Agency Models with Frequent Actions: A Quadratic Approximation Method

Tomasz Sadzik, NYU   Ennio Stacchetti, NYU

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Abstract

The paper analyzes dynamic principal-agent models with short period lengths. The two main contributions are: (i) an analytic characterization of the values of optimal contracts in the limit as the period length goes to 0, and (ii) the construction of relatively simple (almost) optimal contracts for fixed period lengths. Our setting is flexible and includes the pure hidden action or pure hidden information models as special cases. We show how such details of the underlying information structure affect the optimal provision of incentives and the value of the contracts. The dependence is very tractable and we obtain sharp comparative statics results. The results are derived with a novel method that uses a quadratic approximation of the Pareto boundary of the equilibrium value set.

1 Introduction

While the problems of dynamic incentive provision are central to economics, existing methods typically do not provide tractable analytic solutions. In the paper we will consider dynamic contracting problems in which a risk neutral principal interacts repeatedly with a risk averse agent under asymmetric information. These are benchmark models in labor economics, corporate finance (CEO compensation and optimal capital structure), and the literatures on optimal dynamic insurance and taxation. We develop a discrete-time method that allows us to solve such problems *analytically* for a range of contracting environments. We thus also obtain sharp qualitative results about optimal dynamic incentive provision and its dependence on the underlying information structure.

Our models feature frequent decisions and information arrival ("short period length"), with public signals whose variance is large compared to the contribution of the agent's actions. We have in mind situations such as incentivizing the CEO of a company with shares traded on a stock market. The class of models is permissive regarding the precise nature of information structure and distribution of revenue in each period. It embraces, as extreme special cases, *pure hidden action* models, when the agent acts at an ex-ante stage and has private information about his own action only, and *pure hidden information* models, when the agent acts conditionally on the "noise" realization. Respective cases in point are devoting effort to develop a risky project, and diverting funds from realized cash-flow.

More precisely, for each fixed information structure, we look at a sequence of models with shrinking period length. The period length affects the per-period payoffs as well as the variance of the public signals (see Abreu, Milgrom, and Pearce [1991]). On one hand, in a certain sense (explained below) all the sequences of models we consider converge to the same limit. On the other hand, each sequence corresponds to a qualitatively different information structure. First, information structures may differ in the distribution of the public signal and how informative the signal is about the agent's action. Moreover, they may differ in the amount of private information that the agent receives, as in the pure hidden information and pure hidden action models mentioned above. While the differences are crucial for the design of incentives in the standard static models, it is not a priori clear whether and how they should affect the design of incentives in the dynamic model with short period lengths.

In the paper we develop what we call a *quadratic approximation method* to solve those dynamic contracting problems in the following sense. First, for any type of information structure, we characterize the limit of the Pareto frontier of value sets achievable by incentive compatible contracts, as the period length shrinks. Second, we construct relatively simple suboptimal contracts, whose values converge to the Pareto frontier as the period length shrinks. Third, we show that the limit values and the contracts depend on the precise nature of the underlying information structure. The dependence is very tractable and preserves many qualitative features of the static models, which facilitates comparative statics analysis.

In the model, a risk neutral principal and a risk averse agent sign a fully enforceable contract at time zero. Once the contract is signed, in every period the timing is as follows: 1) the agent observes a private signal and then chooses a costly action, 2) both the principal and the agent observe the revenue realization that depends on the agent's action and noise, 3) the principal compensates the agent. The information structure is fully specified by a joint distribution of the private signal and noise, rescaled to account for the period length. For example, the pure hidden action model corresponds to the case when the private signal is completely uninformative of the noise, whereas in the pure hidden information case, the private signal is the same as the noise realization. The contract specifies the agent's consumption for each period contingent on the history of past revenue realizations.

Following standard dynamic programming methods (see Abreu, Pearce, and Stacchetti [1986, 1990] and Spear and Srivastava [1987]), the contracts can be described with the agent's continuation value as a state variable. The continuation value promised last period fully determines the current period action, contingent on the private signal, as well as consumption and new continuation value, contingent on revenue. In order to provide incentives to exert costly effort, the consumption and continuation value must respond to revenue and reward the agent for good outcomes, whereas on the other hand such volatility is inefficient and imposes a "cost of incentives", given the agent's risk aversion. Thus, the model features a tradeoff between dynamic insurance and incentives. The optimal contract resolves this tradeoff and regulates how incentives are provided in the course of the agent's career.

Our method for solving the problem comprises two steps. The first step consists in solving a family of static problems. Given a mean effort and mean cost of effort, roughly, we look for an action scheme with those parameters and a scheme of continuation value "transfers" with minimal variance, which provide local (first-order) incentives for exerting effort. The problem is closely related to possibly the simplest instance of a static principal-agent problem with incentives-insurance tradeoff, in which the agent has mean-variance utility of consumption. For a short period length, it turns out that the per period cost of incentives is proportional to this variance, for a given mean effort and cost of effort. Also, the variance depends on the information structure: intuitively, it is low in the case when the revenue is statistically informative of the agent's action. For example, in some extreme cases when each action gives rise to a different support of revenues, any deviation can be statistically identified and the variance is zero.

The second step consists in solving a Hamilton-Jacobi-Bellman (HJB) differential equation, with the variance of incentive transfers as a parameter. The function F solving this equation provides the principal's value $F(w)$ for the optimal static contract that delivers a given value w to the agent, absent information problems but with explicit cost of incentives, proportional to the variance of incentive transfers. Except for the variance of incentive transfers, the equation is familiar from the continuous-time dynamic programming models (see Sannikov [2007, 2008]). For example, the function that gives the principal's value for the first best optimal contracts (absent any informational problems) solves this equation with the cost of incentives zero.

In the paper we show that the differential equation describes the limit of the values of the optimal contracts, while the policy functions of the two steps can be used to design relatively simple and essentially optimal discrete time contracts.

All the models we consider, independently of the information structure, approximate in distribution the same continuous-time model, in which the agent controls the drift of a noisy revenue process that follows Brownian Diffusion. Nevertheless, the limit solutions are sensitive to the information structure. The solutions depend on the variance of incentive transfers, which in turn depends on the information structure, parametrized by the distribution of signals. With the comparative statics of the contracts reduced to the analysis of static incentive problems, we illustrate how the contracts and their values respond to changes in the "noise" distribution or amount of private information the agent receives. For example, restricting attention to the pure hidden action models when the agent has no private information about the environment, the value of the contracts depends on a single and readily computed parameter of the noise, the *variance of likelihood ratio*. Or, fixing the distribution of revenues to be normal, the case when the agent has no private information about the environment (pure hidden action) results in the most inefficient contracts.

We may compare our results with the optimal contract and its value in the model formulated directly in continuous time (Sannikov [2008], see also Williams [2009]). As regards the value, it agrees with the limit of values for a particular class of information structures, namely pure hidden action models with the variance of likelihood ratio as in the normal distribution. In this sense, we provide a justification of the existing continuous time model. However, the limit is different and new to the literature if we go beyond the pure hidden action case and allow agent some private information about the noise, or just change the variance of likelihood ratio.[1] For example, for the same variance of the revenue, a double-exponential noise results in strictly more efficient contracts, while for some finite support distributions the optimal contracts achieve the first best.

[1] While Williams [2009] considers continuous time models with private information, the models do not allow private information about the signals in the current period.

The relationship with the optimal continuous-time contract itself (as opposed to its value) is even more delicate. In the case when the values converge to the value of the optimal continuous-time contract, the optimal discrete time contracts converge weakly (in distribution) to the optimal continuous time contract. Nevertheless, the discrete-time contracts must be sensitive to the precise details of the noise distribution. For example, a normal distribution requires contracts with continuation values responding linearly to the revenue realizations (see for example Holmstrom and Milgrom [1987]), while for the double-exponential distribution the continuation values take only two values, with the cutoff at a certain revenue level. There is no single contract that would be approximately optimal for both kinds of distributions.[2] In this sense, the optimal continuous-time contract does not contain enough information to construct approximate optimal discrete-time contracts, no matter how short the period length.[3]

While we use our quadratic approximation method on a particular class of principal-agent models, we are convinced that its range of applicability is much broader. In the paper we discuss three extensions. First, the period length might affect the information structure differently. This case includes environments where the information structure is independent of period length, as in Folk Theorem analysis, when we recover the Folk Theorem result as a special case. Second, we look at the case when the agent's action affects not only the mean of the revenue distribution, but also higher moments. Third, we look at a different type of preferences, in which the cost of action to the agent is expressed not in utility, but in monetary terms.

As mentioned above, our results rely on the parametrization of the dynamic contract by the agent's continuation value (Abreu, Pearce, and Stacchetti [1986, 1990], Spear and Srivastava [1987]). This insight leads to a method for computing the optimal contracts (Abreu, Pearce, and Stacchetti [1990]), or more generally Pareto efficient Perfect Public Equilibria, for models with a fixed period length based on the value iteration technique. Phelan and Townsend [1991] show a related method to compute optimal contracts based on the iteration of a linear programming problem. While those approaches are flexible and applicable to a wide variety of problems, they are computationally intense and do not yield analytical solutions. This hinders the comparative statics analysis of how the information structure affects the solutions.

One way to restore analytical tractability is to focus on models with patient players (Radner [1985], Fudenberg, Holmstrom, and Milgrom [1990]). This is equivalent to considering models with short period length, where the period length parametrizes the stage game payoffs, but not the information structure. While this simplifies the analysis, the models have the feature that in the limit, as the period length shrinks, the informational frictions disappear ("Folk Theorem"). Abreu, Milgrom, and Pearce [1991] suggest a more realistic approach where increasing the frequency of actions also affects the information structure. In our benchmark model, short periods come with high variance of public signal, which in particular exacerbates the informational problems and

[2] There is no contradiction between the weak convergence and the fact that no single contract "works" for different noise distributions: all it establishes is that the notion of weak convergence is very weak.

[3] While the results seem to have a flavor of lack of lower hemicontinuity, we would like to treat the statement with a grain of salt. The interpretation is sensitive to which models we consider as being close, i.e., the choice of a topology. The interpretation is correct if one treats weak topology over the models as the right one. We would like to stress that our results are independent of those interpretational issues: we provide full characterization of the limits of the optimal contracts and their values for a range of underlying information structures, whether one interprets the models as convergent or not.

prevents the first-best outcome from being achieved in the limit.

On a technical level, Matsushima [1989] established efficiency results and Fudenberg, Levine, and Maskin [1994] the Folk Theorem for patient players by decomposing continuation values on hyperplanes tangent to the (Pareto frontier of the) set of achievable values. Our method bears a close resemblance to this approach, where we use quadratic instead of the linear approximation of the frontier. The more precise approximation is required by the richer class of processes of public signals we consider. Moreover, the curvature of the boundary is proportional to the cost of incentives; when the process of signals is such that the linear approximation is appropriate, we recover as the special case the result of costless provision of incentives, i.e. Folk Theorem, in our restricted setting of principal-agent problems (see Section 3.1).

Hellwig and Schmidt [2002] is among the first papers to provide an upper hemicontinuity result linking the solutions of discrete-time and existing continuous-time models. It presents a sequence of discrete-time models whose optimal policies converge to the optimal policy of the continuous-time principal-agent model by Holmstrom and Milgrom [1987], in which the agent has CARA utility function and is compensated only at the end of the employment period. Biais, Mariotti, Plantin, and Rochet [2007] established such an upper hemicontinuity result for the principal-agent model of diverting cash-flows by Demarzo and Sannikov [2006], in which the agent is risk-neutral and the efficiency cost from diverting funds is linear, by looking at sequences of binomial revenue processes. Sannikov and Skrzypacz [2007] considers a more general framework of games and limit processes that are an arbitrary mixture of Brownian Diffusion and Poisson processes, and show convergence of the solutions for discrete time games with normal noise and pure hidden action structure, in the case of arbitrarily patient players. Our upper hemicontinuity results are rather general, making no appeal to CARA, infinite patience, risk neutrality of the agent or linearity of costs.

In the opposite direction, Muller [2000] illustrates that, in the context of the model by Holmstrom and Milgrom [1987], for a particular sequence of models the limit of the solutions might disagree with the solution of the continuous-time limit model. For a reputation game with one long-lived player, Fudenberg and Levine [2009] established an analogous result, comparing sequences of models with different numbers of aggregated signal realizations within each time period. The techniques used for the case with one long-lived player, where the Pareto frontier is a point and the optimal policy is stationary, seem to be particular to this setting. By contrast, we look at a broad class of information structures well known from the static incentive literature, and obtain sharp and unified characterizations based on the variance of incentive transfers. We also expect that our method of deriving limit values and essentially optimal contracts (equilibria, or policy functions) based on the quadratic approximation of the Pareto frontier is applicable in general settings, and in particular when the frontier is multi-dimensional and optimal contracts nonstationary.

2 Model

2.1 The Agency Problem

A risk neutral principal contracts with a risk-averse agent. The principal offers the agent an infinite contract specifying a contingent payment for each period as a function of the history of output (gross revenue) in past periods. If the agent accepts it, the contract becomes legally binding and cannot be terminated by either party. In every period of length Δ, the timing is as follows. The agent observes a private signal about the random shock affecting output in the current period, and then chooses an action (effort). The agent's action and the random shock determine the output, which is realized at the end of the period. The principal pays the agent after observing the output and the agent consumes his compensation (the agent can't save or borrow). Note that both the agent's signal and action are his private information, whereas output is publicly observed. Though the agent's actions are unobservable, the principal and the agent also implicitly agree to a full contingent action plan for the agent. Finally, note that we do not allow communication between the agent and the principal.

The agent's per period utility is given by $\Delta[u(c) - h(a)]$, where $a \in \mathcal{A}$ and $c \geq 0$ denotes his consumption. The agent's action and consumption are stated in flow units.[4] The consumption utility function $u : \mathbb{R}_+ \to \mathbb{R}$ is twice continuously differentiable, strictly increasing and strictly concave, with $u(0) = 0$ and $\lim_{c\to\infty} u(c) = \bar{u} < \infty$. The agent's action space is a closed interval $\mathcal{A} \subset \mathbb{R}$ with $0 \in \mathcal{A}$ and right endpoint A. We consider two cases below, either $\mathcal{A} = [0, A]$ or $\mathcal{A} = (-\infty, A]$. The cost of effort function $h : \mathcal{A} \to \mathbb{R}$ is continuous and its restriction to positive actions is strictly convex, strictly increasing and twice continuously differentiable, while $h(a) = 0$ for all $a \leq 0$. We also assume that there exists $\gamma > 0$ such that $h(a) \geq \gamma a$ for all $a \geq 0$. In addition, we assume that $h'(0_+) < u'(0)$, so absent asymmetric information it is efficient to have the agent exert positive effort.

The principal's per period payoff is $\Delta[x + a - c]$, where x is a random shock, a is the agent's action and c is the agent's compensation, again in flow units. We will interpret $y = \Delta[x + a]$ as the output realization. Both the principal and the agent discount future payoffs by the common discount factor $e^{-r\Delta}$, where $r > 0$ is the discount rate.

Let z_n denote the agent's private signal realization in period n. We assume that (x_n, z_n) are randomly distributed and $\{(x_n, z_n)\}$ are i.i.d. We consider two settings: (i) the absolutely continuous case where the conditional distribution of x_n given z_n has a density $g^{\Delta}_{X|Z}(x_n|z_n)$ and z_n has a distribution $G^{\Delta}_{Z}(z_n)$; and (ii) $z_n = x_n$ and the distribution of x_n has density $g^{\Delta}_{X}(x_n)$. We assume that $\mathbb{E}^{\Delta}[x_n] = 0$ and $\mathbb{V}^{\Delta}[x_n] = \sigma^2/\Delta$. The length of the period Δ parametrizes these densities because we assume that the quality of the signals (the inverse of their variances) increases with Δ. Later we make precise assumptions on how these distributions vary with Δ.

Case (i) includes the standard "hidden action" agency model in which z_n is completely uninformative about x_n. Case (ii) corresponds to the "pure hidden information" model where the agent knows exactly the output that will be produced as a function of his own action. We will also investigate intermediate models where, for example, $G^{\Delta}(x_n, z_n)$ is a binormal distribution with a positive correlation between x_n and z_n.

2.2 The Principal's Problem and Important Curves

A *contract* is a process $\{c_n\}$ that for each period n specifies the agent's compensation c_n as a function of the history of outputs $(y_0, \ldots, y_n)$. A *plan* for the agent is another process $\{a_n\}$ that

[4] While our interpretation that consumption, which in principle depends also on the current period's output, flows during the duration of the period seems inconsistent, it is an indirect corollary to our results that consumption independent of current output is with no loss of generality when the period length is small - see the next Section.

specifies the agent's action in each period n as a function of the history $(z_0, y_0, \ldots, z_{n-1}, y_{n-1}, z_n)$. Since the principal's contract does not depend on the agent's private signals and signals across periods are independent, there is no loss of generality in restricting action plans so that a_n is a function of $(y_0, \ldots, y_{n-1}, z_n)$ only.

The principal's expected discounted revenue for a contract-action plan pair $(\{c_n\}, \{a_n\})$ is

$$\Pi(\{c_n\}, \{a_n\}) = \tilde{r}\,\mathbb{E}^{\Delta}\left[\sum_{n=0}^{\infty} e^{-r\Delta n}\left(y_n - \Delta c_n\right)\right] = \tilde{r}\Delta\mathbb{E}^{\Delta}\left[\sum_{n=0}^{\infty} e^{-r\Delta n}\left(a_n - c_n\right)\right],$$

while the agent's expected discounted utility is

$$U(\{c_n\}, \{a_n\}) = \tilde{r}\Delta\mathbb{E}^{\Delta}\left[\sum_{n=0}^{\infty} e^{-r\Delta n}\left(u\left(c_n\right) - h\left(a_n\right)\right)\right],$$

where the factor $\tilde{r}$ is such that $\tilde{r}\Delta = 1 - e^{-r\Delta}$, and normalizes the sums so that $\tilde{r}\Delta \sum e^{-r\Delta n} = 1$.[5] The action plan $\{a_n\}$ is *incentive compatible* (IC) for the contract $\{c_n\}$ if for any other plan $\{a'_n\}$, any N and any realization $(y_0, \ldots, y_{N-1}, z_N)$,

$$\begin{aligned} & \mathbb{E}^{\Delta}\left[\sum_{n=N}^{\infty} e^{-r\Delta n}\left(u\left(c_n\right) - h\left(a_n\right)\right)\Big|(y_0, \ldots, y_{N-1}, z_N)\right] \\ \geq\ & \mathbb{E}^{\Delta}\left[\sum_{n=N}^{\infty} e^{-r\Delta n}\left(u\left(c_n\right) - h\left(a'_n\right)\right)\Big|(y_0, \ldots, y_{N-1}, z_N)\right]. \end{aligned}$$

For a given w, the *principal's problem* consists of finding a contract-action plan $\{c_n\}, \{a_n\}$ that maximizes his expected discounted revenue among all the incentive compatible plans that deliver an expected discounted utility w to the agent. For any $w \in [0, \bar{u})$, let $F^{\Delta}(w)$ be the principal's value from an optimal IC contract-action plan,

$$F^{\Delta}(w) = \sup\{\Pi(\{c_n\}, \{a_n\}) \mid \{a_n\} \text{ is IC for } \{c_n\},\ U(\{c_n\}, \{a_n\}) = w\}.$$

A *wage contract* specifies for each period n a compensation that does not depend on the current output y_n. That is, $\{c_n\}$ is a wage contract if for each period n the agent's compensation c_n only depends on the history of outputs $(y_0, \ldots, y_{n-1})$. Let $F^{\Delta,c} : \mathbb{R}_+ \to \mathbb{R}$ represent the principal's value from an optimal IC wage contract-action plan. An indirect implication of our main result (see Lemma 13) is that when Δ is small, the loss from only allowing wage contracts is small: $F^{\Delta,c}(w) \uparrow F^{\Delta}(w)$ as $\Delta \to 0$, uniformly in $w \in [0, \bar{u})$.

We also define $\overline{F}(w)$ as the principal's value for an optimal feasible ("first best") contract-action plan (not necessarily IC). It is easy to see that such a plan is stationary, and so $\overline{F}(w)$ is just equal to the value of an optimal feasible one period contract-action pair:

$$\overline{F}(w) = \max_{a,c}\{a - c \mid a \in \mathcal{A},\ c \geq 0,\ u(c) - h(a) = w\}.$$

[5] Note that $\tilde{r} \to r$ as $\Delta \downarrow 0$.

One can show that $\overline{F}$ satisfies the following ODE:

$$\overline{F}(w) = \max_{a,c} \left\{ (a-c) + \overline{F}'(w)(w + h(a) - u(c)) \right\}. \tag{1}$$

Finally, let $\underline{F} : [0, \overline{u}) \to \mathbb{R}$ be the *retirement curve*. That is,

$$\underline{F}(w) = -u^{-1}(w).$$

Continuation values $(w, \underline{F}(w))$ with $w \in [0, \bar{u})$ are attained by wage contracts that pay the same $c_n = \underline{F}(w)$ in every period (regardless of history), when the agent chooses action $a_n = 0$ in every period. Since such wage contract-action plan is IC, $F^{\Delta} \geq \underline{F}$.

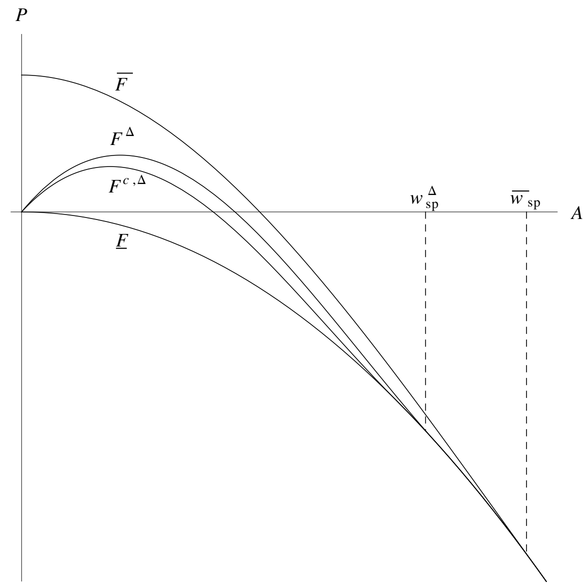

Figure 1: Value functions.

Notice that

$$F^{\Delta}(0) = F^{\Delta,c}(0) = 0.$$

This follows from the limited liability constraint $c \geq 0$ and $u(0) = h(0) = 0$: since the agent can always deviate to exerting no effort, the only way for the agent to receive an expected discounted utility of zero is for the contract to pay zero in every period.[6] Also, there exists $\overline{w}_{sp} \in [0, \bar{u})$ such that $\overline{F}(\overline{w}_{sp}) = \underline{F}(\overline{w}_{sp})$ and $\overline{F}(w) > \underline{F}(w)$ for all $w < \overline{w}_{sp}$. This is because if the agent must receive high expected utility, exerting any positive effort by the agent is too costly for the principal (see Spear and Srivastava [1987], Sannikov [2008]).[7]

[6] Assumption (A2) below guarantees that if agent gets a strictly positive expected continuation value when taking a strictly positive effort (which compensates him for the cost of effort), then he would also get a strictly positive expected continuation value from no effort.

[7] Formally, marginal cost of effort is bounded below by $\gamma > 0$ for positive actions while marginal utility of consumption converges to zero as consumption increases. This excludes any interior solution for $\overline{F}(w)$ for sufficiently high w since such solution must satisfy $h'(a) = u'(c)$, for $c > u^{-1}(w)$ such that $u(c) - h(a) = w$.

