Nash Equilibrium in Discontinuous Games*

Philip J. Reny
Department of Economics
University of Chicago

September 3, 2013

Abstract

We provide several generalizations of the various equilibrium existence results in Reny (1999), Barelli and Meneghel (2013), and McLennan, Monteiro, and Tourky (2011). We also provide an example demonstrating that a natural additional generalization is not possible. All of the theorems yielding existence of pure strategy Nash equilibria here are stated in terms of the players' preference relations over joint strategies. Hence, in contrast to virtually all of the previous work in the area the present results for pure strategy equilibria are entirely ordinal, as they should be.

1. Introduction

A primary objective here is to resolve a nagging problem in the literature on the existence of Nash equilibrium in discontinuous games.[1] Because pure strategy equilibria are invariant to ordinal transformations of payoffs, the "right" pure strategy equilibrium existence result should be stated in purely ordinal terms. Yet, virtually all of the existence theorems in the literature rely on non-ordinal properties of the players' utility functions in the sense that their hypotheses, when satisfied in one game, need not be satisfied in an ordinally equivalent game.[2] All of the conditions introduced here are entirely ordinal and are stated in terms of players' preference relations over the joint strategy space.

A second objective is to better connect the existence results for discontinuous games with the more standard existence results for continuous games that are based upon well-behaved

*I am grateful to Andy McLennan, Roger Myerson and Guilherme Carmona for helpful comments and discussions. Financial support from the National Science Foundation (SES-1227506, SES-0922535, SES-0617884) is gratefully acknowledged.

[1] This literature has grown substantially since the seminal contribution of Dasgupta and Maskin (1986). A sample of papers is Simon (1987), Simon and Zame (1990), Baye, Tian, and Zhou (1993), Reny (1999, 2009, 2011), Jackson, Simon, Swinkels and Zame (2002), Carmona (2005, 2009, 2011), Bagh and Jofre (2006), Monteiro and Page (2007, 2008), Barelli and Soza (2009), Bich (2009), Carbonell-Nicolau (2011), Prokopovych (2011, 2013), De Castro (2011), McLennan, Monteiro, and Tourky (2011), Barelli and Meneghel (2013), Barelli, Govindan, and Wilson (2013), Bich and Laraki (2013), He and Yannelis (2013), Nessah (2013).

[2] Recent exceptions are Barelli and Soza (2009) and Propkopovych (2013). An important practical feature of the hypotheses we shall introduce here is their local nature, which is in keeping with the conditions in most of the literature. This is in contrast to the hypotheses of Barelli and Soza's (2009) Theorem 2.2 and Prokopovych's (2013) Theorem 2, which, because of their global nature, are likely to be rather more difficult to verify in practice.

best-reply correspondences (e.g., Nash (1950, 1951), Glicksberg (1952)). To this end, we introduce the concept of *point-security with respect to a subset of players*, where only the preferences of players within the subset are restricted because players outside the subset are presumed to have well-behaved best-reply correspondences. This idea not only leads to more powerful results, it helps to better connect the ideas introduced by McLennan, Monteiro and Tourky (2011) and Barelli and Meneghel (2013) with those of Reny (1999).

The paper proceeds as follows. Section 2 provides notation and some basic definitions. Section 3 provides a new and ordinal "point security" condition as well as our main result, Theorem 3.4. Section 4 shows how Theorem 3.4 can be used to derive various results from the literature. Section 5 shows a variety of ways that the results obtained in previous sections can be straightforwardly extended and refined, and also contains a related result on the existence of mixed strategy equilibria. Section 6 discusses a natural weakening of the "security" part of the assumptions from previous sections and provides an example showing that, under this weakening, existence of a Nash equilibrium cannot be assured.

2. Preliminaries

Let N be a finite set of players. For each $i \in N$, let X_i be a set of pure strategies for player i and let $\geq_i$ be a binary relation on $X = \times_{i \in N} X_i$. This defines a game $G = (X_i, \geq_i)_{i \in N}$.

The symbol $-i$ denotes "all players but i." In particular, $X_{-i} = \times_{j \neq i} X_j$, and x_{-i} denotes an element of X_{-i}. The product of any number of sets is endowed with the product topology and unless otherwise specified, we restrict attention to the topology relative to X.

A strategy $x^* \in X$ is a (*pure strategy*) *Nash equilibrium* of G if $x^* \geq_i (x_i, x^*_{-i})$ for every player $i \in N$ and every $x_i \in X_i$.[3]

Consider the following assumptions on $G = (X_i, \geq_i)_{i \in N}$. For every $i \in N$,

A.1 X_i is a nonempty, compact, subset of a Hausdorff topological vector space, and $\geq_i$ is complete, reflexive, and transitive.

A.2 X_i is a convex set.

A.3 For every $x \in X$, $\{x'_i \in X_i : (x'_i, x_{-i}) \geq_i x\}$ is a convex set.

When A.3 holds, we will say that the preference realtions $\geq_i$ are *convex,*[4] and when A.2 and A.3 hold we say that the game G is *convex.*

[3]Nash equilibrium will always mean pure strategy Nash equilibrium, although we will include "pure strategy" for emphasis from time to time. We will always say mixed strategy Nash equilibrium when mixed strategies are introduced.

[4]Together with A.1, convexity implies also that $\{x'_i \in X_i : (x'_i, x_{-i}) >_i x\}$ is a convex set. Indeed, suppose that $(x^1_i, x_{-i}) >_i x$ and $(x^2_i, x_{-i}) >_i x$. By completeness, assume without loss that $(x^1_i, x_{-i}) \geq_i (x^2_i, x_{-i})$. Then $(\lambda x^1_i + (1-\lambda)x^2_i, x_{-i}) \geq_i (x^2_i, x_{-i}) >_i x$, where the first inequality follows from convexity. The desired result follows from transitivity.

3. Point Security

In this section, we provide our most basic definitions and results for convex games.

Definition 3.1. *The convex game $G = (X_i, \geq_i)_{i\in N}$ is **point secure** if whenever $x \in X$ is not a Nash equilibrium, there is a point $\hat{x} \in X$ and a neighborhood U of x such that for every $y \in U$ there is a player i for whom,*

$$(\hat{x}_i, x'_{-i}) >_i y, \text{ for every } x' \in U.$$

Point security requires that if some player can profitably deviate from x, then there is a neighborhood U of x and there are deviations $\hat{x}_1, ..., \hat{x}_N$, one for each player, such that for each y in U some player i can "secure" an outcome preferred to y by employing his deviation $\hat{x}_i$. That is, not only must it be true for this player i that unilaterally deviating to $\hat{x}_i$ is profitable at y (i.e., $(\hat{x}_i, y_{-i}) >_i y$), the improvement over y must be "secure" in the sense that it obtains even if the others deviate slightly from y_{-i} to their part of any $x' \in U$.[5]

When each player i's preferences are represented by a *continuous* utility function u_i, and $\hat{x}_i$ is profitable against x, it is clear that $u_i(\hat{x}_i, x'_{-i}) > u_i(y)$ whenever x' and y are in a sufficiently small neighborhood U of x. Consequently, point security is satisfied and, moreover, the *same player* i can be chosen for each y in U. Similar to better-reply security (Reny (1999)), the reason that point security is useful in the presence of discontinuities is that, for each $y \in U$, a different player can be chosen to be the one who can secure for himself a better outcome.[6,7]

The following basic proposition, which generalizes Theorem 3.1 in Reny (1999) and, when A.3 holds, generalizes Proposition 2.7 in McLennan Monteiro and Tourky (2011), is a corollary of our main result, Theorem 3.4 below.

