Bayes Correlated Equilibrium and the Comparison of Information Structures*

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Abstract

The set of outcomes that can arise in Bayes Nash equilibria of an incomplete information game where players may or may not have access to more private information is characterized and shown to be equivalent to the set of an incomplete information version of correlated equilibrium, which we call *Bayes correlated equilibrium*. We describe a partial order on many player information structures - which we call *individual sufficiency* - under which more information shrinks the set of Bayes correlated equilibria. We discuss the relation of the solution concept to alternative definitions of correlated equilibrium in incomplete information games and of the partial order on information structures to others, including Blackwell's for the single player case.

KEYWORDS: Correlated equilibrium, incomplete information, robust predictions, information structure.

JEL CLASSIFICATION: C72, D82, D83.

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1 Introduction

Fix an incomplete information game. What can we say about what might happen in equilibrium if players may or may not have access to more information? We show that behavior corresponds to a Bayes Nash equilibrium for some extra information that the players might observe if and only if it is an incomplete information version of correlated equilibrium that we dub *Bayes correlated equilibrium.* Aumann (1974), (1987) introduced the notion of correlated equilibrium in games with complete information and a number of definitions of correlated equilibrium in games with incomplete information have been suggested, notably in Forges (1993). Our definition is driven by a different motivation from the earlier literature and is weaker than the weakest definition of incomplete information correlated equilibrium (the Bayesian solution in Forges (1993)), because it allows play to be correlated with states that are not known by any player.

While this characterization is a straightforward variation and reinterpretation of existing results in the literature, we believe there are a number of distinct reasons why this characterization is of particular interest. First, it allows the analyst to identify properties of equilibrium outcomes that are going to hold independent of features of the information structure that the analyst does not know; in this sense, properties that hold in all Bayes correlated equilibria of a given incomplete information game constitute robust predictions. Second, it provides a way to partially identify parameters of the underlying economic environment independently of knowledge of the information structure. Third, it provides an indirect method of identifying socially or privately optimal information structures without explicitly working with a space of all information structures. In Bergemann and Morris (2013b), we illustrate these uses of the characterization result in a particular class of continuum player, linear best response games, focussing on normal distributions of types and actions and symmetric information structures and outcomes. While special, these games and equilibria can be used to model many economic phenomena of interest. In this paper, we work with general (finite player, finite action and finite state) games, and illustrate these uses with examples.

We distinguish between the "basic game" from the "information structure" in the definition of the incomplete information game. The basic game refers to the set of actions, the set of payoff states, the utility functions of the players, and the common prior over the payoff states. The information structure refers to the type space of the game, which is generated by a mapping from the payoff states to a probability distribution over types, or signals. The separation between the basic game and the information structure enables us to ask how changes in the information structure affect the equilibrium set for a fixed basic game. A second contribution of the paper and the main formal contribution is that (i) we introduce a natural, statistical, partial order on information structures - called *individual sufficiency* - that captures intuitively when one information structure contains more information than another; and (ii) we show that the set of Bayes correlated equilibria shrinks in all games if and only if the informativeness of the information structure increases.

A one player version of an information structure is an "experiment" in the sense studied by Blackwell (1951), (1953). An experiment consists of a set of signals and a mapping from states to probability distributions over signals. Suppose that we are interested in comparing a pair of experiments. A *combination* of the two experiments is a new experiment where a pair of signals - one from each experiment - is observed and the marginal probability over signals from each of the original experiments corresponds to the original distribution. One experiment is *sufficient* for another if it is possible to construct a combined experiment such that the signal in the former experiment is a sufficient statistic for the signal in the latter experiment. Our partial order on (many player) information structures is a natural generalization of sufficiency. One information structure is *individually sufficient* for another if there exists a combined information structure where each player's signal from the former information structure is a sufficient statistic for his beliefs over both states and others' signals in the combined information structure. This partial order has a couple of key properties - each generalizing well known properties in the one player case - that suggest that it is the "right" ordering on (many player) information structures. First, two information structures are individually sufficient for each other if and only if they are "higher order belief equivalent" in the sense that they correspond to the same probability distribution over beliefs and higher order beliefs about states (for any given prior on states). Second, one information structure is individually sufficient for another if and only if it is possible to start with the latter information structure and then have the players observe some extra signal, so that the expanded information structure is higher order belief equivalent to the former information structure.

Blackwell's theorem showed that if one experiment was sufficient for another, then making decisions based on the former experiment allows a decision maker to attain a richer set of outcomes. In particular, although it was not the form in which Blackwell stated his result, economists have focussed on the implication that one experiment is sufficient for another if and only if it allows a decision maker to attain a higher level of ex ante utility in any decision problem, so that it is "more valuable". Thus we will argue that Blackwell's theorem showed the equivalence of a "statistical ordering" on experiments (sufficiency) and a "feasibility ordering" (more valuable than). In this paper, we introduce (in the many player case) an "incentive ordering" on information structures: we say that an information structure is more *incentive constrained* than another if it gives rise to a smaller set of Bayes correlated equilibria. Our main result, stated in this language, is that one information structure is more incentive constrained than another if and only if the former is individually sufficient for the latter. Thus we show the equivalence between a statistical ordering and an incentive ordering. In order to understand the relation to Blackwell's theorem, we show that our main result, restricted to the one person case, has a natural interpretation and can be seen as an extension of Blackwell's theorem. And we also describe a feasibility ordering on many player information structures which is equivalent to individual sufficiency and more incentive constrained than. Taken together, our main result and discussion of the relation to Blackwell's theorem, highlight the dual role of information. By making more outcomes feasible, more information allows more outcomes to occur. By adding incentive constraints, more information restricts the set of outcomes that can occur. We show that the same partial order - individual sufficiency, reducing to sufficiency in the one player case - captures both roles of information simultaneously.

Our characterization result also has a one player analogue. Consider a decision maker who has access to an experiment, but may have access to more information. What can we say about the joint distribution of actions and states that might result in a given decision problem? We show that they are one person Bayes correlated equilibria. Such one person Bayes correlated equilibria have already arisen in a variety of contexts. Kamenica and Gentzkow (2011) consider the problem of cheap talk with commitment. In order to understand the behavior that a sender/speaker can induce a receiver/decision maker to choose, we might first want to characterize all outcomes that can arise for some committed cheap talk (independent of the objectives of the speaker). This, in our language, is the set of one person Bayes correlated equilibria.[1] Caplin and Martin (2011) study experiments with incomplete perception of a set of physical signals. Since they do not know how the decision maker perceives, they interpret the subject as if she has observed more or less information unknown to the experimenter, and thus outcomes are, in our language, one person Bayes correlated equilibria.

There is a literature studying and comparing alternative definitions of correlated equilibrium under incomplete information, with the papers of Forges (1993), (2006) being particularly important. A standard assumption in that literature - which we dub "join feasibility" - is that play can only depend on the joint information of all the players. This restriction makes sense under the maintained assumption that correlated equilibrium is intended to capture the role of correlation of the players' actions but not unexplained correlation with the state of nature. Our different motivation leads us to allow such unexplained correlation. Liu (2011) also relaxes the join feasibility assumption, but imposes a belief invariance assumption (introduced and studied in combination with join feasibility in Forges (2006)), requiring that, from each player's point of view, the action recommendation that he receives from the mediator does not change his beliefs about others' types and the state. Intuitively, the belief invariant Bayes correlated equilibria of Liu (2011) capture the implications of common knowledge of rationality and a fixed information structure,

[1] As Kamenica and Gentzkow (2011), there is a close connection with work of Aumann and Maschler (1995), on Nash equilibria of infinitely repeated zero sum games with one sided uncertainty and without discounting, as it is as if the informed play can commit to reveal only certain information about the state.

while our Bayes correlated equilibria capture the implications of common knowledge of rationality and the fact that the player have observed at least the signals in the information structure. In particular, the set of Bayes correlated equilibria for a fixed information structure correspond to the set of belief invariant Bayes correlated equilibria of that information structure and all more informed ones. Liu (2011) introduces the notion of an "incomplete information correlating device" and shows that (i) belief invariant Bayes correlated equilibria are invariant to adding a correlation device; and (ii) two information structures are higher order belief equivalent if and only if they can be mapped into each other via correlating devices. Thus two information structures are individually sufficient for each other if and only if they are equivalent to a non-redundant type space plus a correlating device.

Two papers - Lehrer, Rosenberg, and Shmaya (2010), (2011) - have examined comparative statics of how changing the information structure effects the set of predictions that can be made about players' actions, under Bayes Nash equilibrium and definitions of incomplete information correlated equilibrium stronger than Bayes correlated equilibrium. In the language of our paper, they construct statistical orderings on information structures and show how these orderings are relevant for - in Lehrer, Rosenberg, and Shmaya (2010) - feasibility orderings and - in Lehrer, Rosenberg, and Shmaya (2011) - incentive orderings. Our ordering - individual sufficiency - is a more complete variation on the orderings they construct and we will note how some of our results can be proved by adapting their arguments. Two crucial distinctions are the following. First, because they work with solution concepts that maintain join feasibility, the relevant orderings are always refinements of sufficiency, i.e., they require players' joint information in one information structure to be sufficient for their joint information in the other structure, and then impose additional restrictions. By construction, individual sufficiency is a many player analogue of sufficiency but neither implies nor is implied by sufficiency of joint information. Second, because they work with solution concepts that include feasibility restrictions, the results relating information structure orders to incentive constraints in Lehrer, Rosenberg, and Shmaya (2011) are weaker than ours: they characterize when two information structures support the same set of equilibria in all games, but not when one information structure supports a larger or smaller set.

We exposit our results by first presenting at length the one player version of the results in Section 2. As discussed above, we believe that the one player version of our results is of independent interest, relates together a number of results in the literature in an interesting way, and allows us to present an interesting extension and interpretation of Blackwell's theorem. And we can present our results for the one player case in a way that the many player generalization follows easily.

However, this expository material is not necessary for the remainder of the paper, and it is possible to go straight to the general many player analysis in Section 3 on page 19. In Section 3, we provide a complete analysis of the general case with finitely many agents. In Section 4, we offer a many player generalization of the sufficiency ordering, dubbed *individual sufficiency,* for which we can establish an equivalence between the incentive based ordering and the statistical ordering. We thus report results on comparing information structures in many agent environments. In Section 5, we study a class of binary action basic games and binary signal information structures, and use it to illustrate all our results. In Section 6, we explain how the solution concept we dub "Bayes correlated equilibrium" relates to the literature, in particular Forges (1993), (2006). Section 7 concludes and contains a discussion of the relation to the literature on the value of information.

2 The Special Case: One Player

We first preview our results by presenting the one player case. There is a finite set of states, Θ, and we write θ for a typical state. A *decision problem* G consists of (1) a finite set of actions A; (2) a utility function $u : A \times \Theta \to \mathbb{R}$; and (3) a strictly positive prior $\psi \in \Delta_{++}(\Theta)$. Thus $G = (A, u, \psi)$. An *experiment* S consists of (1) a finite set of signals T; and (2) a signal distribution $\pi : \Theta \to \Delta(T)$. Thus $S = (T, \pi)$. Two experiments play an important role. The *null experiment* $\underline{S}$ has $\underline{T} = \{\underline{t}\}$ and $\underline{\pi}(\underline{t}|\theta) = 1$ for all $\theta \in \Theta$. Thus the null experiment provides no information. The *complete experiment* $\overline{S}$ has $\overline{T} = \Theta$ and

$$\overline{\pi}(t|\theta) = \begin{cases} 1, \text{ if } t = \theta, \\ 0, \text{ otherwise,} \end{cases}$$

for all $\theta \in \Theta$. The pair (G, S) is a (one player) game of incomplete information.

Our terminology in this section will be often be non-standard for the one person case as it is chosen to emphasize the link with the many player case that follows.

2.1 Defining Bayes Correlated Equilibrium

A *decision rule* is a mapping

$$\sigma : T \times \Theta \to \Delta(A). \tag{1}$$

Note that it is allowed to depend on states as well as signals. We will be interested in the implications of decision rules if signals are not observed. Call a mapping

$$\nu : \Theta \to \Delta(A) \tag{2}$$

specifying the probability distribution over actions conditional on states a *random choice rule.* Random choice rule ν is *induced* by decision rule σ if

$$\sum_{t \in T} \pi(t|\theta)\,\sigma(a|t,\theta) = \nu(a|\theta) \tag{3}$$

for all $a \in A$ and $\theta \in \Theta$.

Now suppose the action was being chosen by a rational decision maker who was aware of information structure S (i.e., observed a signal $t \in T$), was interested in maximizing expected utility but perhaps knew more. What restrictions beyond consistency would this impose on the decision rule?

Definition 1 (Obedience)
Decision rule σ is obedient for (G, S) if

$$\sum_{\theta \in \Theta} \psi(\theta)\, \pi(t|\theta)\, \sigma(a|t,\theta)\, u(a,\theta) \geq \sum_{\theta \in \Theta} \psi(\theta)\, \pi(t|\theta)\, \sigma(a|t,\theta)\, u\left(a',\theta\right) \tag{4}$$

for all $a, a' \in A$ and $t \in T$.

Definition 2 (Bayes Correlated Equilibrium)
Decision rule σ is a Bayes correlated equilibrium (BCE) of (G, S) if it is obedient for (G, S).

Random choice rule ν is a *BCE random choice rule* for (G, S) if it is induced by a BCE σ. We are interested in the joint distribution of a triple of random variables, (a, t, θ). An elementary property of the conditional probabilities of a triple of random variables was stated as Theorem 7 of Blackwell (1951). This property, a conditional independence property, will play a central role in our analysis, as it did in Blackwell's and other related work. Because this property, and terminology that we will use to describe it, will appear in a variety of different contexts in this paper, we will find it useful to present it abstractly next.

2.2 A Statistical Digression: Blackwell Triples

Suppose that we are interested in a triple of variables, $(x, y, z) \in X \times Y \times Z$, and that we are given a probability distribution on the product space, $P \in \Delta(X \times Y \times Z)$. We will abuse notion by using P to refer to marginal probabilities, writing $P(x)$ for $P(\{x\} \times Y \times Z)$ and $P(x, y)$ for $P(\{x\} \times \{y\} \times Z)$; and conditional probabilities, writing $P(x|y,z)$ for

$$P(x|y,z) = \frac{P(x,y,z)}{P(y,z)}$$

if $P(y,z) > 0$; and

$$P(x|y) = \frac{P(x,y)}{P(y)}$$

if $P(y) > 0$. We will say that the probability of x conditional on y is independent of z (under P) if

$$P(x|y,z) = P(x|y)$$

for all $z \in Z$ whenever $P(y,z) > 0$. Now we have the following statistical fact concerning a triple of random variables.

Lemma 1 (Conditional Independence)
The following statements are equivalent

1. *The probability $P(x|y,z)$ is independent of z.*

2. *The probability $P(z|y,x)$ is independent of x.*

3. $P(x,y,z) = P(y)P(x|y)P(z|y)$ *if* $P(x,y,z) > 0$.

Proof. (3) immediately implies (1) and (2). To see that (1) implies (3), observe that if $P(y,z) > 0$,

$$\begin{aligned} P(y,z)P(x|y,z) &= P(y)P(z|y)P(x|y,z)\text{, by definition} \\ &= P(y)P(z|y)P(x|y)\text{, by (1).} \end{aligned}$$

A symmetric argument shows that (2) implies (3). ■

When these statements are true, we will say that the ordered variables (x,y,z) are a Blackwell triple (under P). Blackwell (1951) observed that the above relationship can be rephrased as saying that a Markov chain, namely $P(x|y,z) = P(x|y)$, is also a Markov chain in reverse, namely $P(z|y,x) = P(z|y)$.[2]

2.3 Foundations of Bayes Correlated Equilibrium

We want to formalize the idea that Bayes correlated equilibria describe all behavior that might arise from a decision maker with access to experiment S but also perhaps more information. First, we review the standard approach to analyzing rational behavior where the information structure is fully described by experiment S.

Definition 3 (Belief Invariance)
A decision rule σ is belief invariant for (G,S) if for all $\theta \in \Theta, t \in T$ such that $\psi(\theta)\pi(t|\theta) > 0$, $\sigma(a|t,\theta)$ is independent of θ.

In other words, if the pair (t,θ) has a strictly positive probability: $\psi(\theta)\pi(t|\theta) > 0$, then the conditional probability $\sigma(a|t,\theta)$ only depends on t and not on θ :

$$\sigma(a|t) \triangleq \sigma(a|t,\theta). \tag{5}$$

[2]For this reason, Torgersen (1991) p. 345, refers to the triple (x,y,z) as a "Markov triple". As the term "Markov triple" is commonly used to refer to solutions of Markov Diophantine equations, we prefer "Blackwell triple".

This property reflects the standard restriction in Bayesian decision making, that a decision maker who has access only to the experiment S, cannot directly condition his actions on the state θ, but rather has to condition his decision on the signal t. The belief invariance condition is imposed as a restriction only on strictly positive probability events, $\psi(\theta)\pi(t|\theta) > 0$, under the information structure S. But as we could always extend the restriction to zero probability events, $\psi(\theta)\pi(t|\theta) = 0$, without affecting either the obedience condition, see (4), or the induced random choice rule, see (3), we shall henceforth omit the positive probability qualifier, $\psi(\theta)\pi(t|\theta) > 0$, when discussing belief invariance.

Now supposed that we fix any prior $\psi \in \Delta_{++}(\Theta)$ and write $\sigma_\psi(\theta|t,a)$ for implied probability of θ conditional on signal t and action a, so that

$$\sigma_\psi(\theta|t,a) = \frac{\psi(\theta)\pi(t|\theta)\sigma(a|t,\theta)}{\sum\limits_{\widetilde{\theta}\in\Theta}\psi\left(\widetilde{\theta}\right)\pi\left(t|\widetilde{\theta}\right)\sigma\left(a|t,\widetilde{\theta}\right)}$$

whenever $\pi\left(t|\widetilde{\theta}\right)\sigma\left(a|t,\widetilde{\theta}\right) > 0$ for some $\widetilde{\theta} \in \Theta$.

Now (t,a,θ) are a Blackwell triple as defined in Section 2.2 and, by Lemma 1, the conditional independence lemma, an equivalent statement of the belief invariance property is then that $\sigma_\psi(\theta|t,a)$ is independent of a. This statement motivates the name: it states that observing the chosen action a does not reveal any information about the state θ beyond that contained in the signal t. This terminology was introduced in the many player context by Forges (2006).

Thus decision rule σ could arise from a decision maker with access only to experiment S if it is belief invariant. Thus a standard description of optimal behavior in this setting corresponds to requiring belief invariance and obedience.

Definition 4 (Bayes Nash Equilibrium)
Decision rule σ is a Bayes Nash Equilibrium (BNE) for (G,S) if it is obedient and belief invariant for (G,S).

Random choice rule ν is a BNE random choice rule if it is induced by a BNE decision rule. We label this a "Bayes Nash equilibrium" as this will be one many player counterpart of this definition. However, we will see that there are multiple ways of extending belief invariance to the many player case, so Bayes Nash equilibria will not be the only natural counterpart of this definition.[3]

We want to ask what can happen if the decision maker observes more information. To discuss this, we must introduce language to talk about comparing and combining experiments. If we have two experiments

[3] In Section 6, we will see that, in the one player case, the set of BNE random choice rules equals the set of Bayesian solutions, the set of belief invariant Bayes correlated equilibria and the set of belief invariant Bayesian solutions.

$S = (T, \pi)$ and $S' = (T', \pi')$, we will say that experiment $S^* = (T^*, \pi^*)$ is a *combination* of experiments S and S' if

$$\begin{aligned} T^* &= T \times T', \\ \sum_{t' \in T'} \pi^*\left(t, t'|\theta\right) &= \pi\left(t|\theta\right) \text{ for each } t \in T \text{ and } \theta \in \Theta, \text{ and} \\ \sum_{t \in T} \pi^*\left(t, t'|\theta\right) &= \pi'\left(t'|\theta\right) \text{ for each } t' \in T' \text{ and } \theta \in \Theta. \end{aligned}$$

Note that this definition places no restrictions on whether signals $t \in T$ and $t' \in T'$ are independent or correlated, conditional on θ, under π^*. Thus any pair of experiments S and S' will have many combined experiments. An experiment S^* is said to be an *expansion* of S if S^* is a combination of S and some other experiment S'.

Now we have the following result motivating our analysis of Bayes correlated equilibria:

Proposition 1 (Epistemic Relationship)
Random choice rule ν is a BNE random choice rule for (G, S^*) for some expansion S^* of S if and only if ν is a BCE random choice rule of (G, S).

We omit formal proofs in this section as they are special cases of the many player analysis that follows.

2.4 Comparing Experiments

Our definition and foundation for Bayes correlated equilibrium clearly suggests that more information must reduce the set of BCE random choice rules by giving rise to more incentive constraints. Thus it is natural to ask what is the right "incentive ordering" on experiments in general that captures how more information reduces the set of possible obedient random choice rules. But what exactly is the right definition of "more information" in this context?

Blackwell (1951), (1953) introduced a famous partial ordering on experiments. His focus was not on incentive constraints. He showed the equivalence between two orderings on experiments, a "statistical" ordering - capturing the statistical relationship between experiments - and a "feasibility" ordering - capturing which experiment allowed the decision maker to attain more outcomes. In this Section, we will describe an incentive ordering and show its equivalence to the statistical and feasibility orderings which Blackwell showed were equivalent. While these distinctions may seem subtle in the single player case, we find this trichotomy into incentive, feasibility and statistical orderings very helpful in the many player case and therefore it is useful to develop intuition here.

An Incentive Ordering of Experiments Writing $BCE(G,S)$ for the set of BCE random choice rules of (G,S), we want to study the following ordering on experiments:

Definition 5 (Incentive Constrained)
Experiment S is more incentive constrained than experiment S' if, for all decision problems G,

$$BCE(G,S) \subseteq BCE(G,S').$$

A Feasibility Ordering of Experiments A key use of information is in enabling more state dependence of behavior. Thus we are interested in which random choice rules are feasible in a given experiment.

Definition 6 (Feasible Random Choice Rule)
A random choice rule ν is feasible for (G,S) if it is induced by a decision rule σ which is belief invariant for (G,S).

Our preferred feasibility ordering directly formalizes this idea. Writing $F(G,S)$ for the set of random choice rules that are feasible for (G,S), we have:

Definition 7 (More Permissive)
Experiment S is more permissive than experiment S' if, for all decision problems G,

$$F(G,S) \supseteq F(G,S').$$

Notice that while we find it useful to define feasibility and thus permissiveness in terms of decision problems, recall that a decision problem $G = (A, u, \psi)$ consists of a set of actions, utility function and prior, and that our definition of feasibility, and thus permissiveness, uses only the action set and thus is a "pure" feasibility ordering that does not refer to utility functions.

Blackwell (1951), (1953) used a different ordering to capture feasibility: one experiment was "more informative" than another if, in any decision problem, it allowed the decision maker to achieve a larger set of mappings from states to state contingent expected utilities. In particular, in our language, any belief invariant decision rule σ induces a random choice rule ν which implies that the expectation of u in state θ, denoted by $w(\theta)$, would be:

$$w(\theta) \triangleq \sum_{a \in A} \nu(a|\theta)\, u(a,\theta).$$

We can write $W(G,S)$ for the set of state-dependent vectors of ex ante expectations that can arise in this way, so

$$W(G,S) \triangleq \bigcup_{\nu \in F(G,S)} \left\{ \left(\sum_{a \in A} \nu(a|\theta)\, u(a,\theta) \right)_{\theta \in \Theta} \right\} \subseteq \mathbb{R}^{|\Theta|},$$

and an element $w \in W(G,S)$ is a vector, denoted by $w = (w(\theta))_{\theta \in \Theta} \in \mathbb{R}^{|\Theta|}$, where $|\Theta|$ is the cardinality of the state space Θ . Each entry in the vector represents the expected utility of the agent in some state $\theta \in \Theta$. Thus Blackwell's definition written in our language is:

Definition 8 (More Informative)
Experiment S is more informative than experiment S' if, for all decision problems G,

$$W(G,S) \supseteq W\left(G,S'\right).$$

This definition does not take a position on whether high expectations of u or low expectations of u are a "good thing;" it only uses the utility function to map feasibility into a different space, namely the space of expected utility vectors. We observe that the definition of "more informative" invokes the action set and the utility function, but it does not use the prior of the decision problem.

Economists have tended to focus on yet another feasibility ordering (see Marschak and Miyasawa (1968)). Write $\overline{w}(G,S)$ for the highest ex ante utility that a decision maker can attain with a belief invariant decision rule (and thus a feasible random choice rule),

$$\overline{w}(G,S) \triangleq \max_{w \in W(G,S)} \left\{ \sum_{\theta \in \Theta} \psi(\theta) w(\theta) \right\} = \max_{\nu \in F(G,S)} \left\{ \sum_{a \in A, \theta \in \Theta} \psi(\theta) \nu(a|\theta) u(a,\theta) \right\}.$$

Definition 9 (More Valuable)
Experiment S is more valuable than S' if, for all decision problems G,

$$\overline{w}(G,S) \geq \overline{w}\left(G,S'\right).$$

This ordering also uses the prior ψ on states in the definition, as well as the action set A and the utility function u used in Blackwell's definition of "more informative" relation. It is important to note that this definition does not refer to the optimality (or obedience) of the decision rules generating it.

Now we have that these three feasibility orderings are all equivalent:

Proposition 2 *The following statements are equivalent*

1. *Experiment S is more permissive than S'*

2. *Experiment S is more informative than S'*

3. *Experiment S is more valuable than S'*

It follows from the definitions that (1) implies (2), which in turn implies (3). A separating hyperplane argument shows that (3) or (2) imply (1). Thus this forms a significant step in the argument for Blackwell's Theorem as usually stated. We choose to emphasize the conceptual similarity between the three orderings, as "more permissive" captures the idea that more outcomes are feasible if there is more information. While not the focus of this paper, for completeness we will state a many player version of Proposition 2 in the Appendix 8.1 and provide an elementary proof.

A Statistical Ordering on Experiments We can state Blackwell's classic statistical ordering on experiments concisely and intuitively using the language of combined experiments:

Definition 10 (Sufficient Experiment)

Experiment $S = (T, \pi)$ is sufficient for experiment $S' = (T', \pi')$ if there exists a combination of experiments $S^* = (T^*, \pi^*)$ with

$$\Pr\left(t'|t, \theta\right) = \frac{\pi^*\left(t', t|\theta\right)}{\sum\limits_{\widetilde{t}' \in T'} \pi^*\left(\widetilde{t}', t|\theta\right)} \tag{6}$$

independent of θ.

We like this definition because it captures the intuitive idea of "sufficiency" and is a version of the definition that extends to the many player case in a natural way. A version of this definition was used in Marschak and Miyasawa (1968). Another way of stating it is that (θ, t, t') form a Blackwell triple, so that, by the conditional independence lemma, an equivalent way of stating (6) is that for some (or, equivalently, all) strictly positive priors $\psi \in \Delta_{++}(\Theta)$,

$$\Pr\left(\theta|t, t'\right) = \frac{\psi(\theta)\,\pi^*\left(t', t|\theta\right)}{\sum\limits_{\widetilde{\theta} \in \Theta} \psi\left(\widetilde{\theta}\right)\pi^*\left(t', t|\widetilde{\theta}\right)}$$

is independent of t'. A third - Markov kernel - way of defining sufficiency is that there exists $\phi : T \to \Delta(T')$ such that

$$\pi'\left(t'|\theta\right) = \sum_{t \in T} \pi(t|\theta)\,\phi\left(t'|t\right)$$

for all $t' \in T'$ and $\theta \in \Theta$.

Finally, notice that the belief invariance property can be expressed in the language of sufficiency. A set of actions A and a random choice rule ν together define an experiment (A, ν). Now random choice rule ν can be induced by a belief invariant decision rule if and only if experiment S is sufficient for experiment (A, ν).

2.5 Equivalence of Three Orderings

Now we can report the equivalence of statistical, incentive and feasibility orderings.

