0$.

has to be predefined.

e) Since the money supply doesn't change in response to ε_t shock

$$\frac{\partial R_t}{\partial \varepsilon_t} = 0$$

cuz elasticity $\frac{1}{\varphi}$.

$$\Rightarrow \frac{\partial L_t}{\partial \varepsilon_t} = \frac{\partial L_t}{\partial \theta_t} = \frac{1}{\varphi+\alpha}\theta_t^{\left(\frac{1}{\varphi+\alpha}-1\right)}\left(\frac{\alpha}{\varphi_0}K^{\alpha}\frac{1}{R_t}\right)^{\frac{1}{1+\varphi}} > 0$$

so labour ↑ when positive tech shock.

Q3 Note 1 person per household, no growth

This is non-stochastic process, no uncertainty $\Rightarrow$ no E_0

Aggregate constraint:

$$K_t^{\alpha}(Ln_t)^{1-\alpha} \geq C_tL + k_{t+1}L - Lk_t(1-\delta) + g_tL \quad \text{where } K_t = k_tL$$

$$\Rightarrow k_t^{\alpha} \cdot n_t^{1-\alpha} \geq c_t + k_{t+1} - k_t(1-\delta) + g_t$$

Social planner problem can either be written as

$$\max_{\{k_{t+1}, C_t, n_t\}} \sum_{t=0}^{\infty} \beta^t u(C_t, n_t)L$$

$$\text{s.t. } K_t^{\alpha}(Ln_t)^{1-\alpha} \geq c_tL + K_{t+1} - K_t(1-\delta) + g_tL$$

or they can do it in the per capita terms as well

$$\max_{\{k_{t+1}, C_t, n_t\}} \sum_{t=0}^{\infty} \beta^t u(C_t, n_t) \quad \text{s.t. } k_t^{\alpha}(n_t)^{1-\alpha} \geq C_t + k_{t+1} - (1-\delta)k_t + g_t$$

Use the 1st.

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(C_t, n_t)L + \lambda_t[K_t^{\alpha}(Ln)^{1-\alpha} - K_{t+1} + (1-\delta)K_t - C_tL - g_tL]$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = -\lambda_t + \lambda_{t+1}(K_{t+1}^{\alpha-1} \cdot \alpha \cdot (Ln_{t+1})^{1-\alpha} + (1-\delta)) = 0 \quad ①$$

$$\frac{\partial \mathcal{L}}{\partial C_t} = \beta^t u_c(C_t, n_t)L - \lambda_t L = 0 \quad ②$$

$$\frac{\partial \mathcal{L}}{\partial n_t} = \beta^t u_n(C_t, n_t)L + \lambda_t K_t^{\alpha} \cdot (Ln)^{\alpha} \cdot L(1-\alpha) = 0 \quad ③$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = K_t^{\alpha}(n_tL)^{1-\alpha} - K_{t+1} + (1-\delta)K_t - C_tL - g_tL = 0 \quad ④$$

So $\beta^t u_c(C_t, n_t) = \lambda_t$ from ②

Subbing into ① we get

$$\beta^t u_c(C_t, n_t) = \beta^{t+1} u_c(C_{t+1}, n_{t+1})\left(\alpha \frac{Y_{t+1}}{K_{t+1}} + 1 - \delta\right) \quad Ⓐ$$

② & ③ combine to get

$$③ \quad -u_n(C_t, n_t) = \left(\frac{Y_t}{Ln_t}\right)(1-\alpha)\, u_c(C_t, n_t)$$

& ④ $Y_t = C_tL + g_tL + K_{t+1} - (1-\delta)K_t$

$$U_c(C_t, n_t) = \mu C_t^{\mu(1-\sigma)-1}(1-n_t)^{(1-\mu)(1-\sigma)}$$
$$U_n(C_t, n_t) = C_t^{\mu(1-\sigma)}(1-n_t)^{(1-\mu)(1-\sigma)-1}(1-\mu)(-1)$$

In steady state $g_t = g$ (steady state level)

