Competitive Bundling*

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Abstract

This paper presents a model of competitive bundling with an arbitrary number of firms. We find that the number of firms matters for the impacts of pure bundling relative to separate sales. The existing literature which focuses on duopoly suggests that bundling reduces consumer valuation heterogeneity and so tends to intensify price competition, harm firms and benefit consumers. However, we show that the opposite can happen beyond the duopoly case: under fairly general conditions, when the number of firms is above some threshold (which can be relatively small), bundling raises market prices, improves profits and harms consumers. Firms' incentives to bundle products and competitive mixed bundling are also investigated.

1 Introduction

Bundling is commonplace in the market. Sometimes firms sell their products in packages only and no individual products are available for purchase. This is called *pure bundling*. Relevant examples include CDs, newspapers, books, TV packages, and education programs. In these cases, firms do not sell songs, articles, chapters, TV channels or courses separately.[1] Sometimes firms sell both separate products and packages, and packages are usually sold at a discount relative to the components.

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[1] The situation, however, is changing in some cases with the development of online market. For instance, consumers nowadays can download single songs from iTune or Amazon. Some websites like www.CengageBrain.com are selling e-chapters of textbooks, and individual articles in many academic journals are also available for purchase.

This is called *mixed bundling*, and relevant examples include software suites, TV-internet-phone, season tickets, and value meals. In many cases, bundling occurs in markets where firms compete with each other.

The usual motivation for bundling is economies of scale in production and selling, or complementarity in consumption. For example, in the traditional market it is perhaps too costly to sell newspaper articles separately. But there are also less obvious but important reasons for bundling. For instance, pure bundling can reduce consumer valuation heterogeneity and facilitate firms extracting consumer surplus (Stigler, 1968). Mixed bundling can be a profitable price discrimination device by offering purchase options to screen consumers (Adams and Yellen, 1976).[2] The economics literature has extensively studied bundling in the monopoly case, and there is also research on competitive bundling. However, the existing works on competitive bundling often focus on the case with two firms and each selling two products (see, e.g., Matutes and Regibeau, 1988, for pure bundling, and Armstrong and Vickers, 2010, for mixed bundling). They use the two-dimensional Hotelling model where consumers distribute on a square and the two firms locate at two opposite corners.

This paper develops a model of competitive bundling with an arbitrary number of firms (and also an arbitrary number of products in the pure bundling part). With more than two multiproduct firms, it is not straightforward to imagine a spatial model with differentiation at the product level.[3] Therefore, we adopt the random utility framework developed in Perloff and Salop (1985) to model product differentiation. More specifically, a consumer's valuation for a firm's product is a random draw from some distribution, and its realization is independent across firms. This reflects, for example, the idea that firms sell products with different styles and consumers often have idiosyncratic tastes. This framework is flexible in accommodating any number of firms and products, and in the case with two firms and two products it can be rephrased as the two-dimensional Hotelling model.

Our research questions are similar to the existing literature: Do firms have incentives to bundle when they face competition? Compared to separate sales, how will bundling affect market prices, profits, and consumer welfare? Our main interest is to investigate whether the analysis and the main insights in the existing works can carry over when there are more firms in the market. In the pure bundling case,

[2] Bundling can also be used by multiproduct incumbent firms to deter the entry of potential competitors (Whinston, 1990, and Nalebuff, 2004).

[3] If there is no product differentiation at all, prices will settle at the marginal costs anyway and so there will be no meaningful scope for bundling. If differentiation is only at the firm level, consumers will one-stop shop even without bundling, which is not very realistic and also makes the study of competitive bundling less interesting.

we find that the number of firms qualitatively matters both for the incentive to bundle and for the impacts of bundling. The economic insights we learned from the duopoly case can be misleading. In the mixed bundling case, having more than two firms generates new challenges in solving the problem. Increasing the number of firms does not qualitatively change firms' incentives to bundle, but it makes the impacts of bundling on firms and consumers less ambiguous. When the number of firms is large, under general conditions firms suffer and consumers benefit from mixed bundling relative to separate sales.

The first part of the paper examines competitive pure bundling. Pure bundling has a broader interpretation than just a pricing strategy. With competition, it can be the outcome of product compatibility choice. Consider a system (e.g., a computer or a stereo system) that consists of several components. If firms make their components incompatible with each other (e.g., by not adopting a common standard, or making it impossible to disassemble the system), then consumers have to buy the whole system from a single firm and cannot mix and match to assemble a system by themselves.[4] Pure bundling can also be the outcome of one-stop shopping due to high shopping costs. For example, if it is too costly for a consumer to visit more than one grocery store, she will buy all the desired products from a single store. Then firms will compete like in a pure bundling situation.

The common idea about pure bundling is that it can reduce consumer valuation heterogeneity. This suggests that if firms bundle in an oligopolistic market, this should intensify price competition and so reduce market prices. The existing works on competitive pure bundling confirm this intuition, and they also suggest that this positive price effect usually outweighs the negative match quality effect (which is caused by the loss of opportunity to mix and match) such that bundling tends to benefit consumers.

However, we will show that this insight is not totally right if we go beyond the duopoly case. Under fairly general conditions, the results will be reversed (i.e., pure bundling raises market prices, benefits firms and harms consumers) when the number of firms is above some threshold (which can be small). The intuition is more transparent to understand when there are a large number of firms. In that case, a firm's marginal consumers (who are indifferent between its product and the best product from its competitors) should have a high valuation for its product almost for

[4]This is actually the leading interpretation taken by early works on competitive pure bundling (e.g., Matutes and Regibeau, 1988, and Economides, 1989). One advantage of adopting this interpretation is that it can naturally justify the assumption that consumers do not buy more than one bundle. If pure bundling is interpreted as a pricing strategy, high production costs are usually needed to justify this assumption as we will discuss more in Section 2.

sure (in other words, they tend to position on the right tail of the valuation density function). With bundling the density function becomes more concentrated and the right tail becomes thinner. This means that compared to the case of separate sales, there are fewer marginal consumers and so the demand becomes less elastic. This induces firms to raise their prices. In terms of consumer welfare, even if bundling reduces market prices, the loss of match quality can become significant when the number of firms increases. Therefore, bundling can harm consumer welfare when the number of firms is relatively large even if bundling reduces market prices.

We also study firms' incentive to bundle. When firms can choose between separate sales and pure bundling, it is always a Nash equilibrium that all firms bundle. (This is simply because if one firm unilaterally unbundles, the market situation does not change at all.) In the duopoly case, we further show that this is the unique equilibrium. However, when the number of firms is above some threshold, separate sales can be an equilibrium outcome as well.

The second part of the paper studies competitive mixed bundling. In this part, we consider an arbitrary number of firms but for tractability we focus on the two-product case. Then a mixed bundling strategy specifies a pair of stand-alone prices for each individual product and a joint-purchase discount. We first show that with any number of firms, starting from separate sales, each firm has a strict incentive to introduce mixed bundling (i.e., offer a joint-purchase discount). Therefore, if mixed bundling is a possibility and doing so is costless, separate sales can never be an equilibrium outcome. Unlike the pure bundling part, solving the pricing game with mixed bundling is not a simple task. The existing analysis of the duopoly case cannot be easily extended to our general model. Despite the complication of the problem, we are able to characterize the equilibrium conditions in a concise way, and we can also show that if the equilibrium joint-purchase discount is small (which is the case, for example, when the number of firms is large), both the stand-alone price and the joint-purchase discount have simple approximations. For example, in the case with a large number of firms, if the production cost is zero, the joint-purchase discount will be approximately half of the stand-along price (i.e., 50% off for the second product). In terms of the welfare impacts of mixed bundling relative to separate sales, it is ambiguous in the duopoly case. But in the limit case with a large number of firms, mixed bundling always benefits consumers and harms firms. [More investigation is needed in this part.]

Related literature.

- Monopoly bundling: Stigler (1968); Adams and Yellen (1976), McAfee, McMillan, and Whinston (1989), Chu, Leslie, and Sorensen (2011), Chen and Riordan

(2013); Schmalensee (1984), Bakos and Brynjolfsson (1999), Fang and Norman (2006)

- Competitive pure bundling: Matutes and Regibeau (1988), Economides (1989), Nalebuff (2000), Hurkens, Joen, and Menicucci (2013)
- Competitive mixed bundling: Matutes and Regibeau (1992), Anderson and Leruth (1993), Thanassoulis (2007), Armstrong and Vickers (2010)
- Inter-firm bundling: Gans and King (2006), Armstrong (2013)
- Bundling and entry deterrence: Whinston (1990), Nalebuff (2004)
- Bundling in auctions: Palfrey (1983), Chakraborty (1999)
- Recent empirical IO papers on bundling: Crawford and Yurukoglu (2012), Ho, Ho, and Mortimer (2012)

The rest of the paper is organized as follows: Section 2 introduces the model, and the benchmark of separate sales is presented in Section 3. Section 4 analyzes the regime of pure bundling and investigates its impacts relative to separate sales. Firms' incentives to bundle are also studied. Section 5 examine similar questions for mixed bundling. We conclude in Section 6, and all omitted proofs are presented in the Appendix.

2 The Model

Consider a market where consumers need to buy $m \geq 2$ products. (They can be m independent products, or m components of a system, depending on what interpretation we will take below for bundling.) There are $n \geq 2$ firms, and each firm supplies all the m products. The unit production cost of any product is normalized to zero (so we can regard the price below as the markup). Each product is horizontally differentiated across firms. We adopt the random utility framework in Perloff and Salop (1985) to model product differentiation. Let x_i^j denote the match utility of firm j's product i. For simplicity, let us consider a setting with symmetric firms and products: x_i^j distributes according to a common cdf F with support $[\underline{x}, \overline{x}]$ and is realized independently across firms, products, and consumers.[5] Let f be the pdf

[5]For simplicity, we have assumed away possible differentiation at the firm level. This can be included, for example, by assuming that a consumer's valuation for firm j's product i is $u^j + x_i^j$, where u^j is also a random variable and it is i.i.d. across firms and consumers but has the same realization for all products in the same firm.

of x_i^j and suppose it is continuous. Suppose consumers have unit demand for each product, and the measure of consumers is normalized to one. In either regime below, we assume that firms choose their prices simultaneously and then consumers make their purchase decisions after observing all prices and match utilities. If a consumer consumes a bundle of m products with match utilities $(x_1, \cdots, x_m)$ and makes a total payment T, she derives surplus $\sum_{i=1}^{m} x_i - T$.

The space of pricing strategies differs across the regimes we are going to investigate: if firm j sells its products separately, it chooses a price vector $(p_1^j, \cdots, p_m^j)$; if it sells its products in a package only, it chooses a bundle price P^j; if it adopts the mixed bundling strategy, then it needs to specify prices P_s^j for all possible subsets s of its m products. (If $m = 2$, firm j's mixed bundling strategy can be framed as a pair of stand-alone prices (ρ_1^j, ρ_2^j) together with a joint-purchase discount δ^j. We will focus on this two-product case in the regime of mixed bundling.) In all the regimes where firms adopt the same selling strategy, we will focus on symmetric pricing equilibrium.

As often assumed in the literature, the market is fully covered in equilibrium. That is, each consumer buys all m products. This will be the case if consumers do not have outside options, or on top of the above match utilities, consumers have a sufficiently high basic valuation for each product (or if $\underline{x}$ is large enough). Alternatively, we can consider a situation where the m products are essential components of a system for which consumers have a high basic valuation. We will relax this assumption in section 4.5.1 and argue that the basic insights about the impacts of pure bundling remain unchanged qualitatively. (However, in the mixed bundling part this assumption is indispensable for tractability.)

In the regime of pure bundling, two additional assumptions are made. First, consumers do not buy more than one bundle. This can be justified if the bundle price is sufficiently high (e.g., due to high production costs) relative to the match utility difference across firms. (If the cost is c for each product, a sufficient condition will be $c > \overline{x} - \underline{x}$.) In addition, if we interpret pure bundling as an outcome of product incompatibility or high shopping costs as we discussed in the introduction, then this assumption is naturally satisfied. Second, when $m \geq 3$ we assume that each firm either bundles all its products or not at all, and there is no finer bundling strategy (e.g., bundle product 1 and 2 but sell product 3 separately) available. (This issue does not arise when $m = 2$.) This assumption excludes the possible situations where firms bundle in asymmetric ways, and the pricing games in those situations are hard to analyze.

3 Separate Sales: Revisit Perloff-Salop Model

We first study the benchmark with separate sales. Firms then compete on each product separately, and so the market for each product is a Perloff-Salop model. Consider the market for product i, and let p be the (symmetric) equilibrium price.[6] Suppose firm j deviates and charges p', while other firms stick to the equilibrium price p. Then the demand for firm j's product i is

$$q(p') = \Pr[x_i^j - p' > \max_{k \neq j}\{x_i^k - p\}] = \int_{\underline{x}}^{\overline{x}} [1 - F(x - p + p')] dF(x)^{n-1} ,$$

where $F(x)^{n-1}$ is the cdf of the match utility of the best product i among the $n-1$ competitors. (In the following, whenever there is no confusion, we will suppress the integral limits $\overline{x}$ and $\underline{x}$.) Firm j's profit from product i is $p'q(p')$, and one can check that the first-order condition for p to be the equilibrium price is

$$\frac{1}{p} = n \int f(x) dF(x)^{n-1} . \quad (1)$$

This first-order condition is also sufficient for defining the equilibrium price if f is logconcave (see, e.g., Caplin and Nalebuff, 1991). A simple observation is that given the assumption of full market coverage, shifting the support of the match utility does not affect the demand function or the equilibrium price.

