Mathematics Review for Economists

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This document presents a review of very basic mathematics for use by students who plan to study economics in graduate school or who have long-ago completed their graduate study and need a quick review of the basics. The first section covers variables and equations, the second deals with functions, the third reviews some elementary principles of calculus and the fourth section reviews basic matrix algebra. Readers can work through whatever parts they think necessary as well as do the exercises provided at the end of each section. Finally, at the very end there is an important exercise in matrix calculations using the freely available statistical program XLispStat.

To obtain XLispStat, download the self extracting zip-file `wxls32zp.exe` from `http://www.economics.utoronto.ca/jfloyd/stats/wxls32zp.exe`, and place it in a directory you create for it called `xlispstat` in the `Program Files` directory on your MS-Windows computer. Then click on `wxls32zp.exe` and all the program files will be extracted into that directory. Finally, right-click on the `wxls32.exe` icon in the directory and drag it to your desktop to create a desk-top icon.

1. Variables and Equations[1]

Mathematical analysis in economics focuses upon variables that are related to each other in various ways. Variables are entities like the price of a good and the quantities of that good produced and consumed, which can be denoted respectively by letters like P, Q_d and Q_s and can take as values positive real numbers. There are two types of real numbers: rational numbers consisting of either integers such as -1, 0, 1, 2, and 3 or fractions like $1/4 = .2500$, $2/6 = .3333333$ and $22/7 = 3.14285714285714$ which are repeating decimal numbers, and irrational numbers such as $\sqrt{2} = 1.4142135623730950$ which are non-repeating decimal numbers. The set of real numbers can be visualized as lying along a straight line running from $-\infty$ and $+\infty$ with infinitely many such numbers lying between any two points on the line represented, for example, by the rational numbers $1/8 = .12500$ and $1/5 = .2000$.

The relationship between the quantity demanded of a product and its price can be expressed in the form of an equation like

$$Q_d = \alpha - \beta P \tag{1}$$

where Q_d is the quantity demanded, P is the price, and the greek letters α and β are parameters that take fixed values such as, for example, $\alpha = 100$ and $\beta = 2$. In this example, the above equation could be rewritten as

$$Q_d = 100 - 2P. \tag{2}$$

It should be emphasized here that it makes no economic sense to have negative prices or quantities, so the condition must be imposed that both Q_d and P be greater than or equal to zero. The quantity demanded would be zero if the price is high enough to exceed the value to consumers of even one unit of the commodity. A zero price would imply that an additional unit of the commodity is of no value although units already purchased would have value. From equation (1), the condition that $Q_d \geq 0$ can be expressed as

$$0 \leq \alpha - \beta P. \tag{3}$$

[1]An appropriate background for the material covered in this section can be obtained by reading the first two chapters of Alpha C.Chiang, *Fundamental Methods of Mathematcial Economics*, McGraw Hill, Third Edition, 1984. The coverage of this material is virtually the same both in earlier editions of this book and in the Fourth Edition, joint with Kevin Wainwright, published in 2005.

Expressions like the one above can be manipulated without violating the equality or inequality by adding the same number to both sides or by multiplying both sides by the same positive number. Multiplying both sides by a negative number will reverse an inequality. Adding βP to both sides of the above inequality yields

$$\alpha \geq \beta P \tag{4}$$

which, after dividing both sides by β, becomes

$$P \leq \frac{\alpha}{\beta}. \tag{5}$$

Keeping in mind that this represents the condition for $Q_d \geq 0$, it specifies the maximum value that P can take when $Q_d = 0$ as equal to the ratio of α over β . Suppose that $-\beta$ is negative, implying a positive value for β, giving the demand curve its negative slope. Since P cannot be less than zero, α must then be positive or zero. It turns out, however, that a zero value of α would imply that Q_d is zero when P is zero and that the good is of no value to consumers even if it is free. So α must be positive—that is, when the price is zero the quantity demanded must be positive. Now assume that β is zero. As can be seen from (1) above, this would imply that $Q_d = \alpha$, independently of the magnitude of P. This could not be ruled out as entirely impossible, although the commodity involved would have to be some absolute necessity like air or water. So we can conclude that, under reasonable circumstances, the condition that α and β both be positive and finite should be imposed upon the demand equation. And the price at which the quantity demanded is zero—that is, the maximum value that P can take—equals α/β .

Actually, the conditions that must be imposed on the demand equation can be more easily determined by simply plotting the equation on a graph. as is done in Figure 1. below.

You can easily see that, if negative price and quantity are not allowed, the demand curve (or better, line) will slope downward to the right from some maximum point equal to the ratio of α over β and cross the quantity axis at a distance from the origin equal to α . As α gets bigger the intersection point with the horizontal axis moves to the right, and as β gets bigger the absolute value of the slope gets smaller and the intersection with the vertical axis moves down closer to the origin. Obviously, as α approaches zero the demand for the commodity disappears. And as β approaches zero the demand curve

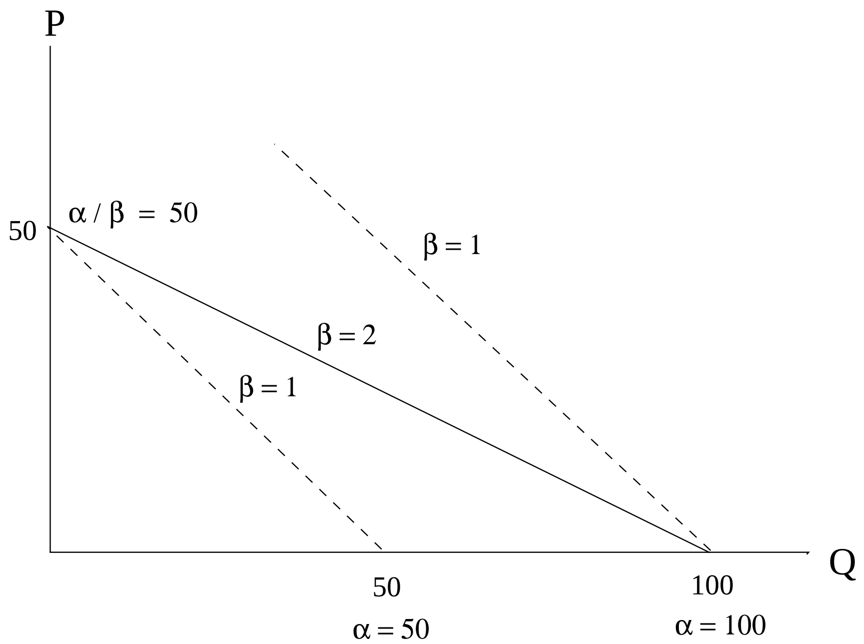

Figure 1: Graphical illustration of Equation (1) where $\alpha = 100$ and $\beta = 2$.

becomes vertical at a point on the horizontal axis to the right of the origin by the amount α. And, as β becomes infinite the demand curve becomes flat, on top of the quantity axis. In that case, the commodity is literally of zero value to consumers but they are willing to consume any amount forced on them. Note that this implies that the commodity is not of negative value—were that the case, consumers would be willing to pay in order to avoid having to consume it.

Another important equation presents the relationship between the quantity supplied of a product and its price. This relationship can be expressed in the form of an equation that gives the value of P associated with each quantity supplied Q_s as follows

$$P = \gamma + \delta Q_s \tag{6}$$

where γ and δ are, as before, parameters. As in the case of the demand relationship, neither the quantity supplied nor the price can be negative. It can be easily seen from this equation that γ is the level that P will take when

Q_s is zero. Therefore, γ has to be a non-negative number. And the level of P has to increase by an amount equal to δ for each one-unit increase in Q_s above zero. It makes little sense to allow P to be zero at low but positive levels of Q_s. This would imply that some amount of the good is available free, after which it has to be produced at increasing cost. Were this the case, γ would have to be negative.

An interesting situation arises in the case of a good such as land, of which a fixed amount is freely available and no increase is possible. In this situation the supply equation would take the form

$$Q_s = Q_0 \tag{7}$$

where Q_0 is a constant that is independent of the level of P. The equilibrium level of P can then be obtained by setting $Q_0 = Q_d$ in equation (1) to obtain

$$Q_0 = \alpha - \beta P. \tag{8}$$

To calculate the equilibrium level of P we first subtract α from both sides to get

$$Q_0 - \alpha = -\beta P. \tag{9}$$

and then divide both sides by $-\beta$ and rearrange the resulting expression to obtain

$$\frac{\alpha - Q_0}{\beta} = P. \tag{10}$$

The condition that P not be negative implies that α must be equal to or greater than Q_0.

The above supply equations are plotted in Figure 2 below under the assumption that $\gamma = 20$, $\delta = .5$ and $Q_0 = 70$.

Another interesting exercise is to solve the demand equation (1) and the supply equation (6) for the equilibrium price and quantity which will henceforth be denoted as Q in the equilibrium condition

$$Q = Q_d = Q_s. \tag{11}$$

To do this, substitute (11) and (6) into (1) to eliminate P, Q_d and Q_s and thereby obtain

$$Q = \alpha - \beta(\gamma + \delta Q) = \alpha - \beta\gamma - \beta\delta Q \tag{12}$$

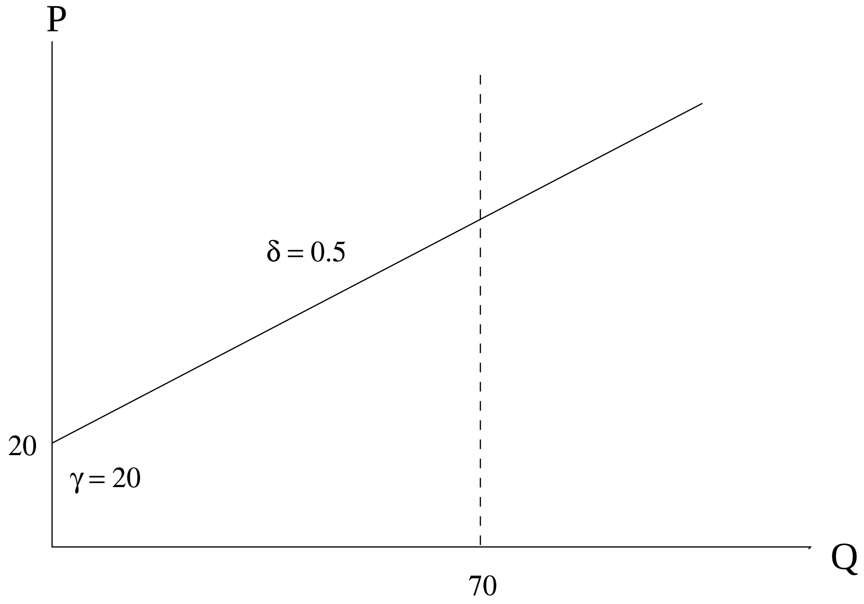

Figure 2: Graphical illustration of Equation (6) where $\gamma = 20$, $\delta = .5$. and $Q_0 = 70$.

which, upon addition of $\beta\delta Q$ to both sides, simplifies to

$$(1 + \beta\delta)\, Q = \alpha - \beta\gamma \tag{13}$$

which, upon division of both sides by $(1 + \beta\delta)$, yields the equilibrium level of Q.

$$Q_e = \frac{\alpha - \beta\gamma}{1 + \beta\delta} \tag{14}$$

Now, to find the equilibrium level of P we can substitute this equation into equation (1) to eliminate Q_d, which now must equal Q_e, to obtain

$$\frac{\alpha - \beta\gamma}{1 + \beta\delta} = \alpha - \beta P \tag{15}$$

which, after subtracting α from both sides, becomes

$$\frac{\alpha - \beta\gamma}{1 + \beta\delta} - \alpha = -\beta P. \tag{16}$$

Now divide both sides by $-\beta$ to obtain the equilibrium price, which equals

$$P_e = \frac{\alpha}{\beta} - \frac{\alpha - \beta\gamma}{\beta(1+\beta\delta)}. \tag{17}$$

This expression can be simplified by multiplying and dividing the right side of the equality by the ratio of α over β or, in other words, factoring out the ratio of α to β. This yields

$$P_e = \frac{\alpha}{\beta}\left[1 - \frac{\alpha - \beta\gamma}{\alpha(1+\beta\delta)}\right]. \tag{18}$$

Substituting the values of the parameters in our numerical example—namely, $\alpha = 100$, $\beta = 2$, $\gamma = 20$ and $\delta = 0.5$ —we obtain an equilibrium level of P equal to 35 and an equilibrium level of Q equal to 30, as shown graphically in the Figure below.

