Efficient estimation of parameters in marginals in semiparametric multivariate models*

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Abstract

Recent literature on semiparametric copula models focused on the situation when the marginals are specified nonparametrically and the copula function is given a parametric form. For example, this setup is used in Chen, Fan and Tsyrennikov (2006) [Efficient Estimation of Semiparametric Multivariate Copula Models, JASA] who focus on efficient estimation of copula parameters. We consider a reverse situation when the marginals are specified parametrically and the copula function is modelled nonparametrically. This setting is no less relevant in applications. We use the method of sieve for efficient estimation of parameters in marginals and show its asymptotic distribution. Simulations suggest that the sieve MLE can be up to 40% more efficient relative to QMLE depending on the strength of dependence between marginals. An application using insurance company loss and expense data demonstrates empirical relevance of this setting.

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1 Introduction

Consider an m-variate random variable Y with joint pdf $h(y_1, \ldots, y_m)$. Let $f_1(y_1), \ldots, f_m(y_m)$ denote the corresponding marginal pdf's. Assume that the marginals are known up to a parameter vector β (β collects the distinct parameters of all marginals). The dependence structure is not given. We observe a random sample $\{\mathbf{y}_i\}_{i=1}^N = \{y_{1i}, \ldots, y_{mi}\}_{i=1}^N$. We are interested in estimating β efficiently without assuming anything about the joint distribution except for the marginals.

As a simple example consider the setting of a standard panel (small T, large N). We have a well specified marginal for each of T cross section (e.g., logit models, duration models, stochastic frontier models, etc.) and we are interested in efficient estimation of the parameters in the marginal distributions without assuming a parametric form on dependence between them. This setting is typical for microeconomic applications. The variable of interest y_t, $t = 1, \ldots, T$, can be the duration of unemployment in period t, or the use of social services in period t. Additional motivation for this problem comes from insurance. In particular, it arises in models of survival of multiple lives, where the two or more durations are dependent (see,e.g., Frees and Valdez, 1998). In life insurance of spouses this effect is known as the "broken heart" syndrome. In finance, a similar setting arises in the so called SCOMDY models (Chen and Fan, 2006a,b), where interest is in estimation of individual conditional distribution parameters and innovations of the univariate GARCH models are allowed to have arbitrary dependence.

We will use the well known representation of log-joint-density in terms of log-marginal-

densities and the log-copula-density:

$$\ln h(y_1, \ldots, y_m; \beta) = \sum_{j=1}^{m} \ln f_j(y_j; \beta) + \ln c(F_1(y_1; \beta), \ldots, F_m(y_m; \beta)), \tag{1}$$

where $c(\cdots)$ is a copula density and F_i denotes the corresponding marginal CDF's. This decomposition is due to Sklar's (1959) theorem which states that any continuous joint distribution can be represented by a unique copula function of the corresponding continuous marginal CDF's.

It is well understood that the parameters of the marginals can be consistently estimated by maximizing the likelihood under the assumption of independence between marginals – this is the so called quasi maximum likelihood estimator, or QMLE. The copula term in (1) is zero in this case. However, QMLE is not efficient if marginals are not independent. For highly dependent marginals, the efficiency loss of QMLE relative to the full likelihood MLE may be quite large. In the context of a two-stage estimation of parametric copula models, Joe (2005) reports that FMLE asymptotic variance estimates for β are up to 93% smaller than those of QMLE. Recently, Prokhorov and Schmidt (2009) investigated the conditions for copula redundancy, that is when using the copula score does not improve efficiency over QMLE. The redundancy conditions they derive are fairly strong so incorporating information about dependence into parametric estimation problem will usually bring efficiency gains.

It is also well understood that, unlike QMLE, FMLE is generally not robust to copula misspecification. That is, the efficiency gains will come at the expense of an asymptotic bias if the joint density is misspecified. Prokhorov and Schmidt (2009) point out that there are

robust parametric copulas, for which pseudo MLE (PMLE) based on an incorrectly specified copula leads to a consistent estimation. But a copula that is robust in one problem may not be robust in another, and some robust copulas are robust because they are redundant. So finding a general class of robust non-redundant parametric copulas is difficult if at all possible.

