

# Costly Self Control and Random Self Indulgence<sup>1</sup>

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## Abstract

We study the *random Strotz* model, a version of the Strotz (1955) model with uncertainty about the nature of the temptation that will strike. We show that the random Strotz representation is unique and characterize a comparative notion of “more temptation averse.” Also, we demonstrate an unexpected connection between the random Strotz model and the Gul–Pesendorfer (2001) (GP) model of temptation. In particular, a preference over menus has a GP representation iff it also has a representation via a sufficiently smooth random Strotz model. We also show that choices of menus combined with choices from menus can distinguish the random GP and random Strotz models.

# 1 Introduction

In this paper, we explore the *random Strotz* model which is a version of the classic Strotz (1955) model of temptation adding uncertainty about the nature of the temptation. Uncertainty is both a plausible and a useful hypothesis regarding temptation. Such uncertainty is frequently a key part of applications of the Strotz model, such as in Battaglini, Benabou, and Tirole (2005), Benabou and Tirole (2004, 2010), Eliaz and Spiegler (2006), and Harris and Laibson (2008). Also, as noted by Caplin and Leahy (2006), for example, uncertainty can “smooth out” the discontinuities present in the usual nonstochastic Strotz model.

The resulting model has some surprising connections to more recent models of temptation. In particular, we show a sense in which the random Strotz model nests the self-control model of Gul and Pesendorfer (2001) (henceforth GP). Specifically, the latter exhibits the same commitment behavior — that is, the same choice of “menus” from which future choices will be made — as a particular class of random Strotz models. In addition, a random generalization of the GP model is equivalent in this sense to the class of all sufficient smooth (in a sense to be made precise) randomizations over Strotz models.

We also show that commitment behavior is sufficient to identify the random Strotz model uniquely. In other words, any commitment behavior is consistent with at most one random Strotz model. Given that the random Strotz model is uniquely identified from such behavior, we can characterize how commitment choices vary as we change aspects of the representation. More specifically, we show that a certain kind of first-order stochastic dominance shift in the uncertainty faced corresponds to a reduction in the agent’s concern about temptation. By contrast, the random GP model is not identified in the same way as random Strotz. When commitment behavior is consistent with at least one random (or nonrandom) GP model, it is consistent with infinitely many distinct random GP models.

Our results are useful for several reasons. First, they clarify the foundations of the random Strotz model and some of its properties. For example, our uniqueness and comparative results may be valuable in the study of temptation with uncertainty as in the papers cited above. Also, the relationship between the random Strotz and random GP models yields a simple axiomatization of a subclass of random Strotz models. Specifically, Stovall (2010) gives an axiomatic characterization of the random GP model for the case where the support of the measure is finite. It is straightforward to show that the connection between random Strotz and random GP we demonstrate implies that Stovall’s axioms characterize a subclass of random Strotz models.<sup>1</sup>

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<sup>1</sup>See the earlier version of this paper, Dekel and Lipman (2010), for more discussion of this point and an axiomatization of random Strotz based on our generalization of Stovall’s result.

Second, the connection between the random Strotz and GP models has significant methodological implications. Most of the work on temptation has focused on using commitment behavior alone as a means of identifying a model, implicitly or explicitly assuming that subsequent choices from menus can be deduced from commitment behavior.<sup>2</sup> Since we show that two very natural models are consistent with the same commitment behavior and different choices from menus, such assumptions are not warranted in general. Instead, our results suggest that we should broaden the set of data considered. In particular, if we consider both commitment choices and choices from menus, then we can separate the two models.

Finally, these results may be helpful in other areas where the random Strotz model can be applied. For example, Olszewski (2007) and Ahn (2007) consider models of ambiguity where an act is viewed not as a function from states to consequences but as a set of lotteries, where this is interpreted as a set of consequences. (See also related work by Gajdos, Hayashi, Tallon, and Vergnaud (2008).)<sup>3</sup> In other words, we interpret a menu not as a set of options that the agent will choose from later but as a set of possible outcomes that “Nature” will choose from later. Under this interpretation, the random Strotz model represents the agent as forming various theories about what guides Nature’s choices. Similarly, any model of control rights necessarily has a Strotzian aspect to it, in that an agent must evaluate his utility from the expected choices by another agent given particular constraints. The relationship between random Strotz and random GP provides a way to axiomatize such representations, as noted above. In addition, our uniqueness and comparative results should be useful for such models.

The basic point that the GP representation can be rewritten in terms of a random determination of which self has control has been made before, though in very different ways. In particular, Benabou and Pycia (2002) note that the GP representation can be written as the equilibrium payoff of a game between the current and future self engaging in a costly battle for control. Also, Chatterjee and Krishna (2007) show that a preference with a GP representation also has a representation where there is a menu-dependent probability that the choice is made by the tempted self, with the choice made by the unttempted self otherwise. Unfortunately, the properties of the function relating menus to probabilities over control make it difficult to interpret in general.<sup>4</sup>

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<sup>2</sup>See, for example, Gul and Pesendorfer (2001) and Dekel, Lipman, and Rustichini (2009).

<sup>3</sup>The Steiner point, which plays a significant role in the analysis of Gajdos, Hayashi, Tallon, and Vergnaud, provides an interesting connection between their work and random Strotz. One definition of the Steiner point of a set of lotteries is that it is the expected value of the lottery chosen by an expected utility preference which is drawn at random from a uniform distribution. Thus it is the expected choice by a particular random Strotz agent.

<sup>4</sup>The published version of Chatterjee and Krishna’s paper, Chatterjee and Krishna (2009), considers only the case where this probability is independent of the menu. While this provides more structure, the constant probability model no longer nests GP. On the other hand, this version of their model is a special case of the random Strotz model we consider.

In both cases, the essence of the argument is the fact that one can find behavior that gives the agent higher utility than in the GP representation and other behavior which gives him lower utility, implying that a certain randomization between the two gives the same utility. Our result provides a tighter connection through a model which is an interesting and natural alternative formulation in its own right.

While not the main purpose of their work, Fudenberg and Levine (2006, 2010a, 2010b)’s dual-selves model also gives a connection between GP and multiple-selves models. Our approach enables us to show an unexpected connection between the Fudenberg–Levine model and random Strotz. Roughly speaking, an adaptation of our result linking GP and random Strotz shows that we can recast the dual-selves model as random Strotz where the tempted self may not maximize expected utility. See Section 3 for a more precise statement.

The next section defines the model and the representations considered. In Section 3, we relate random Strotz representations to (random) GP representations. Section 4 shows the uniqueness and comparative results described above. In Section 5, we discuss choice from menus. Proofs not contained in the text are in the Appendix or supplementary appendix.

## 2 Definitions

Fix a finite set  $Z$  of “prizes” or outcomes, let  $\Delta(Z)$  denote the set of lotteries over  $Z$ , and let  $X$  denote the set of *menus*, the set of compact, nonempty subsets of  $\Delta(Z)$ . The current self has a preference over  $X$ , denoted  $\succeq$ , interpreted as a preference regarding how much commitment to impose on subsequent choices. (In Section 5, we discuss how we represent choices from menus.) Throughout, we assume that  $\succeq$  is *nontrivial* in the sense that there exist  $x, y \in X$  such that  $x \succ y$ .

A function  $w : \Delta(Z) \rightarrow \mathbf{R}$  is an *expected utility (EU) function* if  $w(\lambda\alpha + (1 - \lambda)\beta) = \lambda w(\alpha) + (1 - \lambda)w(\beta)$  for all  $\lambda \in [0, 1]$  and  $\alpha, \beta \in \Delta(Z)$ . Both the Strotz and GP representations use two expected utility functions,  $u, v : \Delta(Z) \rightarrow \mathbf{R}$ . The Strotz representation uses the pair  $(u, v)$  to evaluate a menu  $x$  by

$$V_S(x) = \max_{\beta \in B_v(x)} u(\beta)$$

where  $B_v(x)$  is the set of best elements of  $x$  according to  $v$ . That is,

$$B_v(x) = \{\beta \in x \mid v(\beta) \geq v(\alpha), \quad \forall \alpha \in x\}.$$

Intuitively,  $v$  represents the preference of the future self who will be completely self indulgent, choosing from the menu as he wishes, breaking ties in favor of the current self who has utility function  $u$ .

One unfortunate feature of the Strotz model is that the agent's utility depends discontinuously on the commitments he makes. This occurs because when the choosing self is almost indifferent, the current self may still have strong preferences regarding the choices. A small change in commitments can then create indifference for the chooser. Hence we can find such small changes in commitments that have big effects on the current self's payoff.<sup>5</sup> This discontinuity is both intuitively implausible and analytically inconvenient.

The representation introduced by GP is continuous and hence avoids this problem. In their representation, a menu  $x$  is evaluated by the function

$$V_{GP}(x) = \max_{\beta \in x} [u(\beta) + v(\beta)] - \max_{\alpha \in x} v(\alpha).$$

This representation also has an interesting interpretation. As GP emphasize, the agent chooses from the menu the item which maximizes  $u + v$ , not  $v$ . In this sense, the GP model seems behaviorally richer than the Strotz model as the agent shows partial self control by compromising between  $u$  and  $v$  instead of simply maximizing  $v$ . The term  $[\max_{\alpha \in x} v(\alpha)] - v(\beta)$  can be interpreted as the cost of resisting temptation by choosing  $\beta$  instead of maximizing  $v$ .

As noted, we consider random versions of the GP and Strotz models. Letting  $K$  denote the number of elements of  $Z$ , we identify the set of EU functions with  $\mathbf{R}^K$  since for any such function, we only need to specify the payoffs to the pure outcomes. We use the Borel field over  $\mathbf{R}^K$ . We say that an EU function is *trivial* if it is a scalar times a vector of 1's. We say a measure  $\mu$  over  $\mathbf{R}^K$  is nontrivial if it assigns zero measure to the set of trivial EU functions.

**Definition 1.** A random Strotz representation of  $\succeq$  is a pair  $(u, \mu)$  such that  $u$  is a nontrivial expected utility function and  $\mu$  is a nontrivial measure over expected utility functions such that the function

$$V_{RS}(x) = \int_{\mathbf{R}^K} \max_{\beta \in B_w(x)} u(\beta) \mu(dw)$$

represents the preference.

This is the Strotz representation but where the agent is not sure what his future self's preference will be. It seems quite natural to suppose that an agent may not know

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<sup>5</sup>This difficulty is not eliminated by changing the tie-breaking rule.

exactly what will tempt him in the future or to what extent. Adding uncertainty to the Strotz model also has the potential to resolve the continuity problems noted above since suitably atomless noise ensures that the probability the chooser is indifferent will be zero. Consequently, as Caplin and Leahy (2006) show, such atomlessness can ensure existence of an optimal policy in Strotz's sense.

A random GP representation generalizes the notion of a GP representation in a fashion exactly parallel to the above: specifically, the  $u$  is fixed but there is a probability measure over the "temptations."

**Definition 2.** *A random GP representation is a pair  $(u, \nu)$  such that  $u$  is a nontrivial expected utility function and  $\nu$  is a nontrivial measure over expected utility functions such that the function*

$$V_{RGP}(x) = \int_{\mathbf{R}^K} \left\{ \max_{\alpha \in x} [u(\alpha) + v(\alpha)] - \max_{\alpha \in x} v(\alpha) \right\} \nu(dv)$$

*represents the preference.*

**Remark 1.** For both random Strotz and random GP, the nontriviality of the measure is without loss of generality in the sense that if a representation exists, then one with a nontrivial measure exists. In both cases, the nontriviality of  $u$  is implied by our assumption that  $\succeq$  is nontrivial.