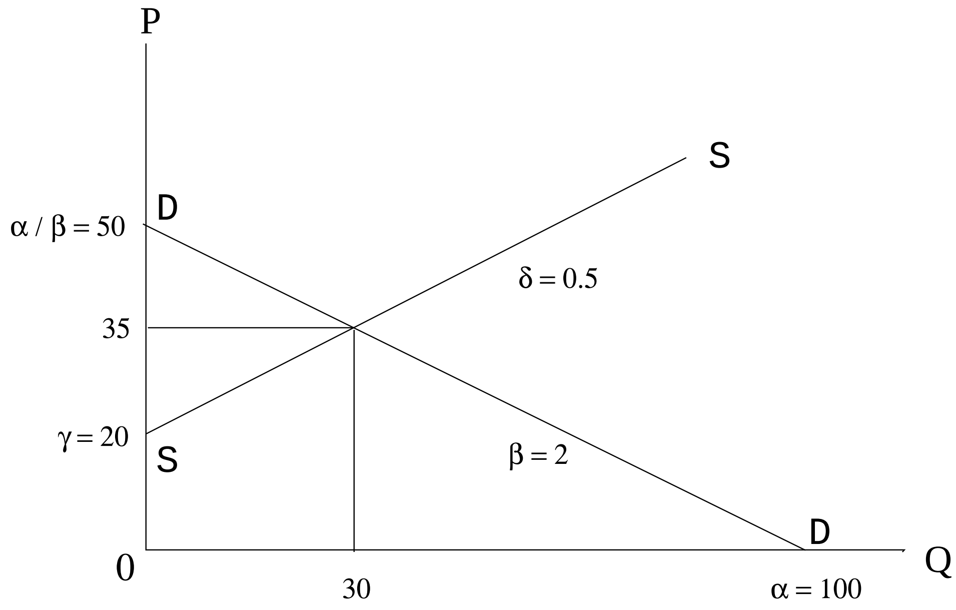

Figure 3: Demand curve [Equation (1)] and supply curve [Equation (6)] and market equilibrium where $\alpha = 100$, $\beta = 2$, $\gamma = 20$ and $\delta = 0.5$.

Exercises

1. What type of real number is each of the following?

$$\sqrt{3} \qquad \sqrt{4} \qquad -1/6 \qquad 21/19$$

And what is the difference between real numbers and rational numbers?

2. Addition, subtraction, multiplication or division of both sides of an equation by the same real number will leave that equation unchanged. Demonstrate this by performing these operations on the equation

$$x = 3 + 4y$$

using arbitrarily chosen values of x and y to check your results.

3. Demonstrate what happens to inequalities when you add, subtract, multiply and divide both sides by the same number, using as an example the inequality $\quad 4 \geq 2$.

4. Solve the following two equations for the equilibrium levels of the variables x and y.

$$\alpha x + \beta y = 32$$
$$\delta x - \gamma y = 9$$

Then demonstrate that when $\alpha = 3$, $\beta = 6$, $\delta = 1$ and $\gamma = 3$, the equilibrium values of x and y are 10 and 1/3.

2. Functions

A function is a relationship between variables. The two equations (1) and (6) of the previous section are examples of functions. The first of these was the demand function that can now be more rigorously expressed as

$$Q_d = D(P) = \alpha - \beta P \tag{1}$$

where Q_d represents the quantity demanded of a commodity and P represents its market price, and the function $D(P)$ states that Q_d depends on P in the sense that if we assign a value to P, $D(P)$ will tell us the resulting value of Q_d. The expression $D(P)$ is a general form which simply states that for every value of P there is an associated value of Q_d. The part of the above equation to the right of the right-most equal sign is a detailed specification of the form of the demand function $D(P)$. It states that the amount of Q demanded is a linear function of P that has the two parameters α and β. Equation (6) of the previous section can be rearranged for expression as a supply function in the following form.

$$Q_s = S(P) = \xi + \epsilon P \tag{2}$$

where $\xi = -\gamma/\delta$ and $\epsilon = 1/\delta$ and Q_s is the quantity of the good supplied. The function $S(P)$ tells us that for every level of P there is a quantity what will be supplied, Q_s and the set of terms on the right gives an exact specification of the response of Q_s to P that is again linear. A complete specification of market equilibrium for this commodity is given by equations (1) and (2) together with the equilibrium condition

$$Q = Q_d = Q_s \tag{3}$$

and the equilibrium price and quantity can be obtained by substituting this equilibrium condition into (1) and (2) and then solving these two equations simultaneously for P and Q.

There are many different functional forms in addition to the linear ones specified above. For example, suppose that a monopolist is interested in the revenue that can be obtained by selling various quantities of a good. The demand function above can be arranged to express P, which is the average revenue, as a function of the quantity supplied by simply rearranging the equation to put P in the left side as follows

$$P = A(Q_s) = \frac{\alpha}{\beta} - \frac{1}{\beta} Q_s \tag{4}$$

where $A(Q_s)$ is the average cost function, the specifics of which are given by the terms on the right. Total revenue is the quantity supplied multiplied by the price and can be expressed as follows as a function of the quantity supplied alone

$$TR = T(Q_s) = Q_s P = Q_s A(Q_s) = Q_s \left[\frac{\alpha}{\beta} - \frac{1}{\beta} Q_s \right] = \frac{\alpha}{\beta} Q_s - \frac{1}{\beta} Q_s^2 \tag{5}$$

where

$$T(Q_s) = \frac{\alpha}{\beta} Q_s - \frac{1}{\beta} Q_s^2. \tag{6}$$

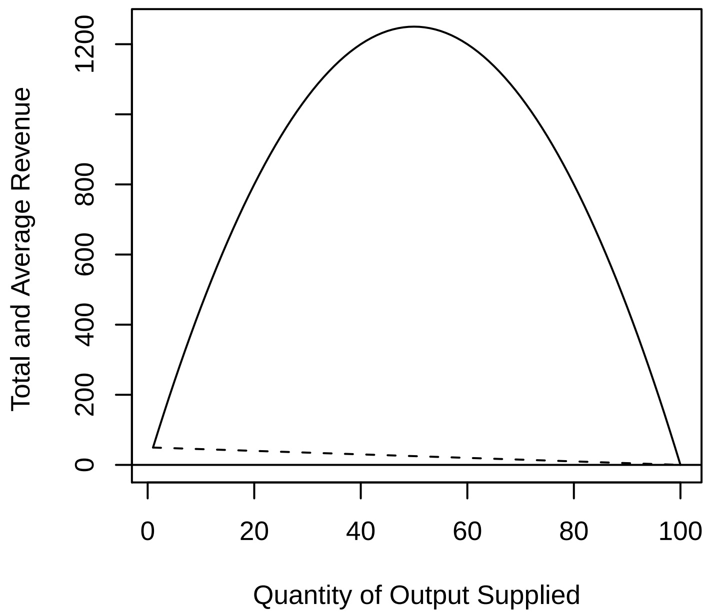

Figure 1: Total revenue (solid line) and average revenue (dashed line) when $\alpha = 100$, $\beta = 2$.

The total revenue function $T(Q_s)$ turns out to be a second degree polynomial in Q_s. The degree of the polynomial is the highest power to which any variable is taken—in this case 2. The linear functions $D(P)$ and $S(P)$ above are therefore first-degree polynomial functions because the variable P in both cases is taken to the first power—that is, P^1 equals P. Using the values $\alpha = 100$ and $\beta = 2$ chosen for the examples in the previous section, the above function becomes

$$TR = 50\, Q_s - 0.5\, Q_s^2, \tag{7}$$

which is plotted in Figure 1 above. It can be seen from this Figure that a second-order polynomial takes the form of a parabola—in this case, an inverted one.

A third-degree polynomial was used in my book to represent the effect on useable output of insufficient liquidity in the economy.[2] The level of output when the optimal quantity of money is in circulation can be represented as

$$Y = mK \tag{8}$$

where Y is the level of output and K is the aggregate stock of capital, broadly defined to include human capital, technology, and knowledge as well as the usual forms of physical capital. When there is insufficient liquidity as a result of there being too small a stock of money in circulation, resources have to be used up making exchange, thereby reducing the quantity of final goods and services available for consumption and investment. The right side of the above equation must then be multiplied by the term

$$\left[1 - \frac{1}{3\lambda}\left(\phi - \lambda\frac{L}{K}\right)^3\right] \tag{9}$$

where L is the stock of liquidity. Equation (8) becomes

$$Y = mK\left[1 - \frac{1}{3\lambda}\left(\phi - \lambda\frac{L}{K}\right)^3\right]. \tag{10}$$

The optimal ratio of the stock of liquidity to the stock of capital is equal to ϕ/λ at which point the cubed term in the square brackets becomes equal to zero, making the term in the square brackets equal to unity, and equation (10) reduces to equation (8). As the stock of liquidity declines the cubed term

[2] See John E. Floyd, *Interest Rates, Exchange Rates and World Monetary Policy*, Springer-Verlag, 2010, pages 24-29.

becomes positive and larger and the term in the square brackets becomes less than unity. As L approaches zero the term in the square brackets becomes equal to

$$\left[1 - \frac{\phi^3}{3\lambda}\right]$$

which gives the fraction of the maximum possible output level remaining when money disappears and all exchange is conducted by barter. The term (9) is plotted in the Figure below for the case where $\phi = 8.94$ and $\lambda = 298.14$.

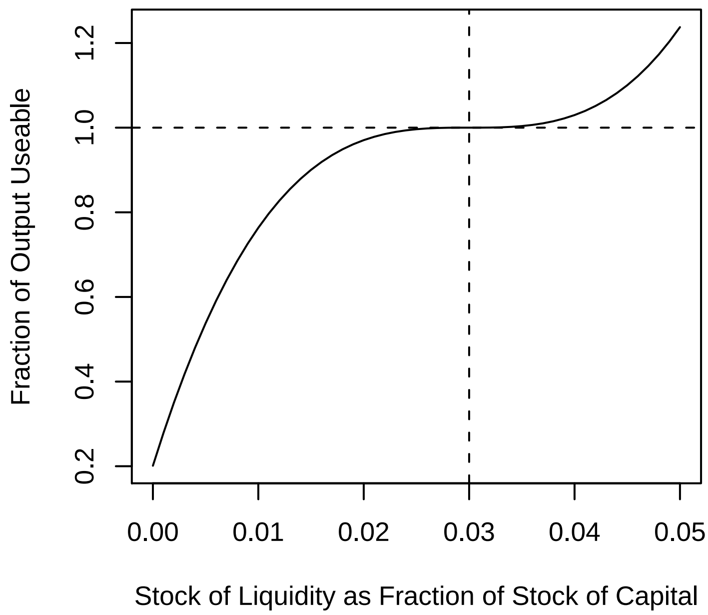

Figure 2: Effect of Provision of Liquidity on Fraction of Output that is Useable

When there is no liquidity, the level of useable output is reduced to 20 percent of maximum possible level and a stock of liquidity equal to 3 percent of the capital stock is turns out to be optimal. Only the levels of the liquidity/capital ratio of 3 percent or below are relevant. For levels of liquidity above that ratio we simply impose the condition that useable output remains at its maximum and the term in the square brackets remains equal to unity. This is another example where a particular function is assumed to apply only over a specific range of values of the independent variable, which in this case is the ratio of the stock of liquidity to the stock of capital.

As will become clear in the next section on differentiation and derivatives, this third order polynomial function was chosen so that the demand function for liquidity (or money) will be represented by the downward-sloping leftward side of a parabola.