Altogether, for any $\Delta > 0$ we have $\underline{F} \leq F^{\Delta,c} \leq F^{\Delta} \leq \overline{F}$ (see Figure 1). In particular, this implies that there exist a minimal agent's value w_{sp}^{Δ}, $0 \leq w_{sp}^{\Delta} \leq \overline{w}_{sp}$, such that:

$$F^{\Delta}(w_{sp}^{\Delta}) = \underline{F}(w_{sp}^{\Delta}).$$

2.3 Frequent Actions: Parametrization and Assumptions

We are interested in solving the principal's problem when the period length Δ is small. We assume that while Δ decreases, (Z, X) are normalized signals generated by a fixed distribution (independent of Δ):

(A1) There exists a distribution function $G(x, z)$ with $\mathbb{E}[x] = 0$ and $\mathbb{V}[x] = \sigma^2$, such that for each $\Delta > 0$,

$$G^{\Delta}(x, z) = G(x\sqrt{\Delta}, z\sqrt{\Delta}).$$

Note that $\mathbb{E}^{\Delta}[x] = 0$, and $\mathbb{V}^{\Delta}[x] = \sigma^2/\Delta$. Consequently, the linear interpolation of the process $\{X_{k\Delta}\}_{k\in\mathbb{N}}$ where $X_{k\Delta} = \frac{\Delta}{\sigma}\sum_{n=1}^{k} x_n$, with $x_n \sim G_X^{\Delta}$, converges in distribution to the Brownian Motion as $\Delta \to 0$ (Invariance Principle, see e.g. Theorem 4.20 in Karatzas and Shreve [1991]). This also implies that if $y_n = \Delta[x_n + a(z_n)]$ and $B_{k\Delta} = \frac{1}{\sigma}\sum_{n=1}^{k}(y_n - \Delta\mathbb{E}^{\Delta}[a(z_n)]) = \frac{\Delta}{\sigma}\sum_{n=1}^{k}(x_n + \xi_n)$, with $a(z) \in [0, A]$, $\xi_n = a(z_n) - \mathbb{E}^{\Delta}[a(z_n)]$ and $(x_n, z_n) \sim G^{\Delta}$, then the linear interpolation of the process $\{B_{k\Delta}\}_{k\in\mathbb{N}}$ converges in distribution to the Brownian Motion as $\Delta \to 0$ (see Whitt [1980])[8]. In other words, the linear interpolations of revenue processes converge in distribution to the continuous time process $\{Y_t\}$ satisfying

$$dY_t = \mathbb{E}^{\Delta}[a_t]dt + \sigma dB_t,$$

where $\{B_t\}$ is a Brownian Motion.

We also assume that one of the following two assumptions is satisfied. The first assumption corresponds to the case when the agent has some imprecise (possibly uninformative) signal about the noise, and the noise has absolute continuous distribution, conditional on the private signal. The available actions are bounded below by zero. The second assumption corresponds to the pure hidden information case when the agent knows the noise realization before taking an action. In this case we assume that the agent's actions are unbounded from below. In a separate note we show that pure hidden action model with compact action set results in the first best contracts as the period length shrinks to zero.

(A2-AC) For any z, the distribution of X conditional on $[Z = z]$ has density function $g_{X|Z}(x|z)$, and there exist $\bar{\delta}, \bar{M} > 0$ such that for all z,

$$\int_{\mathbb{R}} \frac{\sup_{\delta\in[0,\bar{\delta}]} g'_{X|Z}(x-\delta|z)^2}{g_{X|Z}(x|z)} dx < \bar{M} \qquad \text{and} \qquad \int_{\mathbb{R}} \sup_{\delta\in[0,\bar{\delta}]} \left| g''_{X|Z}(x-\delta|z) \right| dx < \bar{M}.$$

Moreover, the set of available actions is $\mathcal{A} = [0, A]$ for some $A \in \mathbb{R}_+$.

[8] Each process $\{B_{k\Delta}\}_{k\in\mathbb{N}}$ is the sum of two continuous path processes, one converging weakly to the Brownian Motion and the other to the process identically equal to zero.

(A2-PHI) $X \equiv Z$ and X has a density function $g_X(x)$. Moreover, the set of available actions is $\mathcal{A} = (-\infty, A]$ for some $A \in \mathbb{R}_+$.

Throughout the paper we use the standard notation $O(\Delta)$ and $o(\Delta)$ to denote arbitrary functions $\alpha(\Delta)$ and $\beta(\Delta)$, respectively, such that

$$\lim_{\Delta \to 0} \left| \frac{\alpha(\Delta)}{\Delta} \right| < \infty \quad \text{and} \quad \lim_{\Delta \to 0} \frac{\beta(\Delta)}{\Delta} = 0.$$

3 Results

3.1 Solution to the Principal's Problem: Values

In this section we focus on determining the principal's value for the optimal contract-action plan, as the period length Δ shrinks to 0. The explicit definition of the contract-action plans that achieve those values is postponed until Section 3.3.

We will see that as $\Delta \to 0$, the solution of the principal's problem can be approximated by the following two step procedure. In the first step we solve a family of static problems. For any "mean action" $\bar{a} \geq 0$ and "mean cost" $\bar{h} \geq 0$, define the *variance of incentive transfers*

$$\begin{aligned} \Theta(\bar{a}, \bar{h}) \quad &= \quad \inf_{a,v} \quad \mathbb{E}\left[v(x)^2\right] \qquad (2) \\ &\text{s.t.} \quad \bar{a} = \mathbb{E}[a(z)], \quad \bar{h} = \mathbb{E}[h(a(z))], \\ &\qquad h'(a(z)) = -\int_{\mathbb{R}} v(x) g'_{X|Z}(x|z)dx \quad \forall z \qquad (\text{FOC}_{\Theta}\text{-AC}) \\ &\qquad h'(a(x)) = v'(x) \quad \forall x \qquad (\text{FOC}_{\Theta}\text{-PHI}) \end{aligned}$$

The infimum is taken over measurable functions in the absolutely continuous case. In the pure hidden information case, the infimum is over piecewise continuously differentiable functions $a(\cdot)$ and continuous functions $v(\cdot)$, and the (FOC_{Θ}-PHI) condition is required everywhere except for finitely many points of discontinuity of $a(\cdot)$.

To interpret the function Θ, consider the following static contracting problem. An agent has a quasilinear utility $v - h(a)$, and chooses an action $a(z)$ that depends on his privately observed signal z. The principal observes a noisy signal $x + \varepsilon a(z)$ and makes transfer v to the agent. His objective is to minimize the variance of transfers that provide first order incentives for an action scheme with mean action $\bar{a}$ and mean cost $\bar{h}$.[9] $\Theta(\bar{a}, \bar{h})$ is the value of the simplified version of this problem, when we substitute $\varepsilon = 0$. In particular, the last two lines in the definition of $\Theta(\bar{a}, \bar{h})$ correspond to the local incentive constraints: the (FOC_{Θ}-AC) applies to the absolutely continuous case, while (FOC_{Θ}-PHI) to the pure hidden information case.

The second step refers to the following optimal control problem:

$$\begin{aligned} F(w) \quad &= \sup_{\{\bar{a}_t, \bar{h}_t, c_t\}} \quad \mathbb{E}\left[\int_0^{\infty} r[\bar{a}_t(W_t) - c_t(W_t)]e^{-rt}dt\right] \\ &\text{s.t.} \quad dW_t = r\left[W_t - u(c_t(W_t)) + \bar{h}_t(W_t)\right]dt + r\sqrt{\Theta(\bar{a}_t(W_t), \bar{h}_t(W_t))}\, dB_t, \quad W_0 = w. \end{aligned}$$

[9] Alternatively, the principal is minimizing expected transfers to an agent with mean-variance utility of transfers, who has an ex ante outside option $-\bar{h}$.

This is a principal's problem similar to that obtained by Sannikov [2008]. The state variable W_t represents the continuation value for the agent. It's drift is proportional to the difference of the continuation value and the expected flow of utility to the agent, who receives payment $c_t(W_t)$ and incurs mean cost $\bar{h}_t(W_t)$. The diffusion coefficient is determined by the variance of incentive transfers. $F(w)$ represents the principal's payoff when the initial continuation value for the agent is w. The corresponding HJB equation for this problem is

$$F(w) = \sup_{\bar{a},\bar{h},c} \left\{ (\bar{a} - c) + F'(w) \left(w + \bar{h} - u(c) \right) + \frac{1}{2} F''(w) r \Theta(\bar{a}, \bar{h}) \right\}, \tag{3}$$

which we solve with the boundary conditions:

$$F(0) = 0 \tag{4}$$

and $F'(0)$ equal to the largest slope such that for some $w_{sp} > 0$

$$F\left(w_{sp}\right) = \underline{F}\left(w_{sp}\right) \quad \text{and} \quad F'\left(w_{sp}\right) = \underline{F}'\left(w_{sp}\right). \tag{5}$$

The first two conditions are analogous to the conditions that must be satisfied by F^{Δ} (see the end of Section 2.2). The last one is the *smooth pasting condition.*

Without any additional restrictions on the information structure, and thus on the variance of incentive transfers function $\Theta(\bar{a}, \bar{h})$, there need not exist a unique solution to the above HJB differential equation with boundary conditions (4) and (5). However, we can get around this by analyzing a perturbed equation, which always has a unique solution and that provides an arbitrarily good approximation to the solution of the principal's problem. For any $\zeta > 0$ consider the following differential equation:

$$F_{\zeta}(w) = \sup_{\bar{a},\bar{h},c} \left\{ (\bar{a} - c) + F_{\zeta}'(w)(w + \bar{h} - u(c)) + \frac{1}{2} F_{\zeta}''(w) r \Theta_{\zeta}(\bar{a}, \bar{h}) \right\}, \tag{6}$$

where

$$\Theta_{\zeta}(\bar{a}, \bar{h}) = \max\{\zeta, \Theta(\bar{a}, \bar{h})\},$$

with the same boundary conditions as above.[10] Let $w_{sp,\zeta}$ be the value at which F_{ζ} satisfies (5).

The following is the first main result of the paper. The proof of Theorem 1 is in Section 4. For a function $f : I \to \mathbb{R}$, we define $|f|_I = \sup_{w \in I} |f(w)|$ and $|f|_I^+ = |\max\{0, f(w)\}|_I$.

Theorem 1 *For any $\zeta > 0$, equation (6) with the boundary conditions (4) and (5) has a unique solution F_{ζ}. The value $w_{sp} = \lim_{\zeta \to 0} w_{sp,\zeta}$ and the function $F = \lim_{\zeta \to 0} F_{\zeta}$ exist. For any agent's promised value $w \in [0, w_{sp}]$, $F(w)$ is the limit of the principal's value from optimal contract as the period length Δ shrinks to zero:*

$$\lim_{\Delta \to 0} \left| F - F^{\Delta} \right|_{[0, w_{sp}]} = 0,$$

[10] Formally, the lower bound ζ on Θ_{ζ} guarantees that the differential equation is uniformly elliptic.

while for $w > w_{sp}$, $F(w)$ provides an upper bound:

$$F(w) \geq \lim_{\Delta \to 0} F^{\Delta}(w) \quad \text{for all} \quad w > w_{sp}.$$

More precisely, for fixed ζ we have

$$\left|F_{\zeta} - F^{\Delta}\right|^{+}_{[0,w_{sp,\zeta}]} = O(\zeta) + \frac{o(\Delta))}{\Delta} \quad \text{and} \quad \left|F^{\Delta} - F_{\zeta}\right|^{+}_{[0,\bar{u}]} = O(\zeta) + \frac{o(\Delta)}{\Delta}.$$

Theorem 1 is proven with just minimal restriction on the information structure (assumptions (A1) and (A2) above). The following proposition shows that with additional assumptions there is no need to consider perturbed equations (while the uniqueness may be lost).

Consider the following assumption:

$$\text{(Cont) } \Theta\left(\bar{a}, \bar{h}\right) \geq \delta\left(\bar{a}\right) \text{ for a continuous } \delta \text{ with } \delta\left(\bar{a}\right) > 0 \text{ when } \bar{a} > 0.$$

Proposition 1 *Let F be as in Theorem 1.*

(i) Assume that $\Theta\left(\bar{a}, \bar{h}\right) \geq \underline{\theta} > 0$ for $\bar{a} > 0$. Then, F is the unique solution of equation (3) with the boundary conditions (4) and (5). Moreover, in (3) one may add the constraint $\bar{a} > 0$.

(ii) Assume (Cont) holds. Then F solves equation (3) with boundary conditions (4) and (5).

Theorem 1 provides an explicit formula for the value of the optimal contract-action plans as the period length converges to zero. The value depends on the underlying information structure, described by the joint distribution $G(x, z)$, and the dependence is fully captured by the variance of incentive transfers function Θ in the differential equation. This facilitates the comparative statics analysis of how the information structure affects the values, as we illustrate in the next section.

The following result is crucial for such analysis. Reducing the variance of incentive transfers, intuitively, decreases the informational rent and the cost of incentives. The proposition shows that this increases the value of the optimal contracts.

For an arbitrary function $\Theta : \mathbb{R}^2_+ \to \mathbb{R}_+ \cup \{\infty\}$ define $D^{\Theta} = \{(\bar{a}, \bar{h}) \in \mathbb{R}^2_+ \mid \Theta(\bar{a}, \bar{h}) < \infty\}$ and $D^{\Theta}_+ = \{(\bar{a}, \bar{h}) \in D^{\Theta} \mid \bar{a} > 0\}$.

Proposition 2 *Consider Θ, $\underline{\Theta}$, and let F^{Θ} and $F^{\underline{\Theta}}$ be as in Theorem 1.*

(i) If $\Theta \geq \underline{\Theta}$ then $F^{\Theta}(w) \leq F^{\underline{\Theta}}(w)$ for all $w \in [0, w^{\underline{\Theta}}_{sp}]$.

(ii) Suppose (Cont) holds. If $\Theta >_{D^{\Theta}_+} \underline{\Theta}$ then $F^{\Theta}(w) < F^{\underline{\Theta}}(w)$ for all $w \in (0, w^{\underline{\Theta}}_{sp})$.

Recall that $\overline{F}$ is the value of an optimal feasible (first best) one period contract-action pair. For $w > 0$, the function $\overline{F}$ solves (1), which is the HJB equation (3) with $\Theta \equiv 0$, while $\overline{F}$ does not satisfy the boundary condition (4). The following proposition shows that when the variance of incentive transfers is identically zero, the functions F_{ζ} from Theorem 1 converge to $\overline{F}$, and so as the period length Δ shrinks to zero the first best value $\overline{F}$ is achievable (see Figure 2).

Proposition 3 *Suppose that $\Theta \equiv 0$ and let F be as in Theorem 1. Then*

$$F = \overline{F}\left(w\right) \quad \text{for all} \quad w > 0.$$

More precisely, for every $\delta > 0$ there is $\overline{\theta}$ such that if $\Theta \leq \overline{\theta}$ then

$$F \geq \overline{F}\left(w\right) - \delta \quad \text{for all} \quad w \geq \delta.$$

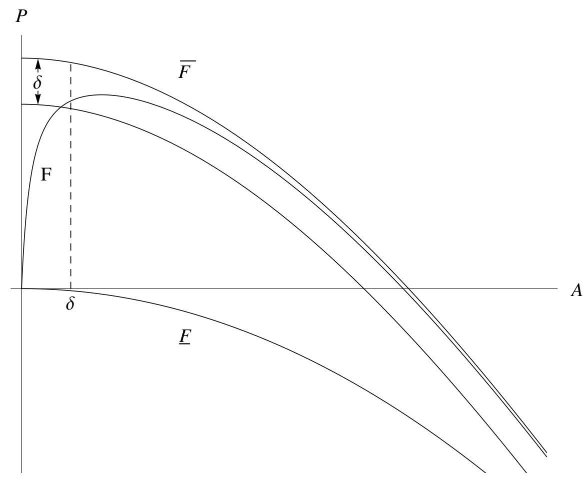

Figure 2: Proposition 3.

3.2 Examples and Comparative Statics

The following example shows that the value of the optimal contract-action plan formulated directly in continuous time (Sannikov [2008]) agrees with the limit of values of discrete time optimal action-plans for a particular signal structure.

Example 1 *Consider the pure hidden action case when X is normally distributed with mean 0 and variance σ^2. For any $\bar{a} \in [0, A]$, since the signal z is uninformative, (FOC$_\Theta$-AC) implies that $a(z) \equiv \bar{a}$. Hence, when $\bar{h} = h(\bar{a})$,*

$$\begin{aligned} \Theta(\bar{a}, h(\bar{a})) &= \min_v \int v(x)^2 g_X(x)dx, \\ s.t. \quad & h'(\bar{a}) = -\int_{\mathbb{R}} v(x) g'_X(x)dx \end{aligned} \tag{7}$$

The optimal solution of this problem is $v(x) = h'(\bar{a})x$ and $\Theta(\bar{a}, h(\bar{a})) = [h'(\bar{a})\sigma]^2$, when $\bar{a} > 0$, and $\Theta(0, 0) = 0$. Moreover, $\Theta(\bar{a}, \bar{h}) = \infty$ for all $\bar{h} \neq h(\bar{a})$. Note that $\Theta(\bar{a}, h(\bar{a}))$ is discontinuous in $\bar{a}$: $\Theta(\bar{a}, h(\bar{a})) \geq [\gamma\sigma]^2 > 0 = \Theta(0, 0)$ for all $\bar{a} > 0$. In this case, the HJB equation (3) with the additional constraint $\bar{a} > 0$ (see Proposition 1) becomes

$$F(w) = \sup_{\bar{a}>0,c} \left\{ (\bar{a} - c) + F'(w)(w + h(\bar{a}) - u(c)) + \frac{1}{2} F''(w) r\sigma^2 h'(\bar{a})^2 \right\},$$

which is exactly Sannikov's equation (5).

The example shows that in the case of pure hidden action models with normal noise the value of the optimal contract depends on the single parameter of the distribution of noise, its variance. We generalize the example in the following way.

Lemma 1 *Consider a pure hidden action model with density* $g_X(x)$. *Then* $\Theta(\bar{a},\bar{h}) = \infty$ *for all* $\bar{h} \neq h(\bar{a})$, $\Theta(0,0) = 0$ *and for* $\bar{a} > 0$

$$\Theta(\bar{a}, h(\bar{a})) = \frac{h'(\bar{a})^2}{VLR(g_X)},$$

where

$$VLR(g_X) = \int \left[\frac{g'_X(x)}{g_X(x)}\right]^2 g_X(x)\, dx = \int \frac{g'_X(x)^2}{g_X(x)}\, dx.$$

Proof. That $\Theta(0,0) = 0$ and $\Theta(\bar{a},\bar{h}) = \infty$ for all $\bar{h} \neq h(\bar{a})$ is clear. Just as in Example 1, the solution of problem (7) for $\bar{a} > 0$, as characterized by the necessary first order conditions, is $v(x) = C\frac{g'(x)}{g(x)}$, where incentive compatibility constraint implies that $C = -\frac{h'(\bar{a})}{VLR(g_X)}$. Consequently, $\Theta(\bar{a}, h(\bar{a})) = \frac{h'(\bar{a})^2}{VLR(g_X)}$, when $\bar{a} > 0$. ■

The variance of incentive transfers, and thus the value of optimal contract-action plans (when Δ shrinks) depends on the single parameter of the underlying noise distribution, the *variance of likelihood ratio* $VLR(g_X)$. The variance of the likelihood ratio is the measure of informativeness of the public signal about the action of the agent: its high value diminishes the informational rent to the agent. Witness the following example.

Example 2 *Consider pure hidden action models for three cases of noise distribution, each with variance* σ^2*: (i) normal distribution, (ii) double exponential distribution and (iii) "linear" distribution, with corresponding densities:*[11]

$$\begin{aligned} g^n_X(x) &= \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\left[\frac{x}{2\sigma}\right]^2}, \quad x \in \mathbb{R} \\ g^{2e}_X(x) &= \frac{\lambda}{2}\, e^{-\lambda|x|}, \quad x \in \mathbb{R} \\ g^l_X(x) &= c - c^2|x|, \quad |x| \leq 1/c, \end{aligned}$$

for $\lambda = \frac{\sqrt{2}}{\sigma}$ *and* $c = \frac{1}{\sigma\sqrt{6}}$. *The corresponding variances of likelihood ratio are:*

$$VLR(g^n_X) = 1/\sigma^2, \quad VLR(g^{2e}_X) = 2/\sigma^2, \quad \text{and} \quad VLR(g^l_X) = \infty.$$

In particular, in the "linear" distribution case the incentives are costless and the first-best is achievable (Proposition 3). Intuitively, with bounded support of the noise, agent's defection from the prescribed action plan gives rise to signals that would not occur otherwise, and those signals have sufficiently high probability (density has sufficient mass at the extremes).

Note that the pure hidden action model satisfies the assumption (Cont) with $\delta(\bar{a}) \equiv \frac{h'(0_+)}{VLR(g_X)}$. Thus, a direct consequence of Lemma 1 and part (ii) of Proposition 2 is the following:

[11] Formally, the "linear" distribution does not satisfy our assumption (A2-AC) as it results in infinite variance of likelihood ratio. But one may consider approximations with the density at the extremes of the support changed to, say, quadratic functions, resulting in finite but arbitrarily large variance of likelihood ratio.

Corollary 1 *In the pure hidden action model with density $g_X(x)$, the limit value of the optimal contract-action plans as Δ shrinks to zero is increasing in the variance of likelihood ratio $VLR(g_X)$.*

Let us now study the settings in which the agent has some private information about the environment.

Lemma 2 *Consider a hidden information model where the range of the agent's signal z is a finite set of values $\{z_1, \ldots, z_K\}$. Then the variance of incentives transfer function is*

$$\Theta(\bar{a}, \bar{h}) = \min_{a_1,\ldots,a_k} \sum_{k=1}^{K} \lambda_k h'(a_k), \quad s.t. \quad \bar{a} = \sum_{k=1}^{K} g_Z(z_k) a_k \quad and \quad \bar{h} = \sum_{k=1}^{K} g_Z(z_k) h'(a_k),$$

where $\lambda = (\lambda_1, \ldots, \lambda_K)^T$ solves the linear system of equations $L \cdot \lambda = (h'(a_1), \ldots, h'(a_K))^T$ with matrix $L = [L_{k.\ell}]$ given by

$$L_{k,\ell} = \int_{\mathbb{R}} \frac{g'_{X|Z}(x|z_k) g'_{X|Z}(x|z_l)}{g_X(x)} dx \quad k, \ell = 1, \ldots, K.$$

Proof. $v(x) = -\frac{1}{g_X(x)} \sum_{k=1}^{K} \lambda_k g'_{X|Z}(x|z_k)$ is the solution to the following auxiliary problem:

$$\begin{aligned} \hat{\Theta}(a_1, \ldots, a_K) &= \inf_{v} \int_{\mathbb{R}} v(x)^2 g_X(x) dx \\ &\quad \text{s.t.} \quad h'(a_k) = -\int_{\mathbb{R}} v(x) g'_{X|Z}(x|z_k) dx \quad k = 1, \ldots, K, \end{aligned}$$

and λ is the vector of Lagrange multipliers for the constraints. Thus $\hat{\Theta}(a_1, \ldots, a_K)$ equal to $\sum_{k=1}^{K} \lambda_k h'(a_k)$ is the variance of incentive transfers for arbitrary actions $a_1, \ldots, a_K$. Minimizining over all actions with mean action $\bar{a}$ and mean cost $\bar{h}$ yields the result. ■

Thus with finitely many actions solving for the cost of incentives reduces to solving a finite dimensional unconstrained minimization problem. Associated to this hidden information model, there is a natural pure hidden action model, where the agent does not observe the signal z. For this pure hidden action model, the corresponding variance of incentives transfer function is given by $\Theta_{\text{PHA}}(\bar{a}, h(\bar{a})) = \hat{\Theta}(\bar{a}, \ldots, \bar{a})$. Since $a_k = \bar{a}$ for all k is always feasible when $\bar{h} = h(\bar{a})$, we obtain the following result.

Corollary 2 *Let Θ be as in Lemma 2 and Θ_{PHA} be the variance of incentive transfers for the associated pure hidden information model. Then $\Theta \leq \Theta_{PHA}$ and $F^{\Theta_{PHA}} \leq F^{\Theta}$.*

In other words, the case when agent has no private information about the environment results in least efficient contracts.

