OPTIMAL DYNAMIC MECHANISM DESIGN WITH DEADLINES

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ABSTRACT. We study the revenue-maximizing sale of an object in a dynamic environment, with buyers that differ in their degree of patience: Besides his privately known valuation, each buyer has a privately known deadline for buying. First, we derive the optimal mechanism, neglecting the incentive constraint for the deadline. Here the seller's desire to extract rents interacts with the dynamic arrival of new information. This can lead to a violation of the neglected incentive constraint. We give sufficient conditions on the type distribution under which the neglected constraint is fulfilled or violated. Next, we consider a model with two periods and two buyers, for the case that the constraint cannot be neglected. Here, the optimal mechanism is implemented by a fixed price in period one and an asymmetric auction in period two. The asymmetry, which is introduced to prevent the first buyer from buying in period one when his deadline is two, leads to pooling of deadlines at the top of the type space.

KEYWORDS: Dynamic Mechanism Design, Multidimensional Signals, Revenue Maximization, Deadlines
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1. INTRODUCTION

In many situations, sellers face a changing population of heterogeneous buyers. Different buyers arrive at different points in time. Some buyers are impatient and want to buy immediately. Others are patient and willing to wait. Patient buyers can act strategically and exploit their flexibility with respect to the time of a purchase in order to get better prices. Typical examples of such scenarios are online auctions, the sale of flight tickets, hotel reservations, or the sale of houses.

To capture heterogeneity in the degree of patience, we assume that buyer's have idiosyncratic deadlines. A deadline can be viewed as an extreme form of time preferences, as in the case of a traveler who needs to buy tickets before a certain date, in order to be able to coordinate with other travel arrangements. Deadlines may also be imposed by third parties. Consider a company that needs to buy a good from a seller in order to enter a contractual relationship with a third party. This could be a physical object, an option

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contract, a license, a patent, etc. It is conceivable that the third party sets a deadline, after which the contractual relationship is no longer available. Therefore, the good is worthless for the company if it is purchased after the deadline.

We analyze the problem of a monopolistic seller with full commitment power, who sells one (or more) unit(s) of a good. The seller wants to maximize revenue in a dynamic environment, where buyers have independent private values, arrive over a finite number of periods, and do not discount future payoffs. Each buyer is characterized by his arrival time, his valuation, and his deadline. A buyer's deadline and valuation are private information. To focus on the effects of private information about time preferences, we assume that arrival times are observable for the seller.

To solve the seller's revenue maximization problem, we adopt a mechanism design approach. When there is no discounting, it is optimal to allocate to a buyer only if his deadline is reached. For this class of mechanisms, incentive compatibility can be characterized by tractable one-dimensional constraints for the deadline and the valuation, respectively. Building on this, the paper makes the following contributions.

First, we derive the *relaxed solution*, which ignores the incentive constraint for the deadline. We show that the neglected incentive constraint for the deadline is violated if the inverse hazard rate of the valuation is convex. If the inverse hazard rate is concave or linear, the neglected constraint is fulfilled automatically. In the latter case, the relaxed solution also solves the general problem. The reason for violations of incentive compatibility is an interaction of the seller's desire to extract rent and the dynamic arrival of new information (due to the arrival of new buyers). Because of new arrivals, the allocation decision at a later deadline is based on more information. We show that this leads to higher rent extraction for later deadlines if the inverse hazard rate is convex. Hence, patient/strategic buyers will mimic types with earlier deadlines to avoid rent extraction—the incentive constraint for the deadline is violated. In the case of a concave inverse hazard rate, rent extraction is higher for earlier deadlines. The impatient buyers, however, cannot profitably mimic later deadlines. Therefore, incentive compatibility is preserved.

Second, we derive the optimal mechanism for the case that the relaxed solution is not incentive compatible. In this case, the seller's ability to extract rent from the patient buyers is limited by the buyers' ability to strategize and deviate to mimicking impatient buyers. For tractability, we restrict the model to two periods and two buyers. The optimal mechanism has a simple structure. In the first period, the seller sets a fixed price. If buyer one does not accept, but indicates that he is patient and would be willing to purchase in the second period, the seller waits and conducts an auction that gathers both buyers. Otherwise, if the first buyer indicates that he is impatient, the object is offered to the second buyer for a fixed price. The seller has two instruments to prevent a patient/strategic type from choosing the fixed price in period one. He can increase the price in the first period, and he can distort the auction format in the second period in favor of buyer one. Both instruments increase the expected payoff from the auction compared to the fixed price and thereby reduce the incentive for the patient/strategic type to deviate. We derive the optimal mechanism and show that the seller always uses both instruments.

The distortion leads to an asymmetric auction, even if both buyers have identically distributed valuations. Moreover, buyer one wins with certainty if his valuation is sufficiently high. In contrast to the relaxed solution, which fully separates buyers with different types for a large class of type distributions, full separation is not optimal if the incentive constraint for the deadline is binding. In particular, buyers with high valuations are not separated with respect to their deadlines. It is not optimal for the seller to separate the strategic/patient type from the impatient/non-strategic type if the valuation is too high. In other words, the model predicts that the optimal mechanism provides incentives for buyers with low valuations to wait until their respective deadlines before they make a purchase. Buyers with higher valuations on the other hand, do not benefit from waiting and may buy earlier.

Finally, the paper makes a methodological contribution. Formally, if the relaxed solution is not incentive compatible, we have to solve an auction problem with a type-dependent participation constraint. Patient/strategic buyers have the "outside option" to buy before their deadlines. This is the first paper that solves such a problem. The solution resembles Jullien (2000), who studies a principal-agent problem with a type-dependent participation constraint. Methodologically, however, the auction problem requires a different approach because of discontinuous winning probabilities and the additional capacity constraint in the auction. We adopt an approach pioneered by Reid (1968), which seems to be new to the mechanism design literature.[1]

1.1. Related Literature

The literature on dynamic revenue maximization can be broadly divided into two types of models. On the one hand, there are models where all buyers are impatient and therefore non-strategic with respect to the purchase time.[2] This is also a standard assumption in the revenue management literature.[3] On the other hand, there are models in which all buyers are assumed to be patient and strategic. In an infinite horizon model, Gallien (2006) shows that under a condition on the inter arrival time distribution, the optimal mechanism in the presence of patient/strategic buyers is the same as with impatient/non-strategic buyers.[4] Patient buyers are only served at the arrival and are never recalled later.

[1]Another problem arises because the usual hazard-rate assumption is not sufficient to guarantee monotonicity of the winning probability in the distorted auction. Reid also shows how a monotonicity constraint can be incorporated in a control problem. Hellwig's (2008) version of Pontriyagin's maximum principle that allows for discontinuities and a monotonicity constraint cannot be applied here because of the capacity constraint.

[2]See for example Das Varma and Vettas (2001); Vulcano et al. (2002); Gershkov and Moldovanu (2009a); Dizdar et al. (2011).

[3]See Elmaghraby and Keskinocak (2003) for a survey. McAfee and te Velde (2007) survey airline pricing. Su (2007) studies a model with patient buyers.

[4]The condition can also be found in the earlier literature on search with recall (Zuckerman, 1986).

Board and Skrzypacz (2010) show that the no recall property fails if the time horizon is finite.

In contrast to both these branches of the literature, we assume that there is heterogeneity in the patience of buyers and that buyers have private information about their degrees of patience in the form of deadlines. This has also been studied by Pai and Vohra (2008b), who focus on sufficient conditions for incentive compatibility of the relaxed solution. These authors also allow for private information about the arrival time and do not make restrictions on the number of periods or objects.[5] For the deadline, they suggest that the incentive constraint is slack if the hazard rate of the valuation is *sufficiently monotone* in the deadline. The condition, however, cannot be applied directly to the the type distribution. For the arrival time, they show that simple monotonicity of the hazard rate in the arrival time is sufficient (cf. Section 5).

This paper is also related to a literature on static mechanism design with two-dimensional private information, in which the second dimension is for example a budget constraint, a minimal capacity, or a quality constraint.[6] Such models are tractable because the second dimension has a special structure:[7] First, deviations are only possible in one direction (e.g. only under-reports of the budget or of the deadline are possible). Second, the second dimension is a constraint that does not enter the utility function as long as it is satisfied. For example, the utility of a buyer is independent of his deadline as long as he gets a unit before the deadline. Except for Szalay (2009), who considers a principal-agent problem, this literature typically makes assumptions that guarantee that the relaxed solution is incentive compatible.

Another interesting feature that has been introduced into dynamic mechanism design models is learning. The literature on sequential screening considers situations in which the seller faces a fixed population of buyers that learn about their valuation over time. Courty and Li (2000) analyze the optimal contract in such a setting. Nocke and Peitz (2007), Möller and Watanabe (2010) and Nocke et al. (2011) analyze optimal pricing schemes and find that advance purchase discounts or clearance sales can be optimal.[8] Recently, Gershkov and Moldovanu (2009b,c, 2010) have studied dynamic mechanism design problems in which the seller learns about future buyers' type distributions from current buyers' types. We abstract from learning both on the buyer- and the seller side by assuming that types are uncorrelated and are fully learned upon arrival.

In a different strand of literature, Said (2008) considers a scheduling model with stochastic arrival and exit of bidders. Pavan et al. (2008) consider a very general dynamic

[5]The results of the present paper can be generalized to many objects if there are only two time periods.
[6]See Beaudry et al. (2009) for an analysis of optimal taxation; Blackorby and Szalay (2008) and Szalay (2009) for regulation; Iyengar and Kumar (2008) and Dizdar et al. (2011) for auction models with capacitated bidders; Che and Gale (2000), Malakhov and Vohra (2005) and Pai and Vohra (2008a) for models with budget constrained buyers.
[7]The models of Rochet and Choné (1998) and Jehiel et al. (1999), in which all dimensions are symmetric, rarely have explicit solutions (see Armstrong, 1996, for an exception).
[8]Related to this are also models where buyer's types change over time. See for example Battaglini (2005), Deb (2011) and references therein.

mechanism design model with agents who receive one-dimensional private information in every period. Finally, there is a literature on efficient dynamic mechanism design. (See Parkes and Singh (2003), Bergemann and Välimäki (2010), and Athey and Segal (2007) for existence results; Mierendorff (2009) for the construction of a simple payment rule). In the context of the present paper, if the goal of the seller is value-maximization, the relaxed solution is always incentive compatible. The interaction of rent extraction and dynamic arrival of information vanishes because the seller is not concerned about rent extraction.

Organization of the Paper

Section 2 describes the model. Section 2.3 characterizes incentive compatibility. Section 2.4 states the seller's problem. Section 3 presents the relaxed solution and conditions for incentive compatibility, formal proofs are in Appendix A. Section 4 presents the general solution for the bunching case. The formal derivation is in Appendix B. Section 5 concludes. The supplementary material in Appendix C contains some further proofs and an extension of the results of Section 3 to multiple objects.

2. The Model

A seller wants to maximize the revenue from selling one indivisible object within $T < \infty$ time periods. The seller's valuation is normalized to zero. In each period, a random number of buyers $N_t \in \mathbb{N}_0$ arrives. The set of buyers who arrive in period t is denoted I_t and we write $I_{\leq t} = \bigcup_{\tau=1}^{t} I_\tau$ for the set and $N_{\leq t} = |I_{\leq t}|$ for the number of buyers that arrive until period t. A buyer $i \in I_t$ is characterized by his arrival time $a_i = t$, his valuation $v_i \in [0, \overline{v}]$, where $\overline{v} > 0$, and his deadline $d_i \in \{t, \ldots, T\}$. The object cannot be sold to a buyer before his arrival time. Utility is quasi-linear. If buyer i has to make a total payment of y_i, then his total payoff is $v_i - y_i$ if he gets the object in periods $a_i, \ldots, d_i$, and $-y_i$ otherwise. Buyers are risk-neutral and maximize expected payoff. Neither the buyers nor the seller discount future payoffs.[9]

The numbers of arrivals in different periods and the types of different buyers are independently distributed. Moreover, to focus on the novel insights that arise due to the dynamic structure of the model, we assume that the deadline and valuation of a buyer are independent. In section 3, we will discuss the consequences of correlations between the deadline and the valuation. $\nu_{t,n}$ denotes the probability that n buyers arrive in period t. To exclude uninteresting cases, we assume that in each period, there is a positive probability of new arrivals ($\forall t : \nu_{t,0} < 1$). For given arrival time a, the probability that the deadline of a buyer equals d is denoted $\rho_{a,d}$. The valuation has distribution function $F_a(v)$ and density $f_a(v)$. This notation implicitly assumes that buyers with the same arrival period are ex-ante identical.

Information realizes over time. In period t, the numbers of future buyers N_{t+1}, $\ldots, N_T$, and their types are not known to anybody. In particular, the decision to sell

[9]If only payments are discounted and all agents have a common discount factor, the results do not change. See Section 5 for a discussion of discounting.

a unit in period t cannot be based on this information. Upon arrival, each buyer privately observes his valuation and his deadline. In order to focus on the incentive issues of private information about deadlines, we assume that the seller observes arrivals.[10] $\nu_{t,n}$, $\rho_{a,d}$ and $F_a(.)$ are commonly known from the first period on.

We assume that for all a, $f_a(v)$ is continuous in v and strictly positive for all $v \in [0, \overline{v}]$, continuously differentiable in v for $v \in (0, \overline{v})$, and that $f_1'(.)$ can be extended continuously to $[0, \overline{v}]$. To avoid additional technicalities, the following assumption is maintained throughout the paper.

Assumption 1. *For all* $a \in \{1, \dots T\}$, *the* virtual valuation $J_a(v) := v - \frac{1-F_a(v)}{f_a(v)}$ *is strictly increasing in* v.

The zero of $J_a(.)$ is denoted v_a^0. For some results in Section 4, we will assume that the monopoly profit from a buyer in the first period is concave:

Assumption 2. $v(1 - F_1(v))$ *is concave for all* $v \in [v_1^0, \overline{v}]$.

2.1. Allocation Rules

A *state* $s_t = (H_t, k_t)$ consists of the history of buyer types $H_t = (a_i, v_i, d_i)_{i \in I_{\leq t}}$, and the variable k_t which indicates if the object is still available, ($k_t = 0$), or has already been allocated to some buyer i, ($k_t = i$). The history of buyer types, excluding the type of buyer i, is denoted H_t^{-i}.

An *allocation rule* defines a winning probability $x_i(s_t) \in [0, 1]$ for each state s_t in which $k_t = 0$, and for each buyer $i \in I_{\leq t}$. If $k_t = 1$, we set $x_i(s_t) = 0$. With only one object available, an allocation rule must satisfy the feasibility constraint

$$\forall t,\ s_t: \quad \sum_{i \in I_{\leq t}} x_i(s_t) \leq 1. \tag{F}$$

The probability that the object is not allocated in state s_t is denoted by $x_0(s_t) = 1 - \sum_{i \in I_{\leq t}} x_i(s_t)$, and we set $x_0(s_t) = 0$ if $k_t \neq 0$.

We say that an allocation rule *allocates only at the deadline* if $x_i(s_t) = 0$ for all $i \in I_{\leq t}$ with $d_i \neq t$. An allocation rule is *symmetric* if for all t, all states s_t with $k_t = 0$, and all $i, j \in I_{\leq t}$ such that $a_i = a_j$, $x_i(s_t) = x_j(\sigma_{i,j}(H_t), k_t)$, where $\sigma_{i,j}$ denotes the permutation that interchanges the i^{th} and the j^{th} element of its argument.