Proposition 3.2. *Suppose that A.1, A.2 and A.3 hold. Then G possesses a pure strategy Nash equilibrium if it is point secure.*

Remark 1. *Definition 3.1 would be equivalent if the phrase "there is a point $\hat{x} \in X$ and a neighborhood U of x such that for every $y \in U$ there is a player i for whom," were replaced with the apparently more permissive phrase "there is a finite subset X^0 of X and a*

[5] We will show by example that this security feature of the definition cannot be dropped.

[6] For example, when Bertrand duopolists choose the same price $x_1 = x_2$ above marginal cost, they each have profitable downward deviations $\hat{x}_1, \hat{x}_2$. But for any pair of prices $y = (y_1, y_2)$ near enough x, the firm i whose downward deviation $\hat{x}_i$ from y_i is profitable depends on whose price in y is higher.

[7] It is not too difficult to show that if each $\geq_i$ is represented by a bounded utility function, then $G = (X_i, u_i)_{i\in N}$ is better-reply secure in the sense of Reny (1999) if and only if whenever x is not a Nash equilibrium, there is an $\varepsilon > 0$, a neighborhood U of x, and $\hat{x} \in X$ such that for every $y \in U$ there is a player i for whom $u_i(\hat{x}_i, x'_{-i}) > u_i(y) + \varepsilon$ for every $x' \in U$. (See McLennan, Monteiro and Tourky (2011) for a closely related characterization of better-reply security). Hence, point security eliminates the ε and the need for a utility representation (whose existence is not obvious when preferences are not continuous).

neighborhood U of x such that for every $y \in U$ there exists $\hat{x} \in X^0$ and a player i for whom." This is because if a set X^0 satisfies the condition in the second phrase, then $\hat{x} = (x_i^i)_{i \in N}$ satisfies the condition in the first, where for each player i, $x^i \in X^0$ is chosen so that for every $x' \in U$ and every $x^0 \in X^0$ there exists $x'' \in U$ s.t.$(x_i^i, x'_{-i}) \geq_i (x_i^0, x''_{-i})$.[8]

3.1. Some Players "Continuous"

If some players have well-behaved best-reply correspondences, then they should not pose any difficulty toward establishing the existence of a Nash equilibrium and we should need to focus only on the other players' preferences. This intuition turns out to be correct and leads to our main result for which we now prepare.

For any subset J of the set of players N, let B_J denote the set of strategies $x \in X$ at which every player $i \in J$ is playing a best reply, i.e., $B_J = \{x \in X : \forall i \in J,\ x \geq_i (x'_i, x_{-i})\ \forall x'_i \in X_i\}$. Note that $B_\emptyset = X$ and that B_N is the set of pure strategy Nash equilibria of G.

Definition 3.3. *The convex game $G = (X_i, \geq_i)_{i \in N}$ is* ***point secure with respect to*** *$I \subseteq N$ if for $J = N \backslash I$, whenever $x \in B_J$ is not a Nash equilibrium there is a neighborhood U of x and a point $\hat{x} \in X$ such that for every $y \in U \cap B_J$ there is a player $i \in I$ for whom,*

$$(\hat{x}_i, x'_{-i}) >_i y, \text{ for every } x' \text{in } U \cap B_J.$$

Note that only the preferences of players in I are restricted by this definition. Moreover, the security requirement is less onerous than it is in Definition 3.1. Indeed, $\hat{x}_i$ need only ensure an outcome better than y for i when the others deviate to $x' \in U \cap B_J$. In particular, the deviations x'_j of players $j \in J$ are not completely free to vary since they must always be playing a best-reply to the others' deviations. This makes the condition easier to satisfy and has some powerful consequences, as will be seen in the next section.

Remark 2. *The definition reduces to point security when $I = N$.*

Say that a player's best reply correspondence is *closed* if it has a closed graph. Our main result, whose proof can be found in the appendix, is the following.

Theorem 3.4. *Suppose that A.1, A.2, and A.3 hold and that G is point secure with respect to $I \subseteq N$. If for each $i \in N \backslash I$, X_i is locally convex and player i's best-reply correspondence is closed and has nonempty and convex values, then G possesses a pure strategy Nash equilibrium.*

Remark 3. *When $I = \emptyset$, Theorem 3.4 reduces to the standard equilibrium existence condition that all players have closed best-reply correspondences with nonempty and convex values.*

[8] Such an x^i exists because $\geq_i$ is complete, reflexive, and transitive.

4. Correspondence-Security

McLennan et. al. (2011) provide an ingenious generalization of Reny's (1999) better-reply security condition by allowing players to secure payoffs with finitely many strategies rather than a single strategy such as $\hat{x}$ in Definition 3.1. Barelli and Meneghel (2013) push this even further by allowing players to secure payoffs by employing correspondences that continuously map others' strategies into subsets of their own.

The purpose of the present section is twofold. First, we provide ordinal and more general versions of some of the results of both McLennan et. al. (2011) and Barelli and Meneghel (2013). Second, we show that these more general results based upon "correspondence-security" conditions are in fact a consequence of Theorem 3.4, which is based on the more basic point-security idea.

Let Y and Z each be subsets of any topological vector space. A correspondence $F : Y \twoheadrightarrow Z$ is *closed* if its graph is closed in the relative topology on $Y \times Z$.

The following definition builds upon Barelli and Meneghel's (2013) Definition 2.1 and is the correspondence analogue of Definition 3.3.

Definition 4.1. *The convex game $G = (X_i, \geq_i)_{i \in N}$ is* ***correspondence secure with respect to*** *$I \subseteq N$ if for $J = N \backslash I$, whenever $x \in B_J$ is not a Nash equilibrium there is neighborhood U of x and closed correspondence $d : U \twoheadrightarrow X$ with nonempty and convex values such that for every $y \in U \cap B_J$ there is a player $i \in I$ for whom,*

$$(z_i, x'_{-i}) >_i y, \text{ for every } x' \in U \cap B_J \text{ and every } z_i \in d_i(x').^{9}$$

Definition 4.1 is strictly more permissive than Definition 3.3 because the former permits z_i to vary with x'_{-i}. Definition 4.1 is also ordinal and, for convex games, strictly more permissive than Barelli and Meneghel's (2013) Definition 2.1.[10] Consequently, in the present convex game setting,[11] Theorem 4.2 below is a strict generalization of Barelli and Meneghel's (2013) Theorem 2.2,[12] and a strict generalization of McLennan, Monteiro and Tourky's (2011) Theorem 3.4 with their "universal restriction operator."

Theorem 4.2. *Suppose that A.1, A.2, and A.3 hold, that each player's pure strategy set is locally convex, and that G is correspondence secure with respect to $I \subseteq N$. If for each*

[9] The i-th coordinate, d_i, of d can always be chosen so that it depends only on x'_{-i}. In particular, choose an open $V \subseteq U$ such that $x \in V = \times V_i$. For any fixed $x^0 \in V$ define $\hat{d}(x') = \times_i d_i(x^0_i, x'_{-i})$ for all $x' \in V$. Then V and $\hat{d}$ satisfy the conditions of the definition.

