Inferring Rationales from Choice:
Identification for *Rational Shortlist Methods*[*]

Rohan Dutta[†] and Sean Horan[‡]

March 2013

Abstract

A wide variety of choice behavior inconsistent with preference maximization can be explained by Manzini and Mariotti's *Rational Shortlist Methods*. Choices are made by sequentially applying a pair of asymmetric binary relations (*rationales*) to eliminate inferior alternatives. Manzini and Mariotti's axiomatic treatment elegantly describes which behavior can be explained by this model. However, it leaves unanswered what can be inferred, from observed behavior, about the underlying rationales. Establishing this connection is fundamental not only for applied and empirical work but also for meaningful welfare analysis. Our results tightly characterize the surprisingly rich relationship between behavior and the underlying rationales. (*JEL* D01)

Keywords: Revealed Preference; Identification; Uniqueness; Two-stage Choice Procedures; Behavioral Industrial Organization; Comparative Advertising; Decoy Marketing; Welfare.

I. Introduction

Manzini and Mariotti [2007] study a choice procedure, called *Rational Shortlist Methods* (RSM), where the decision-maker maximizes in stages using a pair (P_1, P_2) of *rationales* (i.e. asymmetric but not necessarily transitive preferences). Faced with a menu of alternatives, the decision-maker first eliminates any option which is dominated by another alternative according to the rationale P_1 before selecting, from the remaining alternatives, the option which maximizes the rationale P_2.

While they provide an axiomatization of the model, Manzini and Mariotti do not address the important issue of identification: *to what extent and how does choice pin down the underlying rationales?* In this paper, we provide six results related to identification in the RSM model. Together, our results show that identification in this model is at once relatively straightforward and surprisingly rich.

In the spirit of Samuelson's [1938] revealed preference, Proposition 1 defines "revealed rationales" for choice behavior consistent with the RSM model. In contrast to the standard model of preference maximization, the revealed rationales are only partial estimates of the underlying rationales. Indeed, Proposition 2 establishes that the revealed rationales can be used to construct a uniquely "most cautious" and a uniquely "least cautious" estimate for each rationale. This result has significant implications (discussed in Section IV below) for welfare analysis based on the RSM model.

In turn, Propositions 3 and 4 use the revealed rationales to provide, for any choice function consistent with the RSM model, an exact characterization of *all* rationale pairs that lead to the same choice behavior. As a general rule, multiple pairs of rationales induce the same behavior. Effectively, these results pin down the set of valid "completions" of the partial estimates derived in Proposition 1.

Proposition 5 establishes a systematic way to modify a given pair of rationales without affecting the associated choice behavior. Like the "uniqueness up to affine transformations" result for models with cardinal utility representations, this result provides a simple way—which does not involve considering be-

havior directly—to determine whether two different representations (i.e. pairs of rationales) induce the same behavior. Interestingly, this useful "uniqueness result" has no analog in the literature (discussed in Section V below) related to identification in models of procedural decision-making.

Finally, Proposition 6 provides necessary and sufficient conditions for limited choice data to be consistent with the RSM model. Much like the *Strong Axiom of Revealed Preference* (Houthakker [1950]; Samuelson [1950]) does for the standard model of preference maximization, this result provides a straightforward way to test the RSM model against the kind of limited choice data that is frequently encountered in the real world.

Identification results like the ones established in this paper are critical for understanding models of decision-making. Intuitively, they determine the extent to which meaning can be inferred from choice data and, conversely, the extent to which the representation can be given meaning in terms of behavior (Dekel and Lipman [2011]). While this connection is clearly central to the goal of decision theory, it is equally important for empirical and applied work (Spiegler [2008]).

Some Motivating Examples

Some examples will help to illustrate the importance of our results for empirical and applied work on the RSM model. At this point, our goal is simply to highlight the practical significance of our six results. In Section IV, we revisit these examples and discuss, at greater length, the implications of our identification results for welfare analysis and policy making.

The first three examples examine situations where the objective of the "analyst" (whether an economist, a policy maker, or a market participant) is to develop a better understanding of decision-makers who choose according to rational shortlist methods.

Example I (Multi-Criterial Choice) In many choice situations, a variety of product dimensions are important to consumers. Evidence suggests that, instead of aggregating these criteria, consumers tend to evaluate the dimensions lexico-graphically (Tversky et al. [1988]; Dulleck et al. [2011]). After eliminating options which are inferior on the most important product dimension, the consumer selects

3

the product which is most preferred on the second most important dimension. Given purchasing data consistent with this model of choice, what can the analyst infer about the criteria used by consumers?

The multi-criterial choice procedure described in this example is a rational shortlist method where the two rationales reflect the consumer's ranking of alternatives along the most important product dimensions. In some cases, it is fairly straightforward for the analyst to determine which dimensions are most important to consumers. In the experiment of Tversky et al., for instance, the choice scenarios were framed in a way that naturally led subjects to view one of the two specified product dimensions as being more important. In Dulleck et al.'s study of online purchasing behavior, the dimensions likely to be important were limited by the particular structure of the choice environment. Specifically, the shopping website considered in the study only provided information about price, seller reputation, and shipping costs of the products offered for sale.

Even when the analyst cannot directly observe the dimensions that are most important to consumers, Propositions 1 and 2 establish that it is possible to draw inferences about these dimensions from choice behavior. Proposition 1, which defines revealed rationales for choice behavior consistent with the RSM model, establishes lower bounds on the content of each rationale. Proposition 2, in turn, uses the revealed rationales to establish upper bounds for each rationale. Effectively, the lower bounds determine which preference pairs *must* belong to each rationale while the upper bounds determine which preference pairs *may* belong to each rationale (and, consequently, which preference pairs *cannot* belong to each rationale). Although these bounds do not necessarily pin down the two rationales exactly, they do provide a simple way for the analyst to limit the criteria that the consumer might consider important.

For the situation described in Example I, the analyst is interested in both rationales. In other circumstances, the analyst may be more interested in one of the rationales. For example:

Example II (Forced Choice) Decision-makers are frequently "forced" to pick a single alternative. Surveys of consumer preferences, for example, generally require respondents to choose a single product from each of the proposed choice sets.

When preference alone is not sufficient to discriminate among all of the feasible alternatives, the respondent must employ a *tie-breaking rule* to choose among the preference-maximizing alternatives. Provided that respondents use a tie-breaking rule to answer (some of) the survey questions, what can the analyst infer about their "true" (first stage) preferences?

Forced choice is by no means limited to the scenario considered in the example. Indeed, it arises anytime the decision-maker must make a choice but is either *indecisive* or *indifferent* among some of the options (see Eliaz and Ok [2006] for the choice-theoretical distinction between indecisiveness and indifference). For a wide range of *deterministic* (as opposed to *random*) tie-breaking rules, the choice procedure described in Example II is a rational shortlist method.[1]

A different example of forced choice consistent with the RSM model involves a policy maker whose only criterion is efficiency. Since the Pareto relation is generally incomplete and it is only possible to implement a single policy, the policy maker must use a different criterion, such as fairness, to break ties among the Pareto optimal policies (see Houy and Tadenuma [2009]).

Alternatively, the analyst may be more interested in the second rationale. For example:

Example III (Limited Consideration) In a variety of choice situations, consumers focus on a subset of the alternatives before selecting the most preferred alternative among those considered (see e.g. Wright and Barbour [1977]). If the consumer's first stage *consideration set* is the product of a process where alternatives "compete" for the consumer's attention, it is reasonable that:

- an option that is considered on some menu continues to be considered when some of the other options are removed; *and*,

- an option considered on two different menus is also considered when the two menus are "merged" (into a menu consisting of the options available on one of the original menus).[2]

---

[1]Interestingly, Danan [2010] provides an economic justification for decision-makers to favor a deterministic rules.

[2]If the options considered on A are denoted by $\Gamma(A)$, (i) is equivalent to $\Gamma(A) \cap B \subseteq \Gamma(B)$ for $B \subset A$, and (ii) to $\Gamma(A) \cap \Gamma(B) \subseteq \Gamma(A \cup B)$. In the choice theory literature, these properties are known as α and γ (Sen [1971]).

What can the analyst infer about consumers' "true" (second stage) preferences?

It is straightforward to show that the choice procedure described above is a rational shortlist method.[3] While consideration sets have a long history in the marketing literature (see Roberts and Lattin [1997] for a survey), they are only now being studied in economics (Masatlioglu, Nakajima, and Ozbay [2012]; their joint paper with Lleras [2011]; and, Eliaz and Spiegler [2011a]). Formally, the model in Example III is most closely related to the model of Lleras et al. [2011] (see Theorem 4 of their paper), the key difference being that it does not require either rationale to be transitive. With respect to the first rationale, at least, transitivity seems ancillary to the interpretation of the model given in Example III: if product A "outcompetes" another product B for the consumer's attention and B outcompetes a third product C, it is not clear why A *must* outcompete C.

The common feature of Examples II and III is that one of the rationales acts as a *nuisance parameter* which interferes with the analyst's ability to determine the decision-maker's "true" preference (i.e. the other rationale).[4] Proposition 1 establishes simple revealed preference tests for both rationales that are based on the decision-maker's choices from nested sets. These tests provide an easy way for the analyst to identify which preference pairs must be part of the decision-maker's "true" preference (in circumstances where "true" preference coincides with one of the two rationales).

While the examples above highlight the importance of Propositions 1 and 2 for empirical questions related to the RSM model, the next three examples illustrate the importance of Propositions 3 to 6 for applications of the model in the growing field of *behavioral industrial organization* (see Spiegler [2011] for a recent survey).[5] Each examines a situation where the objective of the analyst is to determine the impact of a proposed policy change on the behavior of consumers who choose according to rational shortlist methods.

---

[3] Any choice correspondence satisfying α and γ can be rationalized by an asymmetric preference (Sen [1971]).

[4] In Example II, the nuisance parameter is the tie-breaking rule. In Example III, it is the consideration set.

[5] Some additional applications of the RSM model are discussed in Section IV.b below.

Examples IV and V are concerned with a common advertising practice, known as *comparative advertising*, whose sole purpose is to communicate to consumers that one product is inferior to another. Although comparative advertising has been studied extensively in marketing (see Grewal et al. [1997] for a survey), very few papers in economics (with the exception of Anderson-Renault [2009] and Emons-Fluet [2012]) have examined this practice.

Example IV (Advertising) A firm asks their advertising agency to conduct a market survey in order to determine why the sales of their new laundry detergent are lagging. The agency finds that many consumers are ignorant of the fact that the new product compares favorably with some of its competitors. Based on their findings, the agency recommends an aggressive comparative advertising strategy that targets these competitors. Should the firm follow their recommendation?

When consumers choose by rational shortlist methods, a firm may become more competitive not only by making its product more appealing but also by affecting the composition of the rationales used by consumers. If the first rationale contains the product comparisons which are more salient to consumers and the second rationale contains those which are less salient, a firm may be able to "promote" comparisons to the first rationale with more aggressive advertising or, conversely, "demote" comparisons by obfuscating the relationship with competing products. In order to devise a profitable advertising strategy, the firm must have a good idea of the rationales used by consumers. Propositions 3 and 4 provide a straightforward way to determine, directly from consumer choice data, every pair of rationales that is consistent with behavior.

Example V (Advertising, continued) Another producer of laundry detergent has an accurate picture of the rationales used by consumers (perhaps because it understands how advertising has made certain product comparisons more salient to consumers). Assuming the strategies of its competitors are fixed in the short run, can the firm cut advertising costs without affecting consumer choices?

In the RSM model, different pairs of rationales may lead to the same behavior. Effectively, the firm's problem is to choose the rationale pair which minimizes advertising expenditures among those that leave choice behavior unchanged. In order to carry out this cost minimization exercise, the firm must

first identify all rationale pairs which give rise to the same choice behavior as the original pair. Given an RSM-representation of behavior, Proposition 5 provides a simple way for the analyst to determine all pairs of rationales which lead to the same choice behavior.

Example VI is concerned with a different marketing practice, known as *decoy marketing*, that is related to the *attraction effect* (Huber, Payne, and Puto [1982]). The idea is to introduce a "decoy" product whose sole purpose is to attract the consumer to purchase a different product.

Example VI (Marketing) After losing market share to their competitor *Pep*, a soft-drink company decided to replace their *Kola* with a sweeter version called *New Kola*. Since the new product unanimously beat *Pep* and *Kola* in a blind taste test, the producer was confident that the strategy would increase sales. Unexpectedly, *Pep* customers instead switched to a different brand, *Regal*, after the product swap. Desperate to boost sales, the producer is now considering the possibility of re-introducing *Kola* under the brand *Kola Klassic*. Will this strategy be effective?

Unlike the model of preference maximization, the RSM model accommodates the attraction effect: the addition of a relatively unappealing product may induce the consumer to change her choice. In terms of the example, the model admits the possibility that consumers select a *Kola* product when both versions are available although neither is selected in the absence of the other.

While the behavior of the *Pep* customers who switched to *Regal* is inconsistent with preference maximization[6] and there is evidence to suggest that soft-drink purchasing involves lexicographic multi-criterial decision-making (Leven and Levine [1996]), these factors alone are not enough to recommend the proposed decoy marketing strategy. At a minimum, the producer must also determine that the behavior of former *Pep* customers is consistent with a rational shortlist method which leads them to choose one of the *Kola* products when both are available. Otherwise, there is little to suggest that these consumers might be inclined to switch to a *Kola* product when *Kola Klassic* is re-introduced. In circumstances like these where the analyst has limited access

---

[6]They violate WARP by choosing only *Pep* in the presence of *Regal* and only *Regal* in the presence of *Pep*.

to consumer choice data, Proposition 6 provides a simple way to determine whether the available data is nonetheless consistent with the RSM model.

The remainder of this paper is structured as follows. After presenting the RSM model more formally and illustrating the task of identification with an example in Section II, we detail our six results in Section III. Section IV revisits the examples above and discusses the implications of our results for policy making. Section V concludes with a discussion of related models and extensions.

II. The RSM Model and an Example

A *rationale* is an asymmetric binary relation over a finite choice domain X. A *Rational Shortlist Method* (RSM) is a choice function $c_{(P_1, P_2)} : 2^X \setminus \emptyset \to X$ induced by a pair (P_1, P_2) of rationales. For any menu $A \subseteq X$, the choice induced by (P_1, P_2) is given by

$$c_{(P_1, P_2)}(A) \equiv \max(\max(A; P_1); P_2)$$

where $\max(B; P) = \{a \in B : \text{no } b \in B \text{ s.t. } bPa\}$ denotes the set of P-maximal alternatives in B. Conversely, a choice function c is *RSM-representable* if there exists a pair of rationales (P_1, P_2) such that $c(A) = c_{(P_1, P_2)}(A)$ for any menu of alternatives $A \subseteq X$.