In this paper we address the issue of robust and efficient estimation of β using nonparametric methods. That is, we investigate whether we can obtain a consistent estimator of the parameters of marginals, which is more efficient than the QMLE, by modelling the copula nonparametrically. So the questions we ask are how to estimate β semiparametrically, what is the semiparametric efficiency bound for the estimation of β, and whether we can achieve it. To answer this questions we propose a sieve MLE (SMLE) procedure, which estimates β and $\ln c$ simultaneously (in one-step). Even though other nonparametric methods are available, e.g., kernel, local linear estimators, we choose the linear sieve method because of its simplicity. In effect we are replacing the true copula term in FMLE with its sieve approximator. Given the approximator, the problem becomes essentially identical to regular parametric FMLE. Subject to an approximation error, this produces a generally robust and usually non-redundant copula term, in the sense explained above.

What is not well understood is how such a semiparametric estimator compares to QMLE. Both QMLE and SMLE have the same parametric part – the correctly specified (up to a parameter vector) marginal distributions. They both involve no assumption on the dependence structure and, unlike FMLE, are robust to copula misspecification. However, it is unclear which estimator is more efficient. One the one hand, the SMLE uses dependence information and, as we shall show, is semiparametrically efficient. On the other, SMLE involves estimation

of a number of extra parameters.

The paper is related to the literature on efficient semiparametric estimation of copula parameters with nonparametric marginals (see, e.g., Chen et al., 2006) and on efficient estimation of nonparametric marginals when the copula is fully known (see, e.g., Segers et al., 2008). More generally, it is related to the literature on seive-based estimation of models that contain unknown functions (see, e.g., Ai and Chen, 2003; Newey and Powell, 2003). It is also related to the literature on two-step semiparametric estimation (see, e.g., Newey and McFadden, 1994; Severini and Wong, 1992) and the literature on semiparametric efficiency bound (see, e.g., Severini and Tripathi, 2001; Newey, 1990).

The paper by Chen et al. (2006) considers a problem which is the converse of ours – a sieve MLE estimation when the copula has a known parametric form but the marginals are unknown. In that setting, sieves are employed to approximate univariate marginal densities. We are employing sieves to approximate a multivariate (log-)density. So the main difficulty of our setting is that, in high dimensions, we will suffer from the curse of dimensionality. For low dimensional problems, simulations show that SMLE is feasible and can lead to efficiency gains of up to 40% over QMLE.

We first present theory of SMLE for our problem in Section 2. Section 3 contains simulation results, while Section 4 presents an insurance application. Section 5 contains concluding remarks.

2 Sieve MLE

Denote the true copula density by $c_o(\mathbf{u})$, $\mathbf{u} = (u_1, \ldots, u_m)$, and denote the true parameter vector by β_o. Let $c_o(\mathbf{u})$ belong to an infinite-dimensional space $\Gamma = \{c(\mathbf{u}) : [0,1]^m \to [0,1], \int_{[0,1]^m} c(\mathbf{u}) = 1)$ and β_o belong to $B \subset R^p$. Given a finite amount of data, optimization over the infinite-dimensional space Γ is not feasible. The method of sieves is used to overcome this problem. Define a sequence of approximating spaces Γ_N, called sieves, such that $\bigcup_N \Gamma_N$ is dense in Γ. Optimization is then restricted to the sieve space. Grenander (1981) is credited for observing that the MLE optimization, which is infeasible over an infinite dimensional space, is remedied if we optimize over a subset of the parameter space (i.e. over the sieve space) and then allow the subset to grow with the sample size. See Chen (2007) for a recent survey of sieve methods.

Chen (2007) suggests that a convenient finite dimensional linear sieve for approximating a multivariate log-pdf on $[0,1]^m$ is a tensor product of linear univariate sieves on $[0,1]$:

$$\Gamma_N = \left\{ c_{J_N}(\mathbf{u}) = \exp\left\{ \sum_{k=1}^{J_N} a_{1k} A_k(u_1) \cdot \ldots \cdot \sum_{k=1}^{J_N} a_{mk} A_k(u_m) \right\}, \right. \tag{2}$$

$$\left. \mathbf{u} \in [0,1]^m, \int_{[0,1]^m} c_{J_N}(\mathbf{u}) d\mathbf{u} = 1 \right\}, \tag{3}$$

$$J_N \to \infty \frac{J_N}{N} \to 0, \tag{4}$$

where $\{A_k\}$ contains known basis functions and $\{a_{jk}\}$ contains the unknown sieve coefficients. Specific examples of the basis functions $A_k(u)$ include power series, trigonometric polynomials, splines, wavelets, neural networks and many others. For example, in simulations and

application we use the trigonometric sieve basis functions:

$$A_k(u) = a_k \cos(k\pi u) + b_k \sin(k\pi u),$$

where $u \in [0,1]$ and $a_k, b_k \in R$. The number of sieve elements in the tensor sieve J_N^m is the smoothing parameter analogous to bandwidth in kernel estimation – it sets the quality of sieve approximation.