### 3 Random Strotz and Random GP Models

This section discusses the relationship between the models.

**Theorem 1.** *Any preference with a random GP representation also has a random Strotz representation.*

*Proof.* We first show the claim for an arbitrary GP representation  $(u, \nu)$ . Let  $W$  denote the set of expected utility preferences such that  $w \in W$  iff there exists  $A \in [0, 1]$  with  $w = v + Au$ . Define a measure  $\mu$  over  $W$  by taking the uniform distribution over  $A$ . That is, for a set  $E \subseteq W$ , we have

$$\mu(E) = \Pr[\{A \in [0, 1] \mid v + Au \in E\}],$$

where  $\Pr(\cdot)$  is the uniform distribution. Finally, let  $V_{RS}$  denote the random Strotz representation generated by this measure.

Fix any menu  $x$ . Let  $\beta^*(A)$  denote any element of  $x$  which maximizes  $u$  over the set  $B_{v+Au}(x)$ . Let  $\hat{u}(A) = u(\beta^*(A))$  and let  $\hat{v}(A) = v(\beta^*(A))$ . Note that if multiple elements of  $x$  maximize  $u$  over  $B_{v+Au}(x)$ , the values of  $\hat{u}(A)$  and  $\hat{v}(A)$  do not depend on the particular choice of  $\beta^*(A)$ . Also, it is easy to show that  $\hat{u}$  is nondecreasing in  $A$  and hence measurable. Since  $\hat{u}$  is also bounded, it is integrable. We have

$$V_{RS}(x) = \int_0^1 u(\beta^*(A)) dA = \int_0^1 \hat{u}(A) dA.$$

Define

$$\mathcal{U}(A) = \hat{v}(A) + A\hat{u}(A) = \max_{\bar{A} \in [0,1]} \hat{v}(\bar{A}) + A\hat{u}(\bar{A}).$$

From the usual argument characterizing incentive compatibility with transferrable utility (see, *e.g.*, Mas-Colell, Whinston, and Green (1995), Proposition 23.D.2, page 888, or Milgrom and Segal (2002), Theorem 2),<sup>6</sup> we have

$$\mathcal{U}(s) = \mathcal{U}(0) + \int_0^s \mathcal{U}'(A) dA = \mathcal{U}(0) + \int_0^s \hat{u}(A) dA.$$

Hence

$$\mathcal{U}(1) - \mathcal{U}(0) = \int_0^1 \hat{u}(A) dA = V_{RS}(x).$$

But  $\mathcal{U}(1) = \max_{\beta \in x} [v(\beta) + u(\beta)]$ , while  $\mathcal{U}(0) = \max_{\beta \in x} v(\beta)$ . Hence the left-hand side is the GP representation

To extend to a random GP representation, note that

$$V_{RGP}(x) = \int_{\mathbf{R}^K} \left\{ \max_{\alpha \in x} u(\alpha) + v(\alpha) \right\} - \max_{\alpha \in x} v(\alpha) \Big\} \nu(dv)$$

so

$$V_{RGP}(x) = \int_{\mathbf{R}^K} \left\{ \int_0^1 \max_{\beta \in B_{v+Au}(x)} u(\beta) dA \right\} \nu(dv) \tag{1}$$

which is a random Strotz representation. ■

Clearly, not every random Strotz model is also a random GP. A random Strotz model can be discontinuous since random Strotz includes nonrandom Strotz as a special case, while the random GP model inherits the continuity of GP.

So which random Strotz models are also random GP models? Equation (1) gives a partial answer. If we can write the distribution over  $w$  in the form of a distribution

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<sup>6</sup>For intuition, consider a standard auction problem or other characterization of incentive compatibility with quasi-linear utility. View  $A$  as the type of the agent where this is his valuation for some good. Then  $\bar{A}$  plays the role of the agent's report of his type,  $\hat{u}(\bar{A})$  is the probability the agent obtains the good if his report is  $\bar{A}$ , and  $\hat{v}(\bar{A})$  is the transfer to him when his report is  $\bar{A}$ .



over  $v$ 's and  $A$ 's as in equation (1), then the representation is also a random GP model. Writing a distribution over  $w$  through a distribution of  $v$  and  $A$  where  $w = v + Au$  is not itself restrictive at all. Obviously, any  $w$  can be written as  $v + Au$  for appropriately chosen  $v$  and  $A$ . On the other hand, the requirement that  $A \sim U[0, 1]$  independently of  $v$  is quite special.

We show that a much weaker requirement suffices: we can drop the independence of  $v$  and  $A$  and consider any conditional densities over  $A$  in a large class. More specifically, we generalize the random Strotz representation from the form in equation (1) to what we call *nice* random Strotz representations: those that can be written as

$$\int_{\mathbf{R}^K} \left\{ \int_0^1 \max_{\beta \in B_{v+Au}(x)} u(\beta) f(A | v) dA \right\} \nu(dv)$$

where  $f(\cdot | v)$  is lower semicontinuous.<sup>7</sup>

More precisely, given a random Strotz representation  $(u, \mu)$ , we define a *decomposition* of  $\mu$  to be a set  $\mathcal{V} \subseteq \mathbf{R}^K$ , a probability measure  $\mu_{\mathcal{V}}$  on  $\mathcal{V}$ , and a family of conditional probability measures  $\mu_A(\cdot | v)$  on  $\mathbf{R}$  such that for all measurable  $E \subseteq \mathbf{R}^K$ ,<sup>8</sup>

$$\mu(E) = \int_{\mathcal{V}} \mu_A(\{A \in \mathbf{R}^K | v + Au \in E\} | v) \mu_{\mathcal{V}}(dv).$$

We say that the random Strotz representation  $(u, \mu)$  is *nice* if there exists a decomposition of  $\mu$ , say  $(\mathcal{V}, \mu_{\mathcal{V}}, \mu_A(\cdot | v))$  such that for  $\mu_{\mathcal{V}}$ -almost all  $v \in \mathcal{V}$ ,  $\mu_A$  is representable by a lower semicontinuous density. That is, for almost all  $v \in \mathcal{V}$  and every measurable  $E \subseteq \mathbf{R}$ , we have

$$\mu_A(E | v) = \int_E f(A | v) dA$$

where for every  $a \geq 0$ ,  $\{A \in \mathbf{R} | f(A) > a\}$  is open.

**Theorem 2.** *The preference  $\succeq$  has a random GP representation if and only if it has a nice random Strotz representation.*

*Proof sketch.* The proof is in the Appendix. Here we offer an intuition for why the result holds.

First, it is easy to see from equation (1) that if  $\succeq$  has a random GP representation, it must have a nice random Strotz representation. Turning to the converse, we focus on the case where  $\mathcal{V}$  is a singleton; the extension to the general case is straightforward.

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<sup>7</sup>This imposes two smoothness conditions. First, the measure must have a density. The second condition, lower semicontinuity, seems relatively weak. It strengthens the necessary property of a density that  $\{A | f(A) > a\}$  be measurable to the requirement that such sets are open.

<sup>8</sup>The measures  $\mu_{\mathcal{V}}$  and each  $\mu_A(\cdot | v)$  are defined on the Borel  $\sigma$ -algebras.

So suppose we have a random Strotz representation of the form

$$V(x) = \int_{\mathcal{A}} \max_{\beta \in B_{v+Au}(x)} u(\beta) f(A) dA$$

where  $f$  is lower semicontinuous and  $\mathcal{A}$  is the support of  $f$ . We explain how to rewrite  $V$  in the form of a random GP representation.

It is easy to see how we can rewrite the random Strotz as a random GP when  $A$  is distributed uniformly over some interval other than  $[0, 1]$ . To see this, simply note that

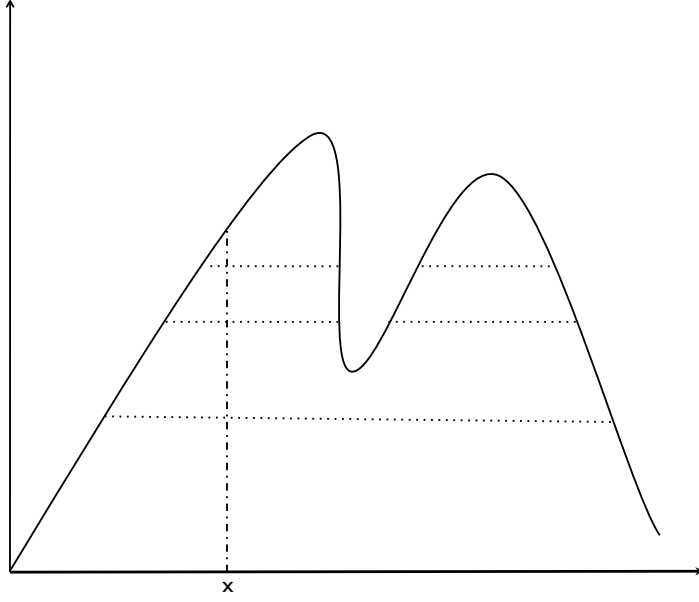
$$\begin{aligned} \int_a^b \max_{\beta \in B_{v+Au}(x)} u(\beta) \frac{1}{b-a} dA &= \int_0^{b-a} \max_{\beta \in B_{v+au+Au}(x)} u(\beta) \frac{1}{b-a} dA \\ &= \int_0^{b-a} \max_{\beta \in B_{(v+au+Au)/(b-a)}(x)} u(\beta) \frac{1}{b-a} dA. \end{aligned}$$

Let  $\bar{v} = (v + au)/(b - a)$ . Substituting:

$$\int_a^b \max_{\beta \in B_{v+Au}(x)} u(\beta) \frac{1}{b-a} dA = \int_0^{b-a} \max_{\beta \in B_{\bar{v}+[A/(b-a)]u}(x)} u(\beta) \frac{1}{b-a} dA = \int_0^1 \max_{\beta \in B_{\bar{v}+\bar{A}u}(x)} u(\beta) d\bar{A}$$

where the last equality follows from the change of variables  $\bar{A} = A/(b - a)$ . From the proof of Theorem 1, we see that this equals the GP representation  $(u, \bar{v})$ .

From this fact, it is not hard to see that if we can rewrite  $f$  as a randomization over uniform distributions over various intervals, then we can rewrite the random Strotz as a randomization over GP representations — that is, as a random GP. It turns out that this can be done if  $f$  is lower semicontinuous. To see the idea, consider the density  $f$  shown in the figure below.



We identify various uniform distributions by taking the supports to correspond to upper contour sets of  $f$  as shown by the dotted lines in the figure. Index these various collections of intervals by  $a$  where this denotes the value of  $f(x)$  to which they correspond. To match the original density at  $x$ , we seek a distribution  $h$  such that

$$f(x) = \int_0^{f(x)} \frac{1}{\lambda(\{x' \mid f(x') \geq a\})} h(a) da$$

where  $\lambda(\cdot)$  denotes Lebesgue measure. For brevity, let

$$\ell(a) = \lambda(\{x' \mid f(x') \geq a\}).$$

Note that the problem above has a trivial solution: Let  $h(a) = \ell(a)$ . Obviously, this makes the equality hold, so the only question is whether this is a legitimate density. It is easy to see that  $h(a) \geq 0$  for all  $a$ , so the only issue is whether it integrates to 1. That is, we need to show that

$$\int_0^{f^*} \lambda(\{x' \mid f(x') \geq a\}) da = 1$$

where  $f^* = \max_x f(x)$ . Note that this integral simply gives another way to compute the area under the density  $f$ . For each  $a$  between 0 and  $f^*$ , it takes the measure of the

horizontal line under  $f$  at height  $a$  and adds these up, getting the area under  $f$  which is obviously 1.