A *payment rule* maps *any* state s_t and the *realized* current allocation decision k_{t+1} to a payment $y_i(s_t, k_{t+1})$ for each buyer $i \in I_{\leq t}$. A payment rule is *symmetric* if for all t, s_t, k_{t+1}, and all $i, j \in I_{\leq t}$ such that $a_i = a_j$, $y_i(s_t, k_{t+1}) = y_j((\sigma_{i,j}(H_t), \tilde{\sigma}_{i,j}(k_t)), \tilde{\sigma}_{i,j}(k_{t+1}))$, where $\tilde{\sigma}_{i,j}(k) = i$ if $k = j$ and vice versa, and $\tilde{\sigma}_{i,j}(k) = k$ if $k \notin \{i, j\}$.[11]

[10] See Section 5 for a discussion of private information about arrival times.

[11] Here, we implicitly assume that the payment in period t only depends on k_t and k_{t+1}. This is without loss of generality and simplifies the definition of the state s_t.

2.2. Mechanisms

The seller's goal is to design a mechanism that has a Bayes-Nash-equilibrium which maximizes his expected revenue. In general, a mechanism can be any game form with T stages, such that only buyers from $I_{\leq t}$ are active in stage t. We assume that the mechanism designer has full commitment power and can choose to conceal any information about the first t stages from the buyers that arrive in stages $t+1, \ldots, T$.[12]

By the revelation principle, the seller can restrict attention to incentive compatible and individually rational direct mechanisms, in which no information is revealed.[13] Since buyers who arrive in the same period are ex-ante identical, it is without loss of generality to restrict attention to symmetric allocation and payment rules.

Definition 1. A *direct mechanism* consists of message spaces $S_1 = [0, \overline{v}] \times \{1, \ldots, T\}$, $\ldots$, $S_T = [0, \overline{v}] \times \{T\}$, and symmetric allocation and payment rules (x, y).

The *interim winning probability* for period t, of a buyer $i \in I_a$ who reports (v', d'), if all other buyers (past, current and future) report their types truthfully, is given by

$$q_a^t(v', d') := E\left[x_i(s_t) \middle| (a_i, v_i, d_i) = (a, v', d')\right].$$

The *interim expected payment* is given by

$$p_a(v', d') := E\left[\sum_{\tau=a}^{T} y_i(s_\tau, k_{\tau+1}) \middle| (a_i, v_i, d_i) = (a, v', d')\right],$$

where we aggregate payments from different periods. (q, p) is called the *reduced form* of (x, y) (explicit expressions can be found in Appendix C.3).

The *interim expected utility* from participating in a mechanism (x, y) with true type (v, d) and report (v', d') is given by

$$U_a(v, d, v', d') := \left[\sum_{\tau=a}^{d} q_a^\tau(v', d')\right] v - p_a(v', d'). \tag{2.1}$$

The expected utility from truth-telling is abbreviated $U_a(v, d) := U_a(v, d, v, d)$.

Definition 2. (i) A direct mechanism (x, y) is *(Bayesian) incentive compatible* if for all $a \in \{1, \ldots, T\}$, $v, v' \in [0, \overline{v}]$, and $d, d' \in \{a, \ldots, T\}$,

$$U_a(v, d) \geq U_a(v, d, v', d'). \tag{IC}$$

(ii) A direct mechanism (x, y) is *individually rational* if for all $a \in \{1, \ldots, T\}$, $v \in [0, \overline{v}]$, and $d \in \{a, \ldots, T\}$,

$$U_a(v, d) \geq 0. \tag{IR}$$

[12] This assumption yields an upper bound on the revenue that can be achieved. We will see that this bound can also be achieved in a periodic ex-post equilibrium, i.e., if buyers observe all information from past and current stages.

[13] The standard revelation principle holds because the seller observes arrival times. Without this assumption, the revelation principle still holds because each buyer could mimic all types with an arrival time greater or equal than his own arrival time. Therefore the *nested range condition* is satisfied and the revelation principle holds (Green and Laffont (1986); Bull and Watson (2007)).

2.3. Characterization of Incentive Compatibility

Since valuations are not discounted, the seller can restrict attention to direct mechanisms that allocate only at the deadline.

Lemma 1. *Let (x, y) be a direct mechanism that satisfies (IC) and (IR). Then, there exists an allocation rule $\hat{x}$ that allocates only at the deadline, such that the direct mechanism $(\hat{x}, y)$ also satisfies (IC) and (IR), and (x, y) and $(\hat{x}, y)$ yield the same expected revenue.*

Proof. The proof can be found in Appendix C. □

Since we restrict attention to mechanisms that allocate only at the deadline, we write $q_a(v, d)$ instead of $q_a^d(v, d)$. For this class of mechanisms, the two-dimensional incentive constraint (IC) is equivalent to two one-dimensional constraints.

Theorem 1. *Let (x, y) be a direct mechanism with reduced form (q, p) that allocates only at the deadline. Then (x, y) is incentive compatible if and only if for all $a \in \{1, \dots, T\}$, all $d \in \{a, \dots, T\}$, and all $v, v' \in [0, \overline{v}]$:*

$$v > v' \Rightarrow q_a(v, d) \geq q_a(v', d), \tag{M}$$

$$U_a(v, d) = U_a(0, d) + \int_0^v q_a(s, d)ds, \tag{PE}$$

$$U_a(v, d) \leq U_a(v, d + 1), \qquad \text{if } d < T, \tag{ICD$^{\mathrm{d}}$}$$

$$\text{and} \quad U_a(0, d) = U_a(0, d + 1), \qquad \text{if } d < T. \tag{ICD$^{\mathrm{u}}$}$$

Sketch of Proof. (M) and (PE) is the standard characterization of one-dimensional incentive compatibility for the valuation (Myerson, 1981). (ICD$^{\mathrm{d}}$) rules out under-reports of the deadline. Together with (M) and (PE), this also rules out simultaneous misreports of an earlier deadline $d' < d$ and a valuation $v' \neq v$. Since the mechanism never allocates after the reported deadline, the constraint takes this simple form because the utility of under-reporting the deadline is independent of the true deadline (cf. (2.1)):

$$d' \leq d \quad \Rightarrow \quad U_a(v, d, v', d') = U_a(v, d', v', d').$$

Incentive compatibility for the valuation implies that $U_a(v, d', v', d')$, and therefore also $U_a(v, d, v', d')$, is maximized by $v' = v$. For $v' = v$, (ICD$^{\mathrm{d}}$) rules out a downward deviation in the deadline. Therefore, simultaneous deviations in the deadline and the valuation are also ruled out. Necessity of (ICD$^{\mathrm{d}}$) is obvious.

For mechanisms that allocate only at the deadline, reporting $d' > d$ can only be profitable if the mechanism pays a subsidy, i.e., if $p_a(v, d') < 0$. (PE) implies that subsidies are non-increasing in the valuation. Therefore, the highest subsidy (if any) is paid for $(0, d')$. By (PE), $v = 0$ is also the valuation for which over-reporting the deadline is most tempting. Hence, to rule out upward deviations, it suffices that $U_a(0, d) = -p_a(0, d) \geq$

$-p_a(0,d') = U_a(0,d,0,d') = U_a(0,d')$. Together with (ICD$^{\mathrm{d}}$) for $v = 0$, this is equivalent to (ICD$^{\mathrm{u}}$).[14] □

Formally, the downward incentive constraint for the deadline resembles a type-dependent participation constraint. A patient/strategic buyer with arrival time a and deadline $d > a$ has the "outside option" to report $d' < d$. He only "participates" voluntarily with a truthful report, if his payoff with $d' = d$ exceeds the payoff of his best "outside option."

2.4. The Seller's Problem

By the revelation principle and Lemma 1, the seller's problem is to choose an incentive compatible and individually rational direct mechanism that allocates only at the deadline, to maximize

$$\sum_{a=1}^{T} E[N_a]E[p_a(v,d)] = \sum_{a=1}^{T}\left[\left(\sum_{N_a=1}^{\infty} N_a\,\nu_{a,N_a}\right)\sum_{d=a}^{T}\rho_{a,d}\int_0^{\overline{v}} p_a(v,d)f_a(v)dv\right].$$

Using (2.1) and (PE) to substitute the payment rule, integrating by parts and setting $U_a(0,d) = 0$ for all $a \in \{1,\ldots,T\}$ and $d \geq a$, the objective of the seller can be rearranged to

$$\sum_{a=1}^{T}\left[\left(\sum_{N_a=1}^{\infty} N_a\,\nu_{a,N_a}\right)\sum_{d=a}^{T}\rho_{a,d}\int_0^{\overline{v}} q_a(v,d)J_a(v)f_a(v)dv\right].$$

Next, we substitute $q_a(v,d)$, and bring the seller's maximization problem into a recursive form. The resulting dynamic program is denoted $\mathcal{R}$:

$$V_T(s_T) := \max_{x(s_T)} \sum_{i\in I_{\leq T}:d_i=T} x_i(s_T)J_{a_i}(v_i), \tag{$\mathcal{R}$}$$

$$\forall t<T:\; V_t(s_t) := \max_{x(s_t)} \sum_{i\in I_{\leq t}:d_i=t} x_i(s_t)J_{a_i}(v_i) + x_0(s_t)E_{s_{t+1}}\left[V_{t+1}(s_{t+1})|s_t, k_{t+1}=0\right],$$

where the reduced form of the optimal policy $x(.)$ must satisfy (M), (ICD$^{\mathrm{d}}$) and (PE), where $U_a(0,d) \equiv 0$. As is common in one-dimensional auction problems, the sellers chooses the virtually efficient allocation policy, i.e., the policy that maximizes the expected virtual valuation of the winning buyer. An additional complication is introduced by the dynamic incentive constraint (ICD$^{\mathrm{d}}$).

3. The Relaxed Solution

In order to derive conditions under which the constraint (ICD$^{\mathrm{d}}$) is binding, we first solve $\mathcal{R}$ subject to (M) only. This is the *relaxed problem* and corresponds to the case where deadlines are observed by the seller. As in the classic optimal auction problem, Assumption

[14]If $v \in [\underline{v},\overline{v}]$ with $\underline{v} > 0$, then the upward incentive compatibility constraint for the deadline would be $U_a(\underline{v},d) \geq -p_a(\underline{v},d+1) = U_a(\underline{v},d+1) - q_a(\underline{v},d+1)\underline{v}$. In this case, a subsidy could be used to separate buyers with different deadlines. One can show, however, that this instrument would not be used in the optimal mechanism, unless the allocation rule is sufficiently distorted. The reason is that the cost of a subsidy is of first order whereas the cost of distorting the allocation rule is of second order.

1 guarantees that (M) is slack at the optimal policy (Myerson, 1981). Therefore, we can ignore (M) in the derivation of the relaxed solution.

For a given state s_t, let $c^t := \max_{i \in I_{\leq t}: d_i = t} J_{a_i}(v_i)$ be the maximal virtual valuation among the buyers with deadlines $d_i = t$. The relaxed solution allocates to a buyer $i \in \arg\max_{i \in I_{\leq t}: d_i = t} J_{a_i}(v_i)$ if the virtual valuation of that buyer exceeds the option value of retaining the object for the next period, formally, if $J_{a_i}(v_i) = c^t \geq E_{s_{t+1}}\left[V_{t+1}(s_{t+1})|s_t, k_{t+1} = 0\right]$. Otherwise, the object is retained for period $t+1$.

This allocation rule defines a *critical virtual valuation* $\zeta_{a_i}^{d_i}(H_{d_i}^{-i})$ for each buyer i, who arrives in a period in which the object is still available. The critical virtual valuation depends on the types of all buyers except buyer i, who arrive until his deadline d_i. Buyer i wins the object if his virtual valuation exceeds the critical virtual valuation and does not win if $J_{a_i}(v_i) < \zeta_{a_i}^{d_i}(H_{d_i}^{-i})$.[15]

Ignoring ties, a buyer i with a_i and d_i wins if and only if:

$$c^t \leq E[V_{t+1}(s_{t+1})|s_t, k_{t+1} = 0] \qquad \text{for all } t \in \{a, \ldots, d_i - 1\},$$

$$\text{and} \qquad J_{a_i}(v_i) = \max\left\{E[V_{d_i+1}(s_{d_i+1})|s_{d_i}, k_{d_i+1} = 0], c^{d_i}\right\},$$

where we set $V_{T+1} \equiv 0$. In words, the second condition ensures that i's virtual valuation is high enough to win in period d_i, given that the object is still available. The first condition ensures that i's virtual valuation is so high that it is optimal to retain the object in all periods before his deadline.

In order to formalize the cutoff for buyer i's virtual valuation, above which the first condition is always fulfilled we make the following definition. Fix some period t and a buyer $i \in I_{\leq t}$. Suppose that $c^t = c$. We define the lowest virtual valuation of buyer i that suffices to delay the allocation in period t by

$$z_t^d(c) := \inf\left\{J \geq 0 \,\middle|\, c \leq E_{s_{t+1}}\left[V_{t+1}(s_{t+1})\middle|H_t = (H_t^{-i}, (a, J_a^{-1}(J), d)), k_{t+1} = 0\right]\right\},$$

where we can select any $a \leq t$. If at H_t^{-i}, the object is not allocated in period t even if i is not present, we have defined $z_d^{t+1}(c^t) = 0$.

With this notation, the critical type of a buyer i is given by

$$\zeta_{a_i}^{d_i}(H_{d_i}^{-i}) = \max\Bigg\{z_{a_i}^{d_i}(c^a), \ldots, z_{d_i-1}^{d_i}(c^{d_i-1}),$$
$$\max_{j \in I_{\leq d_i}: d_j = d_i, j \neq i} J_{a_j}(v_j),\ E[V_{d_i+1}(s_{d_i+1})|s_{d_i}, k_{d_i+1} = 0]\Bigg\}$$

In every period $t = a_i, \ldots, d_i - 1$, i's virtual valuation must exceed $z_{d_i}^{t+1}(c^t)$. Otherwise, the object will be allocated before i's deadline. If i's deadline is reached, he must have a virtual valuation that exceeds the virtual valuation of all other buyers with deadline d_i and the virtual valuation must also exceed the option value of retaining the object for period $d_i + 1$.

[15]All tie-breaking rules yield the same expected revenue.

Theorem 2. *Ignoring ties, the relaxed solution to $\mathcal{R}$ is given by*

$$x_i^{rlx}(s_t) = \begin{cases} 1 & \text{if } d_i = t,\ k_{a_i} = 0 \text{ and } J_{a_i}(v_i) \geq \zeta_{a_i}^{d_i}(H_{d_i}^{-i}), \\ 0 & \text{otherwise.} \end{cases}$$

Together with the following payment rule, this allocation rule constitutes an optimal mechanism if deadlines are observable for the seller:

$$y_i^{rlx}(s_t, k_{t+1}) \quad = \quad \begin{cases} J_{a_i}^{-1}(\zeta_{a_i}^{d_i}(H_{d_i}^{-i})), & \text{if } k_t = 0 \text{ and } k_{t+1} = i, \\ 0 & \text{otherwise.} \end{cases}$$

With the payment rule given in Theorem 2, the payment of a losing buyer is zero. The winner pays the lowest valuation with which he could have obtained the object for a given history of buyer arrivals until period d. Thus, truth-telling is a weakly dominant strategy if the deadline is public and buyers only report their valuations.

Now we turn to the question whether the relaxed solution is incentive compatible if the deadline is privately known. In the relaxed solution, $U_a(0, d) = 0$ for all $a \in \{1, \ldots, T\}$ and $d \in \{a, \ldots, T\}$. Hence, it suffices to check whether the expected payoffs for the payment rule y^{rlx} satisfy (ICD$^{\text{d}}$).

The following observation is crucial for the comparison of expected payoffs for different deadlines.

