Subgame Perfect Cooperation in an Extensive Game

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Abstract

We propose a new concept of core for games in extensive form and label it the γ-core of an extensive game. We show it to be a refinement of the core of the strategic form of the game in the same sense as the set of subgame-perfect Nash equilibria is of the set of Nash equilibria. We rationalize the γ-core payoff vectors as credible contracts, establish additional properties of the γ-core, and study in detail its relationship with the other two concepts of cooperation in an extensive game: the strong and the coalition-proof subgame-perfect Nash equilibria. To further illustrate it and its properties, we introduce three applications: two and three player infinite bargaining games of alternating offers and a dynamic game of global public good provision.

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1. Introduction

Core concepts in dynamic games have attracted the interest of economists for many years. Gale (1978) explores the issue of time consistency in the Arrow-Debreu model with dated commodities when agents distrust the forward contracts signed at the first date. Gale introduces the sequential core which consists of allocations that cannot be improved upon by anyone at any date. Similarly, Forges, Mertens, and Vohra (2002) propose the ex ante incentive compatible core. Becker and Chakrabarti (1995) propose the recursive core as an allocation such that no coalition can improve upon its consumption stream at any time, given its accumulation of assets up to that time. These economic applications show that the bargaining power of coalitions may change as the game proceeds along a history.

In this paper, we propose a new solution concept for general extensive games that incorporates cooperation and also subgame perfection. This new concept, which we label the γ-core of an extensive game, is a refinement of the core of the strategic form of the game in the same sense as the set of subgame-perfect Nash equilibria is of the set of Nash equilibria. More specifically, the γ-core of an extensive game is a subset of the intersection of the γ-cores of a family of strategic games. From its definition and properties, our proposed new concept is a cooperative analog of the non-cooperative subgame-perfect Nash equilibrium. The concept is credible in a sense analogous to that of subgame-perfect equilibria in an extensive game. That is, the concept permits players, including coalitions of players, to take into account the fact that their bargaining power may change during a play of the game.

Fundamental to our notion of subgame perfect cooperation is the idea that a coalition becomes a single player; given a game in extensive form with player set N, when a coalition S forms, a new game is created in which the players in S constitute one single player. Another fundamental idea of our approach is that, at any point in the extensive game, only those players who still have decisions to make can form coalitions and only they can coordinate their decisions from that point onwards.[1] These two ideas are applied to define and study cooperation in

[1] Since the player set of the original game does not include any player who has no decisions to make in the game, we treat a subgame analogously by not including players who no longer have decisions to make in the subgame. Not doing so would be inconsistent with the notion of subgame perfection.

extensive games. They can be applied to games of both perfect and imperfect information; in this paper we treat only games of perfect information and consider game solutions in pure strategies.[2]

An issue that arises in the treatment of cooperation within coalitions in a non-cooperative game is: Given a deviation by a coalition, what is the response of the players in the complementary set? In our approach, since a coalition is simply a player in a game derived from the original game, it is appealing to take the remaining set of players as singletons, especially since that leads to a core concept that relates and can be nicely compared with both the strong and the coalition-proof subgame-perfect Nash equilibria, in which also the remaining players are taken to be singletons.[3] However, as we discuss further in our concluding section, our approach can still be applied even if the remaining players are not assumed to be singletons.

Our proposed new concept of subgame perfect cooperation takes into account interactions of coalitions through the solution concept, analogously to how Nash and subgame-perfect Nash equilibrium (SPNE) concepts take account of interactions of players. Recall that a payoff vector is in the γ-core (Chander, 2010) of a *strategic* game if no coalition can improve upon its part of the payoff vector by deviating from any strategy profile that generates the payoff vector. The subgame-perfect γ-core of an *extensive* game differs; a payoff vector is in the γ-core if there is a history that leads to a terminal node for which the payoff vector is feasible and, along the decision nodes in the history, no coalition (including the grand coalition) can improve upon its part of the payoff vector by deviating.

For the game-theoretic results of this paper, introducing subgame-perfect cooperation, we assume transferable utility so that the utility of a coalition becomes the sum of the utilities of the coalition members. We conclude this introduction with some further discussion of the γ-core, the main results, and applications. Further discussion of the concept and related literature is included in the final section of the paper.

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[2]However, that is not entirely true as the paper includes an application to a dynamic game with simultaneous moves in which a similar strategic game is played at each date. In extensive games with perfect information it makes perfect sense to restrict to pure strategies, but for extension to a general extensive game of imperfect information, correlated strategies of players inside a coalition may have to be considered.

[3] See Chander (2010; 2007) which address this issue in the context of a strategic game. It is shown that forming singletons is a subgame-perfect Nash equilibrium strategy of the remaining players in a game in which the players choose whether to stay alone or to merge. In other words, the players in the complement may actually have incentives to break apart into singletons.

Notice that in an extensive game with transferable utility a feasible payoff vector may involve side-payments and as a contract it can be fulfilled if and only if the game reaches the node for which it is feasible. Therefore, to rationalize the γ-core payoff vectors as contracts we need to assume an environment in which players can write contracts at the beginning of the extensive game which are binding *if and only if* the game reaches the node for which the contract is feasible. Since a node of an extensive game is reached if and only if the players take a certain set of actions (in fact a unique set of actions), this form of contracting is equivalent to assuming that the contract is binding if and only if the players have taken those actions. In other words, the players are free to nullify a contract by deviating from those actions at any point and prevent the game from reaching the node for which the contract is feasible and binding. Accordingly, a contract is *credible* if the players do not have incentives to take actions that can prevent the game from reaching the node for which the contract is feasible and binding.[4]

By definition, the γ-core payoff vectors are credible contracts, and the assumed contracting environment is common to many situations involving sequential trades. E.g., it is implicit in the Arrow-Debreu model with dated commodities in which all trades are agreed upon at date 0, but fulfilled later. Furthermore, as will be shown, our rationalization of the γ-core payoff vectors as credible contracts can be extended to dynamic games in which payoffs occurs at each date and self-sufficient spot contracts are possible.

It is well-known that the conventional notion of the core is “credible” in the sense that if coalitional deviations are limited to those deviations which are immune to further deviations by subcoalitions, the core is unaffected (Ray, 1989). We show that the γ-core of an extensive game is credible, i.e., the γ-core of an extensive game is the same irrespective of whether or not coalitional deviations are required to be immune to further coalitional deviations.

Rubinstein (1980) introduces a notion of “strong perfect equilibrium” for a super game in which a strategic game is played infinitely many times. However, we are not aware of any paper introducing a definition of strong subgame-perfect Nash equilibria (SSPNE) in a *general* extensive game. Using the conceptual framework developed in this paper, we propose such a

[4] Alternatively, in some situations we might regard players as signing binding contracts at the beginning of the game that depend on the terminal node reached in a play of the game.

definition and verify that, as in a strategic game, every SSPNE of an extensive game is coalition-proof. We identify a class of extensive games in which the γ-core is a weaker concept than SSPNE in the sense that if a game in the class admits a SSPNE then it is unique and the γ-core consists of the unique SSPNE payoff vector, but the γ-core may be non-empty even if the game admits no SSPNE.

We illustrate the concept of SSPNE and its relationship with the γ-core by applying these concepts to bargaining games of alternating offers with two and three players. Bargaining games of alternating offers have been proposed in the literature (Rubinstein, 1982) to derive non-cooperative foundations of the Nash bargaining solution. It is well-known that the infinite bargaining game of alternating offers with two players admits a unique SPNE. We show that the unique SPNE is actually a SSPNE and, therefore, a γ-core payoff vector and a credible contract. Furthermore, the same also holds in a three player bargaining game if the players are identical and can use only stationary strategies.

Bernheim, Peleg, and Whinston (1987) introduce the concept of coalition-proof subgame-perfect Nash equilibrium (CPSPNE), which requires that any agreement to deviate must be "self-enforcing". Imposition of this requirement on coalitional deviations leads to a version of the γ-core which is self-enforcing and satisfies subgame perfection - the defining property of our core concept. We refer to this version as the self-enforcing γ-core of an extensive game. Since a deviating coalition can always choose a self-enforcing agreement even if it is not required to do so, the coalitional payoffs are generally lower if only self-enforcing coalitional deviations are permitted. However, that does not necessarily imply a larger core, since the payoff of the grand coalition may also be lower as it too can choose only self-enforcing agreements. In fact, as will be made clear, the concept of self-enforcing γ-core presumes that the game admits a CPSPNE. We identify a class of extensive games in which the two concepts are actually equivalent.

The γ-core payoff vectors, by definition, are efficient outcomes of an extensive game, but the self-enforcing γ-core payoff vectors may not be unless the game admits a unique SSPNE in which case the two cores are equal. In other cases they are disjoint, though both satisfy subgame perfection. Which one of the two versions of the γ-core is the relevant solution concept depends on whether deviating coalitions can write binding agreements (as in a strong Nash equilibrium) or they can write only self enforcing agreements (as in a coalition-proof Nash equilibrium). As noted, the bargaining games of alternating offers admit a unique SSPNE and, therefore, the γ-core and the self-enforcing γ-core are equal. But the game in Example 1 does not admit a SSPNE and thus the two cores are disjoint.

Besides the application to bargaining games of alternating offers, we introduce an application to a dynamic game of global public good provision which further illustrates the γ-core and its various properties. First, since this dynamic game involves simultaneous moves, it demonstrates that the γ-core concept is not restricted to extensive games of perfect information alone. Second, since all players are active in each subgame and payoffs occur at every (decision or terminal) node of the game, self-sufficient spot contracts are possible. Thus, the game allows us to illustrate how our rationalization of the γ-core payoff vectors as credible contracts can be extended to games in which self-sufficient spot contracts are possible. Third, it highlights the distinction between the two versions of the γ-core and their relationship with CPSPNE. Though the game does not admit a SSPNE, both versions of the γ-core are non-empty and the self-enforcing γ-core consists of the unique CPSPNE payoff vector. Fourth, it demonstrates that a γ-core payoff vector, like a SPNE, can be found by backward induction.

Before concluding this section, we remark that our work is distinct from research that seeks to unify cooperative and non-cooperative game theory through underpinning cooperative game theoretic solutions with noncooperative Nash equilibria, the "Nash Program".[5] Numerous papers have contributed to this program including Rubinstein (1982), Perry and Reny (1994), Pérez-Castrillo (1994), Serrano (1995), Compte and Jehiel (2010) and Lehrer and Scarsini (2012), for example. These papers are fundamentally different from ours since they start with a given cooperative game and impose a dynamics or extensive form game whose outcomes are related to cooperative outcomes of the initial game. In contrast, we derive coalitional games from the extensive game as the primitive (Proposition 1). While we do not do so in this paper, instead of using the core as the solution concept for the derived cooperative game, it would be interesting to

[5] See Serrano (2008) for a brief survey and Ray (2007) for a more extended discussion.

use the coalitional Nash bargaining solution, introduced by Compte and Jehiel for this purpose ; we leave this to future research.

The paper is organized as follows. Section 2 presents the definition of γ-core of a *strategic* game. Section 3 motivates and introduces the definition of γ-core of an *extensive* game of perfect information. It establishes its various properties, introduces the concept of an SSPNE of an extensive game, and studies how the γ-core and SSPNE are related. It includes applications to infinite bargaining games of alternating offers with two or three players. It also introduces the self-enforcing γ-core and notes its relationship with CPSPNE. Section 4 introduces a dynamic game of global public good provision and shows that if the payoff functions are quadratic, the γ-core is non-empty and a γ-core payoff vector can be found by backward induction. Section 5 further discusses the γ-core and related literature.

2. The γ-core of a strategic game

It is convenient to first take note of the concept of γ-core of a general *strategic* game (Chander, 2010).[6] We denote a strategic game with transferable utility by (N, T, u) where $N = \{1, \dots, n\}$ is the player set, $T = T_1 \times \cdots \times T_n$ is the set of strategy profiles, T_i is the strategy set of player i, $u = (u_1, \dots, u_n)$ is the vector of payoff functions, and u_i is the payoff function of player i. A strategy profile is denoted by $t = (t_1, \dots, t_n) \in T$. We denote a coalition by S and its complement by $N \backslash S$. Given $t = (t_1, \dots, t_n) \in T$, let $t_S \equiv (t_i)_{i \in S}, t_{-S} \equiv (t_j)_{j \in N \backslash S}$, and $(t_S, t_{-S}) \equiv t = (t_1, \dots, t_n)$.