Another thing to keep in mind when working with polynomial functions is the multiplication rule

$$\begin{aligned}(a\,x + b\,y)(c\,x + d\,y) &= ac\,x^2 + ad\,xy + bc\,xy + bd\,y^2 \\ &= ac\,x^2 + (ad + bc)\,xy + bd\,y^2.\end{aligned}$$

Additional functional forms that often appear either directly in economic analysis or indirectly in statistical testing of economic hypotheses are exponential and logarithmic functions. An exponential function takes the form

$$y = f(x) = b^x \tag{11}$$

where b denotes the fixed base of the exponent. The following exponential function takes the base 10 to the power x.

$$y = f(x) = 10^x \tag{12}$$

The logarithm to the base 10 of y is the number what would have to be raised to the power 10 to obtain y. As you can see from the equation above, that number turns out to be x. So there is a logarithmic function directly related to the above base-10 exponential function—that is,

$$x = log(y) \tag{13}$$

where the expression $log(y)$ means the logarithm to the base 10 of y. Accordingly, by substitution of (13) into (12) we obtain

$$y = 10^{log(y)}. \tag{14}$$

This relationship is illustrated by example in the following table

x	0	1	2	2.5	3
$y = 10^x$	1	10	100	316.228	300
$z = log(y)$	0	1	2	2.5	3

where, as you can see, z always equals x.

The most commonly used exponential function has as its base the irrational number

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = 2.71828182845904523536 \tag{15}$$

and takes the form

$$y = e^x. \tag{16}$$

Its logarithmic counterpart is

$$x = ln(y) \tag{17}$$

where $ln()$ denotes the natural logarithm to the base e —that is, x is the power to which e must be raised to obtain the number y. A table representing examples of the exponents of selected numbers and their logarithms, all to the base e, is presented below.

x	0	1	2	2.5	3
$y = e^x$	1	2.7128	7.3906	12.1825	20.0855
$x = ln(y)$	0	1	2	2.5	3

All these exponents are irrational numbers, here taking the form of non-repeating decimals rounded to four places. From (16) and (17) above, we have

$$y = e^{ln(y)}. \tag{18}$$

There are some important rules to keep in mind when working with exponents, all of which can be derived from the fact that x^n simply equals x multiplied by itself n times.[3]

1) $x^m \times x^n = x^{m+n}$

2) $x^m / x^n = x^{m-n}$

[3]See pages 27 and 28 of the third edition of Chiang's book cited above, or the corresponding pages of any other edition, for a detailed derivation of these rules.

3) $x^{-n} = 1/x^n \qquad (x \neq 0)$

4) $x^0 = 1 \qquad (x \neq 0)$

5) $x^{1/n} = \sqrt[n]{x}$

6) $(x^m)^n = x^{mn}$

7) $x^m \times y^m = (x\,y)^m$

There are also a number of important rules for working with and simplifying logarithmic expressions.[4]

1) $ln(x\,y) = ln(x) + ln(y) \qquad (x, y > 0)$

2) $ln(x/y) = ln(x) - ln(y) \qquad (x, y > 0)$

3) $ln(a\,x^\alpha y^\beta) = ln(a) + ln(x^\alpha) + ln(y^\beta) = ln(a) + \alpha\, ln(x) + \beta\, ln(y)$

4) $ln(a\,e^6 e^3) = ln(a) + ln(e^6) + ln(e^3) = ln(a) + 6 + 3 = ln(a) + 9$

5) $log_{10}\, x = (log_{10}\, e)(log_e\, x) \qquad (x > 0)$

6) $log_{10}\, e = 1/log_e\, 10$

7) $ln(x \pm y) \neq ln(x) \pm ln(y)$

One of the most common uses of logarithms in economics is in the plotting of data. Canada's real exchange rate with respect to the United States, defined as the relative price of Canadian output in terms of U.S. output, is plotted in Figure 3 below. The solid line gives the actual series expressed as a percentage if its initial level in the first quarter of 1957. The dashed line gives the natural logarithm of that series expressed as a percentage of its level in that first quarter. It turns out that use of the logarithm to base 10 instead of the natural logarithm (to the base e) produces an identical series as expressed above. As you can see, the logarithm of the series shows much less variability than the actual level. The reason is that the logarithm of a series, as will become clear in the next section on derivatives, represents the cumulation of the rates of change of the series through time while the level of the series represents the cumulation of the absolute changes through time, where the size of neighboring individual periods over which these respective absolute and relative rates of change are calculated is tiny and the number of such periods over any small interval is therefore extremely large.

[4] For proofs, see pages 284–286 of the Chiang book cited. Use the index to find the appropriate pages in other editions of the book.

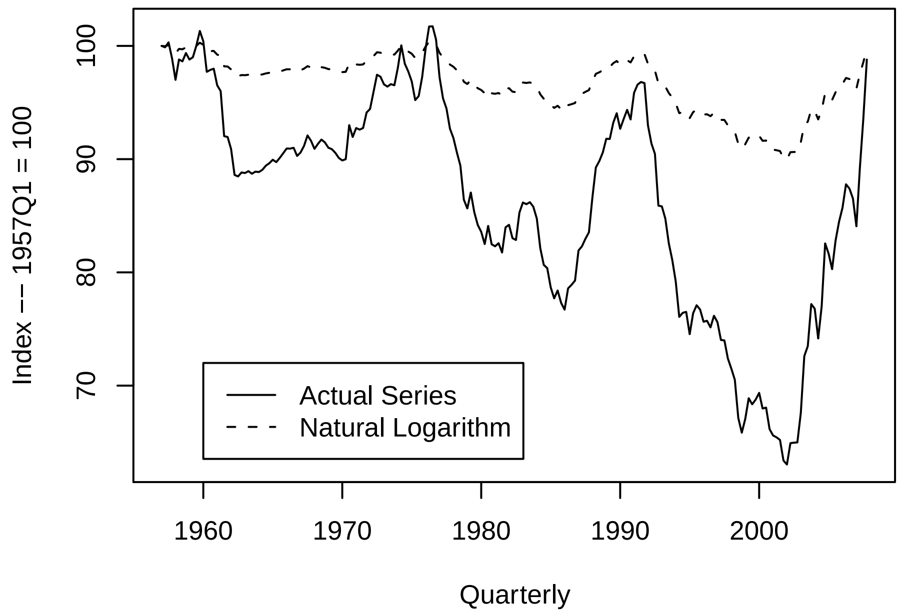

Figure 3: Canadian Real Exchange Rate With Respect to the United States

The question might arise as to whether we distort data by taking logarithms of them. In empirical analysis we attempt to test theories by obtaining quantitative measures of the relationship between different economic variables—in my book cited above, quantitative measures of the effects of various factors determining countries' real exchange rates are obtained. It turns out that, in many cases, the logarithms of relevant series may be more closely related to each other than the unmodified levels of those variables. This provides evidence about the nature of the relationships between the underling variables—the elasticity of series x with respect to series y may appear to be rather constant in the data, and the corresponding slope of series x in terms of series y may be variable, providing important evidence about the nature of the underlying economic relationship between them. Indeed, taking the logarithm of a series does not distort it—it merely represents another of the many ways in which the time-path of the underlying variable can be measured.

In closing this section we must keep in mind that, while it is convenient to explore the nature of functions by examining functions involving only two variables, the emphasis in most economic analysis is on situations in which there are several related variables. For example, we might have three functions

$$\begin{aligned} X &= F(Y, Z) && (19) \\ Y &= G(Z, X) && (20) \\ Z &= L(X, Y) && (21) \end{aligned}$$

that could each take different forms. The problem then is to specify the details of the functional forms $F(\quad)$, $G(\quad)$ and $L(\quad)$, and then to solve the three-equation system for the equilibrium levels of the variables X, Y and Z.

Exercises

1. What is the difference between first, second and third degree polynomial functions?

2. What degree polynomial is the following function?

$$y = (x - 2)(x + 2)$$

3. Expand and simplify the equation

$$0 = (x + 2\,y)^{\,2}$$

to express y as a function of x. What degree of polynomial is this function?

4. What is the difference between an exponential function and a polynomial?

5. Suppose that $ln(y) = 10$. What is the value of y ?

6. What is the reciprocal of $x^{\,-n}$?

7. Show that $x^{\,m} \times x^{\,-n}$ equals 1 when $m = n$.

8. What degree of polynomial is $(x^{\,m})^{\,3}$?

9. Consider the Cobb-Douglas production function

$$X = L^{\alpha}\,K^{\,1-\alpha}$$

where X is the output of a particular product, L and K are the quantities of labour and capital used in its production and α is a parameter. Express this function in logarithmic form.

10. What single number would one have to take the logarithm of to calculate $ln(10) - ln(5)$?

11. Suppose that $log_{10}x = 4$. Calculate the logarithm of x to the base e . You should get a number approximately equal to 9.21.

3. Differentiation and Integration[5]

We now review some basics of calculus—in particular, the differentiation and integration of functions. A total revenue function was constructed in the previous section from the demand function

$$Q = \alpha - \beta P \tag{1}$$

which was rearranged to move the price P, which equals average revenue, to the left side and the quantity Q to the right side as follows

$$P = A(Q) = \frac{\alpha}{\beta} - \frac{1}{\beta}\,Q\,. \tag{2}$$

To get total revenue, we multiply the above function by Q to obtain

$$T(Q) = P\,Q = A(Q)\,Q = \frac{\alpha}{\beta}\,Q - \frac{1}{\beta}\,Q^2. \tag{3}$$

where Q is the quantity supplied by a potential monopolist who, of course, is interested in equating the marginal revenue with the marginal cost. The marginal revenue is the increase in total revenue that results from selling one additional unit which, as can easily be seen from Figure 1, must equal the slope of the total revenue curve—that is, the change in total revenue divided by a one-unit change in the quantity. The marginal revenue from selling the first unit equals both the price and the total revenue from selling that unit. The sale of an additional unit requires that the price be lower (given that the demand curve slopes downward) and, hence, the marginal revenue is also lower. This is clear from the fact that the slope of the total revenue curve gets smaller as the quantity sold increases. In the special case plotted, where $\alpha = 100$ and $\beta = 2$, total revenue is maximum at an output of 50 units and, since its slope at that point is zero, marginal revenue is also zero. Accordingly,

[5] An appropriate background for the material covered in this section can be obtained by reading Alpha C.Chiang, *Fundamental Methods of Mathematcial Economics*, McGraw Hill, Third Edition, 1984, Chapter 6, Chapter 7 except for the part on Jacobian determinants, the sections of Chapter 8 entitled Differentials, Total Differentials, Rules of Differentials, and Total Derivatives, and all but the growth model section of Chapter 13. Equivalent chapters and sections, sometimes with different chapter and section numbers, are available both in earlier editions of this book and in the Fourth Edition, joint with Kevin Wainwright, published in 2005.

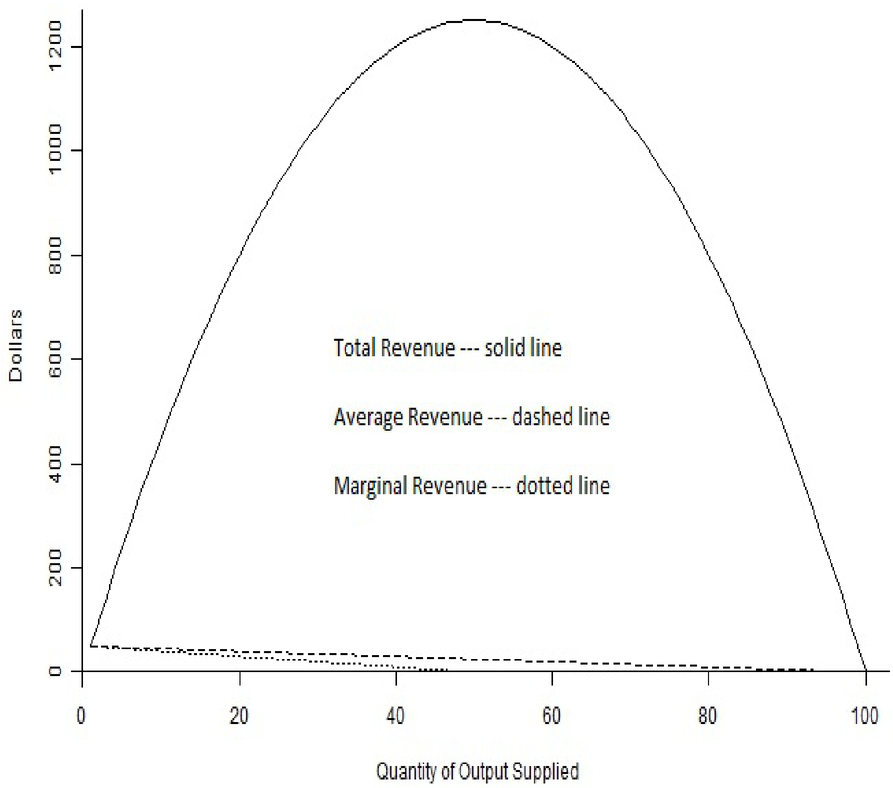

Figure 1: Total, average and marginal revenue when $\alpha = 100$, $\beta = 2$.

the marginal revenue curve plotted as the dotted line in the Figure crosses zero at an output of 50 which turns out to be one-half of the 100 units of output that consumers would purchase at a price of zero, at which point total revenue would also be zero. We do not bother to plot the marginal revenue curve where marginal revenue is negative because no firm would produce and sell output under those conditions, given that marginal cost is always positive. The average revenue, or demand, curve is plotted in the Figure as the dashed line. To make the relationship between the demand curve and marginal revenue curve clearer, these two curves are plotted separately from the total revenue curve in Figure 2. The total revenue associated with any quantity is the area under the marginal revenue curve—that is the sum of the marginal revenues—to the left of that quantity.