Theorem 1 (Equivalence)
The following statements are equivalent:

1. *Experiment S is sufficient for experiment S'*

2. *Experiment S is more incentive constrained than experiment S'*

3. *Experiment S is more permissive than experiment S'*

Blackwell (1951), (1953) showed the equivalence of "is sufficient for" to "is more informative than". Blackwell's theorem is thus implied by Proposition 2 and Theorem 1. Our main result will be a many player generalization of Theorem 1 and we will present a formal proof for that more general case. The proof of the equivalence of "is sufficient for" and "is more permissive than" is straightforward. (1) can be shown to imply (2) by using sufficiency to show that the obedience constraints for (G, S') can always be expressed as averages of obedience constraints for (G, S), and are thus less restrictive. This is how the formal argument will work in the many player generalization. There is also a simple argument to show that (2) implies (3) in the one person case, although this argument does not generalize to the many player case.[4] The simple argument for the one player case goes as follows. If (3) fails, S is not more valuable than S'. Thus there is a decision problem G with $v(G, S') > v(G, S)$. Thus there is a BCE random choice rule $\nu \in BCE(G, S)$ which gives ex ante utility $v(G, S)$. But any BCE random choice rule of $v(G, S')$ gives ex ante utility at least $v(G, S')$, and thus $\nu \notin BCE(G, S')$. Thus $BCE(G, S) \subsetneq BCE(G, S')$ and so (2) fails. The sufficiency ordering is a partial order, with many experiments not comparable. We will later, in particular in Section 4.4, note some elementary and well known properties of this ordering, as these properties we will have interesting many player generalizations.

2.6 Leading Example

We will use the following example to illustrate ideas in this section. There are two states $\Theta = \{\theta_0, \theta_1\}$. Let G be given by $A = \{a_0, a_1\}$, a uniform prior on states and a payoff function such that the payoff is 1 if the action matches the state and 0 otherwise. Thus $u : A \times \Theta \to \mathbb{R}$ is given by the following table:

u	θ_0	θ_1
a_0	1	0
a_1	0	1

(7)

[4] We are grateful to Bruno Strulovici for suggesting this argument.

where the rows correspond to actions and columns correspond to states. Consider a family of information structures S_q with signal space $T = \{t_0, t_1\}$ and where the signal is equal to the state with probability $q \in \left[\frac{1}{2}, 1\right]$. Thus $\pi : \Theta \to \Delta(T)$ can be summarized in the following table where rows correspond to signals and columns correspond to states:

π	θ_0	θ_1
t_0	q	$1-q$
t_1	$1-q$	q

(8)

We can illustrate Bayes correlated equilibrium using this example. A decision rule in the example is a vector of four numbers $(\sigma_{00}, \sigma_{01}, \sigma_{10}, \sigma_{11})$ where σ_{ij} is the probability of choosing action 0 in state θ_i when the signal is t_j. The defining inequality for obedience, given earlier by (4), generates in the present binary decision problem with binary signals four inequalities, namely one for every possible pair (a, t) :

$$\begin{array}{lccc}
(a_0, t_0): & \frac{1}{2}q\sigma_{00} & \geq & \frac{1}{2}(1-q)\sigma_{10} \\
(a_1, t_0): & \frac{1}{2}(1-q)(1-\sigma_{10}) & \geq & \frac{1}{2}q(1-\sigma_{00}) \\
(a_0, t_1): & \frac{1}{2}(1-q)\sigma_{01} & \geq & \frac{1}{2}q\sigma_{11} \\
(a_1, t_1): & \frac{1}{2}q(1-\sigma_{11}) & \geq & \frac{1}{2}(1-q)(1-\sigma_{01})
\end{array} \tag{9}$$

A random choice rule in the example is a pair of numbers (ν_0, ν_1) where ν_i is the probability that action 0 is taken when the state is θ_i. Now (ν_0, ν_1) is a BCE random choice rule if and only if there exist $(\sigma_{00}, \sigma_{01}, \sigma_{10}, \sigma_{11})$ satisfying the obedience inequalities (9) with

$$\begin{aligned}
\nu_0 &= q\sigma_{00} + (1-q)\sigma_{01} \\
\nu_1 &= (1-q)\sigma_{10} + q\sigma_{11}
\end{aligned} \tag{10}$$

Clearly, the obedience conditions imply that $\nu_0 \geq \nu_1$, which we can infer by summing any two inequalities of (9) for a given signal t_j. But if the signal contains any information at all, i.e. $q > 1/2$, then the inequality can be strengthened; using the symmetry of the binary signal, we can sum the above inequalities for (a_0, t_0) and (a_1, t_1), or equivalently, (a_1, t_0) and (a_0, t_1) to find that the set of BCE outcomes with a given information structure S_q is completely described by:

$$BCE(q) = \{(\nu_0, \nu_1) \,|\, \nu_0 \geq \nu_1 + 2q - 1\}.$$

This set is illustrated in Figure 1 for different values of q.

Insert Figure 1 here

Next, we can identify the Bayes Nash equilibrium. In our example, for any $q > \frac{1}{2}$, the unique BNE is to choose action a_0 given signal t_0 and action a_1 given signal t_1, and thus the unique BNE random choice rule has $(\nu_0, \nu_1) = (q, 1-q)$. Note that this is a boundary point of the set of Bayes correlated equilibria.

We see in Figure 1 that the set of BCE random choice rules shrinks as the informativeness of the experiments as represented by q improves. It visually suggests that the incentive ordering of the experiments suggested in Definition 5 indeed captures the idea that more information reduces the set of obedient random choice maps.

It might be helpful to state the results of our leading example in an interpretation taken from Kamenica and Gentzkow (2011). Suppose that states θ_0 and θ_1 correspond to the innocence or guilt of a suspect and that the decision maker is a juror who gets utility 1 from delivering the correct verdict and utility 0 from delivering an incorrect verdict. We noted earlier that a random choice rule (ν_0, ν_1) is a BCE random choice rule under information structure S_q if and only if

$$\nu_0 \geq \nu_1 + 2q - 1. \tag{11}$$

Proposition 1 says that for any such (ν_0, ν_1), we can find a expansion of S_q such that there is a Bayes Nash equilibrium inducing (ν_0, ν_1). For example, suppose that $q = \frac{3}{4}$, $\nu_1 = 0$ and $\nu_0 = \frac{1}{2}$. Consider the expanded experiment under which, in addition to observing $S_{\frac{3}{4}}$, if the juror observed t_0 (i.e., a signal implying that the suspect was innocent with probability $\frac{2}{3}$) and if the suspect was in fact innocent (i.e., the true state was θ_0), then with conditional probability $\frac{2}{3}$ the juror observed an additional announcement that the suspect was in fact innocent. Under this information structure, it is optimal for him to acquit only if he observes the additional signal. If he does not receive an additional signal, Bayes rule calculations show that he attaches probability $\frac{1}{2}$ to the suspect being guilty and it is weakly optimal for him to convict. Under this scenario, the suspect is never acquitted if guilty (i.e., $\nu_1 = 0$) and acquitted with only probability $\frac{1}{2}$ if innocent.

We presented Proposition 1 as a one player version of an incomplete information generalization of the characterization of the correlated equilibrium due to Aumann (1987). This result has many uses and many interpretations, some of which have already been developed and studied.

Suppose (to continue the Kamenica and Gentzkow (2011) interpretation), an unscrupulous district attorney (DA) was interested in acquitting as rarely as possible; the DA would thus want to minimize the ex ante probability of acquittal,

$$\frac{1}{2}\nu_0 + \frac{1}{2}\nu_1. \tag{12}$$

If the district attorney could pick a Bayes correlated equilibrium, his problem would be to minimize (12) subject to (11), and thus, if $q = \frac{3}{4}$, he would set $\nu_1 = 0$ and $\nu_0 = \frac{1}{2}$ and achieve an acquittal rate of $\frac{1}{4}$. Proposition 1 says that if the DA is unable to prevent the juror observing his initial signal under S_q but is able to control what additional information the juror can observe (and commit to this rule ex ante), then it is as if the DA can pick a BCE and we described above the additional disclosure rule that supports his most preferred outcome as a BNE. Thus the set of BCE characterize the set of outcomes that can be achieved by "Bayesian persuasion". Kamenica and Gentzkow (2011) focus on the case where there is no prior information, or, equivalently, the information structure is the null experiment $\underline{S} = (\underline{T}, \underline{\pi})$ where $\underline{T} = \{\underline{t}\}$ and $\underline{\pi}(\underline{t}|\theta) = 1$ for all θ. Note that in this case, the obedience condition becomes (omitting dependence of the decision rule on the null signal)

$$\sum_{\theta \in \Theta} \psi(\theta) \sigma(a|\theta) u(a, \theta) \geq \sum_{\theta \in \Theta} \psi(\theta) \sigma(a|\theta) u(a', \theta) \tag{13}$$

for all $a, a' \in A$. Kamenica and Gentzkow (2011) give both a general characterization of BCE outcomes (in a more general case with infinite actions) and a number of applications.[5]

Caplin and Martin (2011) introduce a theoretical and empirical framework for analyzing imperfect perception. If we, as outside observers, do not know how decision makers interpret cues provided to them, it is as if they may have observed additional information but we do not know what it is. In particular, they test whether "ideal data sets" are consistent with rationality. Ideal data sets are equivalent in our language to random choice rules for the null experiment $\underline{S}$.[6] Caplin and Martin (2011) introduce tests of rationality, i.e., obedience, in this setting

Chwe (2006) studied the implications of incentive constraints without ruling out the possibility of decision makers having access to additional information, and in this sense studies implications of Bayes correlated equilibria. In particular, fix two actions, a and a' and consider two random variables defined on $A \times \Theta$, the payoff gain to choosing action a over a':

$$\Delta_a(a', \theta) = u(a, \theta) - u(a', \theta);$$

and an indicator function for choosing action a:

$$\mathbb{I}_a(a', \theta) = \begin{cases} 1, \text{ if } a' = a; \\ 0, \text{ if } a' \neq a. \end{cases}$$

Chwe (2006) shows (in our language) that if a decision rule σ is obedient under the null experiment (i.e., satisfies (13)), then there is a non-negative covariance between Δ_a and $\mathbb{I}_a$ under the measure on $A \times \Theta$ induced by σ.

We suggested three different orderings of experiments above and indeed we can visually represent them just we represented the incentive ordering. In present example, the set of feasible random choice rules is

[5] Rayo and Segal (2010) analyze a related problem, but one where the decision maker does have prior information. Their analysis does not quite fit our framework because the person designing the information structure does not have access to the decision maker's prior information.

[6] We are grateful to Jonathan Weinstein for bringing this connection to our attention.

equal to set of (ν_0, ν_1) in the convex hull of

$$(0,0), (q, 1-q), \ (1-q, q), (1,1).$$

The feasible random choice rules simply express the possible mappings $\nu : \Theta \to \Delta(A)$ that can be generated from belief invariant decision rule. These four points correspond, respectively, to the decision rules, never acquit, acquit only if innocent, acquit only if guilty and always acquit. In other words, the agent can match the action a_i with the received signal t_i, or reverse match action a_j to signal t_i with $i \neq j$, this generates the points $(q, 1-q)$ and $(1-q, q)$. Alternatively, he can choose not respond to the signal and choose a constant (pure) action, thus generating $(0,0)$ and $(1,1)$. Now, given the experiment, the only remaining degree of freedom is to randomize over these four pure action, thus generating the convex hull. In Figure 2, we plot the set of BCE random choice rules and the set of feasible random choice rules for $q = \frac{5}{8}$.

Insert Figure 2 here.

Observe that these sets intersect at the unique BNE random choice rule. This will be a generic property in the one player case, but we will see that it does not extend to the many player case. In Figure 3, we see how the set of feasible random choice rules expand as q increases to $\frac{7}{8}$ and the set of BCE random choice rules shrinks.

Insert Figure 3 here.

Finally, we notice that in Figure 2 and 3 we visually represented the feasibility constraints and the incentive constraints, thus invoking two of three equivalent criteria of Theorem 1. But in light of the earlier observation that the belief invariance property can be expressed by the sufficiency, and now the above equivalence result, it follows that the feasible set of the random choice rules is at the same time representing the (binary) information structures for which S_q is sufficient. In particular, the random choice rule that represents the BNE in either Figure 2 or 3 can be taken to represent the symmetric (and binary) information structure S_q, interpreting the action as the signal. We can then ask for which binary, but not necessarily symmetric information structures is S_q sufficient. The answer is simply all the binary information structures $S = (T, \pi)$ of the form:

π	θ_0	θ_1
t_0	ν_0	$1-\nu_1$
t_1	$1-\nu_0$	ν_1

where (v_0, v_1) is feasible under S_q. We also see the set of feasible random choice rules is independent of the prior whereas the set of BCE uses the prior in the computation of the boundary. In particular the slope of boundary is determined by the ratio of the probability of the two states, $\Pr(\theta_0)/\Pr(\theta_1)$, which is 1 under the uniform prior considered here.

3 The General Case: Many Players

3.1 Definition of Bayes Correlated Equilibrium

There are I players, $1, 2, ..., I$, and we write i for a typical player. There is a finite set of states, Θ, and we write θ for a typical state. A *basic game* G consists of (1) for each player i, a finite set of actions A_i and a utility function $u_i : A \times \Theta \rightarrow \mathbb{R}$; and (2) a full support prior $\psi \in \Delta_{++}(\Theta)$, where we write $A = A_1 \times \cdots \times A_I$. Thus $G = \left((A_i, u_i)_{i=1}^{I}, \psi\right)$. An *information structure* S consists of (1) for each player i, a finite set of signals (or types) T_i; and (2) a signal distribution $\pi : \Theta \rightarrow \Delta(T)$, where we write $T = T_1 \times \cdots \times T_I$. Thus $S = \left((T_i)_{i=1}^{I}, \pi\right)$.

Together, the basic game G and the information structure S define a standard incomplete information game. While we use different notation, this division of an incomplete information game into the "basic game" and the "information structure" has been used in the literature (see, for example, Lehrer, Rosenberg, and Shmaya (2010)).

Two extreme information structures play an important role. The *null information* structure $\underline{S}$ has $\underline{T}_i = \{\underline{t}_i\}$ for all i and $\underline{\pi}(\underline{t}|\theta) = 1$ for all $\theta \in \Theta$. Thus the null information structure provides no information. The *complete information* structure $\overline{S}$ has $\overline{T}_i = \Theta$ for all i and

$$\overline{\pi}(t|\theta) = \begin{cases} 1, & \text{if } \quad t_i = \theta \text{ for all } i, \\ 0, & \text{otherwise,} \end{cases}$$

for all $\theta \in \Theta$.

A decision rule in the incomplete information game (G, S) is a mapping

$$\sigma : T \times \Theta \rightarrow \Delta(A).$$

One way to mechanically understand the notion of the decision rule in a many player environment is to view the decision rule as the strategy of a mediator who observes the realization of $\theta \in \Theta$ chosen according to ψ and the realization of $t \in T$ according to $\pi(\cdot|\theta)$; and then picked an action profile $a \in A$, and privately announced to player i the draw of a_i. For players to have an incentive to follow the mediator's recommendation in this scenario, it would have to be the case that the recommended action a_i was always preferred to any other action a_i' conditional on the signal t_i that player i had received and his knowledge of the recommended action a_i. This is reflected in the following incentive compatibility condition.

Definition 11 (Obedience)

Decision rule σ is obedient for (G,S) if, for each $i=1,...,I$, $t_i \in T_i$ and $a_i \in A_i$, we have

$$\begin{aligned} & \sum_{a_{-i},t_{-i},\theta} \psi(\theta)\,\pi((t_i,t_{-i})\,|\theta)\,\sigma((a_i,a_{-i})\,|\,(t_i,t_{-i}),\theta)\,u_i((a_i,a_{-i}),\theta) \\ \geq \quad & \sum_{a_{-i},t_{-i},\theta} \psi(\theta)\,\pi((t_i,t_{-i})\,|\theta)\,\sigma((a_i,a_{-i})\,|\,(t_i,t_{-i}),\theta)\,u_i\left((a_i',a_{-i}),\theta\right); \end{aligned} \tag{14}$$

for all $a_i' \in A_i$.

Definition 12 (Bayes Correlated Equilibrium)

A decision rule σ is a Bayes correlated equilibrium (BCE) of (G,S) if it is obedient for (G,S).

If there is complete information, i.e., if Θ is a singleton, and S is the null information structure, then this definition reduces to the Aumann (1987) definition of correlated equilibrium. We provide our motivation for studying this particular generalization next. We postpone until Section 6 a discussion of how this relates to (and why it is weaker than) other definitions in the literature on incomplete information correlated equilibrium.

3.2 Foundations of Bayes Correlated Equilibrium

In this section, we provide our rationale for being interested in Bayes correlated equilibria. Consider an analyst who knows that

1. The basic game G describes actions, payoff functions depending on states, and a prior distribution on states.
2. The players observe at least information structure S, but may observe more.
3. The full, common prior, information structure is common certainty among the players.
4. The players' actions follow a Bayes Nash equilibrium.

What can she deduce about the joint distribution of actions, signals from the information structure S and states? In this section, we will formalize this question and show that all she can deduce is that the distribution will be a Bayes correlated equilibrium of (G,S).

We first review the standard approach to analyzing incomplete information games. A (behavioral) strategy for player i in the incomplete information game (G,S) is $\beta_i : T_i \rightarrow \Delta(A_i)$. The following is the standard definition of Bayes Nash equilibrium in this setting.

Definition 13 (Bayes Nash Equilibrium)
A strategy profile β is a Bayes Nash equilibrium (BNE) of (G,S) if for each $i=1,2,...,I$, $t_i \in T_i$ and $a_i \in A_i$ with $\beta_i(a_i|t_i) > 0$, we have

$$\begin{aligned}&\sum_{a_{-i},t_{-i},\theta} \psi(\theta)\pi((t_i,t_{-i})|\theta)\left(\prod_{j\neq i}\beta_j(a_j|t_j)\right)u_i((a_i,a_{-i}),\theta) \\ \geq &\sum_{a_{-i},t_{-i},\theta} \psi(\theta)\pi((t_i,t_{-i})|\theta)\left(\prod_{j\neq i}\beta_j(a_j|t_j)\right)u_i\left((a_i',a_{-i}),\theta\right),\end{aligned} \tag{15}$$

for each $a_i' \in A_i$.

A Bayes Nash equilibrium β is a profile of strategies. To compare solution concepts, we would like to discuss the decision rule corresponding to a BNE. Decision rule σ is said to be *induced* by strategy profile β if

$$\sigma(a|t,\theta) = \left(\prod_{i=1}^{I}\beta_i(a_i|t_i)\right). \tag{16}$$

for each $a \in A$, $t \in T$ and $\theta \in \Theta$.

Definition 14 (Bayes Nash Equilibrium Decision Rule)
Decision rule σ is BNE decision rule of (G,S) if there exists a Bayesian Nash equilibrium β of (G,S) that induces σ.

We now have the following straightforward but important observation:

Lemma 2 *Every Bayes Nash equilibrium decision rule of (G,S) is a Bayes correlated equilibrium of (G,S).*

Proof. Let σ be induced by BNE strategy profile β. To show obedience, observe that

$$\begin{aligned}&\sum_{a_{-i},t_{-i},\theta} \psi(\theta)\pi((t_i,t_{-i})|\theta)\sigma((a_i,a_{-i})|(t_i,t_{-i}),\theta)u_i((a_i,a_{-i}),\theta) \\ = &\sum_{a_{-i},t_{-i},\theta} \psi(\theta)\pi((t_i,t_{-i})|\theta)\left(\prod_{j=1}^{I}\beta_j(a_j|t_j)\right)u_i((a_i,a_{-i}),\theta), \text{ by (16)}\end{aligned}$$

$$\begin{aligned}= \ &\beta_i(a_i|t_i)\sum_{a_{-i},t_{-i},\theta} \psi(\theta)\pi((t_i,t_{-i})|\theta)\left(\prod_{j\neq i}\beta_j(a_j|t_j)\right)u_i((a_i,a_{-i}),\theta) \\ \geq \ &\beta_i(a_i|t_i)\sum_{a_{-i},t_{-i},\theta} \psi(\theta)\pi((t_i,t_{-i})|\theta)\left(\prod_{j\neq i}\beta_j(a_j|t_j)\right)u_i\left((a_i',a_{-i}),\theta\right), \text{ by (15)}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{a_{-i},t_{-i},\theta} \psi(\theta)\,\pi((t_i,t_{-i})\,|\theta) \left(\prod_{j=1}^{I} \beta_j(a_j|t_j) \right) u_i\left((a_i',a_{-i}),\theta\right) \\
&= \sum_{a_{-i},t_{-i},\theta} \psi(\theta)\,\pi((t_i,t_{-i})\,|\theta)\,\sigma((a_i,a_{-i})\,|\,(t_i,t_{-i}),\theta)\,u_i\left((a_i',a_{-i}),\theta\right), \text{ by (16),}
\end{aligned}
$$

which yields the definition of an obedient decision rule. ■

We will be interested in what can be said about actions and states if signals are not observed. We will call a mapping

$$\nu : \Theta \to \Delta(A),$$

a *random choice rule*, and say ν is induced by decision rule σ if it is the marginal of σ on A. Random choice rule ν is a Bayes Nash (correlated) equilibrium random choice rule of (G,S) if it is induced by a Bayes Nash (correlated) equilibrium decision rule of (G,S).

We want to discuss situations where players observe more information than that contained in a single information structure. To formalize this, we must discuss combinations of information structures. If we have two information structures $S^1 = (T^1,\pi^1)$ and $S^2 = (T^2,\pi^2)$, we will say that information structure $S^* = (T^*,\pi^*)$ is a *combination* of experiments S^1 and S^2 if

$$
\begin{aligned}
T_i^* &= T_i^1 \times T_i^2 \text{ for each } i; \\
\sum_{t^2\in T^2} \pi^*\left(t^1,t^2|\theta\right) &= \pi^1\left(t^1|\theta\right) \text{ for each } t^1 \in T^1 \text{ and } \theta\in\Theta \text{ and} \\
\sum_{t^1\in T^1} \pi^*\left(t^1,t^2|\theta\right) &= \pi^2\left(t^2|\theta\right) \text{ for each } t^2 \in T^2 \text{ and } \theta\in\Theta.
\end{aligned}
$$

Note that this definition places no restrictions on whether signals $t^1 \in T^1$ and $t^2 \in T^2$ are independent or correlated, conditional on θ, under π^*. Thus any pair of information structures S^1 and S^2 will have many combined information structures. An experiment S^* is said to be an *expansion* of S^1 if S^* is a combination of S^1 and some other experiment S^2.

Theorem 2 (Epistemic Relationship)
A decision rule σ is a Bayes correlated equilibrium of (G,S) if and only if, for some expansion S^* of S, it is a BNE decision rule of (G,S^*).

Thus this is an incomplete information analogue of the Aumann (1987) characterization of correlated equilibrium for complete information games. An alternative interpretation of this result - following Aumann (1987) - would be to say that BCE captures the implications of common certainty of rationality (and the common prior assumption) in the game G when player have at least information S, since requiring BNE in some game with expanded information is equivalent to describing a belief closed subset where the game G is being played, players have access to (at least) information S and there is common certainty of rationality.[7] The proof follows a very similar logic to the analogous results of Aumann (1987) for complete information and that of Forges (1993) for the Bayesian solution (discussed in Section 6).

An interesting question that we do not explore is what we can say about the relation between Bayes correlated equilibria and the expansions that are needed to support them as Bayes Nash equilibria. Milchtaich (2012) examines properties of devices needed to implement correlated equilibria, and tools developed in his paper might be useful for this task.

Proof. Suppose that σ is a Bayes correlated equilibrium of (G, S). Thus

$$\begin{aligned} & \sum_{a_{-i},t_{-i},\theta} \psi(\theta)\, \pi((t_i,t_{-i})\,|\theta)\, \sigma((a_i,a_{-i})\,|\,(t_i,t_{-i}),\theta)\, u_i((a_i,a_{-i}),\theta) \\ \geq & \sum_{a_{-i},t_{-i},\theta} \psi(\theta)\, \pi((t_i,t_{-i})\,|\theta)\, \sigma((a_i,a_{-i})\,|\,(t_i,t_{-i}),\theta)\, u_i\left((a_i',a_{-i}),\theta\right) \end{aligned}$$

for each i, $t_i \in T_i$, $a_i \in A_i$ and $a_i' \in A_i$. Let $S^* = \left((T_i^*)_{i=1}^I, \pi^*\right)$ be an expansion of S, and, in particular, a combination of $S = \left((T_i)_{i=1}^I, \pi\right)$ and $S' = \left((T_i')_{i=1}^I, \pi'\right)$, where $T_i' = A_i$ for each i and π^* satisfies

$$\pi^*\left((t_i,a_i)_{i=1}^I \middle|\, \theta\right) = \pi(t|\,\theta)\,\sigma(a\,|t,\theta) \tag{17}$$

for each $a \in A$ and $t \in T$. Now, in the game (G, S^*), consider the "truthful" strategy β_j^* for player j, with

$$\beta_j^*\left(a_j'|t_j,a_j\right) = \begin{cases} 1, & \text{if} \quad a_j' = a_j, \\ 0, & \text{if} \quad a_j' \neq a_j, \end{cases} \tag{18}$$

for all $t_j \in T_j$ and $a_j \in A_j$. Now the interim payoff to player i observing signal (t_i, a_i) and choosing action a_i' in (G, S^*) if he anticipates that each opponent will follow strategy β_j^* is

$$\begin{aligned} & \sum_{a_{-i},t_{-i},a_{-i}',\theta} \psi(\theta)\, \pi^*\left((t_i,t_{-i}),(a_i,a_{-i}')\middle|\,\theta\right) \left(\prod_{j\neq i} \beta_j^*\left(a_j|t_j,a_j'\right)\right) u_i\left((a_i',a_{-i}),\theta\right) \\ = & \sum_{a_{-i},t_{-i},\theta} \psi(\theta)\, \pi((t_i,t_{-i})\,|\theta)\, \sigma((a_i,a_{-i})\,|(t_i,t_{-i}),\theta)\, u_i\left((a_i',a_{-i}),\theta\right), \end{aligned}$$

[7] Aumann and Dreze (2008) extend Aumann (1987) by asking what interim payoff a player might receive consistent with common knowledge of rationality (and the common prior assumption) in a complete information game. They consider the "doubled" game where each action has two identical copies (i.e., leading to the same payoffs). They show that the set of all payoffs that a player might receive in a correlated equilibrium in the doubled game conditional on choosing some action characterizes the set of interim payoffs consistent with common knowledge of rationality. An exactly analogous result holds in our setting: the set of interim payoffs consistent with common knowledge of rationality and observing a particular signal is equal to the set of interim payoffs given that signal and some action in a Bayes correlated equilibrium.

by (17) and (18), and thus Bayes Nash equilibrium optimality conditions for the truth telling strategy profile β^* are implied by the obedience conditions on σ.

Conversely, suppose that β is a Bayes Nash equilibrium of (G, S^*), where S^* is a combined experiment for S and S'. Write $\sigma : T \times \Theta \to \Delta(A)$ for the decision rule induced by β, so that

$$\sigma(a|t,\theta) = \sum_{t' \in T'} \pi^*\left(t, t'|\theta\right) \prod_{j=1}^{I} \beta_j\left(a_j|t_j, t_j'\right).$$

Now $\beta_i\left(a_i|\left(t_i, t_i'\right)\right) > 0$ implies

$$\begin{aligned} &\sum_{a_{-i}, t_{-i}, t'_{-i}, \theta} \psi(\theta)\, \pi^*\left(\left(t_i, t_{-i}\right), \left(t_i', t_{-i}'\right)|\theta\right) \left(\prod_{j \neq i} \beta_j\left(a_j|t_j, t_j'\right)\right) u_i\left(\left(a_i, a_{-i}\right), \theta\right) \\ \geq &\sum_{a_{-i}, t_{-i}, t'_{-i}, \theta} \psi(\theta)\, \pi^*\left(\left(t_i, t_{-i}\right), \left(t_i', t_{-i}'\right)|\theta\right) \left(\prod_{j \neq i} \beta_j\left(a_j|t_j, t_j'\right)\right) u_i\left(\left(a_i', a_{-i}\right), \theta\right), \end{aligned}$$

for each i, $t_i \in T_i$, $t_i' \in T_i'$ and $a_i' \in A_i$. Thus

$$\begin{aligned} &\sum_{t_i'} \beta_i\left(a_i|\left(t_i, t_i'\right)\right) \sum_{a_{-i}, t_{-i}, t'_{-i}, \theta} \psi(\theta)\, \pi^*\left(\left(t_i, t_{-i}\right), \left(t_i', t_{-i}'\right)|\theta\right) \left(\prod_{j \neq i} \beta_j\left(a_j|t_j, t_j'\right)\right) u_i\left(\left(a_i, a_{-i}\right), \theta\right) \\ \geq &\sum_{t_i'} \beta_i\left(a_i|\left(t_i, t_i'\right)\right) \sum_{a_{-i}, t_{-i}, t'_{-i}, \theta} \psi(\theta)\, \pi^*\left(\left(t_i, t_{-i}\right), \left(t_i', t_{-i}'\right)|\theta\right) \left(\prod_{j \neq i} \beta_j\left(a_j|t_j, t_j'\right)\right) u_i\left(\left(a_i', a_{-i}\right), \theta\right), \end{aligned}$$

for each i, $t_i \in T_i$ and $a_i' \in A_i$. But

$$\begin{aligned} &\sum_{t_i'} \beta_i\left(a_i|\left(t_i, t_i'\right)\right) \sum_{a_{-i}, t_{-i}, t'_{-i}, \theta} \psi(\theta)\, \pi^*\left(\left(t_i, t_{-i}\right), \left(t_i', t_{-i}'\right)|\theta\right) \left(\prod_{j \neq i} \beta_j\left(a_j|t_j, t_j'\right)\right) u_i\left(\left(a_i', a_{-i}\right), \theta\right) \\ = &\sum_{a_{-i}, t_{-i}, \theta} \psi(\theta)\, \pi\left(\left(t_i, t_{-i}\right)|\theta\right) \sigma\left(\left(a_i, a_{-i}\right)|\left(t_i, t_{-i}\right), \theta\right) u_i\left(\left(a_i', a_{-i}\right), \theta\right), \end{aligned}$$

and thus BNE equilibrium conditions imply obedience of σ. ■

4 Comparing Information Structures

We now report on our many player generalizations of Blackwell's theorem.