Given a coalition $S \subset N$, the *induced* strategic game (N^S, T^S, u^S) is defined as follows:

- The player set is $N^S = \{S, (j)_{j \in N \backslash S}\}$, i.e., coalition S and all $j \in N \backslash S$ are the players (thus the game has $n - s + 1$ players[7]);
- The set of strategy profiles is $T^S = T_S \times_{j \in N \backslash S} T_j$ where $T_S = \times_{i \in S} T_i$ is the strategy set of player S and T_j is the strategy set of player $j \in N - S$;

[6] In contrast, Maskin's (2003) and Huang and Sjöström's (2006) core concepts are based on a partition function and thereby their concepts abstract away from the strategic interactions that underlie the payoffs of coalitions. Similarly, the conventional α- and β- cores, by definition, rule out interesting strategic interactions between players. See Chander (2007, 2010) for a comparison of the α- β- and γ- cores of a *general* strategic game and Chander and Tulkens (1997) for the same in the context of a *specific* strategic game.

[7] The small letters n and s denote the cardinality of sets N and S, respectively.

- The vector of payoff functions is $u^S = (u_S^S, (u_j^S)_{j\in N\backslash S})$ where $u_S^S(t_S, t_{-S}) = \sum_{i\in S} u_i\,(t_S, t_{-S})$ is the payoff function of player S and $u_j^S(t_S, t_{-S}) = u_j(t_S, t_{-S}\,)$ is the payoff function of player $j \in N\backslash S$, for all $t_S \in T_S$ and $t_{-S} \in \times_{j\in N\backslash S} T_j$.

Observe that if $(\tilde{t}_S, \tilde{t}_{-S}) = \tilde{t}$ is a Nash equilibrium of the induced game (N^S, T^S, u^S), then $u_S^S(\tilde{t}_S, \tilde{t}_{-S}) = \sum_{i\in S} u_i(\tilde{t}_S, \tilde{t}_{-S}) \geq \sum_{i\in S} u_i\,(t_S, \tilde{t}_{-S})$ for all $t_S \in T_S$. Thus, for each $S \subset N$, a Nash equilibrium of the induced game (N^S, T^S, u^S) assigns a payoff to S that it can obtain without cooperation from the remaining players. If the induced game (N^S, T^S, u^S) has multiple Nash equilibria, then any Nash equilibrium with highest payoff for S is selected.[8] In this way, a unique payoff can be assigned to the coalition. (Other selections in the case of multiple equilibria are possible. Our selection of Nash equilibrium with highest payoff makes the conditions for non-emptiness of the γ-core more stringent.)

Given a strategic game (N, T, u), the corresponding game in characteristic form is the function $w^\gamma(S) = \sum_{i\in S} u_i(\tilde{t}_S, \tilde{t}_{-S})\,, S \subset N$, where $(\tilde{t}_S, \tilde{t}_{-S}) \in T$ is a Nash equilibrium of the induced game (N^S, T^S, u^S) with highest payoff for coalition S. The γ-core of a strategic game (N, T, u) or, equivalently, the core of the corresponding game in characteristic function form w^γ is the set of payoff vectors p such that (i) for each $S \subset N, \sum_{i\in S} p_i \geq w^\gamma(S)$ and (ii) $\sum_{i\in N} p_i$ $= w^\gamma(N)$.[9]

3. The γ-core of an extensive game

We denote an extensive game of perfect information by $\Gamma = (N, K, P, u)$ where $N = \{1, \dots, n\}$ is the player set and K is the game tree with origin denoted by 0. Let Z denote the set of terminal nodes of game tree K and let X denote the set of non-terminal nodes, i.e., the set of decision nodes. The player partition of X is given by $P = \{X_1, \dots, X_n\}$ where X_i is the set of all decision nodes of player $i \in N$. The payoff function is $u\colon Z \to R^n$ where $u_i(z)$ denotes the payoff of player i at terminal node z. Since there is a one-one correspondence between the game tree and the strategy sets of the players, we do not explicitly state, as of now, the strategy sets.

[8] Such a payoff will surely exist if the strategy sets are compact (or finite) and the payoff functions are continuous.
[9] Note that it is efficient for the grand coalition to form, since the grand coalition can choose at least the same strategies as the players in any coalition structure (i.e. any partition of the total player set into coalitions).

3.1 *The induced extensive games*

Given an extensive game $\Gamma = (N, K, P, u)$ and a coalition $S \subset N$, the *induced* extensive game $\Gamma^S = (N^S, K^S, P^S, u^S)$ is defined as follows:

- The player set is $N^S = \{S, (i)_{i \in N \setminus S}\}$, i.e., coalition S and all $j \in N \setminus S$ are the players (thus the game has $n - s + 1$ players);
- The game tree is $K^S = K$ (thus the set of decision nodes is X);
- The player partition of X is $P^S = \{X_S, (X_i)_{i \in N \setminus S}\}$ where $X_S = \cup_{j \in S} X_j$;
- The profile of payoff functions is $u^S = (u_S^S, (u_i^S)_{i \in N \setminus S})$ where $u_S^S(z) = \sum_{j \in S} u_j(z)$ is the payoff function of S and $u_i^S(z) = u_i(z)$ is the payoff function of $i \in N \setminus S$, for all $z \in Z$.

Note that if S is a singleton coalition, then $\Gamma^S = \Gamma$. For each $S \subset N$, the induced game $\Gamma^S = (N^S, K^S, P^S, u^S)$ represents the situation in which the players in S have formed a coalition to coordinate their decisions in all their decision nodes. A vector $p = (p_1, p_2, \dots, p_n)$ is a *feasible* payoff vector for terminal node $z \in Z$ if $\sum_{i \in N} p_i = \sum_{i \in N} u_i(z) = u_N^N(z)$. Example 1 below illustrates the definitions so far.

Example 1 Let Γ denote the extensive game depicted in Fig.1. Then, x_1 is the origin of the game tree $K, N = \{1,2\}, Z = \{z_1, z_2, z_3\}, X = \{x_1, x_2\}, P = \{\{x_1\}, \{x_2\}\}$ and $u: Z \to R^2$ is given by $u(z_1) = (2,1), u(z_2) = (4,2)$, and $u(z_3) = (1,3)$.

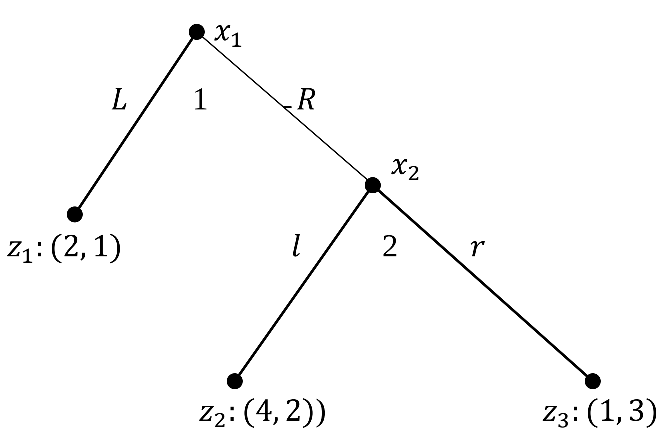

Figure 1

The extensive game Γ^N when players 1 and 2 form a coalition to coordinate their decisions in all their decision nodes is depicted in Fig. 2. The game tree is the same, but now we have a one-player game with player set $\{N\}$. So $P^N = \{\{x_1, x_2\}\}$, $u_N^N(z_1) = 3$, $u_N^N(\mathrm{z}_2) = 6$, and $u_N^N(z_3) = 4$. Notice that each strategy of player N in game Γ^N generates a history of game Γ.

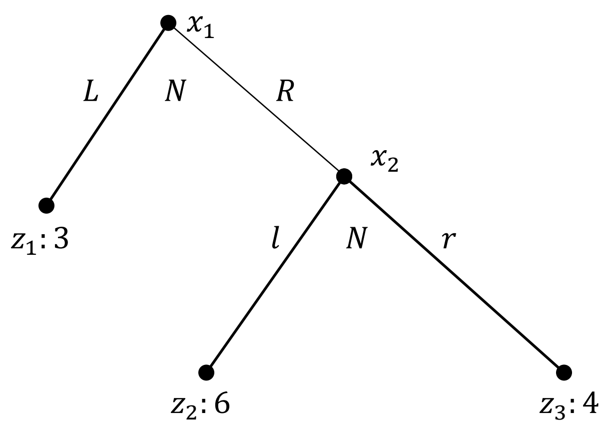

Figure 2

In the following, we will often not distinguish between player i and coalition $\{i\}$. Given $x \in X$, let Γ_x denote the subgame with origin at x. Since the origin of Γ is denoted by 0, $\Gamma_0 = \Gamma$ and if $x \neq 0$, Γ_x is a proper subgame of Γ. Notice that the player set of a proper subgame Γ_x may be smaller than the set N (but is not necessarily so). A player is *active* in subgame Γ_x if some decision node in Γ_x is a decision node of the player. Similarly, a coalition is *active* in subgame Γ_x if *all* its members are active in the subgame Γ_x. Let S be an active coalition in subgame Γ_x. Then, the induced game Γ_x^S is defined from Γ_x in exactly the same way as the induced game Γ^S is defined from Γ. Clearly, $\Gamma_0^S = \Gamma^S$. Since Γ is a game of perfect information, so is each game Γ_x^S, $x \in X$ and S an active coalition in Γ_x. In what follows, it will be often convenient to refer to "a coalition that is active in the subgame with origin at x" simply as "an active coalition at x".

A SPNE of an extensive game induces a Nash equilibrium in each subgame of the extensive game. Therefore, for each coalition S which is active at x, a SPNE strategy of S in the game Γ_x^S prescribes a play that is optimal for S from point x onwards, given the optimal strategies of the remaining active players. Thus, a SPNE payoff of a coalition S in the induced game Γ_x^S is a payoff that it can credibly obtain if the game reaches node x.

The subgame-perfect Nash equilibria of the family of extensive games Γ_x^S, $x \in X$ and S an active coalition in Γ_x, determine the payoffs that coalitions can credibly obtain at each decision node of the game Γ. If the induced game Γ_x^S has more than one SPNE, then a SPNE with highest payoff for coalition S is selected.[10]

We return to Example 1 to illustrate the additional definitions introduced. Since the game Γ in Example 1 has only two players, $\Gamma_{x_1}^{\{1\}} = \Gamma_{x_1}^{\{2\}} = \Gamma$. The SPNE payoff of coalition $\{1\}$ in the induced game $\Gamma_{x_1}^{\{1\}}$ is 2 and its SPNE strategy is L. Similarly, the SPNE payoff of $\{2\}$ in the induced game $\Gamma_{x_1}^{\{2\}}$ is 1 and its SPNE strategy is rR ($\equiv\ r$ if 1 plays R).

The SPNE payoff of player N in the single player game $\Gamma^N (= \Gamma_{x_1}^N)$ in Fig. 2 is 6 and its SPNE strategy is $(R, lR) (\equiv R; l$ if N plays $R)$. Notice that the SPNE strategy (R, lR) of coalition N is not compatible with the SPNE strategies L and rR of coalitions $\{1\}$ and $\{2\}$, respectively. This fact plays a crucial role in what follows.

3.2 *Subgame-perfect cooperation*

In order to highlight the basic idea underlying the γ-core of an extensive game, we return again to Example 1 to show that the bargaining power of coalitions following their SPNE strategies may change as the game unfolds along a history.

If players 1 and 2 form a coalition, the payoff of the coalition is 6, as implied by the SPNE of $\Gamma_{x_1}^N$. If coalition $\{1\}$ decides to deviate in the beginning of the game, its resulting payoff is 2, as implied by the SPNE of $\Gamma_{x_1}^{\{1\}}$. Similarly, if $\{2\}$ decides to deviate in the beginning of the game, its resulting payoff is 1, as implied by the SPNE of $\Gamma_{x_1}^{\{2\}}$. In sum, the coalitions $\{1\}, \{2\}$ and N which all are active at x_1 can obtain payoffs of 2, 1 and 6, respectively. Thus, none of them can improve upon a payoff vector (p_1, p_2) such that $p_1 \geq 2, p_2 \geq 1, p_1 + p_2 = 6$. E. g., given the

[10] Obviously, this is not the only way to handle multiplicity of equilibria. Other selections are possible. As in a strategic game, our selection of equilibrium with highest payoff makes the conditions for non-emptiness of the γ-core of an extensive game more stringent.

payoff vector (3.5, 2.5), no coalition can obtain a higher payoff by deviating from the grand coalition's strategy (R, lR) in the *beginning* of the game.