Manzini and Mariotti [2007] provide an axiomatic characterization of the RSM model (restated in the Appendix).[7] To establish sufficiency, Manzini and Mariotti show that any choice function c satisfying their axioms can be represented by a rationale pair (P_1^*, P^c) where:

- P_1^* is the binary relation defined by aP_1^*b if $c(A) \neq b$ for all $A \supseteq \{a, b\}$; and,

- P^c is the pairwise *revealed preference* defined by aP^cb if $c(x, y) = a$.

---

[7]Rubinstein and Salant [2008] call the model *Post-Dominance Rationality* and give a different axiomatization.

In passing, they observe that there may be additional RSM-representations of behavior (see Remark 1 of their paper). A simple example will serve to illustrate this feature of the RSM model and help provide some intuition for the identification results in Section III below.

Example 1 *Consider a choice function c on $X = \{w, x, y, z\}$ with P^c (pairwise choice) given by:*
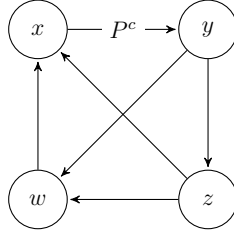
In addition, suppose that $c(w, x, y) = w$ and $c(x, y, z) = x$.[8]

While P^c contains cycles, the choices are consistent with the RSM model (see Remark 4 of the Appendix). As a preliminary point, note that the two rationales (i.e. $P_1 \cup P_2$) must contain P^c. Otherwise, the choices induced by (P_1, P_2) will differ from c on some two-element set(s). To say something more about identification, suppose that (P_1, P_2) represents c.

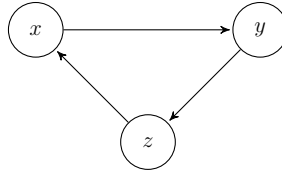Consider the task of assigning the revealed preferences in Table 1 to the rationales P_1 and P_2:

Table 1: Revealed preferences P^c on $\{x, y, z\}$

---

[8]For $\{w, y, z\}$ and $\{w, x, z\}$, P^c is transitive. To be consistent with the RSM model, the P^c-maximal alternative must be chosen. So, it must be that $c(w, y, z) = \max(\{w, y, z\}; P^c) = y$ and $c(w, x, z) = \max(\{w, x, z\}; P^c) = z$.

Since $c(x, y, z) = x$, it follows that yP_1z. Otherwise, z cannot be eliminated in the first stage. This, in turn, precludes $c(x, y, z) = x$. Moreover, $c(x, y, z) = x$ implies that zP_1x cannot occur. Otherwise, x is eliminated in the first stage which again precludes $c(x, y, z) = x$. Since P_1 and P_2 must together contain P^c however, it follows that zP_2x.

Similar reasoning establishes that xP_1y and yP_2w for the 3-cycle $\{w, x, y\}$. This exhausts the inferences that can be drawn from the data provided. By observing the choice from X however, it is possible to draw inferences about how to assign the revealed preference pairs wP^cx and zP^cw. In order for behavior to be consistent with the RSM model, the only possible choices are (i) $c(X) = w$ and (ii) $c(X) = x$ (see Remark 4 of the Appendix).

For (i), it must be that zP_2w. The reasoning is similar to that given for 3-cycles. Because w is chosen in the presence of an alternative z which is revealed preferred, zP_1w cannot occur. Otherwise, w is eliminated from X in the first stage which, in turn, precludes the choice $c(X) = w$. Since P_1 and P_2 must together contain P^c however, it follows that zP_2w. In this case, the choice data cannot help resolve how to assign the revealed preference wP^cx.

For (ii), no such indeterminacy arises. By the same reasoning as the previous case, the fact that x is chosen in the presence of w (which is revealed preferred) implies that wP_2x. To see that that zP_1w, note that $c(X) = x$ requires w to be eliminated from X in the first stage. Otherwise, w eliminates x in the second stage. Since $c(w, x, y) = w$ precludes yP_1w, it must be that zP_1w.

Table 2 summarizes the features of the rationales (P_1, P_2) identified by the analysis above:

Choices \ Rationales	P_1		P_2	
$c(x, y, z) = x$	y	z	z	x
$c(w, x, y) = w$	x	y	y	w
(i) $c(X) = w$	-	-	z	w
(ii) $c(X) = x$	z	w	w	x

Table 2: Inferences drawn from Example 1

An interesting feature of the analysis is that any preference pair which must belong to the second rationale cannot also belong to the first. In fact, the only reason to assign a pair to P_2 is that it induces choices different from c when assigned to P_1. In contrast, the second rationale may repeat any comparison carried out by the first rationale. For $c(X) = x$, this observation, in combination with Table 1, suggests a class of representations $\mathcal{R} = \{(P_1, P_2^i)\}_i$ defined by:

$$\mathcal{R} \quad : \quad P_1 \equiv \{(y,z),(x,y),(z,w)\} \quad \text{and} \quad P_2^i \supseteq P_2 \equiv \{(z,x),(y,w),(w,x)\}$$

It is straightforward to show that any pair (P_1, P_2^i) is an RSM-representation of c (provided that P_2^i is asymmetric). An appealing feature of (P_1, P_2) is that it is the unique minimal representation of c in the sense that $P_2^i \supset P_2$ for any other pair (P_1, P_2^i) that represents c.

For $c(X) = w$, there are two distinct classes of representations

$$\widehat{\mathcal{R}} \quad : \quad \widehat{P}_1 \equiv \{(y,z),(x,y),(w,x)\} \quad \text{and} \quad \widehat{P}_2^i \supseteq \widehat{P}_2 \equiv \{(z,x),(y,w),(z,w)\}$$
$$\widetilde{\mathcal{R}} \quad : \quad \widetilde{P}_1 \equiv \{(y,z),(x,y)\} \quad \text{and} \quad \widetilde{P}_2^i \supseteq \widetilde{P}_2 \equiv \{(z,x),(y,w),(z,w),(w,x)\}$$

which differ only in terms of how they attribute $wP^c x$. Because of the indeterminacy related to $wP^c x$, there are multiple minimal representations of behavior. In particular, both $(\widehat{P}_1, \widehat{P}_2)$ and $(\widetilde{P}_1, \widetilde{P}_2)$ represent c. Any other pair (P_1, P_2) that represents c must contain one (or even both) of these representations. However, neither $(\widehat{P}_1, \widehat{P}_2)$ nor $(\widetilde{P}_1, \widetilde{P}_2)$ contains the other.

III. IDENTIFICATION

In this section, we present six results related to identification in the RSM model. After defining revealed rationales for behavior consistent with the RSM model in part (a), we show in part (b) that these revealed rationales can be used to describe the class of RSM-representations. Part (c) tackles the issue of uniqueness by providing a systematic way to go between RSM-

representations without considering the associated choice behavior. Finally, part (d) addresses the issue of identification with limited choice data.

a. Revealed Rationales

Example 1 suggests that the RSM model is amenable to a simple revealed preference exercise. To fix ideas, suppose that some choice function c is RSM-representable. First, consider the task of identifying the features of the second rationale. Generalizing the analysis from the example, a is revealed preferred to b by the second rationale if a is revealed preferred to b (i.e. $aP^c b$) and there exists a menu $A \supset \{a, b\}$ such that $c(A) = b$. The idea is that b can only be chosen in the presence of a more preferred alternative a if the comparison between a and b occurs after there is an opportunity to eliminate a in the first stage.

Next, consider the task of identifying the features of the first rationale. Generalizing the intuition from the example, a necessary condition for a to be revealed preferred to b by the first rationale is that a is revealed preferred to b (i.e. $aP^c b$) and there is some menu $A \supset \{a, b\}$ such that $bP^c c(A)$. The reasoning is similar to that given in the previous paragraph. For $c(A)$ to be chosen in the presence of a more preferred alternative b, the more preferred alternative must actually be eliminated in the first stage. However, this condition alone is not sufficient. In particular, a must also be the "only way" to eliminate b in the first stage on A. In other words, the comparison between b and any other alternative a' in the *upper contour set* of b

$$UC_{P^c}(b; A) \equiv \{a' \in A : a'P^c b\}$$

must be carried out in the second stage. Formalizing these observations gives the following:

Definition 1 *Let the **revealed 2-rationale** P_2^c be defined by $aP_2^c b$ if:*
  (i) $aP^c b$; and,
  (ii) there exists some $A \supset \{a, b\}$ such that $c(A) = b$.
*Given P_2^c, let the **revealed 1-rationale** P_1^c be defined by $aP_1^c b$ if:*

(i) aP^cb; and,

(ii) there exists some $A \supset \{a,b\}$ such that $bP^cc(A)$ and $a'P_2^cb$ for all $a' \in UC_{P^c}(b; A \setminus \{a\})$.

Although both revealed rationales are sub-relations of the revealed preference (i.e. $P_i^c \subseteq P^c$ for $i = 1, 2$), it is also possible to define them without reference to pairwise choices. In particular:

Proposition 1 *Suppose c is RSM-representable. Then:*

(i) aP_1^cb if and only if $c(A) = b$ and $c(A \cup \{a\}) \notin \{a,b\}$ for some $A \subseteq X$.

(ii) aP_2^cb if and only if $c(A) = b$ and $c(B) = a$ for some $A, B \subseteq X$ such that $\{a,b\} \subseteq B \subset A$.

This result shows that both revealed rationales may be defined in terms of choice on a pair of nested menus. The revealed 1-rationale is identified with the choice of a "third alternative" when a is added to a menu where b is chosen. In turn, the revealed 2-rationale is identified with a "choice reversal" when options are added to a menu where b is chosen in the presence of a.

Proposition 1 also establishes the connection with some other preference relations defined in prior work on the RSM model. In particular, P_1^c coincides with the *dominance* relation defined by Rubinstein and Salant [2008] while P_2^c coincides with the *progressive knowledge* relation defined by Houy [2008] (see also Cherepanov, Feddersen, and Sandroni [2013] who study the same relation for a model generalizing the RSM model). Interestingly, P_2^c also coincides with the revealed preference pairs in P^c that are *not* part of the rationale P_1^* defined by Manzini and Mariotti [2007].

This last observation motivates the following definition. Given a choice function c, define the *c-complement* of a binary relation P to be the binary relation $\overline{P} \equiv P^c \setminus P$ that contains the revealed preference pairs not in P. In other words, P_1^* is the *c*-complement of P_2^c. For concordance, denote the *c*-complement of P_1^c by P_2^*. The next result establishes that the revealed rationales and their *c*-complements act as bounds on the RSM-representations of behavior. To state the result, let $P^{-1} \equiv \{(b,a) \in X^2 : (a,b) \in P\}$ denote the inverse of the binary relation P.

Proposition 2 *Suppose (P_1, P_2) is an RSM-representation of c. Then:*

(i) P_1 *is bounded below by the revealed rationale* P_1^c *and* P_2 *is bounded below by the revealed rationale* P_2^c*; and*

(ii) P_1 *is bounded above by the c-complement* P_1^* *and* $P_2 \setminus (P_1 \cup P_1^{-1})$ *is bounded above by the c-complement* P_2^**.*

Part (i) of the result shows that the revealed rationales capture features which are common to every RSM-representation of behavior. To see this, suppose there are n different rationale pairs $\mathcal{R}(c) \equiv \{(P_1^1, P_2^1), ..., (P_1^n, P_2^n)\}$ that can be used to represent c. Then, Proposition 2(i) implies that $P_1^c \subseteq \cap_{j=1}^n P_1^j$ and $P_2^c \subseteq \cap_{j=1}^n P_2^j$. In words, the revealed rationales serve as "lower bounds" for the rationales which can be used to represent behavior.

Conversely, the "upper bounds" P_1^* and P_2^* capture features which may be part of some RSM-representation. For the first rationale, Proposition 2(ii) implies $\cup_{j=1}^n P_1^j \subseteq P_1^*$. Since $P_1^* \equiv P^c \setminus P_2^c$ by definition, the upper bound also describes revealed preference pairs which are not necessarily part of the second rationale. The upper bound of the second rationale has a similar interpretation. The only difference is that second rationale may contain more than the preference pairs in P_2^*. In particular, it may repeat any comparison carried out in the first stage. Since these repeat comparisons are never actually carried out (one of the alternatives involved must be eliminated in the first stage), they may be resolved differently by the second rationale.

b. Representations of Behavior

The first result concerns minimal representations of behavior. An RSM-representation (P_1, P_2) is said to be minimal if the two rationales: (i) avoid duplication in the sense that aP_1b and aP_2b for no alternatives $a, b \in X$; and, (ii) avoid conflict in the sense that aP_1b and bP_2a for no $a, b \in X$. Formally:

Definition 2 *An RSM-representation (P_1, P_2) of c is **minimal** if*

$$(P_1 \cup P_1^{-1}) \cap (P_2 \cup P_2^{-1}) = \emptyset.$$

To state the result, let $\mathcal{P}_i(c) \equiv \{P : P_i^c \subseteq P \subseteq P_i^*\}$ denote the collection of ith rationales nested between the revealed i-rationale and the c-complement of the other revealed rationale. Given a rationale P_i which is part of an RSM-representation (P_1, P_2), let P_{-i} denote the other rationale.

Proposition 3 *Suppose c is RSM-representable. Then, for $i = 1, 2$:*

(i) $P_i^c \subseteq P_i^*$ so that the interval of rationales $\mathcal{P}_i(c)$ is non-empty; and

(ii) (P_1, P_2) is a minimal representation of c if and only if $P_i \in \mathcal{P}_i(c)$ and $P_{-i} = \overline{P_i}$.

Consequently, any RSM-representable c has a minimal representation—which is unique if and only if $P_1^c \cup P_2^c = P^c$.

The result establishes that there are minimal representations of behavior, namely (P_1^c, P_2^*) and (P_1^*, P_2^c), whose rationales coincide with the bounds identified in Proposition 2. This observation has two implications.[9] For one, it shows that the revealed rationales do indeed capture all of the features shared by the representations and, thus, embody all of the inferences that might be drawn from the data. Formally, $P_1^c = \cap_{j=1}^n P_1^j$ and $P_2^c = \cap_{j=1}^n P_2^j$ (where, recall, $\mathcal{R}(c) \equiv \{(P_1^j, P_2^j)\}_{j=1}^n$ denotes the collection of rationale pairs that represent c).[10] Practically, it also establishes that the revealed rationales can be used directly to construct representations of behavior.

More broadly, Proposition 3 establishes that there is a range of minimal representations between the extremes (P_1^c, P_2^*) and (P_1^*, P_2^c). To clarify, consider the case where P^c is a linear order. Then, c has a variety of minimal representations ranging from $(\emptyset, P^c)$ to $(P^c, \emptyset)$. In particular, any bi-partition (P_1, P_2) of P^c defines a minimal representation. The situation is analogous when the revealed preference is not a linear order. Between (P_1^c, P_2^*) and (P_1^*, P_2^c), any assignment of the "indeterminate" revealed preference pairs

---

[9]Incidentally, this observation also has implications for the special case of the RSM model with acyclic rationales. For that model, it is equally clear that (P_1^*, P_2^c) represents behavior. As such, the revealed 2-rationale P_2^c must be acyclic. Houy [2008] leverages this observation about P_2^c to give an axiomatization of the acyclic RSM model.