Since in general there is no analytic solution for the MLE of the sieve coefficients, the practical implementation of tensor sieves is often complicated. As an alternative we consider using Bernstein polynomials, in particular the Bernstein copula density introduced by Sancetta and Satchell (2004).

$$c_{J_N}(\mathbf{u}) = J_N^m \sum_{v_1=0}^{J_N-1} \cdots \sum_{v_m=0}^{J_N-1} \omega_v \prod_{l=1}^{m} \begin{pmatrix} J_N - 1 \\ v_l \end{pmatrix} u_l^{v_l} (1-u_l)^{J_N - v_l - 1}, \tag{5}$$

where ω_v denotes the parameters of the polynomial indexed by $v = (v_1, \ldots, v_m)$ such that $0 \leq \omega_v \leq 1$ and $\sum_{v_1=0}^{J_N-1} \cdots \sum_{v_m=0}^{J_N-1} \omega_v = 1$. For the initial values of the parameters we may use the multivariate empirical density (histogram) estimator, i.e. $\omega_v = \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}(U_i \in H_v)$, where $U_i = (F_1(y_1), \ldots, F_m(y_m))$, $\mathbb{I}(\cdot)$ is the indicator function and

$$H_v = \left[\frac{v_1}{J_n}, \frac{v_1+1}{J_n}\right] \times \cdots \times \left[\frac{v_m}{J_n}, \frac{v_m+1}{J_n}\right]. \tag{6}$$

Note that the sieve above can be represented by a weighted sum of β-distributions. The relation between the empirical density and the MLE solution for ω still needs to be inves-

tigated but we found this sieve to converge faster in simulations than the tensor product sieve. Sancetta (2007) derives the rates of convergence of Bernstein copula to the true copula. Ghosal (2001), and references therein, discusses the rate of convergence of the sieve MLE based on Bernstein polynomial (only for one-dimensional densities.)

We can now write the sieve for $\Theta = B \times \Gamma$ as $\Theta_N = B \times \Gamma_N$. If we further let $\theta = (\beta', c)'$, the sieve MLE can be written as

$$\hat{\theta} = \arg\max_{\theta \in \Theta_N} \sum_{i=1}^{N} \ln h(\mathbf{y}_i; \theta) \tag{7}$$

This estimator is easy to implement – the estimation problem is in effect a parametric likelihood maximization problem once we replace Θ by Θ_N.

We are interested in the asymptotic distribution of $\hat{\beta}$, the first p elements of $\hat{\theta}$. By the Gramér-Wold device, this distribution is normal if, for any $\lambda \in R^p, \|\lambda\| \neq 0$, the distribution of linear combination $\lambda'\beta$ is normal. Note that $\lambda'\beta$ is a functional of θ, call it $F_\lambda(\theta)$. Its distribution given the sieve estimate $\hat{\theta}$ is known to depend on smoothness of the functional and on the convergence rate of $\hat{\theta}$ (see, e.g., Shen, 1997). In our setting, the functional is very smooth and that will compensate for a slower convergence rate of $\hat{\theta}$. So in establishing asymptotic normality we follow the standard route (see, e.g., Ai and Chen, 2003; Chen et al., 2006). First, we show continuity of $\lambda'\beta$ and then employ the Riesz representation theorem, to show normality of $\sqrt{N}(\hat{\beta} - \beta)$. In showing semiparametric efficiency of the SMLE of β we follow the standard method of looking for the least favorable parametric submodel. A recent simplified version of this approach can be found in Severini and Tripathi (2001).

We first list standard identification and smoothness assumptions used in sieve based estimation (see, e.g., Chen, 2007; Chen et al., 2006).