[10] See footnote 12.

[11] Section 5.2 considers games with non-convex preferences.

[12] It should be noted that the hypotheses of Theorem 2.2 in Barelli and Meneghel (2013) are inadequate to justify the claim on p.823 that the correspondence Φ is convex-valued. One way to correct the deficiency would be to add the assumption that each of the correspondences ϕ_x is convex-valued. But see also footnote 18.

$i \in N \backslash I$, player i's best-reply correspondence is closed and has nonempty and convex values, then G possesses a pure strategy Nash equilibrium.

Apart from the local convexity assumption, Theorem 4.2 generalizes Theorem 3.4 because correspondence security is more permissive than point security. However, the proof below will show that correspondence security is in fact a form of point security. The idea is that one can always construct a surrogate game, G^*, in which each player i chooses a correspondence from X into X_i rather than simply a point in X_i, and such that G^* has a Nash equilibrium only if G has a Nash equilibrium. Since correspondences in G are merely points in the expanded strategy space of G^*, correspondence security in G reduces to point security in G^*.

Proof of Theorem 4.2. Let G be correspondence secure with respect to $I \subseteq N$ and let $J = N \backslash I$. As shown in the proof of Theorem 3.4, the set B_J is nonempty and compact. Suppose by way of contradiction that G has no Nash equilibrium in B_J. Then, for each $x \in B_J$ there is a neighborhood U of x and a closed correspondence $d : U \twoheadrightarrow X$ with nonempty and convex values such that the condition stated in Definition 4.1 holds. Since the collection of all such U's forms an open cover of the compact set B_J, we may extract a finite subcover $U^1, ..., U^K$, together with their associated correspondences $d^1, ..., d^K$. Hence, for every $k = 1, ..., K$ and every $y \in U^k \cap B_J$ there is a player $i \in I$ for whom,

$$(z_i, x'_{-i}) >_i y, \text{ for every } x' \in U^k \cap B_J \text{ and every } z_i \in d_i^k(x'). \tag{4.1}$$

We now define a surrogate game, G^*, and will obtain the desired contradiction by showing that G^* satisfies all the hypotheses of Theorem 3.4 but has no Nash equilibrium.

For each $i \in J$, let $F_i : X \twoheadrightarrow X_i$ denote i's best reply correspondence. For each $k = 1, ..., K$, extend d^k to X by defining $d^k(x) = X$ whenever $x \notin U^k$. Each $d^k : X \twoheadrightarrow X$ is then closed with nonempty and convex values.

Introduce two new players, A and C. The surrogate game G^* has player set $\{A, C\} \cup I$. Player A chooses $a \in X$, player C chooses $c \in X$, and each player $i \in I$ chooses $\alpha_i \in \Delta = \{\lambda \in [0,1]^K : \sum_{k=1}^K \lambda_k = 1\}$, the unit simplex in $\mathbb{R}^K$.

Players A and C have preferences on $X \times X \times \Delta^I$ that are represented by the own-strategy quasiconcave utility functions u_A and u_C, respectively, where $u_A(a, c, \alpha) = 1$ if $a = c$ and 0 otherwise, and $u_B(a, c, \alpha) = 1$ if $c \in d(a, \alpha)$ and 0 otherwise, where $d(a, \alpha) := \left(\times_{i \in I} \sum_{k=1}^K \alpha_{ik} d_i^k(a)\right) \times (\times_{i \in J} F_i(a))$.

Player $i \in I$ has preferences $\geq_i^*$ on $X \times X \times \Delta^I$ defined by $(a, c, \alpha) \geq_i^* (a', c', \alpha')$ if and only if $\forall z_i \in d_i(a, \alpha)$, $\exists z'_i \in d_i(a', \alpha')$ such that $(z_i, a_{-i}) \geq_i (z'_i, a'_{-i})$.[13] Each relation $\geq_i^*$ is complete, reflexive, transitive and convex.[14]

[13] Thus, choosing $\alpha_i \in \Delta$ in the surrogate game is like choosing in the original game the reaction-correspondence $\sum_{k=1}^K \alpha_{ik} d_i^k$ and selecting from it the "worst" reaction when it is multi-valued.

[14] For convexity, suppose $\alpha_i^1, \alpha_i^2 \in \{\alpha'_i : (a, c, \alpha'_i, \alpha_{-i}) \geq_i^* (a, c, \alpha)\}$ and let $\alpha^1 = (\alpha_i^1, \alpha_{-i})$ and $\alpha^2 =$

So defined, the game G^* satisfies A.1-A.3 and players A and C (all players in fact) have locally convex strategy spaces. Note also that players A and C have closed best-reply correspondences whose values are nonempty and convex. We next show that G^* is point secure with respect to I, which will allow us to apply Theorem 3.4.

So as not to confuse the original game with the surrogate game, let $B^*_{\{A,C\}}$ denote the set of strategies (a, c, α) such that players A and C are simultaneously best replying in G^*. Hence, $B^*_{\{A,C\}} = \{(a, c, \alpha) : a = c \text{ and } c \in d(a, \alpha)\}$. Observe that if (a, c, α) is in $B^*_{\{A,C\}}$, then $a \in d(a, \alpha)$ and so $a \in B_J$.

Consider any $(a, c, \alpha) \in B^*_{\{A,C\}}$. Then, as just observed, $a \in B_J$. Hence, there is $\hat{k}$ such that $a \in U^{\hat{k}}$ and so $(a, c, \alpha) \in (U^{\hat{k}} \times U^{\hat{k}} \times \Delta^I) \cap B^*_{\{A,C\}}$. For every $(y, w, \beta) \in (U^{\hat{k}} \times U^{\hat{k}} \times \Delta^I) \cap B^*_{\{A,C\}}$, since (similar to a) $y \in U^{\hat{k}} \cap B_J$, condition (4.1) implies that there is a player $i \in I$ for whom $(z_i, a'_{-i}) >_i y$ for every $a' \in U^{\hat{k}} \cap B_J$ and every $z_i \in d_i^{\hat{k}}(a')$. But then, because $y_i \in d_i(y, \beta)$,

$$(a', c', e_i^{\hat{k}}, \alpha'_{-i}) >_i^* (y, w, \beta) \tag{4.2}$$

holds for every $(a', c', \alpha') \in (U^{\hat{k}} \times U^{\hat{k}} \times \Delta^I) \cap B^*_{\{A,C\}}$, where $e_i^{\hat{k}}$ is the $\hat{k}$-th unit vector in Δ.

Since $(a, c, \alpha) \in B^*_{\{A,C\}}$ was arbitrary, (4.2) holds in particular when $(a, c, \alpha) \in B^*_{\{A,C\}}$ is not a Nash equilibrium and so we have shown that G^* is point-secure with respect to I. But (4.2) also shows that G^* has no Nash equilibrium since any Nash equilibrium (a, c, α) must be in $B^*_{\{A,C\}}$ and so we may set $(a', c', \alpha') = (y, w, \beta) = (a, c, \alpha)$. This contradicts Theorem 3.4 and completes the proof. Q.E.D.