4.1 Incentive Compatibility Ordering

We write $BCE(G, S)$ for the set of BCE random choice rules of (G, S). Our first ordering is:

Definition 15 (Incentive Constrained)
Information structure S is more incentive constrained than information structure S' if, for all basic games G,

$$BCE\left(G,S\right)\subseteq BCE\left(G,S'\right).$$

4.2 Feasibility Ordering

To make the comparison with Blackwell's Theorem, we introduce feasibility orderings on information structures. One natural generalization of feasibility from the single person case would be to look at random choice rules that could arise from players choosing actions independently given their signals, i.e., according to standard incomplete information game strategies. We will focus on an alternative generalization of feasibility from the single person case that is based on the idea of belief invariance. Recall that a *decision rule* is a mapping $\sigma : T \times \Theta \to \Delta\left(A\right)$. We will abuse notation and write $\sigma_i : T \times \Theta \to \Delta\left(A_i\right)$ for the induced mapping looking only at player i's action, so that

$$\sigma_i\left(a_i|\left(t_i,t_{-i}\right),\theta\right) \triangleq \sum_{a_{-i}\in A_{-i}} \sigma\left(\left(a_i,a_{-i}\right)|\left(t_i,t_{-i}\right),\theta\right). \tag{19}$$

Definition 16 (Belief Invariant Decision Rule)
Decision rule σ is belief invariant for (G,S) if, for each i, $\sigma_i\left(a_i|\left(t_i,t_{-i}\right),\theta\right)$ is independent of (t_{-i},θ).

Thus the condition requires that, from each player i's perspective, his strategy depends only on his own signal. However, it allows for there to be correlation among the players' actions. Equivalently, the decision rule is belief invariant if, for each i, the three variables $(a_i, t_i, (t_{-i},\theta))$ are a Blackwell triple under the distribution on $A_i \times T \times \Theta$ induced by any prior $\psi \in \Delta_{++}(\Theta)$, π and σ. Thus, via Lemma 1, we observe that decision rule σ is belief invariant if and only if, for any prior on θ, the probability of (t_{-i},θ) conditional on t_i and a_i is independent of a_i. Thus it says that the action a player chooses cannot reveal more information about the state and others' types than that contained in his type. This statement and motivation of belief invariance has played a central role in the incomplete information correlated equilibrium literature (see, e.g., Forges (2006)), as we will discuss in the next Section. Notice that while it is convenient to identify belief invariance with a basic game G and information structure S, the only connection is that the actions sets of G and signal sets of S are used in defining the space where σ lives.

Now we define feasible random choices to be those that could arise from belief invariant decision rules.

Definition 17 (Feasible Random Choice Rule)
A random choice rule ν is feasible for (G,S) if it is induced by a decision rule which is belief invariant for (G,S).

We write $F(G,S)$ for the set of feasible random choice rules.

Definition 18 (Permissive Information Structure)
Information structure S is more permissive than information structure S' if, for all basic games G,

$$F(G,S) \supseteq F\left(G,S'\right).$$

We discuss and show equivalence to the alternative versions of feasibility orderings we discussed in the single person case, e.g., Blackwell's "more informative" ordering and economists' "more valuable" ordering in Appendix 8.1.

4.3 Statistical Ordering

We will use the following many player generalization of sufficiency.

Definition 19 (Individual Sufficiency)
Information structure $S=(T,\pi)$ is individually sufficient for information structure $S'=(T',\pi')$ if there exists a combined information structure $S^*=(T^*,\pi^*)$ such that, for each i,

$$\Pr\left(t_i'|t_i,t_{-i},\theta\right)=\frac{\sum\limits_{t_{-i}'}\pi^*\left(\left(t_i,t_{-i}\right),\left(t_i',t_{-i}'\right)|\theta\right)}{\sum\limits_{\widetilde{t}_i',t_{-i}'}\pi^*\left(\left(t_i,t_{-i}\right),\left(\widetilde{t}_i',t_{-i}'\right)|\theta\right)} \tag{20}$$

is independent of t_{-i} and θ.

Definition 20 (Mutual Individual Sufficiency)
Information structures S and S' are mutually individually sufficient if S is individually sufficient for S' and S' is individually sufficient for S.

Thus S is individually sufficient for S' if there is a combined experiment under which, for each i, $(t_i', t_i, (t_{-i},\theta))$ is a Blackwell triple. Via Lemma 1, an equivalent way of defining individual sufficiency is that observing t_i' gives no additional information about t_{-i} and θ beyond that contained in t_i. In particular, (20) is equivalent to the claim that, for some (or every) $\psi\in\Delta_{++}(\Theta)$, for each i,

$$\Pr\left(t_{-i},\theta|t_i,t_i'\right)=\frac{\sum\limits_{t_{-i}'}\psi(\theta)\,\pi^*\left(\left(t_i,t_{-i}\right),\left(t_i',t_{-i}'\right)|\theta\right)}{\sum\limits_{\widetilde{t}_{-i},\widetilde{\theta},t_{-i}'}\psi\left(\widetilde{\theta}\right)\pi^*\left(\left(t_i,\widetilde{t}_{-i}\right),\left(t_i',t_{-i}'\right)|\widetilde{\theta}\right)} \tag{21}$$

is independent of t_i'. A third, Markov kernel, way of defining individual sufficiency is to say that S is individually sufficient for S' if there exists $\phi : T \times \Theta \to \Delta(T')$ satisfying the marginal property,

$$\sum_{t \in T} \pi(t|\theta)\, \phi(t'|t, \theta) = \pi'(t'|\theta) \tag{22}$$

for each t' and θ, and the *independence property*, namely that

$$\sum_{t'_{-i} \in T'_{-i}} \phi\left(\left(t'_i, t'_{-i}\right) | \left(t_i, t_{-i}\right), \theta\right) \tag{23}$$

is independent of t_{-i} and θ.

Notice that (23) is a belief invariance property. Thus a random choice rule ν is feasible for (G, S) if and only if information structure $S = (T, \pi)$ is individually sufficient for the information structure $S' = (A, \nu)$.

The Markov kernel version of individual sufficiency will be useful both in arguments and in allowing us to relate individual sufficiency to earlier concepts in the literature, especially those of Liu (2011) and Lehrer, Rosenberg, and Shmaya (2010), (2011). To motivate individual sufficiency, we will first establish its relation to alternative orderings and the earlier literature and then report some general properties that illustrate how it constitutes a natural ordering on information structures. In Section 5, we illustrate individual sufficiency and the properties discussed in this Section using a family of binary signal examples.

Alternative Orderings One can apply the original definition of sufficiency in the many player context, comparing the joint information of all players. Thus - focussing on the Markov kernel formulation - say that information structure S is *sufficient* for S' if there exists $\phi : T \to \Delta(T')$ such that

$$\sum_{t \in T} \pi(t|\theta)\, \phi(t'|t) = \pi'(t'|\theta) \tag{24}$$

for each $t' \in T$ and $\theta \in \Theta$. From the definition, it is not clear if this ordering on information structures is more stringent or less stringent than individual sufficiency. Intuitively, it is more stringent because it requires the Markov kernel ϕ to be independent of θ, which is not required in the Markov kernel formulation of individual sufficiency; but it is less stringent in that it does not require the belief invariance property (23). We will later report examples where information structure S is sufficient for S' but not individually sufficient for S'. Conversely, we will report other examples where information structure S is individually sufficient for S' but not sufficient for S'.

Lehrer, Rosenberg, and Shmaya (2010), (2011) introduce a number of strengthenings of sufficiency for purposes that we will discuss below in Section 7.3. In particular, they say that S' is a *non-communicating garbling* of S if there exists $\phi : T \times \Theta \to \Delta(T')$ satisfying (22) and (23) which is also independent of θ. In order to clarify the relation to individual sufficiency, we will say in this case that S is a a non-communicating *ungarbling* of S'. Now equivalently, S is a non-communicating ungarbling of S' if there exists a combined experiment for S and S' such that the distribution of t' conditional on t and θ is independent of θ and, for each player i, the distribution of t_i' conditional on t_i is independent of t_{-i} and θ. Clearly, if S is a non-communicating ungarbling of S', then S is sufficient for S' and S is individually sufficient for S'. We will show by example in Section 5 below that the converse is not true: we will describe information structures S and S' such that S is sufficient for S', S is individually sufficient for S' but S is not a non-communicating ungarbling of S'.

Liu (2011) introduced a natural definition of a correlation device in a many player context: if you fix an information structure S and signal sets $(T_i')_{i=1}^I$ that players might observe, Liu (2011) said that a mapping $\phi : T \times \Theta \to \Delta(T')$ was a correlation device if it satisfied (23). Thus we can re-state our definition of individual sufficiency in the language of Liu (2011): information structure S is individually sufficient for information structure S' if there exists a combined experiment S^* such that the implied Markov kernel $\phi : T \times \Theta \to \Delta(T')$ is a correlating device.

4.4 Individual Sufficiency and Higher Order Beliefs

Let us write $S \succeq S'$ if information structure S is individually sufficient for S'. The individual sufficiency ordering is a partial order, with many information structures not comparable, just as sufficiency is a partial ordering with many experiments not comparable. This partial order has some natural properties, and which are many player generalizations of well-establish properties for sufficiency, which we establish next. First, we note that the ordering is transitive with a well defined maximal and minimal information structure.

Lemma 3 (Some Basic Properties)

1. *(Largest and smallest) For all* S, $\overline{S} \succeq S \succeq \underline{S}$.
2. *(Transitivity) If* $S \succeq S'$ *and* $S' \succeq S''$, *then* $S \succeq S''$.

An experiment can be identified with the posterior beliefs that it induces. Thus if we fix an experiment $S = (T, \pi)$ and a full support prior $\psi \in \Delta_{++}(\Theta)$, and we write $\pi_\psi(\theta|t)$ for the implied posterior beliefs, Blackwell (1953) said that an experiment was in *standard form* if no two signals gave rise to the same posteriors.

$$\pi_\psi(\theta|t) \triangleq \frac{\pi(t|\theta)\,\psi(\theta)}{\sum_{\theta' \in \Theta} \pi(t|\theta')\,\psi(\theta')}$$

To facilitate comparison with the many player case, we will label this property "non-redundancy". For any information structure $S = (T, \pi)$, we write

$$\pi_\psi(t_{-i}, \theta|t_i) \triangleq \frac{\sum_{t_{-i} \in T_{-i}} \pi(t_i, t_{-i}|\theta)\,\psi(\theta)}{\sum_{t_{-i} \in T_{-i}} \sum_{\theta' \in \Theta} \pi(t_i, t_{-i}|\theta')\,\psi(\theta')}$$

for player i's implied conditional beliefs under prior $\psi \in \Delta_{++}(\Theta)$. We next establish a relationship between individual sufficiency and higher order beliefs and redundancy.

Definition 21 (Higher Order Belief Equivalent)

1. *Information structure S is non-redundant if, for every i and t_i,$t_i' \in T_i$, there exists $t_{-i} \in T_{-i}$ and $\theta \in \Theta$ such that $\pi_\psi(t_{-i}, \theta | t_i) \neq \pi_\psi(t_{-i}, \theta | t_i')$ for some (or all) $\psi \in \Delta_{++}(\Theta)$.*

2. *Two information structures $S^1 = \left((T_i^1)_{i=1}^I, \pi^1 \right)$ and $S^2 = \left((T_i^2)_{i=1}^I, \pi^2 \right)$ are higher order belief equivalent if there exists a non-redundant information structure $S^* = \left((T_i^*)_{i=1}^I, \pi^* \right)$ such that there exist, for each $i = 1, .., I$ and $k = 1, 2$, $f_i^k : T_i^k \to T_i^*$such that:*

   (a) *for each $k = 1, 2$, $t^* \in T^*$ and $\theta \in \Theta$:*
$$\pi^k \left(\left\{ t^k \middle| f^k \left(t^k \right) = t^* \right\} \middle| \theta \right) = \pi^* \left(t^* | \theta \right); \tag{25}$$

   (b) *for each $k = 1, 2$, $i = 1, ., I$, $t_i \in T_i^k$, $t_{-i} \in T_{-i}^*$ and $\theta \in \Theta$*
$$\pi_\psi \left(\left\{ t_{-i}^k \middle| f_{-i}^k \left(t_{-i}^k \right) = t_{-i}^* \right\}, \theta \middle| t_i \right) = \pi^* \left(t_{-i}^*, \theta | f_i^k (t_i) \right). \tag{26}$$

It is easy (but notationally burdensome) to show that two information structures are higher order belief equivalent if and only if, for any prior over states, they generate the same probability distribution over beliefs and higher order beliefs (i.e., Mertens-Zamir types). We present a formal statement of this equivalence in Lemma 10 in Appendix 8.2.

Lemma 4 (Unique Non-Redundant Information Structure)

1. *For every information structure S, there is a unique non-redundant information structure $\widehat{S}$ such that S and $\widehat{S}$ are mutually individually sufficient and higher order belief equivalent*

2. *Any two information structures are mutually individually sufficient if and only if they are higher order belief equivalent.*

Liu (2011) proved (in Theorem 1) that if two information structures - one of which is non-redundant - are higher order belief equivalent if and only if there is a unique correlating device - in the sense he defined and we report in Section 4.3 - that maps the non-redundant information structure into the perhaps redundant one. This implies that the non-redundant information structure is individually sufficient for the perhaps redundant one. Since it is easy to show that the redundant information structure is sufficient for the non-redundant one, we have they are mutually individually sufficient. This implies part 1 of the Lemma. The second part can be proved by adapting arguments in Lehrer, Rosenberg, and Shmaya (2011); for completeness, in Appendix 8.2, we give a proof. Finally, we note that there is a tight connection between the individual sufficiency ordering and the more basic notion of "expansion" as a measure of more information.

Lemma 5

Information structure S is individually sufficient for information structure S' if and only if S is higher order belief equivalent to an expansion of S'.

4.5 Three Orderings

Now we have:

Theorem 3 *The following statements are equivalent:*

1. *Information structure S is individually sufficient for information structure S'.*
2. *Information structure S is more incentive constrained than information structure S'.*
3. *Information structure S is more permissive than information structure S'.*

The equivalence between (1) and (3) is straightforward and is included (in the Theorem and proof) to highlight the connection with Blackwell's Theorem.

A straightforward Corollary of the equivalence of (1) and (2) is:

Corollary 1 *Information structures S and S' are mutually individually sufficient if and only if each is more incentive constrained than the other.*

As we will discuss below, this could have been proved by adapting either the arguments of Liu (2011) or those of Lehrer, Rosenberg, and Shmaya (2011). However, neither result or argument helps prove our main result, which is the equivalence of (1) and (2). The proof that (1) implies (2) is constructive, showing that if S is individually sufficient for S' and ν is a BCE random choice rule of (G, S), we can use the BCE decision rule inducing ν and the Markov kernel establishing individual sufficiency to construct a BCE of (G, S') which induces ν. The novel argument is that (2) implies (1). We do this by constructing a particular basic game G, and a BCE random choice rule ν of (G, S') such that ν is a BCE random choice rule of (G, S) only if S is individually sufficient for S'. A heuristic version of this argument is to consider a game where players have an incentive to truthfully report their beliefs and higher order beliefs, as in Dekel, Fudenberg, and Morris (2007). There is a BCE random choice rule ν^* of (G, S') where they truthfully report their types. We then show that for there to be a BCE of (G, S) which induces ν (and thus has players report the distribution of beliefs and higher order beliefs corresponding to S), S must be individually sufficient for S'. This argument is only heuristic, since we actually want to construct a finite action basic game. Our formal argument uses a large enough finite approximation to the "higher order beliefs" game.

Proof. We first show that (1) implies (2). Suppose that S is individually sufficient for S'. Take any basic game G and any BCE σ of (G, S). We will construct $\sigma' : T' \times \Theta \to \Delta(A)$ which is a BCE of (G, S') which gives rise to the same stochastic map as σ.

Write $V_i(a_i, a_i', t_i)$ for the expected utility for agent i under distribution ν if he is type t_i, receives recommendation a_i but chooses action a_i', so that

$$V_i\left(a_i, a_i', t_i\right) \triangleq \sum_{a_{-i}\in A_{-i}, t_{-i}\in T_{-i}, \theta\in\Theta} \psi(\theta)\, \pi\left((t_i, t_{-i})\,|\theta\right) \sigma\left((a_i, a_{-i})\,|\,(t_i, t_{-i}), \theta\right) u_i\left(\left(a_i', a_{-i}\right), \theta\right).$$

Now - by Definition 11 - for each $i = 1, ..., I$, $t_i \in T_i$ and $a_i \in A_i$, we have

$$V_i\left(a_i, a_i, t_i\right) \geq V_i\left(a_i, a_i', t_i\right) \tag{27}$$

for each $a_i' \in A_i$. Since S is individually sufficient for S', there exists a mapping $\phi : T \times \Theta \to \Delta(T')$ satisfying (22) and (23). Define $\sigma' : T' \times \Theta \to \Delta(A)$ by

$$\pi'\left(t'|\theta\right) \sigma'\left(a|t', \theta\right) = \sum_{t\in T} \pi\left(t|\theta\right) \sigma\left(a|t, \theta\right) \phi\left(t'|t, \theta\right). \tag{28}$$

Symmetrically, write $V_i'(a_i, a_i', t_i')$ for the expected utility for agent i under decision rule σ' if he is type t_i', receives recommendation a_i but chooses action a_i', so that

$$V_i'\left(a_i, a_i', t_i'\right) \triangleq \sum_{a_{-i}\in A_{-i}, t_{-i}'\in T_{-i}', \theta\in\Theta} \psi(\theta)\, \pi'\left(\left(t_i', t_{-i}'\right)|\theta\right) \sigma\left((a_i, a_{-i})\,|\,\left(t_i', t_{-i}'\right), \theta\right) u_i\left(\left(a_i', a_{-i}\right), \theta\right).$$

Now σ satisfies the obedience condition (Definition 11) to be a correlated equilibrium of (G, S') if for each $i = 1, ..., I$, $t_i' \in T_i'$ and $a_i \in A_i$,

$$V_i'\left(a_i, a_i, t_i'\right) \geq V_i'\left(a_i, a_i', t_i'\right)$$

for all $a_i' \in A_i$. But

$$\begin{aligned}
& V_i'\left(a_i, a_i', t_i'\right) \\
= & \sum_{a_{-i}\in A_{-i}, t_{-i}'\in T_{-i}', \theta\in\Theta} \psi(\theta)\, \pi'\left(\left(t_i', t_{-i}'\right)|\theta\right) \sigma\left(\left(a_i, a_{-i}\right) | \left(t_i', t_{-i}'\right), \theta\right) u_i\left(\left(a_i', a_{-i}\right), \theta\right) \\
= & \sum_{a_{-i}\in A_{-i}, t_{-i}'\in T_{-i}', \theta\in\Theta, t\in T} \psi(\theta)\, \pi(t|\theta)\, \sigma\left(\left(a_i, a_{-i}\right)|t, \theta\right) \phi\left(t'|t, \theta\right) u_i\left(\left(a_i', a_{-i}\right), \theta\right) \\
& \text{by the definition of } \nu'\text{, see (28)} \\
= & \sum_{a_{-i}\in A_{-i}, \theta\in\Theta, t\in T} \psi(\theta)\, \pi(t|\theta)\, \sigma\left(\left(a_i, a_{-i}\right)|t, \theta\right) u_i\left(\left(a_i', a_{-i}\right), \theta\right) \sum_{t_{-i}'\in T_{-i}'} \phi_i\left(\left(t_i', t_{-i}'\right)|t, \theta\right) \\
= & \sum_{a_{-i}\in A_{-i}, \theta\in\Theta, t\in T} \psi(\theta)\, \pi(t|\theta)\, \sigma\left(\left(a_i, a_{-i}\right)|t, \theta\right) u_i\left(\left(a_i', a_{-i}\right), \theta\right) \phi_i\left(t_i'|t_i\right), \text{ by (23)} \\
= & \sum_{t_i\in T_i} \phi_i\left(t_i'|t_i\right) \left[\sum_{a_{-i}\in A_{-i}, \theta\in\Theta, t_{-i}\in T_{-i}} \psi(\theta)\, \pi\left(\left(t_i, t_{-i}\right)|\theta\right) \sigma\left(\left(a_i, a_{-i}\right) | \left(t_i, t_{-i}\right), \theta\right) u_i\left(\left(a_i', a_{-i}\right), \theta\right)\right] \\
= & \sum_{t_i\in T_i} \phi_i\left(t_i'|t_i\right) V_i\left(a_i, a_i', t_i\right). \qquad (29)
\end{aligned}$$

Now for each $i = 1, ..., I$, $t_i' \in T_i'$ and $a_i \in A_i$,

$$\begin{aligned}
V_i'\left(a_i, a_i, t_i'\right) &= \sum_{t_i\in T_i} \phi_i\left(t_i'|t_i\right) V_i\left(a_i, a_i, t_i\right), \text{ by (29)} \\
&\geq \sum_{t_i\in T_i} \phi_i\left(t_i'|t_i\right) V_i\left(a_i, a_i', t_i\right), \text{ by (27) for each } t_i \in T_i \\
&= V_i'\left(a_i, a_i', t_i'\right), \text{ by (29)}
\end{aligned}$$

for each $a_i' \in A_i$. Thus σ' is a BCE of (G, S'). By construction σ' and σ induce the random choice rule $\nu : \Theta \to \Delta(A)$. Since this argument started with an arbitrary BCE random choice rule ν of (G, S) and an arbitrary G, we have $BCE(G, S) \subseteq BCE(G, S')$ for all games G.

We now show that (2) implies (1). Suppose that S is more information constrained than S'. We will show that if, in addition, S is non-redundant, then S is individually sufficient for S'. If S was not non-redundant, we could let $\widehat{S}$ be the unique non-redundant information structure which is higher order belief equivalent to S, as shown in Lemma 4. We would show that $\widehat{S}$ is individually sufficient for S'. By Lemma 4, S is individually sufficient for $\widehat{S}$ and so, by Lemma 3, S is individually sufficient for S'.

Write $\lambda_i(t_i) \in \Delta(T_{-i} \times \Theta)$ for type t_i's beliefs

$$\lambda_i\left(t_{-i}, \theta|t_i\right) = \frac{\psi(\theta)\, \pi\left(\left(t_i, t_{-i}\right)|\theta\right)}{\sum\limits_{\widetilde{t}_{-i}, \widetilde{\theta}} \psi(\theta)\, \pi\left(\left(t_i, \widetilde{t}_{-i}\right)|\widetilde{\theta}\right)}.$$

Write Λ_i for the range of λ_i. Thus $\lambda_i : T_{-i} \to \Lambda_i$. By non-redundancy of S, there is well defined inverse map $\lambda_i^{-1} : \Lambda_i \to T_{-i}$, so that $\lambda_i(t_i) = \xi_i$ if and only if $\lambda_i^{-1}(\xi_i) = t_i$.

For any $\zeta : T \times \Theta \to \Delta(T')$, write

$$\lambda_i^{\zeta}\left(t_{-i},\theta|t_i,t_i'\right)=\frac{\sum\limits_{t_{-i}'}\psi(\theta)\,\pi\left((t_i,t_{-i})\,|\theta\right)\zeta\left(\left(t_i',t_{-i}'\right)|\,(t_i,t_{-i}),\theta\right)}{\sum\limits_{\widetilde{t}_{-i},\widetilde{\theta}}\sum\limits_{t_{-i}'}\psi(\theta)\,\pi\left(\left(t_i,\widetilde{t}_{-i}\right)|\widetilde{\theta}\right)\zeta\left(\left(t_i',t_{-i}'\right)|\left(t_i,\widetilde{t}_{-i}\right),\widetilde{\theta}\right)}.$$

Write Z for the set of $\zeta : T \times \Theta \to \Delta(T')$ satisfying

$$\sum_t \pi(t|\theta)\,\zeta\left(t'|t,\theta\right)=\pi'\left(t'|\theta\right)$$

for each t' and θ. Note that Z is a compact set. Now suppose that S is not individual sufficient for S'. Then, for every $\zeta \in Z$, by non-redundancy, there exists i, t_i and t_i' such that

$$\lambda_i^{\zeta}\left(\cdot|t_i,t_i'\right)\neq\lambda_i\left(\cdot|t_i\right).$$

Now define

$$\varepsilon=\frac{1}{2}\inf_{\zeta\in Z}\max_{i,t_i,t_i'}\left\|\lambda_i^{\zeta}\left(\cdot|t_i,t_i'\right)-\lambda_i\left(\cdot|t_i\right)\right\|,$$

where $\|\cdot\|$ represents the Euclidean distance vectors in $\mathbb{R}^{T_{-i}\times\Theta}$. The compactness of the set Z and the continuity of the finite collection of mappings $\lambda_i^{\zeta}\left(\cdot|t_i,t_i'\right)$ with respect to ζ imply that $\varepsilon > 0$.

Now we will construct a basic game $G = \left((A_i,u_i)_{i=1}^I,\psi\right)$ and an action state distribution $\mu^* \in \Delta(A\times\Theta)$ such that $\mu^* \in BCE(G,S)$ but $\mu^* \notin BCE(G,S')$. This will complete the proof of the argument that (2) implies (1).

Recall that Λ_i (the range of λ_i) is a finite subset of $\Delta(T_{-i}\times\Theta)$. Let Ξ_i be any ε-grid of $\Delta(T_{-i}\times\Theta)$, i.e., a finite subset of $\Delta(T_{-i}\times\Theta)$ satisfying the property that, for all $\xi_i \in \Delta(T_{-i}\times\Theta)$, there exists $\xi_i' \in \Xi_i$ with $\left\|\xi_i-\xi_i'\right\|\leq\varepsilon$. Now let $A_i=\Lambda_i\cup\Xi_i$. Let

$$u_i(a,\theta)=\begin{cases}2a_i\left(\left(\lambda_j^{-1}(a_j)\right)_{j\neq i},\theta\right)-\sum\limits_{\widetilde{t}_{-i}\in T_{-i},\widetilde{\theta}\in\Theta}\left(a_i\left(\widetilde{t}_{-i},\widetilde{\theta}\right)\right)^2, \text{ if } a_j\in\Lambda_j,\ \forall j\neq i,\\ 0, \text{ otherwise.}\end{cases}$$

Now suppose player i assigns probability 1 to his opponents choosing $a_{-i}\in\Lambda_{-i}$ and, in particular, for some $\xi_i\in\Delta(T_{-i}\times\Theta)$, assigns probability $\xi_i\left(\left(\lambda_j^{-1}(a_j)\right)_{j\neq i},\theta\right)$ to his opponents choosing a_{-i} and the state being θ.