Yet, we claim that the strategy profile (R, lR) and the payoff vector (3.5, 2.5) are not a sensible prediction of the game. That is because if the strategy profile (R, lR) is followed, the game would reach node x_2 and therefore the strategy profile (R, lR) and the payoff vector (3.5, 2.5) should also be immune to deviations by all active coalitions at x_2. However, it is not. The only active coalition at x_2 is $\{2\}$ and it can obviously obtain a higher payoff of 3 (> 2.5) by taking action r once the game reaches x_2.[11] Thus, the strategy profile (R, lR) is not immune to deviations by all active coalitions along the history generated by the strategy profile.

Note that the set of payoff vectors (p_1, p_2) which are in the γ-core of the corresponding game in strategic form is given by $p_1 + p_2 = 6, p_1 \geq 2, p_2 \geq 1$ and the payoff vector (3.5, 2.5) is in the set. But it is not immune to coalitional deviations as the game unfolds along the history leading to the terminal node for which the payoff vector (3.5, 2.5) is feasible. The set of payoff vectors (p_1, p_2) which are immune to coalitional deviations along the history is smaller and given by $p_1 + p_2 = 6, p_1 \geq 2$, and $p_2 \geq 3$. Furthermore, such payoff vectors can be interpreted as credible contracts. Clearly, as contracts they can be fulfilled if and only if the game reaches terminal node z_2 (for which they are feasible). Thus, in the context of an extensive game, the following form of contracting is sufficient to rationalize such payoff vectors as credible contracts:

(a) Players can write contracts at the beginning of an extensive game.
(b) A feasible payoff vector as a contract is binding *if and only if* the game reaches the terminal node for which it is feasible.
(c) A feasible payoff vector as a contract is *credible* if the players do not have incentives to take actions that can prevent the game from reaching the terminal node for which it is feasible and binding.

[11] Notice that action r is consistent with the SPNE strategy of $\{2\}$ in the game $\Gamma_{x_1}^{\{2\}}$. Still the payoff that $\{2\}$ can obtain at x_2 is higher. That is because the node x_2 is not reached in the history generated by the SPNE of the game $\Gamma_{x_1}^{\{2\}}$.

Since the game can reach terminal node z_2 if and only if the players take a (unique) set of actions, restriction (b) implies that a contract is binding if and only if the players take those actions. In other words, players are free to nullify a contract by taking actions at any point of the game that can prevent the game from reaching terminal node z_2.[12] This form of contracting is common and often necessary in environments involving sequential trades. E.g., it is necessary if the Arrow-Debreu model with dated commodities is to make sense.

To see that, consider a simple Arrow-Debreu economy in which there are two agents, a and b, and two dates, 0 and 1. At each date there is one consumption good available. Each agent's preferences are described by a Cobb-Douglas utility function with equal weights for the two consumption goods. Suppose that the endowments of the two agents are $(\epsilon, 1-\epsilon)$ and $(1-\epsilon, \epsilon)$, respectively, where ϵ is a small positive number. The agents can trade by writing contracts on date 0 which require agent b to give agent a a positive amount of the consumption good at date 0 and agent a to give agent b a positive amount of the consumption good at date 1. However, for such a contract to be effective it is necessary that it be binding *if and only if* on date 1 agent b is found to have given agent a the amount agreed upon on date 0. Only such a form of contracting can facilitate trades that lead to an efficient and individually rational allocation.[13]

Example 1 demonstrates that the relative bargaining power of coalitions following their SPNE strategies may change as the game unfolds along the history generated by a strategy profile. For instance, coalition $\{2\}$ can obtain a payoff of only 1 by deviating at x_1, but a payoff of 3 by deviating at x_2. That is so, despite the fact that coalition $\{2\}$ follows a SPNE strategy in the induced game $\Gamma_{x_1}^{\{2\}}$, because x_2 is not reached in the history generated by the SPNE of the induced game $\Gamma_{x_1}^{\{2\}}$. In more general terms, that is so because a SPNE strategy of a coalition ($\{1\ 2\}$ in Example 1) is not necessarily a SPNE strategy of a proper subcoalition ($\{2\}$ in Example 1).

[12] Clearly, this form of contracting implicitly assumes that a mechanism to enforce contracts which are conditionally binding is in place. E. g., it can be a legal framework which allows imposition of penalties for not fulfilling a contract the conditions of which have been met by all agents concerned. Other enforcement mechanisms that have been considered in the literature include institution of money (Gale, 1978) or transfers of capital (Becker and Chakrabarti, 1995).

[13] It is easily seen that in this example only those contracts which lead to an efficient and individually rational allocation are credible.

To conclude our discussion of Example 1, the relative bargaining power of coalitions following their SPNE strategies may change as the game unfolds along the history generated by a strategy profile and a core concept which takes account of this fact implies a smaller core. The payoff vectors that belong to the so-defined core are credible contracts.

3.3 *Definition of the γ-core of an extensive game*

Given an extensive game Γ and the consequent family of extensive games Γ_x and Γ_x^S, let Ω_x and Ω_x^S denote the corresponding games in strategic form. We shall conveniently denote Ω_0 simply by Ω. Let $w^\gamma(S;x)$ denote the highest SPNE payoff of coalition S in the game Γ_x^S, in which it is active.[14] Then, $w^\gamma(S;x)$ is equal to the SPNE payoff of coalition S in the induced strategic game Ω_x^S.[15] Thus, the function $w^\gamma(S;x)$ is the characteristic function of the strategic game Ω_x and the core of the characteristic function game $w^\gamma(S;x)$ is the γ-core of the strategic game Ω_x.[16] In particular, the function $w^\gamma(S;0), S \subset N$, is the characteristic function of the strategic game Ω and the core of the characteristic function game $w^\gamma(S;0)$ is the γ-core of the strategic game Ω.

Given an extensive game Γ, let z denote the terminal node of the history generated by a strategy profile. Then, a vector $(p_1, \dots, p_n)$ is a *feasible* payoff vector for the strategy profile if $\sum_{i \in N} p_i = u_N^N(z)$. Notice that a payoff vector may be feasible for more than one strategy profile and the histories generated by these strategy profiles may be different. By a history leading to a payoff vector $(p_1, \dots, p_n)$ we mean a history with a terminal node z such that $u_N^N(z) = \sum_{i \in N} p_i$.

Notice that the history generated by any strategy profile begins at the origin of game Γ and all coalitions including coalition N are active at least at the origin. Given the payoffs $w^\gamma(S;x)$, $x \in X$ and S an active coalition at x, the γ-core of the extensive game Γ consists of all payoff vectors with the property that no coalition can improve upon its payoff by deviating not only at

[14] Such a payoff can obviously be found by backward induction in the induced game Γ_x^S, even with infinite strategy sets if the strategy sets are compact.
[15] Thus, $w^\gamma(S;x)$ is the payoff that coalition S can credibly obtain in the strategic game Ω_x^S.
[16] Notice that if coalition S is active at x, then so is every coalition $S' \subset S$, and therefore $w^\gamma(S;x)$ meets the standard definition of a characteristic function.

the origin but also at any decision node along the histories leading to the terminal nodes for which the payoff vectors are feasible.[17]

Definition 1 The γ-core of an extensive game Γ is the set of all payoff vectors $(p_1, \dots, p_n)$ such that for each coalition $S \subset N$, $w^\gamma(S;x) \leq \sum_{i\in S} p_i$ at all decision nodes x along the histories leading to the payoff vector $(p_1, \dots, p_n)$.[18]

Let $z^* \in Z$ be a terminal node such that $u_N^N(z^*) \geq u_N^N(z)$ for all $z \in Z$. Such a terminal node exists if the extensive game Γ is finite or if the strategy sets are compact and the payoff functions are continuous. Definition 1 implies that the γ-core of the extensive game Γ must be a subset of the set of feasible payoff vectors $(p_1, \dots, p_n)$ such that $\sum_{i\in N} p_i = u_N^N(z^*)$. That is because the origin of the extensive game Γ is a decision node along every history of the game and coalition N is active at the origin. Thus, $u_N^N(z^*) = w^\gamma(N;0)$ and there are no other feasible payoff vectors $(p_1, \dots, p_n)$ which are immune to deviations by coalition N.

Notice that Definition 1 takes into account the possibility that the terminal node at which the total payoff $u_N^N(z)$ is highest may not be unique and that the payoffs that coalitions can obtain along the nodes of different histories leading to different terminal nodes with highest total payoff may be different. As we discuss below, this implies a concept which may be considered as "too strong" and the so-defined γ-core may be empty. Therefore, we introduce later a weaker notion of the γ-core such that the weaker γ-core is non-empty if the γ-core is, but the converse is not true.

Let $Z^* \subset Z$ be such that if $z^* \in Z^*$, then $u_N^N(z^*) \geq u_N^N(z)$ for all $z \in Z$. Let $X(z^*)$ denote the set of decision nodes along the history that leads to the terminal node z^*. Let $X^* = \cup X(z^*)$ where the union is taken over all $z^* \in Z^*$.

Definition 1 implies that the γ-core of an extensive game Γ consists of payoff vectors from the set $\{ (p_1, \dots, p_n) \colon \sum_{i\in N} p_i = u_N^N(z^*)\}$ which are immune to deviations by all coalitions which are active at the decision nodes in the set X^*. Since the origin $0 \in X^*$ and all coalitions are active

[17] As Example 1 illustrates, the subgame perfect equilibrium payoffs that a coalition can obtain as the game unfolds along the history generated by a strategy profile may be higher.
[18] Remember that for each coalition S, the function $w^\gamma(S, x)$ is defined for only those decision nodes x at which coalition S is active.

at the origin, the payoff vectors must additionally satisfy at least $w^\gamma(S;0) \leq \sum_{i\in S} p_i$ for all $S \subset N$. Thus, the γ-core of an extensive game Γ is a subset of the γ-core of the corresponding strategic game Ω. Taking account of the subgame perfect payoffs that active coalitions can obtain at decision nodes in the set X^* leads to refinements of the γ-core of the strategic game Ω.

Observe that in Example 1, the γ-core of the strategic game Ω is the set of payoff vectors (p_1, p_2) such that $p_1 + p_2 = 6, p_1 \geq 2, p_2 \geq 1$, but the γ-core of the extensive game Γ is a smaller set of payoff vectors (p_1, p_2) such that $p_1 + p_2 = 6, p_1 \geq 2$, and $p_2 \geq 3$.

3.4 *Properties of the γ-core*

We establish additional properties of the core concept introduced. One of the important properties is that the γ-core of an extensive game is equivalent to the core of a characteristic function game. For each $S \subset N$, let

$$w^\gamma(S) = \max_x \; w^\gamma(S;x)$$

where the maximum is taken over all nodes $x \in X^*$ at which S is active. Since the origin $0 \in X^*$, each coalition S is active at least at some $x \in X^*$. Clearly, $w^\gamma(N) = u_N^N(z^*)$, for all $z^* \in Z^*$. We shall refer to the function $w^\gamma(S)$ as the *γ-characteristic function* of the extensive game Γ.

Proposition 1 The γ-core of the extensive game Γ is equal to the core of the γ-characteristic function game w^γ.

Proof: Let $(p_1, \ldots, p_n)$ be a payoff vector which is in the core of the characteristic function game w^γ. Then, $\sum_{i\in N} p_i = w^\gamma(N)$ and $\sum_{i\in S} p_i \geq w^\gamma(S)$, $S \subset N$. By definition of the characteristic function w^γ, for each $S \subset N, w^\gamma(S) \geq w^\gamma(S;x)$ at each decision node $x \in X^*$ at which S is active and $w^\gamma(N) = u_N^N(z^*)$, for all $z^* \in Z^*$. The above inequalities imply that for each $S \subset N, \sum_{i\in S} p_i \geq w^\gamma(S) \geq w^\gamma(S;x)$ at each $x \in X^*$ at which S is active and the equalities imply $\sum_{i\in N} p_i = u_N^N(z^*)$ for all $z^* \in Z^*$, that is $(p_1, \ldots, p_n)$ is a feasible payoff vector for all terminal nodes z^* with the highest payoff for coalition N. Hence, $(p_1, \ldots, p_n)$ meets all conditions for a payoff vector to be in the γ-core of the extensive game Γ.

