

MA Midterm 2011 Answers

$$1(a) \quad \frac{\partial y^*}{\partial n} \cdot \frac{n}{y^*} = f'(k^*) \frac{\partial k^*}{\partial n} \cdot \frac{n}{f(k^*)} \quad (3 \text{ pts})$$

is gotten from totally differentiating $sf(k^*) = (n+g+\delta)k^*$

$$\text{as } \frac{\partial k^*}{\partial n} \quad \text{as } sf'(k^*) \frac{\partial k^*}{\partial n} = k^* + (n+g+\delta) \frac{\partial k^*}{\partial n} \quad \text{as } \left(\frac{\partial k^*}{\partial n} = \frac{k^*}{sf'(k^*) - (n+g+\delta)} \right)$$

$$\text{as } \frac{\partial y^*}{\partial n} \cdot \frac{n}{y^*} = \frac{f'(k^*)}{f(k^*)} n \left(\frac{k^*}{sf'(k^*) - (n+g+\delta)} \right) \quad \leftarrow 3 \text{ pts}$$

(b) $\delta = 1\%$, $\gamma = 1\%$ $n = 2\%$

elasticity of output with respect to $k = \frac{f'(k^*)k^*}{f(k^*)} = \frac{1}{2}$

how much does y^* change.

$$\frac{dy^*}{dn} \cdot \frac{n}{y^*} = \frac{k^* f'(k^*)}{f(k^*)} \cdot \frac{n}{(n+g+s)k^* \cdot f'(k^*) - (n+g+s)}$$

$$= \frac{1}{2} \cdot \frac{2\%}{\left(\frac{1}{2} - 1\right)(1\% + 1\% + 2\%)} = \frac{1\%}{\left(-\frac{1}{2}\right)4\%} = -0.5$$

4 pts

$$\frac{k_{t+1} - k^*}{k_{t0} - k^*} = e^{-\lambda t} = \frac{y_{t+1} - y^*}{y_{t0} - y^*} = \frac{1}{2} \quad \text{where } \lambda = (1 - \alpha_k)(n + g + \delta)$$

as $\frac{1}{2}(4\%) = 2\%$

as it will take $t = \frac{\ln(\frac{1}{2})}{0.02}$ units of time to move

half way to the balanced growth path 6 pts

(c) $R_{t+1} = \text{Solow residual} = \frac{\dot{Y}_{t+1}}{Y_{t+1}} - \frac{\dot{L}_{t+1}}{L_{t+1}} - \alpha_k \left(\frac{\dot{K}_{t+1}}{K_{t+1}} - \frac{\dot{L}_{t+1}}{L_{t+1}} \right)$ 2 pts

Since perfect competition implies $\alpha_K + \alpha_L = 1$.

(2 pts)

as when $\alpha_K = \frac{1}{2}$, $\frac{\dot{Y}(t)}{Y(t+1)} - \frac{\dot{L}(t)}{L(t+1)} = 8\%$ (growth rate of output per worker)

$\frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)} = 4\%$ (growth rate of capital per worker)

as $R(t) = 8\% - \frac{1}{2}(4\%) = 6\%$

3 pts for final answer.

Question 2

$$(a) \quad U = \int_{t=0}^{\infty} e^{-\rho t} \frac{(C(t)A(t))^{1-\theta}}{1-\theta} dt =$$

$$\int_{t=0}^{\infty} e^{-\rho t} \frac{C(t)^{1-\theta}}{1-\theta} (A(0)e^{g^+t})^{1-\theta} (L(0)e^{nt}) dt$$

$$= B \int_{t=0}^{\infty} e^{-\beta t} \frac{C(t)^{1-\theta}}{1-\theta} dt$$

$$\text{where } B = \frac{A(0)^{1-\theta} L(0)}{H}$$

$$\beta = \rho - (1-\theta)g - n$$

3 pts for proper derivation

(b) State variable is k_{it} (2 pts)
 control variable is c_{it} (2 pts)

$$H = \tilde{B} e^{-\beta t} c_{it+1}^{1-\theta} + \lambda_{it+1} [-k_{it+1}(n+y) - c_{it+1} + f(k_{it+1})] \quad (3 \text{ pts})$$

where $\tilde{B} = B \cdot H$ since social planner maximizes economy wide welfare

$$\frac{\partial H}{\partial c_{it+1}} = \tilde{B} e^{-\beta t} c_{it+1}^{-\theta} - \lambda_{it+1} = 0 \quad (2 \text{ pts})$$

$$\frac{\partial H}{\partial k_{it+1}} = \lambda_{it+1} (f'(k_{it+1}) - (n+y)) = -\dot{\lambda}_{it+1} \quad (2 \text{ pts})$$

$$\frac{\partial H}{\partial \lambda(t)} = -k(t)(n-y) - c(t) + f(k(t+1)) = k'(t) \quad (2 \text{ pts})$$