For this to complete the proof, we need that the set of  $x$  under  $f$  at height  $a$  is a union of (nontrivial) intervals in order to apply our earlier argument. It is not hard to see that this is true if the density is lower semicontinuous.<sup>9</sup> ■

Fudenberg and Levine (2006, 2010a, 2010b) and Noor and Takeoka (2010a, 2010b) give nonlinear generalizations of the GP model which also have interesting connections to the random Strotz model. While these papers present a variety of models, we focus on the following class of such nonlinear extensions. We define a *nonlinear self-control representation* to be a triple consisting of  $(u, v, \psi)$  where  $u$  and  $v$  are EU functions as before,  $\psi : \mathbf{R} \rightarrow \mathbf{R}$  is an increasing and convex function satisfying  $\psi(0) = 0$ , and the preference is represented by

$$V_{NSC}(x) = \max_{\beta \in x} \left[ u(\beta) - \psi \left( \max_{\alpha \in x} v(\alpha) - v(\beta) \right) \right].$$

The convexity of  $\psi$  captures a notion of increasing marginal cost of self-control. As we show in the supplementary appendix, an argument similar to the proof of Theorem 1 establishes that  $V_{NSC}$  can be written as a different kind of random Strotz: specifically,

$$V_{NSC}(x) = \int_0^1 \max_{\beta \in B_{\hat{v}(\cdot, x) + Au}(x)} u(\beta) dA$$

where for each  $x$ ,  $\hat{v}(\cdot, x)$  is a nonlinear transformation of the utility function  $v$  and hence represents the same preference over lotteries as  $v$ . Note that even though  $\hat{v}(\cdot, x)$  represents an EU preference, the fact that it is a nonlinear transformation of  $v$  implies that  $\hat{v}(\cdot, x) + Au$  is *not* an expected utility preference. Thus we have translated a nonlinearity in control costs into a nonlinearity in second-period preferences.

## 4 Properties of Random Strotz Representations

The relationship between random GP and random Strotz representations yields an easy way to see that random GP representations are not well identified. To be more specific,

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<sup>9</sup>To clarify, the key is that there *exists* a lower semicontinuous density. Obviously, if we have a density which is discontinuous at countably many points, we can choose whether the jumps are “up” or “down” without changing the probability distribution. Hence in such cases, it is without loss of generality to assume the density is lower semicontinuous. Thus the bite of this assumption is only in dealing with densities with uncountably many discontinuities. For such cases, the lower semicontinuity assumption guarantees that upper contour sets are unions of open intervals.

any preference with a random GP representation has infinitely many such representations, each with a distinct prediction regarding choices from menus.<sup>10</sup> To see this most simply, consider a GP representation. From Theorem 1, we know that

$$\max_{\beta \in x} [u(\beta) + v(\beta)] - \max_{\beta \in x} v(\beta) = \int_0^1 \max_{\beta \in B_{v+Au}(x)} u(\beta) dA.$$

Fix any partition of  $[0, 1]$  into  $N$  intervals, say  $[0, \Delta_1), [\Delta_1, \Delta_2), \dots, [\Delta_{N-1}, 1]$ . Obviously,

$$\int_0^1 \max_{v+Au} u dA = \sum_{n=1}^N \int_{\Delta_{n-1}}^{\Delta_n} \max_{v+Au} u dA.$$

It is easy to rewrite this as in our proof sketch for Theorem 2 as

$$\sum_{n=1}^N \int_0^1 \max_{v_n+Au} u [\Delta_n - \Delta_{n-1}] dA$$

where

$$v_n = \frac{v + \Delta_{n-1}u}{\Delta_n - \Delta_{n-1}}. \quad (2)$$

Applying Theorem 1 again, this is

$$\sum_{n=1}^N [\Delta_n - \Delta_{n-1}] \left\{ \max_{\beta \in x} [u(\beta) + v_n(\beta)] - \max_{\beta \in x} v_n(\beta) \right\}, \quad (3)$$

a random GP representation. (One can also show the same conclusion directly by substituting for  $v_n$  from equation (2) into equation (3) and rearranging to recover the original GP representation.)

While all these random GP representations predict the same preferences over menus, they predict very different probability distributions over choices from menus. Specifically, as we discuss in more detail in the next section, the random GP above predicts that choice will maximize  $u(\beta) + v_n(\beta)$  with probability  $\Delta_n - \Delta_{n-1}$ . It is easy to see that these choices vary nontrivially with the chosen sequence  $\{\Delta_n\}$ .

By contrast, if a preference has a random Strotz representation, the representation is essentially unique. In particular, the predicted choice from menus is uniquely identified. To see this, suppose we have a preference  $\succeq$  with random Strotz representations  $(u, \mu)$  and  $(\bar{u}, \bar{\mu})$ . It is easy to see that  $u$  and  $\bar{u}$  must be the same up to a positive affine transformation. This follows from the fact that both  $u$  and  $\bar{u}$  must represent the preference over singleton menus. That is,  $\{\alpha\} \succeq \{\beta\}$  iff  $u(\alpha) \geq u(\beta)$ , so  $u(\alpha) \geq u(\beta)$  if and only if

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<sup>10</sup>We thank an anonymous referee for this observation.

$\bar{u}(\alpha) \geq \bar{u}(\beta)$ . Hence, just as with EU representations,  $\bar{u}$  is a positive affine transformation of  $u$ .

Similarly, in a Strotz model,  $v$  is only unique up to positive affine transformations. Likewise, in a random Strotz model, only the choices by a given  $w$  matter for the representation, not the level of utility for  $w$  from these choices. Thus there is no meaningful distinction between a representation that puts probability  $p$  on  $w$  from a representation that puts probability  $p$  on  $2w$ . Hence what is — and what one would naturally want to be — identified is the measure over EU *preferences*, not EU *representations*.

Recall that we identified the space of EU functions with  $\mathbf{R}^K$  where  $K$  is the number of pure outcomes. Given any Borel set  $E \subseteq \mathbf{R}^K$ , let

$$\mathcal{B}(E) = \{w \in \mathbf{R}^K \mid aw + b \in E, \text{ some } a \in \mathbf{R}_+, b \in \mathbf{R}\}.$$

That is,  $\mathcal{B}(E)$  extends  $E$  to the set of all EU functions which represent the same preference over lotteries as some function in  $E$ . We say that  $\mu$  and  $\bar{\mu}$  are *ordinally equivalent* if for every Borel set  $E \subseteq \mathbf{R}^K$ , we have  $\mu(\mathcal{B}(E)) = \bar{\mu}(\mathcal{B}(E))$ . Thus  $\mu$  and  $\bar{\mu}$  are ordinally equivalent if the two measures give the same probability to any given set of EU preferences, ignoring differences between the particular representations chosen for those preferences.

**Theorem 3.** *If  $(u, \mu)$  and  $(\bar{u}, \bar{\mu})$  are random Strotz representations of  $\succeq$ , then  $u$  and  $\bar{u}$  are equal up to a positive affine transformation and  $\mu$  and  $\bar{\mu}$  are ordinally equivalent.*

Given that the measure is identified, it is natural to ask how properties of the measure are related to interpretable properties of the preference it represents. To focus on the role of the measure, we compare two preferences with random Strotz representations that have the same  $u$ . but different  $\mu$ 's. In particular, the behavioral comparison we consider relates to a version of first-order stochastic dominance (FOSD).<sup>11</sup>

We say that  $\succeq_2$  is *more temptation averse* than  $\succeq_1$  if the restriction of  $\succeq_1$  and  $\succeq_2$  to singleton menus are the same and if for all menus  $x$  and lotteries  $\alpha \in x$ , whenever  $\{\alpha\} \succ_1 x$ , we have  $\{\alpha\} \succ_2 x$ . In other words, whenever  $\succeq_1$  strictly prefers committing to a particular choice from the menu rather than leaving the choice open,  $\succeq_2$  does as well.<sup>12</sup>

<sup>11</sup>Gul and Pesendorfer (2001) also give two comparative notions related to temptation for their model. Their comparatives are very different from ours both in spirit and formally.

<sup>12</sup>This definition is equivalent to one used by Ahn (2007) to compare ambiguity aversion, Sarver (2008) to compare regret attitudes, and Higashi, Hyogo, and Takeoka (2009) to compare aversion to commitment. It is also similar in spirit to the way Epstein (1999) and Ghiradato and Marinacci (2002) define comparisons of ambiguity aversion. Since the random Strotz representation is very different from the representations considered in these papers, their characterization results are quite different as well.

One way to think about this comparative is that it is analogous to comparing the agents in terms of their “willingness to pay” for commitment. To see the idea, note that the more an agent would be willing to give up to achieve commitment, the wider the range of options he would prefer committing to. Naturally, in the absence of a common measuring unit such as money, it is difficult to compare two agents in terms of what they are willing to give up to achieve commitment unless they have the same preferences over commitment options. Once we simplify by focusing on preferences with the same commitment preferences, however, the definition captures a natural notion of greater willingness to pay to avoid temptation.

We now explain how this comparative is reflected in the representation. Suppose we have two preferences with random Strotz representations that can be compared according to this definition. Since they have the same preferences restricted to singleton menus, we can take them to have the same  $u$ . So let the random Strotz representation for  $\succeq_i$  be denoted  $(u, \mu_i)$ ,  $i = 1, 2$ .

Suppose we have decompositions of  $\mu_1$  and  $\mu_2$ ,  $(\mathcal{V}_i, \mu_{\mathcal{V}_i}^i, \mu_A^i(\cdot | v))$ ,  $i = 1, 2$ , that are related in the following way. First, we have  $\mathcal{V}_1 = \mathcal{V}_2 \equiv \mathcal{V}$  and  $\mu_{\mathcal{V}}^1 = \mu_{\mathcal{V}}^2 \equiv \mu_{\mathcal{V}}$ . Second, for  $\mu_{\mathcal{V}}$ -almost all  $v \in \mathcal{V}$ , the conditional distribution  $\mu_A^1(\cdot | v)$  first order stochastically dominates  $\mu_A^2(\cdot | v)$ . In this case, we say that  $\mu_2$  *temptation-dominates*  $\mu_1$ .

Intuitively, the different  $v$ ’s represent different types of temptations while the  $A$ ’s measure the “weight” on  $u$  used in determining the lottery chosen from a given menu. It seems natural that it would be difficult to relate different types of temptations — e.g., is a temptation to overeat “worse” than a temptation to oversleep? Thus we should require two preferences to be the same regarding the types of temptations that affect them in order to compare them unambiguously. On the other hand, if one preference has uniformly higher  $A$ ’s in the sense of FOSD, then it has “less trouble” with temptation.

This relationship characterizes our temptation aversion comparison.

**Theorem 4.** *Fix  $\succeq_i$  with random Strotz representation  $(u, \mu_i)$ ,  $i = 1, 2$ . Then  $\succeq_2$  is more temptation averse than  $\succeq_1$  if and only if  $\mu_2$  temptation-dominates  $\mu_1$ .*

## 5 Choice from Menus

To this point, we have focused on the random Strotz and random GP models as representations of preferences over menus. In this sense, we have treated them as models of choice of a menu. As we have seen, we cannot, in general, use choice of menus to distinguish the random GP and random Strotz models.