Assumption 1 *(identification)* $\beta_o \in int(B) \subset R^p$*,* B *is compact and there exists a unique* θ_o *which maximizes* $E[\ln h(\mathbf{Y}_i;\theta)]$ *over* $\Theta = B \times \Gamma$*.*

A common smoothness assumption in nonparametrics is to restrict the class of considered functions by a certain smoothness property (see, e.g., Shen, 1997; Ai and Chen, 2003; Chen et al., 2006). Let g denote a real-valued, J times continuously differentiable function on $[0,1]^m$ whose J-th derivative satisfies the following condition for some $K > 0$ and $r \in (J, J+1]$:

$$|D^J g(x) - D^J g(y)| \leq K|x-y|_E^{r-J}, \text{for all } x, y \in [0,1]^m, \tag{8}$$

where $D^\alpha = \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}}, \alpha = \alpha_1 + \ldots + \alpha_m$ is the differential operator, and $|x|_E = (x'x)^{1/2}$ is the Euclidean norm. Then g is said to belong to the Hölder class on $[0,1]^m$, denoted $\Lambda^r([0,1]^m)$. It is also called r-smooth on $[0,1]^m$. Linear sieves are known to approximate r-smooth functions well.

Assumption 2 *(smoothness - Hölder class for copula; differentiability for marginals)* $\Gamma = \{c = \exp(g) : g \in \Lambda^r([0,1]^m), r > 1/2, \int c(u)du = 1\}$ *and* $\ln f_j(y_j;\beta), j = 1, \ldots, m,$ *are twice differentiable w.r.t.* β

Now we introduce new notation that will be used in proofs of continuity of $F_\lambda(\theta) = \lambda'\beta$ and of asymptotic normality and semiparametric efficiency of $\sqrt{N}(\hat{\beta} - \beta)$. First, we consider the directional derivative of the loglikelihood in direction $\nu = (\nu'_\beta, \nu_\gamma)' \in V$, where V is the

linear span of $\Theta - \{\theta_o\}$.

$$\begin{aligned}
\dot{l}(\theta_o)[\nu] &\equiv \lim_{t\to 0} \frac{\ln h(\theta+t\nu,y)-\ln h(\theta,y)}{t}\Big|_{\theta=\theta_o} \\
&= \frac{\partial \ln h(\theta_o,y)}{\partial \theta'}[\nu] \\
&= \sum_{j=1}^m \left\{ \frac{\partial \ln f_j(y_j,\beta_o)}{\partial \beta'} + \frac{1}{c(F_1(y_1,\beta_o),\ldots,F_m(y_m,\beta_o))} \frac{\partial c(u_1,\ldots,u_m)}{\partial u_j}\Big|_{u_k=F_k(y_k,\beta_o)} \frac{\partial F_j(y_j,\beta_o)}{\partial \beta'} \right\} \nu_\beta \\
&\quad + \frac{1}{c(F_1(y_1,\beta_o),\ldots,F_m(y_m,\beta_o))} \nu_\gamma(u_1,\ldots,u_m)
\end{aligned}$$

Then, we define, on space V, the Fisher inner product $\langle \cdot,\cdot \rangle \equiv E\left[\dot{l}(\theta_o)[\cdot]\dot{l}(\theta_o)[\cdot]\right]$ and the Fisher norm $||\nu|| \equiv \sqrt{\langle \nu,\nu \rangle}$, where expectation is with respect to the true density h. The closed linear span of $\Theta - \{\theta_o\}$ and the inner product $\langle \cdot,\cdot \rangle$ form a Hilbert space, call it $(\bar{V}, ||\cdot||)$.

Since $F_\lambda(\theta) = \lambda'\beta$ is linear on $\bar{V}$, to show its continuity, we only need to establish that it is bounded on $\bar{V}$, i.e. that $\sup_{0\neq\theta-\theta_o\in\bar{V}} \frac{|F_\lambda(\theta)-F_\lambda(\theta_0)|}{||\theta-\theta_o||} < \infty$. This is the case if $\sup_{\nu\neq 0,\nu\in\bar{V}} \frac{|\lambda'\nu_\beta|^2}{||\nu||^2} < \infty$. Note that, similar to Chen et al. (2006) and Ai and Chen (2003), we can write