Let us first study a comparative static question: how does the equilibrium price vary with the number of firms? The equilibrium condition for p can be rewritten as

$$p = \frac{q(p)}{|q'(p)|} = \frac{1/n}{\int f(x) dF(x)^{n-1}} .$$

The equilibrium demand $q(p) = 1/n$ must decrease with n. While the equilibrium demand slope $|q'(p)| = \int f(x) dF(x)^{n-1}$ may increase or decrease with n, depending on the shape of f. For example, if the density function f is increasing, $|q'(p)|$ increases with n. Then p must decrease with n. While if f is decreasing, then $|q'(p)|$ decreases with n, which works against the demand size effect. But as long as $|q'(p)|$ does not decrease with n at a speed faster than $1/n$, the demand elasticity (at any price p) increases with n and so the equilibrium price decreases with n. The following result reports a sufficient condition for that.

Lemma 1 *Suppose $1 - F$ is logconcave (which is implied by logconcave f). Then p decreases with n. Moreover,* $\lim_{n \to \infty} p = 0$ *if and only if* $\lim_{x \to \overline{x}} \frac{f(x)}{1-F(x)} = \infty$.

[6] In the duopoly case, there is no asymmetric pricing equilibrium (see, e.g., section 4 in Perloff and Salop, 1985). Beyond the duopoly case, whether there is asymmetric equilibrium or not is still an unsolved question in general.

Proof. Let $x_{(2)}$ be the second highest order statistics of $\{x_1, \cdots, x_n\}$. Let $F_{(2)}$ and $f_{(2)}$ be its cdf and pdf, respectively. Using

$$f_{(2)}(x) = n(n-1)(1-F(x))F(x)^{n-2}f(x) \ ,$$

we can rewrite (1) as

$$\frac{1}{p} = \int \frac{f(x)}{1-F(x)} dF_{(2)}(x) \ . \tag{2}$$

Since $x_{(2)}$ increases with n in the sense of first order stochastic dominance, a sufficient condition for p to be decreasing in n is that the hazard rate $f/(1-F)$ is increasing (or equivalently, $1-F$ is logconcave). The limit result as $n \to \infty$ follows from (2) immediately. ■

This result improves the relevant discussion in Perloff and Salop (1985) (e.g., they did not derive a clear condition for p to decrease with n). One special case is the exponential distribution which has a constant hazard rate $f/(1-F)$. In that case price is independent of the number of firms. Nevertheless, the logconcavity of $1-F$ is not a necessary condition. That is, even if $1-F$ is not logconcave, price can still decrease with n (if the first-order condition is still sufficient for defining the equilibrium price). One such example is the power distribution with $F(x) = x^{\alpha}$ with $\alpha \in (\frac{1}{n}, 1)$. (In this example, $1-F$ is neither logconcave nor logconvex, and $p = \frac{n\alpha - 1}{n(n-1)\alpha^2}$ decreases with n.)

The condition for $\lim_{n\to\infty} p = 0$ is satisfied if $f(\overline{x}) > 0$. But it can be violated if $f(\overline{x}) = 0$. One example is the standard Gumbel distribution with $F(x) = e^{-e^{-x}}$. Both f and $1-F$ are strictly logconcave in this example. But one can check that $p = \frac{n}{n-1}$, which decreases with n and approaches to 1 in the limit.[7]

Another comparative static question is: if the distribution of match utility becomes more concentrated as illustrated in Figure 1 below (where the density function becomes more "peaked" from the solid one to the dashed one), how will the equilibrium price change? Intuitively, a more peaked density as below means more homogenous consumer valuations (or the product becomes less differentiated across firms). This should induce the market price to decline.

[7]See Gabaix et al. (2013) for a careful and extensive study about the asymptotic behavior of price in this type of random utility models.

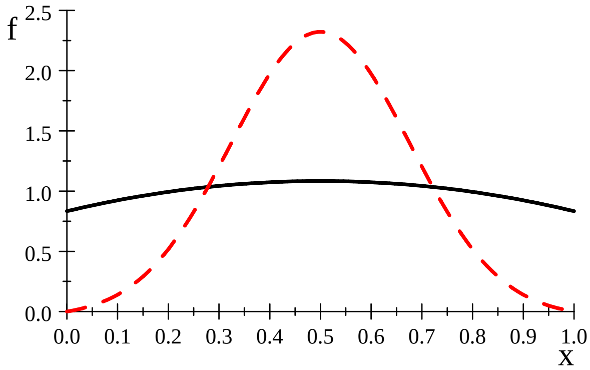


Figure 1: An example where the density function becomes more "peaked"

However, this intuition is not completely right. Consider two densities f_0 and f_1 with the same support, and suppose f_0 is the solid density in Figure 1 and f_1 is the dashed one. Let F_0 and F_1 be the corresponding cdf's. Then the associated equilibrium prices are given by

$$\frac{1}{p_0} = n \int_{\underline{x}}^{\overline{x}} f_0(x) dF_0(x)^{n-1}; \quad \frac{1}{p_1} = n \int_{\underline{x}}^{\overline{x}} f_1(x) dF_1(x)^{n-1} \ , \tag{3}$$

respectively. Each firm is competing with the best product among the $n-1$ competitors. So the measure of a firm's marginal consumers (who are indifferent between its product and the best product among its competitors) in each case is given by the integral terms. They are actually the equilibrium demand slopes. Now consider the case with a large n. Then the match utility of the best product among the competitors is close to $\overline{x}$ almost for sure. So the measure of marginal consumers is approximately determined by $f_i(\overline{x})$, $i = 0, 1$. (More precisely, this requires f_i be continuous and uniformly bounded.) Therefore, if $f_0(\overline{x}) > f_1(\overline{x})$ and n is large, there are actually more marginal consumers in the case of f_0 than the case of f_1. This implies $p_0 < p_1$, even though the density f_1 is more peaked. This argument suggests that when the number of firms is large, the tail behavior, instead of the peakedness, of the density function determines the equilibrium price. The following lemma states this result formally:

Lemma 2 *Consider two density functions f_0 and f_1. Suppose they have the same support $[\underline{x}, \overline{x}]$ and they are continuous and uniformly bounded. If $f_0(\overline{x}) > f_1(\overline{x})$, then there exists $\hat{n}$ such that for $n > \hat{n}$ the associated equilibrium prices as defined in (3) satisfy $p_0 < p_1$.*

This result suggests that in the Perloff-Salop model a mean-preserving contraction of the match utility distribution does not necessarily decrease market prices. (Notice that this is clearly not true if the distribution completely degenerates at the mean (in which case f_1 is not bounded at the mean).[8]) As we will see, this observation is crucial for the price comparison result in next section on pure bundling.

4 Pure Bundling

4.1 Equilibrium prices

Now consider the regime where all firms adopt the pure bundling strategy. Denote by $X^j \equiv \sum_{i=1}^m x_i^j$ the match utility of firm j's bundle. Then if firm j charges a bundle price P' while other firms charge the equilibrium price P, the demand for j's bundle is

$$Q(P') = \Pr[X^j - P' > \max_{k \neq j}\{X^k - P\}] = \Pr[\frac{X^j}{m} - \frac{P'}{m} > \max_{k \neq j}\{\frac{X^k}{m} - \frac{P}{m}\}] \ .$$

Notice that X^j/m has the same support as x_i^j, and it is a mean-preserved contraction of x_i^j. Let $\bar{F}$ and $\bar{f}$ be the cdf and pdf of X^j/m, respectively. (In particular, $\bar{f}(\overline{x}) = 0$.) Then the same logic as in the regime of separate sales implies that the first-order condition for P to be the equilibrium bundle price is

$$\frac{1}{P/m} = n \int \bar{f}(x) d\bar{F}(x)^{n-1} \ . \tag{4}$$

Notice that $\bar{f}$ is logconcave if f is logconcave (see, e.g., Miravete, 2002). So the assumption of logconcave f also implies that the first-order condition above is sufficient for defining the equilibrium bundle price. Also notice that $1 - \bar{F}$ is logconcave if $1 - F$ is logconcave. Hence, similar results as in Lemma 1 also hold here.

Lemma 3 *Suppose* $1 - F$ *is logconcave (which is implied by logconcave* f*). Then the bundle price* P *defined in (1) decreases with* n*. Moreover,* $\lim_{n \to \infty} P = 0$ *if and only if* $\lim_{x \to \overline{x}} \frac{\bar{f}(x)}{1 - \bar{F}(x)} = \infty$.

4.2 Compare prices and profits

From (1) and (4), we can see that the comparison between separate sales and pure bundling is just comparing two Perloff-Salop models with different match utility distributions F and $\bar{F}$ (where the latter is a particular mean-preserving contraction of the former). Bundling leads to lower prices if $P/m < p$.

[8] Also notice that Lemma 2 does not necessarily apply to the case where $f_0(\overline{x}) = f_1(\overline{x})$ but $f_0(x) > f_1(x)$ for x close to $\overline{x}$.

By changing the integral variable from x to $t = F(x)$, we can rewrite (1) and (4), respectively, as

$$\frac{1}{p} = n(n-1)\int_0^1 l(t)t^{n-2}dt \ , \quad \frac{1}{P/m} = n(n-1)\int_0^1 \bar{l}(t)t^{n-2}dt \ , \tag{5}$$

where $l(t) \equiv f(F^{-1}(t))$ and $\bar{l}(t) \equiv \bar{f}(\bar{F}^{-1}(t))$ are two quantile density functions. Then

$$\frac{P}{m} < p \Leftrightarrow \int_0^1 [l(t) - \bar{l}(t)]t^{n-2}dt < 0 \ . \tag{6}$$

Given full market coverage, profit comparison is the same as price comparison.

Proposition 1 *(i) Suppose f is logconcave and $n = 2$. Then bundling reduces market prices and profits for any $m \geq 2$.*
(ii) For a fixed $m < \infty$, if f is uniformly bounded and $f(\bar{x}) > 0$, there exists $\hat{n}$ such that bundling increases market prices and profits for $n > \hat{n}$. If f is further logconcave and $l(t)$ and $\bar{l}(t)$ cross each other at most twice, then bundling increases prices and profits if and only if $n > \hat{n}$.
(iii) For a fixed $n < \infty$, $\lim_{m\to\infty} P/m = 0$ and so there exists $\hat{m}$ such that bundling reduces market prices and profits for $m > \hat{m}$.

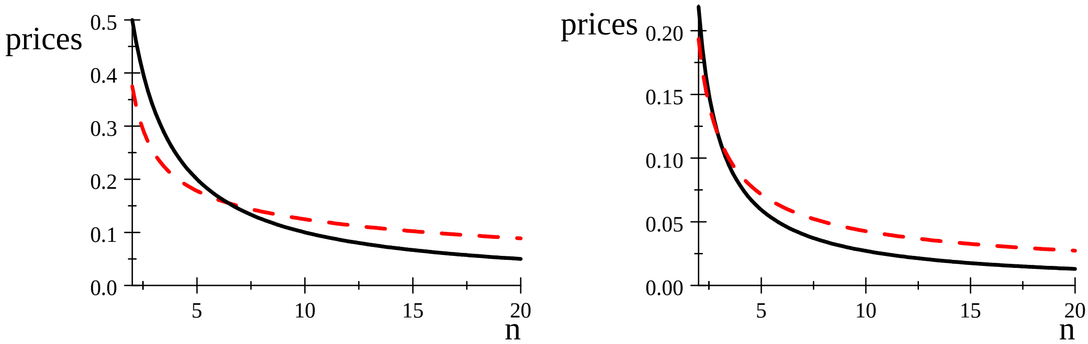
Result (i) generalizes the observation in the existing literature about how pure bundling affects market prices. (Hurkens, Jeon, and Menicucci, 2013, prove a similar result in the two-product case.) In the duopoly case, $\frac{1}{p} = 2\int f(x)^2dx$ and $\frac{1}{P/2} = 2\int \bar{f}(x)^2dx$. The integral in each formula is the measures of marginal consumers. Intuitively, in the duopoly case, the average position of marginal consumers is at the mean, and $\bar{f}$ is more peaked at the mean than f. So there are more marginal consumers in the case of $\bar{f}$. This induces firms in the bundling case to charge lower prices. Result (iii) simply follows from the *law of large numbers*. For a fixed $n < \infty$, X^j/m converges to the mean of x_i^j as $m \to \infty$. In other words, the average valuation for the bundle becomes homogeneous both across consumers and firms. So P/m will converge to zero.

Result (ii) that bundling can soften price competition is more surprising. The intuition just follows the discussion of Lemma 2 given $f(\overline{x}) > \bar{f}(\overline{x}) = 0$.[9,10] To illustrate, consider two examples which satisfy all the conditions in result (ii). In

[9]Under the logconcavity condition, both P/m and p converge to zero as $n \to \infty$. But p converges much faster than P/m if $f(\overline{x}) > 0$, and in that case $\lim_{n\to\infty} \frac{P/m}{p} = \frac{f(\overline{x})}{\bar{f}(\overline{x})} = \infty$.

[10]The logic in result (ii) also implies that the per-product bundle price P/m does not always decrease with m when $f(\overline{x}) > 0$. It also depends on the number of firms. For example, in the uniform example, the per-product bundle price is greater when $m = 3$ than when $m = 2$ if n is above about 32.

the uniform distribution example with $f(x) = 1$, $P/m < p$ when $n \leq 6$ and $P/m > p$ when $n \geq 7$. Figure 2(a) below describes how both prices vary with n (where the solid curve is p and the dashed one is P/m). In the example with an increasing density $f(x) = 4x^3$, as described in Figure 2(b) below $P/m < p$ only when $n = 2$ and $P/m > p$ whenever $n \geq 3$. (This second example also suggests that at least for $m = 2$, beyond the duopoly case we can always find distributions under which bundling increases market prices.)[11]



(a) $f(x) = 1$ (b) $f(x) = 4x^3$

Figure 2: Price comparison with $m = 2$

To illustrate the importance of the assumption $f(\overline{x}) > 0$ for result (ii). Let us consider the following example of normal distribution with $\lim_{x\to\infty} f(x) = 0$.[12] [More general investigation is needed for the case with $f(\overline{x}) = 0$.]