Remark 4. *Local convexity of the X_i for $i \in I$ need not be assumed if for each non-Nash x, the securing correspondence $d : U \twoheadrightarrow X$ can be chosen so that $d : U \twoheadrightarrow X \cap Y$ for some (possibly x-dependent) finite-dimensional affine subspace Y of the ambient vector space (as is the case in McLennan, Monteiro and Tourky (2011)).*

Remark 5. *A more direct proof along the lines of the proof of Theorem 3.4 is possible, but it does not bring out the connection to point security.*

5. Extensions and Refinements

5.1. Symmetric Games

Let $X_i = Z$ for every $i \in N$ and let $(x; y)_i$ denote the strategy vector in which player i chooses $x \in Z$ and every other player chooses $y \in Z$. The Game $G = (X_i = Z, \succeq_i)_{i \in N}$ is *quasi-symmetric* if for every pair of players i and j, $(x; y)_i \succeq_i (y; y)_i$ if and only if $(x; y)_j \succeq_j (y; y)_j$.

$(\alpha_i^2, \alpha_{-i})$. Choose any $\lambda \in [0, 1]$ and any $\bar{z}_i \in d_i(a, \lambda\alpha^1 + (1-\lambda)\alpha^2) = \lambda d_i(a, \alpha^1) + (1-\lambda) d_i(a, \alpha^2)$. Then $\bar{z}_i = \lambda z_i^1 + (1-\lambda) z_i^2$ for some $z_i^1 \in d_i(a, \alpha^1)$, $z_i^2 \in d_i(a, \alpha^2)$. By the definition of α^1, α^2, and $\succeq_i^*$, $\exists z_i', z_i'' \in d_i(a, \alpha)$ such that $(z_i^1, a_{-i}) \succeq_i (z_i', a_{-i})$ and $(z_i^2, a_{-i}) \succeq_i (z_i'', a_{-i})$. Without loss, suppose $(z_i', a_{-i}) \succeq_i (z_i'', a_{-i})$. Then convexity of $\succeq_i$ implies that $(\bar{z}_i, a_{-i}) \succeq_i (z_i'', a_{-i})$, and so $\lambda\alpha_i^1 + (1-\lambda)\alpha_i^2 \in \{\alpha_i' : (a, c, \alpha_i', \alpha_{-i}) \succeq_i^* (a, c, \alpha)\}$.

Thus, we may describe a quasi-symmetric game by a single player's strategy set Z and a single binary relation $\geq$, for player 1, say, on Z^N. A strategy $z \in Z$ is a *symmetric Nash equilibrium* if it is Nash equilibrium for all players to choose z, i.e., $z \geq (z', z, ..., z)$ for all $z' \in Z$.

Definition 5.1. *A quasi-symmetric convex game $G = (Z, \geq)$ is* ***diagonally point secure*** *if whenever $z \in Z$ is not a Nash equilibrium, there is a point $\hat{z} \in Z$ and a neighborhood V of z such that for every $w \in V$, $(\hat{z}, z', ..., z') > (w, ..., w)$ for every $z' \in U$.*

We have the following analogue of Proposition 3.2.

Proposition 5.2. *If $G = (Z, \geq)$ is quasi-symmetric and satisfies A.1-A.3, then it has a symmetric pure strategy equilibrium if it is diagonally point secure.*

For the proof, consider the following two player game in which player 1 chooses $z \in Z$ and player 2 chooses $w \in Z$. Player 1's preference relation is defined by $(z, w) \geq_1 (z', w')$ if and only if $(z, w, ..., w) \geq (z', w', ..., w')$ and player 2's preferences are represented by the quasiconcave utility function $u_2(z, w) = 1$ if $w = z$ and 0 otherwise. It is straightforward to show that under the hypotheses of the proposition, this two-player game is point-secure with respect to player 1 and so we can apply Theorem 3.4 to conclude the existence of a pure strategy Nash equilibrium, $(\hat{z}, \hat{w})$. Since player 2's preferences imply that $\hat{w} = \hat{z}$, we conclude that $\hat{z}$ is a symmetric Nash equilibrium of G.

One can similarly derive symmetric game analogues of the other results above.

5.2. Non-Convex Games

Up to now, we have assumed that the game G is convex, i.e., that both A.2 and A.3 hold. It is straightforward to extend all of our definitions and results to non-convex game settings.

The following definitions extend Definitions 3.3 and 4.1 to non-convex games.[15] For any set A, let coA denote its convex hull.

Definition 5.3. *The game $G = (X_i, \geq_i)_{i \in N}$ is* ***point secure with respect to*** *$I \subseteq N$ if for $J = N \backslash I$, whenever $x \in B_J$ is not a Nash equilibrium there is a neighborhood U of x and a point $\hat{x} \in X$ such that for every $y \in U \cap B_J$ there is a player $i \in I$ for whom,*

$$y_i \notin co\{w_i : (w_i, y_{-i}) \geq_i (\hat{x}_i, x'_{-i})\}, \text{ for every } x' \text{ in } U \cap B_J.$$

Say that a correspondence $F : Y \twoheadrightarrow Z$ is *co-closed* if the correspondence whose value is $coF(y)$ for each $y \in Y$ is closed.[16] Requiring F to be co-closed does not require it to be

[15] These new definitions are equivalent to the previous definitions when G is convex.

[16] For example, a closed correspondence $F : Y \twoheadrightarrow Z$ is co-closed if Z is contained in a finite dimensional subspace of an ambient topological vector space.

either convex-valued or closed.[17]

Definition 5.4. *The game $G = (X_i, \geq_i)_{i \in N}$ is* ***correspondence secure with respect to*** *$I \subseteq N$ if for $J = N \backslash I$, whenever $x \in B_J$ is not a Nash equilibrium there is neighborhood U of x and a co-closed correspondence $d : U \twoheadrightarrow X$ with nonempty values such that for every $y \in U \cap B_J$ there is a player $i \in I$ for whom,*

$$y_i \notin co\{w_i : (w_i, y_{-i}) \geq_i (z_i, x'_{-i})\}$$

holds for every $x' \in U \cap B_J$ and every $z_i \in d_i(x')$.

Remark 6. *Definition 5.4 extends definition 4.1 in several ways. First it permits preferences to be non-convex. Second, it permits the securing correspondences to be non convex-valued. Third, it permits the securing correspondences to be non-closed.*

Remark 7. *When the players' preferences are convex, Definition 5.4 is equivalent to Definition 4.1 because, by convexity, the condition displayed in Definition 5.4 reduces to $(z_i, x'_{-i}) >_i y$ which, if satisfied for all $z_i \in d_i(x')$, is also satisfied for all $z_i \in cod_i(x')$ in which case Definition 4.1 is satisfied for the convex-valued and closed correspondences cod_i.*

Theorem 5.5. *Suppose that A.1 and A.2 hold and that G is point secure with respect to $I \subseteq N$. If for each $i \in N \backslash I$, X_i is locally convex and player i's best-reply correspondence is closed and has nonempty and convex values, then G possesses a pure strategy Nash equilibrium.*

The proof of Theorem 5.5 follows the steps of the proof of Theorem 3.4, except that (a) $(\hat{x}_i^x, x'_{-i}) >_i y$ in (A.1) is replaced with $y_i \notin co\{w_i : (w_i, y_{-i}) \geq_i (\hat{x}_i^x, x'_{-i})\}$, (b) $(\hat{x}_i^k, x'_{-i}) >_i y$ in (A.2) is replaced with $y_i \notin co\{w_i : (w_i, y_{-i}) \geq_i (\hat{x}_i^k, x'_{-i})\}$, and (c) $x^* \geq_i (\hat{x}_i^k, y_{-i}^{i,j_i})$ in the final sentence is replaced with $x_i^* \in co\{w_i : (w_i, x_{-i}^*) \geq_i (\hat{x}_i^k, y_{-i}^{i,j_i})\}$.