[10]Similarly, $P_1^* = \cup_{j=1}^n P_1^j$. In other words, P_1^* captures all of the features that are part of some representation.

not assigned to either revealed rationale defines a representation.[11] In other words, $(P_1^c \cup P_1^{in}, P_2^c \cup P_2^{in})$ represents c for any bi-partition $\{P_1^{in}, P_2^{in}\}$ of $P^{in} \equiv \overline{P_1^c \cup P_2^c}$. Intuitively, P^{in} captures the indeterminacy in how to assign the pairs in P^c. Naturally, c has a unique minimal representation if and only if there is no indeterminacy ($P^{in} = \emptyset$) or, equivalently, the revealed rationales exhaust the revealed preference ($P_1^c \cup P_2^c = P^c$).

Example 1 illustrates behavior which has a unique minimal representation. Even when this is the case, there exist additional representations where the second rationale duplicates or conflicts with the first rationale. In fact, any RSM-representation may be described as a minimal representation with some amount of duplication and/or conflict "added" to the second rationale:

Proposition 4 *The pair (P_1, P_2) is an RSM-representation of c if and only if*

(i) $(P_1, \overline{P_1})$ is a minimal representation of c, and
(ii) $P_2 = \overline{P_1} \sqcup P_1^{as}$ for some rationale $P_1^{as} \subset P_1 \cup P_1^{-1}$.

This generalizes Proposition 3 by establishing that any i^{th} rationale (that can be used to represent behavior) defines a range of representations. At one extreme is a representation involving minimal duplication and conflict. At the other, there is a variety of representations involving maximal duplication and/or conflict. The representation (P_1^*, P^c) proposed by Manzini and Mariotti [2007], for instance, involves maximal duplication. Put in terms of Proposition 4, their representation adds $P_1^* = P^c \setminus P_2^c$ to the second rationale of the minimal representation (P_1^*, P_2^c). Likewise, the representation (P_1^c, P^c) proposed by Rubinstein and Salant [2008] also involves maximal duplication since (P_1^c, P_2^*) is a minimal representation and, moreover, $P^c = P_2^* \sqcup P_1^c$.

c. Uniqueness

Given a pair of rationales (P_1, P_2), the induced choice behavior $c_{(P_1, P_2)}$ may be altered by transformations that move preference pairs from one rationale

---

[11]The reader may suspect that there is even more structure here than indicated. Indeed, it is straightforward to show that the collection of minimal representations $\mathcal{P}^c$ forms a lattice with meet and join operations defined by $(P_1, P_2) \wedge (\widetilde{P}_1, \widetilde{P}_2) \equiv (P_1 \cap \widetilde{P}_1, P_2 \cup \widetilde{P}_2)$ and $(P_1, P_2) \vee (\widetilde{P}_1, \widetilde{P}_2) \equiv (P_1 \cup \widetilde{P}_1, P_2 \cap \widetilde{P}_2)$ for any $(P_1, P_2), (\widetilde{P}_1, \widetilde{P}_2) \in \mathcal{P}^c$.

to the other, add preference pairs to the rationales, or even remove preference pairs. The next result identifies transformations of the representation that leave choice behavior unchanged. The appeal of this result is that these choice-invariant transformations can be characterized entirely in terms of the rationales. Effectively, this is a uniqueness result for representations in the RSM model.

Let $\mathcal{P}$ denote the collection of rationale pairs (P_1, P_2) such that $P_1 \cup P_2$ is a total asymmetric binary relation[12] on X and $P_1 \cap P_2 = \emptyset$. By Proposition 3, any RSM-representable choice function c is minimally represented by some pair $(P_1, P_2) \in \mathcal{P}$.[13] Indeed, $\mathcal{P}$ contains *every* minimal representation of *every* choice function consistent with the RSM model.

Now, suppose $(P_1, P_2) \in \mathcal{P}$ is a minimal RSM-representation of c such that aP_2b. When can (a, b) be moved to the first rationale without affecting behavior? The following condition captures those circumstances where it is "redundant" to include (a, b) in the second rationale:

Definition 3 (Redundancy Condition) *Given $(P_1, P_2) \in \mathcal{P}$, $(a, b) \in P_2$ is (P_1, P_2)-**redundant** if, for all $\{a_i\}_{i=1}^{n} \subseteq X$ such that $a_n = a$ and $(a_i, a_{i+1}) \in P_1$ for $i = 1, ..., n-1$:*

$$(a_j, b) \notin P_1 \text{ for } j = 2, ..., n \text{ implies } (a_1, b) \in P_1 \cup P_2.$$

More plainly, the pair $(a, b) \in P_2$ is (P_1, P_2)-redundant if, for any P_1-chain $(a_1, ..., a)$, the fact that b is not eliminated in the first stage by any member of the sub-chain $(a_2, ..., a)$ implies that b is not preferred to a_1 by either rationale. A simple example will help to illustrate this condition:

Example 2 *Consider the following minimal representation (P_1, P_2) of a choice function c:*

---

[12]A binary relation P on X is total if aPb or bPa for all $a, b \in X$.

[13]It is straightforward to see that $P_1 \cup P_2$ is a total asymmetric binary relation such that $P_1 \cap P_2 = \emptyset$ if and only if (P_1, P_2) is a minimal rationale pair. This establishes the formal connection with Proposition 3.

P_1		P_2	
y	z	v	z
y	v	w	z
x	v	w	x
v	w	w	y
		y	x
		z	x

For this representation of c, the pair (z, x) is redundant. To see this, observe that the only P_1-chain ending in z is (y, z). Since $(y, x) \in P_2$, the redundancy condition is satisfied. Another redundant pair is (y, x). Since there is no P_1-chain that terminates in y, (y, x) vacuously satisfies the redundancy condition. In fact, these are the only redundant pairs in P_2. With (w, z), for instance, the P_1-chain (x, v, w) satisfies the premise of the redundancy condition. Since it is not true that $(x, z) \in P_1 \cup P_2$ however, the redundancy condition is violated.[14]

For this example, it is easy to check that a pair in P_2 can be moved to P_1 without affecting choice behavior if and only if the pair satisfies the redundancy condition. In particular, choice behavior is unaffected by moving the pairs (z, x) and (y, x) to P_1. Conversely, choice is affected by moving any other pair in P_2.

An appealing feature of the redundancy condition is that it also determines which preference pairs can be moved in the other direction. To move a pair (a, b) from P_1 to P_2 without affecting choice, it must be that, once moved to P_2, the pair (a, b) would leave choice behavior unchanged when it is moved back to P_1. In other words, (a, b) must be $(P_1 \setminus (a, b), P_2 \cup (a, b))$-redundant.[15]

In fact, the only other requirement is that $(P_1 \setminus (a, b), P_2 \cup (a, b))$ represents a choice function. In general, there are rationale pairs in $\mathcal{P}$ which induce "empty choice" for some menus. Intuitively, this occurs when there are cycles in P_1 or cycles in P_2 among the alternatives which survive P_1. To rule out

---

[14]The pair (v, z) is not redundant since $(x, v) \in P_1$ but $(x, z) \notin P_1 \cup P_2$. The same kind reasoning establishes that neither (w, y) nor (w, x) is redundant. For both, simply consider the sequence (v, w).

[15]To avoid cluttered notation, we use $P_1 \setminus (a, b)$ and $P_2 \cup (a, b)$ to denote $P_1 \setminus \{(a, b)\}$ and $P_2 \cup \{(a, b)\}$, respectively.

empty choice, it is enough that P_1 is acyclic and, moreover, that P_1 "breaks" any cycle in P_2. Formally, P_2 is said to be P_1-*acyclic* if, for every cycle $\{a_i\}_{i=1}^n$ in P_2, there exist alternatives $a_j, a_k \in \{a_i\}_{i=1}^n$ such that $a_j P_1 a_k$. Let $\mathcal{P}_{RSM} \equiv \{(P_1, P_2) \in \mathcal{P} : P_1 \text{ is acyclic and } P_2 \text{ is } P_1\text{-acyclic}\}$ denote the sub-collection of $\mathcal{P}$ consisting of pairs (P_1, P_2) where P_1 is acyclic and P_1 breaks any P_2-cycle. Lemma 7 of the Appendix shows that these are precisely the pairs in $\mathcal{P}$ which induce choice functions.

Applying the observations above to the representation in Example 2, it is easy to see that (y, z) is $(P_1 \setminus (y, z), P_2 \cup (y, z))$-redundant and $(P_1 \setminus (y, z), P_2 \cup (y, z)) \in \mathcal{P}_{RSM}$. The only other pair in P_1 for which this is true is (y, v). Moreover, it is easy to check that the induced choice behavior is unaffected by moving (y, z) and/or (y, v) to P_1 but not by moving either (x, v) or (v, w).

Proposition 5 *Suppose (P_1, P_2) is a minimal representation of c. Then:*

(i) For $(a, b) \in P_2$, the rationale pair $(P_1 \cup (a, b), P_2 \setminus (a, b))$ is an RSM-representation of c if and only if (a, b) is (P_1, P_2)-redundant.

(ii) For $(a, b) \in P_1$, the rationale pair $(P_1 \setminus (a, b), P_2 \cup (a, b))$ is an RSM-representation of c if and only if (a, b) is $(P_1 \setminus (a, b), P_2 \cup (a, b))$-redundant and $(P_1 \setminus (a, b), P_2 \cup (a, b)) \in \mathcal{P}_{RSM}$.

For minimal representations, this result establishes when a preference pair can be moved without affecting the induced choice behavior. The appeal is that this condition is stated directly in terms of the representation without explicitly referring to behavior. Effectively, the result is the analog of Proposition 3 expressed in the "language" of representations.

To illustrate, consider the example above. The preference pairs in P_2 that cannot be moved without affecting choice behavior are precisely the pairs in the revealed 2-rationale of the induced choice function $c_{(P_1,P_2)}$. Likewise, the pairs in P_1 that cannot be moved are the pairs in the revealed 1-rationale. In other words, $P_1^c = \{(x, v), (v, w)\}$ and $P_2^c = \{(v, z), (w, z), (w, y), (w, x)\}$. The connection between choice-invariance and the revealed rationales highlighted by this example is by no means particular. In fact, it is the combined implication of Propositions 3 and 5.

The preceding analysis addresses only transformations of minimal representations that leave choice behavior unchanged. In general, one might be interested in the choice-invariant transformations of RSM-representations that involve duplication and conflict. It turns out that Proposition 5 can be used to determine the invariant transformations associated with *any* RSM-representation. The basic idea is to exploit the minimal representation described in Proposition 4.

To illustrate, consider the task of moving $(a, b) \in P_2 \setminus P_1$ to the first rationale. Given an RSM-representation (P_1, P_2), the minimal representation $(P_1, \overline{P_1})$ induces the same choice behavior (by Proposition 4). The idea of $\overline{P_1} = P_2 \setminus (P_1 \cup P_1^{-1})$ is to strip away from P_2 any pair that duplicates or conflicts with P_1.[16] Intuitively, the choice-invariance of moving the preference pair (a, b) to P_1 cannot depend on which conflicts and duplications arise in the second rationale. In other words, it is choice-invariant to move (a, b) to P_1 in (P_1, P_2) if and only if $(a, b) \in \overline{P_1}$ and it is choice-invariant to move (a, b) to P_1 in $(P_1, \overline{P_1})$. For the same reason, this condition is necessary and sufficient for the choice-invariance of adding $(a, b) \notin P_1$ to the first rationale.

In fact, similar reasoning shows that moving (respectively, deleting) *from* the first rationale in (P_1, P_2) is linked to moving *to* the second rationale in $(P_1, \overline{P_1})$. Provided $(b, a) \notin P_2$ (respectively, $(a, b) \in P_2$), each of these transformations is choice-invariant for (P_1, P_2) if and only if it is choice-invariant to move (a, b) to $\overline{P_1}$ in $(P_1, \overline{P_1})$. The only other transformations are additions and deletions involving the second rationale. For these, it is easy to see that adding (a, b) to P_2 is choice-invariant if and only if it preserves the asymmetry of P_2 (i.e. $(b, a) \notin P_2$) and deleting (a, b) from P_2 is choice-invariant if and only if $(a, b) \in P_1$ or $(b, a) \in P_1$.

d. Partial Data

Instead of having access to choice data for all possible subsets of alternatives (i.e. $2^X \setminus \emptyset$), the analyst may only observe choices on a sub-domain $\mathcal{D} \subsetneq 2^X \setminus \emptyset$. Given limited data, can the analyst determine whether the observed choice

---

[16]Since $P_2 = \overline{P_1} \sqcup P_1^{as}$, $P_1^{as} \subset P_1 \cup P_1^{-1}$, and $\overline{P_1} \cap (P_1 \cup P_1^{-1}) = \emptyset$, $\overline{P_1} = P_2 \setminus (P_1 \cup P_1^{-1})$ follows by set difference.

behavior is consistent with the RSM model? To answer this question, we first list the set of restrictions that the data imposes on the set of rationale pairs that *might* be used to represent behavior. Consistency with the RSM model can then be determined by verifying whether there exists a pair of rationales which satisfies these restrictions and, moreover, has the structure necessary to represent *some* choice function.

