MFE Midterm Solutions

Note Title 2/28/2011

1. (a) Constant returns to Scale implies

$$c\,F(K(t), A(t)L(t)) = F(c\,K(t),\ c\,A(t)L(t)) \quad \text{for all } c \geq 0$$

$$c\,F(K(t), A(t)L(t)) = c\,\alpha_3\left[K(t)^{\alpha_1}(A(t)L(t))^{\alpha_2}\right] + c\,\alpha_4$$

$$F(c\,K(t),\ c\,A(t)L(t)) = \alpha_3\left[(c\,K(t))^{\alpha_1}(c\,A(t)L(t))^{\alpha_2}\right] + \alpha_4$$

$$= \alpha_3\, c^{\alpha_1+\alpha_2}\left(K(t)^{\alpha_1}(A(t)L(t))^{\alpha_2}\right) + \alpha_4$$

2 pts for knowing def of CRTS

for $c\,F(K(t), A(t)L(t)) = F(c\,K(t),\ c\,A(t)L(t)) \quad \forall c$

$\alpha_4 = 0$, $\alpha_1 + \alpha_2 = 1$, no restriction on α_3

2 pts 2 pts 2 pts

(b) Intensive form of production function

$$y(t) = \frac{Y(t)}{A(t)L(t)} = B\frac{K(t)}{A(t)L(t)} + \frac{K(t)^{\alpha}(A(t)L(t))^{1-\alpha}}{A(t)L(t)} = Bk(t) + k(t)^{\alpha}$$

6 pts

(c)

(i) positive marginal product of k $\Rightarrow$ $f'(k(t)) > 0$

$$f'(k(t)) = B + \alpha k(t)^{\alpha - 1}$$

so $B \geq 0$ and $\alpha > 0$ ensures $f'(k(t)) > 0$

4 pts $\quad$ when $k_{t+1} > 0$

(ii) diminishing marginal returns $\Rightarrow$ $f''(k_{t+1}) < 0$

$f''(k_{t+1}) = \alpha(\alpha-1)\, k_{t+1}^{\alpha-2}$ $\quad$ so $\quad$ $\alpha \in (0,1)$ ensures $f''(k_{t+1}) < 0$ when $k_{t+1} > 0$

4 pts

iii) $\lim_{k \to 0} f'(k) = \infty$ and $\lim_{k \to \infty} f'(k) = 0$ are the Inada conditions

$\lim_{k \to 0} \left(B + \alpha k^{\alpha-1}\right) = \infty$ if $0 < \alpha < 1$ $\quad$ 4 pts

$\lim_{k \to \infty} \left(B + \alpha k^{\alpha-1}\right) = 0$ if $B = 0$ and $0 < \alpha < 1$

4 pts

Question 2

(a) $$U = \int_{t=0}^{\infty} e^{-\rho t} \frac{(c(t)A(t))^{1-\theta}}{1-\theta} \frac{L(t)}{H} dt = \int_{t=0}^{\infty} e^{-\rho t} \frac{c(t)^{1-\theta}}{1-\theta} \left(A(0)e^{gt}\right)^{1-\theta} \frac{L(0)e^{nt}}{H} dt$$

$$= B \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt$$ where $\beta = \rho - n - (1-\theta)g$

and $B = A(0)^{1-\theta} \frac{L(0)}{H}$

4 pts for proper derivation

(b) need $\rho - n - (1-\theta)g > 0$ so the problem is bounded.

(otherwise lifetime utility diverges)

(c) The lifetime budget constraint is

$$\int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t}\, dt \leq k(0) + \int_{t=0}^{\infty} e^{-R(t)} w(t) e^{(n+g)t}\, dt \qquad \text{where } R(t) = \int_{\tau=0}^{t} r(\tau)\, d\tau$$

3pts

$$\mathcal{L} = B \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta}\, dt + \lambda \left[k(0) + \int_{t=0}^{\infty} e^{-R(t)} w(t) e^{(n+g)t}\, dt - \int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t}\, dt \right]$$