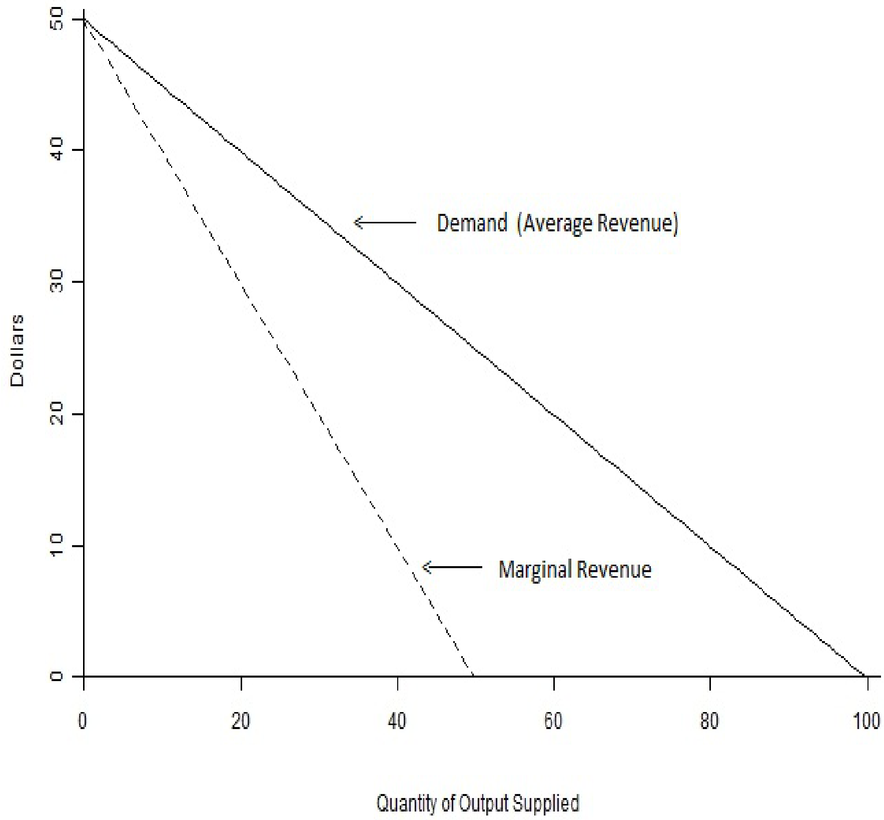

Figure 2: Demand and marginal revenue curves when $\alpha = 100$, $\beta = 2$.

We now state the fact that the derivative of a function $y = F(x)$ is equal to the slope of a plot of that function with the variable x represented by the horizontal axis. This derivative can be denoted in the four alternative ways

$$\frac{dy}{dx} \qquad \frac{dF(x)}{dx} \qquad \frac{d}{dx}F(x) \qquad F'(x)$$

where the presence of the integer d in front of a variable denotes a infinitesimally small change in its quantity. We normally denote the slope of y with respect to x for a one-unit change in x by the expression $\Delta\, y/\Delta\, x$. The expression dy/dx represents the limiting value of that slope as the change in x approaches zero—that is,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \tag{4}$$

The first rule to keep in mind when calculating the derivatives of a function is that the derivative of the sum of two terms is the sum of the derivatives of those terms. A second rule is that in the case of polynomial functions like $y = a\,(b\,x)^n$ the derivative takes the form

$$\frac{dy}{dx} = \frac{d}{dx}\,a\,(b\,x)^n = a\,n\,(b\,x)^{n-1}\,\frac{d}{dx}\,(b\,x) = b\,a\,n\,(b\,x)^{n-1} \tag{5}$$

where, you will notice, a multiplicative constant term remains unaffected. Accordingly, the derivative of equation (3) is the function

$$\begin{aligned} M(Q) &= \frac{d}{dQ}\left[\frac{\alpha}{\beta}\,Q - \frac{1}{\beta}\,Q^{\,2}\right] \\ &= \frac{d}{dQ}\frac{\alpha}{\beta}\,Q - \frac{d}{dQ}\frac{1}{\beta}\,Q^{\,2} \\ &= \frac{\alpha}{\beta} - \frac{2}{\beta}\,Q \end{aligned} \tag{6}$$

which is the marginal revenue function in Figure 2. The slope of that marginal revenue function is its derivative with respect to Q, namely,

$$\begin{aligned} \frac{d\,M(Q)}{dQ} &= \frac{d}{dQ}\frac{\alpha}{\beta} - \frac{d}{dQ}\frac{2}{\beta}\,Q \\ &= 0 - \frac{2}{\beta} \\ &= -\frac{2}{\beta}. \end{aligned} \tag{7}$$

which verifies that, when the demand curve is linear, the marginal revenue curve lies half the distance between the demand curve and the vertical axis. Notice also from the above that the derivative of an additive constant term is zero.

Since marginal revenue is the increase in total revenue from adding another unit, it follows that the total revenue associated with any quantity is the sum of the marginal revenues from adding all units up to and including that last one—that is

$$\sum_0^{Q_0} MR\,\Delta Q = \sum_0^{Q} \frac{\Delta TR}{\Delta Q}\,\Delta Q = TR_{Q_0}\,. \tag{8}$$

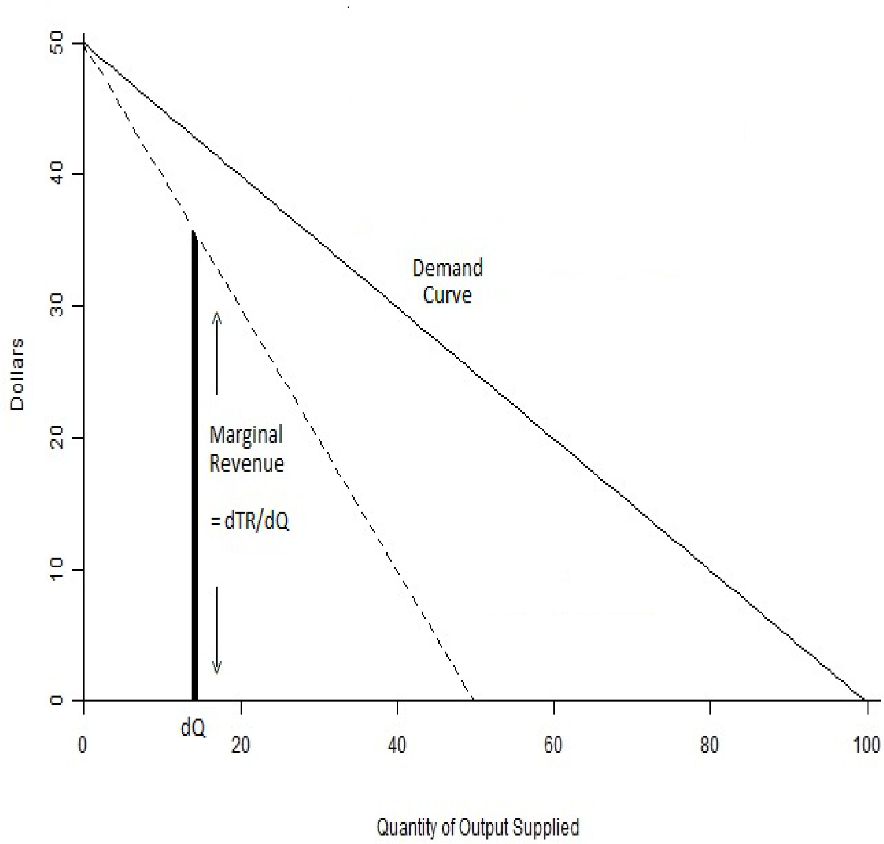

Figure 3: Total revenue visualized as the sum of sucessive marginal revenues.

When we represent marginal revenue as the derivative of the total revenue curve—that is, as

$$\frac{dTR}{dQ} = \lim_{\Delta \to 0} \frac{\Delta TR}{\Delta Q} \tag{9}$$

equation (8) can be rewritten as

$$\int_0^{Q_0} MR \; dQ = \int_0^{Q_0} \frac{dTR}{dQ} \; dQ = TR_{Q_0} \,. \tag{10}$$

Marginal revenue is the length of an infinitesimally narrow vertical slice extending from the marginal revenue curve down to the quantity axis in the reproduction of Figure 2 above and dQ is the width of that slice. We horizontally sum successive slices by calculating the integral of the marginal revenue function.

In the previous section, the effect of the stock of liquidity on the fraction of output that is useable—that is, not lost in the process of making exchange—is a third-degree polynomial term that multiplies the level of the capital stock as follows

$$Y = mK\Omega \tag{11}$$

where

$$\Omega = 1 - \frac{1}{3\lambda}\left(\phi - \lambda\frac{L}{K}\right)^3 . \tag{12}$$

The derivative of income Y with respect to the stock of liquidity L takes the form

$$\frac{dY}{dL} = mK\,\frac{d\Omega}{dL} \tag{13}$$

where, taking into account the fact that ϕ and K are constants,

$$\begin{aligned}
\frac{d\Omega}{dL} &= -3\frac{1}{3\lambda}\left(\phi - \lambda\frac{L}{K}\right)^2 \frac{d}{dL}\left(\phi - \lambda\frac{L}{K}\right) \\
&= -\frac{1}{\lambda}\left(\phi - \lambda\frac{L}{K}\right)^2 \left(\frac{d\phi}{dL} - \lambda\frac{d}{dL}\frac{L}{K}\right) \\
&= -\frac{1}{\lambda}\left(\phi - \lambda\frac{L}{K}\right)^2 \left(0 - \lambda\frac{d}{dL}\frac{L}{K}\right) \\
&= \left(\phi - \lambda\frac{L}{K}\right)^2 \lambda\left(\frac{d}{dL}\frac{L}{K}\right) \\
&= \left(\phi - \lambda\frac{L}{K}\right)^2 \frac{\lambda}{K}\left(\frac{dL}{dL}\right) \\
&= \frac{\lambda}{K}\left(\phi - \lambda\frac{L}{K}\right)^2
\end{aligned} \tag{14}$$

so that

$$\frac{dY}{dL} = m\lambda\left(\phi - \lambda\frac{L}{K}\right)^2 \tag{15}$$

which is a second-degree polynomial function. Since dY/dL is equal to the increase in the final output flow resulting from an increase in the stock of liquidity (or, in a cruder model, money) the above equation can be interpreted as a demand function for liquidity or money when we set $dYdL$ equal to the increase in the income flow to the individual money holder that will result from holding another unit of money. This income-flow comes at a cost of holding an additional unit of money, normally set equal to the nominal interest rate. Also, we need to incorporate the fact that the quantity of money demanded normally also depends on the level of income. It turns out that, under conditions of full-employment, the level of income can be roughly approximated by $m\,K$ so that, treating some measure of the money stock as an indicator of the level of liquidity, we can write the demand function for money as

$$i = m\lambda \left(\phi - \frac{\lambda}{m}\frac{M}{Y}\right)^2 \tag{16}$$

where i is the nominal interest rate. A further modification would have to be made to allow for income changes resulting from changes in the utilization of the capital stock in booms and recessions. In any event, you can see that an increase in i will reduce the level of M demanded at any given level of Y and and increase in Y holding i constant will also increase the quantity of M demanded.