Proposition 1 describes the inferences about the rationales that can be drawn when the choice function *is* RSM-representable. With partial choice data, the same inferences now serve as restrictions on the possible rationale pairs that can be used to represent observed behavior. If, for instance, the analyst observes that adding an alternative a to a menu D_i leads to neither a nor $c(D_i)$ being chosen from the expanded menu, then $aP_1c(D_i)$ for any pair (P_1, P_2) that *might* be used to represent the choice data. Similarly, if the analyst observes that b is chosen in the presence of a on some menu D_i but a is chosen from a larger menu $D_j \supset D_i$, then aP_2b for any pair (P_1, P_2) that *might* be used to represent the choice data. This motivates the following definition, which is analogous to the notion of revealed rationales in Proposition 1:

Definition 4 *Let P_1^D be defined by aP_1^Db if*

$$c(D_i) = b \text{ and } c(D_j) \notin \{a, b\} \text{ for some } D_i, D_j \in \mathcal{D} \text{ such that } D_j = D_i \cup \{a\}.$$

Let P_2^D be defined by aP_2^Db if

$$c(D_i) = b \text{ and } c(D_j) = a \text{ for some } D_i, D_j \in \mathcal{D} \text{ such that } \{a, b\} \subseteq D_j \subset D_i.$$

Nested menus where a and b are chosen allow the analyst to make a strong inference about the relationship between a and b. When the two menus are non-nested, the analyst can only draw a somewhat weaker inference about relationship between the two alternatives. In particular, this kind of choice data rules out the possibility that either alternative is preferred to the other according to the first rationale. This observation motivates the next definition:

22

Definition 5 *Let Q be defined by aQb if $(a, b), (b, a) \notin P_2^D$ and*

$$c(D_i) = b \text{ and } c(D_j) = a \text{ for some } D_i, D_j \in \mathcal{D} \text{ such that } \{a, b\} \subseteq D_j, D_i.$$

*Define a **Q-selection** to be a rationale Q_2 such that (i) $Q_2 \subset Q$ and (ii) for all $(a, b) \in Q$, either $(a, b) \in Q_2$ or $(b, a) \in Q_2$ but not both. Denote the collection of all Q-selections by $\mathcal{Q}$.*

A third possibility is that the data only involves menus where a is sometimes chosen in the presence of b while b is never chosen in the presence of a. Intuitively, this kind of data makes it impossible for b to be preferred to a according to the first rationale:

Definition 6 *Let H be defined by aHb if $(a, b) \notin P_2^D \cup Q$ and*

$$c(D) = a \text{ for some } D \in \mathcal{D} \text{ such that } \{a, b\} \subseteq D.$$

*Define an **H-selection** to be a disjoint pair of rationales (H_1, H_2) such that (i) $H_1 \cup H_2 \subset H \cup H^{-1}$ and (ii) for all $(a, b) \in H$, either $(a, b) \in H_1 \cup H_2$ or $(b, a) \in H_2$ but not both. Denote the collection of all H-selections by $\mathcal{H}$.*

The last possibility is that neither a nor b is ever observed to be chosen in the dataset. Intuitively, this type of data imposes no direct restriction on the relationship between a and b.[17]

Definition 7 *Let I be defined by aIb if*

$$c(D) \neq a, b \text{ for all } D \in \mathcal{D}$$

*Define an **I-selection** to be a disjoint pair of rationales (I_1, I_2) such that (i) $I_1 \cup I_2 \subset I$ and (ii) for all $(a, b) \in I$, either $(a, b) \in I_1 \cup I_2$ or $(b, a) \in I_1 \cup I_2$ but not both. Denote the collection of all I-selections by $\mathcal{I}$.*

---

[17]Having said this, it is not the case that every relationship between the two alternatives need be consistent with *some* RSM-representation.

The four cases above outline the restrictions, which are directly imposed by the choice data, on the relationship between any pair of alternatives. In order for the data to be RSM-representable, there must be some pair of rationales which respects these restrictions and, moreover, represents a choice function. Proposition 3 establishes that any RSM-representable choice function has a minimal representation. By Lemma 7, any minimal representation must be in $\mathcal{P}_{RSM}$. In order to find a pair of rationales that RSM-represents the partial data, it then follows that the analyst must identify a pair of minimal rationales in $\mathcal{P}_{RSM}$ that is consistent with the four restrictions identified above. In light of the preceding discussion, it is straightforward to show the following:

Proposition 6 *The choice data $< c, \mathcal{D} >$ is RSM-representable if and only if there exist selections $Q_2 \in \mathcal{Q}$, $(H_1, H_2) \in \mathcal{H}$, and $(I_1, I_2) \in \mathcal{I}$ with rationales $P_1 \equiv P_1^D \cup H_1 \cup I_1$ and $P_2 \equiv P_2^D \cup Q_2 \cup H_2 \cup I_2$ such that:*

(i) $(P_1, P_2) \in \mathcal{P}_{RSM}$*; and,*

(ii) if $(a, b) \in P_2$ and $c(D) = b$ for some $D \in \mathcal{D}$ such that $\{a, b\} \subset D$, then $(d, a) \in P_1$ for some $d \in D$.

For the standard model of preference maximization, consistency of limited choice boils down to a simple condition, known as the *Strong Axiom of Revealed Preference* (SARP), which requires the revealed preference to be acyclic (Houthakker [1950]; Samuelson [1950]).[18] By comparison, the consistency condition for the RSM model is considerably more involved. This is due to the fact that, unlike the standard model, the RSM model imposes very little structure on the rationales. By comparison with SARP, even the consistency condition for maximization with *one* rationale (i.e. asymmetric preference) is rather involved (see e.g. Theorem 2 of Bossert, Sprumont, and Suzumura [2005]).

IV. Implications: Inferences and Policy

In this section, we examine the implications of our results for drawing inferences from behavior and for using these inferences to evaluate policy. To keep

---

[18]While the formulations due to Houthakker and Samuelson are slightly different, both are equivalent to the acyclicity of the revealed preference for (single-valued) choice functions.

the discussion relatively brief, we focus on the issues raised by Examples I-VI of the Introduction.

a. Inferences from Identification

Examples I to III deal with situations where the analyst is interested in drawing inferences about the rationales consistent with RSM-representable behavior. If the analyst is an economist or a market participant, the objective may simply be to develop a better understanding of a decision-maker who is believed to follow a rational shortlist method. For a policy maker, the exercise is likely to be motivated by welfare concerns.

If there is "good reason" to believe that the decision-maker follows a rational shortlist method and, moreover, that one of the rationales reflects the decision-maker's "true" preference (as posited in Examples II and III), identifying the features of this rationale should help the policy maker to make better policy judgments.[19] Broadly, this is the *model-based* approach to welfare advocated in several recent papers (see e.g. Rubinstein and Salant [2012]). In sharp contrast, others have advocated a *model-free* concept of welfare. One widely discussed proposal, suggested by Bernheim and Rangel [2007, 2009], is to adopt a welfare criterion based on Pareto-dominance. Given the subdomain $\mathcal{D}^W \subseteq 2^X \setminus \emptyset$ of choice situations judged relevant for welfare analysis, the *B-R welfare relation* W is defined by aWb if and only if $c(A) \neq b$ for all $A \in \mathcal{D}^W$ such that $a \in A$.

Recently, there has been considerable debate about which is the more appropriate approach to welfare in situations where the choice data is inconsistent with preference maximization (see e.g. Rubinstein-Salant [2012] and Bernheim [2009] for a discussion of the relevant papers). The debate involves subtle issues that extend far beyond the scope of our work and it is not our intention to argue the position on either side. We merely wish to illustrate how identification results, like those derived in Section III, can help inform the discussion. From Proposition 2:

---

[19]The quotation marks are meant to emphasize that we take no position on what *should* count as good reason to hold these beliefs.

25

Remark 1 (Welfare) *Fix any RSM-representation (P_1, P_2) of c and suppose $\mathcal{D}^W = 2^X \setminus \emptyset$. Then:*

(i) $P_1 \subseteq W = P_1^*$ so that the B-R welfare relation over-estimates P_1; and

(ii) $P_2^c \cap W = \emptyset$ so that the B-R welfare relation need not overlap with P_2.

Part (ii) illustrates that there may be no relationship between the B-R welfare relation and the "true" preference parameter of a plausible model consistent with observed behavior. Several papers (see e.g. Manzini and Mariotti [2012b]; Masatlioglu, Nakajima, and Ozbay [2012]; Rubinstein and Salant [2012]) provide examples of choice behavior which illustrate a similar point for models related to rational shortlist methods. The novelty of Remark 1(ii) is to show that this disconnect is not limited to carefully contrived examples for the RSM model. Indeed, there is no connection between the B-R welfare relation and the revealed 2-rationale for *any* RSM-consistent choice data.

Part (i) makes a related point. One criticism of the model-based approach is that it relies on a particular interpretation of what constitutes the decision-maker's "true" preference in the chosen model (Bernheim [2009]). As illustrated by Examples II and III, there are plausible interpretations of the RSM model, for instance, which support the view that *either* rationale may reflect "true" preference. While this observation is a word of warning against the model-based approach, it should not, in our opinion, be viewed as a tacit endorsement of the model-free approach. To the contrary, Remark 1(i) illustrates that the model-free approach cannot simply escape questions of interpretation by avoiding them. For choice data consistent with the RSM model, using the B-R relation for policy is tantamount to attaching welfare significance to the first rationale.

Setting aside the difficult issue of welfare, Examples I to III suggest that the analyst may have good reason to be interested in the rationales underlying RSM-representable behavior. Even with the identification results in Section III, there is an issue of what inferences the analyst can be justified in drawing from behavior. As is implicit from our analysis of these examples, we advocate a conservative approach. In our view, the analyst should be cautious

about inferring more than what is directly revealed by behavior. The following definition formalizes this conservative view:

Definition 8 *Given alternatives $a, b \in X$, a is said to be P_i-**superior** to b if $aP_i^c b$ for $i = 1, 2$. Moreover, a is said to be **RSM-superior** to b if $aP^c b$.*

Given Proposition 1, a is P_i-superior to b if and only if a is revealed preferred to b by the ith rationale or, equivalently, $(a, b) \in P_i$ for any RSM-representation (P_1, P_2) of behavior. Similarly, a is RSM-superior to b if and only if $(a, b) \in P_1$ or $(a, b) \in P_2 \setminus (P_1 \cup P_1^{-1})$ for any RSM-representation (P_1, P_2) of behavior. In other words, a is RSM-superior to b exactly when choice behavior reveals that a decision-maker following a rational shortlist procedure "uses" the preference of a over b to discriminate between the two alternatives.

While we generally favor making cautious inferences from behavior, there are situations where the analyst may be justified in taking a more liberal approach. One situation is when the analyst believes that "boundedly rational" distortions play only a minimal role in explaining the decision-maker's behavior (see e.g. Cherepanov, Feddersen, and Sandroni [2013]). For example, the analyst in Example II may be confident that the tie-breaking rule has only a limited influence on the survey respondent's behavior. In that case, it might be argued that the upper bound P_1^* provides a more complete understanding of the decision-maker's "true" preference than the revealed rationale P_1^c. Similarly, an analyst who takes a dim view of the role played by consideration sets (in Example III) might favor the upper bound P_2^* over the more cautious P_2^c. The value of Proposition 2 is that it characterizes the range of choice-based inferences that can be drawn for either rationale.

b. Policy from Identification

Examples IV to VI address situations where a seller is interested in determining the impact of a certain marketing or advertising strategy on consumer behavior. The discussion of these examples illustrates how identification results help the seller gain some insight into the potential value of these strategies. Implicit in our analysis was the assumption that the behavior of com-

peting sellers remained fixed. To account for competitive behavior, the conventional approach is to consider an *applied theory* model where the strategic interaction of sellers and the choice behavior of consumers are modeled explicitly. While this type of exercise is far beyond the scope of the paper, it is nonetheless possible to show how identification results are essential for evaluating behavior and policy in applied theory models.

To illustrate, consider the model of decoy marketing in Example VI (which is a little more straightforward than the model of comparative advertising). To simplify, suppose the consumer's "true" preferences are captured by the second rationale and the "distortionary" effect of decoy marketing by the first rationale. If a seller can successfully induce consumers to switch products in equilibrium, what conclusions might the analyst draw about decoy marketing? Given Proposition 1, it is straightforward to establish the following:

Remark 2 (Decoy Marketing) *Suppose the consumer chooses according to a choice function c which is RSM-representable and the seller successfully changes the consumer's choice from $c(A)$ by marketing a decoy product $d \notin A$. Then:*

(i) $c(A \cup \{d\}) = d$ if and only if $c(A \cup \{d\})$ is RSM-superior to $c(A)$; and,

(ii) $c(A \cup \{d\}) \neq d$ if and only if $c(A)$ is P_2-superior to $c(A \cup \{d\})$.

When building an applied theory model, the analyst has flexibility in how to represent the behavior of consumers. As shown by Propositions 3 and 4, a variety of rationale pairs can be used to represent the same behavior. Remark 2 establishes the extent to which the impact of decoy marketing is independent of the representation chosen by the analyst.

In particular, Remark 2(ii) shows that successful decoy marketing which induces the consumer to switch to a "third" option *must be* welfare reducing.[20] Indeed, it is the only change in behavior due to decoy marketing that has unambiguous welfare implications. Remark 2(i) shows that the implications of

---

[20]Manzini and Mariotti [2012b] establish a somewhat weaker result for a generalization of the RSM model.

switching to the decoy alternative d depends, in general, on the decoy alternative in question and how the analyst decides to represent the consumer's behavior. If $(d, c(A)) \in P_2^c$, then the addition is welfare enhancing. If, on the other hand, $(d, c(A)) \in P_1^c \cup P^{in}$, then the welfare implications depend on how the analyst decides to represent the consumer's behavior.[21]

Identification allows the analyst to draw similar conclusions about the model of competitive advertising from Examples V-VI. To simplify the presentation, suppose the analyst decides to represent a consumer who chooses according to the choice function c by the rationale pair (P_1, P_2). Then, as noted in the introduction, comparative advertising could either be used to promote a comparison $(x, y) \in P_2$ to the first rationale or to demote a comparison $(x, y) \in P_1$ to second rationale. Let $c^{(x,y)}(A) \equiv c_{(P_1 \cup (x,y), P_2 \setminus (x,y))}(A)$ denote the product chosen from A when $(x, y) \in P_2$ is promoted. Analogously, let $c_{(x,y)}(A) \equiv c_{(P_1 \setminus (x,y), P_2 \cup (x,y))}(A)$ denote the product chosen when $(x, y) \in P_1$ is demoted.[22]

Suppose the consumer initially chooses a from A and a seller is interested in changing the consumer's choice to another alternative b through comparative advertising. Regardless of the rationale pair used to represent behavior, it is not difficult to see that the seller cannot achieve this goal by advertising which directly compares b to a. However, the seller may successfully change the consumer's choice to b by comparative advertising with a third alternative e. Given Propositions 2 and 3, it is straightforward to show the following:

Remark 3 (Comparative Advertising) *Suppose the consumer chooses according to (P_1, P_2) and the seller successfully changes the consumer's choice from $c(A) = a$ to b by doing comparative advertising with a third alternative $e \in A \setminus \{a, b\}$. Then:*

(i) If $c^{(e,a)}(A) = b$, then a is P_2-superior to b or $(a, b) \in P^{in} \setminus P_1$.
(ii) If $c_{(e,b)}(A) = b$, then b is P_2-superior to a.

---

[21]This analysis is broadly consistent with the wisdom in marketing and psychology which suggests that enlarging the menu may, in some cases, induce the consumer to make a poorer choice (see e.g. Schwartz [2004]).

[22]Technically speaking, the choices $c^{(x,y)}(A)$ and $c_{(x,y)}(A)$ may be empty even though c is RSM-representable.

Remark 3(ii) shows that "obfuscation" has clear implications that do not depend on how the analyst chooses to represent consumer behavior. In the event that obfuscation is successful, (b, a) must be part of the second rationale (and, moreover, neither (b, a) nor (a, b) can be part of the first rationale). On the other hand, Remark 3(i) shows that successful "promotion" has ambiguous implications. While (a, b) must be part of the second rationale used to represent the consumer's behavior, it might not be part of every second rationale that represents behavior. Depending on the interpretation given to the rationales, this result leads to the somewhat surprising conclusion that obfuscation *must* be welfare enhancing while promotion *may* be welfare reducing.