2pts

$$\frac{\partial \mathcal{L}}{\partial c(t)} = B e^{-\beta t} c(t)^{-\theta} - \lambda e^{-R(t)} e^{(n+g)t} = 0 \quad \text{for all } t$$ 2pts

$$\frac{\partial \mathcal{L}}{\partial \lambda} = k(0) + \int_{t=0}^{\infty} e^{-R(t)} w(t) e^{(n+g)t} dt - \int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t} dt = 0$$ 2pts

from $\frac{\partial \mathcal{L}}{\partial c(t)} = 0$ we have

$$\ln B - \beta t - \theta \ln c(t) = \ln \lambda - R(t) + (n+g)t.$$

Since this holds for all t we can take the derivative with respect to t which gives

$$-\beta - \theta \frac{\dot{c}(t)}{c(t)} = -r(t) + (n+g)$$

so $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{1}{\theta}\left(r(t) - \rho - \theta g\right)$

3pts for final derivation

(d) $\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t)$

this says output per unit of effective labour = consumption per unit of effective labour + investment per unit of effective labour.

(this is the resource constraint)

4 pts

(e) k^* is given by $\dot{c} = 0$ $\quad f'(k^*) = \rho + \theta g$

y^* is given by $\quad y^* = f(k^*)$

c^* is given by $\quad c^* = f(k^*) - (n+g)\,k^* \quad (\dot{k} = 0 + k^* \text{ value})$

k^G is given by $\quad f'(k^G) = (n+g)$

2pts each

(f) from $\dot{c} = 0$ curve we get $\quad f''(k^*)\dfrac{\partial k^*}{\partial n} = 0 \quad$ so $\quad \dfrac{\partial k^*}{\partial n} = 0$

since $\quad y^* = f(k^*) \qquad \dfrac{\partial y^*}{\partial n} = f'(k^*)\dfrac{\partial k^*}{\partial n} = 0$

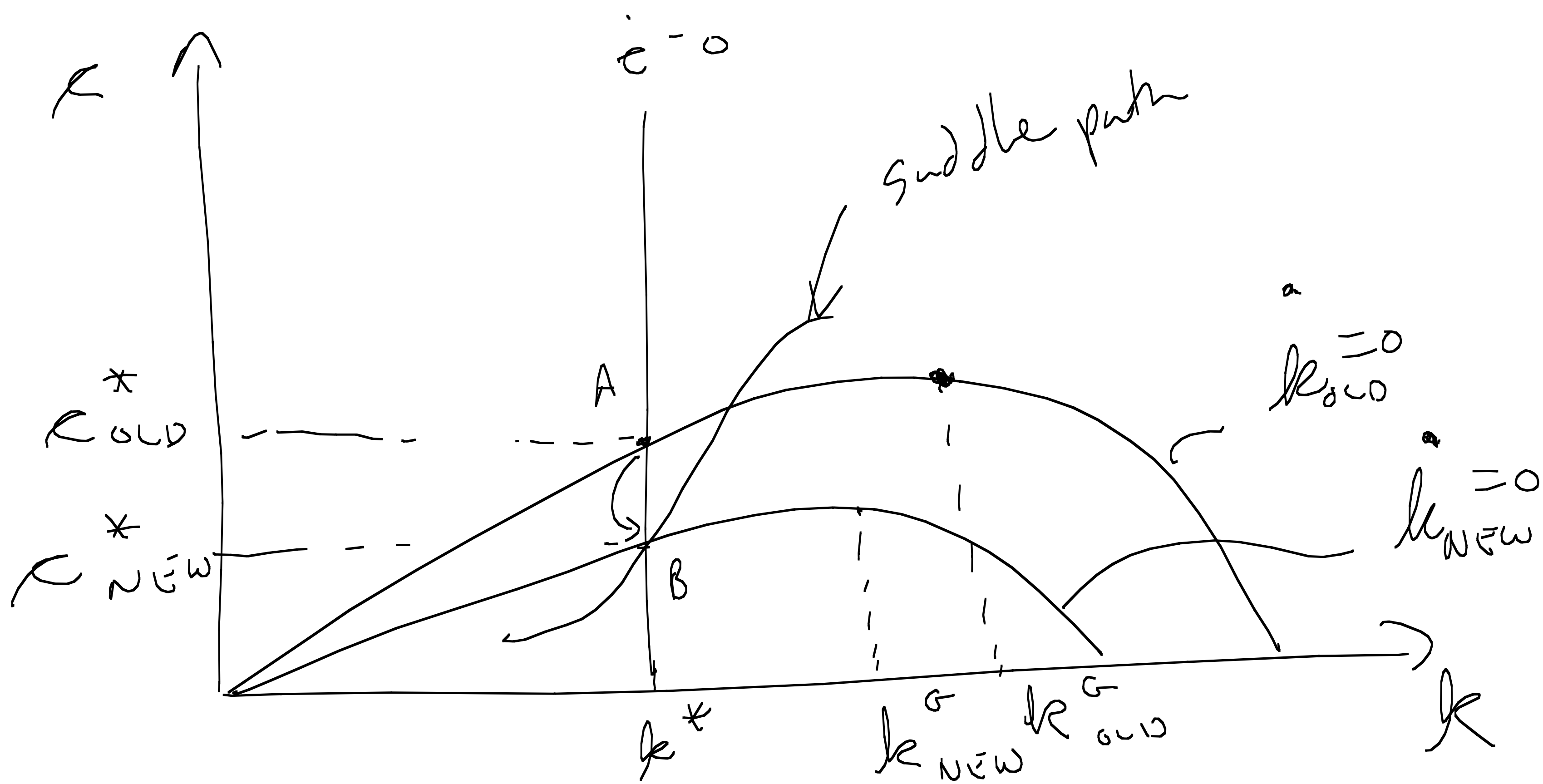
$c^* = f(k^*) - (n+g)k^* \quad$ so $\quad \dfrac{\partial c^*}{\partial n} = \left(f'(k^*) - (n+g)\right)\dfrac{\partial k^*}{\partial n} - k^* = -k^* < 0$