Theorem 5.6. *Suppose that A.1 and A.2 hold, that each player's pure strategy set is locally convex, and that G is correspondence secure with respect to $I \subseteq N$. If for each $i \in N \backslash I$, player i's best-reply correspondence is closed and has nonempty and convex values, then G possesses a pure strategy Nash equilibrium.*

Remark 8. *Because we now permit non-convex preferences, Theorem 5.6 strictly generalizes Barelli and Meneghel's (2013) Theorem 2.2,*[18] *and, because closed correspondences mapping*

[17] Consider, for example, the correspondence mapping each point in $[0, 1]$ into the set of all rational numbers with the usual topology.

[18] See footnote 12. But note that instead of adding the assumption that the ϕ_x correspondences are convex-valued, it would suffice in Barelli and Meneghel's (2011) Theorem 2.2 to replace the assumption that the ϕ_x correspondences are closed with the assumption that they are co-closed.

into subsets of a fixed finite subset of a convex space are co-closed, it also strictly generalizes McLennan, Monteiro and Tourky's (2011) Theorem 3.4 with their "universal restriction operator."

The proof of Theorem 5.6 follows the steps of the proof of Theorem 4.2 except that (a) the correspondences d^k satisfying (4.1) are co-closed, even when extended to all of X, (b) player B's payoff is defined by $u_B(a, c, \alpha) = 1$ if $c \in cod(a, \alpha)$ and 0 otherwise, (c) $\geq_i^*$ is not necessarily convex and so G^* satisfies only A.1 and A.2, and (d) the last three paragraphs of the proof are replaced with the following four paragraphs:

So as not to confuse the original game with the surrogate game, let $B^*_{\{A,C\}}$ denote the set of strategies (a, c, α) such that players A and C are simultaneously best replying in G^*. Hence, $B^*_{\{A,C\}} = \{(a, c, \alpha) : a = c \text{ and } c \in cod(a, \alpha)\}$. Observe that if (a, c, α) is in $B^*_{\{A,C\}}$, then $a \in cod(a, \alpha)$ and so $a \in B_J$.

Consider any $(a, c, \alpha) \in B^*_{\{A,C\}}$. Then, as just observed, a is in B_J. Hence, there is $\hat{k}$ such that $a \in U^{\hat{k}}$ and so $(a, c, \alpha) \in (U^{\hat{k}} \times U^{\hat{k}} \times \Delta^I) \cap B^*_{\{A,C\}}$. For every $(a^1, c^1, \alpha^1) \in (U^{\hat{k}} \times U^{\hat{k}} \times \Delta^I) \cap B^*_{\{A,C\}}$, since (similar to a) $a^1 \in U^{\hat{k}} \cap B_J$, condition (4.1) implies that there is a player $i \in I$ for whom

$$a_i^1 \notin co\{w_i : (w_i, a_{-i}^1) \geq_i (z_i, a'_{-i})\} \tag{5.1}$$

holds for every $a' \in U^{\hat{k}} \cap B_J$ and every $z_i \in d_i^{\hat{k}}(a')$. Because $(a^1, c^1, \alpha^1) \in B^*_{\{A,C\}}$ we have $a_i^1 \in cod_i(a^1, \alpha^1) = \sum_{k=1}^K \alpha_{ik}^1 cod_i^k(a^1) = co \sum_{k=1}^K \alpha_{ik}^1 d_i^k(a^1)$, and so a_i^1 is a convex combination of a_i^{1j}'s such that each $a_i^{1j} = \sum_{k=1}^K \alpha_{ik}^1 \delta_i^{kj}$ and each $\delta_i^{kj} \in d_i^k(a^1)$.

We claim that

$$\alpha_i^1 \notin co\{\gamma_i : (a^1, c^1, \gamma_i, \alpha_{-i}^1) \geq_i^* (a', c', e_i^{\hat{k}}, \alpha'_{-i})\} \tag{5.2}$$

holds for every $(a', c', \alpha') \in (U^{\hat{k}} \times U^{\hat{k}} \times \Delta^I) \cap B^*_{\{A,C\}}$, where $e_i^{\hat{k}}$ is the $\hat{k}$-th unit vector in Δ. Otherwise, for some such (a', c', α'), α_i^1 would be a convex combination of α_i^{1n}'s such that for each n, $(a^1, c^1, \alpha_i^{1n}, \alpha_{-i}^1) \geq_i^* (a', c', e_i^{\hat{k}}, \alpha'_{-i})$. Defining $z_i^{nj} := \sum_{k=1}^K \alpha_{ik}^{1n} \delta_i^{kj}$, we have $z_i^{nj} \in d_i(a^1, \alpha_i^{1n}, \alpha_{-i}^1)$. Hence, because $(a^1, c^1, \alpha_i^{1n}, \alpha_{-i}^1) \geq_i^* (a', c', e_i^{\hat{k}}, \alpha'_{-i})$, there exists for each n and j, $\tilde{z}_i^{nj} \in d_i^{\hat{k}}(a')$ such that $(z_i^{nj}, a_{-i}^1) \geq_i (\tilde{z}_i^{nj}, a'_{-i})$. Let $\tilde{z}_i$ denote a $\tilde{z}_i^{nj}$ that makes $(\tilde{z}_i^{nj}, a'_{-i})$ the least desirable for i as n and j vary. Then, $\tilde{z}_i \in d_i^{\hat{k}}(a')$ and for every n and j we have $(z_i^{nj}, a_{-i}^1) \geq_i (\tilde{z}_i, a'_{-i})$. But, because a_i^1 is evidently a convex combination of the z_i^{nj} and because $(a', c', \alpha') \in (U^{\hat{k}} \times U^{\hat{k}} \times \Delta^I) \cap B^*_{\{A,C\}}$ implies $a' \in U^{\hat{k}} \cap B_J$, this contradicts (5.1) and so establishes (5.2).

Since $(a, c, \alpha) \in B^*_{\{A,C\}}$ was arbitrary (5.2) holds in particular when $(a, c, \alpha) \in B^*_{\{A,C\}}$ is not a Nash equilibrium and so we have shown that G^* is point-secure with respect to I. But (5.2) also shows that G^* has no Nash equilibrium since any Nash equilibrium (a, c, α) must be in $B^*_{\{A,C\}}$ and so we may set $(a', c', \alpha') = (y, r, \beta) = (a, c, \alpha)$. This contradicts Theorem 3.4 and completes the proof. Q.E.D.