Example 3 *An interesting special case of Lemma 2 is when for a given density $g_0(x)$ with mean 0, $g_{X|Z}(x|z_k) = g_0(x - \mu_k)$ for some μ_k, where $\sum_k g_Z(z_k)\mu_k = 0$. For a concrete example, assume that $K = 2$, $g_Z(z_1) = g_Z(z_2) = 1/2$,*

$$g_0(x) = C[1 + cos(x)] \quad for \quad x \in [-\pi, \pi], \quad and \quad -\mu_1 = \mu_2 = \pi/2,$$

where $C = 1/[2\pi]$. Then, $g_{X|Z}(x|z_1) = C[1-sin(x)]$ in $[-3\pi/2, \pi/2]$, and $g_{X|Z}(x|z_2) = C[1+sin(x)]$ in $[-\pi/2, 3\pi/2]$. Hence, $g'_{X|Z}(x|z_1) = -Ccos(x)$ and $g'_{X|Z}(x|z_2) = Ccos(x)$,

$$L = \frac{1}{4}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}, \quad \lambda_1 = \frac{1}{6}[5h'(a_1) + h'(a_2)] \quad and \quad \lambda_2 = \frac{1}{6}[h'(a_1) + 5h'(a_2)],$$

which yields $\hat{\Theta}(a_1, a_2) = \frac{1}{6}[5h'(a_1)^2 + 2h'(a_1)h'(a_2) + 5h'(a_2)^2]$. For generic h, $\bar{a}$ and $\bar{h}$, the constraints $\bar{a} = \frac{1}{2}[a_1 + a_2]$ and $\bar{h} = \frac{1}{2}[h(a_1) + h(a_2)]$ uniquely determine (a_1, a_2) (when the system is compatible) and $\Theta(\bar{a}, \bar{h}) = \hat{\Theta}(a_1, a_2)$. Then, for a fixed $\bar{a}$,

$$\min_{\bar{h}} \Theta(\bar{a}, \bar{h}) = \min \ \frac{1}{6}[5h'(a_1)^2 + 2h'(a_1)h'(a_2) + 5h'(a_2)^2] \quad s.t. \quad \bar{a} = \frac{1}{2}[a_1 + a_2].$$

The optimal solution is $a_1 = a_2 = \bar{a}$ when h'^2 is a convex function, and $a_1 = 0$ and $a_2 = 2\bar{a}$ when h'^2 is a concave function. For example, if $h(a) = \frac{4}{5}a^{5/4} + \gamma a$, where $\gamma > 0$ implies that $h'^2 = [a^{1/4} + \gamma]^2$ is concave. In this case, $\min_{\bar{h}} \Theta(\bar{a}, \bar{h}) = \hat{\Theta}(0, 2\bar{a}) < \hat{\Theta}(\bar{a}, \bar{a}) = \Theta_{PHA}(\bar{a}, h(\bar{a}))$.

The example illustrates that in the hidden information model, in any period, for a fixed mean effort $\bar{a}$ there might arise a tradeoff between two sorts of implementation costs. On one hand, given convex cost function, a “flat” effort scheme $a(z) \equiv \bar{a}$ minimizes the *direct cost of effort* $\mathbb{E}[h(a(z))]$ to be $h(\bar{a})$. On the other hand, the *cost of incentives* $\Theta(\bar{a}, \bar{h})$ might be minimized for the effort schemes that is responsive to the environment and does not satisfy $a(z) \equiv \bar{a}$, which implies $\bar{h} > h(\bar{a})$. How the tradeoff is resolved depends on the relative “prices” of each cost in the differential equation (3), given by F' and F'' respectively.

Consider now the pure hidden information case. The following result allows us to rank the distributions of noise in terms of the cost of incentives that they impose. The condition is a strong form of ranking of signal’s dispersion.[12,13]

Lemma 3 *Consider two signal distributions G and Γ of noise for the pure hidden information case, with corresponding strictly positive densities g and γ. Suppose that:*

$$G(x) = \Gamma(x') \implies g(x) \geq \gamma(x'), \ \forall x, x'$$

Then, for the corresponding variance of incentivizing transfer functions Θ_G and Θ_Γ,

$$\Theta_\Gamma \geq \Theta_G.$$

[12] The condition implies lower variance, but is incomparable to SOSD: a SOS inferior distribution can either dominate or be dominated in terms of our ranking.

[13] Essentially the same proof also shows the strict ranking of variances of incentive transfers, as long as the “inf” in the definition of Θ_Γ can be replaced by “min”. In the cases when the optimal policies for Θ_Γ do not exist, we conjecture that under some additional conditions on the distributions G and Γ the proof can be strengthened to yield strict ranking.

Proof. Fix $(\bar{a}, \bar{h})$ in the domain of Θ_Γ, any $\varepsilon > 0$ and let (a_Γ, v_Γ) be an ε-suboptimal policy for $\Theta_\Gamma(\bar{a}, \bar{h})$. We may assume that $\mathbb{E}_\Gamma [v_\Gamma (x)] = 0$. We will define a policy (a_G, v_G) that is feasible for the problem $\Theta_G(\bar{a}, \bar{h})$ and such that $\mathbb{E}_G \left[v_G^2 (x)\right] \leq \Theta_\Gamma(\bar{a}, \bar{h})$.

For any x let $a_G (x) = a_\Gamma (x')$ where x' is such that $G (x) = \Gamma (x')$. Since both G and Γ are strictly increasing between 0 and 1, a_G is well defined. We have:

$$\int a_G(x)g(x)dx = \int a_G(G^{-1}(\Gamma(x')))\gamma(x')dx' = \int a_\Gamma(x')\gamma(x')dx',$$

where we have used the change of variables $x = G^{-1}(\Gamma(x'))$, so $\frac{dx}{dx'} = \frac{\gamma(x')}{g(x)}$. Similarly we get that $\int h(a_G(x))g(x)dx = \int h(a_\Gamma(x'))\gamma(x')dx'$.

The incentive transfer function v_G is defined via the (FOC$_\Theta$-PHI) condition:

$$v_G'(x) = h'(a_G(x)),$$

except for the finitely many points of discontinuity of $a_G(x)$, where it is extended continuously, together with the condition $\mathbb{E}_G[v_G(x)] = 0$. Choose any points $\overline{x} > \underline{x}$, $\overline{x}' > \underline{x}'$ such that $G(\overline{x}) = \Gamma(\overline{x}')$ and $G(\underline{x}) = \Gamma(\underline{x}')$. We have:

$$\begin{aligned} v_G(\overline{x}) - v_G(\underline{x}) &= \int_{\underline{x}}^{\overline{x}} h(a_G(x))dx \\ &= \int_{\underline{x}'}^{\overline{x}'} h(a_\Gamma(x'))\frac{\gamma(x')}{g(x)}dx' \leq \int_{\underline{x}'}^{\overline{x}'} h(a_\Gamma(x'))dx' = v_\Gamma(\overline{x}') - v_\Gamma(\underline{x}'). \end{aligned}$$

This means that the random variable $v_G (X)$, $X \sim G$, is *less dispersed* than $v_\Gamma (X)$, $X \sim \Gamma$. Since $\mathbb{E}_\Gamma [v_\Gamma (x)] = \mathbb{E}_G [v_G (x)] = 0$, this implies that for the concave function $\phi (x) = x^2$ we have $\mathbb{E}_G \left[v_G^2 (x)\right] \leq \mathbb{E}_\Gamma \left[v_\Gamma^2 (x)\right] \leq \Theta_\Gamma(\bar{a}, \bar{h}) + \varepsilon$ (see Theorem 3.B.2 in Shaked and Shanthikumar [2007], which is taken from Landsberger and Meilijson [1994]). Since $\varepsilon > 0$ is arbitrary, this concludes the proof. ■

The following Lemma is related to Corollary 2.

Lemma 4 *Fix $\sigma^2 > 0$ and any signal distribution with variance of noise σ^2. Let Θ be the corresponding variance of incentive transfers, and let Θ^n correspond to the pure hidden information model with normal noise. Then $\Theta \leq \Theta^n$ and $F^{\Theta^n} \leq F^\Theta$.*

As a corollary, the value of contracts computed directly in continuous time (see Example 1) is the lower bound on the limits of values, for all information structures that approximate the same continuous time limit (in the sense described above).

Proof. The optimal policy function for $\Theta (\bar{a}, h (\bar{a}))$ with $\bar{a} > 0$ in the pure hidden action case with normal distribution has a linear incentivizing transfer function $v(x) = h'(\bar{a})x$, and $\Theta (\bar{a}, h (\bar{a})) = h' (\bar{a})^2 \sigma^2$ (Example 1). This incentivizing transfer function provides not only "ex-ante", but also "ex-post" incentives, thus incentivizing the agent to a constant effort scheme $a (z) = \bar{a}$ under any distribution of signals. As long as the variance of noise X is σ^2, the variance of incentivizing transfers is $h' (\bar{a})^2 \sigma^2$. The Lemma thus follows from Proposition 2, part (i). ■

3.3 Solution to the Principal's Problem: Contract-Action Plans

In this section we show how to construct contract-action plans that are approximately optimal as the period length is short. For this purpose, we first define the Bellman operator associated with the principal's problem, i.e. stage-game maximization problem parametrized by agent's continuation value. We then define policies for the Bellman operator, based on the policies for the variance of incentive transfers function (2) and the policies for the perturbed HJB equation (6). The contract-action plans we construct are defined through a recursive application of those policies, as explained below.

For any interval $I \subset \mathbb{R}$ and any function $f : I \to \mathbb{R}$, define the new function $T_I^\Delta f : \mathbb{R}_+ \to \mathbb{R}$ by

$$\begin{aligned} T_I^\Delta f(w) \quad &= \quad \sup_{a,c,W} \ \Phi^\Delta(a,c,W;f) && (8) \\ \text{s.t.} \quad & a(z) \in \mathcal{A} \quad \forall z, \quad c(y) \geq 0 \quad \text{and} \quad W(y) \in I \quad \forall y \\ & w = \mathbb{E}^\Delta \Big[\tilde{r}\Delta[u(c(\Delta[x+a(z)])) - h(a(z))] + e^{-r\Delta} W(\Delta[x+a(z)]) \Big] && \text{(PK)} \\ & a(z) \in \arg\max_{\hat{a}\in\mathcal{A}} \ \mathbb{E}^\Delta \Big[\tilde{r}\Delta[u(c(\Delta[x+\hat{a}])) - h(\hat{a})] + e^{-r\Delta} W(\Delta[x+\hat{a}]) \mid z \Big] \quad \forall z && \text{(IC-AC)} \\ & a(x) \in \arg\max_{\hat{a}\in\mathcal{A}} \ \tilde{r}\Delta\big[u(c(\Delta[x+\hat{a}])) - h(\hat{a})\big] + e^{-r\Delta} W(\Delta[x+\hat{a}]) \quad \forall x && \text{(IC-PHI)} \end{aligned}$$

where the supremum is taken over measurable functions and

$$\Phi^\Delta(a,c,W;f) = \mathbb{E}^\Delta \Big[\tilde{r}\Delta[a(z) - c(\Delta[x+a(z)])] + e^{-r\Delta} f(W(\Delta[x+a(z)])) \Big]. \qquad (9)$$

In the problem, (PK) is the *promise-keeping constraint* and (IC) is the *incentive constraint*. Here (and similarly below) we write problem $T_I^\Delta f(w)$ with two alternative constraints: (IC-AC) and (IC-PHI), for the absolutely continuous and the pure hidden information cases. The interval I in the problem represents an additional constraint that restricts the range of the continuation values. This constraint ensures that continuation values are in the domain of f.

For $I = [\underline{w}, \overline{w}]$ and $\Delta > 0$ (small), let $I^\Delta = \big[\underline{w} + \Delta^{1/3}, \overline{w} - \Delta^{1/3}\big]$.

Definition 1 *For any $\zeta \geq 0$ and F_ζ solving (6) on an interval I, period length $\Delta > 0$, agent's promised value $w \in I$ and an approximation error $\varepsilon > 0$, define a* simple policy (a, c, W) *as follows. Let $(\bar{a}, \bar{h}, c)$ be an ε-suboptimal policy of (6) at w, and for the corresponding $(\bar{a}, \bar{h})$, let (a, v) be an ε-suboptimal policy of (2).*

If $w \in I^\Delta$ let (see Figure 3)

$$\begin{aligned} & c(y) = c && (10) \\ & W(y) = C + \sqrt{\Delta}\tilde{r}e^{r\Delta} \times v(y/\sqrt{\Delta})\mathbf{1}_{|v(y/\sqrt{\Delta})| \leq M_\varepsilon} && \textit{(AC)}, \\ & W(y) = C + \sqrt{\Delta}\tilde{r}e^{r\Delta} \times \begin{cases} v(-M_\varepsilon) & \textit{if} \quad y/\sqrt{\Delta} < -M_\varepsilon \\ v(y/\sqrt{\Delta}) & \textit{if} \quad |y/\sqrt{\Delta})| \leq M_\varepsilon \\ v(M_\varepsilon) & \textit{if} \quad y/\sqrt{\Delta} > M_\varepsilon \end{cases} && \text{(PHI)}, \end{aligned}$$

$a(z)$ is an action that satisfies the (IC) constraint in (8),

where M_ε is a (large) constant (see Definition 4 in the Appendix) and C is chosen so that the promise keeping (PK) is satisfied.

If $w \notin I^\Delta$ let

$$c(y) = u^{-1}(w), \quad W(y) = w, \quad a(z) = 0. \tag{11}$$

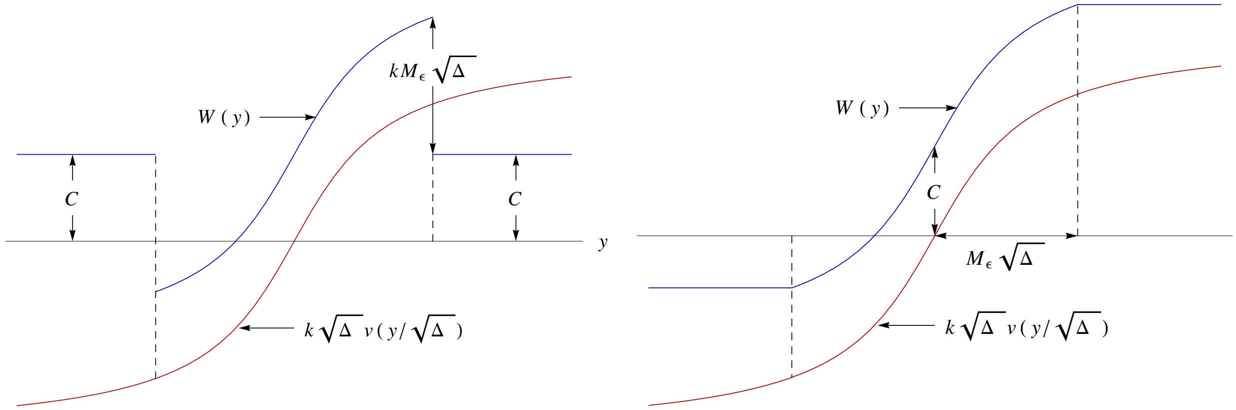

Figure 3: Continuation values for the AC and PHI cases in Definition 1, where $k = \tilde{r}e^{r\Delta}$.

Lemma 11 in Section 4 shows that in the definition of the action scheme a above, the global incentive constraint (IC) is essentially equivalent to the local incentive constraint.

Note that a simple policy depends on F_ζ, ε, Δ and w. A simple policy is feasible for (8): it satisfies the (PK) and (IC) constraints by construction, and $W(y) \in I$ for all y, if the period length Δ is small enough. Finally, note that even for fixed parameters F_ζ, ε, Δ and w there are many simple policies, each corresponding to some ε-suboptimal policies in the problems (2) and (6).[14]

For any initial agent's promised value $w \in [0, \bar{u})$, a set of simple policy functions for all promised values in $[0, \bar{u})$ generates a full (wage) contract-action plan.

Definition 2 *For any $\zeta \geq 0$ and F_ζ solving (6) on an interval I, $\Delta > 0$, $w \in [0, \bar{u})$ and $\varepsilon > 0$ let a* simple contract-action plan *be that defined recursively by a set of simple policies for all $w \in [0, \bar{u})$. That is, if (a, c, W) is a simple policy for w, the plan specifies $(a_0, c_0) = (a, c)$. Then, for each signal y, if $\hat{w} = W(y)$ and $(\hat{a}, \hat{c}, \hat{W})$ is a corresponding simple policy for $\hat{w}$, the plan specifies $(a_1, c_1) = (\hat{a}, \hat{c})$ for the second period. And so on.*

Given a continuation value w for the agent - as a state variable - the algorithm for deriving a simple contract-action plan is as follows. An ε-suboptimal policy $(\bar{a}, \bar{h}, c)$ of the HJB equation (6) at w determines the wage payment and the approximate mean effort and mean cost of effort in

[14]Without additional assumptions, optimal policies for the two problems need not exist.

the current period. Given $(\bar{a}, \bar{h})$, an ε-suboptimal policy (a, v) in the problem of minimizing the variance of transfers (2) is used to construct the scheme of incentivizing continuation values. The continuation value scheme, as a function of the public signal, is equal to function v, which must be linearly rescaled accordingly to the period length and the promised continuation value, as well as truncated at a high absolute value. The action scheme is approximately the same as a; the difference is due to the fact that the truncation of v affects the incentives to a small degree (see Section 4.3).

Remark 1 *In the pure hidden information case, the v in the definition of a simple policy at w is continuous and piecewise twice continuously differentriable (see the definition of Θ). From now on we assume that for any $\varepsilon > 0$, there is a common finite set D_ε such that the set of functions v'' for all $w \in [0, \bar{u})$ are equicontinuous outside of D_ε, which is without loss of generality*[15].

By construction, any simple contract-action plan is incentive compatible. Therefore, its value to the principal is bounded above by F^Δ. The following Theorem, which is the second main result of the paper, shows that any simple contract-action plan is close to optimal (proof is in Section 4).

Theorem 2 *For $\zeta \geq 0$ and F_ζ solving (6) with the boundary conditions (4) and (5) on an interval $I = [0, w_{sp,\zeta}]$, period length Δ, agent's promised value $w \in [0, \bar{u})$ and an approximation error $\varepsilon > 0$ a simple contract-action plan is $[O(\zeta) + O(\varepsilon) + O(\Delta^{1/3})]$-suboptimal.*

In the case of pure hidden action, we follow up on Lemma 1 and Example 2 from the previous section but now in the context of contract-action plans.

Lemma 5 *Consider a pure hidden action model with density $g_X(x)$. Then a simple contract-action plan is based on a process of continuation values that is truncated linear in the likelihood ratio. More precisely, it is based on simple policies (a, c, W) such that for any $w \in [0, w_{sp,\zeta}]^\Delta$ the process of continuation values driven by the public signal (revenue) is given by:*

$$W(y) = C - \sqrt{\Delta}\tilde{r}e^{r\Delta}\lambda(y) \times \mathbf{1}_{|\lambda(y)| \leq M_\varepsilon} \quad \textit{where} \quad \lambda(y) = \frac{h'(\bar{a})}{VLR(g_X)} \times \frac{g'_X(y/\sqrt{\Delta})}{g_X(y/\sqrt{\Delta})}$$

and $\bar{a}$ and C are as in Definition (1).

Example 4 *Consider again pure hidden action models for normal, double exponential and "linear" distribution, all with variance σ^2. Then the simple contract-action plans from Lemma 5 have continuation values processes given by*

$$\begin{aligned} W^n(y) &= C_1 + C_2 \times y \times \mathbf{1}_{|y| \leq C_3}, \\ W^{2e}(y) &= \begin{cases} \overline{C}, \text{ when } y \geq 0 \\ \underline{C}, \text{ when } y < 0 \end{cases}, \\ W^l_X(y) &= C_1 + C_2 \times \frac{sgn(y)}{(1 - |y|)} \times \mathbf{1}_{(1-|y|)^{-1} \leq C_3}, \end{aligned}$$

[15] The proof is immediate from the fact that $(F'_\zeta(w), F''_\zeta(w))$ is continuous on I, and so for a fixed policy $(\bar{a}, \bar{h}, c)$ the right-hand side of equation (6) is contionuous in w. Since I is compact, this shows that for any $\varepsilon > 0$ the set of ε-soboptimal policies $(\bar{a}, \bar{h}, c)$ can be assumed to be finite, which yields the proof.

for appropriate constants, as in Lemma 5.

Holmstrom and Milgrom [1987] show that in the continuous time model it is optimal to use transfers linear in the public signal. The Lemma above generalizes this result, with the standard interpretation of continuation values as "transfers in utility": In the pure hidden action discrete time models and arbitrary noise structure, it is essentially optimal to use transfers linear in the likelihood ratio. (For the normal noise, likelihood ratio is equal to the public signal.) For example, when noise is double exponential the continuation values in every period take only two values, with the threshold signal of zero. We note that while such threshold schemes are often optimal in cases when agent has only two actions available, in our model the agent has continuum of available actions.

In the following we ask the question whether, as period length shrinks, the details of the signal structure matter for the design of approximately optimal contracts. Recall that in the case of values (Theorem 1) the dependence was fully captured by the variance of incentivizing transfers function Θ. However, note that the simple contract-action plans in Lemma 5 and Example 4 look very different for different noise distributions, even if they correspond to the same Θ: witness the case of the normal distribution with variance 1 and a double-exponential distribution with variance 2 (Theorem 1 and Example 4). It is not difficult to establish that, in the pure hidden action model, the contract-action plans *must* be based on continuation value processes that are close to linear in likelihood ratio, as in Lemma 5. Thus, for example, while the continuation values linear in revenue will work for the normal noise, they will be very suboptimal when the noise is double-exponential. One would like to conclude from this that there is no single contract that will work for two different noise structures.

The following Proposition establishes that the conclusion is in fact correct. We note that the conclusion requires a more elaborate argument than the discussion above suggests, as the continuation value process is defined endogenously, relative to the noise structure (the same contract gives rise to different processes, for different noise structures).

Consider two noise distributions with densities g_X and γ_X that have the same variance of likelihood ratio but linearly independent likelihood ratios:

$$VLR(g_X) = VLR(\gamma_X), \quad \inf_C \mathbb{E}_{g,\gamma}\left[\frac{g'_X(x_g)}{g_X(x_g)} - C\frac{\gamma'_X(x_\gamma)}{\gamma_X(x_\gamma)}\right]^2 > 0. \tag{12}$$

Proposition 4 *For any $\Delta > 0$ consider two pure hidden action models with noise densities g_X and f_X that satisfy (12). For every $w_g, w_f \in (0, w_{sp})$ there exists $\delta > 0$, such that for sufficiently small Δ there is no contract $\{c_n\}$ that is δ-suboptimal for the two distributions and delivers values w_g and w_f.*

The Proposition compares the contracts for the special case of signal structures with pure hidden action and the same values of the optimal contracts (as period length shrinks). While this is the most relevant case, as this is exactly when one would suspect the same contract to work, we also comment in the Appendix how to extend the proof to the case of arbitrary two signal structures.

A different way to interpret the result is to say that knowing the optimal continuous-time contract only would provide little guidance as to how the (close to) optimal discrete time contracts look like, no matter how short the period length. Such contracts must depend on the distribution of noise in the discrete-time models, as in Theorem 2.

On the other hand, in the context of pure hidden action models with the same variance of likelihood ratio, the simple contract-action plans determine the unique continuous-time contract. For any continuous time Brownian Motion process $\{B_t\}$ and $VLR(g_X) > 0$, $w \in [0, \bar{u})$ consider a continuous time process $\{W_t\}$ that starts at w and satisfies the stochastic differential equation:

$$dW_t = r\left(W_t - u\left(c\left(W_t\right)\right) + h\left(a\left(W_t\right)\right)\right) dt + r\frac{h'\left(a\left(W_t\right)\right)}{\sqrt{VLR\left(g_X\right)}} dB_t, \tag{13}$$

where $c(W_t)$ and $a(W_t)$ are the minimizers[16] in the solution of (3) with the boundary conditions (4) and (5), together with:

$$W_t = W_\tau, \text{ for } t \geq \tau,$$

where τ is a random time when W_t hits 0 or $w_{sp.}$ The process determines a pair of continuous time processes $(\{c_t\}, \{a_t\})$ such that:

$$a_t = \begin{cases} a\left(W_t\right), \text{ for } t < \tau \\ 0, \text{ for } t \geq \tau \end{cases}, \quad c_t = \begin{cases} c\left(W_t\right), \text{ for } t < \tau \\ -\underline{F}\left(W_\tau\right), \text{ for } t \geq \tau \end{cases}. \tag{14}$$

In the case when $VLR(g_X) = 1$, for any promised value to the agent $w \in [0, \bar{u})$, the pair $(\{c_t\}, \{a_t\})$ is the optimal continuous-time contract derived in Sannikov [2008].[17] The proof follows from the Invariance Principle (see e.g. Theorem 4.20 in Karatzas and Shreve [1991]).