Conversely, let $(p_1, \ldots, p_n)$ be a payoff vector in the γ-core of the extensive game Γ, then for each $S \subset N,\ w^\gamma(S; x) \leq \sum_{i \in S} p_i$ at each decision node x along the history generated by any strategy profile for which the payoff vector $(p_1, \ldots, p_n)$ is feasible. Since the origin 0 is a decision node of the history generated by any strategy profile and coalition N is active at the origin, $\sum_{i \in N} p_i \geq w^\gamma(N; 0)$. Furthermore, since $(p_1, \ldots, p_n)$ is a feasible payoff vector, $\sum_{i \in N} p_i \leq w^\gamma(N, 0) = w^\gamma(N)$. Thus, $\sum_{i \in N} p_i = w^\gamma(N)$ and $(p_1, \ldots, p_n)$ is a feasible payoff vector for any history of the game leading to a $z^* \in Z^*$. Therefore, for each $S \subset N,\ w^\gamma(S; x) \leq \sum_{i \in S} p_i$ at each $x \in X^*$. Thus, by definition, $\sum_{i \in S} p_i \geq\ w^\gamma(S)$ for each $S \subset N$, and the payoff vector $(p_1, \ldots, p_n)$ is in the core of the characteristic function game w^γ. This proves that the core of the characteristic function game w^γ is equal to the γ-core of the extensive game Γ. ■

Proposition 1 has important consequences. There is a vast literature on games in characteristic function form. Proposition 1 implies that the concepts and ideas from that literature can be applied to games in extensive form.[19] However, the fact that the γ-characteristic function is generated by an extensive game makes possible additional analysis. For instance, what is the relationship between the γ-core of an extensive game Γ and the γ-cores of the corresponding family of strategic games $\Omega_x, x \in X^*$? It was noted above that the former is a refinement of the γ-core of the strategic game $\Omega_0 (= \Omega)$. We now make that more precise.

Notice that the family of strategic games $\Omega_x, x \in X^*$, includes at least one game with n players, namely, Ω (=Ω_0). But it may include more. In fact, all games in the family may be games with n players. For example, in a repeated strategic game or the games in the three applications introduced below.

Proposition 2 The γ-core of an extensive game Γ with n players is a subset of the intersection of the γ-cores of the strategic games with n players in the corresponding family of games Ω_x, $x \in X^*$. If all games in the family have n players then it is equal to the intersection and the γ-core of Γ is non-empty if and only if the γ-core of each subgame $\Gamma_x, x \in X^*$, is non-empty.

Proof: Let $(p_1, \ldots, p_n)$ be a payoff vector in the γ-core of Γ. Then, by definition, for each coalition $S \subset N,\ w^\gamma(S) \leq \sum_{i \in S} p_i$ and $w^\gamma(S; x) \leq w^\gamma(S)$ for all $x \in X^*$. Therefore, for

[19] One immediate implication is that an extensive game admits a non-empty γ-core if and only if the derived characteristic function game w^γ is balanced.

each $x \in X^*$, $w^\gamma(S;x) \leq \sum_{i \in S} p_i$ for all $S \subset N$. Furthermore, let $\Omega_x, x \in X^*$, be a game with n players, then x is a decision node at which coalition N is active. Therefore, $w^\gamma(N;x) = w^\gamma(N) = \sum_{i \in N} p_i$. This shows that $(p_1, \dots, p_n)$ belongs to the γ-core of each strategic game with n players in the family of strategic games $\Omega_x, x \in X^*$. Conversely, suppose contrary to the assertion that a payoff vector $(p_1, \dots, p_n)$ is in the γ-core of Γ, but not in the γ-core of some strategic game $\Omega_x, x \in X^*$, with n players. Then, $w^\gamma(S;x) > \sum_{i \in S} p_i$ for some $S \subset N$. But that implies $w^\gamma(S) > \sum_{i \in S} p_i$ for some $S \subset N$, since by definition $w^\gamma(S;x) \leq w^\gamma(S)$ for all $x \in X^*$. Thus, our supposition is wrong and the γ-core of Γ is a subset of the intersection of the γ-cores of the strategic games with n players in the family of strategic games Ω_x, $x \in X^*$.

If $(p_1, \dots, p_n)$ is a payoff vector in the γ-core of Γ, then, by definition, it must satisfy the constraints $w^\gamma(S;x) \leq \sum_{i \in S} p_i$ also at all decision nodes $x \in X^*$ such that not all n players are active at x. Therefore, the set of payoff vectors in the γ-core of Γ is a subset of the intersection of the γ-cores of the strategic games with n players in the family $\Omega_x, x \in X^*$, but equal to the intersection if no game in the family has fewer than n players.

If all games in the family Ω_x, $x \in X^*$ have n players, then the set of decision nodes along any history generated by any strategy profile that maximizes the payoff of coalition N in a subgame $\Gamma_{\bar{x}}, \bar{x} \in X^*$, is a subset of the set X^*. Therefore, the γ-core of the subgame $\Gamma_{\bar{x}}$ is equal the intersection of the γ-cores of a subset of games in the family of games $\Omega_x, x \in X^*$. If this intersection is empty, then the intersection of all strategic games in the family $\Omega_x, x \in X^*$ is also empty. Therefore, the γ-core of Γ is non-empty, only if the γ-core of each subgame $\Gamma_{\bar{x}}, \bar{x} \in X^*$, is non-empty. Conversely, since $\Gamma = \Gamma_0$ and $0 \in X^*$, γ-core of Γ is non-empty if the γ-cores of all subgames $\Gamma_x, x \in X^*$, are non-empty. ■

For each coalition $S \subset N$, the functions $w^\gamma(S;x)$, $x \in X^*$, determine the lower bounds on the γ-characteristic function $w^\gamma(S)$. Since these lower bounds may be attained for different coalitions at different decision nodes x, the γ-characteristic function $w^\gamma(S)$ may not inherit all properties of the functions $w^\gamma(S;x)$ unless the bounds are all attained at the same decision node

$x \in X^*$.[20] Indeed, if the family of strategic games Ω_x, $x \in X^*$, includes a game with n players, say Ω_{x^*} such that for each $S \subset N$, $w^\gamma(S) = w^\gamma(S; x^*)$, then the extensive game Γ and the strategic game Ω_{x^*} have the same properties, since the corresponding characteristic functions of the two games are equal. If $x^* = 0$, the γ-cores of the extensive game Γ and the strategic game Ω $(= \Omega_0)$ are equal and the γ-core of the strategic game does not go through any refinement as the game unfolds along the nodes in the set $x \in X^*$. But if $x^* \neq 0$, the γ-core of the extensive game Γ is smaller than the γ-core of the strategic game Ω.

Ray (1989) shows that the standard core is credible in the sense that if one limits coalitional deviations to those deviations that are immune to further deviations by subcoalitions, the core is unaffected. The γ-core of an extensive game satisfies the same property, i.e., the γ-core of an extensive game is the same whether or not one requires coalitional deviations to be immune to further coalitional deviations.

We need the following definitions. Given a characteristic function game w, deviations by singleton coalitions are credible as no further deviations by subcoalitions are possible. Proceeding recursively, a deviation $(p_1, \dots, p_n)$ by a coalition S of size k, $1 \leq k \leq n$ is credible if $\sum_{i \in S} p_i = w(S)$ and there is no proper subcoalition $S' \subset S$ and a credible deviation q by S' such that $\sum_{i \in S'} q_i = w(S')$ and $q_i > p_i$ for all $i \in S'$.

The core of a characteristic function game w is *credible* if for each core payoff vector p, there is no coalition S and a credible deviation q by S such that $\sum_{i \in S} q_i = w(S)$ and $q_i > p_i$ for all $i \in S$.

Proposition 3 The γ-core of an extensive game Γ is credible.

Proof: In view of Proposition 1, we need to only show that the core of the corresponding game in characteristic function form w^γ is credible. If coalitions can choose only credible deviations, then the so-defined core cannot be smaller than the usual core of the characteristic function game w^γ. Suppose, contrary to the assertion, that it is larger and not identical. Accordingly, suppose

[20] However, that is not peculiar to the γ-core of an extensive game. The same is also true for the SPNE of an extensive game which may not have the same properties as the SPNE of proper subgames unless the extensive game has additional structure in place.

that a payoff vector q belongs to the larger core, but not to the usual core of the characteristic function game w^γ. Thus, there is a coalition $S \subset N$ and a deviation p which is not credible and such that $\sum_{i\in S} p_i = w^\gamma(S)$ and $p_i > q_i$ for all $i \in S$. Since p is not credible, there is a coalition $S' \subset S$ and a credible deviation p' by coalition S' such that $\sum_{i\in S'} p'_i = w^\gamma(S')$ and $p'_i > p_i$ for all $i \in S'$. But that implies there is a coalition $S' \subset N$ and a credible deviation p' by S' such that $\sum_{i\in S'} p'_i = w^\gamma(S')$ and $p'_i > q_i$ which contradicts our supposition that q belongs to the larger core. This proves the proposition. ■

3.5 *An alternative equivalent definition and a weaker γ-core concept*

To motivate the weaker notion of the γ-core, we introduce first an alternative, but equivalent definition of the γ-core of an extensive game.

For each $z^* \in Z^*$, let $w^\gamma_{z^*}(S) = \max_x w^\gamma\,(S; x), S \subset N$, where the maximum is taken over all nodes $x \in X(z^*)$ at which S is active. Since the origin of the game $0 \in X(z^*)$, each coalition S is active at least at some $x \in X(z^*)$, and $w^\gamma_{z^*}(N) = u^N_N(z^*)$. We shall refer to the function $w^\gamma_{z^*}(S), S \subset N$, the γ-characteristic function corresponding to the terminal node z^*, and the core of the characteristic function game $w^\gamma_{z^*}$ as the γ-core corresponding to the terminal node $z^* \in Z^*$. Since $w^\gamma(S) = \max_{z^*} w^\gamma_{z^*}(S)$ where the maximum is taken over all $z^* \in Z^*$, the γ-core of an extensive game in terms of Definition 1 is equal to the intersection of the γ-cores corresponding to the terminal nodes $z^* \in Z^*$.

This equivalence suggests an additional interpretation of the γ-core of an extensive game, namely that the γ-core is a refinement of the set of γ-cores corresponding to the terminal nodes with highest payoff for the grand coalition. However, this refinement, like most others in game theory, though intuitive leads to a concept which in a sense is "too strong", since it implies that the γ-core of an extensive game is empty if the γ-core corresponding to *some* terminal node with highest payoff for the grand coalition is empty.[21] Since we regard the γ-core as the rule for the distribution of gains from coalitional choices, we assume that the grand coalition will not choose a strategy that leads to a terminal node for which the corresponding γ-core is empty; thus, we

[21] Many selection procedures in game theory are motivated by intuitive criteria. However, they can sometimes lead to an empty solution set even though the game has a natural solution. E.g. a strategic game with a unique Nash equilibrium may have no strong Nash equilibrium.

restrict the intersection only to those γ-cores corresponding to terminal nodes which are non-empty. We shall refer to the so-defined weaker notion as the *weak* γ-core of an extensive game. In most applications there is no difference between the two notions because either the set of terminal nodes Z^* with highest payoff for the grand coalition is a singleton or the γ-core corresponding to each terminal node $z^* \in Z^*$ is non-empty.[22] However, in some instances in which the γ-core is empty, the weak γ-core may be non-empty.

It is easily verified that the weak γ-core satisfies the same properties as in propositions 1, 2, and 3. However, it is not really an alternative core concept. It is, in fact, a complementary concept which differs and can be useful *only* in extensive games in which the γ-core is empty.

3.6 *Strong subgame-perfect Nash equilibrium and the γ-core*

Rubinstein (1980) introduces a notion of "strong perfect equilibrium" for a super game in which a strategic game is played infinitely many times. Using the conceptual framework developed in the present paper, we introduce a notion of strong subgame-perfect Nash equilibrium (SSPNE) of a general extensive game.[23] We need some additional notation.

Given game Γ in extensive form, we denote, with some notational inconsistency, the corresponding strategic game Ω by $[\{u_i\}_{i\in N}, \{T_i\}_{i\in N}]$, where T_i is the strategy set of player i, $T = T_1 \times \cdots \times T_n$ is the set of strategy profiles, $t = (t_1, \dots, t_n) \in T$ is a strategy profile, and $u_i(t_1, \dots, t_n)$ is the payoff function of player i.[24] Given $t = (t_1, \dots, t_n) \in T$, let $t_S \equiv (t_i)_{i\in S}, t_{-S} \equiv (t_j)_{j\in N\setminus S}$, and $(t_S, t_{-S}) \equiv t = (t_1, \dots, t_n)$. Similarly, let $T_S \equiv \times_{i\in S} T_i$ and $T_{-S} \equiv \times_{i\in N\setminus S} T_i$. Finally, for each $\bar{t}_S \in T_S$, let $\Gamma/\bar{t}_{-S}$ denote the game restricted to the players in S by the strategies $\bar{t}_{-S}$ for the players in $N\setminus S$ and let $\Omega/\bar{t}_{-S} = [\{\bar{u}_i\}_{i\in S}, \{T_i\}_{i\in S}]$, where $\bar{u}_i(t_S) = u_i(t_S, \bar{t}_{-S})$ for all $i \in S$ and $t_S \in T_S$, denote the corresponding restricted strategic game.