Example of normal distribution. As we have observed before, shifting the support of the match utility distribution does not affect the equilibrium price. So let us suppose $x_i^j \sim \mathcal{N}(0, \sigma^2)$. Then $X^j/m \sim \mathcal{N}(0, \sigma^2/m)$, and so X^j/m is exactly the random variable $x_i^j/\sqrt{m}$. Hence, the demand function in the bundling case is

$$Q(P') = \Pr[\frac{X^j}{m} - \frac{P'}{m} > \max_{k\neq j}\{\frac{X^k}{m} - \frac{P}{m}\}] = \Pr[x_i^j - \frac{P'}{\sqrt{m}} > \max_{k\neq j}\{x_i^k - \frac{P}{\sqrt{m}}\}] \ .$$

Therefore,

$$\frac{P}{\sqrt{m}} = p \Rightarrow \frac{P}{m} = \frac{p}{\sqrt{m}} < p \ . \tag{7}$$

[11] The result that bundling can increase market prices has an implication for collusion: firms may collude on pure bundling even if pure bundling is not an equilibrium choice outcome.

[12] In any example with $\overline{x} - \underline{x} = \infty$, we should keep in mind that if bundling is not caused by product incompatibility or high shopping costs and it is just a pricing strategy, then the assumption that consumers do not buy more than one bundle may not be justified by invoking a high production cost.

That is, with a normal distribution, bundling always reduces market prices (and so profits) regardless of n and m.

4.3 Compare consumer surplus and total welfare

With full market coverage, consumer payment is a pure transfer and so total welfare (which is the sum of firm profits and consumer surplus) only reflects match quality between consumers and products. In symmetric equilibrium, price is the same across products and so it does not distort consumer choices. Since bundling eliminates the opportunity to mix and match for consumers, it must reduce match quality and so total welfare.

However, the comparison of consumer surplus can be more complicated. If pure bundling increases market prices, it must harm consumers. From Proposition 1, we know this is the case at least when $f(\overline{x}) > 0$ and n is large. The trickier situation is when pure bundling decreases market prices. Then there is a trade-off between the negative match quality effect and the positive price effect. Our analysis below suggests that the positive price effect dominates (and so consumers benefit from bundling) only if the number of firms is relatively small.

Denote by v a consumer's expected surplus in the regime of separate sales. Then the per-product surplus is

$$\frac{v}{m} = \mathbb{E}\left[\max_j \{x_i^j\}\right] - p \ . \tag{8}$$

Denote by V a consumer's expected surplus in the regime of pure bundling. Then the per-product surplus is

$$\frac{V}{m} = \mathbb{E}\left[\max_j \left\{\frac{X^j}{m}\right\}\right] - \frac{P}{m} \ . \tag{9}$$

Analytical progress can be made only in the limit case with $m \to \infty$. Suppose μ is the mean of x_i^j. Then we already knew that $\lim_{m\to\infty} X^j/m = \mu$ and $\lim_{m\to\infty} P/m = 0$. So

$$\lim_{m\to\infty} \frac{V}{m} = \mu \ .$$

Therefore, when $m \to \infty$, pure bundling improves consumer welfare if and only if

$$\mathbb{E}\left[\max_j \{x_i^j\}\right] - \mu < p \ . \tag{10}$$

With separate sales, consumers enjoy better matched goods (which is reflected by the left-hand side), but they also pay more (which is reflected by the right-hand side). It is clear that the left-hand side increases with n, while p decreases with n under the logconcavity condition. In the proof of Proposition 2 below, we show that (10) holds for $n = 2$ but fails for a sufficiently large n (the latter is obvious if $\lim_{n\to\infty} p = 0$). Therefore, pure bundling improves consumer welfare if and only if the number of firms is below some threshold.

We summarize the results concerning consumer surplus comparison in the following proposition:

Proposition 2 *(i) For fixed $m < \infty$, if f is uniformly bounded and $f(\bar{x}) > 0$, there exists $\hat{n}$ such that bundling harms consumers if $n > \hat{n}$.*
(ii) Suppose $m \to \infty$ and f is logconcave. Then there exists n^* such that pure bundling benefits consumers if and only if $n \leq n^*$.

Notice that the threshold n^* in result (ii) can be very small. For example, in the uniform distribution case with $f(x) = 1$, condition (10) simplifies to $n^2 - 3n - 2 < 0$. This holds for $n = 2, 3$ but fails for $n \geq 4$, so $n^* = 3$. For a finite m, we have not got any analytical results beyond result (i). [More investigation is perhaps needed here.] However, numerical calculation in the uniform distribution case suggests a similar threshold result as in the case of $m \to \infty$. Figure 3 below describes how consumer surplus varies with n in the uniform case with $m = 2$ (where the dashed curve is for the bundling case), and the threshold is again $n^* = 3$. Moreover, notice that in this example the negative effect when n is relatively large can be much more significant than the positive effect when n is small.

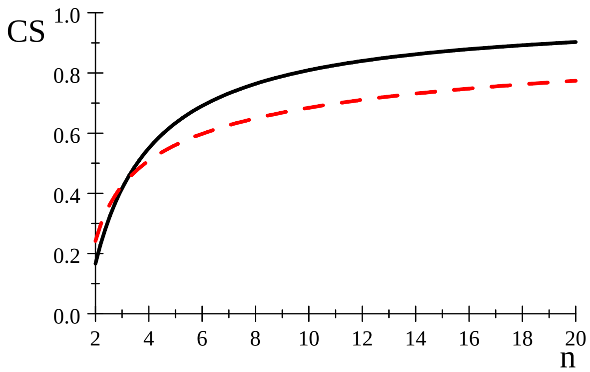


Figure 3: Consumer surplus comparison with uniform distribution and $m = 2$

In the case of normal distribution, we can even get analytical results for any m. A similar threshold result holds and the threshold is independent of m.

Example of normal distribution. Using the definitions of consumer surplus in (8) and (9) and the result (7), we can see that pure bundling improves consumer surplus if and only if

$$p[1 - \frac{1}{\sqrt{m}}] > \mathbb{E}\left[\max_j \{x_i^j\}\right] - \mathbb{E}\left[\max_j \left\{\frac{X^j}{m}\right\}\right] \quad . \tag{11}$$

Consumers pay less in the bundling regime (which is captured by the left-hand side), but they also end up with consuming less well matched products (which is captured by the right-hand side). In the Appendix, we show that

$$\mathbb{E}\left[\max_j\{x_i^j\}\right]=\frac{1}{p}\ , \quad \mathbb{E}\left[\max_j\left\{\frac{X^j}{m}\right\}\right]=\frac{1}{\sqrt{mp}}\ . \tag{12}$$

So the loss of match quality caused by pure bundling equals

$$[1-\frac{1}{\sqrt{m}}]\frac{1}{p}\ .$$

Then (11) simplifies to $p^2>1$. One can check that this holds only for $n=2,3$, and so the threshold is again $n^*=3$.

4.4 Incentive to bundle

We now turn to firms' incentive to bundle. Suppose firms can choose only between separate sales and pure bundling. Then starting from separate sales, does each firm have a unilateral incentive to bundle? Is it an equilibrium outcome that all firms adopt the pure bundling strategy? We will investigate these questions mainly in the case where firms make bundling and pricing decisions simultaneously. We will also briefly discuss the sequential choice case where firms make bundling choices before they engage in price competition.

Simultaneous choices. We start with the relatively simple case with two firms only. In this duopoly case, if one firm chooses to bundle, the situation will be like both firms bundle.

Proposition 3 *Suppose $n=2$, and $P\neq mp$, where p and P are defined in (1) and (4), respectively. Suppose firms make bundling and pricing decisions simultaneously. Then the unique (pure-strategy) Nash equilibrium is that both firms choose to bundle and charge a bundle price P.*

Proof. First of all, both firms bundling is an equilibrium. This is simply because if a firm unilaterally unbundles, the market situation does not change.

Second, it is not an equilibrium outcome that both firms adopt separate sales. Consider the hypothetical equilibrium where both firms sell their products separately at price p. Now suppose firm j unilaterally bundles and sets a bundle price $mp-m\varepsilon$, where ε is a small positive number. The negative (first-order) effect of this deviation on firm j's profits is $\frac{m}{2}\varepsilon$. (Half of the consumers buy from firm j when $\varepsilon=0$, and now they pay $m\varepsilon$ less.) The demand for firm j's bundle is

$$\Pr(X^j+m\varepsilon>X^k)=\int_{\underline{x}}^{\overline{x}}\bar{F}(x+\varepsilon)d\bar{F}(x)\ ,$$

where $k \neq j$. So the positive effect of the deviation on firm j's profit is

$$mp\varepsilon \int_{\underline{x}}^{\overline{x}} \bar{f}(x)^2 dx = \frac{mp}{P} \times \frac{m}{2}\varepsilon \ .$$

(The equality is because of (4) for $n = 2$.) Therefore, the deviation is profitable if $P < mp$. Similarly, one can show that if $P > mp$, unilateral bundling with a price $mp + m\varepsilon$ will be a profitable deviation.

Finally, there are no asymmetric equilibria where one firm bundles and the other does not. Consider a hypothetical equilibrium where firm j bundles and firm k does not. For consumers, this situation is the same as both firms bundle. So in equilibrium it must be the case that firm j offers a bundle price P defined in (4) with $n = 2$, and firm k offers individual prices $\{p_i\}_{i=1}^m$ such that $\sum_{i=1}^m p_i = P$. Suppose now firm j unbundles and offers prices $\{p_i - \varepsilon\}_{i=1}^m$, where ε is a small positive number. The negative (first-order) effect of this deviation on firm j's profit is $\frac{\varepsilon}{2}m$, and the positive effect is

$$\sum_{i=1}^{m} p_i \times \varepsilon \int_{\underline{x}}^{\overline{x}} f(x)^2 dx = \frac{P}{p} \times \frac{\varepsilon}{2} \ .$$

(The equality used (1) for $n = 2$ and $\sum_{i=1}^m p_i = P$.) Therefore, the proposed deviation is profitable if $P > mp$. Similarly, if $P < mp$, setting prices $\{p_i + \varepsilon\}_{i=1}^m$ will be a profitable deviation. ■

The situation is more complicated when there are more firms. But a simple observation is that it is still an equilibrium that all firms bundle. This is again because if one firm unilaterally unbundles, the market situation does not change at all. However, it is no longer clear whether separate sales is an equilibrium outcome or not. Given all other firms offer separate sales prices p, if one firm bundles and sets a bundle price mp, the demand for its bundle will be strictly less than $1/n$ when $n \geq 3$. Of course, the firm may be able to do better by adjusting prices in the same time, but it needs a more intricate analysis.

Starting from separate sales with price p, suppose firm j unilaterally chooses to bundle. Let $y_i \equiv \max_{k \neq j}\{x_i^k\}$ be the maximum match utility of product i among all other firms, and its cdf is $F(y)^{n-1}$. Then firm j is as if competing with a firm that supplies (on average) better products with bundle match utility $Y \equiv \sum_{i=1}^m y_i$. If firm j charges a bundle price mp, its demand will be $\Pr(X^j > Y) < \Pr(X^j > \max\{X^k\}_{k \neq j}) = 1/n$ when $n \geq 3$. If firm j reduces its bundle price to $m(p - \varepsilon)$ in the same time, its demand is

$$\Pr\left(\frac{X^j}{m} + \varepsilon > \frac{Y}{m}\right) \ . \tag{13}$$

Firm j has an incentive to deviate and bundle if there exists ε such that the deviation profit exceeds mp/n.

To make progress, let us consider the limit case with $m \to \infty$. According to *the law of large numbers*, we have

$$\lim_{m\to\infty} \frac{X^j}{m} = \mu; \quad \lim_{m\to\infty} \frac{Y}{m} = \mathbb{E}[y_i] \ .$$

Then (13) will take a step function form. If ε is greater than $\mathbb{E}[y_i] - \mu$, firm j will attract all consumers. Otherwise, it gets zero demand. Therefore, the optimal (per product) deviation price is $p - \{\mathbb{E}[y_i] - \mu\}$. This is greater than the profit p/n in the separate sales regime (and so firm j has an incentive to bundle) if and only if

$$\mathbb{E}[y_i] - \mu = \int_{\underline{x}}^{\overline{x}} \left[F(x) - F(x)^{n-1}\right] dx < p(1 - \frac{1}{n}) \ . \tag{14}$$

This is clearly true for $n = 2$ (as we have seen in Proposition 3). On the other hand, we have $\int_{\underline{x}}^{\overline{x}} F(x)dx > \lim_{n\to\infty} p$ as shown in the proof of Proposition 2. Then (14) must fail when n is sufficiently large. The following proposition reports the result:

Proposition 4 *Suppose $m \to \infty$ and p defined in (1) decreases with n (e.g, when f is logconcave). Then there exists $\tilde{n}$ such that separate sales is also an equilibrium outcome if and only if $n > \tilde{n}$.*

This result implies that when the number of firms is above some threshold, the bundling choice game has two symmetric equilibria. To illustrate the magnitude of $\tilde{n}$, let us consider the uniform distribution example. Then condition (14) becomes $n^2 - 4n + 2 < 0$, and it holds only for $n = 2$ and 3, so in this example $\tilde{n} = 3$. This is the same as the threshold n^* for consumer surplus comparison in result (ii) of Proposition 2. In other words, in this uniform example, separate sales is also an equilibrium outcome if and only if consumers prefer separate sales to pure bundling. So the market outcome (with a proper selection of equilibrium) can always favor consumers.[13]

Sequential choices. Now suppose that firms make their bundling choices first, and then engage in price competition after observing the bundling choice outcome. (This is usually the case when bundling is an outcome of product compatibility choice.) First of all, as in the case of simultaneous choices all firms bundling is always an equilibrium outcome. The issue of whether separate sales is an equilibrium outcome is more complicated. In the duopoly case, if f is logconcave, then bundling leads

[13] The same is true for the case of exponential distribution and normal distribution. However, there exists examples (e.g., $f(x) = 2(1-x)$) where $\tilde{n} \neq n^*$. But we have not found examples where $|\tilde{n} - n^*| \geq 2$.

to a lower price and profit as we have shown in Proposition 1. So no firm has a unilateral incentive to deviate from separate sales. That is, in the duopoly case separate sales is now a Nash equilibrium as well.