Suppose now that we are presented with a fourth-degree polynomial of the form

$$y = F(x) = \alpha + \beta\,x + \gamma\,x^2 + \delta\,x^3 + \epsilon\,x^4\,. \tag{17}$$

We can differentiate this following the rule given by equation (5) together with the facts that the derivative of a constant is zero and the derivative of a sum of terms equals the sum of the respective derivatives to obtain

$$F'(x) = \frac{dy}{dx} = \beta + 2\,\gamma\,x + 3\,\delta\,x^2 + 4\,\epsilon\,x^3\,. \tag{18}$$

Suppose, alternatively, that we are given the function $F'(x)$ in equation (18) without seeing equation (17) and are asked to integrate it. We simply follow for each term the reverse of the differentiation rule by adding the integer 1 to the exponent of that term and then dividing the term by this modified exponent. Thus, for each term

$$\frac{dy}{dx} = \frac{d}{dx}\,a\,x^n = a\,n\,x^{n-1} \tag{19}$$

we obtain

$$\int \frac{dy}{dx}\,dx = \frac{n}{n-1+1}\,a\,x^{n-1+1} = a\,x^n\,. \tag{20}$$

Application of this procedure successively to the terms in equation (18) yields

$$\begin{aligned}\int F'(x)\,dx &= \frac{1}{1}\beta x^{0+1} + \frac{2}{2}\,\gamma\,x^{1+1} + \frac{3}{3}\,\delta\,x^{2+1} + \frac{4}{4}\,\epsilon\,x^{3+1} \\ &= \beta\,x + \gamma\,x^2 + \delta\,x^3 + \epsilon\,x^4 \end{aligned} \tag{21}$$

which, it turns out, differs from equation(17) in that it does not contain the constant term α. Without seeing (17), we had no way of knowing the magnitude of any constant term it contained and in the integration process in (21) inappropriately gave that constant term a value of zero. Accordingly, when integrating functions we have to automatically add to the integral an unknown constant term—the resulting integral is called an indefinite integral. To make it a definite integral, we have to have some initial information about the level of the resulting function and thereby be able to assign the correct value to the constant term. This problem does not arise when we are integrating from some initial x-value x_o and know the value of the function at that value of x.

The derivative of the logarithmic function

$$y = \alpha + \beta\,ln(x) \tag{22}$$

is

$$\frac{dy}{dx} = \frac{d\alpha}{dx} + \frac{d}{dx}\,\beta\,ln(x) = 0 + \beta\,\frac{d\,ln(x)}{dx} = \beta\,\frac{1}{x} = \frac{\beta}{x}\,. \tag{23}$$

As you can see, the derivative of $ln(x)$ is the reciprocal of x

$$\frac{d}{dx}\,ln(x) = \frac{1}{x} \tag{24}$$

which is consistent with the fact that the logarithm of a variable is the cumulation of its past relative changes. The indefinite integral of the function

$$F'(x) = \frac{\beta}{x}$$

is thus simply $\beta\,ln(x)$.

Then there is the exponential function with base e, $y = e^x$, which has the distinguished characteristic of being its own derivative—that is,

$$\frac{dy}{dx} = \frac{d}{dx}\, e^{\,x} = e^{\,x}\,. \tag{25}$$

This property can be verified using the fact, shown in the previous section, that

$$y = e^x \qquad \to \qquad x = ln(y)\,. \tag{26}$$

Given the existance of a smooth relationship between a variable and its logarithm, the reciprocal of the derivative of that function will also exist so that,

$$\frac{dx}{dy} = \frac{d}{dy}\, ln(y) = \frac{1}{y} \qquad \to \qquad \frac{dy}{dx} = y \qquad \to \qquad \frac{dy}{dx} = e^x\,. \tag{27}$$

Of course, we will often have to deal with more complicated exponential functions such as, for example,

$$y = \alpha\, e^{\,\beta\, x\, +\, c} \tag{28}$$

for which the derivative can be calculated according to the standard rules outlined above as

$$\begin{aligned} \frac{dy}{dx} &= \alpha\,(\beta\, x\, +\, c)\, e^{\,(\beta\, x\, +\, c) - 1}\, \frac{d}{dx}\,(\beta\, x\, +\, c) = \alpha\, \beta\,(\beta\, x\, +\, c)\, e^{\,(\beta\, x\, +\, c) - 1} \\ &= \alpha\,(\beta^{\,2}\, x\, +\, \beta\, c)\, e^{\,\beta\, x\, +\, c - 1}\,. \end{aligned} \tag{29}$$

At this point it is useful to collect together the rules for differentiating functions that have been presented thus far and to add some important additional ones.

Rules for Differentiating Functions

1) The derivative of a constant term is zero.

2) The derivative of a sum of terms equals the sum of the derivatives of the individual terms.

3) The derivative of a polynomial where a variable x is to the power n is

$$\frac{d}{dx} x^n = n\, x^{n-1}\,.$$

In the case of a function that is to the power n, the derivative is

$$\frac{d}{dx}\,\alpha\,F(x)^n = n\,\alpha\,F(x)^{n-1}\,F'(x)\,.$$

4) The derivative of the exponential function $y = b^x$ is

$$\frac{d}{dx}\,b^x = b^x\,ln(b)$$

or, in the case of $y = e^x$, since $ln(e) = 1$,

$$\frac{d}{dx}\,e^x = e^x\,.$$

In the case where $y = e^{F(x)}$ the derivative is

$$\frac{d}{dx}\,e^{F(x)} = e^{F(x)}\,F'(x)\,.$$

5) Where y is the logarithm of x to the base e, the derivative with respect to x is

$$\frac{dy}{dx} = \frac{d}{dx}\,ln(x) = \frac{1}{x}\,.$$

6) The chain rule. If

$$y = F_y(z) \qquad \text{and} \qquad z = F_z(x)$$

then

$$\frac{d}{dx}\,F_y(F_z(x)) = F_y'\,(F_z'(x))\,.$$

7) The derivative of the product of two terms equals the first term multiplied by the derivative of the second term plus the second term multiplied by the derivative of the first term.

$$\frac{d}{dx}\,[F_1(x)\,F_2(x)] = F_1(x)\,F_2'(x)\;+\;F_2(x)\,F_1'(x)$$

8) The derivative of the ratio of two terms equals the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{F_1(x)}{F_2(x)}\right] = \frac{F_2(x)\,F_1'(x)\;-\;F_1(x)\,F_2'(x)}{[F_2(x)]^2}$$

9) The total differential of a function of more than a single variable such as, for example,

$$y = F(x_1, x_2, x_3)$$

is

$$dy = \frac{\partial y}{\partial x_1}\, dx_1 + \frac{\partial y}{\partial x_2}\, dx_2 + \frac{\partial y}{\partial x_3}\, dx_3$$

where the term $\partial y/\partial x_i$ is the partial derivative of the function with respect to the variable x_i—that is, the change in y that occurs as a result of a change in x_i holding all other $x-$variables constant.

We end this section with some applications of the total differential rule immediately above. An interesting application is with respect to the standard Cobb-Douglas production function below which states that the level of output of a firm, denoted by X, depends in the following way upon the inputs of labour L and capital K

$$Y = A\, L^{\alpha}\, K^{\,1-\alpha} \tag{30}$$

where α is a parameter and A is a constant representing the level of technology. The total differential of this function is

$$\begin{aligned} dY &= A\,[K^{1-\alpha}\, \alpha L^{\alpha-1}\, dL + L^{\alpha}\,(1-\alpha)\, K^{1-\alpha-1}\, dK] \\ &= (A\, \alpha L^{\alpha-1} K^{1-\alpha})\, dL + (A\,(1-\alpha)\, L^{\alpha} K^{-\alpha})\, dK \end{aligned} \tag{31}$$

where you should note that $(A\, \alpha L^{\alpha-1} K^{1-\alpha})$ and $(A\,(1-\alpha)\, L^{\alpha} K^{-\alpha})$ are the respective marginal products of labour and capital. The increase in output associated with changes in the labour and capital inputs thus equals the marginal product of labour times the change in the input of labour plus the marginal product of capital times the change in the input of capital. Note also that, using a bit of manipulation, these marginal products can also be expressed as

$$\frac{\partial Y}{\partial L} = \frac{L\, A\, \alpha L^{\alpha-1} K^{1-\alpha}}{L} = \frac{\alpha\, A\, L^{\alpha} K^{1-\alpha}}{L} = \alpha\, \frac{Y}{L} \tag{32}$$

and

$$\frac{\partial Y}{\partial K} = \frac{K\, A\,(1-\alpha)\, L^{\alpha}\, K^{-\alpha}}{K} = \frac{(1-\alpha)\ A\, L^{\alpha} K^{1-\alpha}}{K} = (1-\alpha)\, \frac{Y}{K}\,. \tag{33}$$

Substituting the above two equations back into (31), we obtain

$$dY \;=\; \alpha\,\frac{Y}{L}\,dL + (1-\alpha)\frac{Y}{K}\,dK \tag{34}$$

which, upon division of both sides by Y becomes

$$\frac{dY}{Y} \;=\; \alpha\,\frac{dL}{L} + (1-\alpha)\frac{dK}{K}\,. \tag{35}$$

Actually, a simpler way to obtain this equation is to take the logarithm of equation (30) to obtain

$$ln(Y) = ln(A) + \alpha\, ln(L) + (1-\alpha)\, ln(K) \tag{36}$$

and then take the total differential to yield

$$\begin{aligned} d\,ln(Y) &= \alpha\,\frac{\partial ln(L)}{\partial L}\,dL + (1-\alpha)\,\frac{\partial ln(K)}{\partial K}\,dK \\ &= \alpha\,\frac{1}{L}\,dL + (1-\alpha)\,\frac{1}{K}\,dK \\ &= \alpha\,\frac{dL}{L} + (1-\alpha)\,\frac{dK}{K} \end{aligned} \tag{37}$$

An important extension of the total differential analysis in equations (31) and (34) is to the process of maximization. Suppose for example that the wage paid to a unit of labour is ω and the rental rate on a unit of capital is κ. The total cost of production for a firm would then be

$$T\,C = \omega\,L + \kappa\,K \tag{38}$$

and if the quantity of the labour input is changed holding total cost constant we will have

$$dTC = 0 = \omega\,dL + \kappa\,dK$$

which implies that

$$dK = -\frac{\omega}{\kappa}\,dL \tag{39}$$

which, when substituted into the total differential equation (34) yields

$$dY \;=\; \left[\alpha\,\frac{Y}{L} - \frac{\omega\,(1-\alpha)}{\kappa}\,\frac{Y}{K}\right]\,dL\,. \tag{40}$$

Maximization of the level of output producible at any given total cost requires that, starting from a very low initial level, L be increased until a level is reached for which a small change in L has no further effect on output. This will be the level of L for which

$$\alpha \frac{Y}{L} - \frac{\omega (1-\alpha)}{\kappa} \frac{Y}{K} = 0 \quad \rightarrow \quad \alpha \frac{Y}{L} = \frac{\omega (1-\alpha)}{\kappa} \frac{Y}{K} \, .$$

When both sides of the above are divided by $(1-\alpha)\, Y/K$, the expression reduces to

$$\frac{\alpha \, (Y/L)}{(1-\alpha)\, (Y/K)} = \frac{\omega}{\kappa} \, . \tag{41}$$

You will recognize from equations (32) and (33) that the left side of the above equation is simply the ratio of the marginal product of labour over the marginal product of capital—otherwise known as the marginal rate of substitution of labour for capital in production. As noted in the Figure below, this ratio is the slope of the constant output curve. Since the right side of the equation is the ratio of the wage rate to the rental rate on capital, the condition specifies that in equilibrium the marginal rate of substitution must be equal to the ratio of factor prices, the latter being the slope of the firm's budget line in the Figure below. The optimal use of factors in production is often stated as the condition that the wage rate of each factor of production be equal to the value marginal product of that factor, defined as the marginal product times the price at which the product sells in the market. You can easily see that multiplication of the numerator and denominator of the left side of the equation above by the price of the product will leave the optimality condition as stated there unchanged.