[22] An extensive game admits a unique SPNE if no player has the same payoff at two terminal nodes whereas it admits a unique γ-core if the grand coalition does not have the same payoff at two terminal nodes.
[23] Aumann (1959) introduces the notion of strong Nash equilibrium of a *strategic* game.
[24] In terms of the earlier notation, $u_i(t_1, \dots, t_n) = u_i(z)$ where z is the terminal node generated by the strategy profile $(t_1, \dots, t_n)$.

Definition 2 Given an extensive game Γ, a strategy profile $\bar{t} = (\bar{t}_1, \dots, \bar{t}_n) \in T$ is a strong subgame perfect Nash equilibrium of Γ, if $(\bar{t}_S, \bar{t}_{-S}) = \bar{t}$ is a subgame perfect Nash equilibrium of every induced game Γ^S, $S \subset N$.

Unlike the γ-core of an extensive game, a SSPNE requires the *same strategy* $\bar{t}$ to be a SPNE of every induced game. Notice that for each coalition $S \subset N$, if $\bar{t} = (\bar{t}_S, \bar{t}_{-S})$ is a SSPNE of an extensive game Γ, then it is also a SSPNE of every restricted game $\Gamma/\bar{t}_{-S}$. This suggests the following recursive but equivalent definition of SSPNE which is sometimes more convenient to use.

Definition 3 (1) In a single player extensive game Γ, $\bar{t} \in T$ is a SSPNE if and only if $\bar{t}$ is a SPNE of Γ. (2) Let $n > 1$ and assume that SSPNE has been defined for extensive games with fewer than n players. For any extensive game Γ with n players, $\bar{t} \in T$ is a SSPNE of Γ if for all proper subsets $S \subset N$, $\bar{t}_S$ is a SSPNE of the restricted game $\Gamma/\bar{t}_{-S}$ and if there does not exist a strategy $t \in T$ such that $\sum_{i \in N} u_i(t) > \sum_{i \in N} u_i(\bar{t})$.

Proposition 4 In the class of extensive games Γ such that each induced game $\Gamma^S, S \subset N$, admits a unique SPNE, if a game admits a SSPNE, then the SSPNE is unique and the set of γ-core payoff vectors is equal to the unique SSPNE payoff vector. But if the game admits no SSPNE, the γ-core of the game may still be non-empty.

Proof: If Γ admits a SSPNE, then it must be unique. That is because if not, then some induced games Γ^S must admit more than one SPNE, which contradicts our supposition that each induced game Γ^S admits a unique SPNE. Therefore, let $\bar{t} \in T$ denote the unique SSPNE. Then, $\bar{t} = (\bar{t}_S, \bar{t}_{-S})$ is a unique SPNE of every induced game Γ^S, $S \subset N$. and therefore it also induces a unique SPNE in each induced game $\Gamma^S_x, S \subset N, x \in X$.

Since, by supposition, Γ^N admits a unique SPNE, the terminal node with highest payoff for coalition N is unique. Since $\bar{t}$ is the unique SPNE of every induced game $\Gamma^S, S \subset N$, the SPNE of each induced game $\Gamma^S, S \subset N$, generates a history which is identical to the history leading to the terminal node with the highest payoff for coalition N. Let X^* denote the set of decision nodes along the history leading to the terminal node with the highest payoff for N. Then, for each $x \in X^*$, $w^\gamma(S; x) = w^\gamma(S; 0) = \sum_{i \in S} u_i(\bar{t}), S \subset N$, since X^* is the set of decision nodes along the history generated by the unique SPNE of $\Gamma^S, S \subset N$. Thus, $w^\gamma(S) = w^\gamma(S; 0), S \subset N$. Therefore, if $(p_1, \dots, p_n)$ is a γ-core payoff vector, then it must satisfy $\sum_{i \in N} p_i = w^\gamma(N) = \sum_{i \in N} u_i(\bar{t})$ and $\sum_{i \in S} p_i \geq w^\gamma(S) = \sum_{i \in S} u_i(\bar{t})$. Which implies that the SSPNE payoff vector $(u_1(\bar{t}), \dots, u_n(\bar{t}))$ is the unique γ-core payoff vector. This proves the first part of the proposition.

For the second part of the proposition, note that the extensive game in Example 1 belongs to the class of games such that every induced game has a unique SPNE. It does not admit a SSPNE, but, as noted earlier, the γ-core of this game is non-empty. ■

Proposition 4 implies that in the class of extensive games in which each induced game admits a unique SPNE, the γ-core of an extensive game is a weaker concept than SSPNE. Proof of the proposition also illustrates the point made earlier that the γ-characteristic function $w^\gamma(S)$ and the functions $w^\gamma(S; x), x \in X$, may be closely related if the extensive game has additional structure.

Any extensive game which admits a unique SPNE and such that the induced games also have the same structure as the original game belongs to the class of games in Proposition 4. Only in a subset of these games the SPNE may be strong, and the γ-core payoffs may be equal to the SSPNE payoffs. But the subset of games which admit nonempty γ-cores is larger.[25] A more general comparison between SSPNE and the γ-core does not seem possible, but the equivalence between the SSPNE and the γ-core payoffs is not restricted to the class of games in Proposition 4 alone. It can be shown to hold in any extensive game such that the payoff of each coalition $S \subset N$ is highest under the same SPNE strategy $\bar{t}$ than under any other SPNE of the induced game Γ^S. We consider two applications of the proposition.

Example 2 *Two-player bargaining game of alternating offers* (Rubinstein, 1982): Two players, 1 and 2, bargain to split 1 dollar. The rules are as follows: The game denoted by Γ begins in period 1 in which player 1 makes an offer of a split (a real number between 0 and 1) to player 2, which player 2 either accepts or rejects. Acceptance by player 2 ends the game and the proposed split is immediately implemented. If player 2 rejects, nothing happens until period 2. In period 2, the

[25] The game in Example 1 belongs to the class of games in Proposition 4 and admits a nonempty γ-core, but no SSPNE. Similarly the dynamic game in Section 4 belongs to the class and, as will be shown, admits a nonempty γ-core, but no SSPNE.

players' roles are reversed with player 2 making an offer of split to player 1 and player 1 then accepting or rejecting it. The bargaining can potentially go on forever. If that indeed happens, both players get zero. Each player i "discounts" the future using the discount factor $\delta_i \in (0\ 1)$. That is, a dollar received by player i in period t is worth only δ_i^{t-1} in period 1 dollars.

As is well-known, this game has a unique SPNE, in which

- Player 1 always offers p^* and accepts an offer if and only if $q_1 \geq q_1^*$
- Player 2 always offers q^* and accepts a proposal p if and only if $p_2 \geq p_2^*$,

where

$$p^* = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$$

$$q^* = \left(\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}, \frac{1-\delta_1}{1-\delta_1\delta_2}\right).$$

The outcome of this equilibrium strategy profile is that player 1 offers p^* at the start of the game, and player 2 immediately accepts this proposal. Therefore, p^* is a unique SPNE payoff vector of game Γ and since there are only two players the unique SPNE of Γ is also a unique SPNE of both the induced games $\Gamma^{\{1\}}$ and $\Gamma^{\{2\}}$. Observe that $p_1^* + p_2^* = 1$ and thus the unique SPNE is actually a unique SSPNE of Γ. Therefore, by Proposition 4, p^* is a unique γ-core payoff vector of the extensive game Γ.

The example makes use of the fact that there are only two players and therefore the unique SPNE of the game is also a unique SPNE of the two induced games. We now extend this equivalence between the γ-core payoff vectors and the equilibrium payoffs in two-player bargaining game to a three player bargaining game by assuming that the players are identical and can use only stationary strategies.

Example 3 *Three-player bargaining game of alternating offers*: The players are identical, i.e., the discount factors $\delta_1 = \delta_2 = \delta_3 = \delta$. An offer $p_i = (p_{i1}.p_{i2}, p_{i3})$ by player i in period t is first considered by player $i + 1$ (mod 3), who may accept or reject it. If he accepts it, then player $i + 2$ (mod) may accept or reject it. If both accept it, then the game ends and p_i is implemented. Otherwise, player $i + 1$ (mod 3) makes the next offer, in period $t + 1$.

Each player can only use stationary strategies, in which he makes the same offer whenever it is his turn to make an offer and uses the same criterion to accept or reject offers whenever he is the responder. We denote this three-player bargaining game by Γ.

As in the two-player bargaining game, it is easily seen that the three-player bargaining game Γ has a unique SPNE in stationary strategies in which

- Player i always offers p_i^* and accepts a proposal p_{i+1} or p_{i+2} if and only if $p_{(i+1)i} \geq p^*_{(i+1)i}$ or $p_{(i+2)i} \geq p^*_{(i+2)i}$, where

$$p_1^* = \left(\frac{1}{1+\delta+\delta^2}, \frac{\delta}{1+\delta+\delta^2}, \frac{\delta^2}{1+\delta+\delta^2}\right)$$

$$p_2^* = \left(\frac{\delta^2}{1+\delta+\delta^2}, \frac{1}{1+\delta+\delta^2}, \frac{\delta}{1+\delta+\delta^2}\right)$$

$$p_3^* = \left(\frac{\delta}{1+\delta+\delta^2}, \frac{\delta^2}{1+\delta+\delta^2}, \frac{1}{1+\delta+\delta^2}\right).$$

The outcome of this equilibrium strategy is that player 1 offers p_1^* and players 2 and 3 immediately accept it. To verify that the unique SPNE is actually a SSPNE, we need to check that given the strategy of one of the players, say 3, the other two players cannot obtain higher payoffs. Now given the strategy of player 3, unless the stationary strategies p_1 and p_2 of players 1 and 2 offer at least $\frac{\delta^2}{1+\delta+\delta^2}$ and $\frac{\delta}{1+\delta+\delta^2}$, respectively, to player 3, the game will continue forever resulting in payoff of 0 to each. This means that given the strategy of player 3, players 1 and 2 together cannot obtain payoffs which are higher than their current equilibrium payoffs. Hence the unique SPNE of Γ is a SSPNE and the γ-core of the extensive game consists of the unique SPNE payoff vector.

3.6 *Coalition-proof subgame-perfect equilibrium and the γ-core*

The concepts of γ-core and strong Nash equilibrium of an extensive game introduced above assume implicitly that players can make binding commitments. In contrast, Bernheim, Peleg, and Whinston (1987) are concerned with cooperation in situations in which the players cannot make binding agreements. They propose a concept in which the equilibrium strategy profile is immune to "self-enforcing" coalitional deviations. Imposing the same requirement on deviations leads to an alternative version of the γ-core which is self-enforcing and subgame perfect. We shall refer to it as the self-enforcing γ-core of an extensive game. We first note the definition of CPSPNE.

Definition 4 (1) In a single player extensive game Γ, $\bar{t} \in T$ is a CPSPNE if and only if $\bar{t}$ is a SPNE of Γ. (2) Let $n > 1$ and assume that CPSPNE has been defined for extensive games with fewer than n players. For any extensive game Γ with n players, $\bar{t} \in T$ is *self-enforcing* if for all proper subsets $S \subset N$, $\bar{t}_S$ is a CPSPNE of the restricted game $\Gamma/\bar{t}_{-S}$. For any extensive game Γ with n players, $\bar{t} \in T$ is a CPSPNE if it is self-enforcing and if there does not exist another self-enforcing $t \in T$ such that $\sum_{i \in N} u_i(t\,) > \sum_{i \in N} u_i(\bar{t}\,)$. [26]

Observe that if $\bar{t}$ is a CPSPNE of Γ, then $\bar{t}$ is a SPNE of Γ and for every $S \subset N$, $\bar{t}_S$ is a CPSPNE of the restricted extensive game $\Gamma/\bar{t}_{-S}$. We show that, as in strategic games, a SSPNE of an extensive game is a CPSPNE.[27]

Proposition 5 Every SSPNE of an extensive game Γ is a CPSPNE of Γ.