When there are more than two firms, it is complicated to solve the pricing game when one firm bundles and other firms do not. (See the details of how to derive the equilibrium conditions in the appendix.) No general analytical results are available, but in the uniform distribution case numerical simulations suggest that no firm has a unilateral incentive to bundle. For example, in the case with $n = 3$ and $m = 2$, if all three firms adopt separate sales, then each firm charges a price $p = \frac{1}{3}$ and each firm's (per-product) profit is $\frac{1}{9}$. But if one firm, say, firm 1, deviates and bundles, then in the asymmetric pricing game, firm 1 charges a bundle price $P \approx 0.513$ and the other two firms charge an individual price $p \approx 0.317$ for each product. Firm 1 has a slightly higher demand 0.343 but a lower (per-product) profit 0.088. (Each other firm has a lower (per-product) profit 0.104.)[14] Therefore, separate sales is again a Nash equilibrium. [More investigation is needed for possible asymmetric equilibria in both cases.]

4.5 Discussions

4.5.1 Without full market coverage

In this part, we relax the assumption of full market coverage. Then a subtle issue is whether the m products are independent products or perfect complements. This will affect the analysis in the benchmark of separate sales. (With full market coverage, this distinction does not matter.) If the m products are independent products, consumers decide whether to buy each product separately. While if the m products are perfect complements (e.g., they are essential components of a system), then whether to buy a product also depends on how well matched other products the consumer can find. In the following, we first consider the scenario of independent products for simplicity. [Add the case of perfect complements also.]

Suppose now x_i^j is the whole valuation for firm j's product i, and a consumer will buy a product or bundle only if the best offer in the market provides a positive surplus. Without loss of generality let $\underline{x} \leq 0$, and we also assume that x_i^j has a mean greater than the production cost which is normalized to zero.

[14]For larger n, the impact of firm 1's bundling strategy on its own price and profit is similar, but the impact on other firms's prices and profits can be different. For example, when $n = 4$, the separate sales price is $\frac{1}{4}$. But if firm 1 bundles, its bundle price is $P \approx 0.390$ and each other firm's price is $p \approx 0.262$ (which is higher than before). Firm 1's (per-product) profit drops from 0.0625 to 0.0456, while each other firm's (per-product) profit increases from 0.0625 to 0.067.

In the regime of separate sales, if firm j deviates and charges p' for its product i, then the demand for its product i is

$$q(p') = \Pr[x_i^j - p' > \max_{k\neq j}\{0, x_i^k - p\}] = \int_{p'}^{\overline{x}} F(x_i^j - p' + p)^{n-1} dF(x_i^j) \ .$$

One can check that the first-order condition for p to be the (symmetric) equilibrium price is

$$p = \frac{q(p)}{|q'(p)|} = \frac{[1 - F(p)^n]/n}{F(p)^{n-1}f(p) + \int_p^{\overline{x}} f(x)dF(x)^{n-1}} \ . \tag{15}$$

(If f is logconcave, this is also sufficient for defining the symmetric equilibrium price.) In equilibrium, the measure of consumers who leave the market without purchasing product i is $F(p)^n$ (i.e., the probability that each firm's product i has a match utility below the price p). Given the symmetry of firms, the demand for each firm's product i is then the numerator in (15). The demand slope in the denominator now has two parts: (i) The first term is the standard market exclusion effect: when all other firms' match utilities are below p (which occurs with probability $F(p)^{n-1}$), firm j will play as a monopoly, and so raising its price p by ε will exclude $\varepsilon f(p)$ consumers from the market. (ii) The second term is the same competition effect as in the case with full market coverage (up to the adjustment that a marginal consumer's valuation must be greater than the price p).

Similarly, in the bundling case, the equilibrium per-product price P/m is determined by the first-order condition:

$$\frac{P}{m} = \frac{[1 - \bar{F}(P/m)^n]/n}{\bar{F}(P/m)^{n-1}\bar{f}(P/m) + \int_{P/m}^{\overline{x}} \bar{f}(x)d\bar{F}(x)^{n-1}} \ , \tag{16}$$

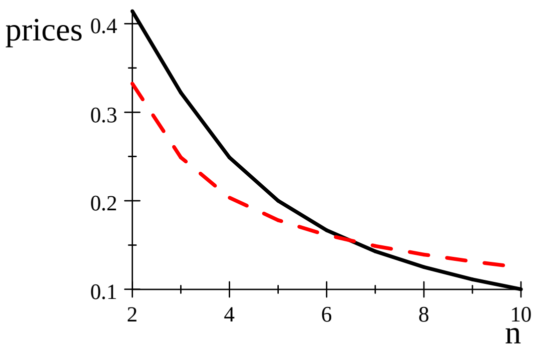
where $\bar{F}$ and $\bar{f}$ are the cdf and pdf of X^j/m as before.

Unlike the case with full market coverage, now the equilibrium price in each regime is implicitly determined in the first-order condition. To ensure the equilibrium existence, both (15) and (16) need to have a solution. The following result reports the required condition:

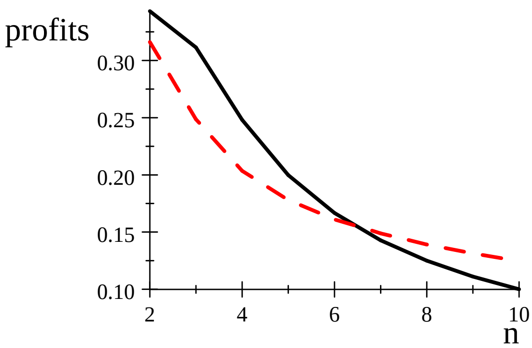
Lemma 4 *Suppose f is logconcave. Then (15) has a unique solution $p \in (0, p_M)$, where p_M is the monopoly price and solves $p_M = [1-F(p_M)]/f(p_M)$, and p decreases with n. Similar results hold for P/m defined in (16).*

A general investigation of the comparison between the two regimes is much harder. However, similar results hold when n is large or when m is large. For a fixed $n < \infty$, we still have $\lim_{m\to\infty} P/m = 0$ since X^j/m converges to the mean and the mean is positive. Hence, for a fixed n pure bundling must induce lower market prices when m is greater than some threshold. For a fixed $m < \infty$, if n is large, then the demand difference between the two numerators in (15) and (16) is negligible, and so is the exclusion effect difference in the denominators. So price comparison is again determined by the comparison of $f(\overline{x})$ and $\bar{f}(\overline{x})$. Thus, if $f(x)$ is uniformly bounded and $f(\overline{x}) > 0$, bundling again raises market prices and harms consumers when n is greater than some threshold.

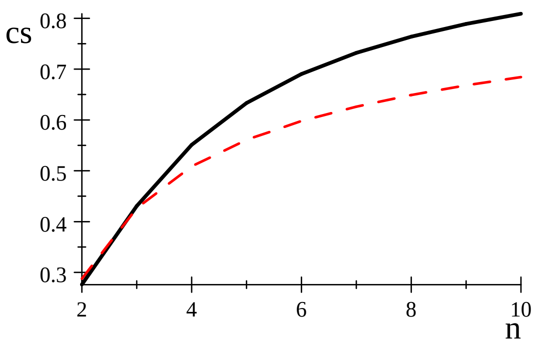
In the uniform example with $f(x) = 1$, numerical simulations described in Figure 4 below suggest that the impacts of pure bundling on market prices, profits, consumer surplus and total welfare are qualitatively similar as in the case with full market coverage. In the normal distribution case with $x_i^j \sim \mathcal{N}(0, \sigma^2)$, one can see that the previous argument for price comparison carries over and so pure bundling always reduces market prices. (But unlike the case with full market coverage, the argument does not work if the mean is not zero. The argument for consumer surplus comparison does not apply here.)



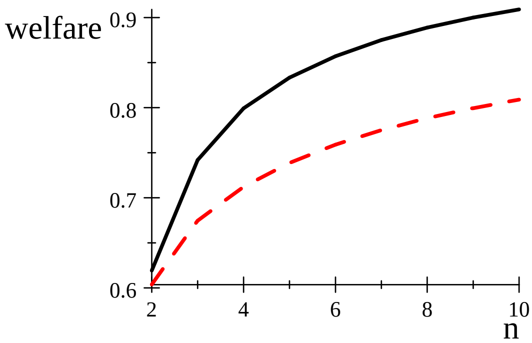
(a): Prices



(b) Profits



(c) Consumer surplus



(d) Total welfare

Figure 4: The impact of pure bundling—uniform example without full market coverage

4.5.2 Buy more than one bundle

Suppose now that bundling is not caused by product incompatibility or high shopping costs and it is simply a pricing strategy. Then if the production cost is relatively small, it will be possible that in the bundling regime the bundle price is low enough such that some consumers buy multiple bundles. For instance, in the uniform example with $m = 2$ which we investigated before, if the production cost is literally zero, the bundle price is actually below $\overline{x} - \underline{x} = 1$ and so some consumers will buy multiple bundles.

Given the possibility of buying more than one bundle, the situation will be actually similar to mixed bundling which we will study in next section. For example, consider the case with $m = 2$. If the bundle price is P, then a consumer faces two options: buy the best bundle and pay price P, or buy two bundles to mix and match and pay $2P$ (the unused products will be disposed freely). For consumers, this is the same as a mixed bundling situation with a stand-alone price P for each product and a joint-purchase discount P. Since analyzing mixed bundling with arbitrary number of firms is not a simple task, we will come back to this issue after we solve the mixed bundling game.

5 Mixed Bundling

In this section, we study competitive mixed bundling. For tractability, we focus on the case with two products and keep the assumption of full market coverage. If a firm adopts a mixed bundling strategy, it offers a pair of stand-alone prices (ρ_1, ρ_2) and a joint-purchase discount $\delta > 0$. (So if a consumer buys both products from this firm, she pays $\rho_1 + \rho_2 - \delta$.) In the following, we first show that starting from separate sales with price p defined in (1), each firm has a unilateral incentive to offer a positive joint-purchase discount. Second, we characterize the symmetric pricing equilibrium with mixed bundling and examine the impact of mixed bundling relative to separate sales.

5.1 Incentive to use mixed bundling

Let firm 1 be the firm in question. As before, let $y_i = \max_{j\geq 2}\{x_i^j\}$ be the match utility of the best product i among all other firms and its cdf is $F(y)^{n-1}$. Suppose all other firms are selling their products separately at price p defined in (1). Does firm 1 have a strict incentive to introduce a joint-purchase discount $\delta > 0$?

Suppose firm 1 offers the same stand-alone price p as before but now introduces a small joint purchase discount $\delta > 0$. How will this small deviation affect its profit?

The negative (first-order) effect is that firm 1 earns δ less from those consumers who buy both products from it. In the regime of separate sales the number of such consumers is $1/n^2$, so this loss is δ/n^2.

The positive effect is that more consumers will buy both products from firm 1. (Some consumers who originally bought only one product from firm 1 will now buy both, and some consumers who originally did not buy from firm 1 at all will now buy both.) For a given realization of (y_1, y_2) from other firms, Figure 5 below depicts how the small deviation affects consumer demand.

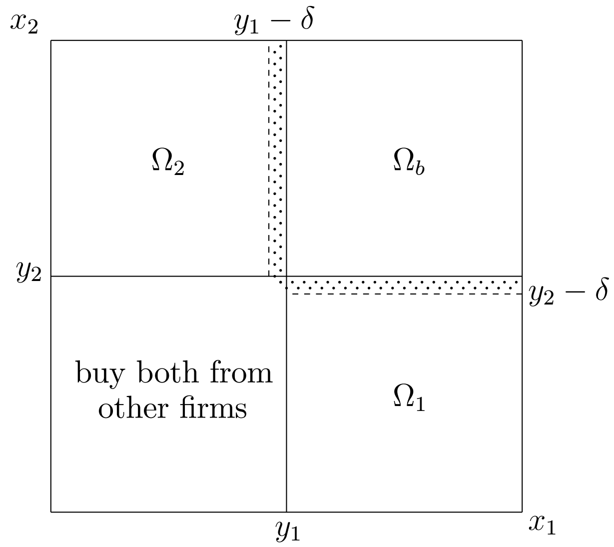


Figure 5: The impact of a joint-purchase discount on demand

Here Ω_i, $i = 1, 2$, indicates the region where the consumer buys only product i from firm 1 and Ω_b indicates the region where the consumer buys both products from firm 1. When firm 1 introduces the discount δ, the region of Ω_b expands but Ω_1 and Ω_2 shrink correspondingly, and the shaded area indicates the increased number of consumers who buy both products from firm 1.