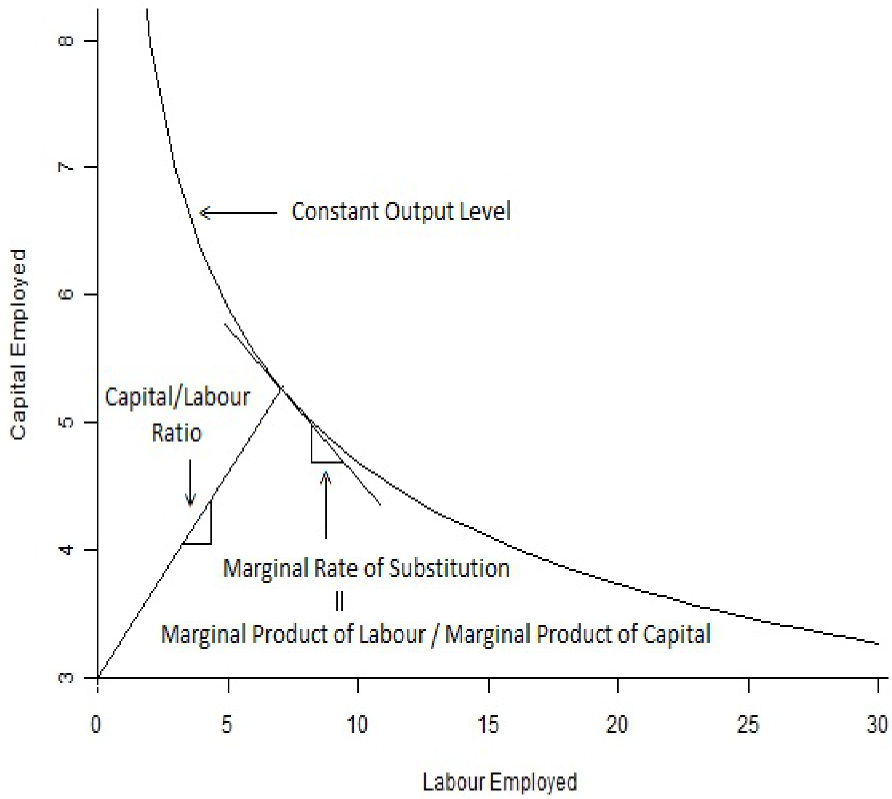

Figure 4: Cobb-Douglas production function $Y = AL^{.75}K^{.25}$ at specific output level.

Another concept of interest is the elasticity of substitution of labour for capital in production. It is defined as the elasticity of the ratio of capital to labour employed with respect to the marginal rate of substitution defined as the marginal product of labour divided by the marginal product of capital—that is, the relative change in the capital/labour ratio divided by the relative change in the ratio of the marginal product of labour over the marginal product of capital. The elasticity of substitution in the Cobb-Douglas production function can be obtained simply by differentiating the left side of equation (41) above with respect to the capital/labour ratio. First, we can cancel out the variable Y and then take the logarithm of both sides and differentiate the logarithm of the marginal rate of substitution with respect to the logarithm of the capital/labour ratio. This yields

$$\begin{aligned}
MRS &= \frac{\alpha\,(Y/L)}{(1-\alpha)\,(Y/K)} = \frac{\alpha}{1-\alpha}\,\frac{K}{L} \\
ln(MRS) &= ln\left(\frac{\alpha}{1-\alpha}\right) + \ln\left(\frac{K}{L})\right) \\
\frac{d\,ln(MRS)}{d\,ln\,(K/L)} &= \frac{d\,ln(\alpha)/(1-\alpha)}{d\,ln(K/L)} + \frac{d\,ln(K/L)}{d\,ln(K/L)} \\
\frac{d\,MRS}{MRS} \Big/ \frac{d\,(K/L)}{K/L} &= 0 + 1 \;=\; 1/\sigma \\
\sigma &= 1
\end{aligned} \tag{42}$$

where σ is the elasticity of substitution, which always equals unity when the production function is Cobb-Douglas.

The above result makes it worthwhile to use functions having constant elasticities that are different from unity. A popular function with this characteristic is the constant elasticity of substitution (CES) function which is written below as a utility function in the form

$$U = A\left[\delta\,C_1^{-\rho} + (1-\delta)\,C_2^{-\rho}\right]^{-1/\rho} \tag{43}$$

where U is the level of utility and C_1 and C_2 are the quantities of two goods consumed. Letting Ψ denote the terms within the square brackets, the marginal utility of C_1 can be calculated as

$$\begin{aligned}
\frac{\partial U}{\partial C_1} &= \frac{-A}{\rho}\left(\Psi^{\,[-(1/\rho)-1]}\right)(-\delta\,\rho)\,C_1^{-\rho-1} \\
&= A\,\delta\left(\Psi^{\,-(1+\rho)/\rho}\right)\,C_1^{-(1+\rho)} \\
&= A\,\delta\left[\delta\,C_1^{-\rho} + (1-\delta)\,C_2^{-\rho}\right]^{-(1+\rho)/\rho}\,C_1^{-(1+\rho)}\,.
\end{aligned} \tag{44}$$

By a similar calculation, the marginal utility of C_2 is

$$\frac{\partial U}{\partial C_2} \;=\; A\,(1-\delta)\left[\delta\,C_2^{-\rho} + (1-\delta)\,C_2^{-\rho}\right]^{-(1+\rho)/\rho}\,C_2^{-(1+\rho)} \tag{45}$$

and the marginal rate of substitution is therefore

$$MRS = \frac{\partial U/\partial C_1}{\partial U/\partial C_2} = \frac{\delta}{(1-\delta)}\left(\frac{C_1}{C_2}\right)^{-(1+\rho)} = \frac{\delta}{(1-\delta)}\left(\frac{C_2}{C_1}\right)^{(1+\rho)} \tag{46}$$

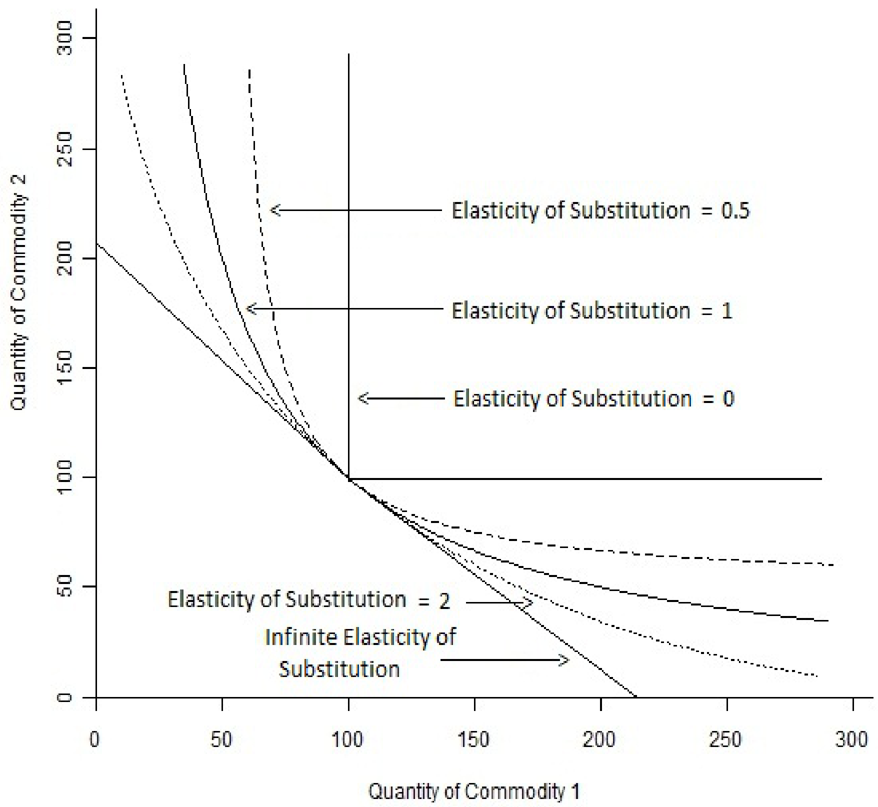

Figure 5: CES utility function $U = A[\delta\, C_1^{-\rho} + (1-\delta)\, C_2{}^{-\rho}]^{-1/\rho}$ at a specific utility level with $\delta = 0.5$ and the elasticity of substitution, equal to $1/(1+\rho)$, set alternatively at 0.5, 1.0, and 2.0 .

which, upon taking the logarithm, becomes

$$ln(MRS) = ln\left(\frac{\delta}{1-\delta}\right) + (1+\rho)\, ln\left(\frac{C_2}{C_1}\right) = (1+\rho)\, ln\left(\frac{C_2}{C_1}\right) . \tag{47}$$

The elasticity of substitution therefore equals

$$\sigma = \frac{d\, ln(C_2/C_1)}{d\, ln(MRS)} = \frac{1}{1+\rho} \tag{48}$$

and becomes equal to unity when $\rho = 0$, less than unity when ρ is positive and greater than unity when $-1 < \rho < 0$, becoming infinite as $\rho \to -1$. Obviously it would make no sense for ρ to be less than minus one. Indifference curves with elasticities of substitution ranging from zero to infinity are illustrated in Figure 5 above where $\delta = .5$.

It turns out that the Cobb-Douglas function is a special case of the CES function where $\rho = 0$, although equation (43) is undefined when $\rho = 0$ because division by zero is not possible. Nevertheless, we can demonstrate that as $\rho \to 0$ the CES function approaches the Cobb-Douglas function. To do this we need to use L'Hôpital's rule which holds that the ratio of two functions $m(x)$ and $n(x)$ approaches the ratio of their derivatives with respect to x as $x \to 0$.

$$\lim_{x\to 0} \frac{m(x)}{n(x)} = \lim_{x\to 0} \frac{m'(x)}{n'(x)} \tag{49}$$

When we divide both sides of equation (43) by A and take the logarithms we obtain

$$ln\left(\frac{Q}{A}\right) = \frac{-ln[\delta C_1^{-\rho} + (1-\delta)C_2^{-\rho}]}{\rho} = \frac{m(\rho)}{n(\rho)} \tag{50}$$

for which $m'(\rho)$ becomes, after using the chain rule and the rule for differentiating exponents with base b,

$$\begin{aligned} m'(\rho) &= \frac{-1}{[\delta C_1^{-\rho} + (1-\delta)C_2^{-\rho}]} \frac{d}{d\rho}[\delta C_1^{-\rho} + (1-\delta)C_2^{-\rho}] \\ &= \frac{-[-\delta\, C_1^{-\rho}\, ln(C_1) - (1-\delta)\, C_2^{-\rho}\, ln(C_2)]}{[\delta C_1^{-\rho} + (1-\delta)C_2^{-\rho}]} \end{aligned}$$

which, in the limit as $\rho \to 0$ becomes

$$m'(\rho) = \delta\, ln(C_1) + (1-\delta)\, ln(C_2)\,. \tag{51}$$

Since $n(\rho) = \rho$, $n'(\rho)$ also equals unity, so we have

$$\begin{aligned} \lim_{\rho\to 0} ln\left(\frac{Q}{A}\right) &= \lim_{\rho\to 0} \frac{m'(\rho)}{n'(\rho)} = \frac{\delta\, ln(C_1) + (1-\delta)\, ln(C_2)}{1} \\ &= \delta\, ln(C_1) + (1-\delta)\, ln(C_2) \end{aligned} \tag{52}$$

This implies that

$$Q = A\, C_1^{\delta}\, C_2^{1-\delta} \tag{53}$$

which is the Cobb-Douglas function.[6]

Exercises

1. Explain the difference between $\Delta y / \Delta x$ and dy/dx .

2. Differentiate the following function.

$$y = a\,x^{\,4} + b\,x^{\,3} + c\,x^{\,2} + d\,x + g$$

3. Integrate the function that you obtained in the previous question, assuming that you are without knowledge of the function you there differentiated.

4. Differentiate the function

$$y = a + b\, ln(x)\,.$$

5. Differentiate the following two functions and explain why the results differ.

$$y = a\, b^x \qquad\qquad y = a\, e^x$$

6. Given that $x = ln(y)$, express y as a function of x.

7. Suppose that

$$y = a + b\, z^{\,2} \qquad\qquad \text{and} \qquad\qquad z = e^{\,x}$$

Use the chain rule to calculate the derivative of y with respect to x .