$$\begin{aligned}
\sup_{\nu\neq 0,\nu\in\bar{V}} \frac{|\lambda'\nu_\beta|^2}{||\nu||^2} &= \sup_{\nu\neq 0,\nu\in\bar{V}} \left\{ |\lambda'\nu_\beta|^2 \left(E\left[\dot{l}(\theta_o)[\nu]^2\right]\right)^{-1} \right\} \\
&= \lambda' \left(ES_\beta S_\beta'\right)^{-1} \lambda,
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
S_\beta' &= \sum_{j=1}^m \left\{ \frac{\partial \ln f_j(y_j,\beta_o)}{\partial \beta'} + \left(\frac{1}{c(\mathbf{u})} \frac{\partial c(u_1,\ldots,u_m)}{\partial u_j} \right)\Big|_{u_k=F_k(y_k,\beta_o)} \frac{\partial F_j(y_j,\beta_o)}{\partial \beta'} \right\} \\
&\quad + \frac{1}{c(F_1(y_1,\beta_o),\ldots,F_m(y_m,\beta_o))} g^*(u_1,\ldots,u_m)
\end{aligned} \tag{10}$$

and $(g_1^*,\ldots,g_p^*)$, which belong to the product space of square integrable zero-mean functions on $[0,1]^m$, are the solutions to the following infinite-dimensional optimization problem for

$q = 1, \ldots, p$:

$$\inf_{g_q} E\left[\sum_{j=1}^{m}\left\{\frac{\partial \ln f_j(y_j,\beta_o)}{\partial \beta_q} + \left(\frac{1}{c(\mathbf{u})}\frac{\partial c(\mathbf{u})}{\partial u_j}\right)\Bigg|_{u_k=F_k(y_k,\beta_o)} \frac{\partial F_j(y_j,\beta_o)}{\partial \beta_q}\right\} + \frac{1}{c(\mathbf{u})}\Bigg|_{u_k=F_k(y_k,\beta_o)} g_q(u_1,\ldots,u_m)\right]^2. \tag{11}$$

So $F_\lambda(\theta) = \lambda'\beta$ is bounded if and only if $ES_\beta S_\beta'$ is finite and positive definite.

Assumption 3 *(nonsingular information) Assume that $ES_\beta S_\beta'$ is nonsingular.*

We can now apply the Riesz representation theorem (see, e.g., Kosorok, 2008, p. 328) to derive the asymptotic distribution of λ. The theorem states that for any continuous linear functional $L(\nu)$ on a Hilbert space there exists a vector ν^* (the Riesz representer of that functional) such that, for any ν

$$L(\nu) = \langle \nu, \nu^* \rangle,$$

and the norm of the functional defined as

$$||L||_* \equiv \sup_{\nu \in V: ||\nu||=1} ||L(\nu)||$$

is equal to $||\nu^*||$.

By the representation theorem, $F_\lambda(\theta - \theta_o) = \lambda'(\beta - \beta_o) = \langle \theta - \theta_o, \nu^* \rangle$ and we can establish asymptotic normality of $\sqrt{N}\lambda'(\hat{\beta} - \beta)$ by examining $\sqrt{N}\langle \hat{\theta} - \theta_o, \nu^* \rangle$.

The last assumption required for asymptotic normality is an assumption on the rate of

convergence for the sieve MLE estimator of the unknown copula function.

Assumption 4 *(convergence of sieve MLE) Assume that* $||\hat{\theta} - \theta_o|| = O_P(\delta_N)$ *for* $(\delta_N)^w = o(N^{-1/2})$ *and there exists* $\Pi_N \nu^* \in V_N - \{\theta_o\}$ *such that* $\delta_N ||\Pi_N \nu^* - \nu^*|| = o(N^{-1/2})$.

A standard result is that under these assumptions, $\hat{\beta}$ is consistent and the asymptotic variance of $N^{1/2}(\hat{\beta} - \beta)$ is $||\nu^*||^2$ (see, e.g. Shen, 1997; Severini and Tripathi, 2001; Chen et al., 2006).

Theorem 1 *Under Assumptions 1-4,* $\sqrt{N}(\hat{\beta} - \beta_o) \Rightarrow N(0, (E[S_\beta S_\beta'])^{-1})$