5.3. Weak Point-Security

The proof of Theorem 3.4 actually proves a stronger result. Indeed, consider the following weakening of the definition of point security with respect to I. For simplicity, we return to the case of convex games.[19]

Definition 5.7. *The convex game* $G = (X_i, \succeq_i)_{i \in N}$ *is* ***weakly point secure with respect to*** $I \subseteq N$ *if for* $J = N \backslash I$*, whenever* $x \in B_J$ *is not a Nash equilibrium there is a neighborhood* U *of* x *and a point* $\hat{x} \in X$ *such that for every* $y \in U \cap B_J$ *and every neighborhood* V *of* y*, there is* $y' \in V \cap B_J$ *and a player* $i \in I$ *for whom,*

$$(\hat{x}_i, x'_{-i}) \succ_i (y_i, y'_{-i}), \text{ for every } x' \text{in } U \cap B_J.$$

Remark 9. *Definition 5.7 is more permissive than Definition 3.3 since if the game is point secure w.r.t.* I *then it is weakly point secure w.r.t.* I *since we may always choose* $y' = y$ *in Definition 5.7.*

The following result is therefore a strengthening of Theorem 3.4.[20]

Theorem 5.8. *Suppose that A.1, A.2, and A.3 hold and that* G *is weakly point secure with respect to* $I \subseteq N$*. If for each* $i \in N \backslash I$*,* X_i *is locally convex and player* i*'s best-reply correspondence is closed and has nonempty and convex values, then* G *possesses a pure strategy Nash equilibrium.*

The proof of Theorem 5.8 follows the steps of Theorem 3.4 except that (A.1) and (A.2) are replaced by their weak point-security counterparts. At the end of the proof, let V be the neighborhood of x^* that is the intersection of all the W^j s.t. $\lambda_j(x^*) > 0$. Then the proof shows (use (A.3) and the convexity of $\succeq_i$) that for every player i and every $x' \in V \cap B_J$, there exists $y^i \in U^k \cap B_J$ s.t. $(x_i^*, x'_{-i}) \succeq_i (\hat{x}_i^k, y^i_{-i})$, contradicting the adjusted (A.2) because $x^* \in U^k \cap B_J$.

Remark 10. *Under weak point-security, the set of Nash equilibria of* G *need not be closed. This is in contrast to all of the previous security conditions.*

A similar weakening of correspondence security to "weak" correspondence security leads to a strengthening of Theorem 4.2. We leave the straightforward details to the reader.

[19] The straightforward extension to non-convex games follows the same pattern as in the previous section.
[20] It also generalizes Theorem 2.2 in Reny (2009).

5.4. Mixed Strategies

Out of the need to calculate expected payoffs, we shall assume throughout this subsection that for each player i, his preference relation is represented by the bounded and measurable utility function u_i. Because the X_i's are compact subsets of a Hausdorff topological vector space, if M_i denotes the set of (regular, countably additive) probability measures on the Borel subsets of X_i, then M_i is compact in the weak* topology.[21] Extend each u_i to $M = \times_{i=1}^N M_i$ by defining $u_i(m) = \int_X u_i(x)dm$ for all $m \in M$.

Obviously, one obtains theorems on the existence of mixed strategy equilibria by applying the results in any of the previous sections to the game's mixed extension. But an additional result can be obtained by considering the following definition (Reny (2009), (2011)).

Definition 5.9. *The game G has the* ***finite deviation property*** *if whenever $m \in M$ is not a Nash equilibrium, there exist $m^1, ..., m^K \in M$ and a neighborhood U of m, such that for all $m' \in U$, there is a player i and a k such that $u_i(m_i^k, m'_{-i}) > u_i(m')$.*

The difference between the finite deviation property and any of the security properties from previous sections is the absence of the "security" requirement. That is, we do not require here that $u_i(m_i^k, m''_{-i}) > u_i(m')$ hold for every $m'' \in U$. It need hold only for $m'' = m'$.

The following result was first reported in Reny (2009); see also Reny (2011).

Theorem 5.10. *If G has the finite deviation property, then G possesses a mixed strategy Nash equilibrium.*

The proof of Theorem 5.10 is straightforward.[22] Nonetheless, it does generalize the mixed strategy existence result in Reny (1999) and it does not follow from any of the theorems above applied to the game's mixed extension.

5.5. Non-Closed Sets of Nash Equilibria

In contrast to Prokopovych (2013), the hypotheses in all of our results above (with the exception of Theorem 5.8) imply that the set of Nash equilibria of G is closed. Hence, all games whose set of Nash equilibria is not closed are ruled out. But there is simple way to generalize all of our results to include some of these games. Simply modify each definition

[21] This follows from the Riesz representation theorem and Alaoglu's theorem. See, for example, Dunford and Schwartz (1988).

[22] Suppose, by way of contradiction that no Nash equilibrium exists. Then for every $m \in M$, each player has finitely many mixed strategies such that for every m' in a neighborhood of m, one of these mixed strategies is a profitable deviation from m' for some player. The resulting open cover of M has a finite subcover, by compactness, and so in fact each player has finitely many mixed strategies – call them *deviation strategies* – such that for every m in M some deviation strategy is a profitable deviation from m for some player. However, by Nash's theorem, the finite game whose pure strategy set is the product of the players' finite sets of deviation strategies has a Nash equilibrium, producing an element of M that no player can profitably deviate from using any of his deviation strategies. This contradiction completes the proof.

above by replacing the phrase "whenever $x \in X$ (or $x \in B_J$, or $z \in Z$, or $m \in M$, or $m \in \bar{B}_J$) is not a Nash equilibrium" by the phrase "whenever $x \in X$ (or $x \in B_J$ or $z \in Z$, or $m \in M$, or $m \in \bar{B}_J$) has a neighborhood containing no Nash equilibrium." Then, the hypotheses no longer imply a closed set of Nash equilibria and the (now stronger) theorems remain correct as stated because their proofs each begin by supposing by way of contradiction that G has no Nash equilibrium, in which case the original phrases and their replacements are equivalent.

Remark 11. *Because of the observation in the last sentence, it is unclear whether the improvement in the theorems obtained here is of significant practical value.*

6. A Conjecture and a Counterexample

One might hope to improve upon the pure strategy results in Sections 3-5 above in various ways. One hope might be to eliminate, analogous to the finite deviation property in Section 5.4, the "security" part of the various definitions. But this does not seem feasible as we now show.

For example, consider weakening the definition of point security to the following.

Definition 6.1. *$G = (X_i, \geq_i)_{i \in N}$ has the **pure-strategy single-deviation property** if whenever $x \in X$ is not a Nash equilibrium, there exists $\hat{x} \in X$ and a neighborhood U of x, such that for all $x' \in U$, there is a player i for whom $(\hat{x}_i, x'_{-i}) >_i x'$.*

One might hope that the following strengthening of Proposition 3.2 is true.

?Conjecture? If $G = (X_i, \geq_i)_{i \in N}$ satisfies A.1, A.2 and A.3, and has the pure-strategy single-deviation property, then G possesses a pure strategy Nash equilibrium.

This conjecture is false. The example below, first reported in Reny (2009), has the single deviation property but possesses no pure strategy Nash equilibrium.[23]

6.1. Counterexample.

There are three players and each player's pure strategy set is $[0, 1]$. The players' payoffs are defined by the following payoff matrices, where player 1's choice of a determines the row, 2's choice of b determines the column and 3's choice of c determines the matrix.