In light of Remarks 2 and 3, we feel that the two applications discussed merit further investigation. In fact, Examples II and III also suggest applications of the RSM model that may be worth analyzing more extensively. For these examples, the tie-breaking rule and the consideration set were viewed as nuisance parameters. For the analyst interested in making inferences about the decision-maker's "true" preference, this is certainly the case. However, sellers may take a more positive view. Like comparative advertising and decoy marketing, both are non-price mechanisms that might be leveraged to "nudge" consumers into making product choices that are more profitable to the seller (Thaler and Sunstein [2008]; Manzini, Mariotti, and Tyson [2011]).

The forced choice model in Example II might be used to study seller behavior in markets where consumers have a limited ability to compare different products. In the proposed model, sellers would influence the tie-breaking rules that consumers use to choose among incomparable products. In our view, the model nicely complements the approach of Piccione and Spiegler [2012] who consider markets where sellers influence consumers' ability to compare products but cannot affect their tie-breaking rules.[23] Likewise, the limited consideration model in Example III might be used to study aspects of competitive advertising not captured by the model discussed in Examples IV-V or the models studied by Eliaz and Spiegler [2011a, 2011b].

---

[23]In their model, the tie-breaking rule is determined by the random assignment of consumers to different sellers.

V. Conclusion

In this paper, we study identification in a model of procedural decision-making first proposed by Manzini and Mariotti [2007]. For this model, we provide simple definitions of revealed preference (Propositions 1-2), a straightforward characterization of choice-equivalent representations (Propositions 3-5), and a complete characterization of testable implications with limited data (Proposition 6). Each of these results corresponds to a standard identification result in the more traditional setting of utility maximization. Part of our broader goal was to show that some models of procedural decision-making, like rational shortlist methods, permit the same kind of identification results as models of decision-making based on utility maximization.

Our work is part of a small but growing literature concerned with identification in models of procedural decision-making (see e.g. Au and Kawai [2011]; de Clippel and Rozen [2012]; Llerars, Masatlioglu, Nakajima, and Ozbay [2011]; Horan [2012]; Masatlioglu, Nakajima, and Ozbay [2012]; and, Tyson [2012]). Similar to our work, each of the cited papers establishes identification results for a model of two-stage choice.[24] At the same time, none of these papers provides analogs for *all* of the identification results established in our paper.

One important difference is that *none* of these papers contains a result like Proposition 5. For models of decision-making with cardinal utility representations, the standard "uniqueness up to affine transformations" result provides a clear understanding of the representations which are consistent with behavior *and* the transformations of representations which leave behavior unchanged. For two-stage models (like rational shortlist methods), the range of representations consistent with behavior (Propositions 3 and 4) provides little insight into the choice-invariant transformations of a given representation. This suggests the need for an additional result which characterizes the class of choice-invariant transformations in models of procedural decision-making.

Of the papers cited above, the most closely related is Horan [2012] (see also Au and Kawai [2011]), who axiomatizes the special case of the RSM model

---

[24]Each of these models is distinct from the RSM model. For the technical details, see Remark 5 of the Appendix.

where both rationales are transitive. Following the same approach taken here, he shows that the addition of transitivity at once simplifies the definition of the revealed rationales and leads to sharper identification of the minimal representations. At the same time, transitivity makes it makes it more difficult to characterize the range of general representations.

Also closely related are the recent paper by Masatlioglu, Nakajima, and Ozbay (MNO) [2012] and their joint work with Lleras [2011]. These papers study two-stage procedures $(\Gamma_1, \succ_2)$ where the decision-maker first "filters" the feasible set using the correspondence $\Gamma_1 : 2^X \to 2^X$ before maximizing the linear order $\succ_2$ on the alternatives in $\Gamma_1(A)$ which survive the first stage. They focus on axiomatizing a variety of procedures where the filter satisfies a particular "choice contraction property" (relating $\Gamma_1(B)$ to $\Gamma_1(A)$ for $B \subset A$). Similar to our approach, the authors first define a second-stage revealed preference P_2^c. They then construct, for every menu A, a lower bound $\Gamma_1^c(A)$ of alternatives that must survive the first stage and an upper bound $\Gamma_1^*(A)$ that might survive the first stage. However, the authors do not use these bounds to characterize the class of filter representations. Technically, the difficulty is that, unlike the RSM model, there may be no representations which attain the bounds Γ_1^c and Γ_1^*.[25]

Another paper related to ours is Tyson [2012], who studies (possibly multi-valued) procedures (Γ_1, P_2) where (i) the filters Γ_1^j consistent with behavior form a lattice, and (ii) the second-stage rationale P_2 is negative transitive. For these procedures, he constructs a revealed filter Γ_1^c and a revealed rationale P_2^c.[26] For the RSM model, the filters $\Gamma_1^j(\cdot) \equiv \max(\cdot; P_1^j)$ (induced by first rationales P_1^j consistent with behavior) form a lattice. Thus, Tyson's result shows how to define the lower bounds $\Gamma_1^c(\cdot) \equiv \max(\cdot; P_1^c)$ and P_2^c when the

---

[25]One issue is that the lower bound $\Gamma_1^c \equiv \cap_{j=1}^n \Gamma_1^j$ need not satisfy the required choice contraction properties even though each filter Γ_1^j associated with a representation does. In particular, some choice contraction properties are not preserved under intersection (see Aizerman and Aleskerov [1995]; Aizerman [1985]). Another issue is that there may be no preference $\succ_2$ so that $(\Gamma_1^*, \succ_2)$ represents c. In particular, the upper bound $\Gamma_1^* \equiv \cup_{j=1}^n \Gamma_1^j$ may be too inclusive to rule out "preference cycles" in the second stage even though each filter Γ_1^j does.

[26]Tyson also shows how to use these objects to provide an axiomatization of the procedure.

second rationale has the structure of a linear order. Because the minimal representations form a lattice (footnote 11), it is possible to characterize, as we do, the revealed rationales and the class of RSM-representations in general.

Incidentally, it is easy to see that the lattice structure of minimal RSM-representations does not depend on the fact that choice is single-valued. Accordingly, our results can be extended to the RSM model with multi-valued choice (Alcantud and García-Sanz [2010]). A natural question is whether our approach can be adapted to any other models. Two natural candidates are the RSM model with acyclic rationales (Houy [2008]) and the extension to more than two rationales (Apesteguia and Ballester [2010]; Manzini and Mariotti [2012a]). For both models, there appear to be technical issues which might preclude a straightforward application of our approach.

A final paper worth noting is de Clippel and Rozen [2012], who study identification for two-stage choice models in circumstances where choice data is limited. They provide testable implications for an extension of the RSM model, called *Categorize then Choose* (CTC), which has been studied by Spears [2011], Manzini and Mariotti [2012b], and Cherepanov, Feddersen, and Sandroni [2013].[27] Although the conditions which they identify are also necessary for choice data to have an RSM-representation, the applicability of their result is somewhat limited. Unlike Proposition 6 (which applies to *any* dataset), their result only applies to datasets that are "sufficiently rich" in terms of behavior.[28] While they also show that a simple acyclicity property is sufficient to ensure that "sparse" choice data (that is not sufficiently rich) is consistent with the CTC model, this condition does not guarantee that the data is consistent with the RSM model. Indeed, it is easy to construct sparse datasets which satisfy their acyclicity condition but fail to be RSM-representable.

---

[27]They also show that the testable implication of the filter procedure proposed by MNO is an acyclicity property.

[28]In particular, they require that the data is rich enough to ensure that $P_2^D \cup P^D$ is total (where the "revealed preference" P^D is defined by xP^Dy if $c(x,y) = x$ and $\{x,y\} \in \mathcal{D}$).

IV. References

Aizerman, Mark. 1985. "New Problems in the General Choice Theory – Review of a Research Trend." *Social Choice and Welfare* 2(4) 235–282.

Aizerman, Mark, and Fuad Aleskerov. 1995. *Theory of Choice.* New York: North-Holland.

Alcantud, José Carlos, and María García-Sanz. 2010. "Rational Choice by Two Sequential Criteria." Universidad de Salamanca working paper.

Anderson, Simon, and Régis Renault. 2009. "Comparative Advertising: Disclosing Horizontal Match Information." *RAND Journal of Economics* 40(3) 558–581.

Apesteguia, José, and Miguel Ballester. 2010. "Choice by Sequential Procedures." Universitat Pompeu Fabra working paper.

Au, Pak Hung, and Keiichi Kawai. 2011. "Sequentially Rationalizable Choice with Transitive Rationales." *Games and Economic Behavior* 73(2) 608–614.

Bernheim, Douglas. 2009. "Behavioral Welfare Economics." *Journal of the European Economic Association* 7(2-3) 267–319.

Bernheim, Douglas, and Antonio Rangel. 2007. "Toward Choice-Theoretic Foundations for Behavioral Welfare Economics." *American Economic Review (Papers and Proceedings)* 97(2) 464–470.

Bernheim, Douglas, and Antonio Rangel. 2009. "Beyond Revealed Preference: Choice-Theoretic Foundations for Behavioral Welfare Economics." *Quarterly Journal of Economics* 124(1) 51–104.

Bossert, Walter, Yves Sprumont, and Kotaro Suzumura. 2005. "Maximal-Element Rationalizability." *Theory and Decision* 58(4) 325–350.

Cherepanov, Vadim, Timothy Feddersen, and Alvaro Sandroni. 2013. "Rationalization." *Theoretical Economics* forthcoming.

de Clippel, Geoffroy, and Kareen Rozen. 2012. "Bounded Rational-

ity and Limited Datasets: Testable Implications, Identifiability, and Out-of-Sample Prediction." Brown University working paper.

Danan, Eric. 2010. "Randomization vs. Selection: How to Choose in the Absence of Preference?" *Management Science* 56(3) 503–518.

Dekel, Eddie, and Barton Lipman. 2010. "How (Not) to Do Decision Theory." *Annual Review of Economics* 2 257–282.

Dulleck, Uwe, Franz Hackl, Bernhard Weiss, and Rudolf Winter-Ebmer. 2011. "Buying Online: An Analysis of Shopbot Visitors." *German Economic Review* 12(4) 395–408.

Eliaz, Kfir, and Efe Ok. 2006. "Indifference or Indecisiveness? Choice-theoretic Foundations of Incomplete Preferences." *Games and Economic Behavior* 56(1) 61–86.

Eliaz, Kfir, and Ran Spiegler. 2011a. "Consideration Sets and Competitive Marketing." *Review of Economic Studies* 79(4) 235–262.

Eliaz, Kfir, and Ran Spiegler. 2011b. "On the Strategic Use of Attention Grabbers." *Theoretical Economics* 6 127–155.

Emons, Winand, and Claude Fluet. 2012. "Non-comparative versus Comparative Advertising of Quality." *International Journal of Industrial Organization* 30(4) 352–360.

Grewal, Dhruv, Sukumar Kavanoor, Edward Fern, Carolyn Costley, and James Barnes. 1997. "Comparative versus Noncomparative Advertising: A Meta-Analysis." *Journal of Marketing* 61(4) 1–15.

Horan, Sean. 2012. "A Simple Model of Two-Stage Maximization." UQAM working paper.

Houthakker, Hendrik. 1950. "Revealed Preference and the Utility Function." *Economica* (New Series) 17(66) 159–174.

Houy, Nicolas. 2008. "Progressive Knowledge Revealed Preferences and Sequential Rationalizability." École Polytechnique working paper.

Houy, Nicolas, and Koichi Tadenuma. 2009. "Lexicographic Compositions of Multiple Criteria for Decision Making." *Journal of Economic Theory* 144(4) 1770–1782.

Huber, Joel, John Payne, and Christopher Puto. 1982. "Adding Asymmetrically Dominated Alternatives: Violations of Regularity and the Similarity Hypothesis." *Journal of Consumer Research* 9(1) 90–98.

Leven, Samuel, and Daniel Levine. 1996. "Multiattribute Decision Making in Context: A Dynamic Neural Network Methodology." *Cognitive Science* 20(2) 271–299.

Llerars, Juan Sebastian, Yusufcan Masatlioglu, Daisuke Nakajima, and Erkut Ozbay. 2011. "When More is Less: Limited Consideration." University of Maryland working paper.

Manzini, Paola, and Marco Mariotti. 2007. "Sequentially Rationalizable Choice." *American Economic Review* 97(5) 1824–1839.

Manzini, Paola, and Marco Mariotti. 2012a. "Choice by Lexicographic Semiorders." *Theoretical Economics* 7 1–23.

Manzini, Paola, and Marco Mariotti. 2012b. "Categorize then Choose: Boundedly Rational Choice and Welfare." *Journal of the European Economic Association* 10(5) 1141–1165.

Manzini, Paola, Marco Mariotti, and Christopher Tyson. 2011. "Manipulation of Choice Behavior." University of St. Andrews working paper.

Masatlioglu, Yusufcan, Daisuke Nakajima, and Erkut Ozbay. 2012. "Revealed Attention." *American Economic Review* 102(5) 2183–2205.

Piccione, Michele, and Ran Spiegler. 2012. "Price Competition under Limited Comparability." *Quarterly Journal of Economics* 127(1) 97–135.

Roberts, John, and James Lattin. 1997. "Consideration: Review of Research and Prospects for Future Insights." *Journal of Marketing Research* 34(3) 406–410.

Rubinstein, Ariel, and Yuval Salant. 2008. "Some Thoughts on the Principle of Revealed Preference." In Caplin, Andrew, and Andrew Schotter (Eds.), *The Foundations of Positive and Normative Economics: A Handbook.* New York: Oxford University Press.

Rubinstein, Ariel, and Yuval Salant. 2012. "Eliciting Welfare Preferences from Behavioural Data Sets." *Review of Economic Studies* 79(1) 375–387.

Samuelson, Paul. 1938. "A Note on the Pure Theory of Consumer's Behaviour." *Economica* (New Series) 5(17) 61–71.

Samuelson, Paul. 1950. "The Problem of Integrability in Utility Theory." *Economica* (New Series) 17(68) 355–385.

Schwartz, Barry. 2004. *The Paradox of Choice: Why More is Less.* New York: Harper Perennial.

Sen, Amartya. 1971. "Choice Functions and Revealed Preferences." *Review of Economic Studies* 38(3) 307–317.

Spears, Dean. 2011. "Intertemporal Bounded Rationality as Consideration Sets with Contraction Consistency." *B.E. Journal of Theoretical Economics* 11(1).

Spiegler, Ran. 2008. "On Two Points of View Regarding Revealed Preferences and Behavioral Economics." In Caplin, Andrew, and Andrew Schotter (Eds.), *The Foundations of Positive and Normative Economics: A Handbook.* New York: Oxford University Press.

Spiegler, Ran. 2011. *Bounded Rationality and Industrial Organization.* New York: Oxford University Press.

Thaler, Richard, and Cass Sunstein. 2008. *Nudge: Improving Decisions about Health, Wealth, and Happiness.* New York: Yale University Press.

Tversky, Amos, Shmuel Sattath, and Paul Slovic. 1988. "Contingent Weighting in Judgment and Choice." *Psychological Review* 95(3) 371–384.

Tyson, Christopher. 2012. "Behavioral Implications of Shortlisting Proce-

dures." *Social Choice and Welfare*, forthcoming.