Given the consistent SML estimates $\hat{\beta}$ and $\hat{c}$, g_q^*'s can be estimated consistently in a sieve minimization problem as follows

$$\arg\min_{g_q \in \mathbf{A}_N} \left[\sum_{i=1}^{N} \sum_{j=1}^{m} \left\{ \frac{\partial \ln f_j(y_{ji}, \hat{\beta})}{\partial \beta_q} + \left(\frac{1}{\hat{c}(\hat{\mathbf{u}}_i)} \frac{\partial \hat{c}(\hat{u}_{1i}, \ldots, \hat{u}_{mi})}{\partial u_j} \right) \Bigg|_{\hat{u}_{ki} = F_k(y_{ki}, \hat{\beta})} \frac{\partial F_j(y_{ji}, \hat{\beta})}{\partial \beta_q} \right\} \right.$$
$$\left. + \sum_{i=1}^{N} \frac{1}{\hat{c}(F_1(y_{1i}, \hat{\beta}), \ldots, F_m(y_{mi}, \hat{\beta}))} g_q(\hat{u}_{1i}, \ldots, \hat{u}_{mi}) \right]^2,$$

where $q = 1, \ldots, p$ and $\mathbf{A}_N$ is one of the sieve spaces discussed above.

3 Simulations

Our initial simulations with linear tensor sieves, including splines, polynomials, and trigonometric polynomials, exhibit slow convergence rates. In contrast, using Bernstein polynomials, we were able to obtain the convergence within reasonable time. We therefore present the results for the latter sieve.

One of the practical problems we face is the choice of the degree of polynomials J_N in finite samples. While some asymptotic results on the rate of convergence and its dependence on J_N are available, they are not informative in the finite sample situation. The literature on sieves suggest using typical model selection techniques, such as BIC, AIC. However, the theoretical implications of using these techniques in the context of sieves are not explored.[1]

The DGP we use in simulations is similar to Joe (2005) who studied asymptotic relative efficiency (ARE) which is the ratio of the asymptotic variance of the Full MLE to QMLE estimators of the parameters of the marginals. Joe (2005) finds that the ARE depends on the specification of the marginals and copula. In particular, the higher is the dependence implied by the copula, the lower is the ARE of the QMLE, i.e. the more efficient is FMLE compared to QMLE. We take the case where the ARE is the lowest and investigate whether we may improve the efficiency of the QMLE by using the semiparametric sieve MLE technique.

We consider bivariate DGP with the marginals which are exponentially distributed with the both mean parameters set to 0.5. The dependence is modelled by the Plackett copula with the parameter 0.002, which implies that we are close the lower Frechet bound. Joe (2005) reports ARE=0.064 of the QMLE in this specific case. In the simulation we use correctly specified marginals up to the 2 parameters to be estimated, while the copula function is modelled using the Bernstein polynomials sieve. We use the BIC to determine the degree of the sieve J_N. The number of observations $N = 1000$. BIC is minimized for $J_N = 8$. Thus we have to estimate 64 nuisance parameters of the sieve and 2 parameters of the marginals.

[1]Similar to bandwidth selection, it should be possible to devise data driven methods for choosing the number of elements in sieve but we do not know of such methods and their derivation is not in the scope of this paper.

The optimization is complicated by the restrictions on the parameters of the sieve and the parameters of the marginals. We used standard constrained maximization routine in Matlab. Because of time constraints we used only 100 simulation runs. This will be extended in the future. We report the simulated mean of the Sieve MLE, QMLE and Full MLE estimators, their simulated variance and the simulated relative efficiency (RE) of the QMLE with respect to the Sieve MLE, i.e. the ratio of the SMLE simulated variance to that of QMLE.

Table 1: Simulated mean and variance for QMLE, SMLE, Plackett copula based FMLE

	μ_1 SMLE	QMLE	FMLE	μ_2 SMLE	QMLE	FMLE
$J_N = 8$						
Mean	0.488679	0.501630	0.499924	0.488633	0.498784	0.499509
Var	0.000126	0.000194	0.000012	0.000159	0.000233	0.000012
RE	0.649485			0.682403		
$J_N = 9$						
Mean	0.489800	0.501900	0.499951	0.489600	0.498300	0.500001
Var	0.000118	0.000194	0.000012	0.000153	0.000234	0.000012
RE	0.607530			0.653698		

The result suggests that in this specific situation we were able to improve the efficiency relatively to the QMLE. It appears that there is some evidence of downward bias in the estimates based on Sieve MLE for $J_N = 8$. Therefore, we try $J_N = 9$ in which case the bias seems to become smaller and the variance is also improved. This may suggest that BIC may not be the optimal procedure to select J_N.