Proof: The proof is by induction. The proposition is true for games with a single player. Suppose the proposition is true for games with k players $1 \leq k < n$. We show that then it is also true for games with $k + 1$ players. Let $\bar{t} \in T$ be a SSPNE of a game Γ with $k + 1$ players. Then, as noted earlier, $\bar{t}_S$ is a SSPNE of $\Gamma/\bar{t}_{-S}$ for each proper subset S of players, and , by definition, there is no $t \in T$ such that $\sum_{i=1}^{k+1} u_i(t) > \sum_{i=1}^{k+1} u_i(\bar{t})$. Since the proposition is true for games with k players, $\bar{t}_S$ is a CPSPNE of $\Gamma/\bar{t}_{-S}$ for each proper subset S of players. Furthermore, there is no self-enforcing $t \in T$ such that $\sum_{i=1}^{k+1} u_i(t) > \sum_{i=1}^{k+1} u_i(\bar{t})$. Hence, $\bar{t} \in T$ is a CPSPNE of game Γ with $k + 1$ players. ■

Given an extensive game Γ and a coalition $S \subset N$, a strategy $\bar{t} = (\bar{t}_S, \bar{t}_{-S}) \in T$ is a *self-enforcing* SPNE of the induced game Γ^S if $(\bar{t}_S, \bar{t}_{-S})$ is a SPNE of Γ and $\bar{t}_S$ is a CPSPNE of the restricted game $\Gamma/\bar{t}_{-S}$. We can similarly define a self-enforcing SPNE for each of the induced games $\Gamma_x^S, x \in X$ and S an active coalition at x. Let $v^\gamma(S; x)$ denote the self-enforcing SPNE payoff of coalition S in the induced game Γ_x^S in which S is active. If a induced game Γ_x^S admits

[26] This definition, which uses the conceptual framework of our paper, is equivalent to the original definition of CPSPNE in Bernheim, Peleg, and Whinston (1987, p. 10). But it is, we believe, less cumbersome.
[27] The converse is obviously not true; the CPSPNE of the game in Example 1 is clearly not a SSPNE.

more than one self-enforcing SPNE, then any one with highest payoff for the coalition is selected to define $v^{\gamma}(S;x)$.

Definition 5 The *self-enforcing* γ-core of an extensive game Γ is the set of all payoff vectors $(p_1, \dots, p_n)$ such that for each coalition $S \subset N$, $v^{\gamma}(S;x) \leq \sum_{i \in S} p_i$ for all decision nodes x along the histories generated by the strategy profiles leading to the terminal nodes for which the payoff vector $(p_1, \dots, p_n)$ is feasible.

Observe that a self-enforcing SPNE of the induced game Γ^N is, by definition, a CPSPNE of Γ. Thus, requiring coalitional deviations to be self-enforcing results in a γ-core concept which, by definition, presumes that the game admits a CPSPNE. Let $\bar{t} \in T$ be a CPSPNE of Γ. Then, the self-enforcing γ-core of the extensive game Γ must be a subset of the set of feasible payoff vectors $(p_1, \dots, p_n)$ such that $\sum_{i \in N} p_i = \sum_{i \in N} u_i\,(\bar{t})$. That is so because the origin of the extensive game Γ is a decision node along every history of the game and there is no other feasible payoff vector that is immune to self-enforcing deviations by coalition N, which is active at least at the origin. In addition, the self-enforcing γ-core must take into account opportunities for higher coalitional payoffs along the histories generated by CPSPNE. Thus, the concept of self-enforcing γ-core is "rather strong" and its existence is restricted to games which admit a CPSPNE. We identify a class of games in which the concepts of self-enforcing γ-core and CPSPNE are equivalent.

Proposition 6 In the class of extensive games Γ such that every induced game $\Gamma^S, S \subset N$, admits a unique self-enforcing SPNE, if a game admits a CPSPNE, then it is unique and the set of self-enforcing γ-core payoff vectors is equal to the unique CPSPNE payoff vector.

<u>Proof</u>: As in the proof of Proposition 4, if Γ admits a CPSPNE, then it must be unique. Let $\bar{t} \in T$ denote the unique CPSPNE. Then, $\bar{t} = (\bar{t}_S, \bar{t}_{-S})$ is a unique self-enforcing CPSPNE of every induced game $\Gamma^S, S \subset N$. Since, by supposition, Γ^N admits a self-enforcing SPNE which is unique, the terminal node with highest self-enforcing SPNE payoff for coalition N is unique. Since $\bar{t}$ is the unique self-enforcing SPNE of every induced game $\Gamma^S, S \subset N$, the self-enforcing SPNE of each induced game $\Gamma^S, S \subset N$, generates a history which is identical to the history leading to the terminal node with the highest self-enforcing payoff for coalition N. Let X^* denote the set of decision nodes along the history leading to the terminal node with the highest self-

enforcing payoff for N. Then, for each $x \in X^*$, $v^\gamma(S;x) = v^\gamma(S;0) = \sum_{i\in S} u_i(\bar{t}\,), S \subset N$, since X^* is the set of decision nodes along the history generated by the unique self-enforcing SPNE of $\Gamma^S, S \subset N$. Thus, $v^\gamma(S) = v^\gamma(S;0), S \subset N$, and the CPSPNE payoff vector $(u_1(\bar{t}\,), \dots, u_n(\bar{t}\,))$ is the unique self-enforcing γ-core payoff vector. ■

Since, as shown, a SSPNE is a CPSPNE, the subset of games in Proposition 4 which admit a (unique) SSPNE also belongs to the class of games in Proposition 6 and the two cores for this subset of games are equal and equivalent to the CPSPNE/SSPNE. But since a CPSPNE is not necessarily a SSPNE, the subset of games in Proposition 6 for which the self-enforcing γ-core is equivalent to the CPSPNE is larger.[28] Since the payoff of the grand coalition in a CPSPNE may be lower, the two cores are disjoint unless the payoff of the grand coalition in the CPSPNE is equal to that in the SSPNE. In the latter case the self-enforcing γ-core may be larger than the γ-core, since the payoffs that deviating coalitions can achieve may be lower if they have to follow self-enforcing strategies. A more general comparison between the two core concepts does not seem possible. Which of them is relevant depends on whether or not the players can write binding agreements.

In Example 1, the self-enforcing γ-core consists of the unique CPSPNE payoff vector $(p_1, p_2) = (2, 1)$ and disjoint with the γ-core. Since, as shown, a SSPNE of an extensive game is also a CPSPNE, examples 2 and 3 also can be viewed as applications of the self-enforcing γ-core. However, the dynamic games in examples 2 and 3 end “too soon” and the full force of the γ-core of an extensive game does not really come into play. Thus, we consider next a dynamic game of global public good provision in which it does.[29]

4. A dynamic game of global public good provision

There are n countries, indexed by $i = 1, \dots, n$. Time is treated as discrete and indexed $t = 1, \dots, T$, where T is finite. The variables $x_{it} \geq 0$ and $y_{it} \geq 0$ denote the consumption and production, respectively, of a composite private good in country i at time t. Similarly $e_{it} \geq 0$ and

[28] The dynamic game in Section 4 belongs to the class of games in Proposition 6 (as well as in Proposition 4) and admits no SSPNE, but, as will be shown, the self-enforcing γ-core of this game is equivalent to the CPSPNE.
[29] Dynamic games of global public good provision have been studied previously by Biran and Forges (2010), Dutta and Radner (2005), and Dockner and Long (1993) among others. Reinganum and Stokey (1985) study a dynamic game of resource extraction with a similar structure.

$z_t \geq 0$ denote, respectively, the level of emissions and the amount of ambient pollution at time t. While x_{it}, y_{it}, and e_{it} are flow variables, z_t is a stock variable as formally defined below.

Production and utility at time *t* are specified as $y_{it} = g_i(e_{it})$ and $u_i(x_{it}, z_t) = x_{it} - v_i(z_t)$, respectively. The function $g_i(e_{it})$ is the production function and $v_i(z_t)$ is the damage function. A consumption stream $(x_{1t}, \dots, x_{nt}; z_t)_{t=1}^{T}$ is *feasible* for an emission profile $(e_{1t,} \dots, e_{nt})_{t=1}^{T}$ if for every $t = 1, \dots, T$,

$$\sum_{i \in N} x_{it} = \sum_{i \in N} g_i(e_{it}) \quad (1)$$

$$z_t = (1 - \delta) z_{t-1} + \sum_{i \in N} e_{it}, \;\; z_0 > 0 \text{ given.} \quad (2)$$

Here $0 \leq \delta < 1$ is the natural rate of decay of the stock z_t. Notice that transfers of the composite private good are allowed across the countries in each period t, but not across the periods. Given the quasi-linearity of the utility functions $u_i(x_{it}, z_t)$, this is not really an assumption as there is no gain from postponing consumption and there is no possibility of borrowing against future consumption. A feasible consumption vector$(x_{1t}, \dots, x_{nt}; z_t)_{t=1}^{T}$ from an emission profile $(e_{1t,} \dots, e_{nt})_{t=1}^{T}$ uniquely determines the aggregate utility $\sum_{t=1}^{T} \beta^{t-1} u_i(x_{it}, z_t) = \sum_{t=1}^{T} \beta^{t-1} [x_{it} - v_i(z_t)]$ of each country i where $0 < \beta \leq 1$ is the discount factor.

In the optimal control literature, the emissions $e_i = (e_{it})_{t=1}^{T}$ are called *control variables* and the resulting stocks $z_t, t = 1, \dots, T$, are the *state variables*. While the latter are not strategies in the dynamic game, they are induced by the former and appear in the payoff functions. In game theoretic terms, z_t is a decision node of the dynamic game and $e_i = (e_{it})_{i=1}^{T}$ is a strategy of player i.

In what follows the production functions, $g_i(e_{it})$ are assumed to be strictly increasing and strictly concave, and the damage functions, $v_i(z_t)$, strictly increasing and convex. In addition we assume that there exists an $e^0 > 0$ such that $g_i'(e^0) < v_i'(e^0)$ for each $i \in N$, $g_i'(e_i) \to \infty$ as $e_i \to 0$, and $v_i'(z_0 + ne^0 T) < \infty$. These assumptions ensure that the emissions e_{it} chosen by each country i are such that $0 \leq e_{it} \leq e^0, t = 1, \dots, T$.

Given $z_0 > 0$ and $T > 1$, the dynamic game of global public good provision, to be denoted by $\Gamma \equiv \Gamma_{z_0}$, or alternatively the corresponding game is strategic form, to be denoted by $\Omega \equiv \Omega_{z_0}$, is the strategic game (N, E, u) where

- $N = \{i = 1, 2, \ldots, n\}$ is the set of players,
- $E = E_1 \times E_2 \times \ldots \times E_n$ is the set of joint strategies and $E_i = \{e_i \equiv (e_{it})_{t=1}^{T}: 0 \leq e_{it} \leq e^0\}$ is the set of strategies of player i.
- $u = (u_1, \ldots, u_n)$ is the vector of payoff functions such that for each $e = ((e_{1t})_{t=1}^{T}, \ldots, (e_{nt})_{t=1}^{T}) \in E,\ u_i(e) = \sum_{t=1}^{T} \beta^{t-1}[g_i(e_{it}) - v_i\,(z_t)]$, where $z_t = (1-\delta)z_{t-1} + \sum_{j \in N} e_{jt}\,, t = 1, \ldots, T$.

A subgame of the dynamic game is denoted by $\Gamma_{z_{t-1}}, t = 1, \ldots T$, and the corresponding game in strategic form by $\Omega_{z_{t-1}} = (N, E^t, u^t)$ where

- $N = \{i = 1, 2, \ldots, n\}$ is the set of players,
- $E^t = E_1^t \times \cdots \times E_n^t$ is the set of joint strategies and $E_i^t = \{e_i^t \equiv (e_{i\tau})_{\tau=t}^{T}: 0 \leq e_{i\tau} \leq e^0\}$ is the set of strategies of player i.
- $u^t = (u_1^t, \ldots, u_n^t)$ is the vector of payoff functions such that for each $e^t = ((e_{1\tau})_{\tau=t}^{T}, \ldots, (e_{n\tau})_{\tau=t}^{T}) \in E^t, u_i^t(e^t) = \sum_{\tau=t}^{T} \beta^{\tau-1}[g_i(e_{i\tau}) - v_i\,(z_\tau)]$, where $z_\tau = (1-\delta)z_{\tau-1} + \sum_{j \in N} e_{j\tau}\,, \tau = t, \ldots, T$.

Notice that the subgame $\Gamma_{z_{t-1}}, t = 1, \ldots T$, depends only on z_{t-1} and not on how the game reached the point z_{t-1}. The "statistic" z_{t-1} summarizes all that has happened before the game reaches the point z_{t-1}.

By definition, a SPNE of the induced game Γ^N leads to a terminal node with highest payoff for coalition N. Therefore, it is a solution of the optimization problem

$$\max_{((e_{it})_{t=1}^{T})_{i=1}^{n}} \left[\, \sum_{t=1}^{T} \beta^{t-1} \sum_{i=1}^{n} (g_i(e_{it}) - v_i(z_t))\right] \tag{3}$$

subject to $z_t = (1-\delta)z_{t-1} + \sum_{i \in N} e_{it}, t = 1, \ldots, T,\ z_0 > 0$ given.