Lemma 2. *For all states s_a, and all $i \in I_a$, $\left(\zeta_a^d(H_d^{-i})\right)_{d=a,\ldots,d_i}$ is a martingale (with respect to $\left(H_d^{-i}\right)_{d=a,\ldots,d_i}$): for all $d \in \{a+1, \ldots, d_i\}$,*

$$E_{H_d^{-i}}\left[\zeta_a^d(H_d^{-i})\middle| H_{d-1}^{-i}\right] = \zeta_a^{d-1}(H_{d-1}^{-i}).$$

Furthermore, for all $d \in \{a, \ldots, d_i - 1\}$,

$$\left[\zeta_a^d(H_d^{-i})\middle| s_a\right] \succ_{SSD} \left[\zeta_a^{d_i}(H_{d_i}^{-i})\middle| s_a\right],$$

where $\succ_{SSD}$ denotes strict second-order stochastic dominance.

Proof. See Appendix A. □

Example 1. To illustrate the lemma, suppose that $T = 2$, $I_1 = \{1, 2\}$ and $I_2 = \{3\}$. For the sake of the example, suppose that buyer one may have deadline one or two, but buyer two is always impatient ($d_2 = 1$). In this case, the critical virtual valuations of buyer $i = 1$ for $d_1 = 1$ and $d_1 = 2$, respectively, are given by

$$\begin{aligned} \zeta_1^1(H_1^{-1}) &= \max\left\{J_1(v_2), E_{v_3}\left[\max\left\{0, J_2(v_3)\right\}\right]\right\}, \\ \text{and} \quad \zeta_1^2(H_2^{-1}) &= \max\left\{z_1^2(J_1(v_2)), J_2(v_3)\right\}, \\ \text{where} \quad z_1^2(J_1(v_2)) &= \min\left\{z \geq 0 \,|\, E_{v_3}\left[\max\left\{z, J_2(v_3)\right\}\right] \geq J_1(v_2)\right\}. \end{aligned}$$

Suppose that buyer two's valuation is high enough so that it is optimal to allocate to him in period one, unless buyer one overbids him, i.e., $J_1(v_2) > E_{v_3}[\max\{0, J_2(v_1)\}]$. If buyer one reports deadline one, his virtual valuation is compared directly with buyer two and the critical virtual valuation is given by $\max\{J_1(v_2), E_{v_3}[\max\{0, J_2(v_3)\}]\}$. If buyer one reports deadline two, however, a lower virtual valuation one suffices to prevent an allocation to buyer two. Buyer one's virtual valuation must exceed $z_1^2(J_1(v_2))$ and $E_{v_3}\left[\max\left\{z_1^2(J_1(v_2)), J_2(v_3)\right\}\right] = J_1(v_2)$ implies $z_1^2(J_1(v_2)) < J_1(v_2)$. On the other hand, buyer one now directly competes with buyer three. His valuation must exceed $J_2(v_3)$ (rather than $E_{v_3}[\max\{0, J_2(v_3)\}]$), to win against buyer three. Lemma 2 implies that in expectation, competition is equally strong but it is more dispersed if a buyer has a later deadline. The latter observation is obvious in this example because, conditional on s_1 the critical virtual valuation for $d_1 = 1$ is constant whereas the critical virtual valuation for $d_1 = 2$ is a random variable. □

The ordering of the critical virtual valuations in terms of second-order stochastic dominance implies that non-linear virtual valuations lead to different expected utilities for different deadlines under the relaxed solution. The following theorem uses this to give a sufficient conditions for incentive compatibility of the relaxed solution.

Theorem 3. *(i) If $J_a(v)$ is weakly convex for all a, then the relaxed solution is incentive compatible and the mechanism from Theorem 2 constitutes an optimal mechanism if deadlines are private information.*

(ii) If $J_a(v)$ is strictly concave for some a, then $(\mathrm{ICD^d})$ *is violated for some type (a, v, d) in the relaxed solution.*

Proof. See Appendix A. □

Intuitively, non-linearity of J_a plays a role because the seller uses J_a^{-1} as a pricing function to determine the payment of the winning bidder. We can interpret the critical virtual valuation as the (virtual) opportunity cost of selling to a buyer. J_a^{-1} is the pricing rule that translates the opportunity cost into the price that the buyer has to pay. This pricing rule takes into account the information rent that the seller has to leave for the buyer. If the pricing rule is non-linear, it matters for the expected price whether the pricing rule is applied to the expected opportunity cost (e.g. in period one) or the actual realization of the opportunity cost (e.g. in period two). In other words, the rent extraction motive of the seller interacts with the dynamic arrival of information and this can leads to violations of incentive compatibility. If the seller maximized value instead of revenue, the pricing rule would be the identity and the relaxed solution would always be incentive compatible. Moreover, if there was no information revelation over time but the seller would maximize revenue, the relaxed solution would also always be incentive compatible. The following example illustrates this point.

Example 2. Suppose that $T = 2$ and in each period, one buyer arrives. The first buyer can have deadline one or two. The relaxed solution looks as follows: If $d_1 = 1$ the seller makes a take-it-or-leave-it offer of $y_1^{\mathrm{rlx}}(v_1, d_1 = 1)$ to buyer one. If the offer is rejected, he waits and makes an offer of $y_2^{\mathrm{rlx}}((v_1, d_1 = 1), v_2)$ to buyer two. The optimal offer to buyer two is determined by $J_2(y_2^{\mathrm{rlx}}) = 0$, which implies that the expected revenue of waiting for period two equals $\zeta_1^1 = y_2^{\text{rlx}}(1 - F_2(y_2^{\text{rlx}})) = E_{v_2}[\max\{0, J_2(v_2)\}]$. The optimal offer in period one is given by $J_1(y_1^{\text{rlx}}(v_1, d_1 = 1)) = \zeta_1^1$. Hence we have

$$y_1^{\text{rlx}}(v_1, d_1 = 1) = J_1^{-1}(E_{v_2}[\max\{0, J_2(v_2)\}]).$$

If $d_1 = 2$, no buyer reaches his deadline in the first period, therefore the mechanism waits for period two. In period two, buyer one wins if $J_1(v_1) \geq \max\{0, J_2(v_2)\} = \zeta_1^2(v_2)$ and the price he has to pay in this case is given by[16]

$$y_1^{\text{rlx}}((v_1, d_1 = 2), v_2) = J_1^{-1}(\zeta_1^2(v_2)) = J_1^{-1}(\max\{0, J_2(v_2)\}).$$

Now suppose that the seller tries to implement the relaxed solution when the deadline is private information. Then, he has to rely on buyer one's claim about the deadline when deciding whether to make the take-it-or-leave-it-offer or to wait for the second period. To understand the sufficient condition for incentive compatibility, suppose that buyer one has the highest possible valuation ($v_1 = \bar{v}$) and deadline two. Under this assumption, the decision to reveal the deadline truthfully only depends on the expected payments, since the buyer will get the object regardless of his report. Expected payments are given by

$$\begin{aligned} & y_1^{\text{rlx}}(\bar{v}, d_1 = 1) = J_1^{-1}(E_{v_2}[\max\{0, J_2(v_2)\}]) && \text{if he reports } d = 1, \\ \text{and} \quad & E_{v_2}[y_1^{\text{rlx}}((\bar{v}, d_1 = 2), v_2)] = E_{v_2}[J_1^{-1}(\max\{0, J_2(v_2)\})] && \text{if he reports } d = 2. \end{aligned}$$

By Jensen's inequality, the expected payment is strictly smaller for $d = 1$ if J_1 is concave (J_1^{-1} is convex). It is greater (equal) if J_1 is convex (linear). Therefore, this type of buyer one will not reveal his deadline truthfully, if J_1 is convex.

If the seller is interested in value-maximization rather than revenue, valuations are not transformed by the virtual valuation function. The take-it-or-leave-it-offer in period one would be $E[v_2]$, and the price for buyer one in the second period would be v_2. Hence, expected payments would be the same for both deadlines and the buyer would not have an incentive to lie about his deadline. This shows that dynamic arrivals alone do not lead to a violation of incentive compatibility.

If the seller maximizes revenue, but no new information arrives in the second period, the violation of incentive compatibility due to a concave virtual valuation also vanishes. To see this, suppose that the valuation of buyer two is already known in the first period. In this case, the mechanism in period two is unchanged, but the take-it-or-leave-it-offer in the first period is now given by $J_1^{-1}(\max\{0, J_2(v_2)\})$ rather than $J_1^{-1}(E_{v_2}[\max\{0, J_2(v_2)\}])$. Therefore, expected payments would be the same for both deadlines. Formally, the *strict* second order stochastic dominance in Lemma 2 depends on new arrivals. Without new arrivals, we have $\zeta_1^1 = \zeta_1^2$ and the expected payment is independent of the deadline. □

[16] If $F_1 = F_2$, this is the payment in a second-price auction. In general, however, the relaxed solution cannot be implemented by a sequence of second-price auctions even in the case of identically distributed valuations. In Example 1, buyer one has to pay $J_1^{-1}(\max\{z_2^2(v_2), J_2(v_3)\})$. Here, the second-price auction is augmented by a backward looking component that increases the price buyer one has to pay whenever he was pivotal for delaying the allocation until period two.

$f(v)$ (support: $[0,1]$)	$J(v)$	$J''(v)$
$2v$	$\frac{1}{2}\frac{3v^2-1}{v}$	$-\frac{1}{v^3}<0$
$1-k+2kv$ ($k\in(0,1]$)	$\frac{2v-2kv+3kv^2-1}{1-k+2kv}$	$-\frac{2k(1+k)^2}{(1-k+2kv)^3}<0$
$(k+1)v^k$ ($k>0$)	$\frac{vk+2v-v^{-k}}{k+1}$	$-v^{-2-k}k<0$
$12(v-\frac{1}{2})^2$	$\frac{2}{3}\frac{v^2(4v-3)}{(2v-1)^2}$	$-\frac{4}{(2v-1)^4}<0$
$\frac{3}{2}-6(v-\frac{1}{2})^2$	$\frac{8v^2-v-1}{6v}$	$-\frac{1}{3v^3}<0$
$2-2v$	$\frac{3v}{2}-\frac{1}{2}$	0
1 (uniform)	$2v-1$	0
$(1+k)(1-v)^k$	$\frac{(k+2)v-1}{k+1}$	0
$1-k+2kv$ ($k\in[-1,0)$)	$\frac{2v-2kv+3kv-1}{1-k+2kv}$	$-\frac{2k(1+k)^2}{(1-k+2kv)^3}>0$

Table 1. Distributions with strictly concave, linear, and strictly convex virtual valuations.

Remark 1. Strict concavity of the virtual valuation is equivalent to

$$\frac{1-F(v)}{(f(v))^2}(f(v)f''(v)-2(f'(v))^2)<f'(v).$$

This implies that all distributions with an increasing density that is not too convex have strictly concave virtual valuations. Conversely, decreasing densities that are not too concave imply weak convexity of the virtual valuation. Table 1 shows densities and virtual valuations for several distributions. For the first group, the virtual valuation is strictly concave wherever it is non-negative. For the second group, it is linear and for the third group it is convex. The relaxed solution violates incentive compatibility for all distributions in the first group and satisfies incentive compatibility for all other examples.

Remark 2. Lemma 2 conditions on the state in the arrival period. This implies that the incentive compatibility result of Theorem 3 also holds if buyers can condition their reports on the state at their arrival time. In other words, under the conditions of part (i) of the theorem, the relaxed solution is periodic ex-post incentive compatible. This shows that the optimal solution does not rely on the seller's ability to conceal information from earlier periods.

Remark 3. Theorem 3 can be generalized in two directions. First, we have assumed so far that the deadline and the valuation of a buyer are independently distributed. If we allow for correlations, the distribution of the valuation will depend on the deadline. By reporting the deadline, the buyer can then choose the distribution he comes from. It is well known that a buyer would prefer to come from a weaker distribution because this leads to less aggressive price setting by the seller. This implies that the relaxed solution is incentive compatible if the valuation distribution is weaker (in the hazard rate order) for higher deadlines.[17]

Second, we show in Appendix C, that for the case of two time periods ($T = 2$), Lemma 2 also holds if the seller has a finite number of units of a good and sells to buyers with unit demand. Therefore, Theorem 3 also holds in this framework. We state the generalizations in the following theorem (where K denotes the number of units, and $J_a(v|d)$ denotes the virtual valuation conditional on the deadline d):

Theorem (3'). *Suppose that $K = 1$ and $T < \infty$, or that $T = 2$ and $K < \infty$.*

(i) The relaxed solution is incentive compatible if deadlines are private information if for all a, d and $d' \in \{a, \ldots, d-1\}$,
  (a) $J_a(v|d') \leq J_a(v|d)$ for all $v \in [0, \overline{v}]$, and
  (b) $J_a(v|d)$ or $J_a(v|d')$ is weakly convex as a function of v.

(ii) (ICD$^{\mathrm{d}}$) is violated in the relaxed solution if for some type type (a, v, d) there exists $d' \in \{a, \ldots, d-1\}$, such that
  (a) $J_a(v|d') \geq J_a(v|d)$ for all $v \in [0, \overline{v}]$, and
  (b) $J_a(v|d)$ or $J_a(v|d')$ is strictly concave as a function of v.

4. Bunching

In cases where the relaxed solution is not incentive compatible, the analysis is significantly more complex. For tractability, we restrict the model to the case of two periods ($T = 2$) and assume deterministic arrival of one buyer in each period ($\nu_{1,1} = \nu_{2,1} = 1$). Furthermore, we will assume that the profit of a monopolist selling to the first buyer is concave (Assumption 2). This assumption ensures that the optimal mechanism does not use lotteries in the first period. For this case, we solve $\mathcal{R}$ subject to (M), (ICD$^{\mathrm{d}}$) and (PE). While Assumption 2 is needed for a complete solution, we will argue that a main property of the optimal solution, namely that deadlines are not separated for high valuations, is robust.

In the following section, we will simplify the notation and decompose the seller's problem into two sub-problems: one for $d_1 = 1$ and one for $d_1 = 2$. These problems are only linked by the incentive compatibility constraint for the deadline (ICD$^{\mathrm{d}}$). In Section 4.2, we show that Assumption 2 rules out lotteries and solve the revenue maximization problem for $d_1 = 1$. Section 4.3 deals with the problem for $d_1 = 2$ in the regular case where the monotonicity constraint is slack. Assumption 1 guarantees that in the optimal solution, (M) is slack for buyer two. For buyer one, however, it is not sufficient for monotonicity. In Section 4.4, we show how the mechanism has to be ironed if (M) is binding at the optimal solution. The reader may want to skip section 4.4 at the first read. Finally, we combine the solutions for $d_1 = 1$ and $d_1 = 2$ to a solution of the general problem.

[17]The observation is not new. It can also be found in the previous literature on static models with two-dimensional private information. See Section 1.1 and Footnote 6.

4.1. Decomposition of the seller's problem

Since $N_1 = N_2 = 1$, we write d instead of d_1 and denote the probability that $d = 1$ by ρ instead of $\rho_{1,1}$. Winning probabilities are written as $x_1(v_1, 1)$, $x_1(v_1, 2, v_2)$, and $x_1(v_1, d, v_2)$, where the second argument is always the deadline of buyer one. $x_1(v_1, 1)$ is the probability that buyer one gets the object if his deadline is one. $x_i(v_1, d, v_2)$ is the probability that buyer i gets the object in period two, conditional on the event that the object has not been allocated in the first period. Note that x is feasible if and only if for all $v_1, v_2 \in [0, \overline{v}]$, $d \in \{1, 2\}$, and $i \in \{1, 2\}$,

$$x_1(v_1, 1), x_i(v_1, d, v_2) \in [0, 1] \quad \text{and} \quad x_1(v_1, 2, v_2) + x_2(v_1, 2, v_2) \leq 1. \tag{F}$$

The feasibility constraint for $d = 1$ is fulfilled automatically because $x_2(v_1, 1, v_2)$ is the winning probability of buyer two conditional on the event that the object has not been allocated in the first period.