On the other hand, each model also makes predictions about choice *from* menus. In the case of random Strotz, it is natural to interpret the representation  $(u, \mu)$  as saying that with probability  $\mu(w)$ , the choice is the one made by  $w$  with ties broken in favor of  $u$  (where this is stated for measures with finite support for simplicity). In the case of a GP representation  $(u, \nu)$ , Gul and Pesendorfer argue that the natural interpretation of the choice from a menu  $x$  is that it is some maximizer of  $u + v$  from that menu. It is natural to interpret a random GP representation  $(u, \nu)$  analogously as saying that with probability  $\nu(v)$ , the choice is that which maximizes  $u + v$ . If we adopt these interpretations as parts of the models and observe both choices of menus and choices from menus, can we distinguish random GP and random Strotz?<sup>13,14</sup>

Formally, fix a random Strotz representation  $(u, \mu)$ . We define a *selection function* for  $(u, \mu)$  to be a measurable function  $\beta^* : X \times \text{supp}(\mu) \rightarrow \Delta(Z)$  such that  $\beta^*(x, w) \in B_u(B_w(x))$  for all  $(x, w) \in X \times \text{supp}(\mu)$ . That is, the selection function  $\beta^*(x, w)$  gives one way that choices could be made from menu  $x$  in the random Strotz representation as a function of  $w$ . Then we can define a *random choice function*  $\rho : X \rightarrow \Delta(\Delta(Z))$  by

$$\rho_x(E) = \mu(\{w \in \mathcal{W} \mid \beta^*(x, w) \in E\}) \quad (4)$$

for every measurable  $E \subseteq x$ . We say that such a random choice function is *rationalized* by  $(u, \mu)$ .

Turning to analogous notions for a random GP representation  $(u, \nu)$ , we define a selection function to be a measurable  $\hat{\beta}^* : X \times \text{supp}(\nu) \rightarrow \Delta(Z)$  such that  $\hat{\beta}^*(x, v) \in B_{u+v}(x)$  for all  $(x, v) \in X \times \text{supp}(\nu)$ . We then define a random choice function  $\rho$  generated by this selection function by

$$\rho_x(E) = \nu\left(\left\{v \in \mathbf{R}^K \mid \hat{\beta}^*(x, v) \in E\right\}\right)$$

every measurable  $E \subseteq x$ . We say such a  $\rho$  is rationalized by  $(u, \nu)$ .<sup>15</sup>

Obviously, the case of no temptation is a special case for both models, so if we are to distinguish the models, we must rule out this common case. We say that a

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<sup>13</sup>Unsurprisingly, if we only observe choices from menus, we cannot distinguish these models in general. Both models predict choice from menus in the form of random expected utility, though with a tie-breaking rule in the case of random Strotz. While the tie-breaking distinction can sometimes separate the two models on the basis of choice from menus, this is not generally possible.

<sup>14</sup>The example at the beginning of Section 4 illustrating the nonuniqueness of random GP representations might suggest that the answer is no. One can use the approach we gave to construct a sequence of random GP representations, all with the same preference over menus, whose choice from menus converges to that of a random Strotz representation with the same preference over menus. Thus if this sequence of random GP representations converged to a random GP representation, it would show that the answer to our question is “no.” One can show that this sequence does not converge to a random GP representation.

<sup>15</sup>These definitions are essentially the same as those used in Gul and Pesendorfer (2006).



preference  $\succeq$  over menus *exhibits temptation* if there exist  $\alpha, \beta \in \Delta(Z)$  with  $\{\alpha\} \succ \{\beta\}$  and  $\{\alpha\} \succ \{\alpha, \beta\}$ .<sup>16</sup> It's not hard to show that if  $(u, \mu)$  is a random Strotz representation of  $\succeq$ , then  $\succeq$  exhibits temptation if and only if there is a  $w \in \text{supp}(\mu)$  such that  $w$  does not represent the same preference over lotteries as  $u$ . Similarly, if  $(u, \nu)$  is a random GP representation of  $\succeq$ , then  $\succeq$  exhibits temptation iff there exists  $v \in \text{supp}(\nu)$  such that  $v$  does not represent the same preference over lotteries as  $u$ .

We show that if a random Strotz representation and a random GP representation generate the same preference over menus, then the random Strotz choices from menus exhibit strictly more temptation in a certain sense. Specifically, we show that the agent would prefer to commit to the expected behavior under the random GP than the expected behavior under the random Strotz. Similarly, one can also show that if a random Strotz representation generates the same choices from menus as a random GP representation, then the random GP agent's preference over menus is more temptation-averse than that of the random Strotz.

**Theorem 5.** *Suppose  $\succeq$  has both a random Strotz representation  $(u, \mu)$  and a random GP representation  $(u, \nu)$ .<sup>17</sup> Let  $\rho^{RS}$  be any random choice function rationalized by this random Strotz representation and let  $\rho^{RGP}$  be any random choice function rationalized by the random GP. Then for any menu  $x$ ,*

$$\left\{ \int \beta \rho_x^{RGP}(d\beta) \right\} \succeq \left\{ \int \beta \rho_x^{RS}(d\beta) \right\}.$$

*This inequality must be strict for some  $x$  if  $\succeq$  exhibits temptation.*

*Proof.* Let  $\hat{\beta}^*$  denote the selection function for  $(u, \nu)$  that generates  $\rho^{RGP}$ . By definition,  $\hat{\beta}^*(x, v) \in B_{u+v}(x)$ . Then for any  $x$ ,

$$\begin{aligned} V_{RGP}(x) &= \int \left\{ \max_x [u(\beta) + v(\beta)] - \max_x v(\beta) \right\} \nu(dv) \\ &= \int \left\{ u(\hat{\beta}^*(x, v)) + v(\hat{\beta}^*(x, v)) - \max_x v(\beta) \right\} \nu(dv) \\ &\leq \int \left\{ u(\hat{\beta}^*(x, v)) + v(\hat{\beta}^*(x, v)) - v(\hat{\beta}^*(x, v)) \right\} \nu(dv) \\ &= \int u(\hat{\beta}^*(x, v)) \nu(dv) = u \left( \int \beta \rho_x^{RGP}(d\beta) \right) \\ &= V_{RGP} \left( \left\{ \int \beta \rho_x^{RGP}(d\beta) \right\} \right) \end{aligned}$$

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<sup>16</sup>One can show that this is equivalent to what GP call  $\succeq$  having a preference for commitment if  $\succeq$  has either a random Strotz or random GP representation.

<sup>17</sup>Note that since the two representations have the same preferences over menus, they must have the same preferences over singleton menus in particular. Hence, up to normalization, they must have the same  $u$ .

Hence  $\{\int \beta \rho_x^{RGP}(d\beta)\} \succeq x$ . But  $\rho_x^{RS}$  must satisfy  $x \sim \{\int \beta \rho_x^{RS}(d\beta)\}$ . (To see this, recall that in the random Strotz model, the agent evaluates the menu  $x$  as if he receives  $\beta$  with probability  $\rho_x^{RS}(\beta)$ .) Therefore,  $\{\int \beta \rho_x^{RGP}(d\beta)\} \succeq \{\int \beta \rho_x^{RS}(d\beta)\}$ .

To see that there must be some menu where the comparison is strict if  $\succeq$  exhibits temptation, consider any  $\alpha$  and  $\beta$  that satisfy  $\{\alpha\} \succ \{\beta\}$  and  $\{\alpha\} \succ \{\alpha, \beta\}$ . It is not hard to show that this implies that there is  $v \in \text{supp}(\nu)$  with  $u(\alpha) > u(\beta)$  and  $v(\alpha) < v(\beta)$  and to use this to show that for the menu  $x = \{\alpha, \beta\}$ , we must have  $V_{RGP}(x) < V_{RGP}(\{\int \beta \rho_x^{RGP}(d\beta)\})$ . Hence the inequality above is strict for such a menu. ■

To see the intuition behind this result most simply, suppose  $\succeq$  has a GP representation and hence also a random Strotz representation. Suppose this preference has  $\{\alpha\} \succ \{\alpha, \beta\} \succ \{\beta\}$ . In the GP case, this is rationalized by having  $u(\alpha) > u(\beta)$ ,  $v(\beta) > v(\alpha)$ , and  $u(\alpha) + v(\alpha) > u(\beta) + v(\beta)$ . These rankings imply that

$$\begin{aligned} V_{GP}(\{\alpha, \beta\}) &= \max\{u(\alpha) + v(\alpha), u(\beta) + v(\beta)\} - \max\{v(\alpha), v(\beta)\} \\ &= u(\alpha) - [v(\beta) - v(\alpha)]. \end{aligned}$$

Thus the predicted choice is  $\alpha$ , the same as the “choice” from the menu  $\{\alpha\}$ , but the menu is ranked lower than  $\{\alpha\}$  because of the self-control cost of  $v(\beta) - v(\alpha)$ . By contrast, the random Strotz representation would have  $V_{RS}(\alpha, \beta) = pu(\alpha) + (1 - p)u(\beta)$  for some  $p \in (0, 1)$ . Thus the random Strotz representation “explains” the fact that  $\{\alpha\} \succ \{\alpha, \beta\}$  not by self-control costs but by a nonzero probability of “self-indulgent” behavior under the latter menu.

In other words, the random Strotz model explains the desire for commitment entirely in terms of a fear of succumbing to temptation, while the random GP model explains it in part by this but in part by a desire to avoid self-control costs. Hence for the ex ante preference over menus to coincide, the choice *from* menus in the random Strotz model must involve succumbing to temptation more frequently.

## A Proof of Theorem 2

*Proof.* As noted in the text, we begin with a random Strotz representation of the form

$$V(x) = \int_{\mathcal{A}} \max_{\beta \in B_{v+Au}(x)} u(\beta) f(A) dA$$

where  $f$  is lower semicontinuous and  $\mathcal{A}$  is the support of  $f$ . We show how to rewrite  $V$  in the form of a random GP representation.

For any  $a \geq 0$ , let  $U(a) = \{A \in \mathbf{R} \mid f(A) > a\}$ . Since  $f$  is lower semicontinuous,  $U(a)$  is open and hence is the union of countably many disjoint open intervals for all  $a$ . Let  $\mathcal{L}(a)$  denote the Lebesgue measure of  $U(a)$ . Then we have

$$\begin{aligned} V(x) &= \int_{\mathcal{A}} \max_{\beta \in B_{v+Au}(x)} u(\beta) \left[ \int_0^{f(A)} \frac{1}{\mathcal{L}(a)} \mathcal{L}(a) da \right] dA \\ &= \int_{\mathcal{A}} \int_0^{f(A)} \max_{\beta \in B_{v+Au}(x)} u(\beta) \frac{1}{\mathcal{L}(a)} \mathcal{L}(a) da dA \end{aligned}$$

Let  $\bar{f} = \sup_{A \in \mathcal{A}} f(A)$ . Then the double integral above is over

$$\{(A, a) \in \mathbf{R}^2 \mid A \in \mathcal{A}, 0 < a < f(A)\} = \{(A, a) \in \mathbf{R}^2 \mid 0 < a < \bar{f}, f(A) > a\}.$$

Thus the integral is equal to

$$\int_0^{\bar{f}} \left[ \int_{U(a)} \max_{\beta \in B_{v+Au}(x)} u(\beta) \frac{1}{\mathcal{L}(a)} dA \right] \mathcal{L}(a) da.$$

Note that  $\int_0^{\bar{f}} \mathcal{L}(a) da = \int_{\mathcal{A}} f(A) dA = 1$ . Hence we can view  $\mathcal{L}(a)$  as a density over  $a \in [0, \bar{f}]$ .