Notice that for a small δ the effect of stealing customers who were purchasing both products from other firms (i.e., the small shaded triangle region) is of second order. The first-order effect is only from the consumers who were originally buying only one product from firm 1 but now buy both. The measure of such consumers is $\delta[f(y_1)(1 - F(y_2)) + f(y_2)(1 - F(y_1))]$, and the benefit from each of them is $p - \delta$. Therefore, given (y_1, y_2), the (first-order) positive effect of introducing a small δ is

$$p\delta[f(y_1)(1 - F(y_2)) + f(y_2)(1 - F(y_1))] \ .$$

Integrating this over all (y_1, y_2) yields

$$\begin{aligned} & p\delta \mathbb{E}_{y_1,y_2}[f(y_1)(1-F(y_2)) + f(y_2)(1-F(y_1))] \\ = \; & p\delta \frac{2}{n} \int f(y) dF(y)^{n-1} \\ = \; & \frac{2\delta}{n^2} \, . \end{aligned}$$

(The first equality used the cdf of y_2, and the second is because of (1).) Thus, the benefit is twice of the loss. As a result, the proposed deviation is indeed profitable, and so separate sales cannot be an equilibrium outcome when mixed bundling is possible.

Proposition 5 *Starting from separate sales with* p *defined in (1), each firm has a strict incentive to introduce mixed bundling.*

Unlike the pure bundling case where a firm's incentive to deviate from separate sales depends on the number of firms, here for any n there is always an incentive to introduce mixed bundling. This generalizes the result in the duopoly case shown by Armstrong and Vickers (2010).

5.2 Equilibrium prices

We now try to characterize the symmetric equilibrium where each firm adopts a mixed bundling strategy (ρ, δ), where ρ is the stand-alone price for each individual product and δ is the joint-purchase discount. Suppose all other firms use the equilibrium strategy, while firm 1 unilaterally deviates and sets (ρ', δ') (suppose the deviation is small). Let $z \equiv \max_{j \geq 2}\{x_1^j + x_2^j\}$ denote the match utility of the best bundle among the other $n-1$ firms.

Then a consumer faces the following five options:

- buy both products at firm 1: $x_1^1 + x_2^1 - 2\rho' + \delta'$
- buy product 1 at firm 1 but product 2 elsewhere: $x_1^1 + y_2 - \rho' - \rho$
- buy product 2 at firm 1 but product 1 elsewhere: $y_1 + x_2^1 - \rho' - \rho$
- buy both products from a firm other than firm 1: $z - 2\rho + \delta$
- buy products from two different firms other than firm 1: $\max_{j\geq 2, k \geq 2, j \neq k}\{x_1^j + x_2^k\} - 2\rho$

Deriving the demand by comparing these five options is a non-trivial problem: there are five random variables in this problem, and z is correlated with y_1 and y_2. The relatively simple case is the duopoly one. In that case, the fourth and fifth options degenerate to $y_1+y_2-2\rho+\delta$. Then the problem can be refrased in a two-dimensional Hotelling model by using two "location" random variables $x_1^1-y_1$ and $x_2^1-y_2$. The general problem with an arbitrary number of firms is more complicated. Let $I(y_i)$ be the identity of the firm that provides the best product i among firm 1's competitors. We need to deal with two cases separately, depending on whether $I(y_1)=I(y_2)$ or not:

(i) $I(y_1)=I(y_2)$. This case occurs with probability $\frac{1}{n-1}$. Since y_1 and y_2 are in the same firm, we have $z=y_1+y_2$ and so the fifth option is dominated (i.e., the consumer will never mix and match among the other $n-1$ firms). In this case, the consumer chooses among the first four options (the superscripts for x_i have been suppressed):

$$x_1+x_2-2\rho'+\delta';\ x_1+y_2-\rho'-\rho;\ y_1+x_2-\rho'-\rho;\ \ y_1+y_2-2\rho+\delta\ .$$

This case is similar to the duopoly one.

(ii) $I(y_1)\neq I(y_2)$. This case occurs only if $n\geq 3$ and the probability is $\frac{n-2}{n-1}$. Then $z<y_1+y_2$ and all the five options are relevant:

$$x_1+x_2-2\rho'+\delta';\ x_1+y_2-\rho'-\rho;\ y_1+x_2-\rho'-\rho;\ \ z-2\rho+\delta;\ \ y_1+y_2-2\rho\ .$$

Relative to the fourth option of buying at one firm other than firm 1, the fifth option can improve match quality by mixing and matching but in the same time the consumer has to give up the joint-purchase discount.

In the second case, z is correlated with y_1 and y_2 as stated in the following lemma:

Lemma 5 *When $n\geq 3$, the cdf of $z\equiv\max_{j\geq 2}\{x_1^j+x_2^j\}$ conditional on (y_1,y_2) and $I(y_1)\neq I(y_2)$ is*

$$L(z)=\frac{F(z-y_1)F(z-y_2)}{[F(y_1)F(y_2)]^{n-2}}\left(F(y_2)F(z-y_2)+\int_{z-y_2}^{y_1}F(z-x)dF(x)\right)^{n-3} \qquad (17)$$

for $z\in[\max\{y_1,y_2\}+\underline{x},y_1+y_2)$.

Proof. We need to calculate the conditional probability of $\max_{j\geq 2}\{x_1^j+x_2^j\}<z$. This event occurs if and only if all the following three conditions hold: (i) $y_1+x_2^{I(y_1)}<z$, (ii) $x_1^{I(y_2)}+y_2<z$, and (iii) $x_1^j+x_2^j<z$ for all $j\neq 1,I(y_1),I(y_2)$. Given y_1 and y_2, condition (i) holds with probability $F(z-y_1)/F(y_2)$, as the conditional cdf of $x_2^{I(y_1)}$ is $F(x)/F(y_2)$. Similarly, condition (ii) holds with probability $F(z-y_2)/F(y_1)$. One can also check (by resorting to a graph, for example) that the probability that condition (iii) holds for a firm other than $I(y_1)$ and $I(y_2)$ is

$$\frac{1}{F(y_1)F(y_2)}\left(F(y_2)F(z-y_2)+\int_{z-y_2}^{y_1}F(z-x)dF(x)\right)\ .$$

Given y_1 and y_2, all the above events are independent of each other, so multiplying those probabilities yields (17). ■

Let us consider the following choice problem:

$$x_1 + x_2 - 2\rho' + \delta';\ x_1 + y_2 - \rho' - \rho;\ y_1 + x_2 - \rho' - \rho;\ A - 2\rho + \delta\ .$$

Then the choice problem in the case of $I(y_1) = I(y_2)$ is this one with $A = y_1 + y_2$, and the choice problem in the case of $I(y_1) \neq I(y_2)$ is this one with $A = \max\{z, y_1 + y_2 - \delta\}$. Let us treat A as a random variable defined as follows:

$$A = \begin{cases} y_1 + y_2 & \text{with prob. } \frac{1}{n-1} \\ \max\{z, y_1 + y_2 - \delta\} & \text{with prob. } \frac{n-2}{n-1} \end{cases}, \tag{18}$$

where z is distributed according to the cdf (17). So $A - 2\rho + \delta$ represents the highest available surplus if the consumer buys the two products from firm 1's competitors.

Given (y_1, y_2, A), the following graph describes the consumer's purchase pattern:

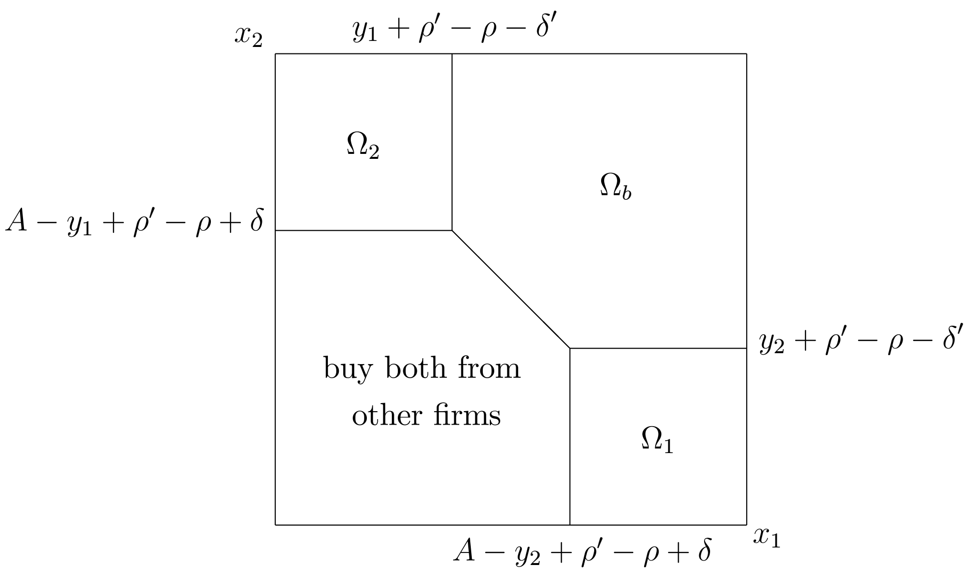


Figure 6: Price deviation and consumer demand

As before, Ω_i, $i = 1, 2$, indicates the region where the consumer buys only product i from firm 1 and Ω_b indicates the region where the consumer buys the bundle from firm 1. Without causing confusion, let $\Omega_k(\delta, y_1, y_2, A)$, $k = 1, 2, b$, be the area of region Ω_k in the equilibrium with $\rho' = \rho$ and $\delta' = \delta$. In the following, we treat A as a random variable. Conditional on (y_1, y_2), A is defined as follows:

It is useful to introduce some notations:

$$\begin{aligned}
\alpha(\delta) &\equiv \mathbb{E}_{y_1,y_2,A}\left[f(y_1 - \delta)\left(1 - F(A - y_1 + \delta)\right)\right]\ , \\
\beta(\delta) &\equiv \mathbb{E}_{y_1,y_2,A}\left[f(A - y_1 + \delta)F(y_1 - \delta)\right]\ , \\
\gamma(\delta) &\equiv \mathbb{E}_{y_1,y_2,A}\Big[\int_{y_1-\delta}^{A-y_2+\delta} f(A - x)f(x)dx\Big]\ , \\
\Omega_k(\delta) &\equiv \mathbb{E}_{y_1,y_2,A}\left[\Omega_k(\delta, y_1, y_2, A)\right]\ .
\end{aligned}$$

(All the expectations are taken over (y_1, y_2, A), and in the following we often suppress the expectation subscripts when there is no confusion.) The economic meaning of α, β and γ will be clear soon, and $\Omega_i(\delta)$ is the equilibrium demand for firm 1's product i only and $\Omega_b(\delta)$ is the equilibrium demand for firm 1's bundle. Due to symmetry of products, $\Omega_1(\delta) = \Omega_2(\delta)$. These notations will help describe the marginal effect of a small price deviation by firm 1 on its demand. Notice that they are all functions of δ only and do not depend on the stand-alone price ρ.[15]

To derive the first-order conditions for ρ and δ, let us consider the following two deviations: First, suppose firm 1 unilaterally deviates and raises its joint-purchase discount to $\delta' = \delta + \varepsilon$ while keeps its stand-alone price unchanged. Then conditional on (y_1, y_2, A), Figure 7(a) below describes how this small deviation affects consumer demand: Ω_b expands because now more consumers buy both products from firm 1. These marginal consumers are distributed on the shaded area.

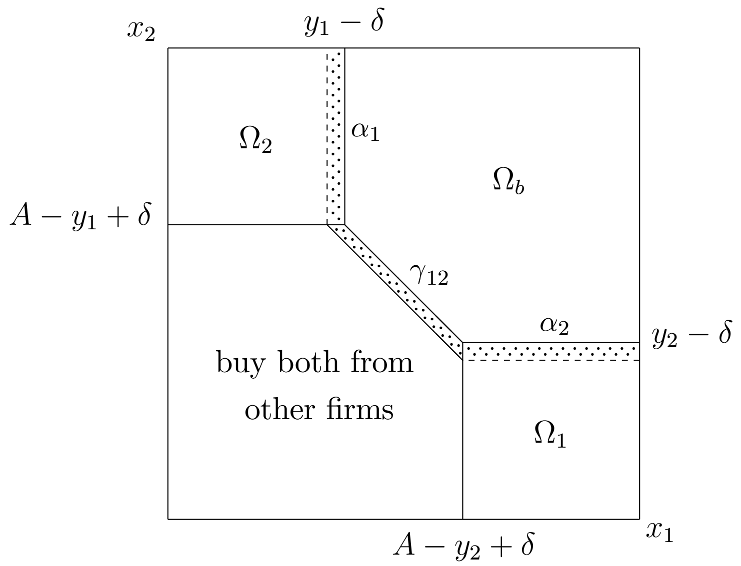


Figure 7(a): Price deviation and consumer demand

Here α_i, $i = 1, 2$, is the line integral $f(y_i - \delta)(1 - F(A - y_i + \delta))$, and γ_{12} is the line integral $\int_{y_1-\delta}^{A-y_2+\delta} f(A - x)f(x)dx$. If we integrate these line integrals over (y_1, y_2, A),

[15] To illustrate how to calculate these functions explicitly, consider a function $\psi(y_1, y_2, A)$. Then

$$\mathbb{E}_{y_1,y_2,A}[\psi(y_1, y_2, A)] = \frac{1}{n-1}\mathbb{E}_{y_1,y_2}[\psi(y_1, y_2, y_1 + y_2)]$$

$$+\frac{n-2}{n-1}\mathbb{E}_{y_1,y_2}\left[L(y_1 + y_2 - \delta)\psi(y_1, y_2, y_1 + y_2 - \delta) + \int_{y_1+y_2-\delta}^{y_1+y_2}\psi(y_1, y_2, z)dL(z)\right],$$

where $L(z)$ is defined in (17). By integration by parts and noticing $L(y_1 + y_2) = 1$, we have

$$\mathbb{E}_{y_1,y_2,A}[\psi(y_1, y_2, A)] = \mathbb{E}_{y_1,y_2}[\psi(y_1, y_2, y_1 + y_2)] - \frac{n-2}{n-1}\mathbb{E}_{y_1,y_2}\left[\int_{y_1+y_2-\delta}^{y_1+y_2}\frac{\partial}{\partial z}\psi(y_1, y_2, z)L(z)dz\right].$$

we get the previously introduced α and γ notations: $\mathbb{E}[\alpha_1] = \mathbb{E}[\alpha_2] = \alpha(\delta)$ and $\mathbb{E}[\gamma_{12}] = \gamma(\delta)$. For those marginal consumers on the horizontal and the vertical shaded regions (which have a measure of $\varepsilon(\alpha_1 + \alpha_2)$), they buy one more product from firm 1 and so firm 1 makes $\rho - \delta$ more money from each of such consumers. For those marginal consumers on the diagonal shaded region (which has a measure of $\varepsilon\gamma_{12}$), they switch from buying both products from other firms to buying both from firm 1. So firm 1 makes $2\rho - \delta$ more money from each of them. The only negative effect of this deviation is that those consumers on Ω_b who were already purchasing both products at firm 1 now pay ε less. Integrating the sum of all these effects over (y_1, y_2, A) should be equal to zero in equilibrium. This yields the following first-order condition:

$$2(\rho - \delta)\alpha(\delta) + (2\rho - \delta)\gamma(\delta) = \Omega_b(\delta) \ . \tag{19}$$

Second, suppose firm 1 unilaterally deviates and raises its stand-along price to $\rho' = \rho + \varepsilon$ and its joint-purchase discount to $\delta' = \delta + 2\varepsilon$ (such that its bundle price does not change). Figure 7(b) below describes how this small deviation affects consumer demand: both Ω_1 and Ω_2 shrink because now fewer consumers buy a single product from firm 1.