The expected payoff to player i with this belief over $A_{-i} \times \Theta$ parameterized by $\xi_i \in \Delta(T_{-i} \times \Theta)$ is

$$\begin{aligned}
& \sum_{t_{-i}\in T_{-i},\theta\in\Theta} \xi_i(t_{-i},\theta)\left(2a_i(t_{-i},\theta) - \sum_{\widetilde{t}_{-i}\in T_{-i},\widetilde{\theta}\in\Theta}\left(a_i\left(\widetilde{t}_{-i},\widetilde{\theta}\right)\right)^2\right) \\
= \; & 2\sum_{t_{-i}\in T_{-i},\theta\in\Theta} \xi_i(t_{-i},\theta)\, a_i(t_{-i},\theta) - \sum_{\widetilde{t}_{-i}\in T_{-i},\widetilde{\theta}\in\Theta}\left(a_i\left(\widetilde{t}_{-i},\widetilde{\theta}\right)\right)^2 \\
= \; & 2\sum_{t_{-i}\in T_{-i},\theta\in\Theta} \xi_i(t_{-i},\theta)\, a_i(t_{-i},\theta) - \sum_{t_{-i}\in T_{-i},\theta\in\Theta}\left(a_i(t_{-i},\theta)\right)^2 \\
= \; & \left(\|\xi_i\|^2 - \|a_i - \xi_i\|^2\right).
\end{aligned}$$

Thus player i with belief ξ_i has a best response to set a_i equal to one of the points in $A_i \subseteq \Delta(T_{-i} \times \Theta)$ with the shortest Euclidean distance to $\widehat{\xi}_i$.

Now the game (G, S) has - by construction - a "truth-telling" BCE where each type t_i always chooses action $\lambda_i(t_i)$. This give rise to action state distribution μ^* where

$$\mu^*(a,\theta) = \begin{cases} \psi(\theta)\,\pi(a|\theta), & \text{if} \quad a \in T, \\ 0, & \text{if} \quad a \notin T. \end{cases}$$

So μ^* is a BCE action state distribution of (G, S). For μ^* to be BCE of (G, S'), there must exist $\zeta \in Z$ such that $\nu' \in \Delta(T \times T' \times \Theta)$ defined by

$$\nu\left(t, t', \theta\right) = \psi(\theta)\,\pi(t|\theta)\,\zeta\left(t'|t,\theta\right)$$

is a BCE. But for any $\zeta \in Z$, we showed that there exist i, t_i and t'_i with

$$\left\|\lambda_i^{\zeta}\left(\cdot|t_i, t'_i\right) - \lambda_i(\cdot|t_i)\right\| \geq 2\varepsilon.$$

But this implies a violation of obedience, since by construction of G, there exists an action $a_i \in A_i$ which is within ε of $\lambda_i^{\zeta}(\cdot|t_i, t'_i)$ and thus closer to $\lambda_i^{\zeta}(\cdot|t_i, t'_i)$ than $\lambda_i(\cdot|t_i)$, and so a player with type t'_i receiving action recommendation t_i would strictly prefer to deviate to a_i.

To show (1) implies (3), observe that if $\nu \in F(G, S')$, then S' is individually sufficient for (A, ν). Now if S is individually sufficient for S' and S' is individually sufficient for (A, ν), then by transitivity (see Lemma 3), S is individually sufficient for (A, ν). This is equivalent to saying that ν is belief invariant for (G, S).

To show (3) implies (1), consider a basic game G with action set T'. If S is not individually sufficient for S', there does not exist $\phi : T \times \Theta \to \Delta(T')$ satisfying (22) and (23). But this implies that does not exist a belief invariant decision rule for (G, S) which induces π'. On the other hand, consider the "truth-telling" decision rule $\sigma' : T' \times \Theta \to \Delta(T')$ for (G, S') with

$$\sigma'\left(\widetilde{t}'|t',\theta\right) = \begin{cases} 1, \text{ if } \widetilde{t}' = t', \\ 0, \text{ otherwise.} \end{cases}$$

This is clearly belief invariant (G, S') and induces random choice rule π'. So $\pi' \in F(G, S')$ but $\pi' \notin F(G, S)$, implying that S is not more permissive that S'. ■

5 Leading Example: Binary Action Games and Binary Signal Information Structures

We will use a family of examples to illustrate our definitions and results. Suppose that there are two players, $i \in \{\text{Ann, Bob}\}$, and two states $\Theta = \{\theta_0, \theta_1\}$. Consider the basic game where each player has two actions $A_i = \{a_0, a_1\}$, the prior is uniform on the two states, and each player gets a payoff of 1 if both players set their actions equal to the state, a payoff of ε if his action is equal to the state but his opponent's is not, and 0 otherwise. Thus the payoff matrices are given by

$\theta = \theta_0$	a_0	a_1
a_0	$1, 1$	$\varepsilon, 0$
a_1	$0, \varepsilon$	$0, 0$

$\theta = \theta_1$	a_0	a_1
a_0	$0, 0$	$0, \varepsilon$
a_1	$\varepsilon, 0$	$1, 1$

, (30)

where the row corresponds to the action of Ann, the column corresponds to the action of Bob, and the matrix corresponds to the state. We will focus on the case where $\varepsilon = 0$ but also discuss the robustness of our results if we let ε vary from 0.

Consider the information structure $S_{q,r}$ where each player has two signals, $T_i = \{t_0, t_1\}$, each player's signal is equal to the true state with probability $q \in \left[\frac{1}{2}, 1\right]$ and both players' signals are equal to the true state with probability $r \in [2q - 1, q]$. We will refer to q as the *accuracy* of the information structure and to r (somewhat loosely) as the *correlation* of the signals. Thus the signal structure is described in the following tables:

$\theta = \theta_0$	t_0	t_1
t_0	r	$q - r$
t_1	$q - r$	$r + 1 - 2q$

$\theta = \theta_1$	t_0	t_1
t_0	$r + 1 - 2q$	$q - r$
t_1	$q - r$	r

, (31)

where the row corresponds to Ann's signal, the column corresponds to Bob's signal and the matrix corresponds to the state. This class of information structures includes all binary signal information structures which are symmetric across players and states.

We will often focus on the information structure with the minimum correlation consistent with accuracy q, i.e., the information structure $S_{q,2q-1}$ with

$\theta=\theta_0$	t_0	t_1
t_0	$2q-1$	$1-q$
t_1	$1-q$	0

$\theta=\theta_1$	t_0	t_1
t_0	0	$1-q$
t_1	$1-q$	$2q-1$

. (32)

Under this information structure, the players never both receive the "wrong" signal. The set of possible information structures $S_{q,r}$ are illustrated in Figure 4 in the (q,r) space. The set of possible information structures are formed by the intersection of three lines, $r=q$, $r=2q-1$ and $q=1/2$, where $r=q$ describes all information structures where the signal of the agents are perfectly correlated, $r=2q-1$, describes the minimal correlation for a given accuracy q, as represented by the information structure $S_{q,2q-1}$ above in (32) and $q=1/2$ describes the set of all information structure that contain zero information regarding the state θ. The set of information structures with conditionally independent signals of the agents is described by $r=q^2$ and is in the interior of the set of possible information structures.

Insert Figure 4 here

The set of actions is isomorphic to the set of signals in our examples, so we can represent symmetric random choice rules using the same notation as for information structures, i.e., there is a random choice rule with accuracy q and correlation r given by

$\theta=\theta_0$	a_0	a_1
a_0	r	$q-r$
a_1	$q-r$	$r+1-2q$

$\theta=\theta_1$	a_0	a_1
a_0	$r+1-2q$	$q-r$
a_1	$q-r$	r

where the row corresponds to Ann's action, the column corresponds to Bob's action and the matrix corresponds to the state. However, while for information structures we assumed $q\geq\frac{1}{2}$, we will need to allow for $q<\frac{1}{2}$ in random choice rules. Thus the set of symmetric random choice rules is formed by the intersection of three lines, $r=q$, $r=2q-1$ and $r=0$, and thus in Figure 4 the triangle in positive orthant that remains after removing the vertical line $q=1/2$.

We will use this two dimensional class of (symmetric) information structures and two dimensional class of (symmetric) decision rules to illustrate our results. We find it useful to restrict attention to these classes because they are easy to visualize and we will extensively use pictures to illustrate what is going on. Calvo-Argengol (2006) showed that - even in complete information games - characterizing and visualizing all correlated equilibria of all two player two actions games is not easy. We emphasize that we will be using this class of examples to illustrate results that apply to general, asymmetric, information structures and general, asymmetric, decision rules.

We now illustrate the set of Bayes correlated equilibria in the example. The set of symmetric BCE random choice rules of $(G_\varepsilon, S_{q,2q-1})$ if $\varepsilon = 0$ and $q \geq \frac{2}{3}$ is the convex hull of the set of four random choice rules with

$$\left(q', r'\right) = (0,0)\,, (1,1), (q, 2q-1) \text{ and } \left(2\,(1-q)\,, 1-q\right). \tag{33}$$

This set is illustrated in Figure 5, for two information structures with minimal correlation. The larger triangle is generated by the information structure: $q = 2/3, r = 1/3$, whereas the smaller trapezoid is generated the information structure $q = 5/6, r = 2/3$. We observe that for $q = 2/3$, the two random choice rules, $(q, 2q-1)$ and $(2\,(1-q)\,, 1-q)$, coincide, and hence reducing the set of BCE to a triangle. We also observe that as the accuracy q of the signals is increasing, the information of each individual agent is improving and thus the incentive constraints tighten and, in turn, the set of Bayes correlated equilibria is shrinking.

Insert Figure 5 here

A full analysis of the example is presented in Appendix 8.3. Here we merely sketch why these points are BCE random choice rules and provide an intuition why they are extremal. First, the random choice rule $(1,1)$ is induced by the decision rule where each player always sets his action equal to the state. This corresponds to the case of complete information, where each player observes the true state and both players set the action equal to the state. Since this is an equilibrium under complete information, a Nash equilibrium, it is also a BCE. Similarly, the random choice rule $(0,0)$ is induced by the decision rule where the players set their actions equal to each other but different to the state. This also corresponds to the case of complete information where each player observes the true state, but where both players follow the dominated but equilibrium strategies of setting their actions different from the state. The random choice rule $(q, 2q-1)$ is induced by the decision rule where each player sets his action equal to his signal. This corresponds to the Bayes Nash equilibrium of the game where players observe no additional information. Finally, consider the random choice rule $(2\,(1-q)\,, 1-q)$. One can show that it is induced by the decision rule described by the following table:

$\theta = \theta_0$	$t_0 a_0$	$t_0 a_1$	$t_1 a_0$	$t_1 a_1$
$t_0 a_0$	$\frac{1-q}{2q-1}$	0	0	1
$t_0 a_1$	0	$\frac{3q-2}{2q-1}$	0	0
$t_1 a_0$	0	0	0	0
$t_1 a_1$	1	0	0	1

$\theta = \theta_1$	$t_0 a_0$	$t_0 a_1$	$t_1 a_0$	$t_1 a_1$
$t_0 a_0$	1	0	0	1
$t_0 a_1$	0	0	0	0
$t_1 a_0$	0	0	$\frac{3q-2}{2q-1}$	0
$t_1 a_1$	1	0	0	$\frac{1-q}{2q-1}$

,

where the matrix determines the state, the row corresponds to the (signal, action) pair of Ann, the column corresponds to the (signal, action) pair of Bob, and the entries describe the conditional probability of that action profile being played given the state and signal profile. Thus, the conditional probabilities in every 2×2 submatrix which is formed by a vector of state θ and signals by Ann and Bob have to sum up to one. We note that the entries in the profile (θ_0, t_1, t_1) and (θ_1, t_0, t_0) are without consequence as each one of these profiles is a zero probability event. This decision rule induces the random choice rule $(2(1-q), 1-q)$ since the probability that any player chooses the action corresponding to the state is

$$(2q-1)\frac{1-q}{2q-1} + (1-q)\,1 = 2(1-q)$$

and the probability that both players choose the action corresponding equal to the state is

$$(2q-1)\frac{1-q}{2q-1} = 1-q.$$

The four extremal BCE random choice rules were given by (33) can be used to illustrate Theorem 2, which established the relationship between the Bayes correlated and Bayes Nash equilibria. Both $(0,0)$ and $(1,1)$ correspond to the Nash equilibria of the complete information game, so the expanded information structure is one where the state becomes common knowledge. Random choice rule $(q, 2q-1)$ corresponds to the Bayes Nash equilibrium with no additional private information beyond the uniform prior over the states. An expansion that generates the BCE random choice $(2(1-q), 1-q)$ is one where, conditional on both players observing the same, correct, signal of the state, with probability $\frac{3q-2}{2q-1}$ there is a public announcement that they have both observed correct signals. Conditional on receiving this public signal, the dominated Nash equilibrium of the complete information game is played. If the public signal is not observed, then Ann is sure that Bob is setting his action equal to the state, and she thinks it equally likely that her signal is correct or incorrect. She is thus indifferent between the two actions. Thus $(2(1-q), 1-q)$ is a BCE where the obedience constraints hold as an equality for both actions. By contrast, in the other three extremal BCE of (33), the obedience constraints are not binding.

We calculated the BCE for a non-generic game (with $\varepsilon = 0$) where there are weakly dominated strategies in the underlying complete information game. However, small changes in ε lead to only small changes in the sets of BCE. In particular, if $\varepsilon > 0$, setting the action equal to the state is a strictly dominant strategy in the complete information game, and $(0,0)$ is no longer be a BCE random choice rule. But there is an extremal BCE random choice rule that is of order ε distant from $(0,0)$. If $\varepsilon < 0$, both setting their actions equal to each other but different to the state is a strict Nash equilibrium of the complete information game, and $(0,0)$ is (just) in the interior of the set of BCE random choice rules.

Next, we illustrate the various statistical orderings that we introduced in Section 4.3 and their relationship to higher order beliefs.

Example 1: Individual Sufficiency and Redundant Types Consider the following binary information structure with signal distribution:

$\theta = \theta_0$	t_0	t_1
t_0	$\frac{1}{2}$	0
t_1	0	$\frac{1}{2}$

$\theta = \theta_1$	t_0	t_1
t_0	0	$\frac{1}{2}$
t_1	$\frac{1}{2}$	0

.

Types here are "redundant" in the sense of Mertens and Zamir (1985): there is common certainty that each player's belief over the states is equal to the prior over the states, and thus types t_0 and t_1 induce exactly the same beliefs and higher-order beliefs. Examples such as this have been leading examples in the literature, see Dekel, Fudenberg, and Morris (2007), Ely and Peski (2006), Liu (2011), Lehrer, Rosenberg, and Shmaya (2011), Forges (2006). Now, importantly in our context, this information structure is mutually individually sufficient with the null information structure. It thus illustrates the fact that individual sufficiency does not depend on redundant types, as established generally in Lemma 4.

Example 2: Individual Sufficiency and Higher Order Beliefs The previous example had "redundant" types with the same beliefs and higher order beliefs. The next example reports two information structures, neither of which is individually sufficient for the other, even though from each player's point of view, they convey the same information about the state θ. Consider the pair of experiments $S_{\frac{2}{3},\frac{1}{3}}$, described by (34), and $S_{\frac{2}{3},\frac{2}{3}}$, described by (35):

$\theta = \theta_0$	t_0	t_1
t_0	$\frac{1}{3}$	$\frac{1}{3}$
t_1	$\frac{1}{3}$	0

$\theta = \theta_1$	t_0	t_1
t_0	0	$\frac{1}{3}$
t_1	$\frac{1}{3}$	$\frac{1}{3}$

(34)

and

$\theta = \theta_0$	t_0	t_1
t_0	$\frac{2}{3}$	0
t_1	0	$\frac{1}{3}$

$\theta = \theta_1$	t_0	t_1
t_0	$\frac{1}{3}$	0
t_1	0	$\frac{2}{3}$

. (35)

Neither is individually sufficient for the other. Intuitively, each player is better informed about the other's information under the latter information structure, but the join of the individual signals gives rise to more extreme posteriors under the first information structure. It thus illustrates that individually sufficiency accounts both for individual information as well as joint information across the agents.

Example 3: Individual Sufficiency vs. Sufficiency Consider the pair of information structures $S_{\frac{2}{3},\frac{1}{3}}$, described by (34), and $S_{\frac{5}{9},\frac{5}{9}}$, described by (36):

$$\begin{array}{|c|c|c|}\hline \theta=\theta_0 & t_0 & t_1 \\ \hline t_0 & \frac{5}{9} & 0 \\ \hline t_1 & 0 & \frac{4}{9} \\ \hline \end{array}\begin{array}{|c|c|c|}\hline \theta=\theta_1 & t_0 & t_1 \\ \hline t_0 & \frac{4}{9} & 0 \\ \hline t_1 & 0 & \frac{5}{9} \\ \hline \end{array}. \tag{36}$$

$S_{\frac{2}{3},\frac{1}{3}}$ is sufficient for $S_{\frac{5}{9},\frac{5}{9}}$ but $S_{\frac{2}{3},\frac{1}{3}}$ is not individually sufficient for $S_{\frac{5}{9},\frac{5}{9}}$. This illustrates that individual sufficiency may be more demanding than sufficiency.

Symmetric Binary Information Structures The previous two examples were comparisons of symmetric binary information structures. We can give a complete ordering of such information structures that verifies these examples:

Lemma 6 (Symmetric Binary Information Structures)

1. *Information structure $S_{q,r}$ is sufficient for information structure $S_{q',r'}$ if and only if*
$$\left(q',r'\right)\in\operatorname{conv}\left\{\left(\frac{1}{2},0\right),\left(\frac{1}{2},\frac{1}{2}\right),\left(q,r\right),\left(q,q\right)\right\}. \tag{37}$$
2. *Information structure $S_{q,r}$ is individually sufficient for information structure $S_{q'r'}$ if and only if*
$$\left(q',r'\right)\in\operatorname{conv}\left\{\left(\frac{1}{2},0\right),\left(\frac{1}{2},\frac{1}{2}\right),\left(q,r\right)\right\}. \tag{38}$$
3. *Information structure $S_{q,r}$ is a non-communicating ungarbling of information structure $S_{q',r'}$ if and only if*
$$\left(q',r'\right)\in\operatorname{conv}\left\{\left(\frac{1}{2},0\right),\left(\frac{1}{2},\frac{1}{2}\right),\left(q,r\right)\right\}. \tag{39}$$

Thus, in the space of symmetric binary information structures, individual sufficiency and non-communicating ungarbling induce the same ordering over the information structures, and in turn sufficiency generates a *more complete* ordering. A necessary and sufficient condition for $S_{q,r}$ to be sufficient for $S_{q',r'}$ is that the accuracy of S is greater than the accuracy of S', i.e. $q \geq q'$. By contrast, for individual sufficiency, greater accuracy is only a necessary condition, but the sufficiency part of the condition requires in addition greater correlation, i.e. $r \geq r'$. The amount by which the correlation of S' can differ from the correlation of S depends on the difference in accuracy between the two information structures. The closer their accuracy, the closer the correlation of the less accurate experiment must be in order for it to be lower in the partial order. The sets of information structures that can be ranked relative to a given information structure S are illustrated in Figure 6 within the (q,r) space for $(q,r) = (4/6, 3/6)$.

Insert Figure 6 here.

We conclude with two other examples to illustrate the relations between orderings. These examples still rely only on binary information structures, but are symmetric *only* across players, but *not* across the states. In particular, the examples illustrate that neither of the two orders, sufficiency or individual sufficiency, is either weaker or stronger than the other.

Example 4: Individual Sufficiency vs. Sufficiency Information structure $S_{\frac{2}{3},\frac{1}{3}}$, described earlier by (34), is individually sufficient for the following binary information structure, but not sufficient for it:

$\theta = \theta_0$	t_0	t_1
t_0	$\frac{4}{9}$	$\frac{1}{9}$
t_1	$\frac{1}{9}$	$\frac{1}{3}$

$\theta = \theta_1$	t_0	t_1
t_0	0	$\frac{4}{9}$
t_1	$\frac{4}{9}$	$\frac{1}{9}$

. (40)

Example 5: Individual Sufficiency and Sufficiency vs. Non-Communicating Ungarbling The final example in this subsection illustrates the relationship between non-communicating ungarbling and (individual) sufficiency. It followed directly from the definition that if S is a non-communicating ungarbling of S', then S is sufficient, and also individually sufficient, for S'. However, S can be sufficient *and* individually sufficient for S', yet S is not a non-communicating ungarbling of S'. Information structure $S_{\frac{2}{3},\frac{1}{3}}$ is sufficient and individually sufficient for the following binary information structure, but not a non-communicating ungarbling of it:

$\theta = \theta_0$	t_0	t_1
t_0	$\frac{5}{18}$	$\frac{5}{18}$
t_1	$\frac{5}{18}$	$\frac{1}{6}$

$\theta = \theta_1$	t_0	t_1
t_0	0	$\frac{4}{9}$
t_1	$\frac{4}{9}$	$\frac{1}{9}$

. (41)

These final two examples are illustrated in Figure 7, where we constrain the binary information structure to be symmetric with respect to the agents, but allow them to be asymmetric with respect to the state. Here, we fix the accuracy of S and S', namely $q = 2/3 > q' = 5/9$, but allow the correlation represented by r' to differ across states, and hence illustrate the set of information structures in the space of $r' = (r'_0, r'_1)$, while $r = (r_0, r_1) = \left(\frac{1}{3}, \frac{1}{3}\right)$. Thus the information structure S' is

$\theta = \theta_0$	t_0	t_1
t_0	r'_0	$\frac{5}{9} - r'_0$
t_1	$\frac{5}{9} - r'_0$	$r'_0 - \frac{1}{9}$

$\theta = \theta_1$	t_0	t_1
t_0	$r'_1 - \frac{1}{9}$	$\frac{5}{9} - r'_1$
t_1	$\frac{5}{9} - r'_1$	r'_1

.

The correlation r_θ is restricted to satisfy $2q - 1 \leq r_\theta \leq q$ by the nonnegativity requirement of the signals and Figure 7 displays the information structures S' for which $S = S_{\frac{2}{3},\frac{1}{3}}$ is sufficient, individually sufficient, respectively a non-communicating ungarbling. We find that the intersection of information structures S' for which S is both sufficient and individually sufficient is strictly larger than the set of non-communicating garblings. In particular, the information structure $S'_{\frac{5}{9},\frac{4}{9},\frac{1}{9}}$, described by (40), is depicted in southeast corner of the square set of information structures for which S is individually sufficient; and the information structure $S'_{\frac{5}{9},\frac{5}{18},\frac{1}{9}}$, described by (41), is depicted in the southeast corner of the hexagon of information structures for which $S = S_{\frac{2}{3},\frac{1}{3}}$ is sufficient. The fact that $S = S_{\frac{2}{3},\frac{1}{3}}$ is individually sufficient for $S'_{\frac{5}{9},\frac{4}{9},\frac{1}{9}}$, but not sufficient for $S'_{\frac{5}{9},\frac{4}{9},\frac{1}{9}}$, may at first appear surprising. But it illustrates, the decentralized assessment of information inherent in individual sufficiency. With the decentralized view, the accuracy q alone determines how informed each individual agent is about the state θ (and all the information is contained in two signals.) However, with sufficiency, the joint signal 01 (and symmetrically 10) can also provide information regarding the state θ. And in particular, if r'_0 is very different from r'_1, then the joint signal will provide information regarding θ, whereas each marginal signal 0 or 1, contains no additional information.

Insert Figure 7 here.

6 Incomplete Information Correlated Equilibrium

Aumann (1974), (1987) introduced a definition of correlated equilibrium for complete information games. A classic paper of Forges (1993) is titled "five legitimate definitions of correlated equilibrium in games with incomplete information." Her title and paper make the point - which we agree with - that there are many natural ways of extending the complete information definition to incomplete information settings and which definition makes sense depends on the purpose for which it is to be used. In this section, we present a way of seeing how our definition of "Bayes correlated equilibrium" relates to other definitions of incomplete information correlated equilibrium, highlighting which definition is relevant for which purpose.

For a fixed basic game G and information structure S, a Bayes correlated equilibrium is a decision rule mapping signal profiles and payoff states to probability distributions over action profiles that satisfies obedience (14), requiring that a player who knows his signal and the action he is supposed to play has no incentive to deviate. We treat payoff states symmetrically with actions and impose no additional restrictions on what is feasible. The role of the information structure, then, is only to impose extra incentive constraints on behavior. Our motive to study this solution concept is Theorem 2: the solution concept captures rational behavior given that players have access to the signals in the information structure, but may have additional information.

If a decision rule is belief invariant for (G, S), players have no less but also no more information than under information structure S. If we impose belief invariance as well as obedience on a decision rule, we get a solution concept that was introduced in Liu (2011).

Definition 22 (Belief Invariant BCE)
A decision rule σ is a belief invariant Bayes correlated equilibrium (BIBCE) of (G, S) if it is obedient and belief invariant for (G, S).

It captures the implications of common knowledge of rationality and that players know exactly the information contained in S, if the common prior assumption is maintained. As explained in Liu (2011), this solution concept can be seen as the common prior analogue of the solution concept of interim correlated equilibrium discussed by Dekel, Fudenberg, and Morris (2007). One can show that the set of Bayes correlated equilibria of (G, S) will consist of all belief invariant BCE of (G, S') for all information structures S' for which S is individually sufficient.

Liu (2011) showed that if two information structures are higher order belief equivalent, then they have the same set of belief invariant Bayes correlated equilibria. His result implies that two information structures that are higher order belief equivalent have the same set of belief invariant Bayes correlated equilibria. This in turn implies that they have the same set of Bayes correlated equilibria, which was Corollary 1 of Theorem 3.

A random choice rule ν is a BIBCE random choice rule if it is induced by a BIBCE decision rule. Observe that a BIBCE random choice rule must therefore be a BCE random choice rule and a feasible random choice rule, but not every random choice rule that is a BCE and feasible is a BIBCE random choice rule. To wit, both the Bayes correlated equilibrium and the belief invariant BCE are defined in terms of a decision rule σ rather than a random choice rule ν. Thus, while the intersection of feasible and obedient decision rules σ equals the set of belief invariant Bayes correlated equilibria, the intersection of feasible and BCE random choice rules ν is a superset, and as evidenced here, sometimes a strict superset, of the belief invariant BCE random choice rules. This is illustrated in Figure 8 where we display the BCE, feasible and BIBCE random choice rules in our leading example with the binary information structure $S_{q,r}$ with $q = 5/6$ and $r = 2/3$.

Insert Figure 8 here.

Much of the literature on incomplete information correlated equilibrium started from the premise that an incomplete information definition of correlated equilibrium should capture what could happen if players had access to a correlation device / mediator under the maintained assumption that the correlation device/mediator did not have access to information that was not available to the players. We can describe the assumption formally as:

Definition 23 (Join Feasible)
Decision rule σ is join feasible for (G, S) if $\sigma(a|t, \theta)$ is independent of θ.

This gives another solution concept:

Definition 24 (Bayesian Solution)
Decision rule σ is a Bayesian solution of (G, S) if it is obedient and join feasible.

This assumption was made implicitly in Forges (1993) and other works, because it was assumed that type or signal profiles exhausted all uncertainty. On the other hand, Lehrer, Rosenberg, and Shmaya (2010), (2011) explicitly impose this assumption. The Bayesian solution was named by Forges (1993) and it is the weakest version of incomplete information correlated equilibrium she studies. It also corresponds to the set of jointly coherent outcomes in Nau (1992), justified from "no arbitrage" conditions. Forges and Koessler (2005) provide a justification if players are able to certify their types to the mediator.

In Figure 9, we display the set of Bayesian solutions and compare them to the BCE and BIBCE random choice rules. In the present binary and symmetric game, it happens that BIBCE $\subseteq$ Bayesian solution $\subseteq$ BCE. More generally, the following inclusions hold for all finite games: BIBCE $\subseteq$ BCE and Bayesian solution $\subseteq$ BCE, but there can be BIBCE which do not form a Bayesian solution. Another special feature of the present binary example is that the Nash equilibria and the Bayes Nash equilibrium under the common prior are either vertices or located on the edges of the BCE set, more generally both Nash and Bayes Nash equilibria can (all) be located in the interior of the BCE set.

Insert Figure 9 here.

Imposing both join feasibility and belief invariance, we get a solution concept that has played an important role in the literature.

Definition 25 (Belief Invariant Bayesian Solution)
Decision rule σ is a belief invariant Bayesian solution of (G, S) if it is obedient, belief invariant and join feasible.

Forges (2006) introduced this name. We will focus our analysis on these four solution concepts. For any fixed game, the definitions imply some relations between the solution concepts illustrated in Figure 10: the set of belief invariant Bayes correlated equilibria is a subset of the set of Bayes correlated equilibria; the set of Bayesian solutions is a subset of the set of Bayes correlated equilibria; the set of belief invariant Bayesian solutions is equal to the intersection of the set of Bayesian solutions and the belief invariant Bayes correlated equilibria.

Insert Figure 10 here.