Lemma 6 *Consider a pure hidden action model with noise density g_X. For any $\varepsilon, \Delta > 0$ and $w \in [0, \bar{u})$ let $(\{c_n^{\Delta,\varepsilon}\}, \{a_n^{\Delta,\varepsilon}\})$ be a simple contract-action plan for F_ε solving (6) with the boundary conditions (4) and (5). Then*

$$\lim_{\varepsilon \to 0} \lim_{\Delta \to 0} (\{c_t^{\Delta,\varepsilon}\}, \{a_t^{\Delta,\varepsilon}\}) = (\{c_t\}, \{a_t\}),$$

where $(\{c_t\}, \{a_t\})$ is the continuous time process defined in (13) and (14) for w, $(\{c_t^{\Delta,\varepsilon}\}, \{a_t^{\Delta,\varepsilon}\})$ is the linear interpolation of $(\{c_n^{\Delta,\varepsilon}\}, \{a_n^{\Delta,\varepsilon}\})$, and the convergence is in the weak topology.*

4 Proof of Theorems 1 and 2

4.1 Bellman Operators

Recall the Bellman operator T_I^Δ associated with the principal's problem, i.e. the stage-game maximization problem parametrized by the agent's continuation value, defined in (8). In particular,

[16] The sets $[0, A]$ and $\left[0, u^{-1}\left(w_{sp}\right)\right]$ are compact, while the right hand side of (3), for the case when $\overline{h} = h\left(\overline{a}\right)$ is continuous in $\overline{a}$ and c, except possibly at $\overline{a} = 0$. Since however in this case the policies satisfy $\overline{a} > 0$ (see Proposition 1), then the right hand side has the same value and policy as the problem for which $\Theta\left(0, 0\right) := \lim_{\overline{a} \to 0} \Theta\left(\overline{a}, h\left(\overline{a}\right)\right)$.

[17] We identify two processes that agree in distribution;

$T^{\Delta}_{[0,\bar{u})}$ is the Bellman operator associated with the principal's optimization problem. The following Proposition is a direct consequence of *self-generation*.

Proposition 5 *F^{Δ} is the largest fixed point f of $T^{\Delta}_{[0,\bar{u})}$ such that $f \leq \overline{F}$.*

More generally, for any interval $I \subset \mathbb{R}$, let F^{Δ}_I be the largest fixed point f of T^{Δ}_I such that $f \leq \overline{F}$.

Proposition 6 *Let $I \subset \mathbb{R}$ be any interval. Then for any two bounded functions $f_1, f_2 : I \to \mathbb{R}$, $|T^{\Delta}_I f_1 - T^{\Delta}_I f_2|^+_I \leq e^{-r\Delta}|f_1 - f_2|^+_I$.*

Proof. The proof is analogous to that of the Blackwell's theorem (Blackwell [1965]). ■

In order to establish our results, it will be useful to also consider other related Bellman operators with a modified objective function and/or constraints. If we restrict the consumption schedule $c(y)$ to be constant, we obtain the operator $T^{\Delta,c}_I$. This is the appropriate operator when the principal is restricted to offer wage contracts only. In the pure hidden information case, let us also analogously define $T^{\Delta,d}_I f(w)$ as $T^{\Delta,c}_I f(w)$ with the additional constraints that $a(\cdot)$ is piecewise continuously differentiable and $W(\cdot)$ is continuous. In the absolutely continuous case, we let $T^{\Delta,d}_I = T^{\Delta,c}_I$.

We also consider a modified Bellman operator with a quadratic objective function and simplified constraints. The continuation value $f(W(y))$ in the objective function is replaced by its quadratic approximation around the agent's promised value w, the feasibility constraint on continuation values is dropped, the consumption schedule is restricted to be constant, only first order conditions are required, and the signal $y = \Delta[x + a(z)]$ is approximated by just Δx:[18]

$$
\begin{aligned}
T^{\Delta,q}f(w) \quad = \quad & \sup_{a,c,W} \; \Phi^{\Delta,q}(a,c,W;f,w) && (15)\\
& \text{s.t.} \quad a(z) \in \mathcal{A} \quad \forall z, \quad c \geq 0, \quad \text{and} \quad W(y) \in \mathbb{R} \quad \forall y \\
& \qquad w = \mathbb{E}^{\Delta}\big[\tilde{r}\Delta[u(c) - h(a(z))] + e^{-r\Delta}W(\Delta x)\big] && (\text{PK}_q)\\
& \qquad \tilde{r}h'(a(z)) = -\frac{e^{-r\Delta}}{\Delta}\int_{\mathbb{R}} W(\Delta x) g^{\Delta\prime}_{X|Z}(x|z)dx \quad \forall z && (\text{FOC}_q\text{-AC})\\
& \qquad \tilde{r}h'(a(x)) = e^{-r\Delta}W'(\Delta x) \quad \forall x && (\text{FOC}_q\text{-PHI}),
\end{aligned}
$$

where

$$
\begin{aligned}
Q(v;f,w) &= f(w) + f'(w)(v-w) + \frac{f''(w)}{2}(v-w)^2 \quad \text{and} \\
\Phi^{\Delta,q}(a,c,W;f,w) &= \mathbb{E}^{\Delta}\Big[\tilde{r}\Delta[a(z) - c] + e^{-r\Delta}Q(W(\Delta x);f,w)\Big]. \qquad (16)
\end{aligned}
$$

Moreover, in the pure hidden information case, the supremum is over piecewise continuously differentiable functions $a(\cdot)$ and continuous functions $W(\cdot)$, and the (FOC_q-PHI) condition is required everywhere except for the (finitely many) points of discontinuity of $a(\cdot)$.

[18] When $a(z) = 0$, the equalities in the IC constraints below should be replaced by the inequality $\tilde{r}h'(0_+) \geq$ r.h.s, and when $a(z) = A$, these equalities should be replaced by $\tilde{r}h'(A_-) \leq$ r.h.s. In the pure hidden information case, when $a(z) = 0$ or $a(z) = A$, at an optimum the inequalities are attained with equality (see e.g. Edmans and Gabaix [2011]).

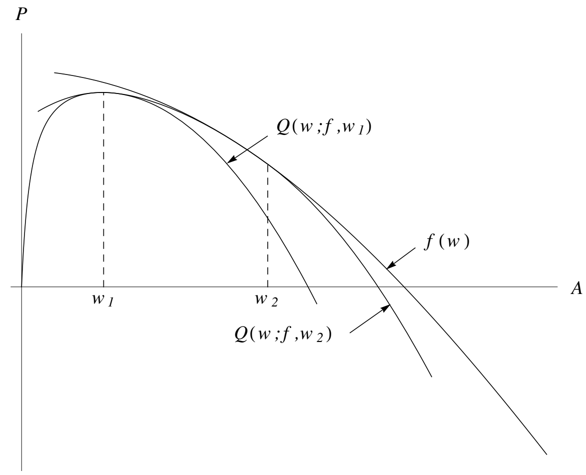

Figure 4: Quadratic approximations.

4.2 From Bellman Operators to HJB Equation

Below we establish the bounds in Theorem 1 and the proof of Theorem 2. Uniqueness of the solutions F_ζ of the differential equations and the existence of the limits $F = \lim F_\zeta$ and $w_{sp} = \lim w_{sp,\zeta}$ in the statement of the Theorem 1 rely solely on the properties of the differential equations (3) and (6) and are established in Section 7.2 (Lemma 18 and Corollary 3), where we more broadly analyze their properties.

The proof is based on the following crucial Proposition 7 and Lemma 7. The Proposition establishes that, roughly, F_ζ is close to being a fixed point of the T_I^Δ operator, and that the simple policies are almost optimal in the problem $T_I^\Delta F_\zeta$. Lemma 7 establishes that if F_ζ is close to a fixed point of T_I^Δ then the largest fixed point must be close to F_ζ, and that the almost optimal policies for $T_I^\Delta F_\zeta$ are almost optimal in the full dynamic programming problem.[19]

Proposition 7 *Fix $\zeta \geq 0$ and F_ζ solving the HJB equation (6) on an interval I with $F''_\zeta < 0$. Then $|T_I^\Delta F_\zeta - F_\zeta|_{I^\Delta} = o(\Delta) + O(\zeta\Delta)$. Moreover, for any $\varepsilon > 0$, $\Delta > 0$ and $w \in I^\Delta$, $\Phi^\Delta(a, c, W; F_\zeta) \geq F_\zeta(w) - O(\varepsilon\Delta) - O(\zeta\Delta)$, where (a, c, W) is a simple policy defined for $(F_\zeta, \varepsilon, \Delta, w)$ by (10) and (11).*

For an interval I, $\Delta > 0$ and a set of feasible policies $p = \{(a, c, W)\}_{w\in I}$ for the Bellman operator T_I^Δ let $T_I^{\Delta,p}$ be the operator defined as $T_I^{\Delta,p} f(w) = \Phi^\Delta(a, c, W; f)$, and let $F_I^{\Delta,p}$ be the value achieved by the policies p. Note that $F_I^{\Delta,p}$ is a fixed point of $T_I^{\Delta,p}$.

Lemma 7 *Let F be defined on an interval I.*

(i) If for some $\varepsilon \geq 0$ and $J \subseteq I$

$$\left|T_I^\Delta F - F\right|_J^+ = o(\Delta) + O(\varepsilon\Delta),$$

[19] See also proof below Lemma 7 in Biais, Mariotti, Plantin, and Rochet [2007].

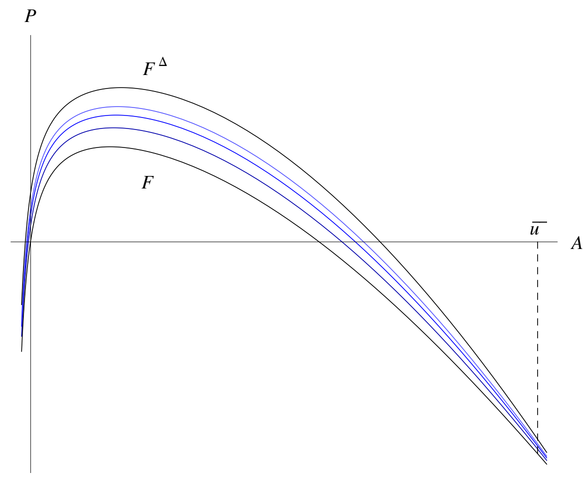

Figure 5: Lemma 7*(i)*. Functions F, $T_I^\Delta F$, $(T_I^\Delta)^2 F$, $(T_I^\Delta)^3 F$, and F^Δ.

then

$$\left|F_J^\Delta - F\right|_J^+ = O(\varepsilon) + \frac{o(\Delta)}{\Delta}.$$

(ii) If for some $\varepsilon \geq 0$ and $J \subseteq I$

$$\left|F - T_I^{\Delta,p} F\right|_J^+ = o(\Delta) + O(\varepsilon\Delta),$$

then

$$\left|F - F_I^{\Delta,p}\right|^+ = O(\varepsilon) + \left|F - F_I^{\Delta,p}\right|_{I\setminus J}^+ + \frac{o(\Delta)}{\Delta}.$$

Proof. (i) Fix $\Delta > 0$ We have

$$\begin{aligned} \left|F_J^\Delta - F\right|_J^+ &\leq \left|F_J^\Delta - T_J^\Delta F_J^\Delta\right|_J^+ + \left|T_J^\Delta F_J^\Delta - T_J^\Delta F\right|_J^+ + \left|T_J^\Delta F - F\right|_J^+ \leq \\ &\leq e^{-r\Delta}\left|F_J^\Delta - F\right|^+ + \left|T_J^\Delta F - F\right|_J^+, \end{aligned}$$

Consequently

$$\left|F_J^\Delta - F\right|^+ \leq \frac{\left|T_J^\Delta F - F\right|_J^+}{r\Delta} \leq \frac{\left|T_I^\Delta F - F\right|_J^+}{r\Delta} = O(\varepsilon) + \frac{o(\Delta)}{\Delta}.$$

(ii) The proof is analogous to case (i). ■

Given Proposition 7 and Lemma 7, the proof of Theorems 1 and 2 is as follows. For any $\zeta, \Delta > 0$ and F_ζ defined on $I = [-\Delta^{1/3}, \bar{u} + \Delta^{1/3}]$ the Proposition implies that $|T_I^\Delta F_\zeta - F_\zeta|_{[0,\bar{u})} = o(\Delta) + O(\zeta\Delta)$, and so part (i) of the Lemma with $J = [0, \bar{u})$ implies that

$$\left|F^\Delta - F_\zeta\right|_{[0,\bar{u})}^+ = O(\zeta) + \frac{o(\Delta)}{\Delta}.$$

On the other hand, for any $\zeta, \Delta > 0$, F_ζ defined on $I = [0, w_{sp,\zeta}]$, an approximation error $\varepsilon > 0$ and the corresponding simple contract-action plan based on the set of simple policies $p = \{(a, c, W)\}_{w \in I}$, we have, from the Proposition, that $\left| F_\zeta - T_I^{\Delta,p} F_\zeta \right|_{I^\Delta}^{+} = o(\Delta) + O(\varepsilon\Delta) + O(\zeta\Delta)$. Thus, part (ii) of the Lemma implies that

$$\begin{aligned} \left| F_\zeta - F_I^{\Delta,p} \right|_I^{+} &= O(\zeta) + O(\varepsilon) + \left| F_\zeta - F_I^{\Delta,p} \right|_{I \setminus I^\Delta}^{+} + \frac{o(\Delta)}{\Delta} = \\ &= O(\zeta) + O(\varepsilon) + \frac{o(\Delta)}{\Delta}. \end{aligned}$$

The last equality follows from the continuity of F_ζ, $F_\zeta(0) = F_I^{\Delta,p}(0)$ and $F_\zeta(w_{sp,\zeta}) = \underline{F}(w_{sp,\zeta}) \leq F_I^{\Delta,p}(w_{sp,\zeta})$. This concludes the proof of both Theorems.

4.3 Proof of Proposition 7

The proof of Proposition 7 is established by a series of Lemmas that relate values of Bellman operators applied to the function F from Theorem 1, as well as their policy functions. As regards the values, the line of argument can be illustrated as follows:

$$F \underset{\text{Lemma 9}}{\sim} T^{\Delta,q} F \underset{\text{Lemma 10}}{\sim} T_I^{\Delta,d} F \underset{\text{Lemma 12}}{\sim} T_I^{\Delta,c} F \underset{\text{Lemma 13}}{\sim} T_I^{\Delta} F.$$

The following Lemma shows the connection between the Bellman operator $T^{\Delta,q}$, for short period length Δ, and the HJB equation (3).

Lemma 8 *Consider any twice differentiable function $F : I \to \infty$ with $F'' < 0$. Then:*

$$T^{\Delta,q} F(w) = \tilde{r}\Delta \sup_{\bar{a},\bar{h},c} \left\{ (\bar{a} - c) + F'(w)[w + \bar{h} - u(c)] + e^{r\Delta} \frac{F''(w)}{2} \tilde{r}\Theta(\bar{a}, \bar{h}) \right\} + e^{-r\Delta} F(w) + O\left(\Delta^2\right).$$

Moreover, for a fixed ε-suboptimal policy $\left(\overline{a}, \overline{h}, c\right)$ in the problem above, together with an ε-suboptimal policy (a, v) in the problem $\Theta\left(\overline{a}, \overline{h}\right)$, for any $\Delta > 0$ the policy $\left(a_q^\Delta, c_q^\Delta, W_q^\Delta\right)$, with

$$\begin{aligned} c_q^\Delta &= c. \\ W_q^\Delta(y) &= w + \tilde{r}\Delta e^{r\Delta}[w + \bar{h} - u(c)] + \tilde{r}\sqrt{\Delta} v(\sqrt{\Delta} y), \\ a_q^\Delta(z) &= a(\sqrt{\Delta} z), \end{aligned} \tag{17}$$

is feasible and $O(\varepsilon\Delta)$-suboptimal for $T^{\Delta,q} F(w)$.

The intuition for the Lemma is as follows. Finding the optimal policy (a, c, W) for the problem $T^{\Delta,q} F(w)$ can be split into two stages: first, choosing optimal mean action $\overline{a}$, mean effort $\overline{h}$ and wage c, and second, choosing optimal action scheme a corresponding to $\overline{a}, \overline{h}$ together with optimal incentivizing continuation value scheme W. As regards the second step, given the quadratic approximation of F, it is only the first two moments of W that are relevant. The first moment is fully pinned down, given $\overline{a}, \overline{h}$ and c, by the promise keeping (PK_q) constraint. Minimization of the second moment is, after renormalization with respect to period length Δ, precisely the definition of Θ.

The following is essentially a corollary to the previous Lemma. We will call the policies defined in (17) *quadratic simple.*

Lemma 9 *Fix $\zeta \geq 0$ and F_ζ solving the HJB equation (6) on an interval I with $F''_\zeta < 0$. Then $|T^{\Delta,q}F_\zeta - F_\zeta|_I = o(\Delta) + O(\zeta\Delta)$. Moreover, for any $\varepsilon, \Delta > 0$, $w \in I$ and corresponding quadratic simple policy (a_q, c_q, W_q), $\Phi^{\Delta,q}(a_q, c_q, W_q; F_\zeta, w) \geq F_\zeta(w) - O(\Delta\varepsilon) - O(\zeta\Delta)$, uniformly in I.*

Next Lemma shows that the simplifications implicit in the definition of $T^{\Delta,q}$ - quadratic approximation of F, possibly unbounded values of W, only local incentive constraints, approximating public signal with just Δx - are negligible when period length Δ is short. (We deal with constant consumption in Lemma 13.) *Simple* policies (Definitions 1 above and 4 in the Appendix) differ from quadratic simple policies above (see Definition 3) in that, essentially, they undo those simplifications: the tails of the continuation values are truncated, the local IC implicit in the quadratic simple policies is replaced by the global IC and the public signal is $\Delta[x + a(z)]$.

Lemma 10 *Let $F : I \to \mathbb{R}$ be twice continuously differentiable with $F'' < 0$. Then $|T_I^{\Delta,d}F - T^{\Delta,q}F|_{I^\Delta} = o(\Delta)$. Moreover, for fixed $\varepsilon > 0$ consider quadratic simple policies (a_q, c_q, W_q) for $T^{\Delta,q}F(w)$, $\Delta > 0$, $w \in I^\Delta$. Then for Δ sufficiently small, $w \in I^\Delta$ and the corresponding simple policies (a, c, W), $\Phi^\Delta(a, c, W; F) \geq \Phi^{\Delta,q}(a_q, c_q, W_q; F, w) - O(\varepsilon\Delta)$, uniformly in w.*

The simple policies achieve similar values to the quadratic simple policies for the following reasons. First, the truncation of continuation values has little effect on the incentives. In the pure hidden information case, agent's incentives, given private signal ("noise"), are determined by the slope of W at nearby revenue realizations. Therefore, giving up slope of the continuation values W at the tails affects agents incentives only for extreme noise realizations. In the absolutely continuous case, given that F'' is negative and bounded away from zero, using W far away from its mean is costly. Under assumption (A2-AC), there are hardly any revenue realizations that are so informative of agent's effort to be worth the cost of such extreme continuation values. Thus the truncation affects agents incentives only slightly.[20] On the other hand, given the truncation, the quadratic approximation of F has little effect on the value of the problem and continuation values are included in I.

Second, in the AC case, since the effect of the agent's effort on the distribution of public signal is small, the optimal incentivizing scheme W under local IC only is such that the agent faces incentives almost constant in his own action (the expected continuation value is almost linear in his own action). Given strict convexity of the cost of effort, the agent's problem is strictly convex and local incentives are sufficient. Also, approximating public signal by just Δx affects incentives, and so the agent's action only slightly. In the PHI case the problem facing the agent could be nonconvex only at the noise realizations close to the finitely many points of discontinuity of W. Thus, we also have the following Lemma.

[20] See also Sannikov and Skrzypacz [2007].

Lemma 11 *Consider any simple policy (a, c, W) defined in (10) for I,F, ε, Δ, w.*

(i) In the AC case, for sufficiently small Δ, $a(z)$ is the unique action that satisfies the local version of (IC), for all z.

(ii) In the PHI case, for sufficiently small Δ, $a(x)$ is one of at most two actions that satisfies the local version of (IC), for all x.

The next Lemma is relevant only for the PHI case. It establishes that the restriction to piecewise continuously differentiable policies a and continuous W is without loss of generality. The proof is closely related to the analogous results in the static mechanism design with quasilinear utilities.

Lemma 12 *Let $Z = X$, and let $F : I \to \mathbb{R}$ be twice continuously differentiable with $F'' < 0$. Then $|T_I^{\Delta,d} F = T_I^{\Delta,c} F|_{I^\Delta} = o(\Delta)$.*

The last Lemma needed to establish the proof of Proposition 7 shows that the restriction to wage contract-action schemes is without loss of generality.

Lemma 13 *If $F : I \to \mathbb{R}$ is twice continuously differentiable and $F'' < 0$, then $\left| T_I^{\Delta,c} F - T_I^{\Delta} F \right|_{I^\Delta} = o(\Delta)$.*

Intuitively, since with short periods the signal about agent's action is weak, in order to provide nonegligible incentives the variation in utility from signal-contingent payments must be of high order $\sqrt{\Delta}$. While the continuation value function may provide such incentives, the direct money payments are only of order Δ. Thus, changing consumption to be constant affects the incentives only slightly.

5 Extensions

5.1 Changing signal structure

Suppose now that for period length Δ the private signal z is distributed with density γ_Z^Δ, while given the private signal z and action a the revenue y is distributed with density $\gamma^\Delta(y|z, a)$. This extends the model in the paper along two dimensions. First, it generalizes the way the period length Δ parametrizes the distribution of signals. Second, it generalizes the way agent's effort affects the distribution of public signal.

For any $\overline{a}$ and $\overline{h} \geq h(\overline{a})$ consider the following problem:

$$\begin{aligned} \Theta^\Delta(\overline{a}, \overline{h}) &= \inf_{a,v} \int v^2(y)\gamma^\Delta(y|z, a(z))\gamma_Z^\Delta(z)dydz, \qquad (18) \\ & \bar{a} = \int a(z)\gamma_Z^\Delta(z)dz \\ & \bar{h} = \int h(a(z))\gamma_Z^\Delta(z)dz \\ & a(z) \in \arg\max_{a\in\mathcal{A}} \left\{ -r\Delta h(a) + e^{-r\Delta} \int_{\mathbb{R}} v(y)\gamma^\Delta(y|z, a(z))dy \right\} \quad \forall z. \end{aligned}$$

Similarly, if z and y take only countably many values and γ_Z^Δ, γ^Δ stand for probabilities, Θ^Δ is defined analogously, with sums replacing integrals.

Suppose that for any such $\overline{h}, \overline{a}$ $\lim_{\Delta \to 0} \frac{\Theta^\Delta(\overline{a}, \overline{h})}{\Delta} = \Theta(\overline{a}, \overline{h})$ for some function Θ, where the convergence is uniform. Then all our results can be extended to this general case (with the obvious changes in the definition of the optimal contracts). Below we consider three examples.

5.1.1 Examples:

General mean shifting effort. We generalize the basic model by allowing the effect of effort on the revenue to vary with noise. In particular, fix a twice continuously differentiable, bounded function ϕ and for any period length $\Delta > 0$ let

$$y = x + \Delta\phi(x/\sqrt{\Delta}, a),$$

where the noise x and the private signal z are distributed with the joint denisty g^Δ as before.