Wright, Peter, and Fredrick Barbour. 1977. "Phased Decision Strategies: Sequels to an Initial Screening." In Starr, Martin, and Milan Zeleny (Eds.), *Studies in Management Sciences, Multiple Criteria Decision Making.* Amsterdam: North-Holland.

V. Mathematical Appendix

For convenience, we restate the main result of Manzini and Mariotti [2007]:

Theorem 1 (Manzini and Mariotti) *A choice function c is RSM-representable if and only if it satisfies:*

Expansion   *If $c(A) = x = c(B)$, then $c(A \cup B) = x$; and*
WWARP   *If $c(A) = x = c(x, y)$ for $A \supset \{x, y\}$, then $c(B) \neq y$ for any B such that $\{x, y\} \subset B \subset A$.*

To establish some claims made in the discussion of Example 1:

Remark 4 *(i) The choice behavior in Example 1 is consistent with the RSM model. (ii) For the choice $c(X)$ to be consistent with the RSM model, $c(X) = x$ or $c(X) = w$.*

Proof. (i) WWARP is trivially satisfied because the only observed choices are from sets of two and three alternatives. To see that c also satisfies Expansion, observe that $c(a, a') = a = c(a, a'')$ if and only if P^c is transitive on $\{a, a', a''\}$. By Expansion, it follows that $c(a, a', a'') = a = \max(\{a, a', a''\}; P^c)$. This only pins down the specified choices on $\{w, x, z\}$ and $\{w, y, z\}$.

(ii) First, suppose that $c(X) = y$. Since $c(w, x, y) = w$ and $c(w, y) = y$, this choice violates WWARP. Next, suppose that $c(X) = z$. Since $c(x, y, z) = x$ and $c(x, z) = z$, this choice likewise violates WWARP. As such, the only possibilities are $c(X) = x$ and $c(X) = w$. It is straightforward to check that neither of these possibilities violates WWARP or Expansion. ∎

To establish the claim made in the first footnote of the conclusion:

Remark 5 *(i) The procedures studied by MNO and MNO (with Lleras) are distinct from the RSM model. (ii) The transitive RSM model is a special case of the acyclic RSM model. (iii) The acyclic RSM model is a special case of the RSM model. (iv) The RSM model is a special case of the model with an arbitrary number of rationales.*

Proof. (i) See MNO and MNO (with Lleras) for examples. **(ii)-(iv)** Each of the stated inclusions follows by definition. For examples establishing that each is strict: (ii) see Example 1; (iii) see Appendix C of Houy [2008]; and, (iv) see Proposition 2 of Apesteguia and Ballester [2010]. ■

Proof of Proposition 1. Since c is RSM-representable, c satisfies WWARP and Expansion.

(i) ($\Rightarrow$) Suppose $xP_1^c y$. By definition of the revealed 1-rationale, $xP^c y$ and there is a $B \supset \{x, y\}$ such that $yP^c c(B) \equiv w$ and $zP_2^c y$ for any $z \in UC \equiv UC_{P^c}(y; B \setminus \{x\})$. By way of contradiction, suppose that $c(A \cup \{x\}) \in \{x, y\}$ for any A such that $c(A) = y$. By definition of the revealed 2-rationale, $zP_2^c y$ implies that there is a B_z such that $\{y, z\} \subset B_z$ and $c(B_z) = y$. Define

$$B^* \equiv [\bigcup_{z \in UC} B_z] \bigcup [B \setminus UC_{P^c}(y; B)].$$

Since $c(y, z') = y$ for all $z' \in B \setminus UC_{P^c}(y; B)$ and $c(B_z) = y$ for all $z \in UC_{P^c}(y; B \setminus \{x\})$, Expansion implies $c(B^*) = y$. First, observe that $B \subseteq B^* \cup \{x\}$ by construction. Next, observe that $x \notin B^*$. Otherwise, $c(y, w) = y = c(B^*)$ and $c(B) = w$ violate WWARP (since $\{y, w\} \subset B \subset B^*$). Since $c(A \cup \{x\}) \in \{x, y\}$ for any A such that $c(A) = y$, $c(B^* \cup \{x\}) \in \{x, y\}$. Finally, observe that $c(B^* \cup \{x\}) = x$. Otherwise, $c(B^* \cup \{x\}) = y$ violates WWARP (by the argument given above).

To complete the proof, observe that $c(x, w) = w$. Otherwise, $c(x, w) = x = c(B^* \cup \{x\})$ and $c(B) = w$ violate WWARP (since $\{x, w\} \subset B \subset B^* \cup \{x\}$). Combining this observation with the assumptions $c(x, y) = x$ and $c(y, w) = w$ establishes that $c(x, y, w) = w$. This follows from the fact that $c(x, y, w) = x$ and $c(x, y, w) = y$ both violate WWARP (by similar arguments to those given above). But, $c(x, y, w) = w$ contradicts the assumption that $c(A \cup \{x\}) \in \{x, y\}$ for any A such that $c(A) = y$ (by setting $A = \{y, w\}$) which, in turn, establishes the result.

($\Leftarrow$) Suppose that $c(B) = y$ and $c(B \cup \{x\}) \notin \{x, y\}$ for some B. First, observe that $c(x, y) = x$. Otherwise, $c(x, y) = y$ and $c(B) = y$ imply $c(B \cup$

$\{x\}) = y$ by Expansion, which contradicts the assumption that $c(B \cup \{x\}) \notin \{x, y\}$. Next, define $c(B \cup \{x\}) \equiv z \notin \{x, y\}$ and observe that $c(y, z) = y$. Otherwise, the choices $c(y, z) = z = c(B \cup \{x\})$ and $c(B) = y$ violate WWARP (since $\{y, z\} \subset B \subset B \cup \{x\}$). The result then follows by defining $A \equiv B \cup \{x\}$. Combining the two observations above with the definition of the revealed 2-rationale and the revealed preference, it follows that $xP^c y$, $yP^c z$, and $z' P_2^c y$ for any $z' \in UC_{P^c}(y; A \setminus \{x\})$. Thus, $xP_1^c y$ as required.

(ii) ($\Rightarrow$) Suppose that $xP_2^c y$. By definition of the revealed 2-rationale, it follows that $c(x, y) = x$ and $c(A) = y$ for some A. The result then follows by defining $B \equiv \{x, y\}$. ($\Leftarrow$) Suppose $c(A) = y$ and $c(B) = x$ for some $\{x, y\} \subseteq B \subset A$. If $c(x, y) = y$, the choices $c(x, y) = y = c(A)$ and $c(B) = x$ violate WWARP (since $\{x, y\} \subset B \subset A$). But, this contradicts the assumption that c is RSM-representable. So, $c(x, y) = x$ which establishes that $xP_2^c y$. ∎

Proof of Proposition 2. Suppose (P_1, P_2) is an RSM-representation of c.

(i) The inclusions $P_1^c \subseteq P_1$ and $P_2^c \subseteq P_2$ follow from the discussion in the text.

(ii) To establish $P_1 \subseteq P_1^*$: Suppose, by way of contradiction, that $xP_1 y$ and $\neg(xP_1^* y)$. From the first relation, it follows that $c(x, y) = x$ so that $xP^c y$. In combination with $\neg(xP_1^* y)$, $xP^c y$ implies that $xP_2^c y$ (since $P_2^c = P^c \setminus P_1^*$). By definition of P_2^c, there exists a menu $A \supset \{x, y\}$ s.t. $c(A) = y$. Since $xP_1 y$ however, it follows that $c(A) \neq y$ which is the desired contradiction.

To establish $P_2 \setminus (P_1 \cup P_1^{-1}) \subseteq P_2^*$: Suppose, by way of contradiction, that $xP_2 y$, $\neg(xP_2^* y)$, $\neg(xP_1 y)$, and $\neg(yP_1 x)$. From the second and third relations, it follows that $yP^c x$. Otherwise, $xP^c y$ implies $xP_2^* y$ or $xP_1^c y$. In the first case, there is a direct contradiction with the assumption that $\neg(xP_2^* y)$. In the second case, the fact that $P_1^c \subseteq P_1$ (proved in part (i) above) establishes a contradiction with the assumption that $\neg(xP_1 y)$. Since $yP^c x$, either $yP_2 x$ or $yP_1 x$ must hold. Otherwise, (P_1, P_2) cannot induce the choice $c(x, y) = y$. Since P_2 is asymmetric and $xP_2 y$, it must be that $yP_1 x$. But, this contradicts the assumption that $\neg(yP_1 x)$. ∎

Lemma 1 *If c is RSM-representable, then $xP_1^c y$ implies $\neg[xP_2^c y]$.*

Proof. Since c is RSM-representable, c satisfies WWARP and Expansion. By way of contradiction, suppose that $xP_1^c y$ and $xP_2^c y$. By definition, $xP_1^c y$ implies that there is a $B \supset \{x,y\}$ such that $yP^c c(B) \equiv w$ and $zP_2^c y$ for any $z \in UC_{P^c}(y; B \setminus \{x\})$. Define B^* as in the proof of Proposition 1(i). By construction, $c(B^*) = y$. Moreover, $xP_2^c y$ implies that there is a $B_x \supset \{x,y\}$ such that $c(B_x) = y$. By Expansion, $c(B^* \cup B_x) = y$. Finally, observe that $c(B^* \cup B_x) = y = c(w,y)$ and $c(B) = w$ contradict WWARP (since $\{w,y\} \subset B \subset B^* \cup B_x$), which completes the proof. ∎

Lemma 2 *If c is RSM-representable, then: (i) $xP_1^c y$ implies $xP_1^* y$; and, (ii) $xP_2^c y$ implies $xP_2^* y$.*

Proof. Since c is RSM-representable, $P_1^c \cap P_2^c = \emptyset$ by Lemma 1. Since $(P_1^c \sqcup P_2^c) \subseteq P^c$ by construction, it then follows that $P_1^c \subseteq (P^c \setminus P_2^c) \equiv P_1^*$. By a similar argument, $P_2^c \subseteq P_2^*$. ∎

Lemma 3 *Given a binary relation P, $x \in \max(A; P)$ implies:*
    (i) $x \in \max(B; P)$ for any B such that $\{x\} \subseteq B \subseteq A$; and
    (ii) $x \in \max(A; \widetilde{P})$ for any $\widetilde{P}$ such that $\widetilde{P} \subseteq P$.

Proof. Both implications follow directly from the definition of $\max(\cdot; \cdot)$. ∎

Lemma 4 *If c is RSM-representable, then it can be represented by (P_1^c, P_2^*) and (P_1^*, P_2^c).*

Proof. First note that $c_{(P_1^c, P_2^*)}$ and $c_{(P_1^*, P_2^c)}$ are not multi-valued. To see this, fix a menu A and suppose that $\{a, a'\} \subseteq c_{(P_1^c, P_2^*)}(A)$ for $aP^c a'$. By definition, $\{a, a'\} \subseteq \max(A; P_1^c)$. In particular, $\neg(aP_1^c a')$. By definition of P_2^*, it follows that $aP_2^* a'$. But, this contradicts $\{a, a'\} \subseteq c_{(P_1^c, P_2^*)}(A)$ and establishes that $c_{(P_1^c, P_2^*)}$ is not multi-valued. A similar argument holds for $c_{(P_1^*, P_2^c)}$.

    Now, fix a menu A and suppose that $c(A) = y$. Define $\overline{A} \equiv \max(A; P_1^c)$ and $\underline{A} \equiv \max(A; P_1^*)$. The result follows by establishing that: (i) $y \in \underline{A}$; and,

(ii) $\max(\overline{A}; P_2^*) = y$. To see that (i) and (ii) are sufficient, observe that: (i) implies (i') $y \in \overline{A}$; and, given (i), (ii) implies (ii') $\max(\underline{A}; P_2^c) = y$. Clearly, (ii) and (ii') establish the result.

To see (i'), note that $P_1^c \subseteq P_1^*$ by Lemma 2. By Lemma 3(ii), it follows that $\underline{A} \subseteq \overline{A}$ so that $y \in \underline{A}$ implies $y \in \overline{A}$. To see (ii'), first observe that $y \in \underline{A} \subseteq \overline{A}$ [which follows from (i) and the argument above]. By Lemma 3(i), $\max(\overline{A}; P_2^*) = y$ implies $y \in \max(\underline{A}; P_2^*)$. Since $P_2^c \subseteq P_2^*$ by Lemma 2, the fact that $y \in \max(\underline{A}; P_2^*)$ implies $y \in \max(\underline{A}; P_2^c)$ by Lemma 3(ii). Finally, $y \in \max(\underline{A}; P_2^c)$ implies $\max(\underline{A}; P_2^c) = y$ because $c_{(P_1^*, P_2^c)}$ is not multi-valued.

To complete the proof, it suffices to establish claims (i) and (ii):

Proof of Claim (i): By way of contradiction, suppose that xP_1^*y for some $x \in A$. By definition, it follows that $\neg(xP_2^cy)$ so that $c(B) \neq y$ for any $B \supset \{x, y\}$. But, this contradicts $c(A) = y$. $\square$

Proof of Claim (ii): As a preliminary observation, note that $y \in \overline{A}$ by (i'). The proof is by induction on $|A|$. For the base case $|A| = 2$, suppose $A = \{x, y\}$ and $c(x, y) = y$. If $\overline{A} = \{y\}$, then $\max(\overline{A}; P_2^*) = y$ trivially. If $\overline{A} = \{x, y\}$, then yP_2^*x by definition so that $\max(\overline{A}; P_2^*) = y$.

Suppose that the claim is true for A such that $|A| = n$. Now, consider some A s.t. $|A| = n + 1$ and $c(A) = y$. First, note that there must be some $x \in A \setminus \{y\}$ such that $c(A \setminus \{x\}) = y$. Otherwise, by the pigeonhole principle, there is an $a \in A \setminus \{y\}$ such that $c(A \setminus \{z\}) = a = c(A \setminus \{z'\})$ for distinct $z, z' \in A$. Then, $c(A) = c([A \setminus \{z\}] \cup [A \setminus \{z'\}]) = a \neq y$ by Expansion, a contradiction.

There are several cases to consider: (1) there are distinct $x, x' \in A \setminus \{y\}$ such that $c(A \setminus \{x\}) = y = c(A \setminus \{x'\})$; or, (2) there is a single $x \in A \setminus \{y\}$ such that $c(A \setminus \{x\}) = y$ where (a) $c(x, y) = y$ or (b) $c(x, y) = x$. To simplify the notation, define $\overline{A}_{-z} \equiv \max(A \setminus \{z\}; P_1^c)$ for any $z \in A$.

Case (1): By the induction step, $\max(\overline{A}_{-x}; P_2^*) = y = \max(\overline{A}_{-x'}; P_2^*)$. By construction, it follows that $\overline{A} = [\overline{A}_{-x} \cap \overline{A}_{-x'}] \cup Z$ where $Z = \{z \in \{x, x'\} : aP_1^cz$ for no $a \in A\}$. If $c(x, y) = x$, then $x \notin Z$. Otherwise, $\max(\overline{A}_{-x'}; P_2^*) \neq y$. Similarly, $x' \notin Z$ if $c(x', y) = x'$.