+ transversality condition (2 pts)

(c) using $\frac{\partial H}{\partial c(t)} = 0$ we get

$$\beta e^{-\rho t} c(t+1)^{-\theta} = \lambda(t) \quad \text{taking logs \& the derivative with}$$

respect to t gives

$$-\beta - \theta \frac{\dot{c}(t)}{c(t)} = \frac{\dot{\lambda}(t)}{\lambda(t)} = - (f'(k(t+1)) - (n-y)) \quad \text{from } \frac{\partial H}{\partial k(t)}$$

as

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta} (f'(k_{t+1}) - n - g - \rho + n + (1-\theta)g)$$

(4 pts)

$$= \frac{f'(k_{t+1}) - \rho - \theta g}{\theta}$$

and $\dot{p} = 0$ so

$$f'(k_{t+1}) = \rho + \theta g$$

$$k_t = 0 \text{ is given by } \frac{\partial H}{\partial \lambda_{t+1}} = 0 \text{ when } \dot{k}_{t+1} = 0$$

(4 pts)

as

$$c_{t+1} = f(k_{t+1}) - (n+g)k_{t+1}$$

d) k^* is given by $f'(k^*) = p + \theta g$ (from $\dot{k} = 0$ equation) (2 pts)

$$y^* = f(k^*)$$

$$\rho^* = f(k^*) - (n + \delta) k^* \quad (2 \text{ pts})$$

k^g is given by $f'(k^g) = (n + \delta)$

(2 pts)

e)

$$\frac{\partial k^*}{\partial p} \text{ is given by } f''(k^*) \frac{\partial k^*}{\partial p} = 1$$

$$\text{or } \frac{\partial k^*}{\partial p} = \frac{1}{f''(k^*)} < 0$$

Since production function concave

(3 pts)

$$\frac{\partial y^*}{\partial p} = f'(k^*) \frac{\partial k^*}{\partial p} = \frac{f'(k^*)}{f''(k^*)} < 0 \quad \text{Since } f'(k^*) > 0 \text{ and } f''(k^*) < 0$$

(3pts)

$$\frac{\partial c^*}{\partial p} = f'(k^*) \frac{\partial k^*}{\partial p} - (n+g) \frac{\partial k^*}{\partial p} = \frac{\partial k^*}{\partial p} [f'(k^*) - (n+g)] < 0$$

(3pts)