With just slight changes in notation in the proof one establishes that

$$\begin{aligned} \Theta(\overline{a}, \overline{h}) \quad &= \quad \inf_{a,v} \quad \mathbb{E}\left[v(x)^2\right] \\ &\bar{a} = \mathbb{E}[\phi(x, a(z))], \quad \bar{h} = \mathbb{E}[h(a(z))], \\ &h'(a(z)) = -\int_{\mathbb{R}} v(x) g'(x|z)\phi_2(x, a)dx \quad \forall z \qquad (\text{FOC}_\Theta\text{-AC}) \\ &h'(a(x)) = v'(x)\phi_2(x, a) \quad \forall x \qquad (\text{FOC}_\Theta\text{-PHI}) \end{aligned}$$

In particular, in the pure hidden action case, for any target action $\overline{a}$ the optimal v function is linear in $\frac{g'(x)\phi_2(x,a)}{g(x)}$, and $\Theta(\overline{a}, h(\overline{a}))$ is equal to $h'(\overline{a}) \times E\left[\left(\frac{g'(x)\phi_2(x,a)}{g(x)}\right)^2\right]$.

Folk Theorem. Consider now the model in which for every period length Δ we have $y = \Delta s$, where s is a (public) signal with distribution that depends on action a and is independent of Δ. The analysis in this case coincides with the analysis of a discrete time model with fixed period length 1 and a fixed distribution of signals γ_S, in which the per period discount factor converges to one. For simplicity let us consider the pure hidden action case and assume that the sets of available actions $\mathcal{A}$ and possible public signals S are finite, with conditional probabilities $p(s|a)$.

In our model only one player's actions affect the public signal, and so the standard identifiability assumptions (see Fudenberg, Levine, and Maskin [1994]) reduce to $\{p(\cdot|a)\}_{a\in\mathcal{A}}$ being linearly independent. Given linear independence, for any $a \in \mathcal{A}$ there exists a function ζ_a such that:

$$\begin{aligned} \sum_{s\in S} \zeta(s)\, p(s|a) \quad &= \quad 0, \qquad (19) \\ \sum_{s\in S} \zeta(s)\, p\left(s|a'\right) \quad &\leq \quad e^r r\left[h\left(a'\right) - h(a)\right]. \end{aligned}$$

In other words, for the period length 1 the policy (a, ζ_a) satisfies the constraints of the problem (18) for $(\overline{a}, \overline{h}) = (a, h(a))$ and so $\Theta^1(a, h(a)) \leq \sum_{s\in S} (\zeta_a(s))^2\, p(s|a)$.

Now, for the period length $\Delta > 0$, for any action a the function $\zeta_a^\Delta(y) = \frac{\Delta e^r}{e^{-r\Delta}}\zeta_a\left(\frac{y}{\Delta}\right)$ satisfies the constraints of the problem (18) for $(\overline{a}, \overline{h}) = (a, h(a))$, since for any action a' we have:

$$-r\Delta h(a') + e^{-r\Delta} \sum_{y \in \Delta S} \zeta_a^\Delta(y) \gamma^\Delta(y|a') = \Delta\left[-rh(a') + e^{-r}\sum_{s\in S}\zeta_a(s)\, p(s|a')\right],$$

and the result follows form the case $\Delta = 1$. Therefore

$$\Theta^\Delta(a, h(a)) \leq \sum_{y\in\Delta S}\left(\zeta_a^\Delta(y)\right)^2 \gamma^\Delta(y|a) = \Delta^2 \frac{e^{2r}}{e^{-2r\Delta}}\sum_{s\in S}(\zeta_a(s))^2\, p(s|a) = O\left(\Delta^2\right).$$

Consequently, the limit of the values of the optimal contracts is described by the solution to the HJB equation (3) with $\Theta(\overline{a}, \overline{h}) = \lim_{\Delta\to 0}\frac{\Theta^\Delta(\overline{a},\overline{h})}{\Delta} = 0$, and so, due to Proposition 3, in the limit the first best outcome is achievable. In other words, for the particular model that we considered we recover the Folk Theorem result.

Shifting Volatility. Suppose now that the agent's action determines the volatility of the revenue process. More specifically, let the set of available actions $\mathcal{A} \subset (0,1)$ be finite and for any period length Δ the revenue be normally distributed with mean 0 and variance $\Delta(1-a)$.[21] Note that unlike in the previous example the signal structure is not independent of Δ, i.e. revenue cannot be written as Δs with distribution of s independednt of Δ.

For the model with $\Delta = 1$ the set of conditional densities $\left\{\gamma^1(\cdot|a)\right\}_{a\in\mathcal{A}}$ is linearly independent and so, just as in the example above, for any $a \in \mathcal{A}$ the exists a function ζ_a satisfying (19). Consequently $\Theta^1(a, h(a)) \leq \int (\zeta_a(y))^2 \gamma^1(y|a)\, dy$.

With the period length $\Delta > 0$, for any action a the function $\zeta_a^\Delta(y) = \frac{\Delta e^r}{e^{-r\Delta}}\zeta_a\left(\frac{y}{\sqrt{\Delta}}\right)$ satisfies the constraints of the problem (18) for $(\overline{a}, \overline{h}) = (a, h(a))$, since for any action a':

$$\begin{aligned}-r\Delta h(a') + e^{-r\Delta}\int \zeta_a^\Delta(y)\gamma^\Delta(y|a')\, dy &= -r\Delta h(a') + e^{-r}\Delta\int\zeta_a\left(\frac{y}{\sqrt{\Delta}}\right)\gamma^\Delta(y|a')\, dy = \\ &= \Delta\left[-rh(a') + e^{-r}\int\zeta_a(y)\gamma^1(y|a')\, dy\right],\end{aligned}$$

and the result follows form the case $\Delta = 1$. Therefore, as above, $\Theta^\Delta(a, h(a)) = O\left(\Delta^2\right)$, and in the limit the first best outcome is achievable.

5.2 Changing payoff structure

The method can also be used to the models with different payoff structure. One important example concerns the problems with diversion of cash-flows (see DeMarzo and Fishman [2007], Demarzo and Sannikov [2006], Biais, Mariotti, Plantin, and Rochet [2007]). The difference with the model in the

[21] The model is trivial in the case when the Principal is risk neutral. If we assume that the Principal has mean-variance preferences $u_p(F_y) = E[y] - Var[y]$, his per period utility is equal to $\Delta(1-a-c)$, and so (up to a constant) the same as considered in the paper.

previous sections is that the "cost" of effort to the agent will not be independent of consumption, but is in fact expressed directly in monetary terms.

The action a of the agent will be interpreted as the amount of money diverted from the cash flow, after the cash flow is observed. To be able to compare the results with the literature we let $a \in \mathcal{A} = [0, \infty)$. Agent's benefit, in monetary terms, from stealing a is $h(a)$, where h is a concave function such that $h' < 1$ and $h'(a) = \gamma < 1$ for $a \geq A$. For any Δ the stage game payoffs are thus:

$$\begin{aligned} u_P(a, c) &= \Delta (drift - a - c) + noise, \\ u_A(a, c) &= \Delta u(c + h(a)). \end{aligned}$$

We thus go beyond the "linear" approach in the literature and allow the h function to be nonlinear as well as agent to be risk averse. Furthermore, as in the literature, we assume that in every period after observing the public signal the principal can break the contract, which will result in a continuation payoffs $w_P, w_A > 0$ for the principal and the agent.

Few remarks. One can show that the payoffs to the principal and the agent cannot fall below w_P and w_A (see DeMarzo and Fishman [2007], Demarzo and Sannikov [2006], Biais, Mariotti, Plantin, and Rochet [2007]). Also, the optimal contracts will never require the agent to divert more than some finite amount A . In contrast to the "linear" case, however, even the optimal contract may require the Agent to divert nonnegative amount. One can show that the values of the optimal contracts converge to F, where F is the maximal function that solves

$$\begin{aligned} F(w) &= \max_{\bar{a}, \bar{u}, c} \left\{ drift - (\bar{a} + c) + F'(w)(w - \bar{u}) + \frac{rF''(w)}{2} \Theta(\bar{a}, \bar{u}, c) \right\}, \\ F(w_A) &= w_P, \quad F \leq \overline{F}, \end{aligned}$$

where

$$\begin{aligned} \Theta(\bar{a}, \bar{u}, c) &= \inf_{a,v} \int v^2(x) g(x) dx, \\ \text{s.t.} \quad & \bar{a} = \int a(x) g(x) dx \\ & \bar{u} = \int u(c + h(a(x))) g(x) dx \\ & v'(x) = u'(c + h(a(x))) h'(a(x)). \end{aligned} \tag{20}$$

Note also that allowing the agent to be risk averse, we can treat this model as a model of insurance: Agent observes own income, decides how much to report (where reporting less than truth might involve some efficiency loss). The principal can decide to end the relationship, which here might mean that he implements a costly perfect monitoring scheme so that the parties get the first best minus the exogenously determined cost of monitoring.

6 Conclusions

We study a rich family of dynamic agency problems that includes the standard hidden action and hidden information models as special cases. We develop a *quadratic approximation method* that, when the period length is short, allows us to characterize the upper boundary of the equilibrium value set by a differential equation, and to construct contracts that are both relatively simple and almost optimal. The quadratic approximation method developed here should be useful in many dynamic settings with asymetric information (for example repeated partnerships and oligopoly games).

The solutions we derive depend on the information structure, including the corresponding densities of signals. Nevertheless our method is very tractable as it involves solving a family of simple static problems and a differential equation. The upper boundary of the equilibrium value set depends on a single parameter of the information structure, the variance of incentive transfers. The simple contracts are built from the optimal solutions of the static problems, which are functions of the likelihood ratios of public signals familiar from the static contracting literature.

In particuar, while easy to construct, the contracts are sensitive to the details of the information structure, for any period length. The upper boundary of the value set of the continuous time model is the limit, as the period length shrinks to 0, of the upper boundary of the value set of discrete time models with particular information structures, whereas the optimal continuous time contract does not provide enough information to construct (approximately) optimal discrete time contracts.

7 Appendix

7.1 Details for Section 4.3

The following Lemma will be used throughout the paper. It says that for any period length $\Delta > 0$ and any of the Bellman operators applied to a strictly concave function F, the continuation value policy function must have variance at most proportional to Δ. Intuitively, this must be the case in order to bound the efficiency loss, due to the high variance and strict concavity of F, by potential per-period gains, which are of order Δ.

Lemma 14 *Let $I = [\underline{w}, \overline{w}]$ and $F : I \to \mathbb{R}$ be twice continuously differentiable with $F'' < 0$. Suppose that the policy (a, c, W) is Δ-suboptimal for the problem $XF(w)$, with $w \leq e^{-r\Delta}\overline{w}$ and X any of the Bellman operators T_I^{Δ}, $T_I^{\Delta,c}$, $T_I^{\Delta,d}$ and $T^{\Delta,q}$. Then for some V that depends on F only we have $\mathbb{V}^{\Delta}[W(\Delta[x + a(z)])] \leq V\Delta$.*

Proof. In each of the above problems the policy $(a, c, W) = (0, 0, e^{r\Delta}w)$ is an available policy that satisfies all the constraints and delivers a value of at least $F(w) + [\min F'] \left(e^{r\Delta} - 1\right) \overline{w} = F(w) + O(\Delta)$. Let $\hat{h} = \mathbb{E}^{\Delta}[h(a(z))]$, $\hat{u} = \mathbb{E}^{\Delta}[u(c(\Delta[x + a(z)]))]$ and $\hat{W} = \mathbb{E}^{\Delta}[W(\Delta[x + a(z)])]$. The promise-keeping constraint implies that

$$\hat{W} - w = \tilde{r}\Delta e^{r\Delta}[w + \hat{h} - \hat{u}] = O(\Delta),$$

since $w \in [\underline{w}, \overline{w}]$, $\hat{h} \in [0, h(A)]$ and $\hat{u} \in [0, \bar{u}]$. Therefore, $W(\Delta[x + a(z)]) - w = (W(\Delta[x + a(z)]) - \hat{W}) + (\hat{W} - w)$ implies

$$\mathbb{E}^{\Delta}[(W(\Delta[x + a(z)]) - w)^2] = \mathbb{V}^{\Delta}[W(\Delta[x + a(z)])] + O(\Delta^2).$$

Consequently, for Y either $\Phi^{\Delta,q}(a,c,W;F,w)$ or $\Phi^{\Delta}(a,c,W;F)$ we have $Y \geq F(w) + O(\Delta)$ and

$$Y \leq \tilde{r}\Delta A + e^{-r\Delta}\Big(F(w) + \tilde{r}\Delta e^{r\Delta}F'(w)[w + \hat{h} - \hat{u}] + \frac{\max F''}{2}\mathbb{V}^{\Delta}\left[W(\Delta[x + a(z)])\right]\Big) + O(\Delta^2),$$

which after rearranging terms gives the result for an appropriate V. ■

Proof. (Lemma 8) Fix $w \in I$ and any feasible policy (a,c,W) for $T^{\Delta,q}F(w)$, let $\bar{a} = \mathbb{E}^{\Delta}[a(z)]$, $\bar{h} = \mathbb{E}^{\Delta}[h(a(z))]$ and $\bar{W} = \mathbb{E}^{\Delta}[W(\Delta x)]$. As in the proof of Lemma 14, the promise-keeping constraint for $T^{\Delta,q}F(w)$ implies that $\bar{W} - w = \tilde{r}\Delta e^{r\Delta}[w + \bar{h} - u(c)]$. Therefore, $T^{\Delta,q}F(w)$ equals

$$
\begin{aligned}
&= \sup_{a,c,W}\left\{\tilde{r}\Delta(\bar{a} - c) + e^{-\Delta r}\mathbb{E}^{\Delta}[F(w) + F'(w)(W(\Delta x) - w) + \frac{1}{2}F''(w)(W(\Delta x) - w)^2]\right\} \\
&= \tilde{r}\Delta \sup_{a,c,W}\left\{(\bar{a} - c) + F'(w)[w + \bar{h} - u(c)] + e^{-r\Delta}\frac{F''(w)}{2\tilde{r}\Delta}\mathbb{V}^{\Delta}[W(\Delta x)]\right\} + e^{-r\Delta}F(w) + O\left(\Delta^2\right) \\
&= \tilde{r}\Delta \sup_{\bar{a},\bar{h},c}\left\{(\bar{a} - c) + F'(w)[w + \bar{h} - u(c)] + e^{r\Delta}\frac{F''(w)}{2}\tilde{r}\Theta(\bar{a},\bar{h}) - F(w)\right\} + e^{-r\Delta}F(w) + O\left(\Delta^2\right),
\end{aligned}
$$

where the last equality follows from the definition of $\Theta(\bar{a},\bar{h})$, as we argue below.

For a given $(\bar{a},\bar{h})$, since $F''(w) < 0$ and $\int g^{\Delta\prime}_{X|Z}(x|z)dx = 0$, the above optimization problem involves the subproblem

$$
\begin{aligned}
\inf_{a,W_0} \quad & \mathbb{E}^{\Delta}[W_0(x)^2] \\
\text{s.t.} \quad & \bar{a} = \mathbb{E}^{\Delta}[a(z)], \quad \bar{h} = \mathbb{E}^{\Delta}[h(a(z))], \quad 0 = \mathbb{E}^{\Delta}[W_0(x)], \\
& \tilde{r}h'(a(z)) = -\frac{e^{-r\Delta}}{\Delta}\int W_0(x)g^{\Delta\prime}_{X|Z}(x|z)dx \quad \forall z \qquad \text{(FOC-AC)}, \\
& \tilde{r}h'(a(x)) = e^{-r\Delta}W_0'(x) \quad \forall x \qquad \text{(FOC-PHI)},
\end{aligned}
$$

where $W_0(x) = W(\Delta x) - \bar{W}$. Note that the constraint $0 = \mathbb{E}^{\Delta}[W_0(x)]$ can be dropped since it will be satisfied by a solution (or infimum sequence) of the relaxed problem. Note also that $g^{\Delta}_X(x) = \sqrt{\Delta}g_X(x\sqrt{\Delta})$, $g^{\Delta}_Z(z)) = \sqrt{\Delta}g_Z(z\sqrt{\Delta})$, and $g^{\Delta}_{X|Z}(x|z)) = \sqrt{\Delta}g_{X|Z}(x\sqrt{\Delta}\,|\,z\sqrt{\Delta})$. Hence, letting $v(x) = e^{-r\Delta}W_0(x/\sqrt{\Delta})/\left[\tilde{r}\sqrt{\Delta}\right]$ and $\tilde{a}(z) = a(z/\sqrt{\Delta})$, the subproblem becomes

$$
\begin{aligned}
\inf_{a,v} \quad & \tilde{r}^2\Delta e^{2r\Delta}\mathbb{E}[v(x)^2] \\
\text{s.t.} \quad & \bar{a} = \mathbb{E}[\tilde{a}(z)], \quad \bar{h} = \mathbb{E}[h(\tilde{a}(z))], \\
& h'(\tilde{a}(z)) = -\int v(x)g'_{X|Z}(x|z)dx \quad \forall z \qquad \text{(FOC-AC)}, \\
& h'(\tilde{a}(x)) = v'(x) \quad \forall x \qquad \text{(FOC-PHI)}.
\end{aligned}
$$

Therefore the value of this last problem is by definition $\tilde{r}^2\Delta e^{2r\Delta}\Theta(\bar{a},\bar{h})$. This justifies the substitution of $\Theta(\bar{a},\bar{h})$ in the equation above and establishes the proof. ■

Proof. (Lemma 9) From Lemma 8 we have

$$\begin{aligned}
& T^{\Delta,q}F_\zeta(w) - F_\zeta(w) \\
= & \sup_{\bar{a},\bar{h},c} \tilde{r}\Delta \left\{ (\bar{a}-c) + F'_\zeta(w)[w+\bar{h}-u(c)] + e^{r\Delta}\frac{F''_\zeta(w)}{2}\tilde{r}\Theta(\bar{a},\bar{h}) - F_\zeta(w) \right\} + O\left(\Delta^2\right) \\
= & O(\zeta\Delta) + O\left(\Delta^2\right).
\end{aligned}$$

The last equality follows because F_ζ satisfies the HJB equation (6). Lemma 8 also yields that $\Phi^{\Delta,q}(a_q, c_q, W_q; F_\zeta, w) \geq F_\zeta(w) - O\left(\Delta^2\right) - O(\Delta\varepsilon) - O(\zeta\Delta)$, establishing the proof. ■

Definition 3 *For a twice differentiable function $F : I \to \infty$ with $F'' < 0$, $\varepsilon > 0$, $w \in I$ and ε-suboptimal policies $\left(\overline{a}, \overline{h}, c\right)$ and (a, v) as in Lemma (8), the policies $\left(a_q^\Delta, c_q^\Delta, W_q^\Delta\right)$ defined in (17), for $\Delta > 0$, will be called* quadratic simple.

Remark 2 *In the pure hidden information case, the v in the definition of a quadratic simple policy at w is continuous and piecewise twice continuously differentriable (see the definition of Θ). We assume that for any $\varepsilon > 0$, there is a common finite set D such that the set of functions v'' for all $w \in I$ are equicontinuous outside of D, which is without loss of generality (see Remark 1).*

The following two technical Lemmas are crucial for the proofs of Lemmas 10 and 11 in the absolutely continuous case. Lemma 15 will be used to show that the incentives provided by the tails of the continuation values are negligible, for short period length. Lemma 16 will establish that the expected continuation value for the agent is almost linear in action, for short period length. This will be used to show strict convexity of agent's problem and so that the local incentives are sufficient, as well as that approximating public signal by Δx hardly affects the incentives, and so agent's action.

Lemma 15 *Suppose $[X|Z=z]$ is absolutely continuous and g satisfies (A2). For any $\varepsilon > 0$ there exist M_ε such that for all $v : \mathbb{R}^2 \to \mathbb{R}$ with $\mathbb{E}^\Delta\left[v(x,z)^2\right] \leq 1$ the following holds*

$$\mathbb{P}\left[\left\{z : \left|\int_{|v|>M_\varepsilon} v(x,z)\, g'_{X|Z}(x|z)\, dx\right| \leq \varepsilon\right\}\right] \geq 1-\varepsilon.$$

Proof. Fix $\varepsilon > 0$ and consider a function v that satisfies $\mathbb{E}[v(x,z)^2] \leq 1$. We have, from the Tshebyshev's inequality,

$$\mathbb{P}\left[\left\{z : \int v(x,z)^2 g_{X|Z}(x|z)dx > 2/\varepsilon\right\}\right] < \frac{\int v(x,z)^2 g(x,z)dxdz}{2/\varepsilon} \leq \varepsilon/2.$$

Due to Assumption (A2-AC) the term

$$\Gamma_N(z) = \int_{|g'_{X|Z}(x|z)|>N g_{X|Z}(x|z)} \frac{g'_{X|Z}(x|z)^2}{g_{X|Z}(x|z)} dx$$

converges pointwise to zero as $N\uparrow\infty$ (Lebesgue's Monotone Convergence Theorem). Also, there exists a set $L_\varepsilon\subset\mathbb{R}$ and $N_\varepsilon>0$ so that $\mathbb{P}_Z[L_\varepsilon]>1-\varepsilon/2$ and for all $z\in L_\varepsilon$ $\Gamma_{N_\varepsilon}(z)\leq\frac{\varepsilon^3}{8}$. Define the set $B_\varepsilon^v=L_\varepsilon\cap\left\{z:\int v(x,z)^2g_{X|Z}(x|z)dx\leq 2/\varepsilon\right\}$, which satisfies $\mathbb{P}_Z\left[B_\varepsilon^v\right]\geq 1-\varepsilon$.

Let $D(z)=\{x\mid |g'_{X|Z}(x|z)|>N_\varepsilon g_{X|Z}(x|z)\}$. If follows from the Cauchy-Shwartz inequality that for all $z\in B_\varepsilon^v$

$$\int_{D(z)}\left|v(x,z)g'_{X|Z}(x|z)\right|dx\leq\left[\int v(x,z)^2g_{X|Z}(x|z)dx\times\int_{D(z)}\frac{g'_{X|Z}(x|z)^2}{g_{X|Z}(x|z)}dx\right]^{\frac{1}{2}}\leq\left[\frac{2}{\varepsilon}\frac{\varepsilon^3}{8}\right]^{\frac{1}{2}}=\frac{\varepsilon}{2}.$$

On the other hand, for $M_\varepsilon>\frac{4N_\varepsilon}{\varepsilon^2}$ and any $z\in B_\varepsilon^v$ we have

$$\int_{\{x\notin D(z):|v|>M_\varepsilon\}}\left|v\left(x,z\right)g'_{X|Z}(x|z)\right|dx\leq\frac{N_\varepsilon}{M_\varepsilon}\int v^2(x,z)g_{X|Z}(x|z)dx<\frac{\varepsilon}{2}.$$

Altogether, the above two expressions establish that for all $z\in B_\varepsilon^v$

$$\int_{|v|>M_\varepsilon}\left|v\left(x,z\right)g'_{X|Z}(x|z)\right|dx\leq\varepsilon,$$

which establishes the proof. ■

Lemma 16 *Suppose $[X|Z=z]$ is absolutely continuous and g satisfies (A2). For any $\varepsilon>0$ and M there exists $\hat\delta>0$ such that for all $\delta(\cdot)$ with $0\leq\delta(\cdot)\leq\hat\delta$ and $v:\mathbb{R}^2\to\mathbb{R}$ the following holds*

$$\left|\int_{|v|\leq M}v\left(x,z\right)g'_{X|Z}\left(x|z\right)dx-\int_{|v|\leq M}v\left(x+\delta(z),z\right)g'_{X|Z}\left(x|z\right)dx\right|\leq\varepsilon,\qquad\forall z\tag{21}$$

$$\mathbb{E}\left[v^2\left(x+\delta(z),z\right)1_{|v|\leq M}\right]\leq\mathbb{E}\left[v^2\left(x,z\right)1_{|v|\leq M}\right]+\varepsilon.$$

Proof. For every x and z, $|g'_{X|Z}(x|z)-g'_{X|Z}(x-\delta(z)|z)|\leq\delta|g''_{X|Z}(x-\xi(x,z)|z)|$ for some $\xi(x,z)\in[0,\delta(z)]\subset[0,\hat\delta]$. Therefore, with $\bar\delta$ and $\bar M$ the constants in (A2), for every $\delta\leq\min\left\{\bar\delta,\frac{\varepsilon}{M\bar M}\right\}$ we have that

$$\int_{|v|\leq M}\left|v(x,z)\left[g'_{X|Z}(x|z)-g'_{X|Z}(x-\delta(z)|z)\right]\right|dx\leq\delta M\int_{\mathbb{R}}\left|g''_{X|Z}(x-\xi(x,z)|z)\right|dx\leq\delta M\bar M\leq\varepsilon,$$

which establishes (21).