[23] Prokopovych (2013) shows that the above conjecture is true for two player games on the unit square.

1: row $a\backslash$2: col b	$b \in [0, 1/3]$	$b \in (1/3, 2/3)$	$b \in [2/3, 1]$
$a \in [0, 1/2]$	$(1-a, b, 1-c)$	$(a, b, 1-c)$	(a, b, c)
$a \in (1/2, 1]$	$(1-a, b, 1-c)$	(a, b, c)	(a, b, c)

3: matrix $c \in [0, 1/2]$

1: row $a\backslash$2: col b	$b \in [0, 1/3]$	$b \in (1/3, 2/3)$	$b \in [2/3, 1]$
$a \in [0, 1/2]$	$(1-a, 1-b, 1-c)$	$(1-a, 1-b, 1-c)$	$(a, 1-b, c)$
$a \in (1/2, 1]$	$(1-a, 1-b, 1-c)$	$(1-a, 1-b, c)$	$(a, 1-b, c)$

3: matrix $c \in (1/2, 1]$

The players' strategy sets are compact and convex and their payoff functions are quasi-concave (in fact linear) in their own strategies.

It is easily verified that if (a, b, c) is such that $b < 2/3$, then either player 1 can profitably deviate by choosing $\hat{a} = 0$ or player 2 can profitably deviate by choosing $\hat{b} = 1$ or player 3 can profitably deviate by choosing $\hat{c} = 0$. Thus, $(\hat{a}, \hat{b}, \hat{c}) = (0, 1, 0)$ serves as a single deviation for any point in the open set U in which player 2's choice is less than 2/3.

Similarly, it is easy to verify that if (a, b, c) is such that $b > 1/3$, then either player 1 can profitably deviate by choosing $\hat{a} = 1$ or player 2 can profitably deviate by choosing $\hat{b} = 0$ or player 3 can profitably deviate by choosing $\hat{c} = 1$. Thus, $(\hat{a}, \hat{b}, \hat{c}) = (1, 0, 1)$ serves as a single deviation for any point in the open set V in which player 2's choice is greater than 1/3.

Because the union of U and V is the entire strategy space, this shows both that the game has the single deviation property and that a Nash equilibrium fails to exist.

A. Appendix

The proof of Theorem 3.4 below follows the basic line of argument developed in McLennan, Monteiro, and Tourky (2011). The main distinction is that we must introduce a well-chosen "dominance" relation to play the role that, in McLennan et. al.'s proof, is played by the players' utility functions, which are of course unavailable here. A secondary distinction is the presence of a subset of players whose best reply correspondences are well-defined.

Proof of Theorem 3.4. Fix any $x^0 \in X$ and let $J = N \backslash I$. Letting F_i denote the best reply correspondence of any player $i \in J$, the correspondence $\times_{i \in J} F_i(x_I^0, \cdot) : \times_{i \in J} X_i \twoheadrightarrow \times_{i \in J} X_i$ is nonempty-valued, convex-valued, and closed, and if x_J^* is any one of its fixed points, then (x_I^0, x_J^*) is a member of B_J, the set of points in X at which players in J are simultaneously best replying. By Glicksberg's (1952) theorem, B_J is nonempty. Moreover, B_J is compact because the best-reply correspondences of the players in J are closed.

Suppose, by way of contradiction, that there is no equilibrium in B_J. Then, by point security with respect to I, for every $x \in B_J$ there is a neighborhood U^x of x and a point $\hat{x}^x \in X$ such that for every $y \in U^x \cap B_J$ there is a player $i \in I$ for whom

$$(\hat{x}_i^x, x'_{-i}) >_i y, \text{ for every } x' \in U^x \cap B_J. \tag{A.1}$$

Thus we have a collection of pairs $\{(U^x, \hat{x}^x)\}_{x \in B_J}$ where the U^x form an open cover of B_J. We may therefore extract a finite sub-collection $\{(U^k, \hat{x}^k)\}$ such that the U^k form a finite open cover of B_J and such that: for each k, and for every $y \in U^k \cap B_J$ there is a player $i \in I$ for whom

$$(\hat{x}_i^k, x'_{-i}) >_i y, \text{ for every } x' \in U^k \cap B_J. \tag{A.2}$$

By construction, $U^k \cap B_J$ is nonempty for every k. Say that k *dominates* k' for i if for every $x \in U^k \cap B_J$ there exists $x' \in U^{k'} \cap B_J$ such that

$$(\hat{x}_i^k, x_{-i}) \geq_i (\hat{x}_i^{k'}, x'_{-i}).$$

This dominance relation, for each player i, inherits from $\geq_i$, completeness, reflexivity, and transitivity.

Because B_J is a compact subset of a Hausdorff space, for each k we may choose a closed set $C^k \subset U^k$ such that resulting finite collection $\{C^k\}$ covers B_J. For each $x \in B_J$, let V^x be the neighborhood of x that is the intersection of the complements of all the C^k not containing x,[24] and for each player i let k_i^x be any member of the (nonempty) set $\{k : x \in U^k\}$ that dominates every other member of the set; such a k_i^x exists because the dominance relation is complete and transitive. Hence, $W^x = V^x \cap (\cap_{i \in N} U^{k_i^x})$ is a neighborhood of x and every C^k intersecting W^x contains x. Consequently, for every player i, for every $x' \in W^x \cap B_J \subseteq U^{k_i^x} \cap B_J$ and for every C^k intersecting W^x, there exists $y^i \in U^k \cap B_J$ such that $(\hat{x}_i^{k_i^x}, x'_{-i}) \geq_i (\hat{x}_i^k, y^i_{-i})$, because $x \in C^k \subset U^k$ implies that k_i^x dominates k for i.[25]

Let $\tilde{x}^x = (\hat{x}_i^{k_i^x})_{i \in N}$. Thus we have a collection of pairs $\{(W^x, \tilde{x}^x)\}_{x \in B_J}$ where the W^x form an open cover of B_J. We may therefore extract a finite subcollection $\{(W^1, \tilde{x}^1), ..., (W^n, \tilde{x}^n)\}$ such that the W^j form an open cover of B_J and such that: for every player i, for every j, for every $x' \in W^j \cap B_J$, and for every C^k intersecting W^j, there exists $y^{i,j} \in U^k \cap B_J$ such that,[26]