since $\quad k^G$ given by $\quad f'(k^G) = n+g \quad$ so $\quad f''(k^G)\dfrac{\partial k^G}{\partial n} = 1$

we have $\quad \dfrac{\partial k^G}{\partial n} = \dfrac{1}{f''(k^G)} < 0$

3 pts each $\times$ 4 = 12 total

(g)



At the time of the change in n the economy jumps from point A to point B and is immediately on its new balanced growth path. Therefore r falls from r^*_{OLD} to r^*_{NEW} immediately, k remains at k^* (and y at $y^* = f(k^*)$). As shown above k^G shifts down.

2 pts for well labelled graph

3 pts for proper movement in k, c and k^G

h) Constant returns to scale implies we can write the firms maximization problem as

$$\max_{K(t),\, L(t)} \; A(t)L(t)f(k(t)) - r(t)K(t) - A(t)L(t)w(t)$$

where $k(t) = \dfrac{K(t)}{A(t)L(t)}$

2 pts

F.O.N.C.

$K(t)$: $A(t)L(t)\,f'(k(t))\cdot\frac{1}{A(t)L(t)} - r(t) = 0$ 4 pts

$\Rightarrow$ $\boxed{f'(k(t)) = r(t)}$

$L(t)$: $A(t)\,f(k(t)) + A(t)L(t)\,f'(k(t))\left(\frac{-K(t)}{A(t)L(t)^2}\right)$ 4 pts

$- A(t)\,w(t) = 0$

$\Rightarrow$ $\boxed{f(k(t)) - k(t)\,f'(k(t)) = w(t)}$

Question 3

a) k^* is given by $s f(k^*) = (n+g+\delta) k^*$

as when $f(k(t)) = k(t)^\alpha$

$s(k^*)^\alpha = (n+g+\delta) k^* \Rightarrow k^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$

$y^* = f(k^*) = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$

3 pts each

$$c^* = (1-s)\,y^* = (1-s)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

b)

k^G given by $f'(k^G) = (n+g+\delta)$

as $\alpha\left(k^G\right)^{\alpha-1} = (n+g+\delta)$ or $k^G = \left(\frac{\alpha}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$