Another way of understanding the solution concepts discussed in this section is to return to the single player environment that we discussed in Section 2. In single player games, belief invariant BCE, Bayesian solutions and belief invariant Bayesian solutions all coincide with what we called the Bayes Nash equilibrium, i.e., standard optimal behavior for the decision problem. It is only the notion of Bayes correlated equilibrium, by imposing neither join feasibility nor belief invariance that allows more information to be reflected in the single player's choice, and thus leads to the larger set of random choice rules.

Forges (1993), (2006) also surveys three stronger solutions concepts for (G, S). We briefly discuss these informally, and for completeness give formal definitions in our language in Appendix 8.4.

1. One could simply look at the agent normal form of (G, S) and consider the correlated equilibria of this complete information game. This is equivalent to requiring that the decision rule could be generated by having a mediator randomize over pure strategies in the incomplete information game, mapping signals to actions, before observing the state or players' signals, and recommendations follow the chosen pure strategies. Forges (1993) calls this "agent normal form correlated equilibrium."

2. If the mediator can make recommendations only based on reports from the players, players must have an incentive to tell the truth. A decision rule is "truth-telling" if players both have an incentive to truthfully report their types and have an incentive to follow their recommendations. This gives the well known solution concept of "communication equilibrium."

3. One could also look at the (non-agent) strategic form of (G, S) and consider the correlated equilibria of this complete information game. This imposes additional incentive constraints, since players now know what the mediator would have recommended if they had been different types. Forges (1993) calls this "strategic form correlated equilibrium."

The relationships between these solution concepts are described in Figure 11 below.

Insert Figure 11 here.

Examples in Forges (1993), (2006) show that the relationships in the Venn diagram are tight.

7 Discussion

7.1 Distributed Certainty

It was important in much of our analysis that we did not assume agents collectively knew all possible payoff relevant information. If they did, "join feasibility" would be automatically satisfied. Formally:

Definition 26 (Distributed Certainty)
Information structure S satisfies distributed certainty if there exists $g : T \to \Theta$ such that $\pi(t|\theta) > 0 \Rightarrow \theta = g(t)$.

An important setting where this condition will always be satisfied is private value environments. This would be modelled in our language by setting $\Theta = \Theta_1 \times \cdots \times \Theta_I$, each $T_i = \Theta_i$ and let

$$\pi(t|\theta) = \begin{cases} 1, & \text{if} \quad t = \theta; \\ 0, & \text{if} \quad t \neq \theta. \end{cases}$$

As an example, in Bergemann, Brooks, and Morris (2012) we study first price auctions where bidders know their own values of a single object. This is a private value environment and thus has distributed certainty.

In our earlier work on robust mechanism design, Bergemann and Morris (2012), we did not assume private values but did assume "distributed certainty" in much of the work; the epistemic foundations we reviewed in Bergemann and Morris (2007) were also based on that assumption. Under distributed certainty, join feasibility is satisfied by any decision rule and we have:

Lemma 7 *If S satisfies distributed certainty then any decision rule σ is join feasible and thus, for any basic game G, any (belief invariant) Bayes correlated equilibrium of (G, S) is a (belief invariant) Bayesian solution of (G, S).*

7.2 Adding Dummy Players

One way to understand our results is to think about basic games and information structures where we add a "dummy player" who knows the state but is otherwise irrelevant.[8] While this is not how we prefer to present our results, it does allow us to make connections with the prior literature and understand formal connections in arguments.

Formally, fix a basic game $G = ((A_i, u_i)_{i=1}^I, \psi)$. Consider a modified basic game with added dummy player 0, $\widetilde{G} = ((\widetilde{A}_i, \widetilde{u}_i)_{i=0}^I, \psi)$ with $\widetilde{A}_0 = \{a_0\}$, $\widetilde{A}_i = A_i$ for $i = 1, ..., I$ and $\widetilde{u}_i((a_0, (a_j)_{j=1}^I), \theta) = u_i((a_j)_{j=1}^I, \theta)$ for $i = 1, ..., I$, and the form of $\widetilde{u}_0$ does not matter since the dummy player 0 has a singleton action set. Fix an information structure $S = ((T_i)_{i=1}^I, \pi)$. Consider a modified information structure with dummy player, $\widetilde{S} = ((\widetilde{T}_i)_{i=0}^I, \widetilde{\pi})$, with $\widetilde{T}_0 = \Theta$, $\widetilde{T}_i = T_i$ for $i = 1, ..., I$ and

$$\widetilde{\pi}\left(\left(t_0, (t_i)_{i=1}^I\right) | \theta\right) = \begin{cases} \pi\left((t_i)_{i=1}^I | \theta\right), & \text{if} \quad t_0 = \theta; \\ 0, & \text{if} \quad t_0 \neq \theta. \end{cases}$$

[8] We are grateful to Atsushi Kajii for suggesting that we pursue this dummy player interpretation of Bayes correlated equilibrium. The taxonomy of incomplete information correlated equilibrium concepts in Milchtaich (2012) discusses the possibility of treating nature as a player.

Finally, given decision rule $\sigma : T \times \Theta \rightarrow \Delta(A)$ for (G, S), call $\widetilde{\sigma} : \widetilde{T} \times \Theta \rightarrow \Delta(\widetilde{A})$ the corresponding decision rule for $(\widetilde{G}, \widetilde{S})$ if

$$\widetilde{\sigma}\left(\left(a_0, (a_i)_{i=1}^I\right) \middle| \left(t_0, (t_i)_{i=1}^I\right), \theta\right) = \sigma\left((a_i)_{i=1}^I \middle| (t_i)_{i=1}^I, \theta\right),$$

and note that the value of $\widetilde{\sigma}(\cdot \left| (t_0, (t_i)_{i=1}^I), \theta\right.)$ when $t_0 \neq \theta$ is not going to be relevant. Now we have:

Lemma 8 *Decision rule $\sigma : T \times \Theta \rightarrow \Delta(A)$ is a (belief invariant) Bayes correlated equilibrium of (G, S) if and only if the corresponding $\widetilde{\sigma} : \widetilde{T} \times \Theta \rightarrow \Delta(\widetilde{A})$ is a (belief invariant) Bayesian solution of the game $(\widetilde{G}, \widetilde{S})$ with added dummy player.*

7.3 Finer Orderings on Information Structures

As we discussed in Section 4.3, Lehrer, Rosenberg, and Shmaya (2010), (2011) introduced an ordering on information structures - non-communicating ungarbling - which - as we showed in examples in Section 5 - is stronger than either individual sufficiency and sufficiency, and in fact is strictly stronger than requiring both sufficiency and individual sufficiency simultaneously.

We will first describe the results from Lehrer, Rosenberg, and Shmaya (2010), (2011) that are closest to ours, and identify the exact connections. Then we will report further results that they generate.

Let us focus on their results for the solution concept of belief invariant Bayesian solution, which is the weakest that they focus on. Lehrer, Rosenberg, and Shmaya (2010) focusses on common interest games where players have identical utility functions and on the belief invariant Bayesian solution of (G, S) which gives players the highest common utility. Theorem 4.5 of Lehrer, Rosenberg, and Shmaya (2010) shows that the maximum utility is higher in (G, S) than in (G, S') for all common interest games if and only if S is a non-communicating ungarbling of S'. We noted in the previous subsection that belief invariant Bayes correlated equilibria are essentially belief invariant Bayesian solutions in the game where a dummy player is added. It is also easy to show that S is individually sufficient for S' if and only if $\widetilde{S}$ is a non-communicating ungarbling of $\widetilde{S}'$, where $\widetilde{S}$ and $\widetilde{S}'$ are the information structures we get if we add a dummy player to S and S' respectively. Thus it is an easy corollary of Theorem 4.5 of Lehrer, Rosenberg, and Shmaya (2010) that if we focus on the belief invariant Bayes correlated equilibrium of common interest games which maximizes players' utility, utility is higher in (G, S) than in (G, S') for all common interest games if and only if S is individually sufficient for S'. Thus the equivalence of feasibility and statistical orderings follows from arguments in Lehrer, Rosenberg, and Shmaya (2010).

Suppose that we write $BIBS(G, S)$ for the set of belief invariant Bayesian solutions of (G, S). Part (c) of Theorem 2.8 of Lehrer, Rosenberg, and Shmaya (2011) shows that $BIBS(G, S) = BIBS(G, S')$ for all basic games G if and only if S and S' are non-communicating garblings of each other. Corollary 1 of

Theorem 3 could have been proved by applying the arguments of Lehrer, Rosenberg, and Shmaya (2011) to basic games and information structures with added dummy players as in the previous sub-section.

However, Lehrer, Rosenberg, and Shmaya (2011) do not have an analogue to our result that if S is individually sufficient for S', then $BCE(G, S) \subseteq BCE(G, S')$ for all basic games G, and we need a new argument to prove it. The analogous claim would be that if S is a non-communicating ungarbling of S', then $BIBS(G, S) \subseteq BIBS(G, S')$ for all basic games G. However, this claim is almost certainly false (even though we haven't constructed an explicit example). The reason is that the BCE solution concept imposes only incentive constraints and no feasibility conditions, so information can only reduce the set of equilibria. However, the $BIBS$ solution concept imposes join feasibility and belief invariance, conditions that become less demanding the more information there is. Thus the classical conflict between incentive and feasibility requirements becomes relevant.

Lehrer, Rosenberg, and Shmaya (2010) and (2011) also propose even stronger orderings on information structures (independent garbling, coordinated garblings) and show that their results on common interest games in Lehrer, Rosenberg, and Shmaya (2010) and general games in Lehrer, Rosenberg, and Shmaya (2011) extend in a natural way to finer solutions concepts (Bayes Nash equilibrium, agent normal form correlated equilibrium, respectively)

7.4 The Value of Information in Games Re-Visited

While Lehrer, Rosenberg, and Shmaya (2010), (2011) are the closest works to ours, there is a large literature on the value of information in games, and we now discuss that work and its relation. Hirshleifer (1971) noted why information might be damaging in a many player context because it removed options to insure ex ante. Our result on the incentive constrained ordering can be seen as a formalization of the idea behind the observation of Hirshleifer (1971): we give a general statement of how information creates more incentive constraints and thus reduces the set of incentive compatible outcomes.

We have highlighted the dual roles of information which are common to the one player and many player cases: increasing feasible outcomes and reducing incentive compatible ones. Neyman (1991) emphasized that within a fixed overall information structure, under Bayes Nash equilibrium, a player was better off with more information. Thus if some of player i's signals are more informative than others, then player i is better off in equilibrium conditional on receiving the more informative signals. In this case, more information makes more outcomes feasible and, because other players do not know if he is more informed or not, does not increase incentive constraints.

Gossner and Mertens (2001) consider Bayes Nash equilibrium and zero sum games and show that a sufficient condition for a player to have a higher value is that he has more information or his opponent has less information. They also showed that two information structures imply the same value in all games if and only if they are higher order belief equivalent. Peski (2008) shows that the sufficient conditions are also necessary. That is, for a fixed information structure, the set of information structures where a player will have a higher value in all zero sum games consists of those where he is more informed and his opponent is less informed. The proof of this result involves an appeal to the separating hyperplane theorem to show that if the condition on information structures is not satisfied, it is possible to construct the zero sum basic game where the player has a lower value. In our main result, we must similarly construct a basic game showing a failure of the incentive constrained ordering if the statistical relation fails. The arguments are quite different, however.

Gossner (2000) considers Bayes Nash equilibrium and general games and characterizes when one information structure supports more BNE outcomes than another. While the bulk of his work focusses on complete information games, in Section 6 and Theorem 19 he considers incomplete information games. His definition that one information structure S' is a *faithful interpretation* of another S translates in our language to the requirement that they are higher order belief equivalent and there is a profile of Markov kernels which are independently mapping each player signals S_i into signals in S'_i. He shows that S supports more BNE outcomes than S' in all games if and only if S' is a faithful interpretation of S. Thus this ordering ranks an information structure higher if it gives more "correlation possibilities", but holds fixed beliefs and higher order beliefs. By contrast, individual sufficiency abstracts from "correlation possibilities" and depends non-trivially on beliefs and higher order beliefs about payoffs.

7.5 An Upper Bound on Information

We have proposed Bayes correlated equilibrium as a solution concept that captures what can happen in Bayes Nash equilibrium in an incomplete information game (G, S) if players have access to information structure S but may also observe additional signals. We can also ask the dual question: suppose players have access to at most the information structure in S but perhaps less. One can offer partial answers to this question using the framework and results of this paper.

Lemma 9 *If random choice rule ν is induced by a decision rule σ which is a Bayes Nash equilibrium decision rule of (G, S') for some S' such that $\overline{S}$ is an expansion of S', then ν is feasible for $\left(G, \overline{S}\right)$.*

Now suppose we fix two information structures $\overline{S}$ and $\underline{S}$ such that $\overline{S}$ is individually sufficient for $\underline{S}$ ($\overline{S} \succeq \underline{S}$) Suppose that we knew that random choice rule ν was induced by a Bayes Nash equilibrium of (G, S) for some information structure S with $\overline{S} \succeq S \succeq \underline{S}$. We know (by Theorem 2) that ν is a BCE random choice rule for $(G, \underline{S})$; and we know from Lemma 9 that ν is a feasible random choice rule for

$\left(G, \overline{S}\right)$. Unfortunately, though, the converse is *not* true, i.e., it is not true that if ν is a BCE random choice rule for $(G, \underline{S})$ and ν is a feasible random choice rule for $\left(G, \overline{S}\right)$, then ν is induced by a Bayes Nash equilibrium of (G, S) for some information structure S with $\overline{S} \succeq S \succeq \underline{S}$. To see why, observe that for this converse to be true, it would have to be true in the special case where $\overline{S} = \underline{S} = S^*$. In this case, we would be requiring that if ν is a BCE and feasible random choice rule for (G, S^*), then ν is induced by a Bayes Nash equilibrium of (G, S) for some information structure S which is higher order belief equivalent to S^*, which in turn implies that ν is a belief invariant Bayes correlated equilibrium of (G, S^*). But we already saw, in Figure 8, there exist random choice rules ν which are BCE and feasible random choice rules but not belief invariant BCE random choice rules. In other words, there may be an obedient decision rule that induces ν and a belief invariant decision rule that induces ν, but there is no decision rule which is simultaneously obedient and belief invariant which induces ν.

8 Appendix

8.1 Feasibility Orderings

We report a many player generalization of the three feasibility orderings described in Section 2.4 for the one player case. In Section 4, we defined the set of feasible random choice rules $F(G,S)$ for (G,S). For purposes of this section, all that matters is that $F(G,S)$ is a compact subset of the set of random choice rules for (G,S).

Say that G is a common interest basic game if there exists a payoff function u^* such that $u_i = u^*$ for all i. This focus on common interest games here follows the work of Lehrer, Rosenberg, and Shmaya (2010) for different solution concepts discussed in Section 7.3. Now we can consider three feasibility orderings for the many player case that reduce to those discussed for the one player case in Section 2.4.

Definition 27 (More Permissive)
Information structure S is more permissive than information structure S' if, for all basic games G,

$$F(G,S) \supseteq F\left(G,S'\right).$$

If G is a common interest basic game, write, as before, $W(G,S)$ for the set of state-dependent vectors of ex ante common ex ante payoffs

$$W(G,S) \triangleq \bigcup_{\nu \in F(G,S)} \left\{ \left(\sum_{a\in A} \nu(a|\theta)\, u^*(a,\theta) \right)_{\theta\in\Theta} \right\} \subseteq \mathbb{R}^{|\Theta|}.$$

Definition 28 (More Informative)
Information structure S is more informative than information structure S' if, for all common interest basic games G,

$$W(G,S) \supseteq W\left(G,S'\right).$$

If G is a common interest basic game, write $\overline{w}(G,S)$ for the highest ex ante utility that (all) players can attain with a belief invariant decision rule (and thus a feasible random choice rule),

$$\overline{w}(G,S) \triangleq \max_{w\in W(G,S)} \sum_{\theta\in\Theta} \psi(\theta)\, w(\theta) = \max_{\nu\in F(G,S)} \sum_{a\in A,\theta\in\Theta} \psi(\theta)\, \nu(a|\theta)\, u^*(a,\theta).$$

Note that while we did not impose obedience in the definition of $\overline{w}(G,S)$, observe that it would not have made any difference if we did. Thus we also have:

$$\overline{w}(G,S) = \max_{\nu\in BIBCE(G,S)} \sum_{a\in A,\theta\in\Theta} \psi(\theta)\, \nu(a|\theta)\, u^*(a,\theta).$$

Definition 29 (More Valuable)

Information structure S is more valuable than information structure S' if, for all common interest basic games G,

$$\overline{w}\left(G,S\right)\geq\overline{w}\left(G,S'\right).$$

Now we have a many player generalization of Proposition 2:

Proposition 3 *The following statements are equivalent*

1. *Information structure S is more permissive than information structure S'*
2. *Information structure S is more informative than information structure S'*
3. *Information structure S is more valuable than information structure S'*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) follows immediately from definitions.

Now suppose that (1) does not hold. Then there exists $G=\left(\left(A_i,u_i\right)_{i=1}^{I},\psi\right)$ and $\nu^*\in F\left(G,S'\right)$ such that $\nu\in F\left(G,S\right)$. Since $F\left(G,S\right)$ is compact and convex, by the separating hyperplane theorem, there exists $x^*:A\times\Theta\rightarrow\mathbb{R}$ such that

$$\sum_{a\in A,\theta\in\Theta}\psi\left(\theta\right)\nu^*\left(a|\theta\right)x^*\left(a,\theta\right)>\sum_{a\in A,\theta\in\Theta}\psi\left(\theta\right)\nu\left(a|\theta\right)x^*\left(a,\theta\right)$$

for all $\nu\in F\left(G,S\right)$. Now let $G^*\left(\left(A_i,u_i^*\right)_{i=1}^{I},\psi\right)$ be the common interest game with $u_i^*=x^*$ for all i. Now

$$\begin{aligned}\overline{w}\left(G^*,S'\right)&\geq\sum_{a\in A,\theta\in\Theta}\psi\left(\theta\right)\nu^*\left(a|\theta\right)x^*\left(a,\theta\right)\\&>\max_{\nu\in F(G,S)}\sum_{a\in A,\theta\in\Theta}\psi\left(\theta\right)\nu\left(a|\theta\right)x^*\left(a,\theta\right)\\&=\overline{w}\left(G^*,S\right)\end{aligned}$$

Thus (3) does not hold. Also

$$\left(\sum_{a\in A}\nu^*\left(a|\theta\right)x^*\left(a,\theta\right)\right)_{\theta\in\Theta}\in W\left(G,S'\right)$$

but

$$\left(\sum_{a\in A}\nu^*\left(a|\theta\right)x^*\left(a,\theta\right)\right)_{\theta\in\Theta}\notin W\left(G,S\right)$$

so (2) does not hold. ■

Note that this proof of the equivalence of the three feasibility orderings does not use any properties of the sets $F(G,S)$ except that they are convex and compact. A separating argument like this lies at the heart of proofs of Blackwell's theorem, see, e.g., the elementary proof of Leshno and Spector (1992) for classic Blackwell's theorem and that of Lehrer, Rosenberg, and Shmaya (2010) for common interest games and different solution concepts.

8.2 Understanding Individual Sufficiency

Proof of Lemma 3. 1. We first show $\overline{S} \succeq S$ where $\overline{S} = \left(\overline{T}, \overline{\pi}\right)$ is the complete information structure and $S = (T, \pi)$ is an arbitrary information structure. Define Markov kernel $\phi : \overline{T} \times \Theta \to \Delta(T)$ by

$$\phi\left(t|\bar{t},\theta\right) \triangleq \pi\left(t|\theta\right). \tag{42}$$

Now the marginal property (22) is satisfied, since

$$\sum_{\bar{t}\in\overline{T}} \overline{\pi}\left(\bar{t}|\theta\right)\phi\left(t|\bar{t},\theta\right) = \pi\left(t|\theta\right)$$

for each $t \in T$ and $\theta \in \Theta$; and the independence property (23) is satisfied since

$$\sum_{t_{-i}\in T_{-i}} \phi\left(\left(t_i,t_{-i}\right)|\left(\bar{t}_i,\bar{t}_{-i}\right),\theta\right) = \sum_{t_{-i}\in T_{-i}} \pi\left(\left(t_i,t_{-i}\right)|\theta\right).$$

Now we show $S \succeq \underline{S}$ where $\underline{S} = (\underline{T}, \underline{\pi})$ is the null information structure and $S = (T, \pi)$ is an arbitrary information structure. Define Markov kernel $\phi : T \times \Theta \to \Delta(\underline{T})$ by $\phi(\underline{t}|t,\theta) = 1$ and the conditions for individual sufficiency are automatically satisfied.

2. If S is sufficient for S', then there exists $\phi : T \times \Theta \to \Delta(T')$ such that

$$\sum_{t} \pi\left(t|\theta\right)\phi\left(t'|t,\theta\right) = \pi'\left(t'|\theta\right)$$

for each t' and θ, and

$$\phi_i\left(t_i'|t_i\right) \equiv \sum_{t_{-i}'} \phi\left(\left(t_i',t_{-i}'\right)|\left(t_i,t_{-i}\right),\theta\right)$$

is independent of t_{-i} and θ. If S' is individually sufficient for S'', then there exists $\phi' : T' \times \Theta \to \Delta(T'')$ such that

$$\sum_{t'} \pi'\left(t'|\theta\right)\phi'\left(t''|t',\theta\right) = \pi''\left(t''|\theta\right)$$

for each t'' and θ, and

$$\phi_i'\left(t_i''|t_i'\right) \equiv \sum_{t_{-i}''} \phi'\left(\left(t_i'',t_{-i}''\right)|\left(t_i',t_{-i}'\right),\theta\right)$$

is independent of t'_{-i} and θ. Define $\phi^* : T \times \Theta \to \Delta(T'')$ by

$$\phi^*\left(t''|t,\theta\right) = \sum_{t'\in T'} \phi\left(t'|t,\theta\right)\phi\left(t''|t',\theta\right).$$

Now

$$\begin{aligned}
\phi_i^*\left(t_i''|t_i\right) &\equiv \sum_{t''_{-i}} \phi^*\left(\left(t_i'',t_{-i}''\right)|\left(t_i,t_{-i}\right),\theta\right) \\
&= \sum_{t''_{-i}}\sum_{t'\in T'} \phi\left(t'|t,\theta\right)\phi'\left(t''|t',\theta\right) \\
&= \sum_{t'\in T'} \phi\left(t'|t,\theta\right)\phi_i'\left(t_i''|t_i'\right) \\
&= \sum_{t_i'\in T_i'} \phi_i\left(t_i'|t_i\right)\phi_i'\left(t_i''|t_i'\right)
\end{aligned}$$

which is independent of t_{-i} and θ. ■

We now present a formal argument that the notion of higher order belief equivalence presented earlier in Definition 21 is equivalent to a definition in terms of the hierarchical belief types of Mertens and Zamir (1985).

Fix Θ. Let $X^0 = \Theta$, and define $X^k = X^{k-1} \times \left[\Delta\left(X^{k-1}\right)\right]^{I-1}$. An element of $\left(\Delta\left(X^k\right)\right)_{k=0}^{\infty} \triangleq H$ is called a hierarchy (of beliefs). For notational simplicity, we will work with a uniform prior on Θ (other full support priors will lead to shifts in posteriors over Θ but no changes in higher order belief equivalence). Fix an information structure $S = \left(\left(T_i\right)_{i=1}^I, \pi\right)$. For each i and $t_i \in T_i$, write $\widehat{\pi}_i^1[t_i] \in \Delta(\Theta) = \Delta\left(X^0\right)$ for his posterior under a uniform prior on Θ, so

$$\widehat{\pi}_i^1[t_i](\theta) = \frac{\sum\limits_{t_{-i}\in T_{-i}} \pi\left(\left(t_i,t_{-i}\right)|\theta\right)}{\sum\limits_{\theta'\in\Theta, t_{-i}\in T_{-i}} \pi\left(\left(t_i,t_{-i}\right)|\theta'\right)}.$$

Write $\widehat{\pi}_i^2[t_i] \in \Delta\left(\Theta \times \left(\Delta(\Theta)\right)^{I-1}\right) = \Delta\left(X^1\right)$ for his belief over Θ and the first order beliefs of other players, so

$$\widehat{\pi}_i^2[t_i]\left(\theta,\pi_{-i}^1\right) = \frac{\sum\limits_{\left\{t_{-i}\in T_{-i}|\widehat{\pi}_j^1(t_j)=\pi_j^1 \text{ for each } j\neq i\right\}} \pi\left(\left(t_i,t_{-i}\right)|\theta\right)}{\sum\limits_{\theta'\in\Theta,\left\{t_{-i}\in T_{-i}|\widehat{\pi}_j^1(t_j)=\pi_j^1 \text{ for each } j\neq i\right\}} \pi\left(\left(t_i,t_{-i}\right)|\theta'\right)}.$$

Proceeding inductively for $k \geq 2$, write $\widehat{\pi}_i^k(t_i) \in \Delta\left(X^{k-1}\right)$ for his belief over Θ and the $(k-1)$th order beliefs of other players, so

$$\widehat{\pi}_i^k \left[t_i\right] \left(\theta, \pi_{-i}^{k-1}\right) = \frac{\sum\limits_{\left\{t_{-i} \in T_{-i} | \widehat{\pi}_j^{k-1}(t_j) = \pi_j^{k-1} \text{ for each } j \neq i\right\}} \pi\left(\left(t_i, t_{-i}\right) | \theta\right)}{\sum\limits_{\theta' \in \Theta, \left\{t_{-i} \in T_{-i} | \widehat{\pi}_j^{k-1}(t_j) = \pi_j^{k-1} \text{ for each } j \neq i\right\}} \pi\left(\left(t_i, t_{-i}\right) | \theta'\right)}.$$

Now we can define $\widehat{\pi}_i : T_i \to H$ by

$$\widehat{\pi}_i \left[t_i\right] = \left(\widehat{\pi}_i^1 \left[t_i\right], \widehat{\pi}_i^2 \left[t_i\right], ...\right)$$

and $\widehat{\pi} : T \to H^I$ by

$$\widehat{\pi}\left[t\right] = \left(\widehat{\pi}_i \left[t_i\right]\right)_{i=1}^I.$$

Now we can identify information structure S with a probability distribution $\chi_{\psi,S} \in \Delta\left(H^I\right)$ defined by

$$\chi_{\psi,S}\left(\left(\pi_i\right)_{i=1}^I\right) = \frac{1}{\#\Theta} \sum_{\left\{t | \widehat{\pi}[t] = \left(\pi_i\right)_{i=1}^I\right\}} \pi\left(t|\theta\right).$$

Lemma 10 (Higher Order Belief Characterization)

The following statements are equivalent:

1. *Information structures S^1 and S^2 are higher order belief equivalent;*

2. $\chi_{S^1} = \chi_{S^2}$.

Proof. We argue that (1) implies (2) by induction. By (26),

$$f_i^k\left(t_i\right) = f_i^k\left(t_i'\right) \Rightarrow \widehat{\pi}_i^{k,1}\left[t_i\right] = \widehat{\pi}_i^{k,1}\left[t_i'\right].$$

Now suppose that

$$f_i^k\left(t_i\right) = f_i^k\left(t_i'\right) \Rightarrow \widehat{\pi}_i^{k,l}\left[t_i\right] = \widehat{\pi}_i^{k,l}\left[t_i'\right].$$

By (26), we have

$$f_i^k\left(t_i\right) = f_i^k\left(t_i'\right) \Rightarrow \widehat{\pi}_i^{k,l+1}\left[t_i\right] = \widehat{\pi}_i^{k,l+1}\left[t_i'\right].$$

But since the premise of the inductive step holds for $l = 1$, we have that for all l

$$f_i^k\left(t_i\right) = f_i^k\left(t_i'\right) \Rightarrow \widehat{\pi}_i^{k,l}\left[t_i\right] = \widehat{\pi}_i^{k,l}\left[t_i'\right].$$

and thus

$$f_i^k\left(t_i\right) = f_i^k\left(t_i'\right) \Rightarrow \widehat{\pi}_i^{k}\left[t_i\right] = \widehat{\pi}_i^{k}\left[t_i'\right].$$

Now suppose that (2) holds. Let T_i^* =range$\left(\widehat{\pi}_i^1\right)$ =range$\left(\widehat{\pi}_i^2\right)$. Let $f_i^k\left(t_i\right) = \widehat{\pi}_i^k\left(t_i\right)$. By construction, properties (25) and (26) hold with respect to information structure $S^* = \left(\left(T_i^*\right)_{i=1}^I, \pi^*\right)$. ■

Proof of Lemma 4. Part (1) of the lemma is stated in Theorem 1 in Liu (2011). Now if information structures $S^1 = \left(T^1, \pi^1\right)$ and $S^2 = \left(T^2, \pi^2\right)$ are higher order belief equivalent, we can show that S^1 is individually sufficient for S^2 by letting

$$\phi\left(t^2|t^1,\theta\right) = \begin{cases} \frac{\pi^2\left(t^2|\theta\right)}{\sum\limits_{\left\{\tilde{t}^2|f^2\left(\tilde{t}^2\right)=f^2\left(t^2\right)\right\}} \pi^2\left(\tilde{t}^2|\theta\right)}, & \text{if} \quad f^2\left(t^2\right) = f^1\left(t^1\right), \\ 0, & \text{otherwise.} \end{cases}$$

One can similarly show that S^2 is individually sufficient for S^1.