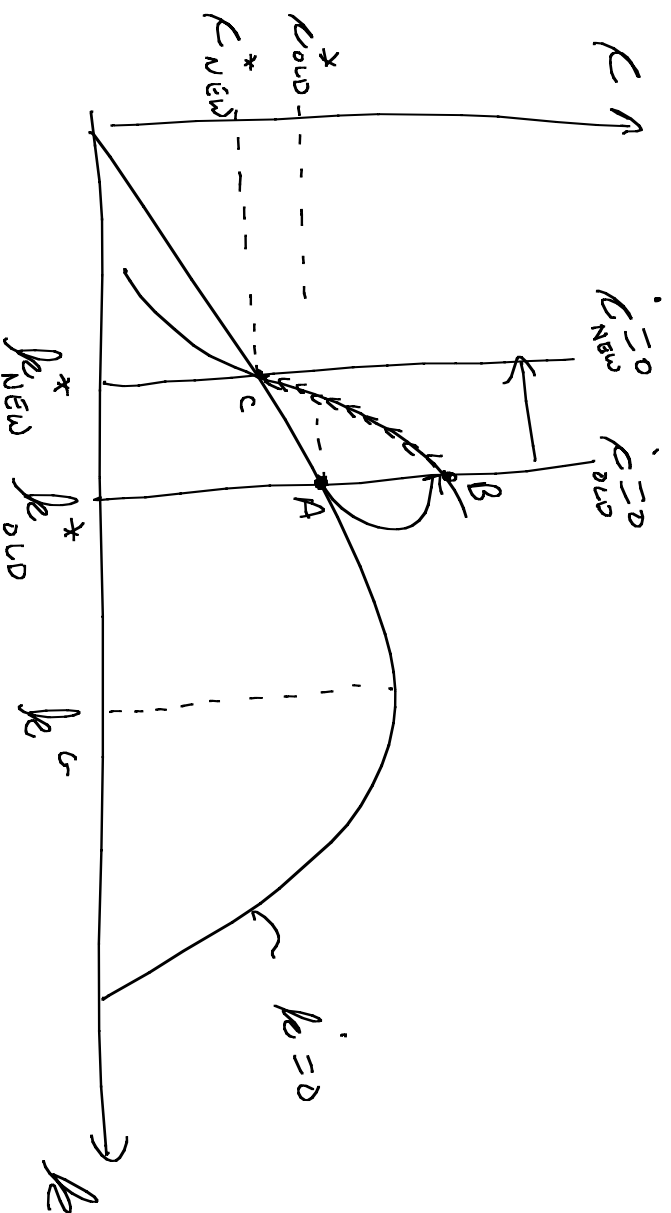
Since $\beta > 0$ implies $p+g > n+g$

Also

$$f''(k^g) \frac{\partial k^g}{\partial p} = 0 \quad \text{as } \frac{\partial k^g}{\partial p} = 0$$

(3pts)

1)



$$P_0 < P_1$$

2 pts for correct shift of $c=0$ curve.
3 pts for correct labelling of k^* , c^* , k^c & correct movement

At the time of the change in P , the economy jumps from point

At to point B (the point at which the new saddle path intersects the old $\dot{y}=0$ curve). Thereafter, the economy moves along the saddle path from point B to point C. Once it reaches the new balanced growth path at C, the economy stays there.

Ypts for explanation of movement depicted in graph

Question 3

(a) $y_{t+1} = \beta k_{t+1}^\alpha$ In steady state $S f(k^*) = (n+g+\delta) k^*$

$$as \quad s(k^*)^\alpha = (n+g+\delta)k^*$$

as

$$k^* = \left(\frac{s}{n+g+\delta} \right)^{\frac{1}{1-\alpha}} \quad \textcircled{3pt5}$$

$$y^* = \left(\frac{s}{n+g+\delta} \right)^{\frac{\alpha}{1-\alpha}} \quad \textcircled{3pt5} \quad \text{and}$$

$$c^* = (1-s)y^* = (1-s) \left(\frac{s}{n+g+\delta} \right)^{\frac{\alpha}{1-\alpha}} \quad \textcircled{3pt5}$$

$$(b) \quad f'(k^c) = (n+g+\delta) \quad \text{as} \quad \alpha(k^c)^{\alpha-1} = (n+g+\delta)$$