3pts

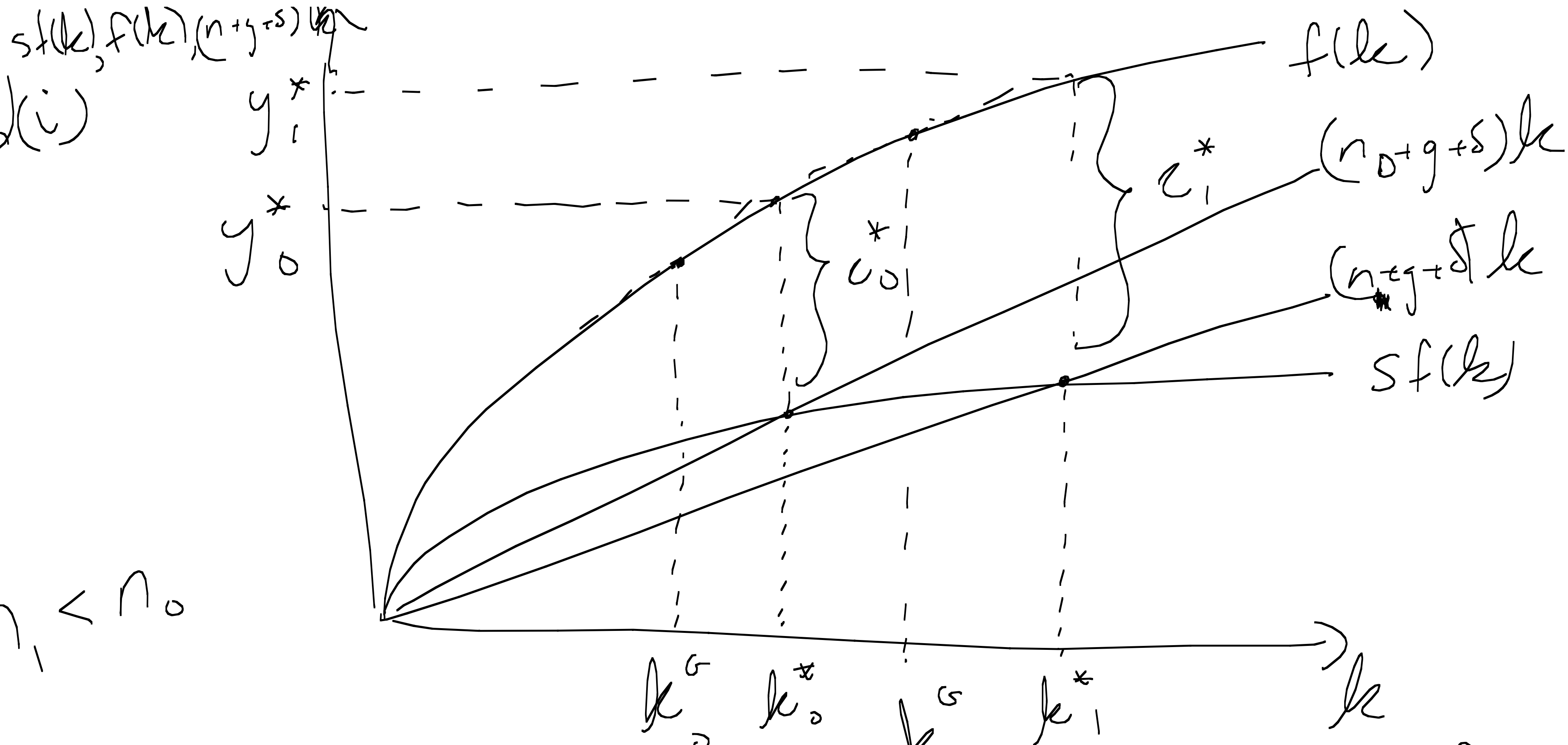
c) Since the economy converges to k^*

for $k^* = k^G$ we need

$$k^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} = \left(\frac{\alpha}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$$ or $\boxed{s = \alpha}$

<u>3 pts</u>

d(i)



$n_1 < n_0$

3 pts each proper movement w/ labels

marks deducted for unlabled axis

d(ii) $$\frac{\partial k^*}{\partial n} = s^{\frac{1}{1-\alpha}} \left(-\frac{1}{1-\alpha}\right) (n+g+\delta)^{-\frac{1}{1-\alpha}-1} < 0$$

$$\frac{\partial y^*}{\partial n} = \overbrace{f'(k^*)}^{+ve} \; \overbrace{\frac{\partial k^*}{\partial n}}^{-ve} < 0$$

$$\frac{\partial c^*}{\partial n} = (1-s) \frac{\partial y^*}{\partial n} < 0$$

$$\frac{\partial k^{*}}{\partial n} = \alpha^{\frac{1}{1-\alpha}} \left(\frac{-1}{1-\alpha}\right) (n+g+\delta)^{-1/1-\alpha - 1}$$

3 pts each.