43

Given these observations, there are four possibilities: (a) $\overline{A} = \overline{A}_{-x} \cap \overline{A}_{-x'}$; (b) $\overline{A} \subseteq \overline{A}_{-x}$, $c(x, y) = y$, and $\overline{A} \setminus \overline{A}_{-x'} = \{x\}$; (c) $\overline{A} \subseteq \overline{A}_{-x'}$, $c(x', y) = y$, and $\overline{A} \setminus \overline{A}_{-x} = \{x'\}$; and, (d) $\overline{A} \setminus \overline{A}_{-x} = \{x'\}$, $\overline{A} \setminus \overline{A}_{-x'} = \{x\}$, and $c(x, y) = y = c(x', y)$. For each, it is a straightforward to check that $\max(\overline{A}_{-x}; P_2^*) = y = \max(\overline{A}_{-x'}; P_2^*)$ implies $y \in \max(\overline{A}; P_2^*)$.

Case (2.a): By the induction step, $\max(\overline{A}_{-x}; P_2^*) = y$. By construction, $\overline{A} \subseteq \overline{A}_{-x} \cup \{x\}$. Since $c(x, y) = y$ and $\max(\overline{A}_{-x}; P_2^*) = y$, then $y \in \max(\overline{A}_{-x} \cup \{x\}; P_2^*)$ so that $y \in \max(\overline{A}; P_2^*)$.

Case (2.b): First, observe that $c(A \setminus \{w\}) = x$ for some $w \in A \setminus \{y\}$. Otherwise, an argument relying on the pigeonhole principle and Expansion (similar to that given above) leads to a contradiction.[29] Next, observe that $c(w, x) = w$. Otherwise, $c(A) = c([A \setminus \{w\}] \cup \{w, x\}) = x \neq y$ by Expansion which again yields a contradiction. By definition, $c(A \setminus \{w\}) = x$ entails $z P_2^c x$ for any $z \in UC_{P^c}(x; A \setminus \{w\})$. Then, by definition, it follows that $w P_1^c x$. Consequently, $\overline{A} \subseteq \overline{A}_{-x}$. Since $\max(\overline{A}_{-x}; P_2^*) = y$ by the induction step, $y \in \max(\overline{A}; P_2^*)$ by Lemma 3(i).

In cases 1, 2.a, and 2.b, $y \in \max(\overline{A}; P_2^*)$. Since $c_{(P_1^c, P_2^*)}$ is not multi-valued, $\max(\overline{A}; P_2^*) = y$. ∎

Lemma 5 *For $i = 1, 2$, $P_i \in \mathcal{P}_i(c)$ implies $\overline{P}_i \in \mathcal{P}_{-i}(c)$.*

Proof. Since $P_i \in \mathcal{P}_i(c)$, $P_i^c \subseteq P_i \subseteq P_i^* \subseteq P^c$. Taking the complement with respect to P^c gives $P_{-i}^c \equiv (P^c \setminus P_i^*) \subseteq (P^c \setminus P_i) \subseteq (P^c \setminus P_i^c) \equiv P_{-i}^*$. Thus, $\overline{P}_i \equiv (P^c \setminus P_i) \in \mathcal{P}_{-i}(c)$ as required. ∎

Lemma 6 *If c is RSM-representable, $(P_1, \overline{P}_1)$ and $(\overline{P}_2, P_2)$ represent c for $P_i \in \mathcal{P}_i(c)$ and $i = 1, 2$.*

Proof. Suppose c is RSM-representable and fix any $P_1 \in \mathcal{P}_1(c)$ and $P_2 \in \mathcal{P}_2(c)$. As a preliminary observation, note that (P_1^c, P_2^*) and (P_1^*, P_2^c) represent

---

[29]If $w = y$, then $c(A) = c([A \setminus \{y\}] \cup \{x, y\}) = x \neq y$ by Expansion, which is a contradiction.

c by Lemma 4. We show that $(P_1, \overline{P_1})$ represents c. The reasoning is similar for $(\overline{P_2}, P_2)$.

Consider any $A \subseteq X$ and suppose that $c(A) = y$. From the fact that $c_{(P_1^*, P_2^c)}(A) = y$, it follows that $y \in \underline{A}$. By Lemma 3(ii), $P_1 \subseteq P_1^*$ implies $y \in \max(A; P_1) \equiv A_1$. Since $P_1^c \subseteq P_1$, Lemma 3(ii) implies $A_1 \subseteq \overline{A}$. Combining these last two results establishes that $y \in A_1 \subseteq \overline{A}$. The result then follows by repeated application of Lemma 3. Since $\max(\overline{A}; P_2^*) = c_{(P_1^c, P_2^*)}(A) = y$ and $y \in A_1 \subseteq \overline{A}$, it follows that $y \in \max(A_1; P_2^*)$. Since $\overline{P_1} \subseteq P_2^*$, this last inclusion establishes that $y \in \max(A_1; \overline{P_1}) = c_{(P_1, P^c \setminus P_1)}(A)$. By the same argument given in Lemma 4, $c_{(P_1, P^c \setminus P_1)}(A)$ is not multi-valued. So, $c_{(P_1, P^c \setminus P_1)}(A) = y$. ∎

Proof of Proposition 3. **(i)** This is established by Lemma 2. **(ii)** ($\Leftarrow$) By Lemma 6, $(P_1, \overline{P_1})$ and $(\overline{P_2}, P_2)$ represent c for $P_1 \in \mathcal{P}_1(c)$ and $P_2 \in \mathcal{P}_2(c)$. By construction, these representations are minimal. ($\Rightarrow$) Suppose (P_1, P_2) is a minimal representation of c. By definition of minimality, $P_1 \cap P_2 = \emptyset$. Moreover, since (P_1, P_2) represents c, it must be that $P_1 \cup P_2 \supseteq P^c$. If $P_1 \cup P_2 \neq P^c$, it must be that there are $x, y \in X$ s.t. (a) $x P_1 y$ and $y P_1 x$, (b) $x P_2 y$ and $y P_2 x$, or (c) $x P_1 y$ and $y P_2 x$. Since (a)-(b) contradict asymmetry and (c) contradicts minimality, it must be that $P_1 \cup P_2 = P^c$. Thus, $P_2 = \overline{P_1}$ and $P_1 = \overline{P_2}$. This, in turn, delivers the desired result. ∎

Proof of Proposition 4. ($\Leftarrow$) Suppose conditions (i) and (ii) are satisfied. Since the repeat comparisons in P_1^{as} are never carried out, it follows that $c_{(P_1, P_2)}(A) = c_{(P_1, \overline{P_1})}(A)$ for any $A \subseteq X$. Since P_1^{as} and $\overline{P_1}$ are asymmetric, it follows that P_2 is asymmetric. Combined with (i), these observations establish that (P_1, P_2) is an RSM-representation of c.

($\Rightarrow$) Suppose (P_1, P_2) represents c. By Proposition 2, $P_1 \in \mathcal{P}_1(c)$. By Proposition 3, it follows that $(P_1, \overline{P_1})$ is a minimal representation of c. This establishes (i). To establish (ii), first note that the asymmetry of P_2 implies that $P_1^{as} \equiv P_2 \setminus \overline{P_1}$ is asymmetric. The fact that $x P_1 y$ or $y P_1 x$ for any $(x, y) \in P_1^{as}$ follows from $P_1 \sqcup \overline{P_1} = P^c$. If $x P^c y$, then $(x, y) \in P_1^{as}$ implies $\neg[x \overline{P_1} y]$. Consequently, $x P_1 y$. If $y P^c x$, then $(x, y) \in P_2$ implies $\neg[y \overline{P_1} x]$ by the asymmetry of P_2. Consequently, $y P_1 x$. ∎

Lemma 7 *If c is RSM-representable, then (P_1, P_2) is a minimal representation of c for some $(P_1, P_2) \in \mathcal{P}_{RSM}$. Conversely, $c_{(P_1, P_2)}$ is an RSM for any $(P_1, P_2) \in \mathcal{P}_{RSM}$.*

Proof. ($\Rightarrow$) Fix some RSM-representable choice function c. It follows directly from Proposition 3 that c has a minimal representation in $\mathcal{P}$, say (P_1, P_2). Therefore (P_1, P_2) never induces empty choice. From the argument in the text, it follows that (P_1, P_2) must belong to $\mathcal{P}_{RSM}$.

($\Leftarrow$) Fix some $(P_1, P_2) \in \mathcal{P}_{RSM}$. From the argument in the text, we know that $c_{(P_1, P_2)}(A)$ is non-empty for any menu $A \subseteq X$. It therefore suffices to show that $c_{(P_1, P_2)}$ is single-valued. Since $\mathcal{P}_{RSM} \subseteq \mathcal{P}$, it follows that $P_1 \cup P_2$ is, by definition, a total binary relation on X. To see that this rules out the possibility of multi-valued choice, suppose $\{x, y\} \subseteq c_{(P_1, P_2)}(A)$ for some $A \subseteq X$ and $x \neq y$. Then, it must be that $(x, y) \notin P_1 \cup P_2$ and $(y, x) \notin P_1 \cup P_2$ which, in turn, contradicts the fact that $P_1 \cup P_2$ is total. $\blacksquare$

Proof of Proposition 5. **(i)** Suppose that (P_1, P_2) is a minimal representation of c.

($\Leftarrow$) By assumption, (x, y) is (P_1, P_2)-redundant. Let $\widetilde{P}_1 = P_1 \cup \{(x, y)\}$ and $\widetilde{P}_2 = P_2 \setminus \{(x, y)\}$. Since (P_1, P_2) is minimal, it follows that $(\widetilde{P}_1 \cup \widetilde{P}_1^{-1}) \cap (\widetilde{P}_2 \cup \widetilde{P}_2^{-1}) = \emptyset$. To prove the result, it suffices to show that $(\widetilde{P}_1, \widetilde{P}_2)$ represents c.

For any $A \subseteq X$ such that $\{x, y\} \not\subset A$, it is easy to see that $c_{(\widetilde{P}_1, \widetilde{P}_2)}(A) = c_{(P_1, P_2)}(A) = c(A)$. Next, consider any A such that $\{x, y\} \subseteq A$. Then, it must be that $\max(A; \widetilde{P}_1) = \max(A; P_1) \setminus \{y\}$. We now show that $c(A) \neq y$. To see this, suppose instead that $c(A) = y$.

First, consider the situation where there is a sequence of n distinct elements $\{a_i\}_{i=1}^n$, with $a_n = x$ such that $a_i \in A$ and $a_i \neq y$ for $1 \leq i \leq n$ where, moreover, $(a_i, a_{i+1})_{1 \leq i \leq n-1} \in P_1$ and $(a_j, y)_{2 \leq j \leq n} \in P_2$.[30] Let a sequence that satisfies these properties be called an α *sequence*.

Since $P_1 \cap P_2 = \emptyset$, $(a_j, y)_{2 \leq j \leq n} \in P_2$ implies $(a_j, y)_{2 \leq j \leq n} \notin P_1$. Since (x, y) is (P_1, P_2)-redundant, the sequence $\{a_i\}_{i=1}^n$ implies that $(a_1, y) \in P_1 \cup P_2$. If

---

[30]To simplify, let $(a_i, a_{i+1})_{j \leq i \leq k} \in P$ (resp. $\notin P$) denote that $(a_i, a_{i+1}) \in P$ (resp. $\notin P$) for all $j \leq i \leq k$.

$(a_1, y) \in P_1$, then clearly $y \notin \max(A; P_1)$. Since $c(A) = y$ by assumption, it must be that $(a_1, y) \in P_2$. Then, for $c_{(P_1, P_2)}(A) = y$ to hold, it must be that $a_1 \notin \max(A; P_1)$. In turn, this implies that there is some $z \in A$ s.t. $(z, a_1) \in P_1$. Note that z must be distinct from the elements of the α *sequence*. If, instead, $z = a_i$ for some $3 \le i \le n$ then $c_{(P_1, P_2)}(\{a_j\}_{j=1}^i) = \emptyset$, which contradicts the fact that (P_1, P_2) represents c. This implies the existence of an α *sequence* of $n + 1$ elements, where the new sequence is $\{b_i\}_{i=1}^{n+1}$ with $b_i = a_{i-1}$ for all $2 \le i \le n + 1$ and $b_1 = z$.

The argument in the previous paragraph establishes that, if $c(A) = y$ for some $A \supseteq \{x, y\}$, the existence of an α *sequence* of n elements in A implies the existence of an α *sequence* of $n + 1$ elements in A. Since A is finite, the existence of an α *sequence* in A yields a contradiction. The proof that $c(A) \ne y$ concludes by showing the existence of an α *sequence* in A. In fact, this follows by the same argument given in the previous paragraph with $x = a_1$. In particular, there is an α *sequence* $\{z, x\}$ such that $(z, x) \in P_1$ and $(x, y) \in P_2$ for some $z \in A \setminus \{x, y\}$. This yields the required contradiction. Therefore, $c(A) \ne y$ for all $A \supseteq \{x, y\}$.

Since (P_1, P_2) is a minimal representation of c, Proposition 3 implies that $P_2 \in \mathcal{P}_2^c$. Since we have established that $c(A) \ne y$ for all $A \supseteq \{x, y\}$, the definition of P_2^c implies that $(x, y) \notin P_2^c$. Therefore, $P_2^c \subseteq \widetilde{P}_2 \subset P_2 \subseteq P_2^*$. Moreover, $\widetilde{P}_1 = P^c \setminus \widetilde{P}_2$ (since $\widetilde{P}_1 \cup \widetilde{P}_2 = P_1 \cup P_2 = P^c$). Given Proposition 3, this implies that $(\widetilde{P}_1, \widetilde{P}_2)$ is a minimal representation of c.

($\Rightarrow$) We prove the contrapositive. In particular, assume that (P_1, P_2) is a minimal representation of c. Fix $(x, y) \in P_2$ such that (x, y) is *not* (P_1, P_2)-redundant. It suffices to show that $(\widetilde{P}_1, \widetilde{P}_2)$ with $\widetilde{P}_1 = P_1 \cup \{(x, y)\}$ and $\widetilde{P}_2 = P_2 \setminus \{(x, y)\}$ does not represent c.

Since (x, y) is not (P_1, P_2)-redundant, there is a sequence $\{a_i\}_{i=1}^n$ of minimal length n s.t. $a_n = x$, $(a_i, a_{i+1})_{1 \le i \le n-1} \in P_1$ and $(a_j, y)_{2 \le j \le n} \in P_2$ but $(a_1, y) \notin P_1 \cup P_2$. Since (P_1, P_2) is a minimal representation, $(a_1, y) \notin P_1 \cup P_2$ implies $(y, a_1) \in P_1 \cup P_2$. The relevant parts of the rationales are depicted in Table 3:

P_1		P_2	
a_1	a_2	x	y
a_2	a_3	a_2	y
a_3	a_4	a_3	y
.		.	
.		.	
a_{n-1}	x	a_{n-1}	y
.		.	
.		.	