as

$$k^c = \left(\frac{\alpha}{n+g+\delta} \right)^{\frac{1}{1-\alpha}}$$

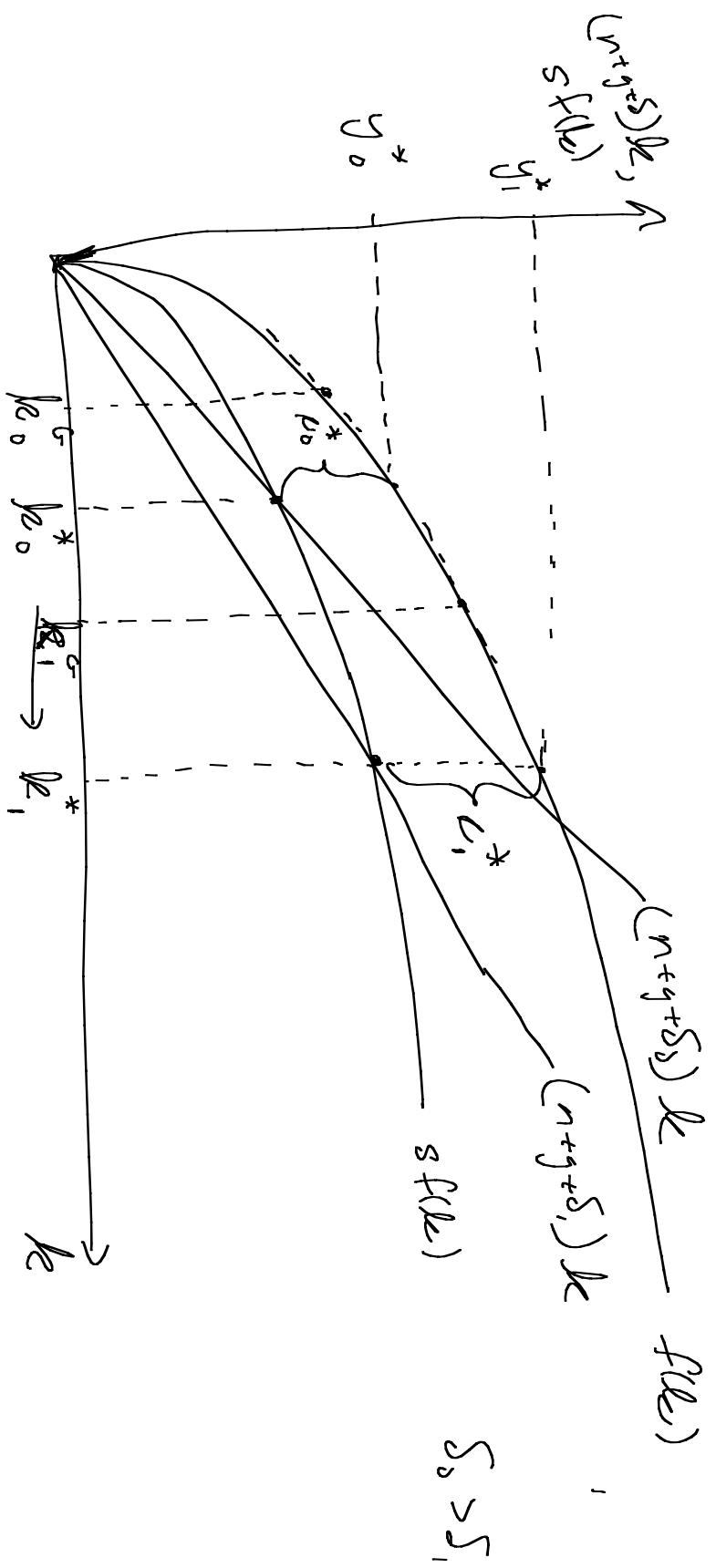
(c) Since the economy converges to k^* , it will converge to

The golden rule level when

3pts

$$\left(\frac{2}{n+y+s}\right)^{\frac{1}{1-\alpha}} = \left(\frac{5}{n+y+s}\right)^{\frac{1}{1-\alpha}} \quad \text{i.e., when } s = \alpha.$$

d) i)



3 pts for each of k^* , f^* , y^* and k^c properly identified
 before and after the change
 marks deducted for poorly labelled graph

(ii)

$$\frac{\partial k^*}{\partial s} = \underbrace{s}_{+ve} \underbrace{\frac{1}{1-\alpha}}_{-ve} \underbrace{(n+y+s)^{-\frac{1}{1-\alpha}-1}}_{-ve} \cdot \underbrace{\left(\frac{-1}{1-\alpha}\right)}_{<0} < 0$$

$$\frac{\partial y^*}{\partial s} = \underbrace{f'(k^*)}_{+ve} \underbrace{\frac{\partial k^*}{\partial s}}_{-ve} < 0$$

$$\frac{\partial c^*}{\partial s} = \underbrace{(1-s)}_{(+ve)} \underbrace{f'(k^*)}_{(+ve)} \underbrace{\frac{\partial k^*}{\partial s}}_{(-ve)} < 0$$

$$\frac{\partial k^c}{\partial s} = \underbrace{\alpha}_{(+ve)} \underbrace{(n+y+s)^{-\frac{1}{1-\alpha}-1}}_{(+ve)} \underbrace{\left(\frac{-1}{1-\alpha}\right)}_{(-ve)} < 0$$

3 points each $\times 4 = 12$ points

e) At the time of the change k_{t+1} decreases. After the initial decrease, actual saving exceeds breakeven investment per unit of effective labour so $k'_{t+1} = s f(k_{t+1}) - (n+g+\delta)k_{t+1} > 0$ until the economy again reaches the balanced growth path again (i.e., k_{t+1} back to original value k^*)

4 pts

Since $y_{t+1} = f(k_{t+1})$ and $k_{t+1} = (1-s)y_t = (1-s)f(k_{t+1})$

and $f'(k_{t+1}) > 0$ at the time of the change
 y_{t+1} and r_{t+1} decrease since k_{t+1} decreases.