Interim winning probabilities of buyer one are given by

$$q_1(v_1, 1) = x_1(v_1, 1), \qquad \text{and} \qquad q_1(v_1, 2) = \int_0^{\overline{v}} x_1(v_1, 2, v_2) f_2(v_2) dv_2.$$

The interim winning probability of buyer two, conditional on the deadline of buyer one and the event that the object has not been allocated in period one, is given by

$$q_2(v_2|1) := \int_0^{\overline{v}} x_2(v_1, 1, v_2) \frac{(1 - x_1(v_1, 1)) f_1(v_1)}{\int_0^{\overline{v}} (1 - x_1(s, 1)) f_1(s) ds} \, dv_1,$$

if $d = 1$ and by

$$q_2(v_2|2) := \int_0^{\overline{v}} x_2(v_1, 2, v_2) f_1(v_1) dv_1,$$

if $d = 2$. Hence, we have

$$q_2(v_2) = \rho \left(\int_0^{\overline{v}} (1 - x_1(v_1, 1)) f_1(v_1) dv_1 \right) q_2(v_2|d = 1) + (1 - \rho)\, q_2(v_2|d = 2).$$

With these definitions, $\mathcal{R}$ subject to (ICD$^{\mathrm{d}}$), (PE), and (M) for buyer one, can be rewritten as the maximization problem $\mathcal{P}$:

$$\max_q \rho \int_0^{\overline{v}} \left[q_1(v_1, 1) J_1(v_1) + (1 - q_1(v_1, 1)) \int_0^{\overline{v}} q_2(v_2|1) J_2(v_2) f_2(v_2) dv_2 \right] f_1(v_1) dv_1$$
$$+ (1 - \rho) \int_0^{\overline{v}} q_1(v, 2)\, J_1(v)\, f_1(v) + q_2(v|2)\, J_2(v)\, f_2(v)\, dv \tag{$\mathcal{P}$}$$

subject to q being the reduced form of a feasible allocation rule,

$$\forall d \in \{1, 2\}, \forall v, v' \in [0, \overline{v}] : \qquad v > v' \Rightarrow q_1(v, d) \geq q_1(v', d), \tag{M$_1$}$$

$$\forall d \in \{1, 2\}, \forall v \in [0, \overline{v}] : \qquad U_1(v, d) = \int_0^v q_1(s, d) ds, \tag{PE$_1$}$$

$$\text{and} \qquad \forall v \in [0, \overline{v}] : \qquad U_1(v, 1) \leq U_1(v, 2), \text{ with equality if } v = 0. \tag{ICD$_1^{\mathrm{d}}$}$$

Except for the incentive constraint for the deadline ($\mathrm{ICD}^{\mathrm{d}}_{1}$), the expected revenues for $d = 1$ (first line in the objective) and $d = 2$ (second line) can be maximized independently. In order to decompose the seller's problem, we introduce a function $U : [0, \overline{v}] \to [0, \overline{v}]$, $U(0) = 0$, that separates $U_1(.,1)$ from $U_1(.,2)$:

$$\forall v \in [0, \overline{v}] : \qquad U_1(v,1) \leq U(v) \leq U_2(v,2), \text{ with equality if } v = 0. \tag{$\mathrm{ICD}^{\mathrm{d}}_{\mathrm{U}}$}$$

Using U as a parameter, the maximization problem can be rewritten as $\mathcal{P}'$:

$$\max_{U} \rho\, \pi_1[U] + (1-\rho)\, \pi_2[U] \tag{$\mathcal{P}'$}$$

$\pi_1[U]$ is defined as the maximal expected revenue that can be achieved if the deadline is one and the expected payoff of the first buyer is constrained by $U_1(v,1) \leq U(v)$ for all $v \in [0, \overline{v}]$. This maximization problem is called $\mathcal{P}_1$:

$$\begin{aligned} \pi_1[U] := \max_{q} \int_0^{\overline{v}} \Bigg[& q_1(v_1,1) J_1(v_1) + \\ & (1 - q_1(v_1,1)) \int_0^{\overline{v}} q_2(v_2|1) J_2(v_2) f_2(v_2) dv_2 \Bigg] f_1(v_1) dv_1 \\ \text{s.t.} \quad & q_1(v,1), q(v|1) \in [0,1],\ (\mathrm{PE}_1),\ (\mathrm{M}_1) \text{ and } (\mathrm{ICD}^{\mathrm{d}}_{\mathrm{U}}) \end{aligned} \tag{$\mathcal{P}_1$}$$

$\pi_2[U]$ is defined as the maximal expected revenue that can be achieved if the deadline is two and the utility of the first buyer is constrained by $U_1(v,2) \geq U(v)$ for all $v \in [0, \overline{v}]$. This maximization problem is called $\mathcal{P}_2$:

$$\begin{aligned} \pi_2(U) := \max_{q} & \int_0^{\overline{v}} q_1(v,2) J_1(v) f_1(v) + q_2(v|2) J_2(v) f_2(v) dv \\ \text{s.t.} \quad & (\mathrm{F}), (\mathrm{PE}_1),\ (\mathrm{M}_1) \text{ and } (\mathrm{ICD}^{\mathrm{d}}_{\mathrm{U}}). \end{aligned} \tag{$\mathcal{P}_2$}$$

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are solved for the same U, we get a solution for $\mathcal{P}$. The following lemma shows that $\mathrm{ICD}^{\mathrm{d}}_{\mathrm{U}}$ has to be checked only for the highest valuation if the seller does not use lotteries in the first period.

Lemma 3. *If $x_1(v_1,1) \in \{0,1\}$ for all $v_1 \in [0, \overline{v}]$, then ($ICD^{d}_{U}$) holds for any v, if it is fulfilled for $v = 0$ and $v = \overline{v}$.*

Proof. $q_1(v,1)$ jumps from zero to one at $v = \overline{v} - U_1(\overline{v},1)$ if the allocation is deterministic. Therefore, the utility schedule for $d = 1$ is the lowest schedule that is consistent with $U_1(0,1)$, $U_1(\overline{v},1)$ and (PE). If $U_1(0,1) = U_1(0,2)$ and $U_1(\overline{v},1) \leq U_1(\overline{v},2)$, then $U_1(v,2)$ must necessarily be greater or equal than $U_1(v,1)$ for all $v \in [0, \overline{v}]$. □

This result is very useful. It implies that the points where the constraint is binding are independent of the solution, as long as the seller does not use lotteries in the first period. In particular, since $U_1(0,1) = U_2(0,2) = 0$ the incentive constraint for the deadline is reduced to a single inequality.

4.2. Solution to $\mathcal{P}_1$

If ($\mathrm{ICD}^{\mathrm{d}}_{\mathrm{U}}$) is ignored, $\mathcal{P}_1$ is equivalent to the problem of finding the optimal selling strategy for a sequence of short-lived buyers. The optimal solution is a sequence of fixed prices (Riley and Zeckhauser, 1983). Optimal prices are determined working backwards in time. If the object was not sold in the first period, the optimal price in the second period is $r_2 = v_2^0$. Hence, the option value of postponing the allocation is $V_2^{\mathrm{opt}} := v_2^0(1 - F_2(v_2^0)) = \int_{v_2^0}^{\overline{v}} J_2(v_2) f_2(v_2) dv_2$. The optimal price in the first period, r_1, is given by $J_1(r_1) = V_2^{\mathrm{opt}}$. This is the relaxed solution of $\mathcal{P}_1$.

If constraint ($\mathrm{ICD}^{\mathrm{d}}_{\mathrm{U}}$) is imposed, the optimal solution to $\mathcal{P}_1$ may involve lotteries.[18] To rule out this possibility we impose Assumption 2.

Lemma 3 implies that if the allocation rule is deterministic in the first period, ($\mathrm{ICD}^{\mathrm{d}}_{\mathrm{U}}$) reduces to $U_1(\overline{v}, 1) \leq \bar{U}$, where we define $\bar{U} := U(\overline{v})$. We will thus treat π_1 as a function of $\bar{U}$ and write $\pi_1(\bar{U})$ instead of $\pi_1[U]$ in this case. The optimal fixed price in period one is now given by the lowest price that satisfies $J_1(r_1) \geq V_2^{\mathrm{opt}}$ and $\overline{v} - r_1 \leq \bar{U}$. The optimal price in period two is not affected by constraint ($\mathrm{ICD}^{\mathrm{d}}_{\mathrm{U}}$).

Theorem 4. *Suppose f_1 satisfies Assumption 2. Then,*

(i) the optimal solution of $\mathcal{P}_1$ does not use lotteries. It is given by

$$q_1(v_1, 1) = \begin{cases} 0, & \text{if } J_1(v_1) < \max\{V_2^{\mathrm{opt}}, J_1(\overline{v} - \bar{U})\}, \\ 1, & \text{otherwise}, \end{cases}$$

$$q_2(v_2, 1) = \begin{cases} 0, & \text{if } J_2(v_2) < 0, \\ 1, & \text{otherwise}. \end{cases}$$

(ii) $\pi_1(\bar{U})$ is continuously differentiable for $\bar{U} \in (0, \overline{v})$ and strictly concave in $\bar{U}$ for $\bar{U} < \overline{v} - J_1^{-1}(V_2^{\mathrm{opt}})$.

Proof. The proof can be found in the Appendix C. □

To understand the role of Assumption 2, note that in the constraint $U(v) \geq \int_0^v q_1(s, 1) ds$, winning probabilities are not weighted in the integral because incentive compatibility constraints are independent of the buyer's own distribution function. In the objective, however, $q_1(v_1, 1)$ is weighted by $(J_1(v_1) - V_2^{\mathrm{opt}}) f_1(v_1)$. Increasing the winning probability $q_1(v_1, 1)$ for valuations in $[v, v + \varepsilon]$, and decreasing it by the same amount on $[v', v' + \varepsilon]$, with $v' + \varepsilon \leq v$, decreases $U_1(v_1, 1)$ for $v_1 \in [v', v + \varepsilon]$ and leaves $U_1(v_1, 1)$ unchanged otherwise. Hence, such a change in q_1 does not destroy incentive compatibility. On the other hand, this shift of winning probability from low to high types increases the seller's revenue if $(J_1(v_1) - V_2^{\mathrm{opt}}) f_1(v_1)$ is increasing. Assumption 2 guarantees that $(J_1(v_1) - V_2^{\mathrm{opt}}) f_1(v_1)$

[18] The no-haggling result of Riley and Zeckhauser (1983) is a consequence of a special structure of the feasible set of the maximization problem. Manelli and Vincent (2007) show that the set of extremal points of the feasible set, which contains the maximizers, is equal to the set of deterministic allocation rules. Due to the additional constraint ($\mathrm{ICD}^{\mathrm{d}}_{\mathrm{U}}$), the set of extremal points changes. Rather than trying to extend the results of Manelli and Vincent here, we use Assumption 2 as a sufficient condition for a deterministic mechanism.

is increasing whenever $J_1(v_1) - V_2^{\text{opt}} \geq 0$. Therefore, the winning probability must jump from zero to one at some point and the allocation is deterministic.

If Assumption 2 does not hold, raising the winning probability for a lower valuation may be more profitable than for a higher valuation because it is sufficiently more likely that buyer one has the low valuation. For this to be the case, the decrease in the density must outweigh the increase in expected revenue, i.e., the virtual valuation.

4.3. Solution to $\mathcal{P}_2$ – The Regular Case

In this section, we solve $\mathcal{P}_2$, imposing ($\text{ICD}_\text{U}^\text{d}$) only for $v = \overline{v}$. By Lemma 3 and Theorem 4, this is sufficient for the general problem if Assumption 2 is fulfilled. In the derivation of the optimal solution of $\mathcal{P}_2$, however, Assumption 2 is not used. Therefore, the results of this and the following section also apply if the mechanism designer is exogenously restricted to set a fixed price in the first period.

To state the optimal solution, we define the *generalized virtual valuation* of buyer one:

$$J_1^{p_U}(v) := J_1(v) + \frac{p_U}{f_1(v)}.$$

The parameter p_U determines the magnitude of the distortion of the allocation rule away from Myerson's (1981) solution for $\mathcal{P}_2$ without ($\text{ICD}_\text{U}^\text{d}$). ($p_U$ is the multiplier of the constraint ($\text{ICD}_\text{U}^\text{d}$) in the underlying control problem.) Suppose we already know the optimal p_U. Then, the optimal allocation rule is given by

$$\begin{aligned} x_1(v_1, 2, v_2) &= \begin{cases} 0, & \text{if } J_1^{p_U}(v_1) < \max\{0, J_2(v_2)\}, \\ 1, & \text{otherwise,} \end{cases} \\ x_2(v_1, 2, v_2) &= \begin{cases} 0, & \text{if } J_2(v_2) \leq \max\{0, J_1^{p_U}(v_1)\}, \\ 1, & \text{otherwise.} \end{cases} \end{aligned} \tag{4.1}$$

For every $\bar{U} \in [0, \overline{v})$, let $p_{\bar{U}}^*$ be the lowest value $p_U \geq 0$, such that the reduced form of (4.1) satisfies $\int_0^{\overline{v}} q_1(v, 2) dv \geq \bar{U}$.

Theorem 5. *Fix $\bar{U}$ and suppose that $J_1^{p_{\bar{U}}^*}(v_1)$ is strictly increasing in v_1. Then*

(i) the reduced form of (4.1) for $p_U = p_{\bar{U}}^*$ is an optimal solution of $\mathcal{P}_2$ subject to (M_1), (PE_1), and (ICD_U^d) for $v = \overline{v}$.

(ii) $p_{\bar{U}}^* = -\pi_2'(\bar{U})$.

(iii) π_2 is weakly concave.

Proof. Theorem 5 is a special case of Theorem 6 below. □

If the relaxed solution is incentive compatible, p_U is zero and valuations (v_1, v_2) tie if $J_1(v_1) = J_2(v_2)$, as in Myerson's solution. If the relaxed solution is not incentive compatible, p_U is strictly positive and valuations tie if $J_1^{p_U}(v_1) = J_2(v_2)$, which is equivalent to

$$(J_1(v_1) - J_2(v_2)) f_1(v_1) = -p_U. \tag{4.2}$$

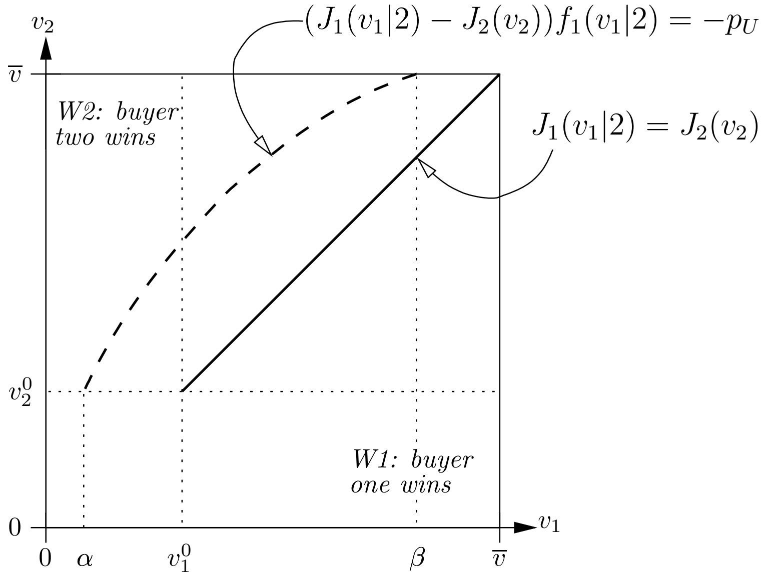

FIGURE 4.1. Optimal allocation rule

Figure 4.1 sketches both cases for identically distributed valuations ($f_1 = f_2$). The solid line is the Myerson-line, at which valuations tie in the relaxed solution. The dashed line is the distorted Myerson-line, at which valuations tie in the general solution. Note that for $p_U > 0$, valuations tie in an area where the (standard) virtual valuation of buyer one is strictly smaller than the virtual valuation of buyer two.