Now suppose that S^1 and S^2 are mutually individually sufficient. If either S^1 or S^2 are redundant, we can replace them with their (by part (1)) unique non-redundant versions, and they will remain mutually individually sufficient. So it is enough to show that if S^1 and S^2 are mutually individually sufficient and non-redundant, then they are higher order belief equivalent. Write ϕ^1 and ϕ^2 for the Markov kernels establishing that, respectively, S^1 is individually sufficient for S^2 and S^2 is individually sufficient for S^1. Define $\widehat{\phi} : T^1 \times \Theta \to \Delta\left(T^1\right)$ by

$$\widehat{\phi}\left(\tilde{t}^1 \left| t^1, \theta\right.\right) = \sum_{t^2 \in T^2} \phi^1\left(t^2 \left| t^1, \theta\right.\right) \phi^2\left(\tilde{t}^1 \left| t^2, \theta\right.\right)$$

for all $t^1, \tilde{t}^1 \in T^1$ and $\theta \in \Theta$. It inherits the properties that

$$\begin{aligned} \pi\left(t|\theta\right) &= \sum_{\tilde{t} \in T} \pi\left(\tilde{t}|\theta\right) \widehat{\phi}\left(t \left| \tilde{t}, \theta\right.\right) \text{ and} \\ \widehat{\phi}_i\left(\tilde{t}_i^1 \left| t_i^1\right.\right) &= \sum_{\tilde{t}_{-i}^1 \in T_{-i}} \widehat{\phi}\left(\left(\tilde{t}_i^1, \tilde{t}_{-i}^1\right) \left| \left(t_i^1, t_{-i}^1\right), \theta\right.\right) \end{aligned}$$

is independent of $\left(t_{-i}^1, \theta\right)$.

Define a partition of T_i^1 by

$$P_i\left(t_i\right) = \left\{ \tilde{t}_i \in T_i^1 \;\middle|\; \begin{array}{l} \text{there exists } \left(t_i^k\right)_{k=1}^K \text{ with} \\ \widehat{\phi}_i\left(\tilde{t}_i^{k+1} \left| t_i^k\right.\right) > 0 \text{ for each } k = 1, ..., K-1 \\ \text{and } \left(t_i^1, t_i^K\right) = \text{either } \left(t_i, \tilde{t}_i\right) \text{ or } \left(t_i, \tilde{t}_i\right) \end{array} \right\}.$$

Now

$$\begin{aligned}
\sum_{\widehat{t}_{-i}\in P_{-i}(t_{-i})} \pi\left(\left(t_i,\widehat{t}_{-i}\right)|\theta\right) &= \sum_{\widetilde{t}\in T}\pi\left(\widetilde{t}|\theta\right) \sum_{\widehat{t}_{-i}\in P_{-i}(t_{-i})} \widehat{\phi}\left(\left(t_i,\widehat{t}_{-i}\right)\left|\left(\widetilde{t}_i,\widetilde{t}_{-i}\right),\theta\right.\right) \\
&= \sum_{\widetilde{t}_i\in P_i(t_i)} \sum_{\widetilde{t}_{-i}\in P_{-i}(t_{-i})} \pi\left(\left(\widetilde{t}_i,\widetilde{t}_{-i}\right)\middle|\,\theta\right) \sum_{\widehat{t}_{-i}\in P_{-i}(t_{-i})} \widehat{\phi}\left(\left(t_i,\widehat{t}_{-i}\right)\left|\left(\widetilde{t}_i,\widetilde{t}_{-i}\right),\theta\right.\right) \\
&= \sum_{\widetilde{t}_i\in P_i(t_i)} \sum_{\widetilde{t}_{-i}\in P_{-i}(t_{-i})} \pi\left(\left(\widetilde{t}_i,\widetilde{t}_{-i}\right)\middle|\,\theta\right) \sum_{\widehat{t}_{-i}\in T_{-i}} \widehat{\phi}\left(\left(t_i,\widehat{t}_{-i}\right)\left|\left(\widetilde{t}_i,\widetilde{t}_{-i}\right),\theta\right.\right) \\
&= \sum_{\widetilde{t}_i\in P_i(t_i)} \sum_{\widetilde{t}_{-i}\in P_{-i}(t_{-i})} \pi\left(\left(\widetilde{t}_i,\widetilde{t}_{-i}\right)\middle|\,\theta\right) \widehat{\phi}_i\left(t_i\left|\widetilde{t}_i\right.\right) \\
&= \sum_{\widetilde{t}_i\in P_i(t_i)} \widehat{\phi}_i\left(t_i\left|\widetilde{t}_i\right.\right) \sum_{\widetilde{t}_{-i}\in P_{-i}(t_{-i})} \pi\left(\left(\widetilde{t}_i,\widetilde{t}_{-i}\right)\middle|\,\theta\right).
\end{aligned}$$

Thus for any $\psi \in \Delta_{++}(\Theta)$,

$$\psi(\theta) \sum_{\widehat{t}_{-i}\in P_{-i}(t_{-i})} \pi\left(\left(t_i,\widehat{t}_{-i}\right)|\theta\right) = \psi(\theta) \sum_{\widetilde{t}_i\in P_i(t_i)} \widehat{\phi}_i\left(t_i\left|\widetilde{t}_i\right.\right) \sum_{\widetilde{t}_{-i}\in P_{-i}(t_{-i})} \pi\left(\left(\widetilde{t}_i,\widetilde{t}_{-i}\right)\middle|\,\theta\right).$$

Writing

$$\lambda_i(t_i) = \sum_{t_{-i}\in T_{-i},\theta\in\Theta} \psi(\theta)\,\pi\left(\left(t_i,t_{-i}\right)|\theta\right),$$

we have

$$\sum_{\widetilde{t}_{-i}\in P_{-i}(t_{-i})} \pi_\psi\left(\widetilde{t}_{-i},\theta|t_i\right) = \frac{1}{\lambda_i(t_i)} \sum_{\widetilde{t}_i\in P_i(t_i)} \widehat{\phi}_i\left(t_i\left|\widetilde{t}_i\right.\right)\lambda_i\left(\widetilde{t}_i\right) \sum_{\widetilde{t}_{-i}\in P_{-i}(t_{-i})} \pi_\psi\left(\widetilde{t}_{-i},\theta|t_i\right).$$

This condition states that posteriors over $(P_{-i}(t_{-i}),\theta)$ for t_i are a weighted sum of posteriors for $\widetilde{t}_i \in P_i(t_i)$. This implies that all have the same beliefs. If the information structure is non-redundant, this implies that each $\widehat{\phi}_i$ must be the identity function. But this implies that ϕ^1 and ϕ^2 are identities and thus S^1 and S^2 are higher order belief equivalent. ■

Proof of Lemma 5. Suppose that S is individually sufficient for S'. Thus there exists $\phi : T\times\Theta \to \Delta(T')$ such that

$$\sum_t \pi(t|\theta)\,\phi\left(t'|t,\theta\right) = \pi'\left(t'|\theta\right) \tag{43}$$

for each t' and θ, and

$$\sum_{t'_{-i}} \phi\left(\left(t'_i,t'_{-i}\right)|\left(t_i,t_{-i}\right),\theta\right) \tag{44}$$

is independent of t_{-i} and θ. Let $S^* = (T^*,\pi^*)$ be the combined information structure with $T_i^* = T_i \times T'_i$ for each i and

$$\pi^*\left(t,t'|\theta\right) = \pi(t|\theta)\,\phi\left(t'|t,\theta\right) \tag{45}$$

for each t, t' and θ.

We will first show that S is individually sufficient for S^*. To do so, define $\phi^* : T \times \Theta \to \Delta(T^*)$ by

$$\phi^*(t^*|t,\theta) = \phi^*\left(\left(\widetilde{t}, t'\right)|t,\theta\right) = \begin{cases} \phi(t'|t,\theta), & \text{if } \widetilde{t} = t, \\ 0, & \text{if } \widetilde{t} \neq t, \end{cases} \tag{46}$$

for each $t^* = \left(\widetilde{t}, t'\right) \in T^*$, t and θ. Observe that

$$\begin{aligned} \sum_t \pi(t|\theta)\,\phi\left(\left(\widetilde{t}, t'\right)|t,\theta\right) &= \pi\left(\widetilde{t}|\theta\right)\phi\left(t'|\widetilde{t},\theta\right), \text{ by (46)} \\ &= \pi^*\left(\widetilde{t}, t'|\theta\right), \text{ by (45).} \end{aligned}$$

for each $\widetilde{t}$, t' and θ. Also observe that

$$\begin{aligned} \sum_{t^*_{-i}} \phi^*\left(\left(t^*_i, t^*_{-i}\right)|\left(t_i, t_{-i}\right),\theta\right) &= \sum_{\widetilde{t}_{-i}, t'_{-i}} \phi^*\left(\left(\left(\widetilde{t}_i, t'_i\right), \left(\widetilde{t}_{-i}, t'_{-i}\right)\right)|\left(t_i, t_{-i}\right),\theta\right) \\ &= \begin{cases} \sum\limits_{t'_{-i} \in T_{-i}} \phi^*\left(\left(t'_i, t'_{-i}\right)|\left(t_i, t_{-i}\right),\theta\right), & \text{if } \widetilde{t} = t, \\ 0, & \text{if } \widetilde{t} \neq t, \end{cases} \end{aligned}$$

is independent of t_{-i} and θ by (44).

We will now show that S^* is individually sufficient for S. To do so, define $\widehat{\phi} : T^* \times \Theta \to \Delta(T)$ by

$$\widehat{\phi}(t|t^*,\theta) = \widehat{\phi}\left(t|\left(\widetilde{t}, t'\right),\theta\right) = \begin{cases} 1, & \text{if } \widetilde{t} = t, \\ 0, & \text{if } \widetilde{t} \neq t, \end{cases} \tag{47}$$

for each $t^* = \left(\widetilde{t}, t'\right) \in T^*$, t and θ. Observe that

$$\begin{aligned} \sum_{t^8 \in T^*} \pi^*(t^*|\theta)\,\widehat{\phi}(t|t^*,\theta) &= \sum_{\left(\widetilde{t}, t'\right) \in T^*} \pi^*\left(\widetilde{t}, t'|\theta\right)\widehat{\phi}\left(t|\left(\widetilde{t}, t'\right),\theta\right) \\ &= \sum_{t' \in T'} \pi^*\left(t, t'|\theta\right), \text{ by (47)} \\ &= \sum_{t' \in T'} \pi(t|\theta)\,\phi\left(t'|t,\theta\right), \text{ by (45)} \\ &= \pi(t|\theta) \end{aligned}$$

for each t and θ. Also observe that

$$\begin{aligned} \sum_{t_{-i}} \widehat{\phi}\left(\left(t_i, t_{-i}\right)|\left(t^*_i, t^*_{-i}\right),\theta\right) &= \sum_{t_{-i}} \widehat{\phi}\left(\left(t_i, t_{-i}\right)|\left(\left(\widetilde{t}_i, t'_i\right), \left(\widetilde{t}_{-i}, t'_{-i}\right)\right),\theta\right) \\ &= \begin{cases} 1, & \text{if } \widetilde{t}_i = t_i, \\ 0, & \text{if } \widetilde{t}_i \neq t_i, \end{cases} \end{aligned}$$

is independent of t_{-i} and θ.

We have now shown that if S is individually sufficient for S' then there exists an expansion of S', S^*, such that S and S^* are mutually individually sufficient. By Lemma 4, S and S^* are higher order belief equivalent.

Conversely, suppose that S is higher order belief equivalent to an expansion of S'. Let us call that expansion $S^* = \left(\left(T_i' \times T_i^+\right)_{i=1}^I, \pi^*\right)$. By Lemma 4, S is individually sufficient for S^*. Thus there exists $\phi^* : T \times \Theta \to \Delta(T^*)$ such that

$$\sum_t \pi(t|\theta)\,\phi^*(t^*|t,\theta) = \pi^*(t^*|\theta) \tag{48}$$

for each t^* and θ, and

$$\sum_{t^*_{-i}} \phi^*\left(\left(t_i^*, t_{-i}^*\right)|\left(t_i, t_{-i}\right), \theta\right) \tag{49}$$

is independent of t_{-i} and θ. Define $\phi : T \times \Theta \to \Delta(T')$ by

$$\phi\left(t'|t,\theta\right) = \sum_{t^+} \phi^*\left(\left(t', t^+\right)|t,\theta\right) \tag{50}$$

for each t, t' and θ. Now

$$\begin{aligned}
\sum_t \pi(t|\theta)\,\phi\left(t'|t,\theta\right) &= \sum_{t^+}\sum_t \pi(t|\theta)\,\phi^*\left(\left(t', t^+\right)|t,\theta\right), \text{ by (50)} \\
&= \sum_{t^+} \pi^*\left(\left(t', t^+\right)|\theta\right), \text{ by (50)} \\
&= \pi'\left(t'|\theta\right), \text{ because } S^* \text{ is an expansion of } S'
\end{aligned}$$

for each t' and θ. Also

$$\begin{aligned}
\sum_{t'_{-i}} \phi\left(\left(t_i', t_{-i}'\right)|\left(t_i, t_{-i}\right), \theta\right) &= \sum_{t^+}\sum_{t'_{-i}} \phi^*\left(\left(t_i', t_{-i}^+\right), \left(t_i', t_{-i}^+\right)|\left(t_i, t_{-i}\right), \theta\right) \\
&= \sum_{t_i^+}\sum_{t^*_{-i}} \phi^*\left(\left(t_i', t_i^+\right), t^*_{-i}|\left(t_i, t_{-i}\right), \theta\right)
\end{aligned}$$

which is independent of t_{-i} and θ by (49). ■

8.3 Leading Example: Binary Action, Binary Information Structure

8.3.1 Bayes Correlated Equilibria in the Leading Example

We first establish the claim that the set of symmetric BCE random choice rules of $(G_\varepsilon, S_{q,2q-1})$ if $\varepsilon = 0$ and $q \geq \frac{2}{3}$ is the convex hull of the set of four random choice rules given earlier by (33):

$$\left(q', r'\right) = (0,0), (1,1), (q, 2q-1) \text{ and } (2(1-q), 1-q).$$

Now, symmetric random choice rules which are parameterized by (q', r') must have

$$\max\left\{0, 2q' - 1\right\} \leq r' \leq q'. \tag{51}$$

Consider the following convenient parameterization of the distribution on $A^2 \times T^2$ conditional on Θ, generated by the information structure $S_{q,2q-1}$ and a symmetric decision rule:

$$\begin{array}{|c|c|c|c|c|} \hline \theta = 0 & 00 & 01 & 10 & 11 \\ \hline 00 & x_1 & x_2 & x_3 & x_4 \\ \hline 01 & x_2 & x_5 & x_6 & x_7 \\ \hline 10 & x_3 & x_6 & 0 & 0 \\ \hline 11 & x_4 & x_7 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|} \hline \theta = 1 & 00 & 01 & 10 & 11 \\ \hline 00 & 0 & 0 & x_7 & x_4 \\ \hline 01 & 0 & 0 & x_6 & x_3 \\ \hline 10 & x_7 & x_6 & x_5 & x_2 \\ \hline 11 & x_4 & x_3 & x_2 & x_1 \\ \hline \end{array}. \tag{52}$$

Here, rows correspond to the signal action pair (t, a) of Ann, the columns correspond to the signal action pair (t, a) of Bob and the matrix corresponds to the state. For this decision rule to induce the random choice rules parameterized by (q', r'), we must have:

$$x_1 + 2x_3 = r', \tag{53}$$

$$x_2 + x_4 + x_6 = q' - r'. \tag{54}$$

For the decision rule to be consistent with information structure $S_{q,2q-1}$, we must have:

$$x_1 + 2x_2 + x_5 = 2q - 1, \tag{55}$$

$$x_3 + x_4 + x_6 + x_7 = 1 - q. \tag{56}$$

The obedience constraint that a player follows his action recommendation when equal to his signal implies that

$$x_1 + x_3 \geq x_4. \tag{57}$$

The obedience constraint that a player follows his action recommendation when not equal to his signal implies that

$$x_2 + x_6 \leq x_3. \tag{58}$$

Adding (57) and (58) gives

$$x_1 + 2x_3 \geq x_2 + x_4 + x_6 \tag{59}$$

Now

$$\begin{aligned} q' - r' &= x_2 + x_4 + x_6, \text{ by } (54) \\ &\leq x_3 + x_4, \text{ by } (57) \\ &\leq x_3 + x_4 + x_6 + x_7 \\ &= 1 - q, \text{ by } (56) \end{aligned} \tag{60}$$

But now the set of (q', r') satisfying (51) and (60) is the convex hull of

$$(q', r') = (0,0), (1,1), (q, 2q-1) \text{ and } (2(1-q), 1-q).$$

We have shown that the set of BCE random choice rules is a subset of this convex hull. To show it is equal to this convex hull, it is enough to show that it is possible to find decision rules described according to (52) satisfying (53) - (58) corresponding to each of these four points. These are, for $(0,0)$:

$\theta = 0$	00	01	10	11
00	0	0	0	0
01	0	$2q-1$	0	$1-q$
10	0	0	0	0
11	0	$1-q$	0	0

$\theta = 1$	00	01	10	11
00	0	0	$1-q$	0
01	0	0	0	0
10	$1-q$	0	$2q-1$	0
11	0	0	0	0

;

for $(1,1)$:

$\theta = 0$	00	01	10	11
00	$2q-1$	0	$1-q$	0
01	0	0	0	0
10	$1-q$	0	0	0
11	0	0	0	0

$\theta = 1$	00	01	10	11
00	0	0	0	0
01	0	0	0	$1-q$
10	0	0	0	0
11	0	$1-q$	0	$2q-1$

;

for $(q, 2q-1)$:

$\theta = 0$	00	01	10	11
00	$2q-1$	0	0	$1-q$
01	0	0	0	0
10	0	0	0	0
11	$1-q$	0	0	0

$\theta = 1$	00	01	10	11
00	0	0	0	$1-q$
01	0	0	0	0
10	0	0	0	0
11	$1-q$	0	0	$2q-1$

;

and for $(2(1-q), 1-q)$:

$\theta=0$	00	01	10	11
00	$1-q$	0	0	$1-q$
01	0	$3q-2$	0	0
10	0	0	0	0
11	$1-q$	0	0	0

$\theta=1$	00	01	10	11
00	0	0	0	$1-q$
01	0	0	0	0
10	0	0	$3q-2$	0
11	$1-q$	0	0	$1-q$

.

8.3.2 Binary Information Structures and Orderings

Proof of Lemma 6. (1.) We first show that if (37) holds, then (q,r) is sufficient for (q',r'). So

$$\left(q',r'\right)=\lambda_1\left(\frac{1}{2},0\right)+\lambda_2\left(\frac{1}{2},\frac{1}{2}\right)+\lambda_3\left(q,r\right)+\lambda_4\left(q,q\right).$$

We define a Markov kernel:

$$\phi\left(t'\left|t\right.\right)\triangleq\frac{1}{2}\lambda_1\mathbb{I}_{t'_1\neq t'_2}\left(t'\right)+\frac{1}{2}\lambda_2\mathbb{I}_{t'_1=t'_2}\left(t'\right)+\lambda_3\mathbb{I}_{t'=t}\left(t,t'\right)+\lambda_4\mathbb{I}_{t'_1=t'_2,t'_1=t_1}\left(t,t'\right). \tag{61}$$

Now we write the information structure S' as, using

$$\pi\left(t'\left|\theta\right.\right)=\sum_t\phi\left(t'\left|t\right.\right)\pi\left(t\left|\theta\right.\right),$$

and so $r'=\pi'\left(00\left|0\right.\right)$ with:

$$\begin{aligned}\pi'\left(00\left|0\right.\right)&=\phi\left(00\left|00\right.\right)\pi\left(00\left|0\right.\right)+\phi\left(00\left|01\right.\right)\pi\left(01\left|0\right.\right)+\phi\left(00\left|10\right.\right)\pi\left(10\left|0\right.\right)+\phi\left(00\left|11\right.\right)\pi\left(11\left|0\right.\right)\\&=\frac{1}{2}\lambda_2+\lambda_3r+\lambda_4\left(r+q-r\right)\\&=\frac{1}{2}\lambda_2+\lambda_3r+\lambda_4q,\end{aligned}$$

and similarly:

$$\begin{aligned}\pi'\left(01\left|0\right.\right)&=\phi\left(01\left|00\right.\right)\pi\left(00\left|0\right.\right)+\phi\left(01\left|01\right.\right)\pi\left(01\left|0\right.\right)+\phi\left(01\left|10\right.\right)\pi\left(10\left|0\right.\right)+\phi\left(01\left|11\right.\right)\pi\left(11\left|0\right.\right)\\&=\frac{1}{2}\lambda_1+\lambda_3\left(q-r\right),\end{aligned}$$

as well as

$$\begin{aligned}\pi'\left(11\left|0\right.\right)&=\phi\left(11\left|00\right.\right)\pi\left(00\left|0\right.\right)+\phi\left(11\left|01\right.\right)\pi\left(01\left|0\right.\right)+\phi\left(11\left|10\right.\right)\pi\left(10\left|0\right.\right)+\phi\left(11\left|11\right.\right)\pi\left(11\left|0\right.\right)\\&=\frac{1}{2}\lambda_2+\lambda_4\left(q-r+\left(r+1-2q\right)\right)\\&=\frac{1}{2}\lambda_2+\lambda_4\left(1-q\right),\end{aligned}$$

and so it follows:

$$\begin{aligned} q' &= \pi'(00\,|0) + \pi'(0\,|01) = \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2 + \lambda_3 q + \lambda_4 q, \\ r' &= \pi'(00\,|0) = \frac{1}{2}\lambda_2 + \lambda_3 r + \lambda_4 q. \end{aligned}$$

More generally, we have

$\pi'(\cdot\|0)$	$t_2' = 0$	$t_2' = 1$
$t_1' = 0$	$\frac{1}{2}\lambda_2 + \lambda_3 r + \lambda_4 q$	$\frac{1}{2}\lambda_1 + \lambda_3(q-r)$
$t_1 = 1$	$\frac{1}{2}\lambda_1 + \lambda_3(q-r)$	$\frac{1}{2}\lambda_2 + \lambda_4(1-q)$

$\pi'(\cdot\|1)$	$t_2' = 0$	$t_2' = 1$
$t_1' = 0$	$\frac{1}{2}\lambda_2 + \lambda_4(1-q)$	$\frac{1}{2}\lambda_1 + \lambda_3(q-r)$
$t_1' = 1$	$\frac{1}{2}\lambda_1 + \lambda_3(q-r)$	$\frac{1}{2}\lambda_2 + \lambda_3 r + \lambda_4 q$

. (62)

We observe that the Markov kernel (61) does not satisfy individual sufficiency, in particular, if $t_1 = t_2' = t_2 = 0$, then

$$\sum_{t_1'} \phi\left(t_1', 0\,|00\right) = \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2 + \lambda_3 + \lambda_4, \tag{63}$$

but if $t_1 = 1, t_2' = t_2 = 0$, then:

$$\sum_{t_1'} \phi\left(t_1', 0\,|10\right) = \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2 + \lambda_3,$$

and hence the sum $\sum_{t_1'} \phi\left(t_1', t_2'\,|t_1, t_2\right)$ *does* depend on t_1, hence a violation of individual sufficiency.

We now show that if S is sufficient for S', then (37) holds. Without loss of generality we can take signal distribution over three rather than four signals as $t = 01$ and $t = 10$ are symmetric. Thus rewriting for the purpose of this proof

$$\begin{array}{cccc} \pi(\cdot|0) & t=00 & t=\{10,01\} & t=11 \\ & r & 2(q-r) & 1+r-2q \end{array} \qquad \begin{array}{cccc} \pi(\cdot|1) & t=00 & t=\{10,01\} & t=11 \\ & 1+r-2q & 2(q-r) & r \end{array},$$

as

$$\begin{array}{cccc} \pi(\cdot|0) & t=0 & t=1 & t=2 \\ & x & 1-x-y & y \end{array} \qquad \begin{array}{cccc} \pi(\cdot|1) & t=0 & t=1 & t=2 \\ & y & 1-x-y & x \end{array}, \tag{64}$$

so that

$$\begin{array}{rcl} x &=& r \\ y &=& 1+r-2q \end{array} \iff \begin{array}{rcl} r &=& x \\ q &=& \frac{1}{2}x - \frac{1}{2}y + \frac{1}{2} \end{array}. \tag{65}$$

Now, for S to be sufficient for S', there has to be a Markov kernel $\phi(t'|t)$ such that:

$$\pi\left(t'|\theta\right) = \sum_{t\in T} \phi\left(t'|t\right)\pi(t|\theta) \tag{66}$$

with

$$\begin{bmatrix} \phi(t'|t) & 0 & 1 & 2 \\ 0 & \phi(0|0) & \phi(1|0) & \phi(2|0) \\ 1 & \phi(0|1) & \phi(1|1) & \phi(2|1) \\ 2 & \phi(0|2) & \phi(1|2) & \phi(2|2) \end{bmatrix}.$$

But with the symmetry of (64), it follows from equalities of (66) that we can write the Markov kernel $\phi(t'|t)$ as:

$$\begin{bmatrix} \phi(t'|t) & 0 & 1 & 2 \\ 0 & a-b & 1-a & b \\ 1 & c & 1-2c & c \\ 2 & b & 1-a & a-b \end{bmatrix}, \tag{67}$$

with the nonnegativity restrictions:

$$0 \leq b \leq a \leq 1, \text{ and } 0 \leq c \leq 1/2. \tag{68a}$$

In turn, it follows that if S is sufficient for the information structure S', then from (66) and (67), the conditional probabilities x', y' have to satisfy:

$$\begin{aligned} x' &= (a-b-c)\,x + (b-c)\,y + c, \\ y' &= (a-b-c)\,y + (b-c)\,x + c. \end{aligned} \tag{69}$$

But with the nonnegativity restrictions (68a), an equivalent restriction to (69) is that:

$$\begin{aligned} x' &= \left(\lambda_2 + \tfrac{1}{2}\lambda_3\right) x - \tfrac{1}{2}\lambda_3 y + \tfrac{1}{2}(\lambda_1 + \lambda_3), \\ y' &= \left(\lambda_2 + \tfrac{1}{2}\lambda_3\right) y - \tfrac{1}{2}\lambda_3 x + \tfrac{1}{2}(\lambda_1 + \lambda_3), \end{aligned} \tag{70}$$

by relabeling

$$a = \lambda_1 + \lambda_2 + \lambda_3, b = \frac{1}{2}\lambda_1, c = \frac{1}{2}(\lambda_1 + \lambda_3),$$

and requiring that:

$$\lambda_1, \lambda_2, \lambda_3 \geq 0 \text{ and } \lambda_1 + \lambda_2 + \lambda_3 \leq 1.$$

In other words (x', y') can be represented as the convex combination of the vertices:

$$\left\{(0,0), \left(\frac{1}{2}, \frac{1}{2}\right), (x,y), \left(x + \frac{1}{2}(1-x-y), y + \frac{1}{2}(1-x-y)\right)\right\},$$

with weights $(1-\lambda_1-\lambda_2-\lambda_3), \lambda_1, \lambda_2, \lambda_3$ respectively. But given (65), this means in terms of the original variables (q', r'), that they can be represented as the convex combination of the vertices

$$\left\{(0,0), \left(\frac{1}{2}, \frac{1}{2}\right), (q,r), (q,q)\right\},$$

with the above weights $(1-\lambda_1-\lambda_2-\lambda_3), \lambda_1, \lambda_2, \lambda_3$, hence establishing (37).