Hereafter, for $k_{t+1} < k^*$, $y_{t+1} = f'(k_{t+1}) k_{t+1} > 0$ and
 $\dot{r}_{t+1} = (1-s) \dot{y}_{t+1} > 0$ because $\dot{k}_{t+1} > 0$

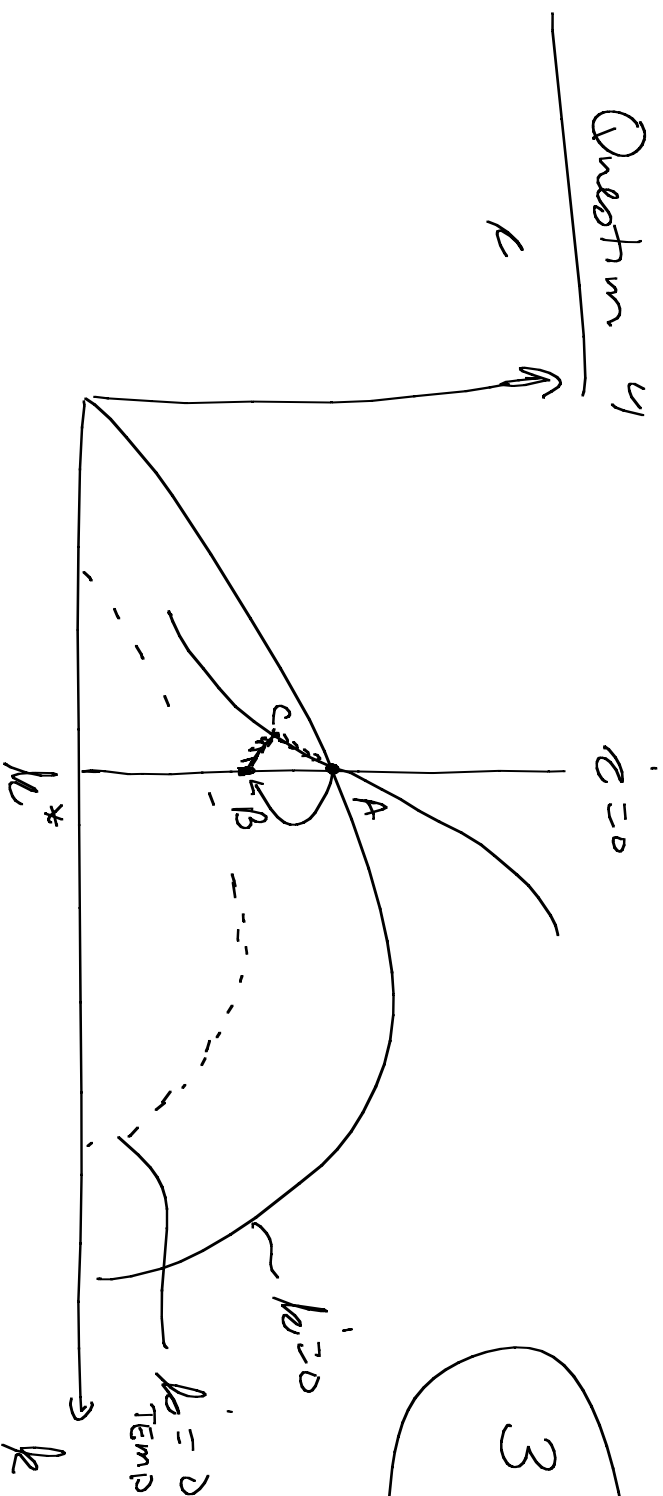
Once k_{t+1} reaches k^* , there is no further change in
 y_{t+1} and r_{t+1} .

4 pts for movement
of r_{t+1} and 4 pts
for movement in y_{t+1}

since k^s is determined by $f'(k^s) = (n+g)$, it is
unaffected by the change in K_{t+1} . It doesn't move

4 pts

Question 4



3 pts for graph

At the time of the increase a energy jumps from point A to B.