To understand condition (4.2), consider the effect on π_2 of an increase of $q_1(.,2)$. Fix any (v_1, v_2) on the distorted Myerson-line, such that $0 \leq J_1^{p_U}(v_1) \leq \overline{v}$. In the figure, this corresponds to $\alpha \leq v_1 \leq \beta$. In order to increase $q_1(v_1, 2)$, the allocation has to be changed from buyer two to buyer one at (v_1, v_2). This leads to a marginal change in π_2 of $J_1(v_1) - J_2(v_2) < 0$ per mass of type profiles for which the allocation is changed. This mass of type profiles is proportional to $f_1(v_1)$. Hence, the left-hand side of (4.2) quantifies the marginal cost of increasing $q_1(v_1, 2)$.

Along the distorted Myerson-line, the marginal cost of increasing $q_1(v_1, 2)$ must be independent of v_1. The reason is that winning probabilities are not weighted in the constraint $\int_0^{\overline{v}} q_1(s, 2)ds \geq \bar{U}$. If the marginal cost of changing $q_1(v_1, 2)$ varied with v_1, we could increase $q_1(v_1, 2)$ where the marginal cost is small and decrease it where the marginal cost is big. If we chose this variation such that $U_1(\overline{v}, 2) = \int_0^{\overline{v}} q_1(s, 2)ds$ were not changed, we could increase the objective function without violating the constraints—a contradiction. Hence, the marginal cost of increasing $q_1(., 2)$ must be constant and equal to p_U for all $v_1 \in [\alpha, \beta]$. As the utility of the highest type is given by $U_1(\overline{v}, 2) = \int_0^{\overline{v}} q_1(s, 2)ds$, p_U can also be interpreted as the marginal cost of the constraint $U_1(\overline{v}, 2) \geq \bar{U}$.

Furthermore, note that the distortion is increasing in p_U, and that by Assumption 1, the marginal cost of a distortion is increasing in the distance from the Myerson solution (the LHS of (4.2) is decreasing in v_2). Therefore, (a) it is optimal to choose the lowest

p_U such that ($\text{ICD}_{\text{U}}^{\text{d}}$) is satisfied, and (b) the cost of distortions is convex, which implies concavity of π_2 in $\bar{U}$.

Finally, (4.2) implies that the distortion of the Myerson-line is bigger for types with lower densities. Intuitively, the expected cost of a distortion is lower for types that are less frequent. This implies that an increasing density can lead to non-monotonicities of the winning-probability.

4.4. Solution to $\mathcal{P}_2$ – The Irregular Case

To ensure an increasing winning probability for buyer one, Theorem 5 requires that $J_1^{p_U^*}$ is strictly increasing. This is a condition on an endogenous object and Assumption 1 does not guarantee monotonicity of $J_1^{p_U}$ for all values of p_U. A decreasing density $f_1(v)$ together with Assumption 1 would be sufficient, but this is quite restrictive and rules out most of the examples of concave virtual valuations in Table 1. To give a complete solution without further assumptions, we show that Myerson's ironing procedure can be used to deal with non-monotonicities of $J_1^{p_U}$.

Definition 3 (Ironing; Myerson, 1981)**.** (i) For every $t \in [0, 1]$, define

$$M_1^{p_U}(t) := J_1(F_1^{-1}(t)) + \frac{p_U}{f_1(F_1^{-1}(t))},$$

as the generalized virtual valuation at the t-quantile of F_1.

(ii) Integrate this function:

$$H^{p_U}(t) := \int_0^t M_1^{p_U}(s)ds.$$

(iii) Take the convex hull (i.e. the greatest convex function G such that $G(t) \leq H^{p_U}(t)$ for all t):

$$\bar{H}^{p_U}(t) := \operatorname{conv} H^{p_U}(t).$$

(iv) Since $\bar{H}^{p_U}$ is convex, it is almost everywhere differentiable and any selection $\bar{M}_1^{p_U}(t)$ from the sub-gradient is non-decreasing.

(v) Reverse the change of variables made in (i) to obtain the *ironed generalized virtual valuation*

$$\bar{J}_1^{p_U}(v_1) := \bar{M}_1^{p_U}(F_1(v_1)).$$

In the irregular case, the optimal allocation rule depends on two parameters, p_U and $\underline{x}_1^0$, and has the following structure:

$$\bar{x}_1(v_1, 2, v_2) = \begin{cases} 1, & \text{if } \bar{J}_1^{p_U}(v_1) > 0 \text{ and } \bar{J}_1^{p_U}(v_1) \geq J_2(v_2) \\ \underline{x}_1^0, & \text{if } \bar{J}_1^{p_U}(v_1) = 0 \text{ and } J_2(v_2) \leq 0, \\ 0, & \text{otherwise,} \end{cases} \tag{4.3}$$

$$\bar{x}_2(v_1, 2, v_2) = \begin{cases} 0, & \text{if } J_2(v_2) \leq \max\{0, \bar{J}_1^{p_U}(v_1)\}, \\ 1, & \text{otherwise.} \end{cases}$$

The optimal parameters are determined as follows. First, let $p^*_{\bar U}$ be the minimal $p_U \geq 0$ such that the reduced form of (4.3) with $\underline{x}^0_1 = 1$ satisfies $\int_0^{\overline{v}} q_1(v,2)dv \geq \bar U$. Second, if $p^*_{\bar U} > 0$, select $\underline{x}^{0*}_1 \in [0,1]$ such that $\int_0^{\overline{v}} q_1(v,2)dv = \bar U$, otherwise set $\underline{x}^{0*}_1 = 1$.

The additional parameter $\underline{x}^0_1$ is only needed if $\bar J_1^{p^*_{\bar U}}(v_1) = 0$ on an interval $[\underline{v}^0_1, \bar v^0_1]$ of positive length. In this case, $\int_{\underline{v}^0_1}^{\bar v^0_1} J_1^{p^*_{\bar U}}(v)dv = 0$ and hence, $U_1(\overline{v},2)$ can be varied at constant marginal cost p_U by changing the winning probability for all valuations in the interval $[\underline{v}^0_1, \bar v^0_1]$. Therefore, a single value of p_U defines the ironed generalized virtual valuation for different values $\bar U$ in a non-empty interval $[a,b]$. $\underline{x}^0_1$ is varied to achieve different values of $U_1(\overline{v},2) \in [a,b]$.

The allocation rule in (4.3) excludes buyer one if his valuation is smaller than $\underline{v}^0_1$. With a valuation in $[\underline{v}^0_1, \bar v^0_1]$, he can win against buyer two if $v_2 \leq v^0_2$, but he gets the object only with probability $\underline{x}^0_1$.[19] To summarize, we have

Theorem 6. *(i) The reduced form of (4.3) for $p^*_{\bar U}$ and $\underline{x}^{0*}_1$ is an optimal solution of $\mathcal{P}_2$ subject to (M_1), (PE_1), and (ICD^d_U) for $v = \overline{v}$.*

(ii) For almost every $\bar U$, $\pi_2'(\bar U) = -p^*_U$.

(iii) π_2 is weakly concave in $\bar U$ and strictly concave if $p_U > 0$ and $\bar J_1^{p_U}(v) = 0$ has a unique solution.

Proof. See Appendix B. □

Note that if $J_1^{p_U}$ is increasing, $\bar J_1^{p_U}$ equals $J_1^{p_U}$. Therefore, Theorem 5 is a special case of Theorem 6.

4.5. Global Solution and Discussion

Under Assumption 2, $\mathcal{P}'$ reduces to the problem of choosing $\bar U$ optimally. The first order necessary condition is

$$\rho\, \pi_1'(\bar U) = -(1-\rho)\, \pi_2'(\bar U).$$

By Theorem 6, π_2 is concave and by Theorem 4 and Assumption 2, π_1 is concave. Therefore, the first-order condition is also sufficient. To determine the optimal distortion, it suffices to compute the unique solution $(\bar U, p_U)$, $p_U \geq 0$ of

$$p_U = \frac{\rho}{1-\rho}\, \pi_1'(\bar U),$$

$$\text{and} \qquad \bar U \leq \int_0^{\overline{v}} q_1^{p_U}(v,2)dv_1, \quad \text{with equality if } p_U > 0,$$

where q^{p_U} is the reduced form of (4.1) for given value of p_U.[20] An explicit form of the solution is not available. However, for given p_U, $U_1(\overline{v},2) = \int_0^{\overline{v}} q_1^{p_U}(v,2)dv_1$ is easy to

[19] It is also possible to construct a deterministic allocation rule with the same reduced form. Choose $\hat v_2$ such that $\underline{x}^0_1 = \frac{F_2(\hat v_2)}{F_2(v^0_2)}$. For $v_1 \in [\underline{v}^0_1, \bar v^0_1]$, set $x_1(v_1,2,v_2) = 1$ if $v_2 \leq \hat v_2$ and $x_1(v_1,2,v_2) = 0$ otherwise. This construction has the disadvantage, however, that the allocation decision for buyer one depends on truthful reports of buyer two in cases when buyer two can never win the object.

[20] We only discuss the global solution for the regular case. The irregular case is similar.

calculate and an explicit expression for π_1' is given in the proof of Theorem 4. Hence, it is easy to compute the optimal p_U numerically. If Assumption 2 is violated, π_1 may fail to be concave and it may be necessary to compute all local maxima to find the global solution. We will now discuss several properties of the general solution.

Monotonicity of q_2. $q_2(v_2|1)$, defined by the fixed price r_2, and $q_2(v_2|2)$, defined by the reduced form of (4.3), are non-decreasing. This follows from Assumption 1. Therefore, $q_2(v_2)$ is also non-decreasing and the optimal solutions of $\mathcal{P}_1$ and $\mathcal{P}_2$ together fulfill all constraints of $\mathcal{P}$.

Distortions in Both Periods. By Theorem 4, $\pi_1(\bar{U})$ is continuously differentiable. Therefore, $p_U > 0$ implies that the allocation for $d = 1$ is distorted. Hence, the general solution involves a distortion for both deadlines, whenever the relaxed solution is not incentive compatible. As distortions are more costly at the deadline which occurs more frequently, the relative magnitude of the distortions depends on ρ. If $d = 1$ is relatively unlikely (ρ small), then the distortion of the fixed price is bigger and the auction is closer to Myerson's solution.

Distortions and Bunching of Deadlines. In the first period, the fixed price is $\max\{\bar{U} - \overline{v}, J_1^{-1}(V_2^{\text{opt}})\}$. It is distorted upwards compared to the relaxed solution, to make the fixed price less attractive. To analyze the distortions in the auction in period two, note that

$$\forall v_1 \in [0, \overline{v}]: \quad J_1^{p_U}(v_1) = J_1(v_1) + \frac{p_U}{f_1(v_1)} > J_1(v_1),$$

if the relaxed solution is not incentive compatible ($p_U > 0$). Therefore, the reserve price for buyer one, which is given by $J_1^{p_U}(r) = 0$, is smaller than in the relaxed solution. Secondly, for all valuations above the reserve price, the winning probability is higher than in the relaxed solution because v_1 ties with a higher valuation v_2. Finally, in contrast to the relaxed solution, the winning probability of bidder two is strictly smaller than one for all $v_2 \in [0, \overline{v}]$. For every $p_U > 0$, there is a non-empty interval $(c, \overline{v}]$ such that $J_1^{p_U}(v_1) > \overline{v}$ for all valuations $v_1 \in (c, \overline{v}]$. Buyer two cannot win against buyer one if $v_1 > c$.

The fact that $q_1(v_1, 2) = 1 = q_1(v_1, 1)$ for high valuations, together with the binding incentive constraint for the deadline at $\bar{v}$ ($U_1(\bar{v}, 1) = U_1(\bar{v}, 2)$) means that the utility of buyer one is independent of the deadline if $v_1 > c$, and that the expected payment in the auction is equal to the fixed price in period one for these valuations. In other words, the optimal allocation does not separate buyers with different deadlines if their valuations are high. It can thus be implemented in a mechanism where buyers with valuations $v_1 > c$ can buy immediately and buyers with lower valuations have to wait until their deadline. This finding also holds without Assumption 2. It can be shown that the utility in period one has to satisfy $U_1(v_1, 1) \in [\max\{0, U_1(\bar{v}, 2) - (\bar{v} - v_1)\}, U_1(v_1, 2)]$ if the relaxed solution is not incentive compatible. This implies that the incentive constraint for the deadline holds with equality of the highest type. Since the relaxed solution is not incentive compatible, an increase of $U_1(v_1, 1)$ for high valuations has a first-order effect on expected revenue.

Therefore, the allocation rule for $d = 2$ must also be distorted for high valuations which implies $q_1(v_1, 2) = 1$ for v_1 sufficiently high.

Dominant Strategies and Indirect Implementation. There are several ways to implement the optimal auction in period two. For example, it can be implemented by a generalized Vickrey auction. In this auction, the winning bidder pays the valuation for which his (generalized) virtual valuation ties with the (generalized) virtual valuation of the losing bidder. For buyer two, this mechanism is incentive compatible in dominant strategies.[21] Hence, the optimal mechanism does not rely on the seller's ability to conceal information about period one.

As in the static auction model, there is also an open format that corresponds to this direct mechanism. Consider the following ascending clock auction. The auctioneer has a clock that runs from zero to $\overline{v}$. For each bidder i, the auctioneer's clock value c_a is translated into a bidder-specific clock value c_i. For bidder one, this is $c_1 = (J_1^{p_U})^{-1}(c_a)$. For bidder two, this is $c_2 = J_2^{-1}(c_a)$. The auctioneer raises c_a continuously and bidders can drop out at any time. If bidder i drops out, the clock stops immediately. Bidder $j \neq i$ wins the object and has to make a payment equal to his bidder-specific clock-value c_j. Given the informational assumptions made in this paper, this auction is strategically equivalent to the generalized Vickrey auction. It has the advantage that the winning bidder does not have to reveal his true valuation to the auctioneer.

5. Conclusion

We have analyzed a dynamic mechanism design model, in which a seller wants to maximize the revenue from selling one (or multiple) unit(s) of a good to buyers that arrive over time, within a finite time horizon. The main innovation of the model is that buyers are privately informed about their deadlines for buying the good. This allows us to study the optimal mechanism in the presence of a heterogeneous population of impatient/non-strategic and patient/strategic buyers.

We found sufficient conditions for full separation. In this case, the incentive compatibility constraint for the deadline is slack in the seller-optimal mechanism. The relaxed solution, which neglects the constraint is fully optimal and the optimal mechanism fully discriminates buyers with respect to their degrees of patience. We also found sufficient conditions for violations of the neglected constraints. Both conditions exploit (a) non-linearities in the virtual valuation function of a buyer, and (b) stochastic dependencies between the deadline and the valuation of a buyer. While the latter effect can also be found in static models with two-dimensional private information, the former effect is due to the dynamic nature of the allocation problem. The critical virtual valuation that a buyer has to overbid in order to get a unit is a martingale with respect to the information about all buyer's types. Therefore, critical virtual valuations for later deadlines are mean preserving spreads of critical virtual valuations for earlier deadlines. This leads to lower

[21] If the auction is considered in isolation, it is also a dominant strategy for buyer one to bid his true valuation. In the dynamic context, however, it is not a dominant strategy to report the deadline truthfully.