(2.) and (3.) We first show that if (39) holds, then S is a non-communicating ungarbling of S'. This in turn implies (from the definitions) that S is individually sufficient for S'. Suppose that

$$(q', r') = \lambda_1 \left(\frac{1}{2}, 0\right) + \lambda_2 \left(\frac{1}{2}, \frac{1}{2}\right) + \lambda_3 (q,r),$$

for some $\lambda \in \Delta(\{1,2,3\})$. Let the Markov kernel be:

$$\phi\left(t'|t\right) = \frac{1}{2}\lambda_1 \mathbb{I}_{t_1' \neq t_2'}\left(t'\right) + \frac{1}{2}\lambda_2 \mathbb{I}_{t_1' = t_2'}\left(t'\right) + \lambda_3 \mathbb{I}_{t'=t}\left(t,t'\right). \tag{71}$$

It is easy to verify that

$$\sum_{t_j' \in T_j} \phi\left(t_i', t_j' | t_i, t_j\right),$$

is independent of t_j, and hence that the Markov kernel satisfies the condition of a non-communicating ungarbling, in particular if $t_i' = t_i$, then $\sum_{t_j' \in T_j} \phi\left(t_i', t_j' | t_i, t_j\right) = \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2 + \lambda_3$ and if $t_i' \neq t_i$, then $\sum_{t_j' \in T_j} \phi\left(t_i', t_j' | t_i, t_j\right) = \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2$. Now, with given Markov kernel $\phi(t'|t)$ we find that the signal structure that can generated from (71) and $\pi(t|\theta)$, or

$$\pi'\left(t'|\theta\right) = \sum_{t \in T} \phi\left(t'|t\right) \pi\left(t|\theta\right)$$

is given by, say $t' = 00$ and $\theta = 0$:

$$\begin{aligned}
\pi'(00|0) &= \phi(00|00)\pi(00|0) + \phi(00|01)\pi(01|0) + \phi(00|10)\pi(10|0) + \phi(00|11)\pi(11|0) \\
&= \frac{1}{2}\lambda_2 + \lambda_3 r,
\end{aligned}$$

and similar:

$$\begin{aligned}
\pi'(01|0) &= \phi(01|00)\pi(00|0) + \phi(01|01)\pi(01|0) + \phi(01|10)\pi(10|0) + \phi(01|11)\pi(11|0) \\
&= \frac{1}{2}\lambda_1 + \lambda_3(q-r),
\end{aligned}$$

and thus we have

$$\begin{aligned}
q' &= \frac{1}{2}(\lambda_1 + \lambda_2) + q\lambda_3, \\
r' &= \frac{1}{2}\lambda_2 + r\lambda_3.
\end{aligned}$$

More generally, we have

$\pi'(\cdot\vert 0)$	0	1
0	$\frac{1}{2}\lambda_2 + r\lambda_3$	$\frac{1}{2}\lambda_1 + (q-r)\lambda_3$
1	$\frac{1}{2}\lambda_1 + (q-r)\lambda_3$	$\frac{1}{2}\lambda_2 + (r+1-2q)\lambda_3$

$\pi'(\cdot\vert 1)$	0	1
0	$\frac{1}{2}\lambda_2 + (r+1-2q)\lambda_3$	$\frac{1}{2}\lambda_1 + (q-r)\lambda_3$
1	$\frac{1}{2}\lambda_1 + (q-r)\lambda_3$	$\frac{1}{2}\lambda_2 + r\lambda_3$

(72)

We now show if S is individually sufficient for S', then condition (38) holds. It then follows from the definitions of individual sufficiency and non-communicating ungarbling that (39) holds as well. After all, the set of information structures $S_{q',r'}$ for which $S_{q,r}$ is individually sufficient is a superset of the set of information structures $S_{q',r'}$ for which $S_{q,r}$ is a non-communicating ungarbling.

For individual sufficiency, the Markov kernel is allowed to depend on θ, or $\phi(t'|t,\theta)$ or $\phi_\theta(t'|t) \triangleq \phi(t'|t,\theta)$. Now by symmetry across states, it will be sufficient to establish the argument for $\theta = 0$, and so we require that there exists $\phi_0 : \{0,1\}^2 \to \{0,1\}^2$ such that

$$\pi'\left(t'|0\right) = \pi(00|0)\,\phi_0\left(t'|00\right) + \pi(01|0)\,\phi_0\left(t'|01\right) + \pi(10|0)\,\phi_0\left(t'|10\right) + \pi(11|0)\,\phi_0\left(t'|11\right)$$

for each t', and

$$\phi_0\left(t'_i, t'_j = 0|t_i, t_j\right) + \phi_0\left(t'_i, t'_j = 1|t_i, t_j\right) = \zeta_{0i}\left(t'_i|t_i\right)$$

is independent of t_j. But by symmetry of the information structure, we have

$$\begin{aligned}
q' &= \pi'(00|0) + \pi'(01|0) \\
&= \pi(00|0)\,\phi_0(00|00) + \pi(01|0)\,\phi_0(00|01) + \pi(10|0)\,\phi_0(00|10) + \pi(11|0)\,\phi_0(00|11) + \\
&\quad \pi(00|0)\,\phi_0(01|00) + \pi(01|0)\,\phi_0(01|01) + \pi(10|0)\,\phi_0(01|10) + \pi(11|0)\,\phi_0(01|11) \\
&= r\phi_0(00|00) + (q-r)\,\phi_0(00|01) + (q-r)\,\phi_0(00|10) + (r+1-2q)\,\phi_0(00|11) + \\
&\quad r\phi_0(01|00) + (q-r)\,\phi_0(01|01) + (q-r)\,\phi_0(01|10) + (r+1-2q)\,\phi_0(01|11) \\
&= r\zeta_{0i}(0|0) + (q-r)\,\zeta_{0i}(0|0) + (q-r)\,\zeta_{0i}(0|1) + (r+1-2q)\,\zeta_{0i}(0|1) \\
&= q\zeta_{0i}(0|0) + (1-q)\,\zeta_{0i}(0|1)\,,
\end{aligned} \tag{73}$$

and likewise:

$$1 - q' = q\zeta_{0i}(1|0) + (1-q)\,\zeta_{0i}(1|1)\,. \tag{74}$$

After observing that $\zeta_{0i}(0|t_i) + \zeta_{0i}(1|t_i) = 1$, we can use (73) and (74) to solve

$$\zeta_{0i}\left(t'_i|t_i\right) = \begin{cases} \frac{q+q'-1}{2q-1}, & \text{if} \quad t_i = t'_i, \\ \frac{q-q'}{2q-1}, & \text{if} \quad t_i \neq t'_i. \end{cases}$$

Thus we can write $\phi_0(t'|t)$ as (where t is represents by a row, and t' by a column) as:

$$\begin{array}{ccccc}
\phi_0(t'|t) & 00 & 01 & 10 & 11 \\
00 & w & \frac{q+q'-1}{2q-1} - w & \frac{q+q'-1}{2q-1} - w & w - \frac{2q'-1}{2q-1} \\
01 & x & \frac{q+q'-1}{2q-1} - x & \frac{q-q'}{2q-1} - x & x \\
10 & y & \frac{q-q'}{2q-1} - y & \frac{q+q'-1}{2q-1} - y & y \\
11 & z - \frac{2q'-1}{2q-1} & \frac{q+q'-1}{2q-1} - z & \frac{q+q'-1}{2q-1} - z & z
\end{array} \tag{75}$$

In the construction of the Markov kernel (75) we use the symmetry in the information structure across agents, namely that, e.g.

$$\phi_0(00|00) + \phi_0(00|01) = \phi_0(00|00) + \phi_0(00|10)\,.$$

Now, non-negativity constraints in the Markov kernel require that:

$$\begin{aligned}\frac{2q'-1}{2q-1} &\leq w \leq \frac{q+q'-1}{2q-1} \\ 0 &\leq x \leq \frac{q-q'}{2q-1} \\ 0 &\leq y \leq \frac{q-q'}{2q-1} \\ \frac{2q'-1}{2q-1} &\leq z \leq \frac{q+q'-1}{2q-1}\end{aligned}$$

Clearly this requires that $q \geq q'$; in this case

$$\begin{aligned}r' &= \pi'(00|0) \\ &= \pi(00|0)\,\phi_0(00|00) + \pi(01|0)\,\phi_0(00|01) + \pi(10|0)\,\phi_0(00|10) + \pi(11|0)\,\phi_0(00|11) \\ &= rw + (q-r)\,x + (q-r)\,y + (r+1-2q)\left(z - \frac{2q'-1}{2q-1}\right)\end{aligned}$$

Now at the lower bound:

$$rw + (q-r)\,x + (q-r)\,y + (r+1-2q)\left(z - \frac{2q'-1}{2q-1}\right) = r\frac{2q'-1}{2q-1},$$

whereas at the upper bound:

$$r\frac{q+q'-1}{2q-1} + (q-r)\frac{q-q'}{2q-1} + (q-r)\frac{q-q'}{2q-1} + (r+1-2q)\left(\frac{q+q'-1}{2q-1} - \frac{2q'-1}{2q-1}\right) = \frac{q-q'}{2q-1} + \left(\frac{q'-\frac{1}{2}}{q-\frac{1}{2}}\right)r$$

and so we have lower and upper bounds for r':

$$\left(\frac{q'-\frac{1}{2}}{q-\frac{1}{2}}\right)r \leq r' \leq \frac{q-q'}{2q-1} + \left(\frac{q'-\frac{1}{2}}{q-\frac{1}{2}}\right)r, \tag{76}$$

which is an equivalent statement of (38), after observing that the convex combination $(\lambda_1, \lambda_2, \lambda_3)$ over the points $\left\{\left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2}\right), (q, r)\right\}$ reaches the lower bound for $\lambda_2 = 0$ and the upper bound for $\lambda_1 = 0$. ■

8.3.3 Examples 4 and 5

We establish a lemma which covers the Examples 4 and 5 of Section 5. Let $\Theta = \{0, 1\}$ and let S be given by $T_1 = T_2 = \{0, 1\}$ with the conditional probabilities:

$\pi(\cdot\|0)$	0	1
0	$\frac{1}{3}$	$\frac{1}{3}$
1	$\frac{1}{3}$	0

$\pi(\cdot\|1)$	0	1
0	0	$\frac{1}{3}$
1	$\frac{1}{3}$	$\frac{1}{3}$

,

which are symmetric across agents and states. However, we compare S with information structures S', which may not be symmetric across states. Thus $T_1' = T_2' = \{0,1\}$ and the conditional probabilities are:

$\pi'(\cdot\|0)$	0	1
0	r_0	$q-r_0$
1	$q-r_0$	r_0+1-2q

$\pi'(\cdot\|1)$	0	1
0	r_1+1-2q	$q-r_1$
1	$q-r_1$	r_1

From each agent's point of view, the information structure S corresponds to observing one signal with (symmetric) accuracy $\frac{2}{3}$, while the second information structure corresponds to observing a signal with (symmetric) accuracy q. In the information structure S, the players' signals are as correlated as possible given the accuracy. In the information structure S', we allow the correlation to depend on the state. Non-negativity also requires that $(r_0, r_1) \in [2q-1, q]^2$. Thus we have S' parameterized by

$$\left\{ (q, r_0, r_1) \left| \frac{1}{2} \leq q \leq 1 \text{ and } (r_0, r_1) \in [2q-1, q]^2 \right. \right\}.$$

Lemma 11 (Binary Information Structures)

1. *Information structure $S_{\frac{2}{3},\frac{1}{3}}$ is sufficient for information structure $S_{q',r_0',r_1'}$ if and only if*

$$q' \leq \frac{2}{3} \text{ and } \left| r_1' - r_0' \right| \leq \min\left\{ \frac{1}{6}, \frac{4}{3} - 2q' \right\}. \tag{77}$$

2. *Information structure $S_{\frac{2}{3},\frac{1}{3}}$ is individually sufficient for information structure $S_{q',r_0',r_1'}$ if and only if*

$$q' \leq \frac{2}{3} \text{ and } 2q' - 1 \leq r_0', r_1' \leq 1 - q'. \tag{78}$$

3. *Information structure $S_{\frac{2}{3},\frac{1}{3}}$ is a non-communicating ungarbling of information structure $S_{q',r_0',r_1'}$ if and only if*

$$q' \leq \frac{2}{3}, \quad 2q' - 1 \leq r_0', r_1' \leq 1 - q', \text{ and } \left| r_0' - r_1' \right| \leq \frac{2}{3} - q'. \tag{79}$$

Proof. (1.) There is sufficiency if there exists $\phi : \{0,1\}^2 \to \Delta\left(\{0,1\}^2\right)$ such that we can express π' as

$\pi'(\cdot\|0)$	0	1
0	$\frac{1}{3}(\phi(00\|00) + \phi(00\|01) + \phi(00\|10))$	$\frac{1}{3}(\phi(01\|00) + \phi(01\|01) + \phi(01\|10))$
1	$\frac{1}{3}(\phi(10\|00) + \phi(10\|01) + \phi(10\|10))$	$\frac{1}{3}(\phi(11\|00) + \phi(11\|01) + \phi(11\|10))$

and

$\pi'(\cdot\|1)$	0	1
0	$\frac{1}{3}(\phi(00\|11) + \phi(00\|01) + \phi(00\|10))$	$\frac{1}{3}(\phi(01\|11) + \phi(01\|01) + \phi(01\|10))$
1	$\frac{1}{3}(\phi(10\|11) + \phi(10\|01) + \phi(10\|10))$	$\frac{1}{3}(\phi(11\|11) + \phi(11\|01) + \phi(11\|10))$

.

Defining $\widehat{\phi}(t') \triangleq \frac{1}{2}\phi(t'|01) + \phi(t'|10)$, we can re-write this as

$\pi'(\cdot\|0)$	0	1
0	$\frac{1}{3}\phi(00\|00) + \frac{2}{3}\widehat{\phi}(00)$	$\frac{1}{3}\phi(01\|00) + \frac{2}{3}\widehat{\phi}(01)$
1	$\frac{1}{3}\phi(10\|00) + \frac{2}{3}\widehat{\phi}(10)$	$\frac{1}{3}\phi(11\|00) + \frac{2}{3}\widehat{\phi}(11)$

$\pi'(\cdot\|1)$	0	1
0	$\frac{1}{3}\phi(00\|11) + \frac{2}{3}\widehat{\phi}(00)$	$\frac{1}{3}\phi(01\|11) + \frac{2}{3}\widehat{\phi}(01)$
1	$\frac{1}{3}\phi(10\|11) + \frac{2}{3}\widehat{\phi}(10)$	$\frac{1}{3}\phi(11\|11) + \frac{2}{3}\widehat{\phi}(11)$

Now a necessary condition for sufficiency is that there exists $\widehat{\phi} \in \Delta(T')$ such that

$$\min\left\{\pi'\left(t'|0\right), \pi'\left(t'|0\right)\right\} \geq \frac{2}{3}\widehat{\phi}\left(t'\right)$$

for all t', and thus

$$\sum_{t'} \min\left\{\pi'\left(t'|0\right), \pi'\left(t'|0\right)\right\} \geq \frac{2}{3}.$$

Suppose first that

$$r_1' \geq r_0' + 2q' - 1.$$

Then

$$\sum_{t'} \min\left\{\pi'\left(t'|0\right), \pi'\left(t'|0\right)\right\} = r_0' + 2\left(q' - r_1'\right) + r_0' + 1 - 2q' \geq \frac{2}{3},$$

which can be re-written as

$$r_1' - r_0' \leq \frac{1}{6}.$$

On the other hand, if

$$r_0' + 2q' - 1 \geq r_1' \geq r_0',$$

then

$$\sum_{t'} \min\left\{\pi'\left(t'|0\right), \pi'\left(t'|0\right)\right\} = r_1' + 1 - 2q' + 2\left(q' - r_1'\right) + r_0' + 1 - 2q' \geq \frac{2}{3},$$

which can be written as

$$r_1' - r_0' \leq \frac{4}{3} - 2q'.$$

Thus if $r_1' \geq r_0'$, it is a necessary condition that either

$$\begin{aligned} r_0' + 2q' - 1 &\leq r_1' \leq r_0' + \frac{1}{6} \\ \text{or } r_1' &\leq \min\left\{r_0' + 2q' - 1, \frac{4}{3} - 2q'\right\} \end{aligned}$$

which implies $q' \leq \frac{2}{3}$ and

$$r_1' - r_0' \leq \min\left\{\frac{1}{6}, \frac{4}{3} - 2q'\right\}.$$

The symmetric argument if $r_0' \geq r_1'$ gives the conditions for sufficiency.

For the converse, suppose that the conditions of (77) are satisfied. Let

$$\phi\left(t'|01\right) = \phi\left(t'|10\right) = \widehat{\phi}\left(t'\right) = \frac{\min\left\{\pi'\left(t'|0\right), \pi'\left(t'|0\right)\right\}}{\sum\limits_{\widetilde{t}'}\min\left\{\widetilde{\pi}'\left(t'|0\right), \pi'\left(\widetilde{t}'|0\right)\right\}},$$

and let

$$\phi\left(t'|00\right) = 3\pi'\left(t'|0\right) - 2\widehat{\phi}\left(t'\right) \quad \text{and} \quad \phi\left(t'|11\right) = 3\pi'\left(t'|1\right) - 2\widehat{\phi}\left(t'\right).$$

By construction, $\phi\left(t'|00\right)$ and $\phi\left(t'|11\right)$ are well defined and the conditions for sufficiency are satisfied.

(2.) Individual sufficiency requires that there exists $\phi_0 : \{0,1\}^2 \to \{0,1\}^2$ such that

$$\pi'\left(t'|0\right) = \frac{1}{3}\phi_0\left(t'|00\right) + \frac{1}{3}\phi_0\left(t'|01\right) + \frac{1}{3}\phi_0\left(t'|10\right)$$

for each t' and

$$\phi_0\left(t_i', t_j' = 0|t_i, t_j\right) + \phi_0\left(t_i', t_j' = 1|t_i, t_j\right) = \zeta_{0i}\left(t_i'|t_i\right)$$

is independent of t_j. But since, for example,

$$\begin{aligned} q' &= \frac{2}{3}\zeta_{0i}\left(0|0\right) + \frac{1}{3}\zeta_{0i}\left(0|1\right) \\ 1 - q' &= \frac{2}{3}\zeta_{0i}\left(1|0\right) + \frac{1}{3}\zeta_{0i}\left(1|1\right) \end{aligned}$$

we have

$$\phi_0\left(t_i', t_j' = 0|t_i, t_j\right) + \phi_0\left(t_i', t_j' = 1|t_i, t_j\right) = \zeta_{0i}\left(t_i'|t_i\right) = \begin{cases} 3q' - 1, \text{ if } t_i' = t_i, \\ 2 - 3q', \text{ if } t_i' \neq t_i. \end{cases}$$

Thus we can write $\phi_0\left(t'|t\right)$ as

$\phi_0\left(t'\vert t\right)$	00	01	10	11
00	x	$3q' - 1 - x$	$3q' - 1 - x$	$x + 3 - 6q'$
01	y	$3q' - 1 - y$	$2 - 3q' - y$	y
10	z	$2 - 3q' - z$	$3q' - 1 - z$	z

Non-negativity constraints require that

$$\begin{aligned} 6q - 3 &\leq x \leq 3q - 1 \\ 0 &\leq y \leq 2 - 3q \\ 0 &\leq z \leq 2 - 3q \end{aligned}$$

These conditions imply that $q' \leq \frac{2}{3}$ and $r'_0 = \frac{1}{3}(x+y+z)$ satisfies

$$2q' - 1 \leq r_0 \leq 1 - q'.$$

The symmetric argument implies that

$$2q' - 1 \leq r'_1 \leq 1 - q'.$$

For the converse, obviously if the conditions in (78) are satisfied, then we can construct ϕ_0 according to the table above. A similar argument would apply for ϕ_1.

(3.) Building on the earlier arguments of (1.) and (2.), we must show that there exists $\phi : \{0,1\}^2 \to \{0,1\}^2$ such that

$\pi'(\cdot\|0)$	0	1
0	$\frac{1}{3}(\phi(00\|00)+\phi(00\|01)+\phi(00\|10))$	$\frac{1}{3}(\phi(01\|00)+\phi(01\|01)+\phi(01\|10))$
1	$\frac{1}{3}(\phi(10\|00)+\phi(10\|01)+\phi(10\|10))$	$\frac{1}{3}(\phi(11\|00)+\phi(11\|01)+\phi(11\|10))$

$\pi'(\cdot\|1)$	0	1
0	$\frac{1}{3}(\phi(00\|11)+\phi(00\|01)+\phi(00\|10))$	$\frac{1}{3}(\phi(01\|11)+\phi(01\|01)+\phi(01\|10))$
1	$\frac{1}{3}(\phi(10\|11)+\phi(10\|01)+\phi(10\|10))$	$\frac{1}{3}(\phi(11\|11)+\phi(11\|01)+\phi(11\|10))$

and also satisfying

$$\phi\left(t'_i, t'_j = 0 | t_i, t_j\right) + \phi\left(t'_i, t'_j = 1 | t_i, t_j\right) = \begin{cases} 3q' - 1, \text{ if } t'_i = t_i, \\ 2 - 3q', \text{ if } t'_i \neq t_i. \end{cases}$$

So we can represent ϕ by

$\phi(t'\|t)$	00	01	10	11
00	x	$3q'-1-x$	$3q'-1-x$	$x+3-6q'$
01	y	$3q'-1-y$	$2-3q'-y$	y
10	z	$2-3q'-z$	$3q'-1-z$	z
11	w	$2-3q'-w$	$2-3q'-w$	$w-3+6q'$

,

where non-negativity constraints imply that

$$\begin{aligned} 6q' - 3 &\leq x \leq 3q' - 1, \\ 0 &\leq y \leq 2 - 3q', \\ 0 &\leq z \leq 2 - 3q', \\ 0 &\leq z \leq 2 - 3q'. \end{aligned}$$

We want to characterize the set of (r_0', r_1') such that we can choose (x, y, z, w) satisfying the above inequalities with

$$r_0' = \frac{1}{3}(x + y + z) \text{ and } r_1' = \frac{1}{3}(y + z + w) - 1 + 2q'.$$

By arguments for individual sufficiency, we must have $2q' - 1 \leq r_0', r_1' \leq 1 - q'$. But in addition observe that

$$y + z = 3r_0' - x,$$

and therefore

$$3r_0' - 3q' + 1 \leq y + z \leq 3r_0' + 3 - 6q'.$$

Now

$$r_1' = \frac{1}{3}w + \frac{1}{3}(y + z) - 1 + 2q',$$

and so

$$\frac{1}{3}\left(3r_0' - 3q' + 1\right) - 1 + 2q' \leq r_1' \leq \frac{1}{3}\left(2 - 3q' + 3r_0' + 3 - 6q'\right) - 1 + 2q',$$

or

$$q' - \frac{2}{3} \leq r_1' - r_0' \leq \frac{2}{3} - q'.$$

Conversely, suppose that these conditions are satisfied. (to be completed) ■

We can then summarize what the conditions of Lemma 11 imply for the case of $q' = \frac{5}{9}$. Now, non-negativity of the conditional probabilities π' then implies that

$$\frac{1}{9} \leq r_0', r_1' \leq \frac{5}{9}.$$

We have that S is sufficient for S' if

$$\left|r_1' - r_0'\right| \leq \frac{1}{6}.$$

We have that S is individually sufficient for S' if

$$\frac{1}{9} \leq r_0', r_1' \leq \frac{4}{9}.$$

We have that S is a non-communicating ungarbling of S' if

$$\frac{1}{9} \leq r_0', r_1' \leq \frac{4}{9}$$

and

$$\left|r_1' - r_0'\right| \leq \frac{1}{9}.$$

8.3.4 Incomplete Information Correlated Equilibrium

In Section 6, we describe the relationship of the a number of prominent solution concepts to each other and to the notion of Bayes correlated equilibrium. In particular, we illustrate the relationship within our leading example of the binary and symmetric game and information structure. These results are graphically represented in Figure 8 and 9. Here we provide the details of the analytical results for the binary and symmetric game and information structure.

Feasible Random Choice Rule We recall that we parametrized the symmetric information structure $S_{q,r}$ with binary signals by (q,r) and for the purpose of the analysis of the present example we restrict attention to the information structure with the minimal positive correlation given accuracy q, and hence $r = 2q-1$, or $S_{q,2q-1}$. We describe the symmetric random choice rule in the binary game by (q', r'). We now consider the feasible random choice rules given the information structure $S_{q,r}$. It follows directly from the definition of a feasible random choice rule, Definition 17, and the binary signal that the set of feasible random choice rules is given by:

$$F\left(G_0, S_{q,r}\right) = \text{conv}\left\{\left(\frac{1}{2},0\right), \left(\frac{1}{2},\frac{1}{2}\right), \left(q, 2q-1\right), \left(1-q, 0\right)\right\}, \tag{80}$$

where $\left(\frac{1}{2},0\right)$ represents uniformly randomized choices over $(0,1)$ and $(1,0)$ and $\left(\frac{1}{2},\frac{1}{2}\right)$ represents uniformly randomized choices over $(0,0)$ and $(1,1)$. The remaining feasible random choices represent the case when each agent either always follows the signal, $a = t$, namely $(q, 2q-1)$, or never follows the signal, $a \neq t$, namely $(1-q, 0)$.

Belief Invariant Bayes Correlated Equilibrium With $q \geq \frac{2}{3}$, the set of belief invariant Bayes correlated equilibria is the convex hull of

$$\left\{\left(\frac{2-q}{3},\frac{1}{3}\right), \left(\frac{1}{2},\frac{1}{2}\right), \left(q, 1-2q\right)\right\}. \tag{81}$$

We continue to use parametrization (52). We recall that a decision rule is defined by:

$$\sigma : \Theta \times T_1 \times T_2 \to \Delta\left(A_1 \times A_2\right),$$

and a belief invariant decision rule, see Definition (16) required that:

$$\sigma_i\left(a_i \left| t_i, t_{-i}, \theta\right.\right) = \sum_{a_{-i} \in A_{-i}} \sigma\left(a_i, a_{-i} \left| t_i, t_{-i}, \theta\right.\right)$$

is independent of t_{-i} and θ. So, in the binary game it is sufficient to consider $a_i = 0$ and $t_i = 0$, then we have:

$$\frac{x_1 + x_2}{x_1 + 2x_2 + x_5} = \frac{x_3 + x_4}{x_3 + x_4 + x_6 + x_7} = \frac{x_4 + x_7}{x_3 + x_4 + x_6 + x_7}. \tag{82}$$

In other words if we write α for the probability that a player chooses action a if he observes signal $t = a$, then:

$$q' = q\alpha + (1-q)(1-\alpha) \Leftrightarrow \alpha = \frac{q' + q - 1}{2q - 1}.$$

where we note that, by symmetry assumptions, this must be the same for all players and signals. This gives us the following conditions for belief invariance:

$$\begin{aligned} x_1 + x_2 &= \left(\frac{q' + q - 1}{2q - 1}\right)(x_1 + 2x_2 + x_5), \\ x_3 + x_4 &= \left(\frac{q' + q - 1}{2q - 1}\right)(x_3 + x_4 + x_6 + x_7), \\ x_4 + x_7 &= \left(\frac{q' + q - 1}{2q - 1}\right)(x_3 + x_4 + x_6 + x_7). \end{aligned} \tag{83}$$

Now, we find that

$$\begin{aligned} x_1 + x_2 &= \left(\frac{q' + q - 1}{2q - 1}\right)(x_1 + 2x_2 + x_5), \text{ by (83)} \\ &= \left(\frac{q' + q - 1}{2q - 1}\right)(2q - 1), \text{ by (55)} \\ &= q' + q - 1, \end{aligned}$$

and thus

$$x_2 = q' + q - 1 - x_1$$

Also

$$\begin{aligned} x_3 + x_6 &= \left(1 - \frac{q' + q - 1}{2q - 1}\right)(x_3 + x_4 + x_6 + x_7), \text{ by (83)} \\ &= \left(\frac{q - q'}{2q - 1}\right)(1 - q), \text{ by (56)} \end{aligned}$$

and thus

$$x_6 = \left(\frac{q - q'}{2q - 1}\right)(1 - q) - x_3.$$

From (58), we have the requirement that

$$x_2 + x_6 - x_3 \leq 0.$$

Using the above expressions for x_2 and x_6, we have

$$q' + q - 1 - x_1 + \left(\frac{q - q'}{2q - 1}\right)(1 - q) - x_3 - x_3 \leq 0.$$

or

$$q' + q - 1 + \left(\frac{q - q'}{2q - 1}\right)(1 - q) \leq x_1 + 2x_3 = r'.$$

But this condition states that (q', r') is above the straight line connecting $(1 - q, 1 - q)$, $\left(\frac{2-q}{3}, \frac{1}{3}\right)$ and $(q, 2q - 1)$. But we also have

$$\begin{aligned} x_3 + x_4 &= \left(\frac{q' + q - 1}{2q - 1}\right)(x_3 + x_4 + x_6 + x_7), \text{ by (83)} \\ &= \left(\frac{q' + q - 1}{2q - 1}\right)(1 - q), \text{ (56).} \end{aligned}$$

Combining with earlier inequalities we have:

$$x_3 \leq \min\left(\left(\frac{q - q'}{2q - 1}\right)(1 - q), \left(\frac{q' + q - 1}{2q - 1}\right)(1 - q)\right) = \begin{cases} \left(\frac{q-q'}{2q-1}\right)(1 - q), \text{ if } q' \geq \frac{1}{2}, \\ \left(\frac{q'+q-1}{2q-1}\right)(1 - q), \text{ if } q' \leq \frac{1}{2}, \end{cases}.$$

and

$$x_1 \leq q' + q - 1,$$

and thus

$$r' = x_1 + 2x_3 \leq \begin{cases} q' + q - 1 + 2\left(\frac{q-q'}{2q-1}\right)(1 - q), \text{ if } q' \geq \frac{1}{2}, \\ q' + q - 1 + 2\left(\frac{q'+q-1}{2q-1}\right)(1 - q), \text{ if } q' \leq \frac{1}{2}. \end{cases}$$

But this condition states that (q', r') is below, first, the straight line first connecting $(1 - q, 0)$ to $\left(\frac{2-q}{3}, \frac{1}{3}\right)$ to $\left(\frac{1}{2}, \frac{1}{2}\right)$ and then connecting $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $(q, 2q - 1)$. These lower and upper bounds establish the necessity of the above conditions.