(so r decreases, k doesn't move). Since $r_{t+1} = f'(k_{t+1})$, r_{t+1} doesn't initially change. After the initial movement until the time gov't expenditure return to zero, the economy moves from point B to point C (arriving at C at the moment Δ gov't expenditures return to zero). During the time

k_{t+1} decreases, r_{t+1} increases and $r_{t+1} = f'(k_{t+1})$ increases (since $f''(k_{t+1}) < 0$). After the gov't expenditure returns to

0, the economy moves along the saddle path from C to A.

(so k_{t+1} increases, r_{t+1} increases and r_{t+1} decreases). Once the economy returns to its balanced growth path, $k_{t+1} = k^*$,

$c(t) = c^*$ and $r(t) = f'(k^*)$, the values do not change.

9 points for movements at $c(t), k(t)$ and $r(t)$
Total 12

Question 5

$$(a) \quad \mathcal{L} = E \left(\sum_{t=0}^{\infty} \left(\frac{1}{1+p} \right)^t u(c_t) + \lambda_t [r_t k_{t+1} + y_t - k_{t+1} + k_t - c_t] \right) \quad (2 \text{ pts})$$

$$\frac{\partial \mathcal{L}}{\partial c_t} = E_t \left[\left(\frac{1}{1+p} \right)^t u'(c_t) - \lambda_t \right] = 0 \quad (2 \text{ pts}) \quad A_t \text{ and all states of the world}$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = E_t \left[-\gamma_t + \gamma_{t+1} (1+r_{t+1}) \right] \stackrel{2 \text{ pts}}{=} 0$$

$$\frac{\partial \mathcal{L}}{\partial \gamma_{t+1}} = r_t k_{t+1} + \gamma_t - k_{t+1} + k_t - c_t \stackrel{2 \text{ pts}}{=} 0$$

} t and all states at the world

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial K_{t+1}} = 0 \quad \text{gives us}$$

7 pts for derivation

$$\left(\frac{1}{1+\rho}\right)^t u'(c_{t+1}) = E_t \left(\left(\frac{1}{1+\rho}\right)^{t+1} u'(c_{t+1}) (1+r_{t+1}) \right)$$

as

$$u'(c_{t+1}) = \left(\frac{1}{1+\rho}\right) E_t (u'(c_{t+1}) (1+r_{t+1}))$$

$$= \frac{1}{1+r} \left\{ E_t(u'(C_{t+1})) E_t(1+r_{t+1}) + Cov_t(u'(C_{t+1}), 1+r_{t+1}) \right\}$$

(b) Since $u'(C) > 0$ and $u''(C)$

$$\text{when } \left. \begin{array}{l} Cov_t(C_{t+1}, r_{t+1}) > 0, \quad Cov_t(u'(C_{t+1}), r_{t+1}) < 0 \\ \text{and } Cov_t(C_{t+1}, r_{t+1}) < 0 \Rightarrow Cov_t(u'(C_{t+1}), r_{t+1}) > 0 \end{array} \right] \text{opts}$$

Let $\tilde{C}_t$ be current consumption when $cov_t(C_{t+1}, r_{t+1}) > 0$
 and $\hat{C}_t$ be current consumption when $cov_t(C_{t+1}, r_{t+1}) < 0$

from the first order conditions we have

$$u'(c_t^{\sim}) > u'(c_t^1) \quad \text{as} \quad c_t^{\sim} < c_t^1$$

Since $u''(c) < 0$

Give 9 points for proper derivation

c)

Consumption follows a random walk when

$$C_t = E_t(C_{t+1}) \quad \text{or} \quad C_{t+1} = C_t + e_t$$

where e_t is white noise

5 pts

The first order necessary condition shows that consumption follows a random walk when

$$\textcircled{1} \quad 1 + \rho = 1 + r_t \quad \text{for all } t$$

$$\textcircled{s_{pt}^-}$$

$$\textcircled{2} \quad u'(C_t) \text{ is linear in } C_t \quad \text{i.e., when utility is quadratic} \quad \textcircled{s_{pt}^-}$$

$$\text{In this case } \text{Cov}_t(1 + r_{t+1}, u'(C_{t+1})) = 0 \quad \text{as}$$

$$\left. \begin{aligned} u'(C_t) &= E_t(u'(C_{t+1})) \\ u'(C_t) &= 1 - a C_t \quad \text{as we get } C_t = E_t C_{t+1} \end{aligned} \right\} s_{pt}^-$$