(higher) payoffs for later deadlines in the case of concave (convex) virtual valuations and destroys (guarantees) incentive compatibility.

We have also studied the case of bunching. If the relaxed solution is not incentive compatible, the incentive constraint for the deadline is binding in the optimal mechanism. Therefore, we have to solve a mechanism design problem with two-dimensional private information. The fact that the second dimension is a deadline puts some structure on the model. The two-dimensional problem is similar to a standard one-dimensional mechanism design problem with a type-dependent outside option. We solved this problem for the case of two time periods and deterministic arrival of one buyer in each period. We have shown that the optimal mechanism has a very similar structure as the relaxed solution, but the allocation rule is distorted in favor of buyers with later deadlines and earlier arrival. This provides incentives to report the deadline truthfully. Our analysis also shows that it in contrast to the relaxed solution, it is not optimal to separate types with different degrees of patience if their valuations are large.

The analysis in this paper has several limitation which we will discuss in the following. Some assumptions were made to ensure tractability, others merely to simplify the exposition.

Discounting. The assumption of no discounting can be relaxed. If only payments are discounted and buyers and the seller use a common discount factor, the analysis is almost identical. On the other hand, if valuations are discounted, Lemma 1 may not be valid. For example, it may be optimal to allocate a unit in the first period even if the deadline of the winner is two, because the waiting cost due to discounting is too high. In this case, it is more complicated to rule out upward deviations in the deadline.

The appropriate modeling choice depends on the application. In the example given in the introduction, the buyer's valuation is the present discounted value of the revenue stream from the contractual relationship with the third party. This could for example be a production contract. If production starts after the deadline and is independent of the time at which the firm obtains the object (as long as it gets it before the deadline), it seems reasonable that the firm only discounts payments. Similar arguments apply in any situation where the buyer plans to use the good at a fixed time after the deadline as in the case of flight tickets of hotel reservations.

Stochastic Exit. We have implicitly assumed that buyers are available until their deadline. In some situations, however, buyers may find other opportunities to purchase a similar object if the seller does not sell in the period of arrival. Therefore, stochastic exit, random participation as in Rochet and Stole (2002) or competition with other sellers would be interesting extensions for future research.

Incentive Compatibility of the Relaxed Solution with Many Objects. For more than two time periods, the proof of the martingale property of the critical virtual valuation uses a property of the optimal allocation rule that is shown in Mierendorff (2009) for the case of a single object. With one object, there is a unique bidder in each period that has a positive probability of winning. This greatly simplifies the analysis because in each state, the type of only one buyer is relevant for the allocation rule and buyers who are irrelevant in period t will not be recalled in the future. While a proof is not available, I conjecture, that the ordering of critical virtual valuations w.r.t. second order stochastic dominance generalizes to the case of many objects. If this conjecture is true, then the conditions for incentive compatibility of the relaxed solution carry over to the case of multiple objects and more than two time periods.

Privately Known Arrival Times. The arrival time has similar properties as the deadline. Misreports are only feasible in one direction and the arrival time does not enter the utility function directly. Therefore, the analysis of a model with private arrival times is similar to the analysis in the present paper. Pai and Vohra (2008b) show that the relaxed solution is incentive compatible with respect to the arrival time if virtual valuations are decreasing in the arrival time. This result is driven by correlations between valuations and deadlines. The arrival time does not influence the time of the allocation, and therefore the amount of information available to the seller is independent of the arrival time. This implies that non-linearities of the virtual valuation do not distort the incentives to report the arrival time truthfully as is the case of deadlines. We also note that there is an additional effect that relaxes the incentive compatibility constraint for the arrival time. By delaying the report of his arrival, a buyer runs the risk that units are allocated to buyers that he could have overbid if he had reported his arrival truthfully. Therefore, a virtual valuation that increases in the arrival time, does not automatically destroy incentive compatibility.

Generalizing the Bunching Case: More Bidders. Introducing more bidders who arrive in the second period is straight forward. The assumption that there is only one bidder in the first period is more important. It was used to show that the object is offered to buyer one for a fixed price if he reports deadline one. We have shown that in this case, misreporting deadline one instead of deadline two is most profitable for the buyer with the highest valuation. Hence, we know exactly where the incentive compatibility constraint for the deadline binds. If more than one buyer arrives in the first period, a fixed price is no longer optimal and the incentive compatibility constraint for the deadline may bind for interior types. The exact points where it binds arise endogenously in the optimal solution.

Generalizing the Bunching Case: Number of Periods. Increasing the number of periods introduces several complications. Consider for example a model with three periods. Suppose that in each period a single bidder arrives, whose deadline can be any period after his arrival. Now, from period two onwards, there is more than one bidder who participates in the mechanism. This introduces similar problems as the introduction of more bidders in the first period, as discussed in the preceding paragraph. Additional complications will arise because buyers from different periods will have to be treated asymmetrically. In the third period, the mechanism designer has to design an optimal auction with three different bidders, two of which have type-dependent participation constraints. In the case of two periods and two bidders, the feasibility constraint could be used to eliminate the winning probability of one bidder (see Appendix B). A generalization of this approach to three bidders is not obvious.

Appendix A. Proofs of Lemma 2 and Theorem 3

Proof of Lemma 2. To simplify notation, define $c_a^\tau := \max_{j\in\{i\in I_a|d_i=\tau\}} J_a(v_j)$ and $c_{\le a}^\tau := \max\{c_1^\tau,\dots,c_a^\tau\}$. With the definition of c^t in the main text, we have $c^t = c_{\le t}^t$. For fixed $i\in I_{\le\tau}$ define $c_a^{\tau,-i} := \max_{j\in\{\ell\in I_a\setminus\{i\}|d_\ell=\tau\}} J_a(v_j)$ and $c_{\le a}^{\tau,-i} := \max\{c_1^{\tau,-i},\dots,c_a^{\tau,-i}\}$. Since the seller maximizes virtual surplus, we reformulate his problem in terms of virtual valuations. We replace the type of a buyer (a_i,v_i,d_i) by (a_i,c_i,d_i) where $c_i = J_{a_i}(v_i)$ is the virtual valuation. Furthermore, instead of H_t and s_t we use $\tilde H_t = (a_i,c_i,d_i)_{i\in I_{\le t}}$ and $\tilde s_t = (\tilde H_t, k_t)$. With this notation, the seller's maximization problem becomes

$$\tilde V_T(\tilde s_T) = \max_{x(\tilde s_T)} \sum_{i\in I_{\le T}:d_i=T} x_i(\tilde s_T)c_i,$$

$$\tilde V_t(\tilde s_t) = \max_{x(\tilde s_t)} \left[\sum_{i\in I_{\le t}:d_i=t} x_i(\tilde s_t)c_i + x_0(\tilde s_t)E\left[\tilde V_{t+1}(\tilde s_{t+1})\middle|\tilde H_t, k_{t+1}=0\right]\right].$$

Mierendorff (2009) has shown that for this problem, for each state $\tilde s_t$ in which the object is still available, there is a unique period $\theta_t \ge t$, in which the object will be allocated if it is allocated to a buyer $i\in I_{\le t}$. Therefore $\tilde V(\tilde s_t) = \tilde V((\theta_t, c_{\le t}^{\theta_t}), k_t)$. The expected surplus at the relaxed solution only depends on the highest virtual valuation of buyers with deadlines θ_t. Moreover, (up to ties) there is a unique buyer $i\in I_{\le t}$ who can possibly win the object. This buyer is characterized by $d_i=\theta_i$ and $c_i = c_{\le t}^{\theta_t}$. We call this buyer the *tentative winner at state $\tilde s_t$.*

Since we want to prove a statement about critical virtual valuations for different deadlines, we first establish that the following claim is true:

Claim 1. For given $\tilde H_{d-1}^{-i}$, $z_t^d(c_{\le t}^t) = z_{d-1}^d(z_t^{d-1}(c_{\le t}^t))$.

Proof of Claim 1. Note first that $z_t^\tau(c_{\le t}^t)=0$ for some $\tau>t$ implies that $z_t^\tau(c_{\le t}^t)=0$ for all $\tau>t$. To see this, note that $z_t^\tau(c_{\le t}^t)=0$ if and only if $c_{\le t}^t \le E_{\tilde s_{t+1}}[V_{t+1}(s_{t+1})|\tilde H_t = (\tilde H_t^{-i},(a_i,0,d_i)), k_{t+1}=0]$, where a_i and $d_i>t$ can be selected arbitrarily since $c_i=0$. Since the second condition is independent of τ, the first must also hold independently of τ. Hence, it remains to show the claim for the case that $z_t^d(c_{\le t}^t)>0$ and $z_t^{d-1}(c_{\le t}^t)>0$.

$z_t^d(c_{\le t}^t)>0$ and $z_t^{d-1}(c_{\le t}^t)>0$ implies

$$E_{\tilde s_{t+1}}\left[V_{t+1}(\tilde s_{t+1})\middle|\tilde H_t = (a, z_t^{d-1}(c_{\le t}^t), d-1), k_{t+1}=0\right]$$
$$= c_{\le t}^t = E_{s_{t+1}}\left[V_{t+1}(\tilde s_{t+1})\middle|\tilde H_t = (a, z_t^d(c_{\le t}^t), d), k_{t+1}=0\right].$$

The expected values are independent of $\tilde H_t^{-i}$ because $c_{\le t}^t > E_{\tilde s_{t+1}}[V_{t+1}(\tilde s_{t+1}) \mid \tilde H_t = (\tilde H_t^{-i},(a_i,0,d_i)), k_{t+1} = 0]$. This implies that a buyer with type $(a, z_t^d(c_{\le t}^t), d)$ has a positive probability of winning the object if $\tilde H_t = (\tilde H_t^{-i},(a,z_t^d(c_{\le t}^t),d))$. Therefore, $(a,z_t^d(c_{\le t}^t),d)$ is the tentative winner at this state and all other buyers $j\in I_{\le t}$ have a zero winning probability conditional on $\tilde{H}_t = (\tilde{H}_t^{-i}, (a, z_t^d(c_{\leq t}^t), d))$. Their types are thus irrelevant for the seller's expected revenue.

If we insert $V_{t+1}, V_{t+2}, \dots, V_{d-1}$ we obtain

$$\begin{aligned}
&E_{\tilde{s}_{t+1}}\left[V_{t+1}(\tilde{s}_{t+1})\middle|\tilde{H}_t = (a, z_t^{d-1}(c_{\leq t}^t), d-1), k_{t+1} = 0\right] \qquad \text{(A.1)}\\
=&E_{\tilde{s}_{d-1}}\Bigg[\sum_j x_j^{\text{rlx}}(\tilde{s}_{t+1})c_j + x_0^{\text{rlx}}(\tilde{s}_{t+1})\Bigg[\sum_k x_k^{\text{rlx}}(\tilde{s}_{t+2})c_k + x_0^{\text{rlx}}(\tilde{s}_{t+2})\dots\Bigg[\sum_\ell x_\ell^{\text{rlx}}(\tilde{s}_{d-2})c_\ell +\\
&\qquad + x_0^{\text{rlx}}(\tilde{s}_{d-2})\underbrace{\max\left\{c_{\leq d-1}^{d-1,-i}, z_t^{d-1}(c_{\leq t}^t), E_{s_d}[\tilde{V}_d(\tilde{s}_d)|\tilde{H}_{d-1} = \tilde{H}_{d-1}^{-i}, k_d = 0]\right\}}_{=:A}\Bigg]\\
&\qquad\qquad \dots \Bigg]\Bigg|\tilde{H}_t = (a, z_t^{d-1}(c_{\leq t}^t), d-1), k_{t+1} = 0\Bigg],
\end{aligned}$$

and

$$\begin{aligned}
&E_{\tilde{s}_{t+1}}\left[V_{t+1}(\tilde{s}_{t+1})\middle|\tilde{H}_t = (a, z_t^d(c_{\leq t}^t), d), k_{t+1} = 0\right] \qquad \text{(A.2)}\\
=&E_{\tilde{s}_{d-1}}\Bigg[\sum_j x_j^{\text{rlx}}(\tilde{s}_{t+1})c_j + x_0^{\text{rlx}}(\tilde{s}_{t+1})\Bigg[\sum_k x_k^{\text{rlx}}(\tilde{s}_{t+2})c_k + x_0^{\text{rlx}}(\tilde{s}_{t+2})\dots\Bigg[\sum_\ell x_\ell^{\text{rlx}}(\tilde{s}_{d-2})c_\ell + x_0^{\text{rlx}}(\tilde{s}_{d-2})\times\\
&\times\underbrace{\max\left\{c_{\leq d-1}^{d-1,-i}, E_{s_d}\left[\tilde{V}_d(\tilde{s}_d)\middle|\tilde{H}_{d-1} = (a, z_t^d(c_{\leq t}^t), d), k_d = 0\right], E_{s_d}\left[\tilde{V}_d(\tilde{s}_d)\middle|\tilde{H}_{d-1} = \tilde{H}_{d-1}^{-i}, k_d = 0\right]\right\}}_{=:B}\Bigg]\\
&\qquad\qquad \dots \Bigg]\Bigg|\tilde{H}_t = (a, z_t^d(c_{\leq t}^t), d), k_{t+1} = 0\Bigg].
\end{aligned}$$

In the second derivation we have used that because of the uniqueness of the tentative winner,

$$\begin{aligned}
&E_{\tilde{s}_d}\left[\tilde{V}_d(\tilde{s}_d)\middle|\tilde{H}_{d-1} = (\tilde{H}_{d-1}^{-i}, (a, z_t^d(c_{\leq t}^t), d)), k_d = 0\right]\\
=&\max\left\{E_{\tilde{s}_d}\left[\tilde{V}_d(\tilde{s}_d)\middle|\tilde{H}_{d-1} = \tilde{H}_{d-1}^{-i}, k_d = 0\right], E_{\tilde{s}_d}\left[\tilde{V}_d(\tilde{s}_d)\middle|\tilde{H}_{d-1} = (a, z_t^d(c_{\leq t}^t), d), k_d = 0\right]\right\}.
\end{aligned}$$

Now suppose by contradiction, that $z_t^d(c_{\leq t}^t) > z_{d-1}^d(z_t^{d-1}(c_{\leq t}^t))$. By the definition of z_{d-1}^d this implies

$$E_{\tilde{s}_d}\left[V_d(\tilde{s}_d)\middle|\tilde{H}_{d-1} = (a, z_t^d(c_{\leq t}^t), d), k_d = 0\right] > z_t^{d-1}(c_{\leq t}^t),$$

and hence $A \leq B$ for all s_{d-1}. This implies that

$$\prod_{\tau=t+1}^{d-1} x_0^{\text{rlx}}\left(\tilde{s}_\tau\middle|\tilde{H}_t = (a, z_t^{d-1}(c_{\leq t}^t), d-1)\right) \leq \prod_{\tau=t+1}^{d-1} x_0^{\text{rlx}}\left(\tilde{s}_\tau\middle|\tilde{H}_t = (a, z_t^d(c_{\leq t}^t), d)\right).$$

Moreover, with strictly positive probability $A = z_t^{d-1}(c_{\leq t}^t) < B$. Hence (A.1) is strictly smaller than (A.2), which is a contradiction. Similarly, $z_t^d(c_{\leq t}^t) < z_{d-1}^d(z_t^{d-1}(c_{\leq t}^t))$ leads to a contradiction which proves the claim. $\square$