$$(\tilde{x}_i^j, x'_{-i}) \geq_i (\hat{x}_i^k, y^{i,j}_{-i}). \tag{A.3}$$

Let W be the union of the W^j. By Munkres (1975, Theorem 5.1) for each j, there is a continuous function $\lambda_j : W \to [0, 1]$ such that $\sum_{j=1}^n \lambda_j(x) = 1$ for all $x \in W$ and such that $\lambda_j(x) > 0$ implies $x \in W^j$.[27] Consequently, for each player i, the function on W defined by $g_i(x) = \sum_j \lambda_j(x)\tilde{x}_i^j$, is continuous. The correspondence mapping the compact, convex and locally convex space $(\times_{i \in I} co\{\tilde{x}_i^1, ..., \tilde{x}_i^n\}) \times (\times_{i \in J} X_i)$ into subsets of $co\{\tilde{x}_i^1, ..., \tilde{x}_i^n\}$ defined by $G_i(x) = \{g_i(x)\}$ if $x \in W$ and $G_i(x) = co\{\tilde{x}_i^1, ..., \tilde{x}_i^n\}$ if $x \in X \backslash W$ is therefore nonempty-valued, convex-valued and closed. Hence, by Glicksberg's (1952) theorem, there exists $x^* \in X$ such that $(x_i^*)_{i \in J} \times (x_i^*)_{i \in I} \in (\times_{i \in J} F_i(x^*)) \times (\times_{i \in I} G_i(x^*))$. Consequently, (i) $x_i^* \in F_i(x^*)$ for every $i \in J$ and (ii) $x_i^* \in G_i(x^*)$ for every $i \in I$. But (i) implies that $x^* \in B_J \subset W$ so that $G_i(x^*) = \{g_i(x^*)\}$ which together with (ii) implies that, (iii) $x_i^* = \sum_j \lambda_j(x^*)\tilde{x}_i^j$ for every $i \in I$.

Because the C^k cover B_J, we may choose k such that $x^* \in C^k \subset U^k$. Then, for any

[24] $V^x = X$ if every C^k contains x.

[25] The reflexivity of the dominance relation is needed here because the statement must hold when, in particular, $k = k_i^x$.

[26] The dependence of $y^{i,j}$ on x' and k is suppressed.

[27] That is, the finite collection of functions $\{\lambda_j\}$ is a partition of unity subordinated to the cover $\{W^j\}$.

j with $\lambda_j(x^*) > 0$, we have $x^* \in W^j \cap B_J$, and C^k intersects W^j (both sets contain x^*). Consequently, by (A.3), there exists for each player i, $y^{i,j} \in U^k \cap B_J$ such that,

$$(\tilde{x}_i^j, x_{-i}^*) \geq_i (\hat{x}_i^k, y_{-i}^{i,j}) \geq_i (\hat{x}_i^k, y_{-i}^{i,j_i}),$$

where, for each player i, j_i is the value of j giving the least preferred outcome for i among all the $(\hat{x}_i^k, y_{-i}^{i,j})$ such that $\lambda_j(x^*) > 0$. Together with (iii) this implies, for each player $i \in I$, that $x^* \geq_i (\hat{x}_i^k, y_{-i}^{i,j_i})$, contradicting (A.2) because x^* and each y^{i,j_i} are in $U^k \cap B_J$. Q.E.D.

References

Bagh, A., and A. Jofre (2006): "Reciprocal Upper Semicontinuity and Better Reply Secure Games: A Comment," *Econometrica*, 74, 1715–1721.

Barelli, P., and I. Soza (2009): "On the Existence of Nash Equilibrium in Discontinuous and Qualitative Games," mimeo, University of Rochester.

Barelli, P. and I. Meneghel (2013): " A Note on the Equilibrium Existence Problem in Discontinuous Games," *Econometrica,* 81, 813–824.

Barelli P., Govindan S. and Wilson R. (2013): "Competition for a Majority," mimeo, University of Rochester and Stanford University.

Baye, M. R., G. Tian, and J. Zhou (1993): "Characterizations of the Existence of Equilibria in Games with Discontinuous and Non-quasiconcave Payoffs," *Review of Economic Studies*, 60, 935-948.

Bich, P. (2009): "Existence of Pure Nash Equilibria in Discontinuous and Non Quasiconcave Games," *International Journal of Game Theory*, 38, 395–410.

Bich, P. and R. Laraki (2013): "On the Existence of Approximated Equilibria and Sharing-Rule Equilibria in Discontinuous Games," mimeo Paris School of Economics and Ecole Polytechnique.

Carbonell-Nicolau, O. (2011): "On the Existence of Pure-Strategy Perfect Equilibrium in Discontinuous Games," *Games and Economic Behavior*, 71, 23–48.

Carmona, G. (2005): "On the Existence of Equilibria in Discontinuous Games: Three Counterexamples," *International Journal of Game Theory*, 33, 181–187.

Carmona, G. (2009): "An Existence Result for Discontinuous Games," *Journal of Economic Theory*, 144, 1333–1340.

Carmona G. (2011): "Understanding Some Recent Existence Results for Discontinuous Games," *Economic Theory*, 48(1), 31-45.

Dasgupta, P. and E. Maskin (1986): "The Existence of Equilibrium in Discontinuous Economic Games, I: Theory," *Review of Economic Studies* 53, 1-26.

De Castro, L. (2011): "Equilibrium Existence and Approximation of Regular Discontinuous Games," *Economic Theory*, 48, 67-85.

Dunford, N., and J. T. Schwartz (1988): *Linear Operators Part I: General Theory.* New York: John Wiley and Sons.

Glicksberg, I.L. (1952): "A Further Generalization of the Kakutani Fixed Point Theorem," *Proceedings of the American Mathematical Society,* 170-174.

He., W. and N. C. Yannelis (2013): "On Discontinuous Games with Asymmetric Information," mimeo, National University of Singapore and The University of Iowa.

Jackson M., Simon L.K., Swinkels J.M. and Zame W.R. (2002): "Communication and Equilibrium in Discontinuous Games of Incomplete Information," *Econometrica*, 70(5), 1711-1740.

McLennan, A., P. K. Monteiro, and R. Tourky (2011): "Games with Discontinuous Payoffs: A Strengthening of Reny's Existence Theorem," *Econometrica* 79, 1643–1664.

Monteiro, P. and F. Page (2007): "Uniform Payoff Security and Nash Equilibrium in Compact Games," *Journal of Economic Theory*, 134, 566–575.

Monteiro, P. and F. Page (2008): "Catalog Competition and Nash Equilibrium in Nonlinear Pricing Games," *Economic Theory*, 34, 503–524.

Munkres (1975): *Topology, A First Course.* New Jersey: Prentice-Hall.

Nash J. (1950): "Equilibrium Points in N-Person Games," *Proceedings of the National Academy of Sciences*, 36, 48-49.

Nash J. (1951): "Non-cooperative Games," *Annals of Mathematics*, 54, 286-295.

Nessah, R. (2013): "Weakly Continuous Security in Discontinuous and Nonquasiconcave Games: Existence and Characterization," mimeo, IESEG School of Management.

Prokopovych, P. (2011) "On Equilibrium Existence in Payoff Secure Games," *Economic Theory*, 48, 5-16.

Prokopovych, P. (2013): "The single deviation property in games with discontinuous payoffs," *Economic Theory*, 53, 383–402.

Reny, P. J. (1999): "On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games," *Econometrica*, 67, 1029-1056.

Reny, P. J. (2009): "Further Results on the Existence of Nash Equilibria in Discontinuous Games," http://home.uchicago.edu/~preny/papers/further-results-05-30-09.pdf.

Reny P.J. (2011): "Strategic Approximations of Discontinuous Games," *Economic Theory*, 48, 17-29.

Simon, L. (1987): "Games with Discontinuous Payoffs," *Review of Economic Studies*, 54, 569-597.

Simon, L. and W. Zame (1990): "Discontinuous Games and Endogenous Sharing Rules," *Econometrica* 58, 861-872.