Lemma Optimization problem (3) has a unique solution $((e^*_{1t})^T_{t=1}, \dots, (e^*_{nt})^T_{t=1})$ which is characterized by the equations $g_i^{'}(e^*_{it}) = \sum_{\tau=t}^T [\beta(1-\delta)]^{\tau-t} \sum_{j\in N} v_j^{'}(z^*_\tau)$, $z^*_t = (1-\delta)z^*_{t-1} + i\in Neit*$, $i\in N, t=1,\dots,T$, and $z0*=z0$.

Proof of the lemma is relegated to the appendix to the paper. The lemma implies that the induced game Γ^N admits a unique SPNE and the unique SPNE is characterized by a system of $(n+1)T$ equations. On closer examination, these equations also imply that $((e^*_{1\tau})^T_{\tau=t}, \dots, (e^*_{n\tau})^T_{\tau=t})$ is the unique SPNE of every induced game $\Gamma^N_{z^*_{t-1}}, t = 1, \dots, T$, as it should be.

A payoff vector $(p_1, \dots, p_n)$ = ($\sum_{t=1}^T \beta^{t-1}$ $((x_{1t} - v_1(z^*_t), \dots, \sum_{t=1}^T \beta^{t-1}$ $((x_{nt} - v_n(z^*_t))$ is feasible for the strategy profile$((e^*_{1t})^T_{t=1}, \dots, (e^*_{nt})^T_{t=1})$, if $(x_{1t}, \dots, x_{nt}; z^*_t)^T_{t=1}$ is a feasible consumption stream for $((e^*_{1t})^T_{t=1}, \dots, (e^*_{nt})^T_{t=1})$. Notice that if $(p_1, \dots, p_n)$ is a feasible payoff vector for the strategy profile $((e^*_{1t})^T_{t=1}, \dots, (e^*_{nt})^T_{t=1})$ of the extensive game Γ, then for each $t = 1, \dots, T$, $(p_{1t}, \dots, p_{nt})$ = ($\sum_{\tau=t}^T \beta^{\tau-1}$ $((x_{1\tau} - v_1(z^*_\tau), \dots, \sum_{\tau=t}^T \beta^{\tau-1}$ $((x_{n\tau} - v_n(z^*_\tau))$ is a feasible payoff vector for the strategy profile $((e^*_{1\tau})^T_{\tau=t}, \dots, (e^*_{n\tau})^T_{\tau=t})$ of the extensive game $\Gamma_{z^*_{t-1}}$. By definition, $(p_{11}, \dots, p_{n1}) = (p_1, \dots, p_n)$. [30]

Let X^* denote the set $\{z_0, z^*_1, \dots, z^*_{T-1}\}$. The γ-core of the dynamic game Γ is the set of payoff vectors $(p_1, \dots, p_n)$ which are feasible for the strategy profile$((e^*_{1t})^T_{t=1}, \dots, (e^*_{nt})^T_{t=1})$ and such that the vector $(p_{1t}, \dots, p_{nt})$ is a γ-core payoff vector of the subgame game $\Gamma_{z^*_{t-1}}, t = 1, \dots, T$. Such a payoff vector clearly meets the requirement that no coalition can be better off by deviating at any decision node along the history generated by the strategy profile for which the payoff vector $(p_1, \dots, p_n)$ is feasible.

Observe that two alternative consumption streams$(x^{'}_{1t}, \dots, x^{'}_{nt}; z^*_t)^T_{t=1}$ and $(x^{''}_{1t}, \dots, x^{''}_{nt}; z^*_t)^T_{t=1}$ which are feasible for the strategy profile $((e^*_{1t})^T_{t=1}, \dots, (e^*_{nt})^T_{t=1})$ can be such that they lead to the same total payoff summed over all periods for each player, i.e.,, $\sum_{t=1}^T \beta^{t-1}$ $((x^{'}_{1t} - v_1(z^*_t), \dots, \sum_{t=1}^T \beta^{t-1}$ $\left(\left(x^{'}_{nt} - v_n(z^*_t)\right) = \sum_{t=1}^T \beta^{t-1}\right.$ $((x^{''}_{1t} - v_1(z^*_t), \dots, \sum_{t=1}^T \beta^{t-1}$ $((x^{''}_{nt} -$

[30] If a consumption stream $(x_{1t}, \dots, x_{nt}; z^*_t)^T_{t=1}$ is feasible for $((e^*_{1t})^T_{t=1}, \dots, (e^*_{nt})^T_{t=1})$, then it is Pareto efficient and $z^*_1, \dots, z^*_T$ is the Pareto efficient sequence of ambient pollution.

$v_n(z_t^*))$, but for some coalition $S \subset N$ and $t = \bar{t}$, $(\sum_{i \in S} \sum_{\tau=\bar{t}}^{T} \beta^{\tau-1} ((x_{i\tau}^{'} - v_i(z_\tau^*) <$ $\sum_{i \in S} \sum_{\tau=\bar{t}}^{T} \beta^{\tau-1} ((x_{i\tau}^{''} - v_i(z_\tau^*)$. In words, the consumption stream $(x_{1t}^{'}, \dots, x_{nt}^{'}; z_t^*)_{t=1}^{T}$ assigns higher amounts of the private good to coalition S in the early periods, but lower amounts in the later periods though the total payoff is the same. Therefore, coalition S may deviate in later periods if the consumption stream is $(x_{1t}^{'}, \dots, x_{nt}^{'}; z_t^*)_{t=1}^{T}$ instead of $(x_{1t}^{''}, \dots, x_{nt}^{''}; z_t^*)_{t=1}^{T}$. The γ-core payoff vectors are such that no coalition will have incentive to deviate in any period.

In order to keep the derivations simple, we assume henceforth that $\beta = 1$ and $\delta = 0$. The proof for $\beta \leq 1$ and $\delta \geq 0$ is analogous.

Proposition 7 The dynamic game Γ and each induced game $\Gamma^S, S \subset N$, admit a unique SPNE if $g_i^{'''} = v_i^{'''} = 0$ for each $i = 1, \dots n$.

Proof: We show that backward induction leads to a unique SPNE. Begin with a subgame in period T. A strategy profile $(e_{1T}, \dots, e_{nT})$ is a SPNE of a subgame $\Gamma_{z_{T-1}}$ if each e_{iT} maximizes $g_i(e_{iT}) - v_i\big(z_{T-1} + \sum_{j \in N} e_{jT}\big)$, given $e_{jT}, j \neq i$. Therefore, by the first order conditions for optimization,

$$g_i^{'}(e_{iT}) = v_i^{'}\big(z_{T-1} + \textstyle\sum_{j \in N} e_{jT}\big), i = 1, \dots, n. \qquad (4)$$

We claim these equations have a unique solution. Suppose not and let $(\bar{e}_{1T}, \dots, \bar{e}_{nT})$ and $(\bar{\bar{e}}_{1T}, \dots \bar{\bar{e}}_{nT})$ be two different solutions such that $\sum_{i \in N} \bar{e}_{iT} = (>) \sum_{i \in N} \bar{\bar{e}}_{iT}$. Then, since each v_i is convex and g_i is strictly concave, (4) implies $\bar{e}_{iT} = (<) \bar{\bar{e}}_{iT}$ for $i = 1, \dots, n$, which contradicts our supposition. Hence, $\Gamma_{z_{T-1}}$ admits a unique SPNE. Let $(e_{1T}(z_{T-1}), \dots, e_{nT}(z_{T-1}))$ denote the unique SPNE of $\Gamma_{z_{T-1}}$. By differentiating (4),

$$g_i^{''}\big(e_{iT}(z_{T-1})\big)e_{iT}^{'}(z_{T-1}) = v_i^{''}(z_{T-1} + \textstyle\sum_{j \in N} e_{jT}(z_{T-1}))(1 + \sum_{j \in N} e_{jT}^{'}(z_{T-1}), i = 1, \dots, n. \qquad (5)$$

Since $g_i^{''} < 0$ and $v_i^{''} \geq 0$, equations (5) imply $e_{iT}^{'}(z_{T-1}) \leq 0$ and $(1 + \sum_{j \in N} e_{jT}^{'}(z_{T-1}) \geq 0$. By differentiating (5), $g_i^{'''}\big(e_{iT}(z_{T-1})\big)(e_{iT}^{'}(z_{T-1})^2 + g_i^{''}\big(e_{iT}(z_{T-1})\big)e_{iT}^{''}(z_{T-1}) =$ $v_i^{'''}(z_{T-1} + \sum_{j \in N} e_{jT}(z_{T-1}))(1 + \sum_{j \in N} e_{jT}^{'})^2 + v_i^{''}(z_{T-1} + \sum_{j \in N} e_{jT}(z_{T-1})) \sum_{j \in N} e_{jT}^{''}(z_{T-1}), i =$ $1, \dots, n$. Therefore,

$$g_i''\big(e_{iT}(z_{T-1})\big)e_{iT}''(z_{T-1}) = v_i''(z_{T-1} + \sum_{j\in N} e_{jT}(z_{T-1}))\sum_{j\in N} e_{jT}''(z_{T-1}), i = 1, \dots, n, \qquad (6)$$

since $g_i''' = v_i''' = 0, i = 1, \dots n$. Since $g_i'' < 0$ and $v_i'' \geq 0$, equations (6) imply $e_{iT}''(z_{T-1}) = 0$.

Let $q_{iT}(z_{T-1}) \equiv g_i\big(e_{iT}(z_{T-1})\big) - v_i\big(z_{T-1} + \sum_{j\in N} e_{jT}(z_{T-1})\big), i = 1, \dots, n$. Then, $q_{iT}'(z_{T-1}) = g_i'\big(e_{iT}(z_{T-1})\big)e_{jT}'(z_{T-1}) - v_i'(z_{T-1} + \sum_{j\in N} e_{jT}(z_{T-1}))(1 + \sum_{j\in N} e_{jT}'(z_{T-1}) \leq 0$,

since $g_i' > 0, v_i' > 0$, and, as shown, $e_{jT}'(z_{T-1}) \leq 0$ and $(1 + \sum_{j\in N} e_{jT}'(z_{T-1}) \geq 0$. Furthermore,

$q_{iT}''(z_{T-1}) = g_i''\big(e_{iT}(z_{T-1})\big)(e_{iT}'(z_{T-1}))^2 - v_i''(z_{T-1} + \sum_{j\in N} e_{jT}(z_{T-1}))(1 + \sum_{j\in N} e_{jT}'(z_{T-1}))^2 +$
$g_i'\big(e_{iT}(z_{T-1})\big)e_{iT}''(z_{T-1}) - v_i'(z_{T-1} + \sum_{j\in N} e_{jT}(z_{T-1}))\sum_{j\in N} e_{jT}''(z_{T-1})$

$= g_i''\big(e_{iT}(z_{T-1})\big)(e_{iT}'(z_{T-1}))^2 - v_i''(z_{T-1} + \sum_{j\in N} e_{jT}(z_{T-1}))(1 + \sum_{j\in N} e_{jT}'(z_{T-1}))^2 \leq 0,$

since, as shown, $e_{iT}''(z_{T-1}) = 0, i = 1, \dots, n$.

Thus, each $q_{iT}(z_{T-1}), i = 1, \dots, n$, is a non-increasing concave function of z_{T-1}. Using backward induction, a strategy profile $((e_{1T-1}, e_{iT}) \dots, (e_{nT-1}, e_{nT})$ is a SPNE of the subgame $\Gamma_{z_{T-2}}$if each e_{iT-1} maximizes $g_i(e_{iT-1}) - [v_i\big(z_{T-2} + \sum_{j\in N} e_{jT-1}\big) - q_{iT}\big(z_{T-2} + \sum_{j\in N} e_{jT-1}\big)]$. Since $q_{iT}(z_{T-1}), i = 1, \dots, n$, is a non-increasing concave function of z_{T-1}, the subgame $\Gamma_{z_{T-2}}$has essentially the same structure as the game $\Gamma_{z_{T-1}}$. Therefore, $\Gamma_{z_{T-2}}$ admits a unique SPNE and the SPNE payoffs $q_{iT-1}(z_{T-2}), i = 1, \dots, n$, are similarly non-increasing and concave functions of z_{T-2}. Continuing in this manner, the backward induction leads to a unique SPNE of the extensive game Γ.