4 Application from insurance

We demonstrate the use of SMLE with an insurance application. We have data on 1,500 insurance claims. For each claim, we have the amount of claim payment, or loss, (Y_1) and

the amount of claim-related expenses (Y_2). The claim-related expenses known as ALAE (allocated loss adjustment expense) include the insurance company expenses attributable to an individual claim, e.g. the lawyers' fees and claim investigation expenses. The claim amount variable is censored – there is a dummy variable, d, which is equal to one if a given claim has surpassed the policy limit and zero if not. For details of the data set, see Frees and Valdez (1998).

The claim amount and ALAE are assumed to be distributed according to the Pareto distribution with parameters (λ_1, θ_1) and (λ_2, θ_2), respectively:

$$F_j(Y_j) = 1 - \left(\frac{\lambda_j + Y_j}{\lambda_j}\right)^{-\theta_j}, \quad j = 1, 2. \tag{12}$$

Interest lies in efficient estimation of the marginal distribution parameters $(\lambda_1, \theta_1, \lambda_2, \theta_2)$, making efficient use of the strong dependence between the claim amount and ALAE. Additional complications arise due to censoring of Y_1. The likelihood contributions for censored observations will not be the same as for the uncensored ones and we need to account for that.

Define the marginal pdfs $f_j(y_j), j = 1, 2$. The QMLE log-likelihood contribution of an uncensored observation is $\ln f_j(y_j), j = 1, 2$. For a censored observation, the contribution is $\ln(1 - F_1(y_1)) = \theta_1(\ln(\lambda_1) - \ln(\lambda_1 + y_1))$. So for QMLE, the log-likelihood contribution of claim i is

$$l_i^Q = (1 - d_i) \ln f_1(y_{1i}) + d_i \ln(1 - F_1(y_{1i})) + \ln f_2(y_{2i}).$$

Now consider the joint likelihood. Define the joint cdf $H(y_1, y_2)$ and joint pdf $h(y_1, y_2)$.

The FMLE contribution of an uncensored observation is $\ln h(y_1, y_2) = \ln f_1(y_1) + \ln f_2(y_2) + \ln c(F_1(y_1), F_2(y_2))$. To derive the contribution of a censored observation we follow Frees and Valdez (1998) in observing that $Prob(Y_1 \geq y_1, Y_2 \leq y_2) = F_2(y_2) - H(y_1, y_2)$. So the log-likelihood contribution of a censored observation is $f_2(y_2) - H_2(y_1, y_2)$, where $H_2(y_1, y_2) = \frac{\partial H(y_1,y_2)}{\partial y_2}$. But $H(y_1, y_2) = C(F_1(y_1), F_2(y_2))$ so $H_2(y_1, y_2) = C_2(F_1(y_1), F_2(y_2))\, f_2(y_2)$, where $C_2(u_1, u_2) = \frac{\partial C(u_1,u_2)}{\partial u_2}$. Therefore the full likelihood contribution for observation i can be written as

$$\begin{aligned} l_i^F &= (1-d_i)[\ln f_1(y_1) + \ln f_2(y_2) + \ln c(F_1(y_1), F_2(y_2))] \\ &\quad + d_i[\ln f_2(y_2) + \ln(1 - C_2(F_1(y_1), F_2(y_2)))]. \end{aligned}$$

The main difficulty imposed by censoring is that we need to evaluate an additional term involving a copula derivative. For the SMLE, the term is approximated along with $\ln c$. For the FMLE, the term can be derived analytically for a given copula family or evaluated numerically.

The extra term will carry over to the variance problem (11) and a consistent estimate of the SMLE variance, $\hat{V}$, will now be

$$\begin{aligned} \arg\min_{g_q \in \mathbf{A}_N} \Bigg[&\sum_{i=1}^{N} (1-d_i) \left\{ \sum_{j=1}^{2} \left(\frac{\partial \ln f_j(y_{ji}, \hat{\beta})}{\partial \beta_q} + \frac{1}{\hat{c}(\hat{\mathbf{u}}_i)} \frac{\partial \hat{c}(\hat{\mathbf{u}}_i)}{\partial u_j} \frac{\partial F_j(y_{ji}, \hat{\beta})}{\partial \beta_q} \right) + \frac{1}{\hat{c}(\hat{u}_{1i}, \hat{u}_{2i})} g_q(\hat{u}_{1i}, \hat{u}_{2i}) \right\} \\ &+ \sum_{i=1}^{N} d_i \left\{ \frac{\partial \ln f_2(y_{2i}, \hat{\beta})}{\partial \beta_q} - \frac{1}{1 - \hat{C}_2(\hat{u}_{1i}, \hat{u}_{2i})} \left(\sum_{j=1}^{2} \frac{\partial \hat{C}_2(\hat{\mathbf{u}}_i)}{\partial u_j} \frac{\partial F_j(y_{ji}, \hat{\beta})}{\partial \beta_q} + \int_0^1 g_q(s, \hat{u}_{2i})\, ds \right) \right\} \Bigg]^2, \end{aligned}$$