$C_t = C_{t+1} = C$, $Y_t = Y_{t+1} = Y$, $n_t = n_{t-1} = n$

$K_t = K_{t+1} = K$

Start with the saving Euler eq (A)

$$\cancel{\mu C^{\mu(1-\sigma)-1}}\ \cancel{(1-n)^{(1-\mu)(1-\sigma)}} = \beta \cancel{\mu C^{\mu(1-\sigma)-1}}\ \cancel{(1-n)^{(1-\mu)(1-\sigma)}}\left(\alpha\frac{Y}{K}+1-\delta\right)$$

$$\Rightarrow \left[\frac{1}{\beta}-(1-\delta)\right] = \alpha\frac{Y}{K} = \alpha\left(\frac{K}{Ln}\right)^{\alpha-1}$$

$$\Rightarrow \frac{K}{Ln} = \left[\frac{\frac{1}{\beta}-(1-\delta)}{\alpha}\right]^{\frac{1}{\alpha-1}} \qquad (A) \qquad U_{Ct} \leftrightarrow U_{C,t+1}$$

$$\Rightarrow \frac{Y}{K} = \frac{K^\alpha(Ln)^{1-\alpha}}{K} = \left(\frac{Ln}{K}\right)^{1-\alpha} = \frac{\frac{1}{\beta}-(1-\delta)}{\alpha}$$

in steady state we also have

$$Y = cL + gL + \delta K \quad \Rightarrow \frac{Y}{K} = \frac{cL}{K} + \frac{gL}{K} + \delta \qquad (B)$$

and the labour - consumption trade off eq'n is given by

$$(1-\alpha)\left(\frac{Y}{nL}\right)\mu C^{\mu(1-\sigma)-1}(1-n)^{(1-\mu)(1-\sigma)} = (1-\mu)C^{\mu(1-\sigma)}(1-n)^{(1-\mu)(1-\sigma)-1}$$

$$\Rightarrow (1-\alpha)\left(\frac{Y}{nL}\right) = \left(\frac{1-\mu}{\mu}\right)\left(\frac{c}{1-n}\right) \qquad U_{Ct} \leftrightarrow U_{nt}$$

Notice we can write this as

$$(1-\alpha)\left(\frac{Y}{K}\right)\left(\frac{1}{nL}\right) = \left(\frac{1-\mu}{\mu}\right)\left(\frac{c}{K}\right)\cdot\left(\frac{1}{1-n}\right)$$

$$\Rightarrow \frac{c}{K} = (1-\alpha)\frac{Y}{K}\cdot\left(\frac{1}{nL}\right)\left(\frac{\mu}{1-\mu}\right)(1-n) \qquad (C)$$

sub into (B) to get

$$\frac{Y}{K} = (1-\alpha)\frac{Y}{K}\left(\frac{1}{n\cancel{L}}\right)\left(\frac{\mu}{1-\mu}\right)(1-n)\cancel{L} + \frac{gL}{K} + \delta$$

(so is $\left(\frac{K}{Ln}\right)$)

also notice that this can be solved for n given that we know $\frac{Y}{K}$ is only a function of parameters, g is given, L is given, δ is given, and $\frac{gL}{K} = g\left(\frac{Ln}{K}\right)\frac{1}{n}$

$$\Rightarrow \left(\frac{Y}{K}\right)n = (1-\alpha)\frac{Y}{K}\left(\frac{\mu}{1-\mu}\right)(1-n) + g\left(\frac{Ln}{K}\right) + \delta n$$

$$\Rightarrow n = \frac{(1-\alpha)\frac{Y}{K}\frac{\mu}{1-\mu} + g\left(\frac{Ln}{K}\right)}{\frac{Y}{K} + (1-\alpha)\left(\frac{Y}{K}\right)\left(\frac{\mu}{1-\mu}\right) - \delta}$$

then given n, we find
$$K = \left(\frac{K}{Ln}\right)\cdot L \cdot n$$
$$Y = K^{\alpha}\cdot(Ln)^{1-\alpha}$$
$$C = (1-\alpha)Y\left(\frac{\mu}{1-\mu}\right)\left(\frac{1-n}{n}\right)\frac{1}{L}$$

all are a function of parameters

How does $\frac{Y}{Ln}$ vary with g?

$$\frac{Y}{Ln} = \left(\frac{K}{Ln}\right)^{\alpha} = \left(\frac{\frac{1}{\beta} - (1-\delta)}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}$$ so it does vary with g.