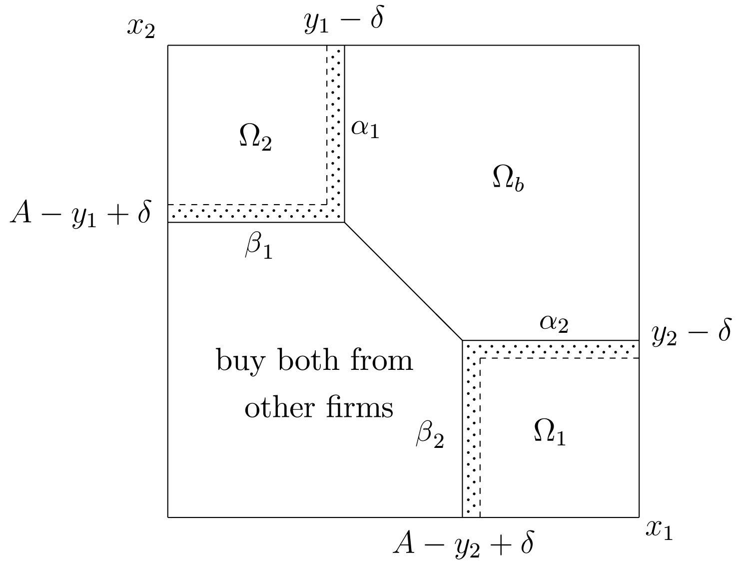


Figure 7(b): Price deviation and consumer demand

Here, β_i, $i = 1, 2$, is the line integral $f(A - y_i + \delta)F(y_i - \delta)$. If we integrate these line integrals over (y_1, y_2, A), we get the previously introduced β notation: $\mathbb{E}[\beta_1] = \mathbb{E}[\beta_2] = \beta(\delta)$. For those marginal consumers with a measure of $\varepsilon(\alpha_1 + \alpha_2)$, they switch from buying only one product from firm 1 to buying both from firm 1. So the firm makes $\rho - \delta$ more money from each of them. For those marginal consumers with a measure of $\varepsilon(\beta_1 + \beta_2)$, they switch from from buying one product from firm 1 to buying both from other firms. So the firm loses ρ from each of them. The direct revenue effect of this deviation is that firm 1 makes ε more from those consumers on Ω_1 and Ω_2 who were originally buying one product from firm 1. Integrating the sum of these effects over (y_1, y_2, A) should be equal to zero in equilibrium. This yields another first-order condition:

$$(\rho - \delta)\alpha(\delta) + \Omega_1(\delta) = \rho\beta(\delta) \ . \tag{20}$$

It is useful to notice that if all firms charge the same stand-alone price ρ, then for any common joint-purchase discount $\tilde{\delta} < \rho$ we have

$$\Omega_1(\tilde{\delta}) + \Omega_b(\tilde{\delta}) = \frac{1}{n} \ . \tag{21}$$

With full market coverage, every consumer buys product i, but with $\tilde{\delta} < \rho$ (i.e., the bundle is more expensive than a single product) no consumer buys more than one product i. Since all firms are *ex ante* symmetric, the demand for each firm's product i must be $1/n$.

Notice that both (19) and (20) are linear in ρ. Together with (21) at $\tilde{\delta} = \delta$, one can solve:

$$\rho = \frac{1/n + \delta(\alpha(\delta) + \gamma(\delta))}{\alpha(\delta) + \beta(\delta) + 2\gamma(\delta)} \ . \tag{22}$$

Substituting this into (20) yields an equation of δ:

$$\frac{1/n + \delta(\alpha(\delta) + \gamma(\delta))}{\alpha(\delta) + \beta(\delta) + 2\gamma(\delta)} (\beta(\delta) - \alpha(\delta)) = \Omega_1(\delta) - \delta\alpha(\delta) \ . \tag{23}$$

[More investigation is needed for the existence of solution to this equation.] We summarize the analysis so far in the following proposition:

Proposition 6 *If a symmetric mixed bundling equilibrium exists, then (ρ, δ) solve the system of (22) and (23).*

In general, it is hard to investigate whether the first-order conditions are also sufficient for defining a symmetric equilibrium. But in the numerical examples below, this will be verified numerically. To make analytical progress, we will study two extreme cases: the duopoly case and the case with a large number of firms. For convenience, let $H(\cdot)$ be the cdf of $x_i - y_i$. Then

$$H(t) \equiv \int_{\underline{x}}^{\overline{x}} F(x+t)dF(x)^{n-1}; \quad h(t) \equiv \int_{\underline{x}}^{\overline{x}} f(x+t)dF(x)^{n-1} \ .$$

In particular, when $n = 2$, the pdf $h(t)$ is symmetric around zero, and so $h(-t) = h(t)$ and $H(-t) = 1 - H(t)$. One can also check that for any $n \geq 2$, $H(0) = 1 - \frac{1}{n}$. By using this h notation, the price (1) in the regime of separate sales can be written as $p = \frac{1}{nh(0)}$.

In the duopoly case, we have $\alpha(\delta) = \beta(\delta) = h(\delta)[1 - H(\delta)]$ and $\Omega_1(\delta) = [1 - H(\delta)]^2$. Thus, (23) simplifies to[16]

$$\delta = \frac{1 - H(\delta)}{h(\delta)} \ . \tag{24}$$

If $1 - H$ is logconcave (which is implied, e.g., by the logconcavity of f), this equation has a unique positive solution. Meanwhile, (22) becomes

$$\rho = \frac{\delta}{2} + \frac{1}{4(\alpha(\delta) + \gamma(\delta))} \tag{25}$$

with $\gamma(\delta) = 2\int_0^\delta h(t)^2 dt$.

Corollary 1 *In the duopoly case, the equilibrium prices (ρ, δ) are determined by (24) and (25). Compared to the regime of separate sales, the bundle price becomes lower (i.e., $2\rho - \delta < 2p$) if f is logconcave, and the stand-alone price becomes higher (i.e., $\rho > p$) if $\eta'(t) > -\frac{1}{2}$ for $t \geq 0$, where $\eta(t) = \frac{1-H(t)}{h(t)}$.*

In the uniform example, one can check that $\delta = 1/3$, $\rho \approx 0.572$ and the bundle price is $2\rho - \delta \approx 0.811$. Compared to the regime of separate sales where the single product price is 0.5, now each single product is more expensive now but the whole bundle is cheaper. In the normal distribution example, one can check that $\delta \approx 1.063$, $\rho \approx 1.846$ and the bundle price is $2\rho - \delta \approx 2.629$. Compared to the regime of separate sales where the single product price is about 1.773, the same results hold.

The complication in dealing with the case with more than two firms mainly comes from the fact that $\alpha(\delta)$, $\beta(\delta)$, $\gamma(\delta)$ and $\Omega_1(\delta)$ do not have simple expressions due to the complication of the A random variable defined in (18). However, if δ is small, they all have relatively simple approximations. Since it is natural to predict that δ is small when n is large, we turn to that case now.

Corollary 2 *Suppose $\left|\frac{f'(\overline{x})}{f(\overline{x})}\right| < \infty$ and $\lim_{n\to\infty} p = 0$ (where p is the price in the regime of separate sales). Then when n is large, the equilibrium prices can be approximated as*

$$\rho \approx \frac{1}{nh(0)} = p; \quad \delta \approx \frac{\rho}{2} \ .$$

Compared to the regime of separate sales, the bundle price is lower.

Two observations follow from this result: First, when n is large, the stand-alone price is approximately equal to the price in the regime of separate sales. Second, the joint-purchase discount is approximately half of the stand-alone price. [Numerical investigation is needed beyond these two extreme cases.]

[16] Alternatively, (24) can be written as $\Omega_1(\delta) + \frac{1}{2}\delta\Omega_1'(\delta) = 0$. Armstrong and Vickers (2010) derived the same formula in the duopoly case with potentially asymmetric products.

5.3 Impacts of mixed bundling

Given the assumption of full market coverage, total welfare is determined by the match quality between consumers and products. Since the joint-purchase discount induces consumers to one-stop shop too often, mixed bundling, relative to separate sales, must lower total welfare. In the following, we discuss the impacts of mixed bundling on profit and consumer surplus.

Let $\pi(\tilde{\rho}, \tilde{\delta})$ be the industry profit when the stand-alone price is $\tilde{\rho}$ and the joint-purchase discount is $\tilde{\delta}$. Then

$$\pi(\tilde{\rho}, \tilde{\delta}) = 2\tilde{\rho} - n\tilde{\delta}\Omega_b(\tilde{\delta}) \ .$$

Thus, relative to separate sales the impact of mixed bundling on profit is

$$\pi(\rho, \delta) - \pi(p, 0) = 2(\rho - p) - n\delta\Omega_b(\delta) \ .$$

Let $v(\tilde{\rho}, \tilde{\delta})$ be the consumer surplus when the stand-alone price is $\tilde{\rho}$ and the joint-purchase discount is $\tilde{\delta}$. Given full market coverage, the standard argument implies that $v_1(\tilde{\rho}, \tilde{\delta}) = -2$ and $v_2(\tilde{\rho}, \tilde{\delta}) = n\Omega_b(\tilde{\delta})$, where the subscripts indicate partial derivatives. Then relative to separate sales, the impact of mixed bundling on consumer surplus is

$$\begin{aligned} v(\rho, \delta) - v(p, 0) &= v(\rho, \delta) - v(p, \delta) + v(p, \delta) - v(p, 0) \\ &= \int_p^\rho v_1(\tilde{\rho}, \delta) d\tilde{\rho} + \int_0^\delta v_2(p, \tilde{\delta}) d\tilde{\delta} \\ &= -2(\rho - p) + n \int_0^\delta \Omega_b(\tilde{\delta}) d\tilde{\delta} \ . \end{aligned}$$

In the duopoly case, Armstrong and Vickers (2010) have derived a condition under which compared to separate sales, consumers get better off and firms get worse off in the mixed bundling equilibrium. In the case with large n, given our approximation result, this must be the case. [More investigation is needed beyond the two extreme cases.]

5.4 Buy more than one bundle in pure bundling

As we pointed out before, if consumers can buy more than one bundle in the pure bundling case, the situation for consumers is just like mixed bundling with a stand-alone price P and a joint-purchase discount P. Then the analysis of mixed bunding applies as long as we impose the condition $\rho = \delta$. Suppose firm 1 unilaterally deviates to $P' = P + \varepsilon$. Figure 6 then implies that the impact of this small deviation on firm 1's demand is as follows (conditional on (y_1, y_2, A)):

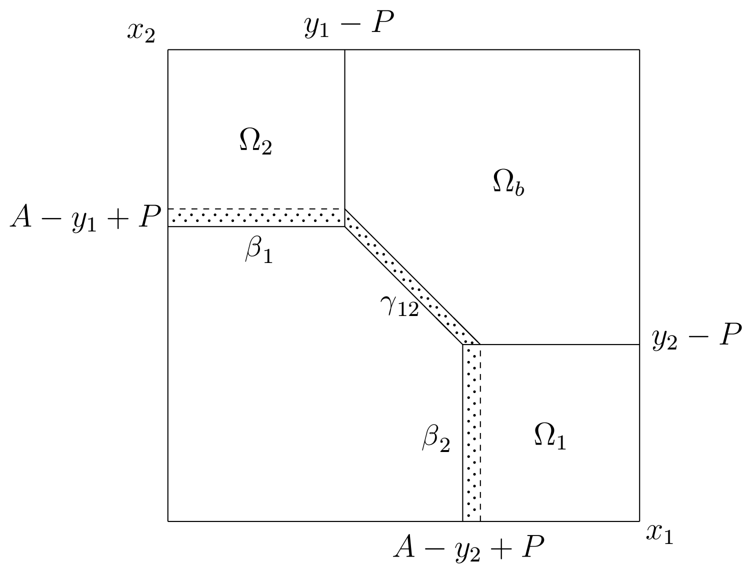


Figure 8: Price deviation and consumer demand

The only diffence in this graph is that in the region of Ω_i, $i = 1, 2$, consumers also buy the whole bundle from firm 1 at price P' as in the region of Ω_b, but they only consumes its product i. (This means that in Ω_i they also buy a bundle from one of the other firms and consume its product $j \neq i$.) Hence, the demand for firm 1's bundle is $\Omega_1 + \Omega_2 + \Omega_b$.