where $\beta = (\lambda_1, \theta_1, \lambda_2, \theta_2)'$, $\hat{u}_{ki} = F_k(y_{ki}, \hat{\beta})$ and $q = 1, \ldots, 4$. And we will need to evaluate both g_q and its integral over u_1.

The three estimators, QMLE, FMLE and SMLE, and their standard errors are given in Table 2. The QMLE is an estimator based on the assumption of independence. It is known to be robust in the sense that it is consistent even if independence is a false assumption but to obtain the correct standard errors a "sandwich" formula for variance is needed. We report the robust standard errors in the table. The FMLE estimator is based on a fully parametric specification of the joint likelihood function. We follow Frees and Valdez (1998) and assume the Frank copula with dependence parameter α, which along with the Pareto marginals completely parameterizes the model. Consistency of this estimator, sometimes called Pseudo-MLE, relies on correctness of the assumed copula family. If Frank is an incorrect copula family the FMLE results in a bias. The SMLE estimator is also robust in the sense that it does not rely on a correctly specified parametric copula family. But it is not as efficient as any fully parametric model. So we should expect SMLE to be close to QMLE in terms of the estimates and to be between FMLE and QMLE in terms of standard errors.

Estimation results support this intuition. Our FMLE estimates using the Frank copula (which turn out almost identical to those in Frees and Valdez (1998)) provide evidence of a estimation bias that is not present in QMLE and SMLE, both of which are very close. This is an indication of robustness of QMLE and SMLE versus FMLE against a copula misspecification. While the FMLE standard errors are usually smaller than those of QMLE, which is an indication of higher efficiency – a compensation for the lack of robustness. The point we wish to stress is that the SMLE standard errors are smaller than those of QMLE and this gain comes at no robustness cost but at some computational cost. To obtain the SMLE, we used the cosine sieve with three elements in the sieve ($J_N = 3$). The choice was

based on the BIC.

Table 2: Estimates and standard errors of QMLE, SMLE, Frank copula based FMLE for insurance application

	QML Est. (Rob.St.Er.)	SML Est. (St.Er.)	FML Est. (St.Er.)
λ_1	14,442.57	14,438.91	14,561.68
	(2,385.31)	(1,434.87)	(1,392.08)
θ_1	1.135	1.136	1.115
	(0.127)	(0.067)	(0.065)
λ_2	15,133.78	15,133.78	16,708.93
	(2,172.04)	(1,549.66)	(1,833.18)
θ_2	2.223	2.223	2.312
	(0.246)	(0.142)	(0.188)
α			3.158
			(0.175)
LogL	-31,950.80	-31,813.60	-31,778.41

5 Concluding Remarks

We have proposed an efficient semiparametric estimator of marginal distribution parameters. This is a sieve maximum likelihood estimator based on a finite-dimensional approximation of the unspecified part of the joint distribution. As such, the estimator inherits the costs and benefits of the multivariate sieve MLE. The major benefit permitted by sieve MLE is the increased relative asymptotic efficiency compared to quasi-MLE. We showed that the efficiency gains are non-trivial. In some simulations the relative efficiency with respect to QMLE was about 0.6 – a 40% improvement.

The gains come at an increased computational expense. The MLE convergence is slow for the traditional sieves we considered. We found that the Bernstein polynomial is preferred to

other sieves in simulations. The running times are greater than QMLE or full MLE assuming a parametric copula family but they are still reasonable (at least for the two dimensional problems we consider). Moreover, simulations reveal a downward bias in SMLE, which seems to be caused by the sieve approximation error – it decreases as the number of sieve elements increases.

A simple alternative to the proposed method is a fully parametric ML estimation problem. Although simpler computationally, it imposes an assumption on the dependence structure, which, if violated, renders the ML estimates inconsistent. In this respect, the semiparametric approach is more robust but clearly no more efficient than any parametric alternative.

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