Fix any  $a \in (0, \bar{f})$ . Since  $U(a)$  is a union of disjoint open intervals, we can write

$$\int_{U(a)} \max_{\beta \in B_{v+Au}(x)} u(\beta) \frac{1}{\mathcal{L}(a)} dA = \sum_{k=1}^{\infty} \int_{b_k}^{c_k} \max_{\beta \in B_{v+Au}(x)} u(\beta) \frac{1}{\mathcal{L}(a)} dA,$$

where  $(b_k, c_k)$ ,  $k = 1, 2, \dots$ , is the collection of intervals defining  $U(a)$  and we suppress the dependence of the  $b_k$ 's and  $c_k$ 's on  $a$  for notational simplicity. Rewriting, this is

$$= \sum_{k=1}^{\infty} \frac{c_k - b_k}{\mathcal{L}(a)} \int_{b_k}^{c_k} \max_{\beta \in B_{v+Au}(x)} u(\beta) \frac{1}{c_k - b_k} dA.$$

As shown in the text,

$$\int_{b_k}^{c_k} \max_{\beta \in B_{v+Au}(x)} u(\beta) \frac{1}{c_k - b_k} dA = \max_{\beta \in x} [u(\beta) + v_k(\beta)] - \max_{\beta \in x} v_k(\beta)$$

where  $v_k = (v + b_k u)/(c_k - b_k)$ . Hence for any  $a$ , we have

$$\int_{U(a)} \max_{\beta \in B_{v+Au}(x)} u(\beta) \frac{1}{\mathcal{L}(a)} dA = \sum_{k=1}^{\infty} \frac{c_k - b_k}{\mathcal{L}(a)} \left\{ \max_{\beta \in x} [u(\beta) + v_k(\beta)] - \max_{\beta \in x} v_k(\beta) \right\}.$$

Since  $\sum_k (c_k - b_k)/\mathcal{L}(a) = 1$ , this is a random GP representation. Hence we have established that the random Strotz representation is a randomization over random GP representations and hence is a random GP representation. ■

## B Proof of Theorem 3

We prove this result by showing that if we restrict attention to measures on a particular subspace of  $\mathbf{R}^K$  which includes one EU function for each nontrivial EU preference, then the measure is unique. The particular space we use is

$$\mathcal{W} = \{w \in \mathbf{R}^K \mid w \cdot \mathbf{1} = 0, w \cdot w = 1\}$$

where  $\mathbf{1}$  is a  $K$  vector of 1's. It is easy to see that any nontrivial EU preference is represented by exactly one  $w \in \mathcal{W}$ .

Theorem 3 is concerned only with the measure of sets of EU functions which are closed under equivalence in the sense that if  $w$  is in the set, then every  $w'$  equivalent to  $w$  is in the set as well. Hence we may as well take our measures to be over  $\mathcal{W}$ . For the  $\sigma$ -algebra on  $\mathcal{W}$ , we use the Borel  $\sigma$ -algebra using as our topology on  $\mathcal{W}$  the (relativized) usual Euclidean topology on  $\mathbf{R}^K$ .

The proofs of Lemmas 1 and 2 are straightforward algebra and hence omitted.

**Lemma 1.** *Fix  $w, \bar{w} \in \mathcal{W}$ . Then  $w \cdot \bar{w} \in [-1, 1]$ . Furthermore,  $w \cdot \bar{w} = 1$  iff  $w = \bar{w}$  and  $w \cdot \bar{w} = -1$  iff  $w = -\bar{w}$ .*

Let  $\mathcal{V} = \{v \in \mathcal{W} \mid v \cdot u = 0\}$ .

**Lemma 2.** *For every  $w \in \mathcal{W}$ , there exists  $v \in \mathcal{V}$  and  $A \in [-1, 1]$  such that  $w = v\sqrt{1 - A^2} + Au$ . If  $w = u$ , then this holds for every  $(A, v) \in \{1\} \times \mathcal{V}$ , while if  $w = -u$ , it holds for every  $(A, v) \in \{-1\} \times \mathcal{V}$ . For every other  $w \in \mathcal{W}$ , the  $(A, v)$  is unique.*

Define an order over  $\mathcal{W}$  by  $w C_u \hat{w}$  (read “ $w$  is closer to  $u$  than  $\hat{w}$ ”) if

$$u(\alpha) > u(\beta), \hat{w}(\alpha) \geq \hat{w}(\beta) \text{ implies } w(\alpha) \geq w(\beta).$$

In other words,  $w$  is willing to “go along with”  $u$  at least as often as  $\hat{w}$ . Define a set  $W \subseteq \mathcal{W}$  to be *closed under  $C_u$*  if  $w' \in W$  and  $w C_u w'$  implies  $w \in W$ .

For brevity, let  $\eta(A) = \sqrt{1 - A^2}$ . The proof of the following lemma is in the supplementary appendix.

**Lemma 3.**  $w_1 C_u w_2$  if and only if there exists  $v \in \mathcal{V}$  such that  $w_i = v\sqrt{1 - A_i^2} + A_i u$ ,  $i = 1, 2$ , with  $A_1 \geq A_2$ .

Given a function  $A^* : \mathcal{V} \rightarrow [-1, 1]$ , let

$$W(A^*) = \bigcup_{v \in \mathcal{V}} \{w \in \mathcal{W} \mid w = v\sqrt{1 - A^2} + Au, \text{ for some } A \geq A^*(v), A \neq -1\}.$$

Note that by excluding  $A = -1$ , the definition of  $W(A^*)$  ensures that  $-u \notin W(A^*)$  for any  $A^*$ .

**Lemma 4.** A set  $W \subseteq \mathcal{W}$ ,  $W \neq \mathcal{W}$ , is closed and closed under  $C_u$  if and only if there exists a lower semi-continuous function  $A^*$  such that  $W = W(A^*)$  and  $A^*(v) > -1$  for all  $v \in \mathcal{V}$ .

*Proof.* Fix any lower semi-continuous function  $A^*$  such that  $A^*(v) > -1$  for all  $v \in \mathcal{V}$ . Let  $W = W(A^*)$ . Since the definition of  $W(A^*)$  prevents  $-u \in W(A^*)$ ,  $W \neq \mathcal{W}$ . From Lemma 3, it is easy to see that  $W$  is closed under  $C_u$  if and only if  $v\sqrt{1 - A^2} + Au \in W$  implies  $v\sqrt{1 - \hat{A}^2} + \hat{A}u \in W$  for all  $\hat{A} \in (A, 1]$ . Hence the definition of  $W(A^*)$  obviously implies  $W$  is closed under  $C_u$ . Finally, to show that  $W$  is closed, fix any sequence  $w_n$  converging to  $w$  such that  $w_n \in W$  for all  $n$ . We can write  $w_n = \sqrt{1 - A_n^2}v_n + A_n u$  for each  $n$ . Since  $w_n \in W$  for all  $n$ , we have  $A_n \geq A^*(v_n)$  for all  $n$ . Let  $v$  denote the limit of  $v_n$  and  $A$  the limit of  $A_n$ . It is easy to see from the proof of Lemma 2 that the  $v$  and  $A$  associated with a given  $w$  depend continuously on  $w$ , so we must have  $w = \sqrt{1 - A^2}v + Au$ . Hence  $w \in W(A^*)$  if and only if  $\lim_{n \rightarrow \infty} A^*(v_n) \geq A^*(\lim_{n \rightarrow \infty} v_n)$ . Since  $A^*$  is lower semi-continuous, this holds. Hence  $W$  is closed.

For the converse, suppose  $W \neq \mathcal{W}$  is closed and closed under  $C_u$ . For each  $v$ , let

$$A^*(v) = \min\{A \in [-1, 1] \mid v\sqrt{1 - A^2} + Au \in W\}.$$

Since  $W$  is closed,  $A^*(v)$  is well-defined. Since  $w C_u -u$  for all  $w$ , the fact that  $W \neq \mathcal{W}$  implies  $-u \notin W$ . Hence  $A^*(v) > -1$  for all  $v$ . Since  $W$  is closed under  $C_u$ , for every

$A \geq A^*(v)$ , we have  $v\sqrt{1-A^2} + Au \in W$ , implying that  $v\sqrt{1-A^2} + Au \in W$  if and only if  $A \geq A^*(v)$ . Hence  $W = W(A^*)$ .

Finally, to see that  $A^*$  is lower semi-continuous, again, consider the sequence constructed above. As noted, for any such sequence,  $w \in W$  if and only if  $\lim_{n \rightarrow \infty} A^*(v_n) \geq A^*(\lim_{n \rightarrow \infty} v_n)$ . Since  $W$  is closed, we must have  $w \in W$ . Hence any jumps in  $A^*$  must be downward, so  $A^*$  is lower semi-continuous. ■

We note that if  $A^*$  is lower semi-continuous, then it is measurable.

**Lemma 5.** *Fix any measurable function  $A^* : \mathcal{V} \rightarrow [-1, 1]$  such that  $W(A^*)$  is closed. Then there exists a sequence of positive numbers  $\{\varepsilon_n\}$  and a sequence of menus  $\{x_n\}$  such that for every random Strotz representation with commitment preference represented by  $u$ ,*

$$\lim_{n \rightarrow \infty} \frac{V(x_n)}{\varepsilon_n} = \mu(W(A^*)).$$

*Proof.* Fix such an  $A^*$  function.

**Part 1.** First, suppose that  $A^*$  is bounded in the sense that  $A^*(v)/\sqrt{1-(A^*(v))^2}$  is bounded from above and below.

For each  $v \in \mathcal{V}$ , let

$$\alpha_v = \frac{1}{K} \mathbf{1} + \varphi v$$

$$\beta_v(\varepsilon) = \alpha_v + \varphi \varepsilon \left[ u - \frac{A^*(v)}{a^*(v)} v \right]$$

where  $a^*(v) = \sqrt{1-(A^*(v))^2}$ . By the boundedness of  $A^*/a^*$ , there exists  $\varphi > 0$  such that for all sufficiently small  $\varepsilon > 0$ , every  $\alpha_v$  and  $\beta_v(\varepsilon)$  is a lottery.

Let  $L(v) = \{w \in \mathcal{W} \mid w = av + Au, \text{ some } a \geq 0, A \in -[-1, 1]\}$ . Suppose  $w = L(v)$  and consider some  $\bar{v}$  which may or may not equal  $v$ . Then

$$w \cdot \alpha_{\bar{v}} = a\varphi v \cdot \bar{v}$$

while

$$w \cdot \alpha_v = a\varphi.$$

Since  $v \cdot \bar{v} \leq 1$ , strictly if  $\bar{v} \neq v$ , we see that  $w \cdot \alpha_{\bar{v}} \leq w \cdot \alpha_v$ , strictly so for any  $\bar{v} \neq v$ . Hence if  $w$  picks any  $\alpha$ , he must pick  $\alpha_v$ .