To establish sufficiency, it is enough to construct BIBCE corresponding to the extreme points as follows. For $\left(\frac{2-q}{3}, \frac{1}{3}\right)$, the BIBCE is:

$\theta = 0$	00	01	10	11
00	$\frac{1}{3}(2q - 1)$	0	$\frac{1}{3}(1 - q)$	0
01	0	$\frac{2}{3}(2q - 1)$	$\frac{1}{3}(1 - q)$	$\frac{1}{3}(1 - q)$
10	$\frac{1}{3}(1 - q)$	$\frac{1}{3}(1 - q)$	0	0
11	0	$\frac{1}{3}(1 - q)$	0	0

For $\left(\frac{1}{2}, \frac{1}{2}\right)$, the BIBCE is

$\theta = 0$	00	01	10	11
00	$q - \frac{1}{2}$	0	$\frac{1}{2}(1 - q)$	0
01	0	$q - \frac{1}{2}$	0	$\frac{1}{2}(1 - q)$
10	$\frac{1}{2}(1 - q)$	0	0	0
11	0	$\frac{1}{2}(1 - q)$	0	0

For $(q, 2q-1)$, the BIBCE is the BNE with

$\theta = 0$	00	01	10	11
00	$2q-1$	0	0	$1-q$
01	0	0	0	0
10	0	0	0	0
11	$1-q$	0	0	0

Bayesian Solutions The set of Bayesian solutions (if $q \geq \frac{2}{3}$) will be the convex hull of

$$\left\{\left(1-q, \frac{1}{2}(1-q)\right), (1-q, 1-q), (q, q), (q, 2q-1), (2(1-q), (1-q))\right\} \tag{84}$$

Now, join feasibility, under our prior symmetric assumptions and the special structure of the experiment, $r = 2q-1$, means that the only potentially binding constraint is that:

$$x_3 = x_7.$$

This in turn implies that

$$2x_3 + x_4 + x_6 = x_3 + x_4 + x_6 + x_7 = 1 - q,$$

and thus

$$2x_3 + x_4 + x_6 \leq 1 - q.$$

Also since

$$x_1 + 2x_2 + x_5 = 2q - 1,$$

we have

$$x_1 + x_2 \leq 2q - 1,$$

and so

$$\begin{aligned} q' &= x_1 + 2x_3 + x_2 + x_4 + x_6 \\ &\leq 2q - 1 + 1 - q \\ &= q. \end{aligned}$$

Conversely, obedience requires that

$$2x_3 \geq x_4 + x_6.$$

Combined with

$$2x_3 + x_4 + x_6 = 1 - q,$$

we have

$$2x_3 \geq \frac{1}{2}(1-q),$$

and thus

$$q' \geq 1-q.$$

These two bounds on q' $(1-q \leq q' \leq q)$ together with the BCE restrictions imply that it is necessary for a Bayesian solution to be in the convex hull described above.

For sufficiency, we show that each extreme points corresponds to a Bayesian solution. For $\left(1-q, \frac{1}{2}(1-q)\right)$, we have

$\theta = 0$	00	01	10	11
00	0	0	$\frac{1}{4}(1-q)$	$\frac{1}{4}(1-q)$
01	0	$2q-1$	$\frac{1}{4}(1-q)$	$\frac{1}{4}(1-q)$
10	$\frac{1}{4}(1-q)$	$\frac{1}{4}(1-q)$	0	0
11	$\frac{1}{4}(1-q)$	$\frac{1}{4}(1-q)$	0	0

.

For $(1-q, 1-q)$, we have

$\theta = 0$	00	01	10	11
00	0	0	$\frac{1}{2}(1-q)$	0
01	0	$2q-1$	0	$\frac{1}{2}(1-q)$
10	$\frac{1}{2}(1-q)$	0	0	0
11	0	$\frac{1}{2}(1-q)$	0	0

.

For (q, q), we have

$\theta = 0$	00	01	10	11
00	$2q-1$	0	$\frac{1}{2}(1-q)$	0
01	0	0	0	$\frac{1}{2}(1-q)$
10	$\frac{1}{2}(1-q)$	0	0	0
11	0	$\frac{1}{2}(1-q)$	0	0

.

For $(q, 2q-1)$, we have

$\theta = 0$	00	01	10	11
00	$2q-1$	0	0	$1-q$
01	0	0	0	0
10	0	0	0	0
11	$1-q$	0	0	0

.

For $(2(1-q), 1-q)$, we have

$\theta = 0$	00	01	10	11
00	$1-q$	0	0	$1-q$
01	0	$3q-2$	0	0
10	0	0	0	0
11	$1-q$	0	0	0

.

Belief Invariant Bayesian Solutions It is easy to verify that the extreme points of the set of belief invariant Bayes correlated equilibria satisfy join feasibility. This means that the set of belief invariant Bayesian solutions equals the set of belief invariant BCE in the current example.

8.4 More Definitions of Incomplete Information Correlated Equilibrium

We briefly discuss how further definitions of incomplete information correlated equilibrium reviewed by Forges (1993), (2006) appear in our framework. A recent paper of Milchtaich (2012) gives an even richer taxonomy of possible definitions of incomplete information correlated equilibrium.

Fix (G, S), write B_i for the finite set of pure strategies $b_i : T_i \to A_i$, b for a profile of pure strategies and $B = B_1 \times ... \times B_I$. The following is a feasibility restriction which says that a mediator could have generated the decision by picking a profile of pure strategies without knowing the state or players' types, and then making action recommendations according to the pure strategies after somehow learning the players' types:

Definition 30 *Decision rule σ is agent normal form feasible for (G, S) if there exists $q \in \Delta(B)$ such that*

$$\sigma(a|t,\theta) = \sum_{\{b \in B | b_i(t_i) = a_i, \forall i\}} q(b)$$

One can show that agent normal form feasibility implies belief invariance.[9] This restriction is added to give the second stronger solution concept:

Definition 31 *Decision rule σ is an agent normal form correlated equilibrium of (G, S) if it is join feasible, agent normal form feasible (and thus belief invariant) and obedient.*

This is the solution concept discussed in Section 4.2 of Forges (1993) and Section 2.3 of Forges (2006). It corresponds to applying the complete information definition of correlated equilibrium to the agent normal

[9] Forges (2006) cites an example due to Lehrer, Rosenberg and Shmaya showing that belief invariance does not imply agent normal form feasibility as incorrectly claimed in Forges (1993).

form of the reduced incomplete information game. It was also studied by Samuelson and Zhang (1989) and Cotter (1994). The solution concept only makes sense on the understanding that the players receive a recommendation for each type but do not learn what recommendation they would have received if they had been different types. If they did learn the whole strategy that the mediator choose for them in the strategic form game, then an extra incentive compatibility condition would be required:

Definition 32 *Decision rule σ is strategic form incentive compatible for (G, S) if there exists $q \in \Delta(V)$ such that*

$$\sigma(a|t,\theta) = \sum_{\{b \in B | b_i(t_i) = a_i \text{ for each } i\}} q(b) \tag{85}$$

for each $a \in A$, $t \in T$ and $\theta \in \Theta$; and, for each $i = 1, ..., I$, $t_i \in T_i$, $a_i \in A_i$ and $b_i \in B_i$ such that $b_i(t_i) = a_i$, we have

$$\begin{aligned} & \sum_{a_{-i} \in A_{-i}, t_{-i} \in T_{-i}, \theta \in \Theta} \psi(\theta)\pi(t|\theta) \left(\sum_{\{b_{-i} \in B_{-i} | b_{-i}(t_{-i}) = a_{-i}\}} q(b_i, b_{-i}) \right) u_i((a_i, a_{-i}), \theta) \\ \geq & \sum_{a_{-i} \in A_{-i}, t_{-i} \in T_{-i}, \theta \in \Theta} \psi(\theta)\pi(t|\theta) \left(\sum_{\{b_{-i} \in B_{-i} | b_{-i}(t_{-i}) = a_{-i}\}} q(b_i, b_{-i}) \right) u_i\left(\left(a_i', a_{-i}\right), \theta\right) \end{aligned} \tag{86}$$

Note that this condition implies both agent normal form feasibility and obedience. This restriction gives the third stronger solution concept:

Definition 33 *Decision rule σ is a strategic form correlated equilibrium of (G, S) if it is join feasible and strategic form incentive compatible (and thus agent normal form feasible, belief invariant and obedient).*

This is the solution concept discussed in Section 4.1 of Forges (1993) and Section 2.2 of Forges (2006). This solution concept was studied by Cotter (1991).

Thus far we have simply been adding restrictions, so that the solution concept have become stronger as we go from Bayesian solution, to belief invariant Bayesian solution, to agent normal form correlated equilibrium, to strategic form correlated equilibrium. For the Bayesian solution, an omniscient mediator who observes players' types for free is assumed. For agent normal form and strategic form correlated equilibrium, the players' types cannot play a role in the selection of recommendations to the players. An intermediate assumption is that the players can report their types to the mediator, but will do so truthfully only if it is incentive compatible to do so.

Definition 34 *Decision rule σ is truth telling for (G,S) if, for each $i=1,...,I$ and $t_i \in T_i$, we have*

$$\begin{aligned} & \sum_{a\in A, t_{-i}\in T_{-i}, \theta\in\Theta} \psi(\theta)\,\pi((t_i,t_{-i})\,|\theta)\,\sigma((a_i,a_{-i})\,|\,(t_i,t_{-i}),\theta)\,u_i((a_i,a_{-i}),\theta) \\ \geq & \sum_{a\in A, t_{-i}\in T_{-i}, \theta\in\Theta} \psi(\theta)\,\pi((t_i,t_{-i})\,|\theta)\,\sigma\left((a_i,a_{-i})\,|\,\left(t_i',t_{-i}\right),\theta\right)u_i((\delta_i(a_i),a_{-i}),\theta)\,; \end{aligned} \tag{87}$$

for all $t_i' \in T_i$ and $\delta_i : A_i \to A_i$.

Note that this condition implies obedience (Definition 14). One can show that this condition is implied by strategic form incentive compatibility. Now we have the fifth solution concept:

Definition 35 *Decision rule σ is a communication equilibrium of (G,S) if it is join feasible and truth-telling (and thus obedient).*

This is the solution concept discussed in Section 4.3 of Forges (1993) and Section 2.4 of Forges (2006), and developed earlier in the work of Myerson (1982) and Forges (1986).

Finally, we discuss the "universal Bayesian approach" in Section 6 of Forges (1993). She considers a prior "information scheme" (in our language, prior on Θ and information structure) is not taken as given. Thus her "universal Bayesian solution" is defined for $(A_i,u_i)_{i=1}^I$. Expressing her ideas in the language of random choice rules, she studies the following solution concept.

Definition 36 *A prior $\psi \in \Delta(\Theta)$ and a random choice rule $\nu : \Theta \to \Delta(A)$ for a universal Bayesian solution of $(A_i,u_i)_{i=1}^I$ if they satisfy obedience for the the null information structure, i.e., for each $i=1,...,I$ and $a_i \in A_i$, we have*

$$\begin{aligned} & \sum_{a_{-i},\theta} \psi(\theta)\,\nu((a_i,a_{-i})\,|\theta)\,u_i((a_i,a_{-i}),\theta) \\ \geq & \sum_{a_{-i},\theta} \psi(\theta)\,\nu((a_i,a_{-i})\,|\theta)\,u_i\left(\left(a_i',a_{-i}\right),\theta\right); \end{aligned} \tag{88}$$

for all $a_i' \in A_i$.

Thus Bayes correlated equilibria of $(G,\underline{S})$ correspond to universal Bayesian solutions of G.

9 Illustrations

1. Figure

2. Figure

3. Figure

4. Figure

5. Figure

6. Figure

7. Figure

8. Figure

9. Figure

10. Figure

11. Figure

10 Missing Proofs

Lemma 12 *If S satisfies distributed certainty, then join feasibility is satisfied by all decision rules $\sigma : T \times \Theta \to \Delta(A)$.*

Proof. Distributed certainty implies that there is one θ such that $\pi(t|\theta) > 0$. Thus σ is vacuously independent of θ. ■

10.1 Proof of Lemma 8.

Fix a decision rule $\sigma : T \times \Theta \to \Delta(A)$ for (G, S). Let $\widetilde{\sigma} : \widetilde{T} \times \Theta \to \Delta(\widetilde{A})$ be the corresponding decision rule for $(\widetilde{G}, \widetilde{S})$ with

$$\widetilde{\sigma}\left(\left(a_0, (a_i)_{i=1}^I\right) \middle| \left(t_0, (t_i)_{i=1}^I\right), \theta\right) = \sigma\left((a_i)_{i=1}^I \middle| (t_i)_{i=1}^I, \theta\right).$$

Join feasibility of $\widetilde{\sigma}$ is automatically satisfied since for each t, $\pi(t|\theta) > 0$ only if $\theta = g(t)$ and thus $\widetilde{\sigma}\left(\left(a_0, (a_i)_{i=1}^I\right) \middle| \left(t_0, (t_i)_{i=1}^I\right), \theta\right)$ is independent of θ. Obedience of $\widetilde{\sigma}$ for $(\widetilde{G}, \widetilde{S})$ reduces to obedience of σ for (G, S). Belief invariance of $\widetilde{\sigma}$ for $(\widetilde{G}, \widetilde{S})$ reduces to belief invariance of σ for (G, S).

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ILLUSTRATIONS

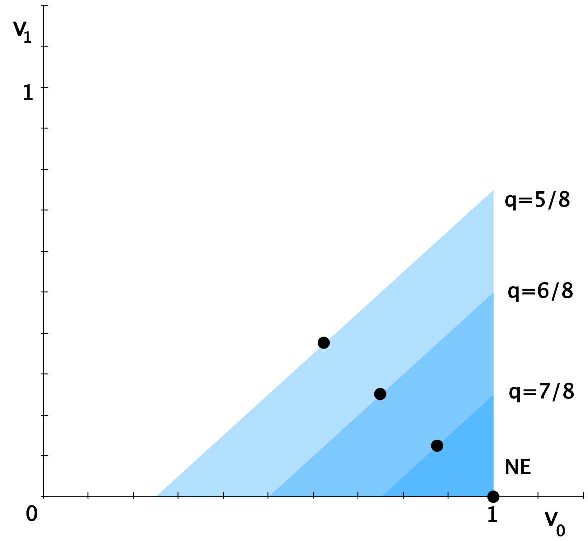

Figure 1: Single Agent BCE random choice rule and varying accuracy q

one bce

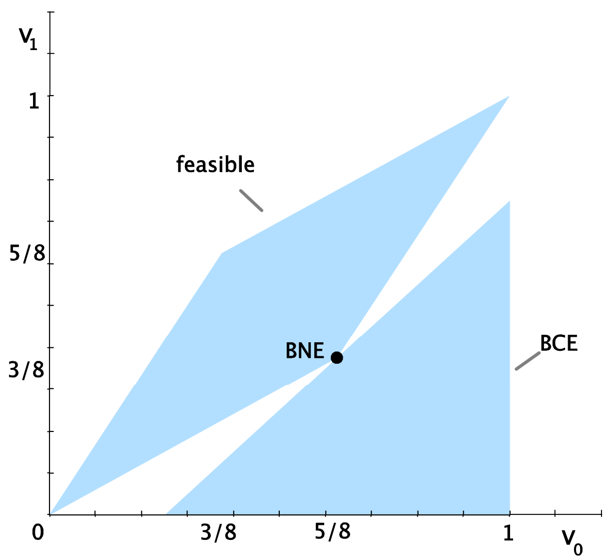

Figure 2: Single Agent BCE, BNE and feasible random choice rules for accuracy $q = 5/8$.

q5/8

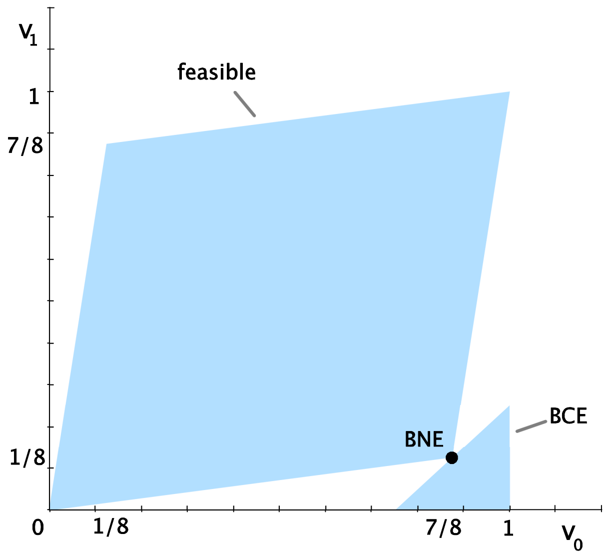

Figure 3: Single Agent BCE, BNE and feasible

q7/8 random choice rules for accuracy $q = 7/8$.

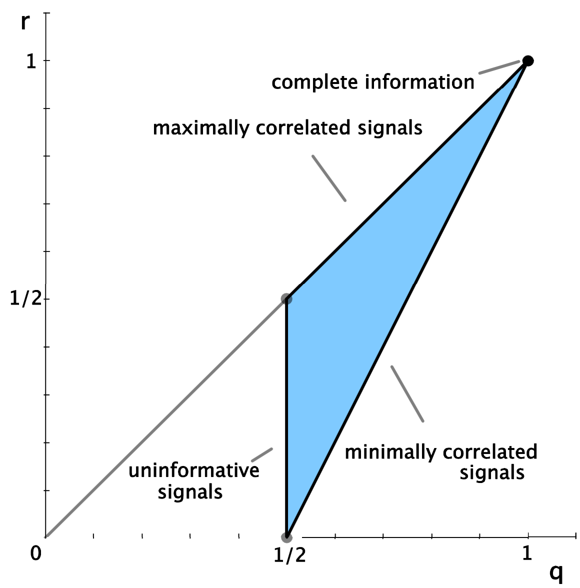

Figure 4: Binary Symmetric Information

pi Structures with accuracy q and correlation r.

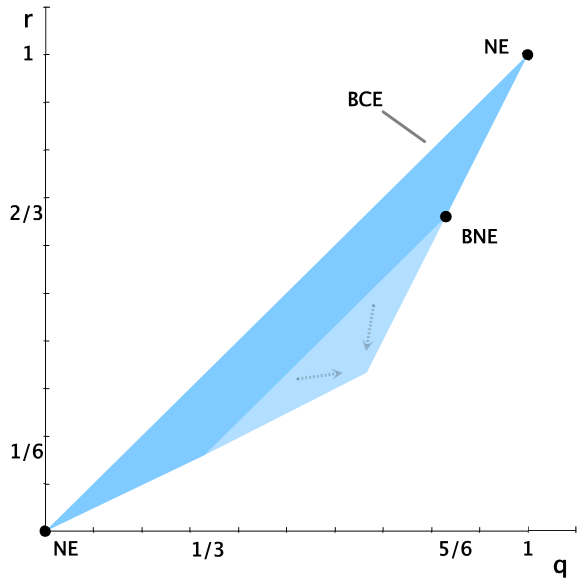

Figure 5: BCE random choice rules for

ai $(q,r) = (2/3, 1/3)$ and $(q,r) = (5/6, 2/3)$.

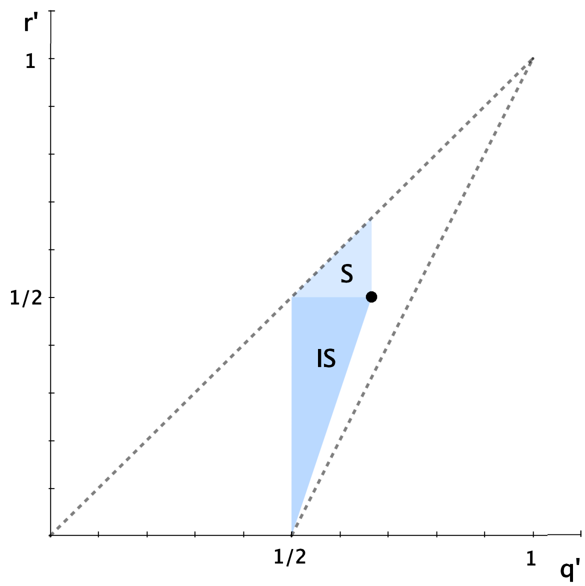

Figure 6: The set of binary symmetric information structures for which $S_{q,r}$ with $(q,r) = (4/6, 3/6)$ is individually sufficient

sis (IS) and sufficient sufficient $(S \cup IS)$.

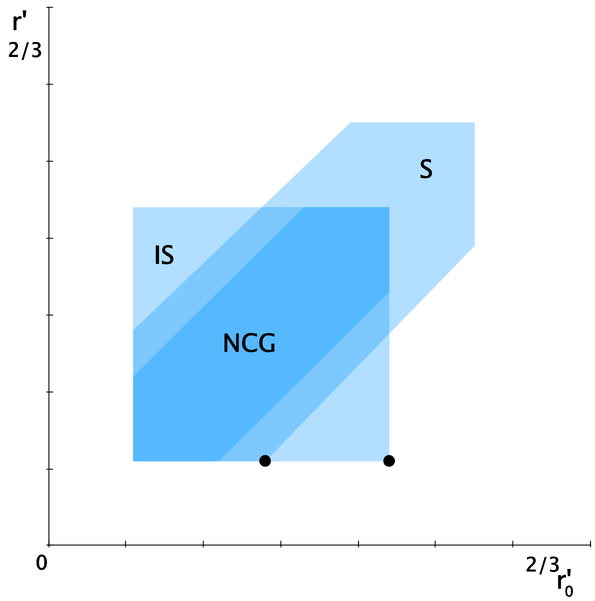

Figure 7: The set of asymmetric information structures S_{q',r'_0,r'_1} with $q\prime = 5/9$ for which $S_{q,r}$ with $(q,r) = (2/3, 1/3)$ is sufficient (S), individually sufficient (IS) and a non-communicating ungarbling (NCG).

sit

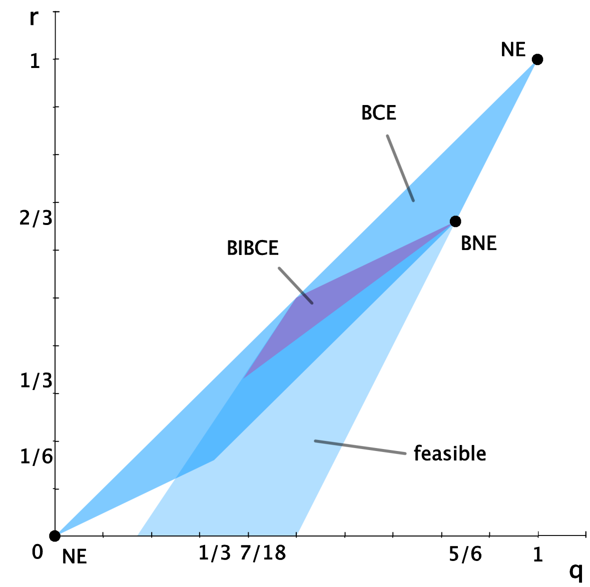

Figure 8: The set of BCE, BIBCE and feasible random choice rules under $S_{q,r}$ with $(q,r) = (5/6, 2/3)$.

aj

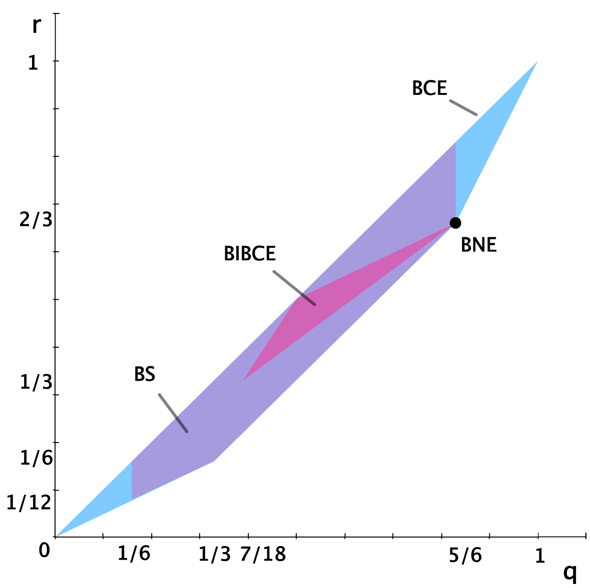

Figure 9: The set of BCE, BIBCE and Bayes solution random choice rules under $S_{q,r}$ with $(q,r) = (5/6, 2/3)$.

ak

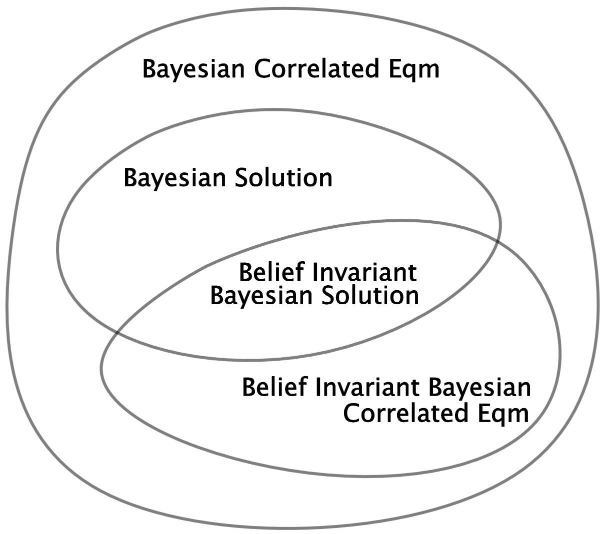

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Figure 10: The Solution Concepts

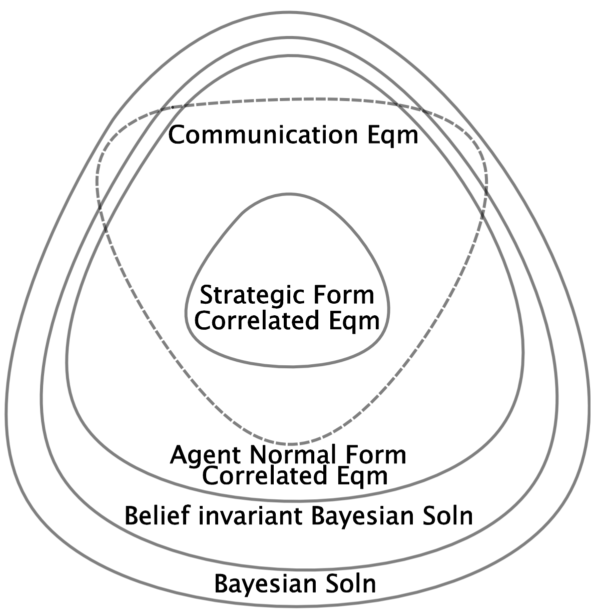

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Figure 11: More Solution Concepts