8. Using the two functions in the previous question, find

$$\frac{d}{dx}\,(yz) \qquad\qquad \text{and} \qquad\qquad \frac{d}{dx}\left(\frac{y}{z}\right)\,.$$

[6] Every use of L'Hôpital's rule brings to mind the comments of a well-known economist when asked about the quality of two university economics departments, neither of which he liked. His reply was "You'd have to use L'Hôpital's rule to compare them".

9. Let $U(C_1)$ be the utility from consumption in period one and $U(C_2)$ be the utility from consumption in year two, where the functional form is identical in the two years. Let the utility from consumption in both periods be the present value in year one of utilities from consumption in the current and subsequent years where the discount rate is ω, otherwise known as the individual's rate of time preference. Thus,

$$U = U(C_1) + \frac{1}{1+\omega}\, U(C_2)\,.$$

Take the total differential of the present value of utility U. Let the interest rate r be the relative increase in year two consumption as a result of the sacrifice of a unit of consumption in year one, so that

$$\Delta C_2 = -(1+r)\,\Delta C_1$$

Assuming that the individual maximizes utility, calculate the condition for optimal allocation of consumption between the two periods.

10. Calculate the marginal products of labour and capital arising from the following production function.

$$Q = A[\delta\, L^{-\rho} + (1-\delta)\, K^{-\rho}]^{-1/\rho}$$

Then calculate the elasticity of substitution between the inputs. How do the results differ then you use the Cobb-Douglas production function instead of the CES production function above?

4. Matrix Algebra[7]

Finally, we review some basics of matrix algebra. Consider the three-equation model

$$\begin{aligned} a_{11}\,x_1 + a_{12}\,x_2 + a_{13}\,x_3 &= d_1 \\ a_{21}\,x_1 + a_{22}\,x_2 + a_{23}\,x_3 &= d_2 \\ a_{31}\,x_1 + a_{32}\,x_2 + a_{33}\,x_3 &= d_3 \end{aligned} \tag{1}$$

which, under appropriate conditions, can be solved for the three variables x_1, x_1 and x_1 in terms of the parameters a_{ij} and d_i. This system of equations can also be written in the following form where we chose i and j above to refer respectively to the rows and columns with the row denoted first.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

This system of equations can also be expressed as

$$\mathbf{A}\,\mathbf{x} = \mathbf{d} \tag{2}$$

where $\mathbf{A}$ is the 3×3 matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and the two column vectors,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

which can also be described as 3×1 matrices, are denoted as $\mathbf{x}$ and $\mathbf{d}$ respectively.

[7] An excellent detailed presentation of the principles of matrix algebra reviewed in the discussion that follows can be found in Alpha C.Chiang, *Fundamental Methods of Mathematcial Economics*, McGraw Hill, Third Edition, 1984, Chapters 4 and 5. Equivalent material can be found in the same chapters in the Fourth Edition, joint with Kevin Wainwright, published in 2005.

The matrix $\mathbf{A}$ can be transposed by interchanging its rows and columns as follows, where the transpose is denoted as $\mathbf{A}'$,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \mathbf{A}' = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

and, accordingly, the transposes of $\mathbf{x}$ and $\mathbf{d}$ convert them from column vectors into the row vectors

$$\mathbf{x}' = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \quad \text{and} \quad \mathbf{d}' = \begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix}.$$

Obviously, the transpose of the transpose of a matrix will be equal to the original matrix.

The sum of two matrices can be calculated only if both matrices have the same number of rows and columns—that is, are of the same dimension. The elements of the resulting matrix are the sums of the corresponding elements in the two matrices being summed, as noted below.

$$\mathbf{B} + \mathbf{C} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} = \begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} \\ b_{21}+c_{21} & b_{22}+c_{22} \\ b_{31}+c_{31} & b_{32}+c_{32} \end{bmatrix}$$

Multiplication of a matrix by a single number, which in this context is called a scalar, produces a new matrix whose elements are those of the original matrix after each element is multiplied by the scalar—that is,

$$k\,\mathbf{A} = \begin{bmatrix} k\,a_{11} & k\,a_{12} & k\,a_{13} \\ k\,a_{21} & k\,a_{22} & k\,a_{23} \\ k\,a_{31} & k\,a_{32} & k\,a_{33} \end{bmatrix}$$

where k is the scalar.

The multiplication of two matrices is much more complicated. As a start, we can review the multiplication of $\mathbf{A}$ and $\mathbf{x}$ in equation (2) which gives us the vector on the left side of the equality in equation (2).

$$\mathbf{A}\,\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}\,x_1 + a_{12}\,x_2 + a_{13}\,x_3 \\ a_{21}\,x_1 + a_{22}\,x_2 + a_{23}\,x_3 \\ a_{31}\,x_1 + a_{32}\,x_2 + a_{33}\,x_3 \end{bmatrix}$$

The result is a column vector which is obtained by multiplying the elements of each row of $\mathbf{A}$ by the corresponding element of the column vector $\mathbf{x}$ and then summing those products to yield the corresponding element of the resulting column vector.

Suppose now that we try to multiply the two matrices $\mathbf{B}$ and $\mathbf{C}$ which are reproduced below.

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \qquad\qquad \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

This cannot be done because the two matrices are not conformable—each column of $\mathbf{C}$ has three elements while each row of $\mathbf{B}$ has only two elements. To multiply two matrices, the left-most one must have the same number of columns as the right-most one has rows—that is, if the left-most matrix has the dimension $n \times m$ the the right-most matrix has the dimension $p \times q$, the two matrices are conformable for multiplication only if $m = p$. The multiplication process can be illustrated by multiplying the 2×4 matrix $\mathbf{H}$ by the 3×2 matrix $\mathbf{B}$.

$$\mathbf{B}\,\mathbf{H} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}\,h_{11} + b_{12}\,h_{21} & b_{11}\,h_{12} + b_{12}\,h_{22} & b_{11}\,h_{13} + b_{12}\,h_{23} & b_{11}\,h_{14} + b_{12}\,h_{24} \\ b_{21}\,h_{11} + b_{22}\,h_{21} & b_{21}\,h_{12} + b_{22}\,h_{22} & b_{21}\,h_{13} + b_{22}\,h_{23} & b_{21}\,h_{14} + b_{22}\,h_{24} \\ b_{31}\,h_{11} + b_{32}\,h_{21} & b_{31}\,h_{12} + b_{32}\,h_{22} & b_{31}\,h_{13} + b_{32}\,h_{23} & b_{31}\,h_{14} + b_{32}\,h_{24} \end{bmatrix}$$

The resulting matrix has dimension equal to the number of rows of the matrix on the left and the number of columns of the matrix on the right. This multiplication process produces a new matrix whose $(i,j)^{th}$ element is equal to the sum of products of the corresponding elements of the i^{th} row of $\mathbf{B}$ and j^{th} column of $\mathbf{H}$. Notice that we cannot take the product $\mathbf{H}\,\mathbf{B}$ because the number of columns of $\mathbf{H}$ is different from the number of rows of $\mathbf{B}$—that is we can only pre-multiply $\mathbf{H}$ by $\mathbf{B}$ or post-multiply $\mathbf{B}$ by $\mathbf{H}$.

Consider now the two vectors

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The inner product of these two vectors is

$$\mathbf{a}'\,\mathbf{b} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \mathbf{b}'\,\mathbf{a} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1\,b_1 + a_2\,b_2 + a_3\,b_3 \end{bmatrix}$$

and their outer product is

$$\mathbf{a}\,\mathbf{b}' = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \mathbf{b}\,\mathbf{a}' = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1\,b_1 & a_1\,b_2 & a_1\,b_3 \\ a_2\,b_1 & a_2\,b_2 & a_2\,b_3 \\ a_3\,b_1 & a_3\,b_2 & a_3\,b_3 \end{bmatrix}$$

where, in both cases, the number of rows in the left-most matrix, which is $1\,\times 3$ in the inner-product case and $3\,\times 1$ in the outer-product case, equals the number of columns in the right-most matrix, which is 3×1 in the inner-product case and $1\,\times 3$ in the outer-product case.

Suppose now that we want to solve the three equation system (1) which is reproduced below.

$$\begin{aligned} a_{11}\,x_1 + a_{12}\,x_2 + a_{13}\,x_3 &= d_1 \\ a_{21}\,x_1 + a_{22}\,x_2 + a_{23}\,x_3 &= d_2 \\ a_{31}\,x_1 + a_{32}\,x_2 + a_{33}\,x_3 &= d_3 \end{aligned} \qquad (1)$$

It is well known that for a three-equation system like this to have a solution, none of the equations can be a multiple of any other equation or a linear combination of the other equations—in that case, we would really have only a two-equation system above and the redundant equation would have to be dropped. In terms of matrix algebra, determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

would be zero in that case. The determinant of the matrix $\mathbf{A}$ is expressed as $|\mathbf{A}|$ and calculated as follows.

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\begin{aligned}
&= a_{11}(a_{22}\, a_{33} - a_{23}\, a_{32}) - a_{12}(a_{21}\, a_{33} - a_{23}\, a_{31}) + a_{13}(a_{21}\, a_{32} - a_{22}\, a_{31}) \\
&= (a_{11}\, a_{22}\, a_{33} - a_{11}\, a_{23}\, a_{32}) - (a_{12}\, a_{21}\, a_{33} - a_{12}\, a_{23}\, a_{31}) + (a_{13}\, a_{21}\, a_{32} - a_{13}\, a_{22}\, a_{31}) \\
&= a_{11}\, a_{22}\, a_{33} - a_{11}\, a_{23}\, a_{32} - a_{12}\, a_{21}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32} - a_{13}\, a_{22}\, a_{31} \qquad (3)
\end{aligned}$$

The calculations are more complicated in cases where the dimension of the matrix in question exceeds 3×3. The first condition to keep in mind here is that only square matrices have determinants. And the second condition is that if the determinant of a matrix is zero the matrix is singular and the underlying system of equations does not have a unique solution because at least one equation is a a multiple of another equation or a linear combination of other equations—and, therefore, one row (or column) of the matrix is not independent of other rows (or columns).

It turns out that for most non-numeric problems it is easier to try to solve a system of equations by substitution, and thereby determine if a solution exists, than to calculate the relevant determinant and then solve the system using matrix algebra. When the analysis is entirely numeric, however, the matrix approach is often easiest because one can use the computer in calculating determinants and in solving the equation system, especially when there are a large number of equations involved.

If $|\mathbf{A}|$ non-zero, and $\mathbf{A}$ is therefore non-singular, one needs to calculate the inverse of $\mathbf{A}$ to solve the system of equations. The inverse of this matrix is another square matrix denoted by $\mathbf{A}^{-1}$ for which

$$\mathbf{A}^{-1}\,\mathbf{A} = \mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I} \qquad (4)$$

where

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an identity matrix—that is a square matrix whose diagonal elements are 1 and all other elements are 0 . Obviously, the resulting identity matrix must have the same dimensions as the matrix whose inverse is being calculated.

The solution of the system

$$\mathbf{A}\,\mathbf{x} = \mathbf{d}$$

is therefore

$$\mathbf{A}^{-1}\,\mathbf{A}\,\mathbf{x} \quad = \quad \mathbf{A}^{-1}\,\mathbf{d}$$

which is the column vector

$$\mathbf{I}\,\mathbf{x} = \mathbf{x} = \mathbf{A}^{-1}\,\mathbf{d}\,. \tag{5}$$

Finally, we should note that the rank of a matrix is defined as the number of independent rows and columns of that matrix. Accordingly, the rank of an $m \times n$ matrix can never be greater than the smaller of m or n—rectangular matrices will have a rank equal to, at most, the lesser of the number of their rows or number of columns.