(e) One time decrease in $L(t)$ implies

$k(t) = \frac{K(t)}{A(t)L(t)}$ increases above its steady state level, k^*. As such, at the time of the change, $y(t)$ increases and $c(t) = (1-s)\,y(t)$ increases since $k(t)$ rises. Thereafter $0 > \dot{k}(t) = s\,f(k(t)) - (n+g+\delta)k(t)$ since break even investment exceeds actual investment per unit of effective labour, and $c(t)$ and $y(t)$ decrease since $\dot{y}(t) = f'(k(t))\,\dot{k}(t) < 0$ and

$\dot{c}(t) = (1-s)\,f'(k(t))\,\dot{k}(t) < 0$

This occurs until $k(t)$ again reaches its steady state value k^*. when it does $c = c^*$ and $y(t) = y^*$ as before.

k^G is unaffected by the one time

decrease in $c(t)$ since it is given by

$$f'(k^G) = (n + g + \delta)$$

4 points each $* 4 = 16$ pts

Question 4

(a) $$\mathcal{L} = E\left\{ \sum_{t=0}^{T} \beta^t \left[c_t - \frac{a c_t^2}{2} \right] + \lambda_t \left[r_t k_t + y_t - k_{t+1} + k_t - c_t \right] \right\}$$

3 pts

$$\frac{\partial \mathcal{L}}{\partial C_t} = E_t\left\{ \beta^t (1 - aC_t) - \lambda_t \right\} = 0$$

where $1 - aC_t = u'(C_t)$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = E_t\left\{ -\lambda_t + \lambda_{t+1}(1 + r_{t+1}) \right\} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = r_t K_t + Y_t - K_{t+1} + K_t - C_t = 0$$

for all t and all states of the world

3 pts each.

from $\frac{\partial \mathcal{L}}{\partial C_t} = 0$ and $\frac{\partial \mathcal{L}}{\partial K_{t+1}} = 0$ we get

$$\beta^t u'(C_t) = E_t \left\{ \beta^{t+1} u'(C_{t+1})(1+r_{t+1}) \right\}$$

or

$$u'(C_t) = \beta E_t \left\{ u'(C_{t+1})(1+r_{t+1}) \right\}$$

$$= \beta \left\{ E_t(u'(c_{t+1}))\, E_t(1 + r_{t+1}) + \mathrm{Cov}_t(1 - a\, c_{t+1},\, 1 + r_{t+1}) \right\}$$

$$= \beta \left\{ E_t(u'(c_{t+1}))\, E_T(1 + r_{t+1}) - a\, \mathrm{Cov}_t(c_{t+1},\, 1 + r_{t+1}) \right\} \quad *$$

6 points for derivation

(b) let $\tilde{c}_t$ be current consumption

when $Cov_t(c_{t+1}, r_{t+1}) > 0$

and

let $\hat{c}_t$ be current consumption when

$Cov_t(c_{t+1}, r_{t+1}) < 0$

from (*) it follows

$$u'(\tilde{C}_t) < u'(\hat{C}_t)$$

Since $u''(C) < 0$ this implies

$$\tilde{C}_t > \hat{C}_t$$

15 pts for proper answer w/ justification

(c) Consumption follows a random walk when

$$C_t = E_t(C_{t+1})$$

5pts

or $C_{t+1} = C_t + e_t$ where e_t is white noise

Notice the first order conditions

give

$$1 - a C_t = \beta E_t \left((1 - a C_{t+1}) (1 + r_{t+1}) \right)$$

therefore Consumption will follow a random walk when

$$\frac{1}{\beta} = 1 + r_{t+1} \quad \text{for all } t$$

since then

$$1 - aC_t = \beta E_t\left(\left(1 - aC_{t+1}\right)\left(\frac{1}{\beta}\right)\right)$$

$$= E_t\left(1 - aC_{t+1}\right)$$

So

$$1 - aC_t = 1 - aE_t\left(C_{t+1}\right)$$

or

$$C_t = E_t\left(C_{t+1}\right)$$

12 pts for condition & justification