Similarly, for any $\delta\leq\min\left\{\bar\delta,\frac{\varepsilon}{M^2\sqrt{\bar M}}\right\}$ we have that

$$\int_{\mathbb{R}}\int_{|v|\leq M}\left|v(x,z)^2\left(g(x,z)-g(x-a(z),z)\right)\right|dxdz\leq\delta M^2\int_{\mathbb{R}}\int_{\mathbb{R}}\left|g'_{X|Z}(x-\xi(x,z)|z)g_Z(z)\right|dxdz$$

$$\leq\quad\delta M^2\int_{\mathbb{R}}\left[\int_{\mathbb{R}}\frac{g'_{X|Z}(x-\xi(x,z)\,|\,z)^2}{g_{X|Z}(x|z)}\,dx\right]^{\frac{1}{2}}g_Z(z)dz\leq\delta M^2\sqrt{\bar M}\leq\varepsilon,$$

with the second inequality following from the Cauchy-Schwarz inequality, which establishes the Lemma. ■

Definition 4 *For a twice differentiable function* $F : I \to \infty$ *with* $F'' < 0$*,* $\varepsilon > 0$*,* $\Delta > 0$*,* $w \in I^{\Delta}$ *and quadratic simple policies* (a_q, c_q, W_q) *in the problem* $T^{\Delta,q}F(w)$*, define the* simple policy (a, c, W) *for* $T_I^{\Delta,c}F(w)$ *as*

$$
\begin{aligned}
&c = c_q, \\
&W(y) = C + W_q(y)\mathbf{1}_{|W_q(\Delta x) - \mathbb{E}^{\Delta}[W_q(\Delta x)]| \le \sqrt{\Delta} M_\varepsilon} && \text{(AC)}, \\
&W(y) = C + \begin{cases} W_q(-\sqrt{\Delta} M_\varepsilon) & \text{if } \Delta x < -\sqrt{\Delta} M_\varepsilon \\ W_q(\Delta x) & \text{if } |\Delta x| \le \sqrt{\Delta} M_\varepsilon \\ W_q(\sqrt{\Delta} M_\varepsilon) & \text{if } \Delta x > \sqrt{\Delta} M_\varepsilon \end{cases} && \text{(PHI)}, \\
&a(z) \text{ is an action that satisfies the (IC) constraint in (8),}
\end{aligned}
$$

where M_ε *is the constant defined in Lemma 15,*[22] *when* $X|Z$ *is AC, and such that* $\mathbb{P}_X([-M_\varepsilon, M_\varepsilon]) \ge 1 - \varepsilon$*, when* $X \equiv Z$*.* C *is chosen to satisfy the (PK) constraint in (8).*

Note that in the case when F solves the HJB equation (6) the above definition of a simple policy agrees with the one in Definition 1.

Proof. (Lemma 10) Fix $\varepsilon > 0$ and $\Delta > 0$ such that $\sqrt{\Delta} < \delta/A$, for δ as in Lemma 16 (with $M = M_\varepsilon$) and $w \in I^{\Delta}$.

Step 1: We first show that $\Phi^{\Delta}(a, c, W; F) \ge \Phi^{\Delta,q}(a_q, c_q, W_q; F, w) - O(\varepsilon\Delta)$, uniformly in w. Since ε is arbitrary, in view of Lemma 8, this establishes $|T^{\Delta,q}F - T_I^{\Delta,d}F|^+_{I^\Delta} = o(\Delta)$.

By Taylor series expansion, for $y = \Delta[x + a(z)]$

$$F(W(y)) = F(w) + F'(w)(W(y) - w) + \frac{1}{2}F''(w)(W(y) - w)^2 + o([W(y) - w]^2).$$

(PK) implies that $w - \mathbb{E}^{\Delta}[W(y)] = O(\Delta)$ and $\left|W(y) - \mathbb{E}^{\Delta}[W(y)]\right| = O(\sqrt{\Delta})$ by construction (note that when $Z = X$ the IC constraint implies $W'(y) \le \tilde{r}h'(A)$), and so $|w - W(y)| = O(\sqrt{\Delta})$ for all y. Therefore, for Δ small enough the policy (a, c, W) is feasible when $w \in I^{\Delta}$ and

$$
\begin{aligned}
\Phi^{\Delta}(a, c, W; F) \quad \ge \quad & \tilde{r}\Delta(\mathbb{E}^{\Delta}[a(z)] - c) + e^{-r\Delta}\Big[F(w) + F'(w)\mathbb{E}^{\Delta}[W(y) - w] \\
& + \frac{F''(w)}{2}\mathbb{E}^{\Delta}[(W(y) - w)^2]\Big] + o(\Delta)
\end{aligned}
\tag{22}
$$

Let us bound from below the terms in the second line of the above expression by the corresponding terms in $\Phi^{\Delta,q}(a_q, c_q, W_q; F, w)$.

Case 1: $X|Z$ has continuous density. Given the definition of W, the necessary local version of (IC) take the following form:

$$-h'(a(z)) = \frac{e^{-r\Delta}}{\tilde{r}\Delta}\int_{\mathbb{R}} W(\Delta[x + a(z)])g^{\Delta\prime}_{X|Z}(x|z)dx = \frac{1}{\Delta}\int_{|v| \le M_\varepsilon} \sqrt{\Delta}v(\sqrt{\Delta}[x + a(z)])g^{\Delta\prime}_{X|Z}(x|z)dx, \tag{23}$$

[22] We assume, without loss of generality, that $\mathbb{V}_X[v(x)]] \le 1$ - else work with rescaled v.

whereas the definition of W_q and (FOC$_q$-AC) imply

$$-h'(a_q(z)) = \frac{e^{-r\Delta}}{\tilde{r}\Delta} \int_{\mathbb{R}} W(\Delta x) g^{\Delta\prime}_{X|Z}(x|z)dx = \frac{1}{\Delta} \int_{\mathbb{R}} \sqrt{\Delta} v(\sqrt{\Delta} x) g^{\Delta\prime}_{X|Z}(x|z)dx.$$

Lemma 14 implies that $\mathbb{V}^{\Delta}\left[\sqrt{\Delta} v(\sqrt{\Delta} x)\right] \leq V\Delta$, and so Lemma 15 applied to $v(x)$ yields

$$\mathbb{P}^{\Delta}\left[\left|\frac{1}{\Delta}\int_{|v|>M_\varepsilon} \sqrt{\Delta} v(\sqrt{\Delta} x) g^{\Delta\prime}_{X|Z}(x|z)dx\right| \leq \varepsilon\right] = \mathbb{P}\left[\left|\int_{|v|>M_\varepsilon} v(x) g'_{X|Z}(x|z)dx\right| \leq \varepsilon\right] \geq 1-\varepsilon.$$

On the other hand, from Lemma 16 follows that for sufficiently small Δ and every z

$$\begin{aligned} & \left|\int_{|v|\leq M_\varepsilon} \sqrt{\Delta} v(\sqrt{\Delta} x) g^{\Delta\prime}_{X|Z}(x|z)dx - \int_{|v|\leq M_\varepsilon} \sqrt{\Delta} v(\sqrt{\Delta}[x + a\,(z)]) g^{\Delta\prime}_{X|Z}(x|z)dx\right| \\ = & \left|\int_{|v|\leq M_\varepsilon} v(x) g'_{X|Z}(x|z)dx - \int_{|v|\leq M_\varepsilon} v(x + \sqrt{\Delta} a(z)) g'_{X|Z}(x|z)dx\right| \leq \varepsilon \end{aligned}$$

Consequently, with probability greater than $1-\varepsilon$, $|h'(a_q(z) - h'\,(a(z))\,| \leq 2\varepsilon$, and so

$$\mathbb{P}^{\Delta}[|a_q(z) - a(z)| \leq \frac{2\varepsilon}{\inf h''}] \geq \mathbb{P}^{\Delta}[|h'(a_q(z) - h'(a(z))| \leq 2\varepsilon] \geq 1-\varepsilon. \tag{24}$$

Since also $c(y) \equiv c_q$ and actions are bounded we have

$$\tilde{r}\Delta \left|(\mathbb{E}^{\Delta}\left[a_q(z)\right] - c_q) - (\mathbb{E}^{\Delta}\left[a(z)\right] - c)\right| = O(\varepsilon\Delta). \tag{25}$$

Subtracting (PK$_q$) for problem $T^{\Delta,q}F$ from (PK) for problem $T^{\Delta,d}F$ and using (24), we obtain

$$e^{-r\Delta} F'(w) \left|\mathbb{E}^{\Delta}\left[W(y)\right] - \mathbb{E}^{\Delta}\left[W_q(\Delta x)\right]\right| = O(\varepsilon\Delta). \tag{26}$$

Finally

$$\begin{aligned} \mathbb{E}^{\Delta}[(W\,(y) - w)^2] &= \tilde{r}^2 e^{2r\Delta}\mathbb{E}^{\Delta}[\Delta v^2(\sqrt{\Delta}[x + a\,(z)])\mathbf{1}_{|v|\leq M_\varepsilon}] + O\left(\Delta^2\right) \\ &\leq \tilde{r}^2 e^{2r\Delta}\mathbb{E}^{\Delta}[\Delta v^2(\sqrt{\Delta} x)\mathbf{1}_{|v|\leq M_\varepsilon}] + O\left(\varepsilon\Delta\right) \\ &\leq \tilde{r}^2 e^{2r\Delta}\mathbb{E}^{\Delta}[\Delta v^2(\sqrt{\Delta} x)] + O\left(\varepsilon\Delta\right) = \mathbb{E}^{\Delta}[(W_q\left(\Delta x\right) - w)^2] + O\left(\varepsilon\Delta\right). \end{aligned} \tag{27}$$

The first inequality follows from Lemma 15. The equalities follow from the definitions of W_q and W and the fact that $\mathbb{E}^{\Delta}[W\,(\Delta[x + a(z)]) - w] = O(\Delta)$ and $\mathbb{E}^{\Delta}[W_q\,(\Delta x) - w] = O(\Delta)$, from (PK) and (PK$_q$). Inequalities (25)–(27) establish the proof of Case 1.

Case 2: $X \equiv Z$. Given the definition of W, the necessary local version of (IC) take the following form:

$$\tilde{r}h'(a(x)) = e^{-r\Delta}W'\,(y) = \tilde{r}v'(\sqrt{\Delta}[x + a\,(x)]), \tag{28}$$

whereas, given the definition of W_q and (FOC$_q$-PHI), we have

$$\tilde{r}h'(a_q(x)) = e^{-r\Delta}W_q'(\Delta x) = \tilde{r}v'(\sqrt{\Delta}x).$$

Let D be the finite set of points such that each v in the definition of the policy is twice continuously differentiable on $\mathbb{R}\backslash D$ (see Remark 2) and consider the set

$$N_\varepsilon^\Delta = [-M_\varepsilon/\sqrt{\Delta}, M_\varepsilon/\sqrt{\Delta} - A]\backslash \bigcup_{d\in D}\{d/\sqrt{\Delta} + \zeta : \zeta \in [0, A]\}.$$

For sufficiently small Δ, $\mathbb{P}^\Delta\left[N_\varepsilon^\Delta\right] \geq 1 - \varepsilon$. Moreover, for any $x \in N_\varepsilon^\Delta$, v' is continuously differentiable on $[\sqrt{\Delta}x, \sqrt{\Delta}[x + a(x)]]$. Consequently, for all such x $|h'(a_q(x) - h'(a(x))| \leq \sqrt{\Delta}\max v''$, where the maximum is taken over the set $[-M_\varepsilon, M_\varepsilon]$, and hence

$$|a_q(x) - a(x)| \leq \frac{\sqrt{\Delta}\max v''}{\inf h''}.$$

Since $\mathbb{P}^\Delta\left[N_\varepsilon^\Delta\right] \geq 1 - \varepsilon$, we have that the inequalities (25) and (26) hold. Moreover, by taking the maximum over $\max v''$ over $[-M_\varepsilon, M_\varepsilon]$ for all w (which is well defined, due to the assumption of equicontinuity) we establish that the bounds in those inequalities are uniform in $w \in I^\Delta$. Finally, (27) follows from Lemma 15 just as in the previous case. This establishes the proof.

Step 2: We will show that $\left|T_I^{\Delta,d}F(w) - T^{\Delta,q}F(w)\right|_{I^\Delta}^+ = o(\Delta)$.

Case 1: **$X|Z$ has continuous density.** The proof is fully analogous to the previous step. For a policy (a, c, W) that is $\varepsilon\Delta$-suboptimal in the problem $T_I^{\Delta,d}F(w)$ define (a_q, c_q, W_q) as in Definition 1: $c_q = c$, $W_q(\Delta x) = C + W(\Delta x)$ if $\left|W(\Delta x) - \mathbb{E}^\Delta[W(y)]\right| \leq \sqrt{\Delta}M_\varepsilon$ and $W_q(\Delta x) = C$ otherwise, while $a_q(z)$ is defined by the (FOC$_q$-AC) condition and C is chosen to satisfy (PK$_q$). By construction (a_q, c_q, W_q) is feasible $T^{\Delta,q}F(w)$ and we prove as in step 1 that $\Phi^{\Delta,q}(a_q, c_q, W_q; F, w) \geq \Phi^\Delta(a, c, W; F) - O(\varepsilon\Delta)$.

Case 2: $X \equiv Z$.[23] For a policy (a, c, W) that is $\varepsilon\Delta$-suboptimal in the problem $T_I^{\Delta,d}F(w)$ define (a_q, c_q, W_q) as follows. Let $c_q = c$, $a_q(x) = a(x)$ for $x \in [-M_\varepsilon/\sqrt{\Delta}+1, M_\varepsilon/\sqrt{\Delta}-1]$, $a_q(x) = 0$ for $x \notin [-M_\varepsilon/\sqrt{\Delta}, M_\varepsilon/\sqrt{\Delta}]$ and a_q piecewise continuously differentiable. W_q is defined by the local IC in (15), continuity and PK. The policy (a_q, c_q, W_q) is feasible by construction, and we must prove that $\Phi^{\Delta,q}(a_q, c_q, W_q; F, w) \geq \Phi^\Delta(a, c, W; F) - O(\varepsilon\Delta)$.

On the one hand, $\mathbb{P}^\Delta[a_q(x) = a(x)] \geq 1 - 2\varepsilon$ for sufficiently small Δ, which implies the analogues of (25) and (26). On the other hand, for all $\underline{x}, \overline{x} \in [-M_\varepsilon/\sqrt{\Delta}, M_\varepsilon/\sqrt{\Delta}]$

$$\begin{aligned}
W_q(\Delta\bar{x}) - W_q(\Delta\underline{x}) &= \tilde{r}e^{r\Delta}\int_{\underline{x}}^{\bar{x}} \Delta h'(a_q(x))dx \;=\; \tilde{r}e^{r\Delta}\int_{\underline{x}}^{\bar{x}} \Delta h'(a(x))dx \\
&= \tilde{r}e^{r\Delta}\Big[\int_{\underline{x}}^{\overline{x}} \Delta h'(a(x))(1 + a'(x))dx - \Delta(h(a(\overline{x})) - h(a(\underline{x})))\Big] \\
&= W(\Delta[\overline{x} + a(\overline{x})]) - W(\Delta[\underline{x} + a(\underline{x})]) + O(\Delta),
\end{aligned}$$

[23] In Step 1 we used the fact that the quadratic simple policies, for all Δ, are based on the same set of v functions from the definition of Θ. In particular, the W_q functions have the same number of points of discontinuity, for all Δ. In this Step, without additional proofs we cannot assume such uniformity, and so the construction is different.

Consequently $\mathbb{V}^{\Delta}[W_q(\Delta x)] \leq \mathbb{V}^{\Delta}\left[W(y)\mathbf{1}_{|x|\leq M_\varepsilon/\sqrt{\Delta}}\right] + O(\Delta^2)$. Moreover, since $\mathbb{V}^{\Delta}[W(y)] \leq V\Delta$ (Lemma 14) and $W' \in [0, h'(A)]$, there is K_ε such that for any Δ, $|x| \leq M_\varepsilon/\sqrt{\Delta}$ implies $y \in B$, where $B = \{y \mid |W(y) - \mathbb{E}^{\Delta}[W(y)]| \leq \sqrt{\Delta}K_\varepsilon\}$.

Altogether

$$\begin{aligned}\Phi^{\Delta,q}\left(a_q, c_q, W_q; F, w\right) &= \tilde{r}\Delta(\mathbb{E}^{\Delta}[a(x)] - c) + e^{-r\Delta}\Big[F(w) + F'(w)\mathbb{E}^{\Delta}[W(y) - w] \\ &\quad + \frac{1}{2}F''(w)\,\mathbb{V}^{\Delta}\left[W(y)\mathbf{1}_B\right]\Big] + O\left(\varepsilon\Delta\right) \leq \Phi^{\Delta}\left(a, c, W; F\right) + O\left(\varepsilon\Delta\right),\end{aligned}$$

which establishes the Lemma. ■

Proof. (Lemma 11) Fix $w \in I$ and a simple policy (a, c, W).

(i) Consider the necessary local version of (IC) in (23). Since h is strictly convex, Lemma 16 implies that for sufficiently small Δ and any z there is a unique solution $a\left(z\right)$ to this equation.

(ii) Consider now the necessary local version of (IC) in (28). Since h' is strictly convex it follows that if v' is continuously differentiable on $(\sqrt{\Delta}[x + a_1], \sqrt{\Delta}[x + a\left(x\right)])$ for some $a_1 < a\left(x\right)$ (without loss of generality) then a_1 cannot be the solution to this equation. Therefore, since for sufficiently small Δ there may be at most one point of discontinuity of v'' on an interval of length $\sqrt{\Delta}A$, for every x there are only two actions that satisfy (28). ■

The following Lemma 17 is relevant only for the proof of Lemma 12. It is related to the standard results in the static mechanism design.

Lemma 17 *Suppose $X \equiv Z$. For any $\Delta > 0$ and $w \in I^{\Delta}$, if (a, c, W) satisfies (IC) in $T_I^{\Delta,c}F(w)$ then $x + a(x)$ is nondecreasing. Conversely, if (a, c, W) satisfies the local version of (IC) almost everywhere and $x + a(x)$ is nondecreasing, then (a, c, W) satisfies the IC.*

Proof. The proof is standard, but we provide it for completeness. Suppose first that (a, c, W) is incentive compatible. Therefore for any $x' > x$

$$\begin{aligned}-\tilde{r}h(a(x')) + e^{-r\Delta}W(\Delta[x' + a(x')]) &\geq -\tilde{r}h(a(x) - (x' - x)) + e^{-r\Delta}W(\Delta[x + a(x)]), \\ -\tilde{r}h(a(x)) + e^{-r\Delta}W(\Delta[x + a(x)]) &\geq -\tilde{r}h(a(x') + (x' - x)) + e^{-r\Delta}W(\Delta[x' + a(x')]).\end{aligned}$$

Hence,

$$h(a(x')) - h(a(x) - (x' - x)) \leq h(a(x') + (x' - x)) - h(a(x)).$$

Since h is convex, this implies that $a(x') \geq a(x) - (x' - x)$.

Conversely, we argue by contradiction. Assume that (a, c, W) satisfies the local IC and $x + a(x)$ is nondecreasing. Let

$$V(x, x') = -\tilde{r}h(a(x') + (x' - x)) + e^{r\Delta}W(\Delta[x' + a(x')]).$$

By local IC, $V_2(x, x) = 0$ for all x. Suppose that for some $x' > x$ we have $V(x, x') > V(x, x)$. Then

$$0 < V(x, x') - V(x, x) = \int_x^{x'} V_2(x, s)ds = \int_x^{x'} [V_2(x, s)) - V_2(s, s)]ds = -\int_x^{x'}\int_x^{s} V_{12}(z, s)dzds.$$

But

$$V_{12}(z,s) = \tilde{r}h''(a(s)+(s-z))(1+a'(s)) \geq 0.$$

which is a contradiction. The case $V(x,x') > V(x,x)$ with $x' < x$ is analogous. ■

Proof. (Lemma 12) Fix $\Delta, \varepsilon > 0$ and consider any Δ-suboptimal policy (a,c,W) for $T^{\Delta,c}F(w)$. Let M_ε be such that $\mathbb{P}_X^\Delta[[-M_\varepsilon/\sqrt{\Delta}, M_\varepsilon/\sqrt{\Delta}]] \geq 1-\varepsilon$. We construct a policy (a_d, c_d, W_d) as follows. The function $a_d(\cdot)$ is derived from the function $a(\cdot)$ below so that $a_d(\cdot)$ is piecewise continuously differentiable and $x + a_d(x)$ is nondecreasing. Then we let $c_d = c$, and W_d be such that it satisfies the local version of (IC):

$$\tilde{r}h'(a_d(x)) = e^{-r\Delta}W_d'(\Delta[x+a_d(x)]),$$

is continuous and the constant of integration is adjusted so that it satisfies the PK condition. By Lemma 17, the policy (a_d, c_d, W_d) is feasible by construction.

Below we will define a_d so that $a_d(x) = 0$ if $x \notin [-M_\varepsilon/\sqrt{\Delta}, M_\varepsilon/\sqrt{\Delta} + A]$, $x + a_d(x)$ is nondecreasing and

$$\int_{-M_\varepsilon/\sqrt{\Delta}}^{M_\varepsilon/\sqrt{\Delta}} |a_d(x) - a(x)|\, dx \leq \varepsilon \quad \text{and} \quad \int_{-M_\varepsilon/\sqrt{\Delta}}^{M_\varepsilon/\sqrt{\Delta}} \left|a_d'(x) - a'(x)\right| dx \leq \varepsilon. \tag{29}$$

Recall that if f is nondecreasing, then f is differentiable a.e. and $\int_a^b f'(x)dx \leq f(b) - f(a)$.[24] Since

$$\begin{aligned} & h'(a_d(x))(1+a_d'(x)) - h'(a(x))(1+a'(x)) \\ = \; & h'(a_d(x))(a_d'(x) - a'(x)) + (h'(a_d(x)) - h'(a(x)))(1+a'(x)), \end{aligned}$$

(29) implies that for any $\underline{x}, \overline{x} \in [-M_\varepsilon/\sqrt{\Delta}, M_\varepsilon/\sqrt{\Delta}]$,

$$\begin{aligned} & W_d(\Delta[\overline{x} + a_d(\overline{x})]) - W_d(\Delta[\underline{x} + a_d(\underline{x})]) = \tilde{r}e^{r\Delta}\Delta \int_{\underline{x}}^{\overline{x}} h'(a_d(x))(1+a_d'(x))dx \\ \leq \; & W(\Delta[\overline{x} + a(\overline{x})]) - W(\Delta[\underline{x} + a(\underline{x})]) + \tilde{r}e^{r\Delta}\Delta\left[h'(A)\varepsilon + \max h''\left[\frac{2M_\varepsilon}{\sqrt{\Delta}} + a(\overline{x}) - a(\underline{x})\right]\right] \end{aligned}$$

The rest of the proof will follow as in last step of Lemma 10 to establish that $\Phi^\Delta(a_d, c_d, W_d; F) \geq \Phi^\Delta(a, c, W; F) - O(\varepsilon\Delta)$.