The elasticity of n wrt g is

$$\frac{\partial n}{\partial g}\cdot\frac{g}{n} = \left[\frac{\frac{Ln}{K}}{\left[\frac{Y}{K} + (1-\alpha)\left(\frac{Y}{K}\right)\left(\frac{\mu}{1-\mu}\right) - \delta\right]}\right]\cdot\frac{g}{n} = \frac{\frac{Ln}{K}}{\frac{Y}{K} + (1-\alpha)\frac{Y}{K}\left(\frac{\mu}{1-\mu}\right) - \delta}\cdot\frac{g}{\left[\frac{(1-\alpha)\frac{Y}{K}\left(\frac{\mu}{1-\mu}\right) + g\left(\frac{Ln}{K}\right)}{\frac{Y}{K} + (1-\alpha)\frac{Y}{K}\left(\frac{\mu}{1-\mu}\right) - \delta}\right]}$$

$$= \frac{g\left(\frac{Ln}{K}\right)}{(1-\alpha)\frac{Y}{K}\left(\frac{\mu}{1-\mu}\right) + g\left(\frac{Ln}{K}\right)} < 1 \quad \text{since} \quad (1-\alpha)\frac{Y}{K}\left(\frac{\mu}{1-\mu}\right) > 0$$

Q4. a) $U=\int_{t=0}^{\infty} e^{-\rho t} \frac{c(t)^{1-\theta}}{1-\theta} \frac{L(t)}{H}\, dt$

$$= \int_{t=0}^{\infty} e^{-\rho t} \frac{(A(t)c(t))^{1-\theta}}{1-\theta} \frac{L(0)e^{nt}}{H} dt$$

$$= \int_{t=0}^{\infty} e^{-\rho t} A(0)^{1-\theta} e^{g(1-\theta)t} \frac{c(t)^{1-\theta}}{1-\theta} \cdot \frac{L(0)e^{nt}}{H} dt$$

$$= B\int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt$$

where $B = \frac{A(0)^{1-\theta}}{H} L(0)$ & $\beta = \rho - (1-\theta)g - n$

b) $\mathcal{L} = B\int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt + \lambda\Big[k(0) + \int_{t=0}^{\infty} e^{-R(t)} w(t) e^{(n+g)t} dt$

$- \int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t} dt\Big]$

$$\frac{\partial \mathcal{L}}{\partial c(t)} = Be^{-\beta t} c(t)^{-\theta} - e^{-R(t)} \lambda e^{(n+g)t} = 0 \implies \frac{\dot{c}}{c} = \frac{r(t)-\rho-\theta g}{\theta}$$

$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \implies$ budget constraint at equality

$\Pi = K^{\alpha}(AL)^{1-\alpha} - rK - WAL$

$K(t)$: $\alpha K^{\alpha-1}(AL)^{1-\alpha} - r = 0 \implies r = \alpha\left(\frac{K}{AL}\right)^{\alpha-1} = \alpha k^{\alpha-1}$

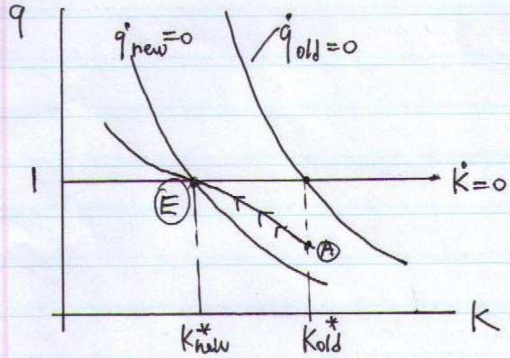
$L(t)$: $(1-\alpha)\, K(t)^{\alpha} A(t)^{1-\alpha} L(t)^{-\alpha} - w(t)A(t) = 0$

$$(1-\alpha)\left(\frac{K(t)}{A(t)L(t)}\right)^{\alpha} = w(t) \implies (1-\alpha)\, k(t)^{\alpha} = w(t)$$

Q5. permenant increase in r

$\dot{q}(t) = r q(t) - \pi(K(t)) \Rightarrow \dot{q} = 0 \Rightarrow q(t) = \dfrac{\pi(K(t))}{r}$

& $\dot{K}(t) = f(q(t)) = N C'^{-1}(q-1)$ so $\dot{K} = 0 \Rightarrow q = 1$



A permanent increase in r shifts the $\dot{q} = 0$ curve down (and changes the slope)

A change in r has no impact on $\dot{K} = 0$ as immediately falling the stick q decreases to point Ⓐ on the new saddle path. There after K decreases and q increases so the economy moves along the saddle path to a new steady state at Ⓔ with lower K and $q = 1$.

⑤