Exercises

1. Subtract the matrix

$$\begin{bmatrix} 4 & 0 & 8 \\ 6 & 0 & 2 \\ 8 & 2 & 3 \end{bmatrix} \qquad \text{from the matrix} \qquad \begin{bmatrix} 8 & 1 & 3 \\ 4 & 0 & 1 \\ 6 & 0 & 3 \end{bmatrix}.$$

2. Calculate the inner-product and outer-product of the vectors

$$\begin{bmatrix} 8 \\ 4 \\ 6 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}.$$

3. Pre-multiply and post-multiply the matrix

$$\begin{bmatrix} 4 & 8 \\ 6 & 2 \\ 8 & 3 \end{bmatrix} \qquad \text{by the matrix} \qquad \begin{bmatrix} 8 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}.$$

4. Pre-multiply and post-multiply the matrix

$$\begin{bmatrix} 8 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix} \qquad \text{by the matrix} \qquad \begin{bmatrix} 4 & 8 \\ 6 & 2 \end{bmatrix}.$$

5. Calculate the determinants of the two matrices

$$\begin{bmatrix} 4 & 0 & 8 \\ 6 & 0 & 2 \\ 8 & 2 & 3 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 4 & 2 & 2 \\ 6 & 2 & 3 \\ 8 & 4 & 2 \end{bmatrix}$$

and explain why they are singular or non-singular.

6. Put the three equations

$$\begin{aligned} 8\,x + y + 3\,z &= d \\ 4\,x + z &= e \\ 6\,x + 3\,z &= f \end{aligned}$$

into matrix form and then solve by substitution to obtain the equilibrium values of x , y and z in terms of the parameters d , e and f . Write the solution in matrix form and then extract the inverse of the original matrix that pre-multiplied the (x, y, z) column-vector. Assuming that $d = e = f = 1$, what are the equilibrium values of x , y and z ?

Computer Exercise

Work through the attached XLispStat matrix-calculations exercise and follow the programing instructions to obtain the correct answers to the above questions and thereby check your work. You need no initial knowledge of computer programming or the program XLispStat to work through the exercise—everything you will need to know is explained there.

Matrix Calculations Using XLispStat

While XLispStat is a program using the programming language Lisp (which means List Processing) for statistical calculations, we can very easily use it for computational matrix programing and analysis. To begin, click on the XLispStat icon on your desktop. A screen will appear with the following text at the top.

XLISP-PLUS version 3.03 Portions Copyright (c) 1988, by David Betz. Modified by Thomas Almy and others. XLISP-STAT Release 3.52.8 (Beta). Copyright (c) 1989-1998, by Luke Tierney.
>

The > character is the program's request for your input to tell it what to do.

Lisp commands have a very simple form. You enclose in a single set of brackets **( )** first the command or function you want the program to execute and then a series of words giving the arguments—that is, relevant information—the function requires in executing the command.

The first thing we need to do is give the XLispStat interpreter instructions to make some lists of numbers. The function we need to execute is named (not surprisingly) **list** and requires as arguments the group of numbers to appear in the list in the order in which you want them to appear. Also we need to give each list a name. This we do by using the function **def** which takes as its two arguments in this case the name of the list and the command to execute to produce the list. So let us make the following lists of numbers by entering the code

```
> (def LIST1 (list 4 6 8))
> (def LIST2 (list 0 0 2))
> (def LIST3 (list 8 2 3))
> (def LIST4 (list 8 4 6))
> (def LIST5 (list 1 0 0))
> (def LIST6 (list 3 1 3))
```

Notice that the **list** function is embedded in the **def** function.

To find out what items are in the work space we simply enter at the prompt the **variables** function, which takes no arguments, and to check the content of the lists we have just defined, we simply type the name of the list-object at the prompt. For example,

```
> (variables)
(LIST1 LIST2 LIST3 LIST4 LIST5 LIST6)
> LIST1
(4 6 8)
> LIST2
(0 0 2)
> LIST3
(8 2 3)
> LIST4
(8 4 6)
> LIST5
(1 0 0)
> LIST6
(3 1 3)
>
```

If you have a look at exercise 1 above, you will note that these lists are the columns of the two matrices there defined. To construct the matrices from these lists we use the **bind-columns** function, which takes as its arguments the names of the lists that are to form the columns. Again we have to use the **def** function to give these matrices names.

```
> (def MATA (bind-columns LIST1 LIST2 LIST3))
MATA
> (def MATB (bind-columns LIST4 LIST5 LIST6))
MATB
>
```

To have a look at these matrices, we use the **print-matrix** function, which takes as its single argument the name of the matrix to be printed.

```
> (print-matrix MATA)
#2a(
   (4   0   8   )
   (6   0   2   )
   (8   2   3   )
   )
NIL
```

```
> (print-matrix MATB)
#2a(
   (8   1   3  )
   (4   0   1  )
   (6   0   3  )
   )
NIL
>
```

The term **NIL** tell us that no new objects have been created in the workspace by our command.

Subtraction of **MATA** from **MATB** simply involves appropriate use of the **-** function which subtracts the second matrix listed from the first.

```
> (print-matrix (- MATB MATA))
#2a(
   ( 4   1  -5  )
   (-2   0  -1  )
   (-2  -2   0  )
   )
NIL
>
```

Suppose now that we want to take the transpose of **MATA**. To do this we use the function **transpose**, feeding it the name of the matrix and using the **def** function to give the transpose the name **MATAT**.

```
>(def MATAT (transpose MATA))
MATAT
>
```

And we can have a look at this transposed matrix using the command

```
> (print-matrix MATAT)
#2a(
   (4   6   8  )
   (0   0   2  )
   (8   2   3  )
   )
```

```
NIL
>
```

Another way to calculate the transpose of **MATA** is to bind the lists **LIST1 LIST2** and **LIST3** together as rows, not columns using the **bind-rows** function.

```
> (def MATC (bind-rows list1 list2 list3))
MATC
> (print-matrix MATC)
#2a(
   (4   6   8   )
   (0   0   2   )
   (8   2   3   )
   )
NIL
>
```

You will recognize that **MATC** and **MATAT** are identical.

Question 2 asked for calculation of the inner-and outer-products of two column vectors, which can be created as 3x1 matrices using the **bind-columns**, **list** and **def** functions.

```
> (def vec1 (bind-columns (list 8 4 6)))
VEC1
> (def vec2 (bind-columns (list 2 3 3)))
VEC2
> (print-matrix VEC1)
#2a(
   (8   )
   (4   )
   (6   )
   )
NIL
> (print-matrix VEC2)
#2a(
   (2   )
   (3   )
   (3   )
   )
```

NIL
>

Notice that we embedded the **list** function in the **bind-columns** function which was then embedded in the **def** function.

The inner-product and the outer-product can be calculate using the function **matmult**, which pre-multiplies the right-most matrix by the left-most matrix given as its two arguments.

```
> (def IPROD (matmult (transpose VEC1) VEC2))
IPROD
> (print-matrix IPROD)
#2a(
   ( 46.0000 )
   )
NIL
> (def OPROD (matmult VEC1 (transpose VEC2)))
OPROD
> (print-matrix OPROD)
#2a(
   (  16.0000  24.0000  24.0000  )
   (   8.0000  12.0000  12.0000  )
   (  12.0000  18.0000  18.0000  )
   )
NIL
>
```

Next we construct the two matrices in Question 3, embedding alternatively the **list** and the **bind-columns** and **list** and **bind-rows** functions in the **def** function.

```
> (def MATD (bind-columns (list 4 6 8)(list 8 2 3)))
MATD
> (def MATE (bind-rows (list 8 1 3)(list 4 2 1)))
MATE
> (print-matrix MATD)
#2a(
   (4  8  )
   (6  2  )
   (8  3  )
```

```
    )
NIL

> (print-matrix MATE)
#2a(
    (8   1   3   )
    (4   2   1   )
    )
NIL
>
```

Now when we pre- and post-multiply **MATD** by **MATE** we get

```
> (def MATED (matmult MATE MATD))
MATED
> (def MATDE (matmult MATD MATE))
MATDE
> (print-matrix MATED)
#2a(
    (   62.0000   75.0000   )
    (   36.0000   39.0000   )
    )
NIL
> (print-matrix MATDE)
#2a(
    (   64.0000   20.0000   20.0000   )
    (   56.0000   10.0000   20.0000   )
    (   76.0000   14.0000   27.0000   )
    )
NIL
>
```

Notice that the two product matrices are completely different.

Question 4 asks you to pre- and post-multiply **MATD** above by the 2×2 matrix constructed below.

```
> (def MATF (bind-columns (list 4 6)(list 8 2)))
MATF
```

```
> (print-matrix MATF)
#2a(
   (4  8  )
   (6  2  )
   )
NIL
>
```

Pre-multiplication of **MATF** by **MATD** yields

```
> (def MATDF (matmult MATD MATF))
MATDF
> (print-matrix MATDF)
#2a(
   (  64.0000  48.0000  )
   (  36.0000  52.0000  )
   (  50.0000  70.0000  )
   )
NIL
>
```

while post-multiplication of **MATF** by **MATD** produces the error message

```
> (def MATFD (matmult MATF MATD))
Error: dimensions do not match
Happened in: #<Byte-Code-Closure-MATMULT: #1418c34>
>
```

which results, of course, because the matrices are not conformable for multiplication in that order—**MATF** has 2 columns while matrix **MATD** has 3 rows.

Question 5 asks for calculation of the determinant of **MATA** and the determinant of the following new matrix.

```
> (def MATG (bind-columns (list 4 6 8)(list 2 2 4)(list 2 3 2)))
MATG
> (print-matrix MATG)
#2a(
   (4  2  2  )
   (6  2  3  )
   (8  4  4  )
```

)
NIL
>

To calculate determinants we use the function **determinant** which takes as its sole argument the matrix for which the determinant is to be calculated. Applying this function to the two matrices above yields

> **(def DETMATA (determinant MATA))**
DETMATA
> **DETMATA**
80.0
>

which is non-singular and

> **(def DETMATG (determinant MATG))**
DETMATG
> **DETMATG**
0.0
>

which is singular because, as you can see, the third column is the first column multiplied by 2 and the two columns are therefore not independent.

Finally, we need to do the computations relevant for Question 6. The three-equation system is as follows in matrix form.

$$\begin{bmatrix} 8 & 1 & 3 \\ 4 & 0 & 1 \\ 6 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

Solving the sytem by substitution produces the following results.

$$\begin{aligned} x &= (1/2)\, e - (1/6)\, f \\ y &= d - e - (2/3)\, f \\ z &= -e + (2/3)\, f \end{aligned}$$

which can be presented in matrix form as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.0000 & 0.5000 & -1.6666 \\ 1.0000 & -1.0000 & -0.6666 \\ 0.0000 & -1.0000 & 0.3333 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

where the square matrix is the inverse of the original one. We construct the original matrix in XLispStat using the following code.

```
> (def ORIGMAT (bind-columns (list 8 4 6)(list 1 0 0)(list 3 1 3)))
ORIGMAT
> (print-matrix ORIGMAT)
#2a(
    (8   1   3   )
    (4   0   1   )
    (6   0   3   )
    )
NIL
>
```

The inverse of this matrix is

```
> (def INVOMAT (inverse ORIGMAT))
INVOMAT
> (print-matrix INVOMAT)
#2a(
    (-6.938894E-18    0.50000   -0.166667   )
    ( 1.00000        -1.00000   -0.666667   )
    ( 2.775558E-17   -1.00000    0.666667   )
    )
NIL
>
```

which is the same as the one calculated by hand using substitution, except for the fact that XLispStat makes mathematically perfect calculations which recognize that the zero-numbers are only approximate and must therefore be presented in scientific notation. To obtain conventional decimal numbers, the decimal point of the number in the upper left corner of the matrix has to be moved 18 positions to the left, yielding the conventional decimal number −.0000000000000000006938894, and 17 positions to the left in case of the number in the bottom left corner, yielding .000000000000000002775558.

Finally, assuming that $d = e = f = 1$, the solution of the system can be obtained in XLispStat as follows.

> **(def XYZEQ (matmult INVOMAT (bind-columns (list 1 1 1))))**
XYZEQ
> **XYZEQ**
#2A((0.33333333333333337) (-0.6666666666666666) (-0.33333333333333337))
>

You are advised that the calculations here are performed in the XLispStat batch file **matrix.lsp** and the output you will obtain from running that file is in the file **matrix.lou**. You will notice in the batch file the functions **princ** and **terpri** that are not discussed here. The **princ** function tells the program to print whatever is included as an argument. When printing text, the material must be encased in quotation marks while printing objects in the workspace requires only the name of the object. The **terpri** function, which is placed in the usual brackets without any arguments included, merely tells the interpreter to start a new line.