Furthermore, each induced game $\Gamma^S, S \subset N$, has essentially the same mathematical structure as the game Γ. Beginning with the game in period T, a strategy profile $(e_{1T}, \dots, e_{nT})$ is a SPNE of the subgame $\Gamma^S_{z_{T-1}}$if

$$g_i'(e_{iT}) = \sum_{j\in S} v_{jT}'\big(z_{T-1} + \sum_{j\in N} e_{jT}\big), i \in S,$$

and

$$g_j'\big(e_{jT}\big) = v_{jT}'(z_{T-1} + \sum_{i\in N} e_{iT}), j \in N\backslash S. \qquad (7)$$

Since the sum of increasing and convex functions is increasing and convex, these equations have the same essential properties as equations (4). The rest of the proof is analogous and hence omitted. ■

Since Γ admits a unique SPNE, the SPNE is a CPSPNE as there is no room for cooperation which is self-enforcing. Therefore, as Proposition 6 shows, the self-enforcing γ-core of the extensive game Γ is non-empty and consists of the unique CPSPNE payoff vector. Let $((\bar{e}_{1t})_{t=1}^{T}, \dots, (\bar{e}_{nt})_{t=1}^{T})$ denote the unique CPSPNE, and let $\bar{z}_t = \bar{z}_{t-1} + \sum_{i\in N} \bar{e}_{it}, t = 1, \dots, T$. As the lemma shows, $g_i^{'}(e_{iT}^{*}) = \sum_{j\in N} v_j^{'}(z_T^{*})\,, z_T^{*} = z_{T-1}^{*} + \sum_{i\in N} e_{iT}^{*}, i = 1, \dots, n.$ Comparing with (4) after substituting $e_{iT} = \bar{e}_{iT}$, implies that either $\bar{z}_{T-1} \neq z_{T-1}^{*}$ or $\bar{e}_{iT} \neq e_{iT}^{*}, i = 1, \dots, n,$ which implies that the public good provision is not Pareto efficient under the CPSPNE.

The γ-core of the dynamic game of public good provision can be shown to be non-empty in a variety of cases. To mention a few, the γ-core of the dynamic game is generally non-empty if the players are identical or if the damage functions are linear. Given restrictions on space, it is not possible to discuss all these cases here. But the one below illustrates more simply how a γ-core payoff vector, like a SPNE, of an extensive game can be found by backward induction. We assume that the production functions $g_i(e_{it})$ are quadratic,

$$g_i(e_{it}) = c_i e_{it} - \frac{1}{2} e_{it}^2, \tag{8}$$

where $c_i > 0$ is sufficiently large, and the damage functions

$$v_i(z_t) = \frac{1}{2} z_t^2. \tag{9}$$

Proposition 7 shows that for these specific production and damage functions, the dynamic game Γ and each induced game $\Gamma^S, S \subset N$, admit a unique SPNE.

Proposition 8 The dynamic game Γ admits a non-empty γ-core if the production and damage functions are quadratic as in (8) and (9).

Proof: In view of Proposition 2, it is sufficient to show that the γ-core of each subgame $\Gamma_{z_{t-1}^{*}}, t = 1, \dots, T$ is non-empty. We begin with the subgame $\Gamma_{z_{T-1}^{*}}$ in the last period T. Since the one stage game $\Gamma_{z_{T-1}^{*}}$ is essentially a strategic game, a result in Chander and Tulkens (1997;

Theorem 2) implies that a specific payoff vector $(p_{1T}, \ldots, p_{nT})$ such that $p_{iT} = x_{iT} - v_i(z^*_{T-1} + \sum_{j\in N} e^*_{jT})$ where $(e^*_{1T}, \ldots, e^*_{nT})$ satisfy equations (1) and (2) and $(e^*_{1T}, \ldots, e^*_{nT})$ are as defined in the lemma (after substituting $\beta = 1$ and $\delta = 0$) belongs to the γ-core of $\Gamma_{z^*_{T-1}}$. The γ-core of $\Gamma_{z^*_{T-1}}$ is therefore non-empty. The payoff vector $(p_{1T}, \ldots, p_{nT})$ depends on z^*_{T-1} in the following manner. Given an arbitrary z_{T-1}, the unique SPNE of the subgame $\Gamma_{z_{T-1}}$ and the induced game $\Gamma^N_{z_{T-1}}$ are given by the first order condition $c_i - e_{iT} = z_{t-1} + \sum_{j\in N} e_{jt}$ and $c_i - e_{it} = n(z_{t-1} + \sum_{j\in N} e_{jt})$, respectively. Therefore, the SPNE of $\Gamma_{z_{T-1}}$ and $\Gamma^N_{z_{T-1}}$ are

$$\bar{e}_{iT}(z_{T-1}) = c_i - \frac{1}{1+n}\left(\sum_{j\in N} c_j + z_{T-1}\right) \text{ and } e^*{}_{iT}(z_{T-1}) = c_i - \frac{n}{1+n^2}\left(\sum_{j\in N} c_j + z_{T-1}\right),$$

respectively. Then, as in Chander and Tulkens (1997; Theorem 2), the payoff vector

$$p_{iT}(z_{T-1}) = g_i\big(\bar{e}_{iT}(z_{T-1})\big) - \frac{v_i'}{\sum_{j\in N} v_j'}\left(\sum_{j\in N} g_j\big(\bar{e}_{iT}(z_{T-1})\big) - \sum_{j\in N} g_j\left(e^*_{iT}(z_{T-1})\right) - v_i\left(z_{T-1} + \sum_{j\in N} e^*_{jT}(z_{T-1})\right)\right), i = 1, \ldots, n,$$

belongs to the γ-core of $\Gamma_{z_{T-1}}$. Using (8) and (9) and after substitution one obtains

$$p_{iT}(z_{T-1}) = \frac{1}{2}c_i^2 - \frac{1}{1+n^2}\left(1 + \frac{1}{2}\frac{n^2}{1+n^2}\right)\left(z_{T-1} + \sum_{j\in N} c_j\right)^2, i \in N.$$

Given the γ-core payoff vector $(p_{1T}(z_{T-1}), \ldots, p_{nT}(z_{T-1}))$ of $\Gamma_{z_{T-1}}$, consider the reduced form of the subgame $\Gamma_{z^*_{T-2}}$ in which the payoff of player i is given by

$$c_i e_{iT-1} - \frac{1}{2}e^2_{iT-1} - \frac{1}{2}\left(z^*_{T-2} + \sum_{j\in N} e_{jT-1}\right)^2 + p_{iT}\left(z^*_{T-2} + \sum_{j\in N} e_{jT-1}\right).$$

Since each $p_{iT}(z_{T-1}), i \in N$, is quadratic in z_{T-1}, the payoff functions in this reduced subgame have essentially the same functional form as the payoff functions in the game $\Gamma_{z^*_{T-1}}$. Therefore, the reduced form of the game $\Gamma_{z^*_{T-2}}$ has a non-empty γ-core. Let $\widehat{\Gamma}_{z^*_{T-2}}$ denote the reduced form of the game $\Gamma_{z^*_{T-2}}$. Then, $(e^*_{1T-1}, \ldots, e^*_{nT-1})$ as defined in the lemma (after substituting $\beta = 1$ and $\delta = 0$) is the unique SPNE of the induced game $\widehat{\Gamma}^N_{z^*_{T-2}}$ and $z^*_{T-2} + \sum_{j\in N} e^*_{jT-1} = z^*_{T-1}$. That is because only the unique strategy $(e^*_{1T-1}, \ldots, e^*_{nT-1})$ leads to the highest payoff for coalition N. Thus, the γ-core of both $\Gamma_{z^*_{T-1}}$ and $\Gamma_{z^*_{T-2}}$ are non-empty. We can similarly prove that the γ-core of the subgame $\Gamma_{z^*_{T-3}}$ is non–empty and so on. ■

5. Concluding remarks

This paper brings together two of the most important solution concepts of game theory: subgame-perfect Nash equilibrium of a non-cooperative game and the core of a cooperative game.[31] Our approach can be extended to the case in which when a coalition deviates the remaining players may form one or more non-trivial coalitions. Papers taking this approach include Ray and Vohra (1997) and Maskin (2003). Ray and Vohra address the question of what properties might be expected of binding agreements but treat strategic rather than extensive games; thus subgame perfection plays no role. Maskin (2003) proposes a core concept in which if a coalition deviates, the remaining players form one single coalition. Our approach can be used to extend the idea underlying Maskin's core concept to extensive games. More specifically, the induced subgames will now have only two players: if S is the set of *all* active players at a decision node $x,$ then for each $S^{'} \subset S,$ the player set of the induced subgame with origin at x consists of $\{S^{'}, \; S - S^{'}\}$. As in the case of γ-core, the highest SPNE payoffs of the induced subgame can be used to assign payoffs to all coalitions $S^{'} \subset S$. However, unlike the γ-core, it does not seem possible to relate the so-defined Maskin's core to SSPNE and CPSPNE.[32]

Our approach differs from that in Chander (2007; 2010) and other literature in that we consider extensive games and subgame perfection. Our approach rests on two fundamental ideas discussed in the introduction: coalitions become players and, at the origin of any subgame, only those players who still have decisions to make can become part of a coalition. Possibilities for coalition actions are taken into account through the equilibrium notion – in this, paper, the γ-core.

[31] A link between the two is apparently missing in the current literature.
[32] Maskin introduces his core concept in the primitive framework of a partition function. Our approach can be extended to derive a partition function from the extensive game. For each partition of the total player set, consider the induced games in which each coalition in the partition becomes one single player. Then, the payoff of a coalition in a partition is equal to its highest subgame-perfect Nash equilibrium payoff in the game induced by the partition. Maskin's core of the so derived partition function, however, may not be equal to the one proposed above unless the game is such that all players are active at each decision node.

Appendix

Proof of the lemma: First note that the optimum is in the interior of the interval $0 \leq e_{it} \leq e^0$ for each $i \in N$ and $t = 1, \dots, T$. If $e_{it} \geq e^0$, equation (2) implies $z_t \geq e^0$. Since, by assumption, $v_i'(z)$ is non-decreasing and $g_i'(e_{it})$ in non-increasing, $g_i'(e_{it}) < v_i'(z_t)$ if $g_i'(e^0) < v_i'(e^0)$. Thus, decreasing e_{it} can increase the value of the objective function. This proves that $e_{it} < e^0$ for each $i \in N$ and $t = 1, \dots, T$; which also implies $z_t \leq z_0 + ne^0T$ and, therefore, $v_i'(z_t) < \infty, t = 1, \dots, T$. Since $g_i'(e_{it}) \to \infty$ as $e_{it} \to 0$, $g_i'(0) > \sum_{i \in N} v_i'(z_t)$. Thus, increasing e_{it} if it is sufficiently close to zero increases the value of the objective function. Hence, $0 < e_{it} < e^0$, $i \in N$, $t = 1, \dots, T$. The Langrangian associated with the optimization problem is given by

$$L = \sum_{t=1}^{T} \beta^{t-1} \sum_{i=1}^{n} (g_i(e_{it}) - v_i(z_t)) + \sum_{t=1}^{T} \lambda_t [z_t - (1-\delta) z_{t-1} - \sum_{i \in N} e_{it}]$$

where the variables λ_t are the Lagrange multipliers associated with the *T* constraints. Since as shown the solution is interior, the first order conditions for $(e_{it}^*)_{t=1}^T, (z_t^*)_{t=1}^T$ and $(\lambda_t^*)_t^T$ to be an optimum are:

$$\frac{\partial L}{\partial e_{it}} = \beta^{t-1} g_i'(e_{it}^*) - \lambda_t^* = 0, t = 1, \dots, T, i \in N,$$

$$\frac{\partial L}{\partial z_t} = -\beta^{t-1} \sum_{i \in N} v_i'(z_t^*) + \lambda_t^* - \lambda_{t+1}^*(1-\delta) = 0, t = 1, \dots, T-1,$$

$$\frac{\partial L}{\partial z_T} = -\beta^{T-1} \sum_{i \in N} v_i'(z_T^*) + \lambda_T^* = 0,$$

$$z_t = (1-\delta) z_{t-1} + \sum_{i \in N} e_{it}, t = 1, \dots, T, z_0 = z_0^*.$$

After substitution, one obtains $g_i'(e_{it}^*) - \beta(1-\delta) g_i'(e_{it+1}^*) = \sum_{j \in N} v_j'(z_t^*), z_t^* = (1-\delta) z_{t-1}^* + \sum_{i \in N} e_{it}^*, i \in N, t = 1, \dots, T$. These equations can be rewritten as

$$. \; g_i'(e_{it}^*) = \sum_{\tau=t}^{T} [\beta(1-\delta)]^{\tau-t} \sum_{j \in N} v_j'(z_\tau^*), z_t^* = (1-\delta) z_{t-1}^* + \sum_{i \in N} e_{it}^*, i \in N, t = 1, \dots, T.$$

These are $(n+1)T$ equations in $(n+1)T$ variables. The solution is unique because the objective function is strictly concave and the constraints are linear. . ■

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