Also, for any  $v$ ,  $u \cdot \alpha_v = 0 < \varphi \varepsilon = u \cdot \beta_v(\varepsilon)$ . So  $u$  is indifferent among the  $\alpha$ 's, indifferent among the  $\beta$ 's, and prefers the  $\beta$ 's to the  $\alpha$ 's. Hence, letting  $x_\varepsilon$  denote the

menu consisting of all the  $\alpha$ 's and  $\beta$ 's, we see that  $V(x_\varepsilon) = \varphi\varepsilon\mu(W_\varepsilon)$ , where

$$W_\varepsilon = \bigcup_{v \in \mathcal{V}} \{w \in L(v) \mid w \cdot \beta_{\bar{v}}(\varepsilon) \geq w \cdot \alpha_v, \text{ for some } \bar{v} \in \mathcal{V}\}.$$

We now show that  $\lim_{\varepsilon \downarrow 0} \mu(W_\varepsilon) = \mu(W(A^*))$ . Note that if  $w = av + Au$ , then  $a = \sqrt{1 - A^2}$  and

$$w \cdot \beta_v(\varepsilon) = w \cdot \alpha_v + \varphi\varepsilon \left[ A - \frac{A^*(v)}{a^*(v)} a \right],$$

so  $w \cdot \beta_v(\varepsilon) \geq w \cdot \alpha_v$  iff  $(A/a) \geq (A^*(v)/a^*(v))$ . It is not hard to show that this holds iff  $A \geq A^*(v)$ . Hence for every  $\varepsilon$ , we have  $W(A^*) \subseteq W_\varepsilon$ .

Next, we show that if  $w \notin W(A^*)$ , then there is  $\bar{\varepsilon} > 0$  such that for all  $\varepsilon \in (0, \bar{\varepsilon})$ , we have  $w \notin W_\varepsilon$ . To show this, suppose not. Then there exists a sequence  $\varepsilon_n$  converging to zero such that  $w \in W_{\varepsilon_n} \setminus W(A^*)$  for all  $n$ .

Write  $w = av + Au$ . Then there exists a sequence  $\bar{v}_n$  such that

$$w \cdot \beta_{\bar{v}_n}(\varepsilon_n) \geq w \cdot \alpha_v$$

or

$$a\varphi v \cdot \bar{v}_n + \varphi\varepsilon_n \left[ A - \frac{A_n}{a_n} av \cdot \bar{v}_n \right] \geq a\varphi$$

where  $A_n = A^*(\bar{v}_n)$  and  $a_n = \sqrt{1 - A_n^2}$ . Rearranging yields

$$\varepsilon_n \left[ \frac{A}{a} - \frac{A_n}{a_n} v \cdot \bar{v}_n \right] \geq 1 - v \cdot \bar{v}_n.$$

Since  $v \cdot \bar{v}_n \leq 1$  and  $A_n/a_n$  is bounded from below, we must have  $v \cdot \bar{v}_n \rightarrow 1$  as  $n \rightarrow \infty$ . Note for future use that this implies  $\bar{v}_n \rightarrow v$ . Also, the fact that the right-hand side is nonnegative for all  $n$  implies that

$$\frac{A}{a} \geq \frac{A_n}{a_n} v \cdot \bar{v}_n$$

for all  $n$ . Recall that  $w \in L(v)$  and  $w \notin W^*$ . Hence  $A < A^*(v)$ . So we have

$$\frac{A^*(v)}{a^*(v)} > \frac{A^*(\bar{v}_n)}{a^*(\bar{v}_n)} v \cdot \bar{v}_n.$$

Since  $v \cdot \bar{v}_n \rightarrow 1$ , we have

$$\frac{A^*(v)}{a^*(v)} \geq \lim_{n \rightarrow \infty} \frac{A_n}{a_n},$$

or  $A^*(v) \geq \lim_{n \rightarrow \infty} A_n$ .

By Lemma 4, the fact that  $W(A^*)$  is closed that  $A^*$  is lower semi-continuous. Hence we have the opposite weak inequality, so  $A^*(v) = \lim_{n \rightarrow \infty} A_n$ .

But recall that

$$\frac{A}{a} \geq \frac{A_n}{a_n} v \cdot \bar{v}_n$$

for all  $n$ . Hence

$$\frac{A}{a} \geq \lim_{n \rightarrow \infty} \frac{A_n}{a_n} v \cdot \bar{v}_n = \frac{A^*(v)}{a^*(v)}.$$

But this implies  $A \geq A^*(v)$  or  $w \in W(A^*)$ , a contradiction.

Hence  $\lim_{n \rightarrow \infty} \mu(W_\varepsilon) = \mu(W^*)$ . Therefore,

$$\lim_{n \rightarrow \infty} \frac{V(x_\varepsilon)}{\varphi \varepsilon_n} = \lim_{n \rightarrow \infty} \mu(W_\varepsilon) = \mu(W^*).$$

Taking the sequence referred to in the statement of the lemma to be  $\{\varphi \varepsilon_n\}$  gives the desired conclusion.

**Part 2.** Now we drop the assumption that  $A^*$  is bounded. Fix a sequence  $\{\varepsilon_n\}$  with  $\varepsilon_n > 0$  for all  $n$  such that  $\varepsilon_n \rightarrow 0$ . Define a new function

$$A_n^*(v) = \begin{cases} A^*(v), & \text{if } -1 + \varepsilon_n \leq A^*(v) \leq 1 - \varepsilon_n \\ -1 + \varepsilon_n, & \text{if } A^*(v) < -1 + \varepsilon_n \\ 1 - \varepsilon_n, & \text{if } A^*(v) > 1 - \varepsilon_n \end{cases}$$

Clearly,  $A_n^*$  is bounded for every  $n$ . It is tedious but not difficult to show that the fact that  $W(A^*)$  is closed implies that  $W(A_n^*)$  is closed for every  $n$ . Hence for each  $n$ , we can find sequences  $\{\varepsilon_m^n\}$  and  $\{x_m^n\}$  such that

$$\lim_{m \rightarrow \infty} \frac{V(x_m^n)}{\varepsilon_m^n} = \mu(W(A_n^*)).$$

That is, for any  $\delta > 0$ , there exists  $M_n(\delta)$  such that

$$\left| \frac{V(x_m^n)}{\varepsilon_m^n} - \mu(W(A_n^*)) \right| < \delta, \quad \forall m \geq M_n(\delta).$$

Rewriting,

$$\left| \frac{V(x_m^n)}{\varepsilon_m^n} - \mu(W(A^*)) + \mu(W(A^*) \setminus W(A_n^*)) - \mu(W(A_n^*) \setminus W(A^*)) \right| < \delta, \quad \forall m \geq M_n(\delta).$$



Clearly, though, if  $w \in W(A^*) \setminus W(A_n^*)$ , then there must be some  $N$  such that  $w \in W(A^*) \cap W(A_n^*)$  for all  $n \geq N$ . The analogous statement is true for  $W(A_n^*) \setminus W(A^*)$ .

So fix any sequence  $\{\delta_n\}$  converging to zero. For each  $n$ , fix  $m_n \geq M_n(\delta_n)$ . Consider the sequence  $\{\hat{\varepsilon}_n\} = \{\varepsilon_{m_n}^n\}$  and  $\{\hat{x}_n\} = \{x_{m_n}^n\}$ . Clearly, for every  $n$ ,

$$\left| \frac{V(\hat{x}_n)}{\hat{\varepsilon}_n} - \mu(W(A^*)) + k\mu(W(A^*) \setminus W(A_n^*)) - \mu(W(A_n^*) \setminus W(A^*)) \right| < \delta_n.$$

So

$$\lim_{n \rightarrow \infty} \frac{V(\hat{x}_n)}{\hat{\varepsilon}_n} = \mu(W(A^*)). \blacksquare$$

To complete the proof of Theorem 3, suppose  $(u, \mu)$  and  $(\bar{u}, \bar{\mu})$  are random Strotz representations of  $\succeq$  where  $\mu$  and  $\bar{\mu}$  are measures over  $\mathcal{W}$ . Let  $V$  and  $\bar{V}$  denote the utility functions over menus generated by  $(u, \mu)$  and  $(\bar{u}, \bar{\mu})$  respectively. As explained in the text,  $u$  and  $\bar{u}$  must be the same up to a positive affine transformation. For convenience, we transform so that  $u = \bar{u}$ .

Since  $V$  and  $\bar{V}$  are random Strotz representations of the same preference, there exists  $a > 0$  and  $b$  such that  $V(x) = a\bar{V}(x) + b$  for all menus  $x$ . For  $x = \{\alpha\}$ , then,  $V(\{\alpha\}) = a\bar{V}(\{\alpha\}) + b$  or  $u(\alpha) = a\bar{u}(\alpha) + b$ . Since  $u = \bar{u}$ , then,  $a = 1$  and  $b = 0$ . In other words, we must have  $V = \bar{V}$ .

Note that the sequence of menus constructed in the proof of Lemma 5 is independent of the representation. Hence for any set  $W$  which is closed and closed under  $C_u$ , we must have  $\mu(W) = \bar{\mu}(W)$ .

Fix any measurable set  $E \subseteq \mathcal{W}$  which is closed under  $C_u$ . By Theorem 12.3 of Billingsley (1995, page 174),

$$\mu(E) = \sup_{W \subseteq E | W \text{ closed}} \mu(W)$$

and similarly for  $\bar{\mu}$ . It is easy to see that if  $W \subseteq E$ , then if we close  $W$  under  $C_u$ , the resulting set will be contained in  $E$ . That is,

$$W^* \equiv \{w \in \mathcal{W} \mid w C_u w', \text{ for some } w' \in W\} \subseteq E.$$

Obviously,  $\mu(W^*) \geq \mu(W)$ . Hence

$$\mu(E) = \sup_{W \subseteq E | W \text{ closed, closed under } C_u} \mu(W).$$

Since  $\mu(W) = \bar{\mu}(W)$  for any  $W$  which is closed and closed under  $C_u$ , this implies  $\mu(E) = \bar{\mu}(E)$ . Thus  $\mu$  and  $\bar{\mu}$  coincide for any measurable set which is closed under  $C_u$ .

Let  $\mathcal{P}$  denote the collection of measurable sets  $E$  which are closed under  $C_u$ . So we have established that  $\mu$  and  $\bar{\mu}$  coincide on  $\mathcal{P}$ . It is easy to show that  $\mathcal{P}$  is a  $\pi$ -system. To see this, suppose  $E_1, E_2 \in \mathcal{P}$ . Then  $E_1$  and  $E_2$  are measurable, so  $E_1 \cap E_2$  is measurable. Also, fix any  $w \in E_1 \cap E_2$  and any  $w'$  such that  $w' C_u w$ . Since  $w \in E_i$  and  $E_i$  is closed under  $C_u$ , we must have  $w' \in E_i$ ,  $i = 1, 2$ . Hence  $w' \in E_1 \cap E_2$ . So  $E_1 \cap E_2$  is closed under  $C_u$  and hence is an element of  $\mathcal{P}$ . Hence  $\mathcal{P}$  is a  $\pi$ -system.

Theorem 3.3 of Billingsley (1995) then implies  $\mu = \bar{\mu}$  on  $\sigma(\mathcal{P})$ , the  $\sigma$ -algebra generated by  $\mathcal{P}$ . We now show that  $\sigma(\mathcal{P})$  is the Borel  $\sigma$ -algebra, completing the proof of uniqueness.