Now, consider the critical virtual valuation for deadline $d_i - 1$:

$$\zeta_{a_i}^{d_i-1}(H_{d_i-1}^{-i}) = \max\Big\{ z_{a_i}^{d_i-1}(c_{\leq a_i}^{a_i,-i}), \dots, z_{d_i-2}^{d_i-1}(c_{\leq d_i-2}^{d_i-2,-i}), c_{\leq d_i-1}^{d_i-1,-i},$$
$$E_{s_{d_i}}\left[V_{d_i}(s_{d_i})\Big| H_{d_i-1}^{-i}, k_{d_i}=0\right]\Big\}.$$

Claim 1 allows us to replace the cutoff values $z_\tau^{d_i-1}$. $\forall \tau \in \{a_i, \dots, d_i-2\}$:

$$z_\tau^{d_i-1}(c_{\leq\tau}^{\tau,-i}) = E_{s_{d_i}}\left[V_{d_i}(s_{d_i})\Big|(a_i, z_{d_i-1}^{d_i}(z_\tau^{d_i-1}(c_{\leq\tau}^{\tau,-i})), d_i), k_{d_i}=0\right],$$
$$= E_{s_{d_i}}\left[V_{d_i}(s_{d_i})\Big|(a_i, z_\tau^{d_i}(c_{\leq\tau}^{\tau,-i}), d_i), k_{d_i}=0\right].$$

Hence,

$$\begin{aligned}
&\zeta_{a_i,}^{d_i-1}(H_{d_i-1}^{-i}) = \\
=&\max\Big\{E_{s_{d_i}}\left[V_{d_i}(s_{d_i})\Big|(a_i, z_{a_i}^{d_i}(c_{\leq a_i}^{a_i,-i}), d_i), k_{d_i}=0\right], \dots \\
&\dots, E_{s_{d_i}}\left[V_{d_i}(s_{d_i})\Big|(a_i, z_{d_i-1}^{d_i}(c_{\leq d_i-1}^{d_i-1,-i}), d_i), k_{d_i}=0\right], E_{s_{d_i}}\left[V_{d_i}(s_{d_i})\Big|H_{d_i-1}^{-i}, k_{d_i}=0\right]\Big\} \\
=&E_{s_{d_i}}\Big[\max\Big\{z_{a_i}^{d_i}(c_{\leq a_i}^{a_i,-i}), \dots, z_{d_i-1}^{d_i}(c_{\leq d_i-1}^{d_i-1,-i}), \\
&c_{\leq d_i}^{d_i}, E_{s_{d_i+1}}\left[V_{d_i+1}(s_{d_i+1})\Big|H_{d_i}^{-i}, k_{d_i+1}=0\right]\Big\}\Big|H_{d_i-1}^{-i}\Big] \\
=&E_{H_{d_i}}\left[\zeta_{a_i}^{d_i}(H_{d_i}^{-i}, 1)\Big|H_{d_i-1}^{-i}\right].
\end{aligned}$$

As $\left[\zeta_{a_i}^{d_i-1}(H_{d_i-1}^{-i})\Big|H_{d_i-1}^{-i}\right]$ is deterministic for each H_{d_i-1},

$$\left[\zeta_{a_i}^{d_i-1}(H_{d_i-1}^{-i})\Big|H_{d_i-1}^{-i}\right] \succ_{\mathrm{SSD}} \left[\zeta_{a_i}^{d_i}(H_{d_i}^{-i})\Big|H_{d_i-1}^{-i}\right]$$

and the lemma follows. □

Proof of Theorem 3. Consider a buyer i with type (a, v, d), where $a < d \leq T$ and let $d' \in \{a, \dots, d-1\}$. Fix the state in the arrival period s_a, and let

$$G(\zeta) = \mathrm{Prob}\left\{\zeta_a^d(H_d, k_a) \leq \zeta\Big|s_a\right\},$$
$$\text{and} \qquad G'(\zeta) = \mathrm{Prob}\left\{\zeta_a^{d'}(H_{d'}, k_a) \leq \zeta\Big|s_a\right\}.$$

Lemma 2 implies that G and G' have the same mean and $G' \succ_{\mathrm{SSD}} G$.

(i) Suppose that $J_a(v)$ is convex. Conditional on s_a we have for $v > v_0^a$

$$\begin{aligned}
U_a(v,d) &= \int_0^{J_a(v)} (v - J_a^{-1}(\zeta))dG(\zeta), \\
&= (v - J_a^{-1}(0))G(0) + \int_0^{J_a(v)} \frac{d}{d\zeta}J_a^{-1}(\zeta)\, G(\zeta)d\zeta, \\
&\geq (v - J_a^{-1}(0))G'(0) + \int_0^{J_a(v)} \frac{d}{d\zeta}J_a^{-1}(\zeta)\, G'(\zeta)d\zeta = U_a(v,d')
\end{aligned}$$

The last line follows because (a) $v \geq J_a^{-1}(0) = v_0^a$ and $G'(0) \leq G(0)$ by SSD, and (b) because $\frac{d}{d\zeta} J_a^{-1}(v)$ is non-negative and non-increasing and for all non-negative and non-increasing functions $\phi : [0, \overline{v}] \to \mathbb{R}$, we have

$$\forall x \in [0, \overline{v}] : \qquad \int_0^x \phi(s) G'(s) ds \leq \int_0^x \phi(s) G(s) ds.$$

For $\phi(s) = \mathbf{1}_{\{s \leq x\}}$ this follows directly from SSD and since any non-increasing function $\phi : [0, \overline{v}] \to \mathbb{R}$ can be uniformly approximated by non-increasing step functions the result follows.

(ii) Suppose that $v = \overline{v}$ and that $J_a(v)$ is strictly concave. Conditional on s_a the expected payoff of i is given by

$$\begin{aligned} U_a(\overline{v}, d) &= \int_0^{\overline{v}} (\overline{v} - J_a^{-1}(\zeta)) dG(\zeta), \\ &< \int_0^{\overline{v}} (\overline{v} - J_a^{-1}(\zeta) dG'(\zeta) = U_a(\overline{v}, d'). \end{aligned}$$

In the second line we have used strict convexity of $J_a^{-1}(\zeta)$ as a function of ζ. □

Appendix B. Proof of Theorem 6

It will be convenient to make the changes of variables $t_1 = F_1(v_1)$ and $t_2 = F_2(v_2)$. Defining $v_1(t_1) := F_1^{-1}(t_1)$ and $v_2(t_2) := F_2^{-1}(t_2)$, we have

$$\begin{aligned} t_i &\sim U[0,1] \quad \text{for } i = 1, 2, \\ v_1'(t_1) &= \frac{1}{f_1(v_1(t_1))}, \\ \text{and} \quad v_2'(t_2) &= \frac{1}{f_2(v_2(t_2))}, \end{aligned}$$

Furthermore, for $i = 1, 2$ we introduce

$$\begin{aligned} q_i(t) &= q_i(v_i(t), 2), \\ U(t) &= U_1(v_1(t), 2), \\ M_1(t) &= J_1(v_1(t)) = v_1(t) - (1-t) v_1'(t) \\ M_2(t) &= J_2(v_2(t)) = v_2(t) - (1-t) v_2'(t) \\ t_1^0 &= F_1(v_1^0). \\ \text{and} \qquad t_2^0 &= F_2(v_2^0). \end{aligned}$$

The objective of the seller becomes

$$R[q_1, q_2] := \int_0^1 q_1(t) M_1(t) + q_2(t) M_2(t) dt. \tag{B.1}$$

The following Theorem formulates the feasibility constraint in terms of q.[22]

[22] The characterization is a generalization of Border's (1991) characterization for symmetric allocation rules. Matthews (1984) conjectured the result proved by Border (see also Chen, 1986). For an early application of a special case of the result see Maskin and Riley (1984).

Theorem 7 (Mierendorff, 2011). *For $i = 1, 2$, let $q_i : [0,1] \to [0,1]$ be non-decreasing. (q_1, q_2) is the reduced form of a feasible allocation rule if and only if for all $t_1, t_2 \in [0,1]$,*

$$\int_{t_1}^{1} q_1(t)dt + \int_{t_2}^{1} q_2(t)dt \leq 1 - t_1 t_2.$$

Now we can restate $\mathcal{P}_2$ as $\mathcal{P}_2'$:

$$\pi_2(\bar{U}) = \sup_{(q_1,q_2)} R[q_1, q_2] \tag{$\mathcal{P}_2'$}$$

subject to

$$\begin{aligned}
&\forall t \in [0,1]: && q_i(t) \in [0,1], && \text{(B.2)}\\
&\forall t > t', && q_i(t) \geq q_i(t'), && \text{(B.3)}\\
&\forall t_1, t_2 \in [0,1]: && \int_{t_1}^{1} q_1(\theta)d\theta + \int_{t_2}^{1} q_2(\theta)d\theta \leq 1 - t_1 t_2, && \text{(B.4)}\\
&\forall t \in [0,1]: && U(t) = \int_0^t q_1(\theta)v_1'(\theta)d\theta, && \text{(B.5)}\\
&\text{and} && U(1) \geq \bar{U}. && \text{(B.6)}
\end{aligned}$$

Using $q_i(F_i(v_i)) = q_i(v_i, 2)$, a solution to $\mathcal{P}_2$ can be derived easily from a solution to $\mathcal{P}_2'$.

We can use the (non-standard) constraint (B.4) to eliminate q_2 from the objective function. For $q_1 : [0,1] \to [0,1]$ non-decreasing, define the *inverse* as

$$q_1^{-1}(t) := \begin{cases} 1 & \text{if } q_1(1) < t, \\ \inf\{\theta \in [0,1] \mid q_1(\theta) \geq t\} & \text{otherwise.} \end{cases}$$

Lemma 4. *Let $q_1 : [0,1] \to [0,1]$ be non-decreasing. Then an optimal solution to*

$$\sup_{q_2} \int_0^1 q_2(t)M_2(t)dt \qquad \textit{subject to (B.2)–(B.4)},$$

is given by

$$q_2^*(t) = \begin{cases} q_1^{-1}(t) & \textit{if } t \geq t_2^0, \\ 0 & \textit{otherwise.} \end{cases}$$

The solution is unique for almost every t.

Proof. The proof can be found in the supplementary appendix. □

Using Lemma 4, (B.1) becomes

$$\int_0^1 q_1(t)M_1(t)dt + \int_{t_2^0}^1 q_1^{-1}(t)M_2(t)dt. \tag{B.7}$$

If q_1 is absolutely continuous, substituting $s = q_1(t)$ in the second integral yields

$$\int_0^1 q_1(t)M_1(t) + tq_1'(t)\tilde{M}_2(q_1(t))dt + \int_{q(1)}^1 \tilde{M}_2(t)dt, \tag{B.8}$$

where we define $\tilde{M}_2(t) := \max\{0, M_2(t)\}$.

Monotonicity implies some regularity of q_1. In particular $q_1 = q_1^C + q_1^J$ where q_1^C is a continuous function and q_1^J is a pure jump function. This leaves two problems unresolved. Firstly, we have to deal with jumps and secondly, absolute continuity of q_1^C is not guaranteed. To deal with this we restrict q_1 to be globally Lipschitz continuous with constant K,

$$q_1 \in \mathcal{L}^K := \{q : [0,1] \to [0,1] \mid \forall t, t' \in [0,1] : \; |q(t) - q(t')| \leq K|t - t'|\}.$$

We define the maximization problem $\mathcal{P}_2^K$ as $\mathcal{P}_2'$ subject to the additional constraint $q_1 \in \mathcal{L}^K$. The set of winning probabilities that satisfy (B.4) is weakly-compact (cf. Mierendorff (2011) and Border (1991)). Since $\mathcal{L}^K$ is sequentially compact standard arguments can be used to prove existence.

Theorem 8. *(a) An optimal solution of $\mathcal{P}_2'$ exists.*
(b) For every $K > 0$, an optimal solution of $\mathcal{P}_2^K$ exists.

Proof. The proof can be found in the supplementary appendix. □

The next step is to show that Lipschitz solutions converge to the general solution if K tends to infinity. The proof is based on Reid (1968).

Lemma 5. *Let $(q_1^n, q_2^n)_{n\in\mathbb{N}}$ a sequence of optimal solutions of $\mathcal{P}_2^{K_n}$ where $K_n \to \infty$ as $n \to \infty$. Then, there exists a solution (q_1, q_2) of $\mathcal{P}_2'$ and a sub-sequence $(q_1^{n_j}, q_2^{n_j})_{j\in\mathbb{N}}$ such that $q_i^{n_j}(t) \overset{j\to\infty}{\longrightarrow} q_i(t)$ for almost every t and $R[q_1, q_2] = \pi_2(\bar{U})$.*

Proof. After taking a sub-sequence, we can assume that (q_1^n, q_2^n) converges a.e. to a solution $(\hat{q}_1, \hat{q}_2)$ of $\mathcal{P}_2'$ (see proof of Theorem 8). To show optimality of $(\hat{q}_1, \hat{q}_2)$, let (q_1, q_2) be an optimal solution of $\mathcal{P}_2'$. We can extend q_1 to $\mathbb{R}$ by setting $q_1(t) = 0$ if $t < 0$ and $q_1(t) = 1$ if $t > 1$. Define $q_{d,1} : \mathbb{R} \to [0,1]$ as

$$q_{d,1}(t) := \frac{1}{2d} \int_{t-d}^{t+d} q_1(s)ds.$$

By the Lebesgue differentiation theorem $q_{d,1}(t) \to q_1(t)$ for almost every $t \in [0,1]$ as $d \to 0$. Since q_1 is non-decreasing and $q_1(t) \in [0,1]$, $q_{d,1}$ also has these properties. Furthermore $q_{d,1} \in \mathcal{L}^{\frac{1}{2d}}$:

$$\begin{aligned}
\forall t > t' : \qquad 0 \leq q_{d,1}(t) - q_{d,1}(t') &= \frac{1}{2d}\left(\int_{t-d}^{t+d} q_1(s)ds - \int_{t'-d}^{t'+d} q_1(s)ds\right) \\
&= \frac{1}{2d}\left(\int_{t'+d}^{t+d} q_1(s)ds - \int_{t'-d}^{t-d} q_1(s)ds\right) \\
&\leq \frac{1}{2d}\int_{t'+d}^{t+d} q_1(s)ds \\
&\leq \frac{1}{2d}(t - t')
\end{aligned}$$

Since $q_{d,1}$ may violate $\int_0^1 q_{d,1}(t)v_1'(t)dt \geq \bar{U}$, we define $\tilde{q}_{d,1} := \lambda_d + (1-\lambda_d)q_{d,1}$ and

$$\tilde{q}_{d,2}(t) := \begin{cases} \tilde{q}_{d,1}^{-1}(t), & \text{if } M_2(t) \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\lambda_d := \max\left\{0, \frac{\bar{U}-\int_0^1 q_{d,1}(t)v_1'(t)dt}{\overline{v}-\int_0^1 q_{d,1}(t)v_1'(t)dt}\right\}$. For every d, $(\tilde{q}_{d,1}, \tilde{q}_{d,2})$ is a solution of $\mathcal{P}_2^{\frac{1}{2d}}$. λ_d converges to zero as $d \to 0$. By Lemma 4, $q_2(t) = q_1^{-1}(t)$ for a.e. t such that $M_2(t) \geq 0$ and $q_2(t) = 0$ otherwise. Hence, for $i = 1, 2$, $\tilde{q}_{d,i} \to q_i$ almost everywhere as $d \to 0$. By the dominated convergence theorem, $R[\tilde{q}_{d,1}, \tilde{q}_{d,2}] \to R[q_1, q_2]$ and $R[q_1^n, q_2^n] \to R[\hat{q}_1, \hat{q}_2]$. Define $d_n = \frac{1}{2K_n}$. Then, $R[\tilde{q}_{d_n,1}, \tilde{q}_{d_n,2}] \leq R[q_1^n, q_2^n]$ and we have $R[q_1^n, q_2^n] \to R[q_1, q_2]$ and hence $R[\hat{q}_1, \hat{q}_2] = R[q_1, q_2]$. $\square$

In the next section, we derive properties of the Lipschitz solution. Finally, we show that there is a limiting solution that yields the same expected revenue as the solution proposed in Theorem 6.