We now construct an a_d satisfying (29) and $x + a_d(x)$ is nondecreasing. First, note that since for any $y > x$ we have $a(x) \geq a(y) - \frac{y-x}{\Delta}$, a may not discontinuously decrease. Therefore, the set of points $D \subset [-M_\varepsilon/\sqrt{\Delta}, M_\varepsilon/\sqrt{\Delta}]$ at which a may be discontinuous is at most countable. Moreover, if $J = \sum_{x\in D}(a(x_+) - a(x_-))$, then

$$J + \int_{-M_\varepsilon/\sqrt{\Delta}}^{M_\varepsilon/\sqrt{\Delta}} (1+a'(x))dx = \frac{2M_\varepsilon}{\sqrt{\Delta}} + a(\overline{x}) - a(\underline{x}) \leq A + \frac{2M_\varepsilon}{\sqrt{\Delta}}.$$

Since $1 + a'(x) \geq 0$, this implies that $J \leq A + \frac{2M_\varepsilon}{\sqrt{\Delta}}$. Let D_f be a finite set of points where a is discontinuous such that $\sum_{x\in D_f}(a(x_+) - a(x_-)) \geq J - \varepsilon/2$, and let $\delta = \min_{x\in D_f}(a(x_+) - a(x_-))$.

[24] See, for example, Theorem 2 in Chapter 5 of Royden [1988].

For any $n \in \mathbb{N}$ and $x \in [-M_\varepsilon/\sqrt{\Delta}, M_\varepsilon/\sqrt{\Delta}]$ let

$$a'_n(x) = \frac{n}{2} \int_{x-1/n}^{x+1/n} a'(s)\, ds.$$

The function a'_n is differentiable and for any x, $a'_n(x) \geq -1$ (since $a'(x) \geq -1$). From the Lebesgue's Density Theorem it follows that for sufficiently large n, $\int_{-M_\varepsilon/\sqrt{\Delta}}^{M_\varepsilon/\sqrt{\Delta}} |a'_n(x) - a'(x)| dx \leq \delta$.

Finally, for $D_f = \left\{d_1, ..., \dot{d}_n\right\}$, $d_0 = -M_\varepsilon/\sqrt{\Delta}$, $d_{n+1} = M_\varepsilon/\sqrt{\Delta}$, and for any $x \in [d_i, d_{i+1})$ let

$$a_d(x) = a(d_i) + \int_{d_i}^{x} a'_n(s)\, ds.$$

The function a_d satisfies (29) and $x + a_d(x)$ is nondecreasing by construction, which establishes the proof. ■

Proof. (Lemma 13) Fix $\varepsilon, \Delta > 0$ and any $w \in I^\Delta$, and let (a, c, W) be a policy function that is $\varepsilon\Delta-$suboptimal in the problem $T_I^\Delta F(w)$. Using Lemma 15 and arguments as in the proof of Proposition 10 we may assume without loss of generality that for every public signal $y = \Delta[x + a(z)]$ $\left|W(y) - \mathbb{E}^\Delta[W(y)]\right| = O\left(\sqrt{\Delta}\right)$, and that $\Phi^\Delta(a, c, W; F)$ is equal, up to $o(\Delta)$

$$\tilde{r}\Delta\left(\mathbb{E}^\Delta[a(z)] - \mathbb{E}^\Delta[c(y)]\right) + e^{-r\Delta}\left[\begin{array}{l} F(w) + F'(w)\left(\mathbb{E}^\Delta[W(y)] - w\right) \\ + \frac{F''(w)}{2}\mathbb{V}^\Delta[W(y)] \end{array}\right]. \tag{30}$$

Let the policy (a_c, c_c, W_c) with constant consumption be defined so that $a_c \equiv a$, $\mathbb{E}^\Delta[u(c(y))] = u(c_c)$ and

$$e^{-r\Delta}W_c(y) = \Delta\tilde{r}\left[u(c(y)) - \mathbb{E}^\Delta[u(c(y))]\right] + e^{-r\Delta}W(y).$$

We will compare the terms in (30) with the analogous terms for the policy (a_c, c_c, W_c). We have $\mathbb{E}^\Delta[a(z)] = \mathbb{E}^\Delta[a_c(z)]$, $\mathbb{E}^\Delta[W(y)] = \mathbb{E}^\Delta[W_c(y)]$ and, from concavity of u, $c_c \leq \mathbb{E}^\Delta[c(y)]$. Letting $\zeta(y) := \tilde{r}e^{r\Delta}[u(c(y)) - u(c_c)]$ the following establishes the proof:

$$\mathbb{V}^\Delta[W_c(y)] - \mathbb{V}^\Delta[W(y)] = \mathbb{E}^\Delta\left[\left(W(y) + \Delta\zeta(y) - \mathbb{E}^\Delta[W(y)]\right)^2\right] - \mathbb{E}^\Delta\left[\left(W(y) - \mathbb{E}^\Delta[W(y)]\right)^2\right]$$
$$= \Delta^2\mathbb{E}^\Delta\left[\zeta^2(y)\right] + \Delta\mathbb{E}^\Delta\left[\left(W(y) - \mathbb{E}^\Delta[W(y)]\right)\zeta(y)\right] \leq \Delta^2(\tilde{r}e^{r\Delta}\overline{u})^2 + \Delta^{3/2}\tilde{r}e^{r\Delta}\overline{u} = o(\Delta).$$

■

7.2 The HJB Equation

We start with a Lemma that establishes some basic properties of the solution of the HJB equation.

Lemma 18 *Suppose that* $\Theta \geq \underline{\theta} > 0$.

(i) For any initial conditions $F(\underline{w})$ *and* $F'(\underline{w})$ *the HJB equation (3) has a unique solution* F *in any interval* $[\underline{w}, \bar{w}] \subset \mathbb{R}$.

(ii) F is twice continuously differentiable and (F, F') depends continuously on the initial conditions.

(iii) F' is monotone with respect to $F'(\underline{w})$. That is, if F_1 and F_2 are two solutions of the HJB equation in an interval $[\underline{w}, \bar{w}] \subset \mathbb{R}$ with $F_1(\underline{w}) = F_2(\underline{w})$ and $F_1'(\underline{w}) > F_2'(\underline{w})$, then $F_1'(w) > F_2'(w)$ (and hence $F_1(w) > F_2(w)$) for all $w > \underline{w}$.

Proof. See Sannikov [2008]. ■

Note that the definition of Θ_ζ for any $\zeta > 0$ guarantees that $\Theta_\zeta \geq \zeta > 0$. Lemma 18 thus implies that for any $\zeta > 0$ the HJB equation (6) with the boundary conditions (4) and (5) has a unique solution F_ζ. Moreover, the uniqueness, continuity and monotonicity in the initial slope suggest the natural procedure for computing F_ζ.

The following Lemma is crucial for the proof of the Theorems.

Lemma 19 *Suppose that $\Theta \geq \underline{\theta} > 0$. The solution F of the HJB equation (3) with the boundary conditions (4) and (5) is strictly concave.*

Proof. See Sannikov [2008]. ■

The following "single crossing" Lemma will be used in the proof of Proposition 2.

Lemma 20 *Consider two functions $\Theta \geq \underline{\Theta} \geq 0$, and suppose that F^{Θ} and $F^{\underline{\Theta}}$ solve the corresponding HJB equations (3) with $F^{\Theta\prime\prime} \leq 0$.*

(i) If for some w, $F^{\Theta}(w) = F(w)$ and $F^{\Theta}(w') > F^{\underline{\Theta}}(w')$ in a right neighborhood of w, then $F^{\Theta\prime}(w') > F^{\underline{\Theta}\prime}(w')$ for all $w' > w$.

(ii) Assume $\Theta > \underline{\Theta}$. If for some w, $F^{\Theta}(w) = F^{\underline{\Theta}}(w)$ and $F^{\Theta\prime}(w) \geq F^{\underline{\Theta}\prime}(w)$, then $F^{\Theta\prime}(w') > F^{\underline{\Theta}\prime}(w')$ for all $w' > w$.

Note that the precondition of part (i) is implied by (but is not equivalent to) $F^{\Theta}(w) = F^{\underline{\Theta}}(w)$ and $F^{\Theta\prime}(w) > F^{\underline{\Theta}\prime}(w)$.

Proof. We prove only part (i) (the proof of part (ii) is analogous). First, by assumption, $F^{\Theta\prime}(w') > F^{\underline{\Theta}\prime}(w')$ for all $w' > w$ sufficiently close to w. Suppose now that there exists $w' > w$ with $F^{\Theta\prime}(w') \leq F^{\underline{\Theta}\prime}(w')$ - we now assume that w' is the smallest with this property. Since $F^{\Theta\prime} >_{(w,w')} F^{\underline{\Theta}\prime}$, we have that $F^{\Theta}(w') > F^{\underline{\Theta}}(w')$. Therefore, it must be the case that $F^{\Theta\prime\prime}(w') > F^{\underline{\Theta}\prime\prime}(w')$: otherwise, since $F^{\Theta\prime\prime}(w') \leq 0$ and $\Theta \geq \underline{\Theta}$, every policy $(\bar{a}, \bar{h}, c)$ would yield a weakly higher value of the right-hand side of HJB equation (3) for $F^{\underline{\Theta}}(w')$ than for $F^{\Theta}(w')$. But $F^{\Theta\prime\prime}(w') > F^{\underline{\Theta}\prime\prime}(w')$ implies that $F^{\Theta\prime}(w'') < F^{\underline{\Theta}\prime}(w'')$ for w'' in a left neighborhood of w', contradicting the minimality of w'. ■

7.2.1 Proof of Proposition 2, part (i)

For any $\zeta > 0$, let F_ζ^{Θ} and $F_\zeta^{\underline{\Theta}}$ be as in Theorem 1. By Lemma 18, F_ζ^{Θ} and $F_\zeta^{\underline{\Theta}}$ exist. Let us show that $F_\zeta^{\Theta}(w) \leq F_\zeta^{\underline{\Theta}}(w)$ for all $w \in [0, w_{sp,\zeta}^{\Theta}]$. Part (i) of Lemma 20 implies that there is no point $w \in [0, w_{sp,\zeta}^{\Theta})$ such that $F_\zeta^{\Theta}(w) = F_\zeta^{\underline{\Theta}}(w)$ and $F_\zeta^{\Theta}(w') > F_\zeta^{\underline{\Theta}}(w')$ in a right neighborhood of

w: otherwise $F^{\Theta}_{\underline{\zeta}}(w^{\Theta}_{sp,\zeta}) < F^{\Theta}_{\zeta}(w^{\Theta}_{sp,\zeta}) = \underline{F}(w^{\Theta}_{sp,\zeta})$, contradicting $F^{\Theta}_{\underline{\zeta}} \geq \underline{F}$. The result attains since $F^{\Theta}_{\zeta}(0) = F^{\Theta}_{\underline{\zeta}}(0)$.

The result also implies monotonicity in ζ: for $\underline{\zeta} \leq \zeta$ $F^{\Theta}_{\zeta}(w) \leq F^{\Theta}_{\underline{\zeta}}(w)$ for all $w \in [0, w^{\Theta}_{sp,\underline{\zeta}}]$ and $F^{\Theta}_{\zeta}(w) \leq F^{\underline{\Theta}}_{\underline{\zeta}}(w)$ for all $w \in [0, w^{\underline{\Theta}}_{sp,\underline{\zeta}}]$. Altogether, we obtain $F^{\Theta}(w) \leq F^{\underline{\Theta}}(w)$ for all $w \in [0, w^{\Theta}_{sp}]$ by taking pointwise limits as $\zeta \downarrow 0$. This establishes the proof.

The following Corollary establishes the last missing part of the proof of Theorem 1.

Corollary 3 *For any Θ the Function F and w_{sp} in Theorem 1 exist.*

7.2.2 Proof of Proposition 1

We will use the following Lemma.

Lemma 21

(i) Suppose that $\Theta(\bar{a}, \bar{h}) \geq \underline{\theta} > 0$ for $\bar{a} > 0$. Then for all $\zeta > 0$ the F_{ζ} as in Theorem 1 solves equation (6) with an additional constraint $\bar{a} > 0$.

(ii) Suppose (Cont) holds. Then for any $[\underline{w}, \overline{w}] \subset (0, w_{sp})$ there exists $\gamma > 0$ such that for all sufficiently small ζ, the F_{ζ} as in Theorem 1 solves equation (6) on $[\underline{w}, \overline{w}]$ with an additional constraint $\bar{a} \geq \gamma$.

Proof. For any $\lambda \in [\underline{F}'(w_{sp}), \infty)$ let H_{λ} be the linear function tangent to the retirement curve $\{(w, \underline{F}(w)) : w \in [0, w_{sp}]\}$ with the slope λ (if $\lambda \geq \underline{F}'(0)$, $H_{\lambda}(w) = \lambda w$).

(i) Fix any $\zeta > 0$. Since F_{ζ} and $\underline{F}$ are concave and $F_{\zeta} \geq \underline{F}$, for any $w \in [0, w_{sp,\zeta}]$ we have $F_{\zeta}(w) - H_{F'_{\zeta}(w)}(w) \geq 0$. On the other hand, for any ζ and $w \in [0, w_{sp}]$, if we restrict the policy on the right hand side of equation (6) to satisfy $\bar{a} = 0$, we have $\max_c\{-c + F'_{\zeta}(w)(w - u(c)) + \frac{1}{2}F''_{\zeta}(w)r\zeta\} < \max_c\{-c + F'_{\zeta}(w)(w - u(c))\} = \underline{F}(w') + \underline{F}'(w')(w - w') = H_{F'_{\zeta}(w)}(w)$, where w' is such that $\underline{F}'(w') = F'_{\zeta}(w)$. Hence, choosing $\bar{a} = 0$ is never optimal and without any loss the additional constraint $\bar{a} > 0$ can be included in (6) to compute $F_{\zeta}(w)$.

(ii) We may assume $w_{sp} > 0$. Note also that for any $\zeta > 0$ and F_{ζ} as in Theorem 1 we have

$$\overline{F}'(\overline{w}_{sp}) \leq F'_{\zeta}(w) \leq \overline{F}(\underline{w}) / \underline{w},$$

for all $w \in [\underline{w}, \overline{w}]$. We will establish that there is $\alpha > 0$ such that for any ζ and $w \in [\underline{w}, \overline{w}]$, $F_{\zeta}(w) - H_{F'_{\zeta}(w)}(w) \geq \alpha$. If not, then let $\{w_n\}, \{w'_n\}, \{\zeta_n\}$ and $\{\alpha_n\}$ with $w_n \in [\underline{w}, \overline{w}]$, $w'_n \leq w_{sp}$, $\zeta_n \downarrow 0$, $\alpha_n \downarrow 0$ be such that $F_{\zeta_n}(w_n) - H_{F'_{\zeta_n}(w_n)}(w_n) \leq \alpha_n$ (where w'_n is such that $\underline{F}'(w'_n) = F'_{\zeta_n}(w_n)$). We consider three cases.

(Case 1) Suppose that for some $\delta > 0$ and all n, $w'_n \in [\delta, w_{sp} - \delta]$. The concavity of F_{ζ_n} and $\underline{F}$ imply that $F_{\zeta_n}(w_n) - H_{F'_{\zeta_n}(w_n)}(w_n) \geq F_{\zeta_n}(w'_n) - H_{F'_{\zeta_n}(w_n)}(w'_n) = F_{\zeta_n}(w'_n) - \underline{F}(w'_n)$. But, since F_{ζ_n} is increasing as $\zeta_n \downarrow 0$ (Proposition 2, part (i)), $F_{\zeta_n}(w'_n) - \underline{F}(w'_n) \geq \inf_{w \in [\delta, w_{sp} - \delta]} F_{\zeta_1}(w) - \underline{F}(w) > 0$, a contradiction.

(Case 2) If $w'_n \downarrow 0$ (we might assume so by choosing a subsequence), then we would have $F_{\zeta_n}(w_n) \to H_{F'_{\zeta_n}(w_n)}(w_n) \to \underline{F}'(0) \times w_n$. By concavity of all F_{ζ_n} this would imply that, first,

$F_{\zeta_n}(w) \to \underline{F}'(0) \times w$ for all $w \in [0, w_n]$, and second, that there is a sequence $\{w_n''\}$, $w_n'' \in [0, w_n]$, such that $F_{\zeta_n}'(w_n'') \to \underline{F}'(0)$ and $F_{\zeta_n}''(w_n'') \to 0$. But then

$$\begin{aligned} F_{\zeta_n}\left(w_n''\right) &\to \max_{a,c}\left\{(a-c) + \underline{F}'(0)\left(w_n'' + h(a) - u(c)\right)\right\} \\ &= \max_{a}\left\{a + \underline{F}'(0)\left(w_n'' + h(a)\right)\right\} > \underline{F}'(0)\, w_n'', \end{aligned}$$

where the equality follows from the fact that $\underline{F}'(0) = \frac{1}{u'(0)}$ and strict concavity of u, while the inequality follows from $h_+'(0) < u'(0)$. This establishes the required contradiction.

(Case 3) If $w_n' \uparrow w_{sp}$, we derive the contradiction in the analogous way as in case 2.

We have established that for all ζ and $w \in [\underline{w}, \overline{w}]$, $F_\zeta(w) - H_{F_\zeta'(w)}(w) \geq \alpha > 0$. On the other hand, for any ζ and $w \in [\underline{w}, \overline{w}]$, if we restrict the policy on the right hand side of equation (6) to satisfy $\bar{a} = 0$, we have $\max_c\{-c + F_\zeta'(w)(w - u(c)) + \frac{1}{2}F_\zeta''(w)\, r\zeta\} \leq H_{F_\zeta'(w)}(w) \leq F_\zeta(w) - \alpha$. Since F_ζ' are uniformly bounded on $[\underline{w}, \overline{w}]$ and $\bar{h} \leq \frac{\bar{a}}{A} h(A)$, we see that the policy for $F_\zeta(w)$, for any ζ and $w \in [\underline{w}, \overline{w}]$, satisfies the additional constraint $\bar{a} \geq \gamma$ for appropriate $\gamma > 0$. ■

We are now ready to prove Proposition 1.

(i) Given Lemma 21, part (i), all the functions F_ζ for $\zeta < \underline{\theta}$ are the same, and solve the equation (3). The Proposition then follows from Theorem 1.

(ii) Choose any $[\underline{w}, \overline{w}] \subset (0, w_{sp})$. Lemma 21, part (ii), guarantees that for sufficiently small ζ all F_ζ satisfy the constraint $\bar{a} \geq \gamma$ on $[\underline{w}, \overline{w}]$, for some $\gamma > 0$. Therefore, for sufficiently small ζ all F_ζ satisfy on $[\underline{w}, \overline{w}]$:

$$F''(w) = \inf_{\bar{a} \geq \gamma, \bar{h}, c}\left\{\frac{F(w) - (\bar{a} - c) - F'(w)\left(w + \bar{h} - u(c)\right)}{r\Theta\left(\bar{a}, \bar{h}\right)/2}\right\},$$

with the right-hand side Lipschitz continuous in $(w, F(w), F'(w))$, since $\Theta \geq \delta(\gamma) > 0$ for $\bar{a} \geq \gamma$.

Part (i) of Proposition 2 guarantees that F_ζ converge in the supremum norm as $\zeta \downarrow 0$ to a function F. Since F_ζ' are uniformly bounded on $[\underline{w}, \overline{w}]$, it follows that all F_ζ'' and F_ζ' are Lipschitz continuous with the same Lipschitz constant, and so F_ζ' converge to F' not only in L^1 but in the supremum norm, by the Arzela-Ascoli Theorem. Uniform Lipschitz continuity guarantees also that $F' = \frac{d}{dw}F$, that $F'' := \lim_{\zeta \downarrow 0} F_\zeta''$ exists and F satisfies the above equation (all on $[\underline{w}, \overline{w}]$). Since the set $[\underline{w}, \overline{w}]$ is arbitrary, this proves that F solves (3) in $(0, w_{sp})$.

7.2.3 Proof of Proposition 2, part (ii)

Part (ii) of Proposition 1 shows that $F^{\underline{\Theta}}$ and F^{Θ} satisfy the HJB equation (3). Part (ii) of Lemma 21 guarantees that for $F^{\underline{\Theta}}$ and F^{Θ} restricted to $[\underline{w}, \overline{w}] \subset (0, w_{sp,\Theta})$ as well as the domains of Θ and $\underline{\Theta}$ restricted to $\left\{\left(\bar{a}, \bar{h}\right) : \bar{a} \geq \gamma\right\}$, the preconditions for part (ii) of Lemma 20 are satisfied, with $\underline{\theta} = \delta(\gamma)$ for γ as in Lemma 21. Moreover, part (i) of the Proposition implies that $F^{\underline{\Theta}} \geq F^{\Theta}$. Therefore, if $F^{\underline{\Theta}}(w) = F^{\Theta}(w)$ for some $w \in (0, w_{sp}^{\underline{\Theta}})$, it must be that $F^{\underline{\Theta}\prime}(w) = F^{\Theta\prime}(w)$, and so $F^{\underline{\Theta}\prime}(w') < F^{\Theta\prime}(w')$ for $w' > w$, contradicting $F^{\underline{\Theta}} \geq \underline{F}$. This completes the proof.

7.2.4 Proof of Proposition 3

The proof follows from the following Lemma.

Lemma 22 *Suppose that $\Theta \equiv 0$. For any $\delta > 0$ and sufficiently small ζ the solution F_ζ of the HJB equation (6) with initial conditions*

$$F_\zeta(\widetilde{w}) = \overline{F}(\widetilde{w}) - \delta,\ F'_\zeta(\widetilde{w}) = \overline{F}'(\widetilde{w})$$

for some $\widetilde{w} \in [0, \overline{w}_{sp}]$ satisfies

$$F''_\zeta \leq_{[0,\overline{w}_{sp}]} -\frac{2\delta}{\zeta}.$$

Proof. For any $\lambda \in [\overline{F}'(\overline{w}_{sp}), \infty)$ let G_λ be the linear function tangent to the first-best frontier $\{(w, \overline{F}(w)) : w \in [0, \overline{w}_{sp}]\}$ with the slope λ. We will show that if for an arbitrary $w \in [0, w_{sp}]$

$$G_{F'_\zeta(w)}(w) - F_\zeta(w) \geq \delta, \tag{31}$$

then $F''_\zeta(w) \leq -\frac{2\delta}{\zeta}$. Note that then as long as $-\frac{2\delta}{\zeta} \leq \min_{w\in[0,\overline{w}_{sp}]} \overline{F}''(w)$ the above condition will be satisfied over the whole interval $[0, \overline{w}_{sp}]$, which will establish the Lemma.

Since $\Theta \equiv 0$, the HJB equation (6) takes the form

$$F''_\zeta(w) = \frac{2}{\zeta} \min_{a,h,c} \{F_\zeta(w) - (a - c) - F'_\zeta(w)(w + h - u(c))\}. \tag{32}$$

Let w' be such that $F'_\zeta(w) = \overline{F}'(w')$. For the policy $(a(w'), c(w'))$ in the problem (1) at w' we have:

$$\begin{aligned}
&F_\zeta(w) - (a(w') - c(w')) - F'_\zeta(w)(w + h(a(w')) - u(c(w'))) \ = \\
&\overline{F}(w') - (a(w') - c(w')) - \overline{F}'(w')(w' + h(a(w')) - u(c(w'))) \\
&+ \ [F_\zeta(w) - \overline{F}(w') + F'_\zeta(w)(w' - w)] \ = \ [F_\zeta(w) - \overline{F}(w') + F'_\zeta(w)(w' - w)] \leq -\delta,
\end{aligned}$$

where the last equality follows from (1), while the last inequality follows from (31). Since $(a(w'), h(a(w')), c(w'))$ is an available policy in the problem (32), this establishes that $F''_\zeta(w) \leq -\frac{2\delta}{\zeta}$. ■

Given the Lemma, for any $\delta > 0$ and sufficiently small $\zeta > 0$ the solution F_ζ of the HJB equation (6) with initial conditions $F_\zeta(\widetilde{w}) = \overline{F}(\widetilde{w}) - \delta$, $F'_\zeta(\widetilde{w}) = \overline{F}'(\widetilde{w})$ with $\widetilde{w} \in [\delta, \overline{w}_{sp}]$ will satisfy $F_\zeta(\underline{w}) = \underline{F}(\underline{w})$ and $F_\zeta(\overline{w}) = \underline{F}(\overline{w})$ for some $0 < \underline{w} < \overline{w} < \overline{w}_{sp}$. This together with Proposition 7 and part (ii) of Lemma 7 establishes the proof of the Proposition.