Fix any open set  $W \subseteq \mathcal{W}$  and any  $w \in W \setminus \{-u, u\}$ . It is not hard to see that there exists a closed  $\hat{\mathcal{V}} \subseteq \mathcal{V}$  and rational numbers  $r_1, r_2 \in (-1, 1]$  such that  $w \in W(A_1^*) \setminus W(A_2^*) \subseteq W$  where

$$A_i^*(v) = \begin{cases} r_i, & \text{for } v \in \hat{\mathcal{V}} \\ 1, & \text{otherwise} \end{cases}$$

Obviously,  $W(A_1^*), W(A_2^*) \in \mathcal{P}$  implies  $W(A_1^*) \setminus W(A_2^*) \in \sigma(\mathcal{P})$ . Note that  $\{u\} \in \mathcal{P}$  and that  $W \setminus \{-u\} \in \mathcal{P}$  implies  $\{-u\} \in \sigma(\mathcal{P})$ . Hence  $W$  is the union of a countable collection of sets in  $\sigma(\mathcal{P})$  so  $W \in \sigma(\mathcal{P})$ . Hence  $\sigma(\mathcal{P})$  contains all open sets and so contains the Borel  $\sigma$ -algebra. Since all sets in  $\mathcal{P}$  are in the Borel  $\sigma$ -algebra,  $\sigma(\mathcal{P})$  cannot be larger than the Borel field, so it must equal it. ■

## C Proof of Theorem 4

**Lemma 6.** *Suppose  $\succeq_i$  has a random Strotz representation  $(u, \mu_i)$ ,  $i = 1, 2$ . Then  $\succeq_2$  is more temptation averse than  $\succeq_1$  if and only if for every menu  $x$ ,*

$$V_2(x) \equiv \int \max_{\beta \in B_w(x)} u(\beta) \mu_2(dw) \leq \int \max_{\beta \in B_w(x)} u(\beta) \mu_1(dw) \equiv V_1(x).$$

*Proof of Lemma.* Suppose  $\succeq_2$  is more temptation averse than  $\succeq_1$  but that  $V_2(x) > V_1(x)$  for some menu  $x$ . Without loss of generality, assume  $x$  is closed and convex. Then there exists  $\alpha \in x$  such that  $\{\alpha\} \sim_2 x$ . So  $u(\alpha) = V_2(x) > V_1(x)$ . Hence  $\{\alpha\} \succ_1 x$  but we do not have  $\{\alpha\} \succ_2 x$ , contradicting  $\succeq_2$  more temptation averse. For the converse, suppose  $V_1(x) \geq V_2(x)$  for all  $x$ . Then whenever  $u(\alpha) > V_1(x)$ , we have  $u(\alpha) > V_2(x)$ , so  $\succeq_2$  is more temptation averse than  $\succeq_1$ . ■

First, we show that if  $\mu_2$  temptation-dominates  $\mu_1$ , then  $\succeq_2$  is more temptation averse than  $\succeq_1$ . Fix any menu and any  $v \in \mathcal{V}$ . Since the utility  $u$  gets from the choice is weakly increasing in  $A$ , we know that the expected utility of the menu conditional on  $v$  is higher

under  $\mu_A^1$  than under  $\mu_A^2$  for (almost) every  $v$ . Hence the same is true when we take expectations over  $v$  since the marginals are the same. Thus  $V_1(x) \geq V_2(x)$  for all  $x$ , implying  $\succeq_2$  is more temptation averse than  $\succeq_1$  by Lemma 6.

For the converse, suppose  $\succeq_2$  is more temptation-averse than  $\succeq_1$ . We construct decompositions of  $\mu_1$  and  $\mu_2$  which show that  $\mu_2$  temptation-dominates  $\mu_1$ . We define  $\mathcal{V}$  as in the proof of Theorem 3. To begin constructing the relevant measures, we fix a partition  $\mathcal{V}_1, \dots, \mathcal{V}_N$  of  $\mathcal{V}$  with the property that each  $\mathcal{V}_i$  is measurable. We refer to such a partition as a *measurable partition*. For any  $\mathcal{V}_n$  and any  $A_n \in [-1, 1]$ , let

$$\mu_i(A_n, \mathcal{V}_n) = \mu_i \left( \{w \in \mathcal{W} \mid w = v\sqrt{1-A^2} + Au, \text{ for } 1 > A \geq A_n, A \neq -1, v \in \mathcal{V}_n\} \right).$$

Note that  $\mu_i(A_n, \mathcal{V}_n)$  is defined so that it does not include the measure of  $u$  or  $-u$ . In particular,  $\mu_i(1, \mathcal{V}_n) = 0$  and

$$\mu_i(-1, \mathcal{V}_n) = \mu_i(L(\mathcal{V}_n)) - \mu_i(\{u\}) - \mu_i(\{-u\})$$

where

$$L(\mathcal{V}_n) = \left\{ w \in \mathcal{W} \mid w = v\sqrt{1-A^2} + Au, \text{ for } A \in [-1, 1], v \in \mathcal{V}_n \right\}.$$

Fix any  $A_1, \dots, A_N$  and define

$$W^* = \bigcup_{n=1}^N \bigcup_{v \in \mathcal{V}_n} \{w \in \mathcal{W} \mid w = v\sqrt{1-A^2} + Au, A \geq A_n, A \neq -1\}.$$

First, suppose  $W^*$  is closed. By Lemma 5, we know that there is a sequence of menus  $x_m$  and numbers  $\varepsilon_m$  such that

$$\lim_{m \rightarrow \infty} \frac{V_i(x_m)}{\varepsilon_m} = \mu_i(W^*).$$

Since this sequence is independent of the preference (given that both have commitment utility  $u$ ) and since  $V_1(x_m) \geq V_2(x_m)$  for all  $m$ , we have  $\mu_1(W^*) \geq \mu_2(W^*)$ .

Now suppose that  $W^*$  is not closed. In this case, Theorem 12.3 of Billingsley (1995, page 174) implies that

$$\mu_2(W^*) = \sup_{W \subseteq E \mid W \text{ closed}} \mu_2(W).$$

As shown in the proof of Theorem 3, we can rewrite this as

$$\mu_2(W^*) = \sup_{W \subseteq W^* \mid W \text{ closed, closed under } C_u} \mu_2(W).$$

But we know that for every  $W$  which is closed and closed under  $C_u$ , we have  $\mu_2(W) \leq \mu_1(W)$ . Hence

$$\begin{aligned}\mu_2(W^*) &= \sup_{W \subseteq E | W \text{ closed}} \mu_2(W) \\ &\leq \sup_{W \subseteq W^* | W \text{ closed, closed under } C_u} \mu_1(W) \\ &= \mu_1(W^*).\end{aligned}$$

Summarizing, we have

$$\mu_1(u) + \sum_{n=1}^N \mu_1(A_n, \mathcal{V}_n) \geq \mu_2(u) + \sum_{n=1}^N \mu_2(A_n, \mathcal{V}_n). \quad (5)$$

for any  $(A_1, \dots, A_N) \in [-1, 1]^N$ .

Note that this implies  $\mu_1(u) \geq \mu_2(u)$  and  $\mu_1(-u) \leq \mu_2(-u)$ . The former is implied by taking  $A_n = 1$  for all  $n$  and the latter by  $A_n = -1$  for all  $n$ .

Let  $\mu^*(u) = \mu_1(u) - \mu_2(u)$ . First, assume  $\mu^*(u) > 0$ . For each  $n$ , let

$$\lambda_n^* = \sup_{A_n \in [-1, 1]} \frac{\mu_2(A_n, \mathcal{V}_n) - \mu_1(A_n, \mathcal{V}_n)}{\mu^*(u)}.$$

By assumption,  $\mu^*(u) > 0$ , so this is well-defined. Also, note that for  $A_n = 1$ , the difference on the right-hand side is zero. Hence  $\lambda_n^* \geq 0$ .

Also,

$$\sum_n \lambda_n^* = \frac{1}{\mu^*(u)} \sum_n \sup_{A_n \in [-1, 1]} [\mu_2(A_n, \mathcal{V}_n) - \mu_1(A_n, \mathcal{V}_n)].$$

Suppose this is strictly greater than 1. Then for each  $n$ , there is a sequence  $\{A_n^m\}$  such that

$$\lim_{m \rightarrow \infty} \sum_n \mu_2(A_n^m, \mathcal{V}_n) > \mu^*(u) + \lim_{m \rightarrow \infty} \sum_n \mu_1(A_n^m, \mathcal{V}_n).$$

Substituting for  $\mu^*(u)$  and rearranging,

$$\lim_{m \rightarrow \infty} \left[ \mu_2(u) + \sum_n \mu_2(A_n^m, \mathcal{V}_n) \right] > \lim_{m \rightarrow \infty} \left[ \mu_1(u) + \sum_n \mu_1(A_n^m, \mathcal{V}_n) \right].$$

But this contradicts equation (5). Hence  $\sum_n \lambda_n^* \leq 1$ .

Fix any  $\lambda_1^1, \dots, \lambda_N^1$ , summing to 1, such that  $\lambda_n^1 \geq \lambda_n^*$  for all  $n$ . Obviously, such a  $\lambda^1$  exists. Then the definition of  $\lambda_n^*$  and the fact that  $\lambda_n^1 \geq \lambda_n^*$  implies

$$\lambda_n^1 \mu^*(u) + \mu_1(A_n, \mathcal{V}_n) \geq \mu_2(A_n, \mathcal{V}_n), \quad \forall A_n \in [-1, 1], \quad \forall n.$$

Substituting for  $\mu^*(u)$ , then,

$$\lambda_n^1 \mu_1(u) + \mu_1(A_n, \mathcal{V}_n) \geq \lambda_n^1 \mu_2(u) + \mu_2(A_n, \mathcal{V}_n), \quad \forall A_n \in [-1, 1], \quad \forall n. \quad (6)$$

Next, suppose  $\mu^*(u) = 0$ . In this case, define  $\lambda_n^1 = 1/N$  for  $n = 1, \dots, N$ . Then equation (5) evaluated at any fixed  $A_n$  with  $A_m = 1$  for all  $m \neq n$  implies equation (6).

Next, define  $\mu^*(-u) = \mu_2(-u) - \mu_1(-u)$ . First, assume  $\mu^*(-u) > 0$ . Then define  $\lambda_1^2, \dots, \lambda_N^2$  by

$$\lambda_n^1 \mu_1(u) + \mu_1(-1, \mathcal{V}_n) = \lambda_n^1 \mu_2(u) + \mu_2(-1, \mathcal{V}_n) + \lambda_n^2 \mu^*(-u). \quad (7)$$

By equation (6) at  $A_n = -1$ ,  $\lambda_n^2 \geq 0$  for all  $n$ . Also, summing both sides over  $n$  and using  $\sum_n \lambda_n^1 = 1$ , we obtain

$$\mu_1(u) + \sum_n \mu_1(-1, \mathcal{V}_n) = \mu_2(u) + \sum_n \mu_2(-1, \mathcal{V}_n) + \mu^*(-u) \sum_n \lambda_n^2.$$

The left-hand side is  $\mu_1(\mathcal{W}) - \mu_1(-u) = 1 - \mu_1(-u)$ . The right-hand side is  $1 - \mu_2(-u) + \mu^*(-u) \sum_n \lambda_n^2$ . Hence we have

$$\mu^*(-u) = \mu^*(-u) \sum_n \lambda_n^2.$$

Since  $\mu^*(-u) > 0$  by assumption, we must have  $\sum_n \lambda_n^2 = 1$ .