Table 3: A smallest sequence violating the redundancy condition

Notice that (y, a_1) has to be added to one of the rationales. Consider the menu $A = \{a_i\}_{i=1}^n \cup \{y\}$. Then, $\max(A; P_1)$ must be $\{a_1, y\}$ or $\{y\}$ (depending on whether (y, a_1) is in P_1 or P_2). In either case, $c_{(P_1, P_2)}(A) = y$. Since (P_1, P_2) represents c, it must be that $c(A) = y$. On the other hand, $y \notin \max(A; \widetilde{P_1})$ since $x \in A$ and $(x, y) \in \widetilde{P_1}$. Therefore, $(\widetilde{P_1}, \widetilde{P_2})$ does not represent c.

(ii) Suppose (P_1, P_2) is a minimal representation of c. Let $\widetilde{P_1} = P_1 \setminus \{(x, y)\}$ and $\widetilde{P_2} = P_2 \cup \{(x, y)\}$.

($\Leftarrow$) By Lemma 7, $(\widetilde{P_1}, \widetilde{P_2}) \in \mathcal{P}_{RSM}$ implies that $(\widetilde{P_1}, \widetilde{P_2})$ represents some RSM, say c'. Since (x, y) is $(\widetilde{P_1}, \widetilde{P_2})$-redundant, by part (i), it must be that (P_1, P_2) represents the same choice function as $(\widetilde{P_1}, \widetilde{P_2})$. Therefore, $(\widetilde{P_1}, \widetilde{P_2})$ represents $c = c'$. ($\Rightarrow$) Suppose that c is represented by $(\widetilde{P_1}, \widetilde{P_2})$. By Lemma 7, it must be that $(\widetilde{P_1}, \widetilde{P_2}) \in \mathcal{P}_{RSM}$. It remains to show that (x, y) is $(\widetilde{P_1}, \widetilde{P_2})$-redundant. By way of contradiction, suppose not. Then, by part (i), it must be that (P_1, P_2) does not represent c, which contradicts our initial premise. ∎

Proof of Proposition 6. Denote generic menus in $\mathcal{D}$ by D_i, D_j. Without loss of generality, the relevant domain for the analysis is $\widetilde{X} = \cup_{\mathcal{D}} D_i$. (The alternatives in $X \setminus \widetilde{X}$ can be made to play no part by supposing that $c(x, y) = x$ for any $x \in \widetilde{X}$ and $y \in X \setminus \widetilde{X}$.)

($\Rightarrow$) By assumption, there exists a pair of rationales that is consistent with the observed data $< c, \mathcal{D} >$. By Proposition 3, any RSM-representable choice function has a minimal representation. By Lemma 7, any rationale pair that is

a minimal representation of some choice function belongs to $\mathcal{P}_{RSM}$. So, there exists a pair $(P_1, P_2) \in \mathcal{P}_{RSM}$ such that $c_{(P_1,P_2)}(D_i) = c(D_i)$ for all $D_i \in \mathcal{D}$.

It suffices to show that there exist selections $Q_2 \in \mathcal{Q}$, $(H_1, H_2) \in \mathcal{H}$, and $(I_1, I_2) \in \mathcal{I}$ such that $P_1 = P_1^D \cup H_1 \cup I_1$ and $P_2 = P_2^D \cup Q_2 \cup H_2 \cup I_2$. Recall that $(P_1, P_2) \in \mathcal{P}_{RSM}$ implies that, for all $a, b \in \widetilde{X}$, $\{(a,b), (b,a)\} \cap (P_1 \cup P_2) \neq \emptyset$, $P_1 \cap P_2 = \emptyset$, P_1 is acyclic and P_1 breaks any P_2-cycle.

First observe that if $(a,b) \in P_1^D$, then $(a,b) \in P_1^{c(P_1,P_2)}$ so that $P_1^D \subset P_1^{c(P_1,P_2)}$. By similar reasoning, $P_2^D \subset P_2^{c(P_1,P_2)}$. By Proposition 2(i), it then follows that $P_1^D \subset P_1$ and $P_2^D \subset P_2$.

Next, consider some $(a,b) \in Q$. By definition, there exist $D_i, D_j \in \mathcal{D}$ with $\{a, b\} \subset D_i, D_j$ such that $c(D_i) = a$ and $c(D_j) = b$. Moreover, $\{(a,b), (b,a)\} \cap P_2^D = \emptyset$. Since $c_{(P_1,P_2)}(D_i) = c(D_i) = a$ with $b \in D_i$, it must be that $(b,a) \notin P_1$. Similarly, $c_{(P_1,P_2)}(D_j) = c(D_j) = b$ with $a \in D_j$ implies $(a,b) \notin P_1$. Since $(P_1, P_2) \in \mathcal{P}_{RSM}$ and $a, b \in \widetilde{X}$, it must be that $\{(a,b), (b,a)\} \cap (P_1 \cup P_2) \neq \emptyset$. As such, either $(a,b) \in P_2$ or $(b,a) \in P_2$ but not both. Since (a,b) was chosen arbitrarily, this establishes that $Q_2 \subset P_2$ for some $Q_2 \in \mathcal{Q}$.

Third, consider some $(a,b) \in H$. By definition, there exists a $D_i \in \mathcal{D}$ with $\{a, b\} \in D_i$ such that $c(D_i) = a$. Moreover, $(a,b) \notin P_2^D \cup Q$. Since $c_{(P_1,P_2)}(D_i) = c(D_i) = a$ with $b \in D_i$, it must be that $(b,a) \notin P_1$. Again, $(P_1, P_2) \in \mathcal{P}_{RSM}$ implies that one and only one of $(a,b) \in P_1$, $(a,b) \in P_2$, or $(b,a) \in P_2$. Since (a,b) was chosen arbitrarily, this establishes that $H_1 \subset P_1$ and $H_2 \subset P_2$ for some $(H_1, H_2) \in \mathcal{H}$.

Finally, consider some $(a,b) \in I$. By definition $c(D_i) \notin \{a, b\}$ for all $D_i \in \mathcal{D}$. In this case, $a, b \in \widetilde{X}$ and $(P_1, P_2) \in \mathcal{P}_{RSM}$ implies that one and only one of $(a,b) \in P_1$, $(a,b) \in P_2$, $(b,a) \in P_1$, or $(b,a) \in P_2$. In other words, there exists a selection (I_1, I_2) from I such that $I_1 \subset P_1$ and $I_2 \subset P_2$.

The four previous paragraphs show that if $(P_1, P_2) \in \mathcal{P}_{RSM}$ represents a choice function that coincides with observed data, then there exist selections $Q_2 \in \mathcal{Q}$, $(H_1, H_2) \in \mathcal{H}$, and $(I_1, I_2) \in \mathcal{I}$ such that $(P_1^D \cup H_1 \cup I_1) \subset P_1$ and $(P_2^D \cup Q_2 \cup H_2 \cup I_2) \subset P_2$.

To establish equality, note that for any $a, b \in \widetilde{X}$ if $\{(a,b), (b,a)\} \cap (P_1^D \cup P_2^D \cup Q_2 \cup H) = \emptyset$, then $(a,b) \in I$. It follows that if $(a,b) \in P_1 \setminus (P_1^D \cup H_1 \cup I_1)$,

then either $(b, a) \in (P_1^D \cup H_1 \cup I_1) \subset P_1$ or $\{(a, b), (b, a)\} \cap (H_2 \cup Q_2 \cup I_2) \neq \emptyset$. Either possibility violates $(P_1, P_2) \in \mathcal{P}_{RSM}$ (which requires that $P_1 \cup P_2$ is a total binary relation on $\widetilde{X}$ and $P_1 \cap P_2 = \emptyset$). A similar argument rules out the possibility that $P_2 \setminus (P^D \cup H_2 \cup Q_2 \cup I_2) \neq \emptyset$. Therefore, $P_1 = P_1^D \cup H_1 \cup I_1$ and $P_2 = P_2^D \cup Q_2 \cup H_2 \cup I_2$. Combined with the fact that $(P_1, P_2) \in \mathcal{P}_{RSM}$, this completes the proof for part (i).

The proof of (ii) is by contradiction. Suppose that $(d, a) \notin P_1$ for all $d \in D_i$ and some $(a, b) \in P_2$ such that $\{a, b\} \subset D_i \in \mathcal{D}$ and $c(D_i) = b$. By definition, it follows that $a \in \max(D_i; P_1)$. Since $(a, b) \in P_2$, $b \neq c_{(P_1, P_2)}(D_i) = c(D_i) = b$ which delivers the required contradiction.

($\Leftarrow$) By assumption, there exist selections $Q_2 \in \mathcal{Q}$, $(H_1, H_2) \in \mathcal{H}$, and $(I_1, I_2) \in \mathcal{I}$ which define a pair of rationales (P_1, P_2) where $P_1 \equiv P_1^D \cup H_1 \cup I_1$ and $P_2 \equiv P_2^D \cup Q_2 \cup H_2 \cup I_2$ such that:

(i) $(P_1, P_2) \in \mathcal{P}_{RSM}$; and,

(ii) if $(a, b) \in P_2$ and $\{a, b\} \subset D_i \in \mathcal{D}$ with $c(D_i) = b$, then $(d, a) \in P_1$ for some $d \in S_i$.

Given $(P_1, P_2) \in \mathcal{P}_{RSM}$, Lemma 7 implies that (P_1, P_2) represents some choice function. It suffices to show that $c_{(P_1, P_2)}(D_i) = c(D_i)$ for all $D_i \in \mathcal{D}$. Since $c_{(P_1, P_2)}$ is single valued, it suffices to show that $c(D_i) = a$ implies $a \in c_{(P_1, P_2)}(D_i)$.

Fix any menu $D_i \in \mathcal{D}$ and suppose $c(D_i) = a$. By way of contradiction, suppose there exists some $b \in D_i$ such that $(b, a) \in P_1$ (so that $a \notin c_{(P_1, P_2)}(D_i)$). Since $\{a, b\} \subset D_i$, $(a, b) \notin I$. Since $c(D_i) = a$, either $(a, b) \in P_2^D \cup Q$, $(b, a) \in P_2^D \cup Q$, or $(a, b) \in H$. If $\{(a, b), (b, a)\} \cap (P_2^D \cup Q) \neq \emptyset$ then $\{(a, b), (b, a)\} \cap P_2 \neq \emptyset$. Given $(b, a) \in P_1$, this contradicts the assumption that $(P_1, P_2) \in \mathcal{P}_{RSM}$. If instead $(a, b) \in H$, then by construction one and only one of $(a, b) \in H_1 \subset P_1$, $(a, b) \in H_2 \subset P_2$, or $(b, a) \in H_2 \subset P_2$. Given $(b, a) \in P_1$, any of these possibilities contradicts the assumption that $(P_1, P_2) \in \mathcal{P}_{RSM}$. Therefore, $c(D_i) = a$ implies $(b, a) \notin P_1$ for any $b \in D_i$ so that $a \in \max(D_i; P_1)$.

Now suppose there exists some $b \in \max(D_i; P_1)$ such that bP_2a. Again, this would imply that $a \notin c_{(P_1, P_2)}(D_i)$. Since $\{a, b\} \subset D_i$ and $c(D_i) = a$, part (ii) implies that dP_1b for some $d \in D_i$. But, this contradicts the claim that $b \in \max(D_i; P_1)$ and establishes that $a \in c_{(P_1, P_2)}(D_i)$. ∎

Proof of Remark 1. **(i)** By definition, $P_1^* \equiv \overline{P_2^c} = P^c \setminus P_2^c$. Thus, $W = P_1^*$ by definition. By Proposition 2, $P_1 \subseteq P_1^*$ which establishes the desired result. **(ii)** First, suppose $xP_2^c y$. By definition, there is some $A \in D^W$ such that $c(A) = y$ and $x \in A$. So, $\neg(xWy)$. Next, suppose xWy. By definition, there is no $A \in D^W = 2^X \setminus \emptyset$ such that $c(A) = y$ and $x \in A$. So, $\neg(xP_2^c y)$. ∎

Proof of Remark 2. To simplify, suppose $c(A) = x$ and $c(A \cup \{d\}) = y \neq x$.

(i) ($\Rightarrow$) If $y = d$, then $c(x, d) = d$. Otherwise, $c(x, d) = x = c(A)$ implies $c(A \cup \{d\}) = x$ by Expansion which, in turn, contradicts the assumption that $c(A \cup \{d\}) \neq x$. Thus, $c(x, d) = d$ which, by definition, gives the desired result that $dP^c x$. ($\Leftarrow$) By definition of RSM-superiority, $c(x, y) = y$. If $y \neq d$, then $c(x, y) = y = c(A \cup \{d\})$ and $c(A) = x$ violate WWARP (since $\{x, y\} \subset A \subset A \cup \{d\}$). So, $y = d$ as required.

(ii) ($\Rightarrow$) By Proposition 1(ii), $c(A) = x$ and $c(A \cup \{d\}) = y$ for $y \in A$ imply $xP_2^c y$. ($\Leftarrow$) By way of contradiction, suppose that $y = d$. Then, by definition of P_2-superiority, $c(x, y) = x$. Since $c(x, y) = x = c(A)$, Expansion implies $c(A \cup \{d\}) = x$ which contradicts the assumption that $c(A \cup \{d\}) \neq x$ and establishes the result. ∎

Proof of Remark 3. Suppose c is RSM-representable and $c(A) = x$.

(i) Suppose $c^{(e,x)}(A) = y \neq x$ for some $e \in A \setminus \{y\}$. There are four cases to consider: (a) $xP_1 y$; (b) $xP_2 y$; (c) $yP_1 x$; and, (d) $yP_2 x$. Case (a) contradicts $c^{(e,x)}(A) = y$, case (c) contradicts $c(A) = x$, and case (d) contradicts the joint choices $c(A) = x$ and $c^{(e,x)}(A) = y$ (since $c^{(e,x)}(A) = y$ implies that y must survive $P_1 \cup (e, x)$ on A and, hence, P_1 as well). Thus, $(x, y) \in P_2 \setminus (P_1 \cup P_1^{-1})$ or, equivalently, $(x, y) \in P_2^c \sqcup P^{in}$ by Proposition 2. This delivers the desired result.

(ii) Suppose $c_{(e,y)}(A) = y \neq x$ for some $e \in A \setminus \{x\}$. As in part (i), there are four cases. Cases (a)-(b) contradict $c_{(e,y)}(A) = y$ (for case (b), $c(A) = x$ implies that x must survive the first stage) while case (c) contradicts $c(A) = x$. Thus, $(y, x) \in P_2 \setminus (P_1 \cup P_1^{-1})$ or, equivalently $(y, x) \in P_2^c \sqcup P^{in}$ by Proposition 2. If $yP^{in}x$, then $c(A) \neq x$ by Proposition 3. So, $yP_2^c x$ as required. ∎