7.3 Proof of Proposition 4

Fix period length $\Delta > 0$, densities g_X and γ_X satisfying (12) and any $w_g, w_\gamma \in [0, \bar{u})$. Consider the problem of finding a contract $\{c_n\}$ and action plans $\{a_{g,n}\}, \{a_{\gamma,n}\}$ that maximize the sum of principal's expected discounted revenues under noise densities g_X and γ_X, such that $\{c_n\}, \{a_{g,n}\}$ is incentive compatible under g_X and $\{c_n\}, \{a_{\gamma,n}\}$ is incentive compatible under γ_X, and they deliver expected discounted utilities w_g and w_γ to the agent. Let $F^{\Delta}_{g,\gamma}(w_g, w_\gamma)$ be the value to the principal from the optimal contract:

$$\begin{aligned} F^{\Delta}_{g,\gamma}(w_g, w_\gamma) \;=\; \sup \; & \Big\{ \Pi_g(\{c_n\},\{a_{g,n}\}) + \Pi_\gamma(\{c_n\},\{a_{\gamma,n}\}) \;\Big|\; \\ & \{a_{g,n}\} \text{ is IC for } \{c_n\},\; U_g(\{c_n\},\{a_{g,n}\}) \;=\; w \text{ under density } g_X, \\ & \{a_{\gamma,n}\} \text{ is IC for } \{c_n\},\; U_\gamma(\{c_n\},\{a_{\gamma,n}\}) \;=\; w \text{ under density } \gamma_X \Big\} \end{aligned}$$

To establish the Proposition we show that if $w_g, w_\gamma \in (0, w_{sp})$ then there is $\delta > 0$ such that for sufficiently small Δ $F^{\Delta}_{g,\gamma}(w_g, w_\gamma) + \delta \le F(w_g) + F(w_\gamma) =: F_2(w_g, w_\gamma)$, where F is as in Theorem 1.

First, consider the following Bellman operator:

$$\begin{aligned} T^{\Delta}_{g,\gamma} f(w_g, w_\gamma) \;&=\; \sup_{a_g, a_\gamma, c, W_g, W_\gamma} \; \Phi^{\Delta}_g(a_g, c, W_g; f) + \Phi^{\Delta}_\gamma(a_\gamma, c, W_\gamma; f) \\ \text{s.t.} \quad & a_\phi \in \mathcal{A}, \quad c(y) \ge 0 \quad \text{and} \quad W_\phi(y) \in [0, \bar{u}) \quad \forall y \\ & w_\phi = \mathbb{E}^{\Delta}_\phi \Big[\tilde{r}\Delta[u(c(\Delta[x + a_\phi])) - h(a_\phi)] + e^{-r\Delta} W_\phi(\Delta[x + a_\phi]) \Big] \qquad (\text{PK}_2) \\ & a_\phi \in \arg\max_{\hat{a} \in \mathcal{A}} \; \mathbb{E}^{\Delta}_\phi \Big[\tilde{r}\Delta[u(c(\Delta[x + \hat{a}])) - h(\hat{a})] + e^{-r\Delta} W_\phi(\Delta[x + \hat{a}]) \Big] \quad (\text{IC}_2\text{-AC}) \end{aligned}$$

where the supremum is taken over measurable functions and and $\Phi^{\Delta}_\phi(a, c, W; f)$ is as in (9), for $\phi \in \{g, \gamma\}$. The following is an analogue of Proposition 5:

Proposition 8 *$F^{\Delta}_{g,\gamma}$ is the largest fixed point f of $T^{\Delta}_{g,\gamma}$ such that $f(w_g, w_\gamma) \le \overline{F}(w_g) + \overline{F}(w_\gamma)$.*

For a set of feasible policies $p = \{(a_g, a_\gamma, c, W_g, W_\gamma)\}_{(w_g, w_\gamma) \in [0,\bar{u})^2}$ for the Bellman operator $T^{\Delta}_{g,\gamma}$ let $T^{\Delta,p}_{g,\gamma}$ be the operator defined as $T^{\Delta,p}_{g,\gamma} f(w) = \Phi^{\Delta}_g(a_g, c, W_g; f) + \Phi^{\Delta}_\gamma(a_\gamma, c, W_\gamma; f)$, and let $F^{\Delta,p}_{g,\gamma}$ be the value achieved by the policies p. Note that $F^{\Delta,p}_{g,\gamma}$ is a fixed point of $T^{\Delta,p}_{g,\gamma}$. Also, policies p together with an initial point $(w_g, w_\gamma) = (w^p_{g,0}, w^p_{\gamma,0})$ determine a stochastic process $\{(w^p_{g,n}, w^p_{\gamma,n})\}$ of continuation values.

For the proof of the Proposition we use the following five claims. Claim 1 is related to Lemma 7. It shows that for a fixed set of policies p for the Bellman operator $T^{\Delta}_{g,\gamma}$, how far the value of the contract built up recursively from those policies falls short of F_2 ($F_2 - F^{\Delta,p}_{g,\gamma}$) can be expressed as a discounted expected sum of how far each policy applied to F_2 falls short of F_2 ($F_2 - T^{\Delta,p}_{g,\gamma} F_2$).

The idea behind the construction in the remaining four claims is as follows. For any $\varepsilon > 0$ consider the set $S_\varepsilon = \{(w_g, w_\gamma) \in [\varepsilon, w_{sp} - \varepsilon]^2 : |w_g - w_\gamma| > \varepsilon, \max\{w_g, w_\gamma\} > w_0 + \varepsilon\}$, where w_0 is such that $F'(w_0) = \underline{F}'(0) = -\frac{1}{u'(0)}$. Claim 2 shows that once the two continuation values are in the set, $F_2 - T^{\Delta,p}_{g,\gamma} F_2$ must be negative: The reason is that to achieve $F_2(w_g, w_f) = F(w_g) + F(w_\gamma)$ the wages paid in the separate two optimal policies for each continuation value must be different (such that $-1/u'(c_g) = F'(w_g)$, and $-1/u'(c_f) = F'(w_f)$), whereas $T^{\Delta,p}_{g,\gamma}$ restricts the wage to be the same.

Claim 3 shows that if $F_2 - T^{\Delta,p}_{g,\gamma} F_2$ is to remain small, it must be that the variances of continuation values W_g, W_f and $W_g - W_f$ must be bounded away from zero, and not too big. This follows from the results in the paper: for the policy p to fare well, the continuation values for each noise must be approximately linear in likelihod ratio. Also, since the likelihood ratios are linearly independent by assumption, $W_g - W_f$ can't be too small. Using Claim 3, Claim 4 shows that under policies p once the process of continuation values (w_g, w_γ) enters set S_ε, it must stay there for a while with nonnegligible probability; Claim 5 shows that starting at any interior point of continuation values the process enters S_ε in finite time with nonnegligible probabilities. Those results, together with Claim 2 establish the Proposition.

Fix a set of policies p for the Bellman operator $T^{\Delta}_{g,\gamma}$.

Claim 1 *Consider function $F : [0, \bar{u})^2 \to \mathbb{R}$ and $(w^p_{g,0}, w^p_{\gamma,0}) \in [0, \bar{u})^2$. Then for any $N \in \mathbb{N}$*

$$\begin{aligned} F_2\left(w_g, w_\gamma\right) - F^{\Delta,p}_{g,\gamma}\left(w_g, w_\gamma\right) \;=\; & \mathbb{E}^{\Delta}_{g,\gamma}\Big[\sum_{n=0}^{N} e^{-rn\Delta}(F_2(w^p_{g,n}, w^p_{\gamma,n}) - T^{\Delta,p}_{g,\gamma}F_2(w^p_{g,n}, w^p_{\gamma,n})) + \\ & + e^{-r(N+1)\Delta}(F_2(w^p_{g,N+1}, w^p_{\gamma,N+1}) - F^{\Delta,p}_{g,\gamma}(w^p_{g,N+1}, w^p_{\gamma,N+1}))\Big]. \end{aligned}$$

Proof. For any $(w_g, w_\gamma) \in [0, \bar{u})^2$ we have

$$\begin{aligned} & F_2\left(w_g, w_\gamma\right) - F^{\Delta,p}_{g,\gamma}\left(w_g, w_\gamma\right) = F_2\left(w_g, w_\gamma\right) - T^{\Delta,p}_{g,\gamma}F^{\Delta,p}_{g,\gamma}\left(w_g, w_\gamma\right) = \\ = \; & F_2\left(w_g, w_\gamma\right) - T^{\Delta,p}_{g,\gamma}F_2\left(w_g, w_\gamma\right) + T^{\Delta,p}_{g,\gamma}F_2\left(w_g, w_\gamma\right) - T^{\Delta,p}_{g,\gamma}F^{\Delta,p}_{g,\gamma}\left(w_g, w_\gamma\right) \\ = \; & \mathbb{E}^{\Delta}_{g,\gamma}\Big[F_2\left(w_g, w_\gamma\right) - T^{\Delta,p}_{g,\gamma}F_2\left(w_g, w_\gamma\right) + e^{-r\Delta}(F_2(w^p_{g,1}, w^p_{\gamma,1}) - F^{\Delta,p}_{g,\gamma}(w^p_{g,1}, w^p_{\gamma,1}))\Big], \end{aligned}$$

and using the equality recursively yields the proof. ■

Claim 2 *Fix $\varepsilon > 0$ and $(w_g, w_\gamma) \in S_\varepsilon$. Then there is δ_1 such that for sufficiently small $\Delta > 0$:*

$$F_2(w_g, w_\gamma) - T^{\Delta}_{g,\gamma}F_2(w_g, w_\gamma) > \delta_1\Delta.$$

Proof. In analogy to $T^{\Delta,q}$ we also define a simplified "quadratic" operator $T^{\Delta,q}_{g,\gamma}$:

$$\begin{aligned} T^{\Delta,q}_{g,\gamma}f(w_g, w_\gamma) \;=\; & \sup_{a_g, a_\gamma, c, W_g, W_\gamma} \Phi^{\Delta,q}_{g}(a_g, c, W_g; f, w_g) + \Phi^{\Delta}_{\gamma}(a_\gamma, c, W_\gamma; f, w_\gamma) \\ \text{s.t.} \quad & a_\phi(z) \in \mathcal{A}, \quad c \geq 0, \quad \text{and} \quad W_\phi(y) \in \mathbb{R} \quad \forall y \\ & w_\phi = \mathbb{E}^{\Delta}_{\phi}\Big[\tilde{r}\Delta[u(c(\Delta[x + a_\phi])) - h(a_\phi)] + e^{-r\Delta}W_\phi(\Delta[x + a_\phi])\Big] \quad (\text{PK}_2q) \\ & \tilde{r}h'(a_\phi) = -\frac{e^{-r\Delta}}{\Delta}\int_{\mathbb{R}} W(\Delta x)\phi^{\Delta\prime}_X(x)dx \quad (\text{FOC}_2q\text{-AC}) \end{aligned}$$

where the supremum is taken over measurable functions and $\Phi^{\Delta}_{\phi}(a, c, W; f, w_\phi)$ is defined in (16), for $\phi \in \{g, \gamma\}$. Using analogues to Lemmas 10 and 13 we establish:

$$|T^{\Delta,q}_{g,\gamma}F_2 - T^{\Delta}_{g,\gamma}F_2|_{[0,\bar{u})^2} = o(\Delta).$$

Fix $\varepsilon > 0$ and $(w_g, w_\gamma) \in [\varepsilon, w_{sp} - \varepsilon]^2$ such that $|w_g - w_\gamma| \geq \varepsilon$. In view of the above bound, it is sufficient to establish that $F_2(w_g, w_\gamma) - T^{\Delta,q}_{g,\gamma} F_2(w_g, w_\gamma) > \delta_1 \Delta$, and so, due to Proposition 7 and Lemmas 10 and 13, it is sufficient to show that

$$T^{\Delta,q}_g F(w_g) + T^{\Delta,q}_\gamma F(w_\gamma) - T^{\Delta,q}_{g,\gamma} F_2(w_g, w_\gamma) > \delta_1 \Delta,$$

where $T^{\Delta,q}_g$ and $T^{\Delta,q}_\gamma$ stand for operator $T^{\Delta,q}$ under the respective noise densities.

We have

$$\begin{aligned} T^{\Delta,q}_\phi F(w_\phi) &= \sup_c -\tilde{r}\Delta \left\{c + F'(w_\phi) u(c)\right\} + \sup_{a,W} \Phi^\Delta_\phi (a, W; F, w_\phi), \\ T^{\Delta,q}_{g,\gamma} F_2(w_g, w_\gamma) &= \sup_c -\tilde{r}\Delta \left\{2c + F'(w_g) u(c) + F'(w_\gamma) u(c)\right\} + \\ &\quad + \sup_{a_g, W_g} \Phi^\Delta_g (a_g, W_g; F, w_g) + \sup_{a_\gamma, W_\gamma} \Phi^\Delta_\gamma (a_\gamma, W_\gamma; F, w_\gamma), \end{aligned}$$

where

$$\Phi^\Delta_\phi (a, W; F, w_\phi) = e^{-\Delta r} F(w_\phi) + \tilde{r}\Delta \left\{a + F'(w_\phi)[w_\phi + h(a_\phi)]\right\} + e^{-\Delta r} \mathbb{E}^\Delta_\phi [\frac{1}{2} F''(w_\phi) (W(\Delta x) - w_\phi)^2],$$

$\phi \in \{g, \gamma\}$. The proof follows from the fact that F'' is bounded away from 0 on $[0, w_{sp})^2$ and so $|F'(w_g) - F'(w_\gamma)| > \varepsilon_1$ for some $\varepsilon_1 > 0$, which implies that for some δ_1:

$$\sup_c - \left\{c + F'(w_g) u(c)\right\} + \sup_c - \left\{c + F'(w_\gamma) u(c)\right\} > \sup_c - \left\{2c + F'(w_g) u(c) + F'(w_\gamma) u(c)\right\} + \delta_1.$$

■

Claim 3 *Fix $\varepsilon > 0$ and $(w_g, w_\gamma) \in [\varepsilon, w_{sp} - \varepsilon]^2$. Then there is $\delta_2 > 0$ such that for sufficiently small Δ and any feasible policy $(a_g, a_\gamma, c, W_g, W_\gamma)$ for $T^\Delta_{g,\gamma} F_2(w_g, w_\gamma)$ if*

$$\Phi^\Delta_g(a_g, c, W_g; F_2) + \Phi^\Delta_\gamma(a_\gamma, c, W_\gamma; F_2) > F_2(w_g, w_\gamma) - \delta_2 \Delta$$

then

$$\begin{aligned} \mathbb{V}^\Delta_g[W_g(\Delta(x_g + a_g))], \mathbb{V}^\Delta_\gamma[W_\gamma(\Delta(x_\gamma + a_\gamma))] &> \delta_2 \Delta, \\ \mathbb{V}^\Delta_{g,\gamma}[W_g(\Delta(x_g + a_g)) - W_\gamma(\Delta(x_\gamma + a_\gamma))] &> \delta_2 \Delta. \end{aligned}$$

On the other hand,

$$\begin{aligned} & F_2(w_g, w_\gamma) - \Phi^\Delta_g(a_g, c, W_g; F_2) + \Phi^\Delta_\gamma(a_\gamma, c, W_\gamma; F_2) \\ > \; & \delta_2 \big(\mathbb{V}^\Delta_g[W_g(\Delta(x_g + a_g))] - \Delta \frac{(rh'(A))^2}{VLR(g_X)}\big) + \delta_2 \big(\mathbb{V}^\Delta_\gamma[W_\gamma(\Delta(x_\gamma + a_\gamma))] - \Delta \frac{(rh'(A))^2}{VLR(g_\gamma)}\big) \end{aligned}$$

Proof. Lemmas 21 part ii) and 8 imply that for certain $\delta_2 > 0$ and sufficiently small Δ if $\Phi_g^\Delta(a_g, c, W_g; F_2) + \Phi_\gamma^\Delta(a_\gamma, c, W_\gamma; F_2) > F_2(w_g, w_\gamma) - \delta_2\Delta$, then $a_g, a_\gamma > \underline{a} > 0$. But then Lemmas 8 and 1 imply that $\mathbb{V}_\phi^\Delta[W_\phi(\Delta(x_\phi + a_\phi))] \approx \Delta\frac{(rh'(a_\phi))^2}{VLR(\phi_X)}$, for $\phi \in \{g, \gamma\}$, which yields the first inequality. The same Lemmas imply that $W_\phi(\Delta(x_\phi + a_\phi)) \approx \mathbb{E}_\phi^\Delta[W_\phi(\Delta(x_\phi + a_\phi))] + \sqrt{\Delta} D\frac{g'(x_\phi)}{g(x_\phi)}$ (in $L_1(\phi_X^\Delta)$), for $\phi \in \{g, \gamma\}$, and so the second inequality follows from the linear independence of likelihood ratios (12). Finally, F'' bounded away from zero immediately implies the third inequality. ■

Fix a set of policies p for the Bellman operator $T_{g,\gamma}^\Delta$, $T > 0$, $\varepsilon_1 > \varepsilon_2 > 0$.

Claim 4 *Fix an initial point $(w_g, w_\gamma) \in S_\varepsilon$. Then there are $\delta_3, T > 0$ such that for sufficiently small Δ*

$$\mathbb{E}_{g,\gamma}^\Delta\Big[\sum_{n=0}^{T/\Delta} e^{-rn\Delta}(F_2(w_{g,n}^p, w_{\gamma,n}^p) - T_{g,\gamma}^{\Delta,p}F_2(w_{g,n}^p, w_{\gamma,n}^p))\Big] \le \delta_3$$

implies

$$\mathbb{P}_{g,\gamma}^\Delta[(w_{g,n}^p, w_{\gamma,n}^p) \in S_{\varepsilon/2}, n = 0, ..., T/\Delta] > \delta_3.$$

Proof. If the precondition is satisfied, then Claim 3 implies that, for $\phi \in \{g, \gamma\}$,

$$\begin{aligned}
\mathbb{V}_\phi^\Delta[(w_{\phi,t}^p - w_{\phi,0}^p)] &\le T\frac{(rh'(A))^2}{VLR(g_X)} + \delta/\delta' =: C_{T,\delta}, \text{ for } t \le T/\Delta, \\
\mathbb{E}_\phi^\Delta[\mathbb{V}_\phi^\Delta[(w_{\phi,T/\Delta}^p - w_{\phi,t}^p)]|w_{\phi,0}^p] &\le C_{T,\delta}, \text{ for } t \le T/\Delta,
\end{aligned}$$

with $C_{T,\delta} \to 0$ as $T, \delta \to 0$. We also have

$$\mathbb{E}_\phi^\Delta[(w_{\phi,t}^p - w_{\phi,t'}^p)]|w_{\phi,t'}^p] \le D_{T,\delta}, \text{ for } t' < t' \le T/\Delta,$$

with $D_{T,\delta} \to 0$ as $T, \delta \to 0$. It therefore follows that for $\alpha = \frac{\varepsilon}{4} > 0$ and τ the stopping time of reaching the set $[\alpha, \infty)$

$$\begin{aligned}
&\mathbb{P}_\phi^\Delta[\max_{t \le T/\Delta}\{w_{\phi,t}^p - w_{\phi,0}^p\} \ge \alpha] = \\
= \; &\mathbb{P}_\phi^\Delta[\max_{t \le T/\Delta}\{w_{\phi,t}^p - w_{\phi,0}^p\} \ge \alpha, w_{\phi,T/\Delta}^p - w_{\phi,0}^p \ge \alpha/2] + \\
&\mathbb{P}_\phi^\Delta[\max_{t \le T/\Delta}\{w_{\phi,t}^p - w_{\phi,0}^p\} \ge \alpha, w_{\phi,T/\Delta}^p - w_{\phi,0}^p < \alpha/2] \\
\le \; &\mathbb{P}_\phi^\Delta[w_{\phi,T/\Delta}^p - w_{\phi,0}^p \ge \alpha/2] + \mathbb{P}_\phi^\Delta[w_{\phi,T/\Delta}^p - w_{\phi,\tau}^p < -\alpha/2] \le 2\frac{C_{T,\delta}}{(\alpha/2 - D_{T,\delta})^2} \to 0,
\end{aligned}$$

as $T, \delta \to 0$. This establishes the proof. ■

Claim 5 *Fix an initial point $(w_g, w_\gamma) \in [\varepsilon, w_{sp} - \varepsilon]^2$. Then there are $\delta_4, T > 0$ such that for sufficiently small Δ*

$$\mathbb{E}_{g,\gamma}^\Delta\Big[\sum_{n=0}^{T/\Delta} e^{-rn\Delta}(F_2(w_{g,n}^p, w_{\gamma,n}^p) - T_{g,\gamma}^{\Delta,p}F_2(w_{g,n}^p, w_{\gamma,n}^p))\Big] \le \delta_4$$

implies

$$\mathbb{P}_{g,\gamma}[(w^p_{g,T/\Delta+1}, w^p_{\gamma,T/\Delta+1}) \in S_\varepsilon] > \delta_4.$$

Proof. The proof is similar to the proof of the previous claim and so is omitted. ■

Given the claims, the rest of the proof is as follows. If $(w_g, w_\gamma) \in S_\varepsilon$ then for the constants as in the claims

$$\begin{aligned} F_2(w_g, w_\gamma) - F^{\Delta,p}_{g,\gamma}(w_g, w_\gamma) &\geq \mathbb{E}^{\Delta}_{g,\gamma}\Big[\sum_{n=0}^{T/\Delta} e^{-rn\Delta}(F_2(w^p_{g,n}, w^p_{\gamma,n}) - T^{\Delta,p}_{g,\gamma}F_2(w^p_{g,n}, w^p_{\gamma,n}))\Big] \\ &\geq \min\left\{\delta_3, \frac{1-e^{-rT}}{1-e^{-r\Delta}}\delta_3\delta_1\right\}, \end{aligned}$$

where the first inequality follows from Claim 1 and the second inequality follows from Claims 2 and 4.

If on the other hand $(w_g, w_\gamma) \in [\varepsilon_1, w_{sp} - \varepsilon]^2 \setminus S_\varepsilon$ then

$$\begin{aligned} F_2(w_g, w_\gamma) - F^{\Delta,p}_{g,\gamma}(w_g, w_\gamma) &\geq \mathbb{E}^{\Delta}_{g,\gamma}\Big[\sum_{n=0}^{T/\Delta} e^{-rn\Delta}(F_2(w^p_{g,n}, w^p_{\gamma,n}) - T^{\Delta,p}_{g,\gamma}F_2(w^p_{g,n}, w^p_{\gamma,n})) + \\ &\quad + e^{-rT}(F_2(w^p_{g,T/\Delta+1}, w^p_{\gamma,T/\Delta+1}) - T^{\Delta,p}_{g,\gamma}F_2(w^p_{g,T/\Delta+1}, w^p_{\gamma,T/\Delta+1}))\Big] \\ &\geq \min\left\{\delta_4, e^{-rT}\delta_4 \min\left\{\delta_3, \frac{1-e^{-r(T+\Delta)}}{1-e^{-r\Delta}}\delta_3\delta_1\right\}\right\}, \end{aligned}$$

where the first inequality follows from Claim 1 and the second inequality follows from Claim 5 and the inequalities above. This establishes the proof of the Proposition.

We note that the proof can be extended beyond the pure hidden action case and $VLR(g_X) = VLR(\gamma_X)$. As regards the equality of variances of likelihood ratios, this guaranteed that the limits of the values of contracts F_g and F_γ for two noise distributions are the same function F (Lemma 1). Because of that, as long as the continuation values w_g and w_γ are not the same the derivatives $F'_g(w_g)$ and $F'_\gamma(w_\gamma)$ differ as well, which is crucial for Claim 2. Dropping the assumption $VLR(g_X) = VLR(\gamma_X)$ the proof would be analogous, yet the computation of the set of continuation values (w_g, w_f) for which $F'_g(w_g) \neq F'_\gamma(w_\gamma)$ would be cumbersome.

On the other hand, the assumption of pure hidden action models was also not crucial for the proof: For two different information structures the proof will work as long as, roughly, the optimal policies in the problem of minimizing variance of incentive transfers are sufficiently different (see Claim 3).

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