Second, suppose  $\mu^*(-u) = 0$ . In this case, the definition of  $\lambda^1$  implies

$$\lambda_n^1 \mu_1(u) + \mu_1(-1, \mathcal{V}_n) \geq \lambda_n^1 \mu_2(u) + \mu_2(-1, \mathcal{V}_n) \quad (8)$$

for every  $n$ . Summing both sides over  $n$  and using  $\sum_n \lambda_n^1 = 1$ , we obtain

$$\mu_1(u) + \sum_n \mu_1(-1, \mathcal{V}_n) \geq \mu_2(u) + \sum_n \mu_2(-1, \mathcal{V}_n).$$

But since  $\mu^*(-u) = 0$ , we have  $\mu_1(-u) = \mu_2(-u)$ , so

$$\mu_1(u) + \sum_n \mu_1(-1, \mathcal{V}_n) + \mu_1(-u) \geq \mu_2(u) + \sum_n \mu_2(-1, \mathcal{V}_n) + \mu_2(-u).$$

But both each side of this inequality must equal 1. Hence equation (8) must hold with equality for all  $n$ . In light of this, we can define  $\lambda_n^2 = 1/n$  for all  $n$  and equation (7) still holds.

This implies that we can rewrite  $\mu_i$  as a measure  $\hat{\mu}_i$  over  $[-1, 1] \times \mathcal{V}$ ,  $i = 1, 2$ , as follows. For any measurable  $E \subseteq (-1, 1) \times \mathcal{V}$ , let

$$\hat{\mu}_i(E) = \mu_i \left( \left\{ w \in \mathcal{W} \mid w = v\sqrt{1 - A^2} + Au, (A, v) \in E \right\} \right).$$

For  $E = \{1\} \times \mathcal{V}_n$ , let

$$\hat{\mu}_i(E) = \lambda_n^1 \mu_i(u)$$

and for  $E = \{-1\} \times \mathcal{V}_n$ , let

$$\hat{\mu}_i(E) = \lambda_n^2 \mu_i(-u).$$

To see that such a measure exists, for each  $n$ , choose an arbitrary  $v_n \in \mathcal{V}_n$  and assign probability  $\lambda_n^1 \mu_i(u)$  to  $\{1\} \times v_n$  and probability  $\lambda_n^2 \mu_i(-u)$  to  $\{-1\} \times v_n$ . Extend this to the Borel field on  $[-1, 1] \times \mathcal{V}$  in the obvious manner. That is, for each  $E \subseteq [-1, 1] \times \mathcal{V}_n$ , let

$$\begin{aligned} \hat{\mu}_i(E) &= \hat{\mu}_i(E \cap [(-1, 1) \times \mathcal{V}]) + \hat{\mu}_i(E \cap (\{1\} \times \{v_i \mid i = 1, \dots, N\})) \\ &\quad + \hat{\mu}_i(E \cap (\{-1\} \times \{v_i \mid i = 1, \dots, N\})). \end{aligned}$$

The key point to observe about these measures is that for every  $n$  and every  $A_n \in [-1, 1]$ , we have

$$\hat{\mu}_1([A_n, 1] \times \mathcal{V}_n) = \lambda_n^1 \mu_1(u) + \mu_1(A_n, \mathcal{V}_n) \geq \lambda_n^1 \mu_2(u) + \mu_2(A_n, \mathcal{V}_n) = \hat{\mu}_2([A_n, 1] \times \mathcal{V}_n)$$

and

$$\begin{aligned} \hat{\mu}_1([-1, 1] \times \mathcal{V}_n) &= \lambda_n^1 \mu_1(u) + \mu_1(-1, \mathcal{V}_n) + \lambda_n^2 \mu_1(-u) \\ &= \lambda_n^1 \mu_2(u) + \mu_2(-1, \mathcal{V}_n) + \lambda_n^2 \mu_2(-u) \\ &= \hat{\mu}_2([-1, 1] \times \mathcal{V}_n). \end{aligned}$$

Generalizing, given any finite measurable partition  $\Pi$  of  $\mathcal{V}$ , let  $\mathcal{M}_\Pi$  be the set of pairs of measures  $(\hat{\mu}_1, \hat{\mu}_2)$  over  $[-1, 1] \times \mathcal{V}$  such that

$$\begin{aligned} \hat{\mu}_i(E) &= \mu_i \left( \left\{ w \in \mathcal{W} \mid w = v\sqrt{1 - A^2} + Au, (A, v) \in E \right\} \right), \\ \forall \text{ measurable } E, \quad i &= 1, 2, \end{aligned} \tag{9}$$

$$\hat{\mu}_1([A_n, 1] \times \mathcal{V}_n) \geq \hat{\mu}_2([A_n, 1] \times \mathcal{V}_n), \quad \forall A_n \in [-1, 1], \quad \forall n, \tag{10}$$

and

$$\hat{\mu}_1([-1, 1] \times \mathcal{V}_n) = \hat{\mu}_2([-1, 1] \times \mathcal{V}_n), \quad \forall n. \quad (11)$$

We have shown that for every finite measurable partition  $\Pi$ ,  $\mathcal{M}_\Pi$  is nonempty. It is also not hard to see that it must be closed. Clearly, if  $\Pi'$  is a refinement of  $\Pi$ , then  $\mathcal{M}_{\Pi'} \subseteq \mathcal{M}_\Pi$ .

Each  $\mathcal{M}_\Pi$  is a closed nonempty subset of the space of pairs of measures over  $\mathcal{V}$ , obviously a compact set. Fix a finite collection of finite measurable partitions, say  $\Pi_1, \dots, \Pi_T$ . Let  $\Pi$  be the coarsest common refinement of these partitions. Then  $\mathcal{M}_\Pi \subseteq \mathcal{M}_{\Pi_t}$  for all  $t$ . Since  $\mathcal{M}_\Pi$  must be nonempty, we see that  $\cap_t \mathcal{M}_{\Pi_t} \neq \emptyset$ . By Kelly (1955, Chapter 5, Theorem 1), this implies that  $\cap_\Pi \mathcal{M}_\Pi$  is nonempty where the intersection is taken over the set of all finite measurable partitions. Hence there is at least one pair of measures which satisfies equations (9), (10), and (11) for every finite measurable partition.

Hence we have shown that we can rewrite  $\mu_1$  and  $\mu_2$  as distributions  $\hat{\mu}_1$  and  $\hat{\mu}_2$  over  $(A, v) \in [-1, 1] \times \mathcal{V}$  with the following properties. First, equation (9) implies that for every menu  $x$ ,

$$\int_w \max_{\beta \in B_w(x)} u(\beta) \mu_i(dw) = \int_{(A,v)} \max_{\beta \in B_{v\sqrt{1-A^2}+Au}(x)} u(\beta) \hat{\mu}_i(d(A, v)), \quad i = 1, 2.$$

This holds since we have only specified how mass at  $u$  and  $-u$  is spread across the sets  $\{1\} \times \mathcal{V}$  and  $\{-1\} \times \mathcal{V}$  respectively. Since  $(1, v)$  and  $(1, v')$  both correspond to utility function  $u$ , this has no effect on the calculation of the utility of any menu.

Second, equation (10) implies that for every measurable function  $A^* : \mathcal{V} \rightarrow [-1, 1]$ ,

$$\int_v \hat{\mu}_1([A^*(v), 1] \times \{v\}) dv \geq \int_v \hat{\mu}_2([A^*(v), 1] \times \{v\}) dv.$$

To see this, simply note since  $A^*$  is bounded and measurable, there exists an increasing sequence of simple functions  $A_n^*$  converging to  $A^*$  pointwise from below.<sup>18</sup> Letting  $W^* = \{(A, v) \mid A \geq A^*(v)\}$  and  $W_n = \{(A, v) \mid A \geq A_n^*(v)\}$ , we see that  $W^* = \cap_n W_n$ , so  $\hat{\mu}_i(W^*) = \lim_{n \rightarrow \infty} \hat{\mu}_i(W_n)$ . Hence

$$\hat{\mu}_1(W^*) = \lim_{n \rightarrow \infty} \hat{\mu}_1(W_n) \geq \lim_{n \rightarrow \infty} \hat{\mu}_2(W_n) = \hat{\mu}_2(W^*),$$

where the inequality follows from equation (10).

Third, it is easy to see that equation (11) implies that the marginals of  $\hat{\mu}_1$  and  $\hat{\mu}_2$  over  $\mathcal{V}$  are the same.

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<sup>18</sup>It is straightforward to modify the proof of Theorem 13.5 in Billingsley (1995, page 185) to show this.

Letting  $\mu_{\mathcal{V}}$  denote the (common) marginal of  $\hat{\mu}_1$  and  $\hat{\mu}_2$  on  $\mathcal{V}$  and  $\mu_A^i(\cdot | v)$  a regular version of the conditional for  $\hat{\mu}_i$ , we see that  $(\mathcal{V}, \mu_{\mathcal{V}}^i, \mu_A^i(\cdot | v))$  is a decomposition of  $\mu_i$  for  $i = 1, 2$ .

Now we complete the proof by showing that  $\mu_A^1(\cdot | v)$  FOSD  $\mu_A^2(\cdot | v)$  for almost all  $v$ .

Let  $\mathcal{V}(\bar{A})$  denote the set of  $v$  such that  $\mu_A^1([\bar{A}, 1] | v) < \mu_A^2([\bar{A}, 1] | v)$  and let  $\mathcal{V}^* = \cup_{\bar{A} \in [-1, 1]} \mathcal{V}(\bar{A})$ . We now show that  $\mathcal{V}^*$  has  $\mu_{\mathcal{V}}$  measure zero.

First, note that if there is an  $\bar{A}$  such that  $\mu_A^1([\bar{A}, 1] | v) < \mu_A^2([\bar{A}, 1] | v)$ , then there must be a rational  $\bar{A}$  with this property. This is obviously true if the distributions are continuous in a neighborhood of  $\bar{A}$ . If a distribution has a mass point at  $\bar{A}$ , then we can perturb the  $\bar{A}$  slightly in one direction and maintain the inequality. Hence  $\mathcal{V}^* = \cup_{\bar{A} \in \mathcal{R}} \mathcal{V}(\bar{A})$  where  $\mathcal{R}$  denotes the rationals. For any  $\bar{A}$ ,  $\mathcal{V}(\bar{A})$  is measurable, so, as a countable union of measurable sets,  $\mathcal{V}^*$  is measurable. Clearly,

$$\mu_{\mathcal{V}}(\mathcal{V}^*) \leq \sum_{\bar{A} \in \mathcal{R}} \mu_{\mathcal{V}}(\mathcal{V}(\bar{A})).$$

To show that the right-hand side is zero, suppose it is positive. Then there must be some rational  $\bar{A}$  such that  $\mu_{\mathcal{V}}(\mathcal{V}(\bar{A})) > 0$ . For every  $v \in \mathcal{V}(\bar{A})$ , we have

$$\mu_A^1([\bar{A}, 1] | v) \mu_{\mathcal{V}}(v) < \mu_A^2([\bar{A}, 1] | v) \mu_{\mathcal{V}}(v).$$

Integrating over  $v \in \mathcal{V}(\bar{A})$ , we get

$$\hat{\mu}_1([\bar{A}, 1] \times \mathcal{V}(\bar{A})) < \hat{\mu}_2([\bar{A}, 1] \times \mathcal{V}(\bar{A})),$$

a contradiction to equation (10). ■



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