Compared to the equilibrium situation with $P' = P$, the demand regions shrink by the shaded area $\varepsilon(\beta_1 + \beta_2 + \gamma_{12})$. (The line integrals are defined the same as before except that δ is replaced by P.) Therefore, the negative effect of the deviation is that firm 1 makes P less from each consumer in the shaded area, and the positive effect is that it makes ε more from each consumer in the whole demand region who was already buying. Integrating the sum of these two effects over (y_1, y_2, A) and using $\Omega_1(P) + \Omega_2(P) + \Omega_b(P) = 1/n + \Omega_1(P)$, we get the first-order condition for P to be the equilibrium price:

$$P = \frac{1/n + \Omega_1(P)}{2\beta(P) + \gamma(P)} \ . \tag{26}$$

[Does this equation have a solution $P < \overline{x} - \underline{x}$?]

In the duopoly case, the first-order condition becomes:

$$P = \frac{1/2 + [1 - H(P)]^2}{2\left(h(P)[1 - H(P)] + \int_0^P h(t)^2 dt\right)} \ , \tag{27}$$

where H is the cdf of $x_i^1 - x_i^2$. This equation must have a solution $P \in (0, \overline{x} - \underline{x})$. If $P = 0$, the left-hand side of (26) is clearly less than the right-hand side. If $P = \overline{x} - \underline{x}$, the left-hand side is greater if and only if $(\overline{x} - \underline{x}) \int_0^{\overline{x} - \underline{x}} h(t)^2 dt > \frac{1}{4}$. But this inequality is implied by the Schwarz's Inequality (i.e., $\int f^2 \int g^2 \geq \left(\int fg\right)^2$ and the equality holds if and only if $f = kg$ for some constant k) and the fact that $h(t)$ cannot be a constant.

In the uniform distribution example,[17] one can numerical solve $P \approx 0.745\,43 < 2p = 1$. (This P is close to the bundle price 0.75 when consumers buy one bundle only.) In the normal distribution example, we have $P \approx 2.\,487\,5 < 2p \approx 3.\,544\,9$. (This P is also close to the bundle price 2.5066 when consumers buy one bundle only.) So in both of the duopoly examples bundling reduces market prices as before.

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6 Conclusion

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Appendix

Proof of Proposition 1: (i) The duopoly model can be converted into the standard Hotelling setting. Let $d_i = x_i^1 - x_i^2$, and let H and h be its cdf and pdf, respectively. It is clear that d_i has a support $[\underline{x} - \overline{x}, \overline{x} - \underline{x}]$ and it is symmetric around the mean zero. Since x_i^j has a logconcave density, z_i has a logconcave density also. Then the equilibrium price p in the regime of separate sales is given by

$$\frac{1}{p} = 2h(0) \; .$$

Let $\bar{H}$ and $\bar{h}$ be the cdf and pdf of $\sum_{i=1}^m d_i/m$. Then the equilibrium price in the pure bundling regime is given by

$$\frac{1}{P/m} = 2\bar{h}(0) \; .$$

Given that d_i is logconcave and symmetric, $\sum_{i=1}^m d_i/m$ is more peaked than each d_i in the sense $\Pr(|\sum_{i=1}^m d_i/m| \leq t) \geq \Pr(|d_i| \leq t)$ for any $t \in [0, \overline{x} - \underline{x}]$. (See, e.g., Theorem 2.3 in Proschan, 1965.) In particular, this implies that $\bar{h}(0) \geq g(0)$. Therefore, $P/m \leq p$.[18]

[17] In the uniform example, we have

$$H(t) = \begin{cases} \frac{1}{2}(1+t)^2 & \text{if } t \in [-1,0] \\ 1 - \frac{1}{2}(1-t)^2 & \text{if } t \in [0,1] \end{cases} \quad \text{and} \quad h(t) = \begin{cases} 1+t & \text{if } t \in [-1,0] \\ 1-t & \text{if } t \in [0,1] \end{cases} .$$

[18] In the duopoly case with $m = 2$ and $f(x) = e^{-x}$, one can check that

$$h(z) = \begin{cases} \frac{1}{2}e^{-z} & \text{if } z > 0 \\ \frac{1}{2}e^{z} & \text{if } z < 0 \end{cases} \quad \text{and } \bar{h}(z) = \begin{cases} (\frac{1}{2}+z)e^{-2z} & \text{if } z > 0 \\ (\frac{1}{2}-z)e^{2z} & \text{if } z < 0 \end{cases} .$$

Thus, we actually have $h(0) = \bar{h}(0)$ and so $P/m = p$. (Notice that h is strictly logconcave in this example, so strict logconcavity does not gurantee that $\bar{h}(0) > h(0)$.)

(ii) Since f is uniformly bounded, so is $\left|l(t) - \bar{l}(t)\right|$ for $t \in [0, 1]$. Then if n is sufficiently large, the sign of $\int_0^1 [l(t) - \bar{l}(t)]t^{n-2}dt$ is determined by the sign of $l(t) - \bar{l}(t)$ for t close to 1. $f(\bar{x}) > 0$ implies that $l(1) > 0$. While it must be true that $\bar{l}(1) = \bar{f}(\overline{x}) = 0$. Hence, $l(1) - \bar{l}(1) > 0$ and the opposite of (6) holds for a sufficiently large n.

To prove the second part of the result, we resort to one version of the *Variation Diminishing Theorem* (see Theorem 3.1 in Karlin, 1968). We need first to introduce two concepts.

A real function $K(x, y)$ of two variables ranging over linearly ordered sets X and Y, respectively, is said to be *totally positive of order* r if for all $x_1 < \cdots < x_m$ and $y_1 < \cdots < y_m$ with $x_i \in X$, $y_i \in Y$ and $1 \leq m \leq r$, we have

$$\begin{vmatrix} K(x_1, y_1) & \cdots & K(x_1, y_m) \\ \vdots & & \vdots \\ K(x_m, y_1) & \cdots & K(x_m, y_m) \end{vmatrix} \geq 0 \ .$$

We also need to introduce one way to count the number of sign changes of a function. Consider a function $f(t)$ for $t \in A$ where A is an ordered set of the real line. Let

$$S(f) = \sup S[f(t_1), \cdots, f(t_m)] \ ,$$

where the supremum is extended over all sets $t_1 \leq \cdots \leq t_m$ ($t_i \in A$), m is arbitrary but finite, and $S(x_1, \cdots, x_m)$ is the number of sign changes of the indicated sequence, zero terms being discarded.

Theorem 1 (Karlin,1968) *Consider the following transformation*

$$g(x) = \int_Y K(x, y)f(y)d\mu(y) \ ,$$

where $K(x, y)$ is a two-dimensional Borel-measurable function and μ is a sigma-finite regular measure defined on Y. Suppose f is Borel-measurable and bounded, and the integral exists. Then if K is totally positive of order r and $S(f) \leq r - 1$, then

$$S(g) \leq S(f) \ .$$

Now consider

$$\gamma(n) = \int_0^1 [l(t) - \bar{l}(t)]t^{n-2}dt \ .$$

Our assumption implies that $S[l(t) - \bar{l}(t)] \leq 2$. The lemma below show that $K(t, n) = t^{n-2}$ is totally positive of order 3. Therefore, the theorem implies that $S(\gamma) \leq 2$. That is, $\lambda(n)$ changes its sign at most twice as n varies. When f is logconcave, bounded, and $f(\overline{x}) > 0$, we already knew that $\lambda(2) < 0$ and $\lambda(n) > 0$ for sufficiently large n. Hence, $\lambda(n)$ cannot changes its sign twice, and so it must change its sign once.

Lemma 6 *Let $t \in (0,1)$ and $n \geq 2$ be integers. Then t^{n-2} is strictly totally positive of order* 3.

Proof. We need to show that for all $0 < t_1 < t_2 < t_3 < 1$ and $2 \leq n_1 < n_2 < n_3$, we have $t_1^{n_1-2} > 0$,

$$\begin{vmatrix} t_1^{n_1-2} & t_1^{n_2-2} \\ t_2^{n_1-2} & t_2^{n_2-2} \end{vmatrix} > 0 \text{ and } \begin{vmatrix} t_1^{n_1-2} & t_1^{n_2-2} & t_1^{n_3-2} \\ t_2^{n_1-2} & t_2^{n_2-2} & t_2^{n_3-2} \\ t_3^{n_1-2} & t_3^{n_2-2} & t_3^{n_3-2} \end{vmatrix} > 0 \; .$$

The first two inequalities are easy to check. The third one is equivalent to

$$\begin{vmatrix} t_1^{n_1} & t_1^{n_2} & t_1^{n_3} \\ t_2^{n_1} & t_2^{n_2} & t_2^{n_3} \\ t_3^{n_1} & t_3^{n_2} & t_3^{n_3} \end{vmatrix} > 0 \; .$$

Dividing the i_{th} row by $t_i^{n_1}$ $(i = 1,2,3)$ and then dividing the second column by $t_1^{n_2-n_1}$ and the third column by $t_1^{n_3-n_1}$, we can see the determinant has the same sign as

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & r_2^{\delta_2} & r_2^{\delta_3} \\ 1 & r_3^{\delta_2} & r_3^{\delta_3} \end{vmatrix} = (r_2^{\delta_2} - 1)(r_3^{\delta_3} - 1) - (r_2^{\delta_3} - 1)(r_3^{\delta_2} - 1) \; ,$$

where $\delta_j \equiv n_j - n_1$ and $r_j \equiv t_j/t_1$, $j = 2,3$. Notice that $0 < \delta_2 < \delta_3$ and $1 < r_2 < r_3$. To show that the above expression is positive, it suffices to show that $x^y - 1$ is log-supermodular for $x > 1$ and $y > 0$. One can check that the cross partial derivative of $\log(x^y - 1)$ has the same sign as $x^y - 1 - \log x^y$. This must be strictly positive for $x^y > 1$ because $\log z < z - 1$ for $z \neq 1$. ■

(iii) Suppose x_i^j has a mean μ and variance σ^2. When m is large, by *the central limit theorem*, $X^j/m = \sum_{i=1}^m x_i^j$ distributes (approximately) according to a normal distribution $\mathcal{N}(\mu, \sigma^2/m)$. Then

$$\frac{P}{m} \approx \frac{\sigma/\sqrt{m}}{n(n-1)\int_{-\infty}^{\infty} \Phi(x)^{n-2}\phi(x)^2 dx} \; ,$$

where Φ and ϕ are the cdf and pdf of the standard normal distribution, respectively. (So the right-hand side is just $\frac{\sigma}{\sqrt{m}}p$, where p is the linear price when x_i^j follows the standard normal distribution.) The bundle price P increases with m in the speed of $\sqrt{m}$ but in the separate sales case mp increases in the speed of m. Therefore, for a sufficiently large m, we must have $P/m < p$. The limit result is also clear.

Proof of Proposition 2: Result (i) simply follows from result (ii) in Proposition 1. To prove (ii), notice that the left-hand side of (10) increases with n while the right-hand side decreases with n given f is logconcave. So it suffices to prove two things: (a) (10) holds for $n = 2$, and (b) the opposite is true for a sufficiently large n.

Condition (b) is relatively easy to show. If $\lim_{n\to\infty} p = 0$, this is clearly true. But we already knew that $\lim_{n\to\infty} p$ can be strictly positive even if f is logconcave. The left-hand side of (10) approaches $\overline{x} - \mu = \int_{\underline{x}}^{\overline{x}} F(x)dx$ as $n \to \infty$. So we need to show that

$$\lim_{n\to\infty} p = \frac{1 - F(\overline{x})}{f(\overline{x})} < \int_{\underline{x}}^{\overline{x}} F(x)dx \ , \tag{28}$$

where the equality is implied by (2). (When $f(\overline{x}) > 0$ or $\overline{x} = \infty$, this is obviously true.) Note that logconcave f implies logconcave $1 - F$ (or decreasing $(1 - F)/f$). So

$$\begin{aligned} \int_{\underline{x}}^{\overline{x}} F(x)dx &= \int_{\underline{x}}^{\overline{x}} \frac{1 - F(x)}{f(x)} \frac{F(x)}{1 - F(x)} dF(x) \\ &> \frac{1 - F(\overline{x})}{f(\overline{x})} \int_{\underline{x}}^{\overline{x}} \frac{F(x)}{1 - F(x)} dF(x) \\ &= \frac{1 - F(\overline{x})}{f(\overline{x})} \int_0^1 \frac{t}{1 - t} dt \ . \end{aligned}$$

The integral term is infinity, so condition (28) must hold.

We then prove condition (a). Using (1) and the notation $l(t) \equiv f(F^{-1}(t))$, we can rewrite (10)

$$\int_0^1 \frac{t - t^n}{l(t)} dt \int_0^1 t^{n-2} l(t)dt < \frac{1}{n(n-1)} \ .$$

When $n = 2$, this becomes

$$\int_0^1 \frac{t(1-t)}{l(t)} dt \int_0^1 l(t)dt < \frac{1}{2} \ . \tag{29}$$

To prove this inequality, we need the following technical result:[19]

Lemma 7 *Suppose* $\varphi : [0, 1] \to \mathbb{R}$ *is a nonnegative function such that* $\int_0^1 \frac{\varphi(t)}{t(1-t)} dt < \infty$*, and* $r : [0, 1] \to \mathbb{R}$ *is a concave pdf. Then*

$$\int_0^1 \frac{\varphi(t)}{r(t)} dt \le \max\left(\int_0^1 \frac{\varphi(t)}{2t} dt, \int_0^1 \frac{\varphi(t)}{2(1-t)} dt \right) \ .$$

Proof. Since r is a concave pdf, it is a mixture of triangular distributions and admits a representation of the form

$$r(t) = \int_0^1 r_\theta(t) \lambda(\theta) d\theta \ ,$$

[19] I am grateful to Tomás F. Móri in Budapest for helping me to prove this lemma.

where $\lambda(\cdot)$ is a pdf defined on $[0,1]$, $r_1(t) = 2t$, $r_0(t) = 2(1-t)$, and for $0 < \theta < 1$

$$r_\theta(t) = \begin{cases} 2\dfrac{t}{\theta} & \text{if } 0 \le t < \theta \\ \\ 2\dfrac{1-t}{1-\theta} & \text{if } \theta \le t \le 1 \end{cases} .$$

(See, for instance, Example 5 in Csiszár and Móri, 2004.)

By Jensen's Inequality we have

$$\frac{1}{r(t)} = \frac{1}{\int_0^1 r_\theta(t)\lambda(\theta)d\theta} \le \int_0^1 \frac{1}{r_\theta(t)}\lambda(\theta)d\theta .$$

Then

$$\int_0^1 \frac{\varphi(t)}{r(t)}dt \le \int_0^1 \varphi(t)\left(\int_0^1 \frac{1}{r_\theta(t)}\lambda(\theta)d\theta\right)dt = \int_0^1 \left(\int_0^1 \frac{\varphi(t)}{r_\theta(t)}dt\right)\lambda(\theta)d\theta \le \sup_{1\le\theta\le1} \int_0^1 \frac{\varphi(t)}{r_\theta(t)}dt .$$

Notice that

$$\int_0^1 \frac{\varphi(t)}{r_\theta(t)}dt = \frac{\theta}{2}\int_0^\theta \frac{\varphi(t)}{t}dt + \frac{1-\theta}{2}\int_\theta^1 \frac{\varphi(t)}{1-t}dt .$$

This is a convex function of θ, because its derivative

$$\frac{1}{2}\int_0^\theta \frac{\varphi(t)}{t}dt - \frac{1}{2}\int_\theta^1 \frac{\varphi(t)}{1-t}dt$$

increases in θ. Hence, its maximum is attained at one of the endpoints of the domain $[0,1]$. This completes the proof of the lemma. ∎

Now let $\varphi(t) = t(1-t)$ and

$$r(t) = \frac{l(t)}{\int_0^1 l(t)dt} .$$

Since $l(t)$ is concave if and only if f is logconave, the defined $r(t)$ is indeed a concave pdf. (The integral in the denominator is finite since $l(t)$ is nonnegative and concave.) Then Lemma 7 implies that the left-hand side of (29) is actually no greater than 1/4.[20]

Consumer surplus comparison with normal distribution. To derive (12), it suffices to have the following result:

[20] For the exponential density $f(x) = e^{-x}$, the left-hand side of (29) equals 1/4. So our result is not tight. However, if f is not logconcave, it is easy to find counterexamples. For instance, (29) fails to hold for a power distribution $F(x) = x^k$ with k close to 1/2, or for a Weibull distribution $F(x) = 1 - e^{-x^k}$ with a small $k \in (0,1)$.

Lemma 8 *Consider a sequence of i.i.d. random variables* $\{x^j\}_{j=1}^n$ *with* $x^j \sim \mathcal{N}(0, \sigma^2)$. *Let* p *be the separate sales price as defined in (1) when each product's match utility follows the distribution of* x^j. *Then*

$$\mathbb{E}\left[\max_j \{x^j\}\right] = \frac{\sigma^2}{p} .$$

Proof. Let $F(\cdot)$ denote the cdf of x^j. Then the cdf of $\max_j \{x^j\}$ is $F(\cdot)^n$, and so

$$\mathbb{E}\left[\max_j \{x^j\}\right] = \int_{-\infty}^{\infty} x dF(x)^n = n \int_{-\infty}^{\infty} x F(x)^{n-1} f(x) dx .$$

For a normal distribution with zero mean, we have $f'(x) = -xf(x)/\sigma^2$. Therefore,

$$\begin{aligned}\mathbb{E}\left[\max_j \{x^j\}\right] &= -\sigma^2 n \int_{-\infty}^{\infty} F(x)^{n-1} f'(x) dx \\ &= \sigma^2 n(n-1) \int_{-\infty}^{\infty} F(x)^{n-2} f(x)^2 dx \\ &= \frac{\sigma^2}{p} .\end{aligned}$$

(The second step is from integration by parts, and the last step used (1).) ■

Proof of Proposition 4: Using the notation $l(t) = f(F^{-1}(t))$, we rewrite (14) as

$$\int_0^1 \frac{t - t^{n-1}}{l(t)} dt < p(1 - \frac{1}{n}) .$$

The left-hand side increases with n, while the right-hand side may increase with n as well (even if p decreases with n). Given the observations for $n = 2$ and $n \to \infty$, it suffices to show that the left-hand side increases with n faster than the right-hand side. Denote the left-hand side by $L(n)$ and the right-hand side by $R(n)$.

On the one hand, we have

$$L(n+1) - L(n) = \int_0^1 \frac{t^{n-1}(1-t)}{l(t)} dt > \frac{1}{n^2(n+1)^2} \left(\int_0^1 l(t) t^{n-1}(1-t) dt \right)^{-1} .$$

The inequality is from Jensen's Inequality (notice that $\int_0^1 t^{n-1}(1-t)dt = \frac{1}{n(n+1)}$, and so $n(n+1)t^{n-1}(1-t)$ is a pdf on $[0, 1]$). On the other hand, we have

$$R(n+1) - R(n) \leq \frac{p_n}{n(n+1)} = \frac{1}{n^2(n^2-1)} \left(\int_0^1 l(t) t^{n-2} dt \right)^{-1} ,$$

where p_n is the separate sales price with n firms. The inequality is from the assumption that p_n decreases in n, and the equality is from the definition of p_n in (5).

Therefore, a sufficient condition for our result is that

$$(n+1)^2 \int_0^1 l(t) t^{n-1}(1-t) dt < (n^2 - 1) \int_0^1 l(t) t^{n-2} dt .$$

Since $t(1-t) \leq \frac{1}{4}$ for $t \in [0,1]$, this condition holds if $\frac{n+1}{4} < n-1$. But this true for any $n \geq 2$.

Proof of Lemma 4: Since the logconcavity of f implies the logconcavity of $\bar{f}$, we only need to show the results for p. To prove the first result, we actually only need $1-F$ to be logconcave (which is implied by the logconcavity of f). When $p=0$, it is clear that the left-hand side of (15) is less than the right-hand side. We can also show the opposite when $p = p_M$. By using the $(n-1)_{\text{th}}$ order statistic as in the proof of Lemma 1, the right-hand side of (15) equals

$$\frac{1-F(p)^n}{nF(p)^{n-1}f(p) + \int_p^{\overline{x}} \frac{f(x)}{1-F(x)} dF_{(n-1)}(x)}$$

$$< \frac{1-F(p)^n}{nF(p)^{n-1}f(p) + \frac{f(p)}{1-F(p)}[1-F_{(n-1)}(p)]} = \frac{1-F(p)}{f(p)} \ .$$

(The inequality is because $f/(1-F)$ is increasing, and the equality used $F_{(n-1)}(p) = F(p)^n + nF(p)^{n-1}[1-F(p)]$.) Then the fact that $p_M = \frac{1-F(p_M)}{f(p_M)}$ implies the result we want. This shows that (15) has a solution $p \in (0, p_M)$.

To show the uniqueness, we prove that the right-hand side of (15) decreases with p. One can verify that its derivative with respect to p is negative if and only if

$$[1-F(p)^n]f'(p) + nf(p)\left[F(p)^{n-1}f(p) + \int_p^{\overline{x}} f(x)dF(x)^{n-1}\right] > 0 \ .$$

Using $[1-F]f' + f^2 > 0$ (which is implied by the logconcavity of $1-F$), one can check that the above inequality holds if

$$n\int_p^{\overline{x}} f(x)dF(x)^{n-1} > [1-F(p)^n]\frac{f(p)}{1-F(p)} - nf(p)F(p)^{n-1} \ .$$

The left-hand side equals $\int_p^{\overline{x}} \frac{f(x)}{1-F(x)} dF_{(n-1)}(x)$, and the right-hand side equals $\frac{f(p)}{1-F(p)}[1- F_{(n-1)}(p)]$. So the inequality is implied by $\frac{f(x)}{1-F(x)} > \frac{f(p)}{1-F(p)}$ for $x > p$.

To prove the second comparative static result, let us first rewrite (15) as

$$\frac{1}{p} = \frac{f(\overline{x}) - \int_p^{\overline{x}} f'(x)F(x)^{n-1}dx}{[1-F(p)^n]/n} = \frac{nf(\overline{x})}{1-F(p)^n} - \int_p^{\overline{x}} \frac{f'(x)}{f(x)} d\frac{F(x)^n - F(p)^n}{1-F(p)^n} \ . \quad (30)$$

(The first step is from integration by parts.) First of all, $\frac{n}{1-F(p)^n}$ increases with n.[21] Second, the logconcavity of f implies $-\frac{f'}{f}$ is increasing. Third, notice that $\frac{F(x)^n - F(p)^n}{1-F(p)^n}$

[21] For $x \in [0,1)$, $\frac{n}{1-x^n}$ strictly increases with n. To see that, notice

$$\frac{n+1}{1-x^{n+1}} < \frac{n}{1-x^n} \Leftrightarrow \frac{1}{n+1}\sum_{i=0}^{n} x^i < \frac{1}{n}\sum_{i=0}^{n-1} x^i \ .$$

is cdf of the highest order statistic of $\{x_i\}_{i=1}^n$ conditional on it being greater than p, and so $\frac{F(x)^{n+1}-F(p)^{n+1}}{1-F(p)^{n+1}}$ first-order stochastically dominates $\frac{F(x)^n-F(p)^n}{1-F(p)^n}$. These three observations imply that the right-hand side of (30) increases with n. So the unique solution p must decrease with n.

Proof of Corollary 1: From (24) and (25), we know that the bundle price is $2\rho - \delta = 1/[2(\alpha+\gamma)]$ (the variable δ in $\alpha(\delta)$ and $\gamma(\delta)$ has been suppressed). And the bundle price in the regime of separate sales is $1/h(0)$. The former is lower if

$$\alpha + \gamma = h(\delta)[1 - H(\delta)] + 2\int_0^\delta h(t)^2 dt > \frac{1}{2}h(0) \ .$$

Notice that the left-hand side of the inequality equals $h(0)/2$ at $\delta = 0$. So it suffices to show that the left-hand side is increasing in δ. Its derivative is $h(\delta)^2 + h'(\delta)[1 - H(\delta)]$. This is positive if $h/(1-H)$ is increasing or equivalently $1-H$ is logconcave. The logconcavity of $1-H$ is implied by the logconcavity of $1-F$ (which is further implied by the logconcavity of f).

The individual product price becomes higher in the regime of mixed bundling if

$$\frac{\delta}{2} + \frac{1}{4(\alpha+\gamma)} > \frac{1}{2h(0)} \ .$$

Notice that the equality holds at $\delta = 0$. So it suffices to show that the left-hand side increases in δ (which requires $\alpha + \gamma$ increase not too fast), or equivalently

$$2(\alpha+\gamma)^2 > \alpha' + \gamma' \ .$$

Notice that $\alpha'+\gamma' = -h(\delta)^2\eta'(\delta)$ and $\alpha+\gamma > H(\delta)h(\delta) > h(\delta)/2$ (the first inequality is because $h(\delta)$ is decreasing in $\delta > 0$ and the second is because $H(\delta) > 1/2$). Then the above inequality holds if $\eta'(\delta) > -1/2$.

Proof of Corollary 2: When $\delta \approx 0$, by using the result in footnote 15 and the Taylor expansion, one can check that

$$\begin{aligned}
\alpha(\delta) &\approx \frac{1}{n}h(0) - \delta\left(\frac{1}{n}h'(0) + \frac{1}{n-1}h(0)^2\right) \ , \\
\beta(\delta) &\approx \left(1 - \frac{1}{n}\right)h(0) + \delta\left(\frac{1}{n}h'(0) - h(0)^2\right) \ , \\
\gamma(\delta) &\approx \frac{n\delta}{n-1}h(0)^2 \ , \\
\Omega_1(\delta) &\approx \frac{1}{n}\left(1 - \frac{1}{n}\right) - \frac{2\delta}{n}h(0) \ ,
\end{aligned}$$

where $h(0) = \int f(x)dF(x)^{n-1}$ and $h'(0) = \int f'(x)dF(x)^{n-1}$. Substituting them into (22) and (23) and discarding all higher order effects, one can solve

$$\rho = \frac{1}{nh(0)}\frac{1+\delta h(0)}{1+\frac{n}{n-1}\delta h(0)} \ ,$$

and

$$\delta = \frac{1}{2\frac{h'(0)}{h(0)} + \frac{2n^2-3n+2}{n^2-n} nh(0)} \ .$$

Notice that $\lim_{n\to\infty} p = 0$ implies $\lim_{n\to\infty} nh(0) = \infty$, and $\left|\frac{f'(\overline{x})}{f(\overline{x})}\right| < \infty$ implies $\lim_{n\to\infty} \frac{h'(0)}{h(0)} < \infty$. Then for large n, we can deduce the approximations in the statement.

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