B.1. Solution on the class $\mathcal{L}^K$

Using Lemma 4, we rewrite $\mathcal{P}_2^K$ as a control problem. The state variables are the expected utility of bidder one, denoted $U(t)$, and the winning probability, denoted $q(t)$, (in the control problem we write q instead of q_1). As q is absolutely continuous, we can use $u(t) = q'(t)$ as a control variable. The objective is defined as

$$R_c[U, q, u] := \int_0^1 q(t)M_1(t) + tu(t)\tilde{M}_2(q(t))dt + \int_{q(1)}^1 \tilde{M}_2(t)dt.$$

where u is a measurable control

$$u : [0,1] \to [0, K]. \tag{B.9}$$

The evolution of the state variables is governed by

$$U'(t) = q(t)v_1'(t), \tag{B.10}$$
$$q'(t) = u(t). \tag{B.11}$$

We impose the state constraint

$$\forall t \in [0,1]: \qquad q(t) \leq 1. \tag{B.12}$$

Furthermore, we impose the following constraints on the start- and endpoints:

$$U(0) = 0, \tag{B.13}$$
$$q(0) \geq 0, \tag{B.14}$$
$$U(1) \geq \bar{U}, \tag{B.15}$$

To summarize, we have the following control problem:

$$\max_{(U,q,u)} R_c[U, q, u], \qquad \text{subject to (B.9)–(B.15).} \tag{$\mathcal{P}_C^K$}$$

(B.10) is (B.5) in differential form. (B.9) and (B.11) ensure that $q \in \mathcal{L}^K$ and non-decreasing. (B.9), (B.11) and (B.14) imply $q(t) \geq 0$ for all t. Hence, we can dispense with a second state constraint.

The Pontriyagin maximum principle yields the following necessary conditions for an optimum.

Theorem 9 (Clarke (1983), pp. 210-212). *Let (U, q, u) be a solution of $\mathcal{P}_C^K$. If (U, q, u) is optimal, there exists $\omega \in \{0, 1\}$, an absolutely continuous function $p : [0, 1] \to \mathbb{R}^2$, the components of which we denote by (p_U, p_q), and a non-negative measure μ on $[0, 1]$, such that the following conditions hold:*

(i) For almost every $t \in [0, 1]$,

$$p_U'(t) = 0, \tag{B.16}$$

$$p_q'(t) = -\omega \left[M_1(t) + tu(t)\tilde{M}_2'(q(t)) \right] - p_U v_1'(t). \tag{B.17}$$

(ii) For almost every $t \in [0, 1]$, $u(t)$ maximizes

$$\left[\omega t \tilde{M}_2(q(t)) + p_q(t) + \mu[0, t) \right] u.$$

(iii) μ is supported on $\{q(t) = 1\}$,

(iv) p satisfies the transversality conditions

$$p_q(0) \leq 0, \qquad (\textit{with equality if } q(0) > 0,)$$

$$p_U(1) \geq 0, \qquad (\textit{with equality if } U(1) > \underline{U},)$$

$$p_q(1) = -\omega \tilde{M}_2(q(1)) - \mu[0, 1].$$

(v) $\omega + \|p\| + \|\mu\| > 0$.

Note that (B.16) implies that p_U is constant. First, we show that trivial solutions do not occur.

Lemma 6 (Non-triviality). *If $\bar{U} < \overline{v}$, $\omega = 1$.*

Proof. Suppose that $\omega = 0$. By (B.17), $p_q'(t) = -p_U v_1'(t)$. By the transversality conditions, $p_U \geq 0$. $p_U = 0$ implies, $p_q'(t) = 0$ and $p_q(t) = p_q(0)$ for all t. $p_U > 0$ implies, $p_q'(t) < 0$ and $p_q(t) < 0$ for all $t > 0$.

Suppose $p_U > 0$. By, the transversality condition this implies $U(1) = \bar{U}$. By (ii), $u(t)$ maximizes $(p_q(t) + \mu[0, t))u$. If $q(0) < 1$, $\mu[0, t) = 0$ for t close to zero and hence $u(t) = 0$. As $\mu[0, t)$ cannot become positive we must have $q(t) = q(0) < 1$ for all t and consequently $\mu[0, 1] = 0$. The transversality condition therefore requires $p_q(1) = 0$, a contradiction. If, however, $q(0) = 1$ we would have $U(1) = \overline{v} > \bar{U}$. Again a contradiction.

Now suppose that $p_U = 0$. If $q(1) < 1$, $\mu[0, 1] = 0$ and by the transversality conditions, $p(t) = 0$ for all t. This implies $\omega + \|p\| + \|\mu\| = 0$, in contradiction to (v). Hence, $q(1) = 1$. Since $p_q(t) = p_q(1)$, we have $p_q(t) = -\mu[0, 1]$. To fulfill (v) we must have $\mu[0, 1] > 0$. $u(t)$ maximizes $(\mu[0, t) - \mu[0, 1])u$. This implies that $u(t) = 0$ if $q(t) < 1$. Hence, we must have $q(t) = 1$ for all $t \in [0, 1]$. This implies $U(1) = \overline{v}$, which cannot be optimal if $\bar{U} < \overline{v}$. $\square$

Defining $M_1^{p_U}(t) := M_1(t) + p_U v_1'(t)$, we can rewrite (B.17) as

$$-p_q'(t) = M_1^{p_U}(t) + tu(t)\tilde{M}_2'(q(t)), \text{ for a. e. } t \in [0,1]. \tag{B.18}$$

Condition (ii) implies that for almost every $t \in [0,1]$,

$$u(t) = K \qquad \text{if } t\tilde{M}_2(q(t)) + p_q(t) > 0, \tag{B.19}$$

$$u(t) \in [0,K] \qquad \text{if } t\tilde{M}_2(q(t)) + p_q(t) + \mu[0,t) = 0, \tag{B.20}$$

$$u(t) = 0 \qquad \text{if } t\tilde{M}_2(q(t)) + p_q(t) + \mu[0,t) < 0. \tag{B.21}$$

In (B.19), $\mu[0,t)$ was omitted because $q(t) < 1$ if $u(t) = K$. Integrating (B.18) yields for $s,t \in [0,1]$:

$$\begin{aligned} p_q(t) &= p_q(s) - \int_s^t M_1^{p_U}(\theta) + \theta u(\theta)\tilde{M}_2'(q(\theta))d\theta \\ &= p_q(s) - \int_s^t M_1^{p_U}(\theta) - \tilde{M}_2(q(\theta))d\theta - t\tilde{M}_2(q(t)) + s\tilde{M}_2(q(s)). \end{aligned} \tag{B.22}$$

If we substitute (B.22) in (B.19)–(B.21) and define $H^{p_U}(t) = \int_0^t M_1^{p_U}(\theta)d\theta$ and $m_q(t) = \int_0^t \tilde{M}_2(q(\theta))d\theta$, we have that for almost every $t \in [0,1]$,

$$u(t) = K \qquad \text{if } p_q(0) + m_q(t) > H^{p_U}(t), \tag{B.23}$$

$$u(t) \in [0,K] \qquad \text{if } p_q(0) + m_q(t) + \mu[0,t) = H^{p_U}(t), \tag{B.24}$$

$$u(t) = 0 \qquad \text{if } p_q(0) + m_q(t) + \mu[0,t) < H^{p_U}(t). \tag{B.25}$$

Lemma 7 (Reid (1968))**.** *Suppose $p_q(0) + m_q(t) = H^{p_U}(t)$ for $t \in \{\underline{t}, \bar{t}\}$, $\underline{t} < \bar{t}$ and $q(t) < 1$ for $t < \bar{t}$. Let $\alpha, \beta \in \mathbb{R}$ and $l(t) = \alpha + \beta t$. If $l(t) \leq H^{p_U}(t)$ for all $t \in [\underline{t}, \bar{t}]$, then $p_q(0) + m_q(t) \geq l(t)$ for all $t \in [\underline{t}, \bar{t}]$.*

Proof. Suppose that $m_q(s) + p_q(0) < l(s)$ for some $s \in (\underline{t}, \bar{t})$. Then there exists $\varepsilon > 0$ and $\underline{t} < t_1 < t_2 < \bar{t}$ such that $m_q(t) + p_q(0) < l(t) - \varepsilon$ for $t \in (t_1, t_2)$, $m_q(t_1) + p_q(0) = l(t_1) - \varepsilon$, and $p_q(0) + m_q(t_2) = l(t_2) - \varepsilon$. This implies that $m_q'(t) = \tilde{M}_2(q(t))$ cannot be constant on (t_1, t_2). On the other hand, $m_q(t) + p_q(0) + \mu[0,t) = m_q(t) + p_q(0) < l(t) - \varepsilon < H(t)$ and hence $u(t) = 0$ for $t \in (t_1, t_2)$, which implies that $m_q'(t)$ is constant, a contradiction. □

An immediate implication of the Lemma is that $p_q(0) + m_q(t) \geq \bar{H}^{p_U}_{[\underline{t},\bar{t}]}(t)$, where $\bar{H}^{p_U}_{[\underline{t},\bar{t}]}(t)$ denotes the convex hull of H^{p_U} restricted to $[\underline{t}, \bar{t}]$, i.e. the greatest convex function $G : [\underline{t}, \bar{t}] \to \mathbb{R}$ such that $G(t) < H^{p_U}(t)$ for all $t \in [\underline{t}, \bar{t}]$. Furthermore, $p_q(0) + m_q(t)$ is convex because q and $\tilde{M}_2$ are non-decreasing. This yields the following

Corollary 1. *Suppose $p_q(0) + m_q(t) \leq H^{p_U}(t)$ for all $t \in [\underline{t}, \bar{t}]$, with equality at the endpoints of the interval and $q(t) < 1$ for $t < \bar{t}$. Then $p_q(0) + m_q(t) = \bar{H}^{p_U}_{[\underline{t},\bar{t}]}(t)$, for all $t \in [\underline{t}, \bar{t}]$.*

If $M_1^{p_U}$ is non-decreasing on $[\underline{t}, \bar{t}]$, then $H^{p_U}(t) = \bar{H}^{p_U}_{[\underline{t},\bar{t}]}(t)$. Differentiating $p_q(0) + m_q(t) = \bar{H}^{p_U}_{[\underline{t},\bar{t}]}$ yields $M_1^{p_U} = \tilde{M}_2(q(t))$ for $t \in [\underline{t}, \bar{t}]$.

If, however, $M_1^{p_U}$ is not monotonic on $[\underline{t},\overline{t}]$, differentiating yields $\bar{M}_{[\underline{t},\overline{t}]}^{p_U}(t) = \tilde{M}_2(q(t))$, where $\bar{M}_{[\underline{t},\overline{t}]}^{p_U} = \frac{d\bar{H}_{[\underline{t},\overline{t}]}^{p_U}(t)}{dt}$ is non-decreasing. Hence, Reid's Lemma provides a control theoretic technique to show that Myerson's ironing procedure can be used to solve irregular instances of mechanism design problems.

Now we establish some properties of the optimal solution. Define

$$x_{p_U}(t) = \begin{cases} 0, & \text{if } M_1^{p_U}(t) < M_2(0), \\ M_2^{-1}(M_1^{p_U}(t)), & \text{if } M_1^{p_U}(t) \in [M_2(0), \overline{v}], \\ 1, & \text{if } M_1^{p_U}(t) > \overline{v}, \end{cases}$$

$$\text{and} \qquad x_{p_U}^{[\underline{t},\overline{t}]}(t) = \begin{cases} 0, & \text{if } \bar{M}_{[\underline{t},\overline{t}]}^{p_U}(t) < M_2(0), \\ M_2^{-1}(\bar{M}_{[\underline{t},\overline{t}]}^{p_U}(t)), & \text{if } \bar{M}_{[\underline{t},\overline{t}]}^{p_U}(t) \in [M_2(0), \overline{v}], \\ 1, & \text{if } \bar{M}_{[\underline{t},\overline{t}]}^{p_U}(t) > \overline{v}. \end{cases}$$

The derivative of x_{p_U} is given by

$$x'_{p_U}(t) = \frac{M_1'(t) + p_U v''(t)}{M_2'(x_{p_U}(t))}.$$

The assumptions on f_i and F_i guarantee that $x'_{p_U}(t)$ is continuous on $[0,1]$. Let $K^{p_U} := \max_{t\in[0,1]} x'_{p_U}(t)$. Then $x_{p_U} \in \mathcal{L}^{K^{p_U}}$. In what follows, we write $\bar{H}^{p_U}$ for $\bar{H}_{[0,1]}^{p_U}$ and $\bar{M}_1^{p_U}$ for $\bar{M}_{[0,1]}^{p_U}$.

Lemma 8 (interior solution)**.** *Suppose $u(t) \in (0,K)$ for a.e. $t \in [\underline{t},\overline{t}]$, $\underline{t} < \overline{t}$. Then for all $t \in [\underline{t},\overline{t}]$,*

(i) $q(t) = x_{p_U}(t)$ if $q(t) \geq t_2^0$,

(ii) $M_1^{p_U}(t) = 0$ if $q(t) < t_2^0$.

Proof. If $u(t) > 0$, we must have $\mu[0,t) = 0$. (B.23) – (B.25) imply that $p_q(0) + m_q(t) = H^{p_U}(t)$ for all$t \in (\underline{t},\overline{t})$. Differentiating this w.r.t. t yields

$$\tilde{M}_2(q(t)) = M_1^{p_U}(t).$$

If $q(t) \geq t_2^0$, $\tilde{M}_2(q(t)) = M_2(q(t))$ and hence that $q(t) = x_{p_U}(t)$. If $q(t) < t_2^0$, $\tilde{M}_2(q(t)) = 0$ and hence $M_1^{p_U}(t) = 0$. By continuity, the results extend to $\underline{t}$ and $\overline{t}$. □

Next, we derive necessary conditions for intervals where $u(t)$ is in $\{0, K\}$.

Lemma 9 (constant q)**.** *Suppose $q(t) = a \in [0,1]$ on $[\underline{t},\overline{t}]$, $\underline{t} < \overline{t}$, and let $[\underline{t},\overline{t}]$ be chosen maximally. Then*

$$p_q(t) + t\tilde{M}_2(q(t)) = 0,$$
$$p_q(0) + m_q(t) = H^{p_U}(t),$$

for $t = \underline{t}$ if $\underline{t} > 0$ and for $t = \overline{t}$ if $\overline{t} < 1$, and furthermore

$$M_1^{p_U}(\underline{t}) \geq \tilde{M}_2(a), \qquad \text{if } \underline{t} > 0, \tag{B.26}$$

$$\text{and} \qquad M_1^{p_U}(\bar{t}) \leq \tilde{M}_2(a), \qquad \text{if } \bar{t} < 1. \tag{B.27}$$

Proof. If $q(t)$ is constant, then for almost every $t \in (\underline{t}, \bar{t})$, $u(t) = 0$ and therefore $p_q(t) + t\tilde{M}_2(q(t)) + \mu[0,t) \leq 0$ and $p_q(0) + m_q(t) + \mu[0,t) \leq H^{p_U}(t)$. As $\mu \geq 0$ and by continuity, $p_q(t) + t\tilde{M}_2(q(t)) \leq 0$ and $p_q(0) + m_q(t) \leq H^{p_U}(t)$ for $t \in \{\underline{t}, \bar{t}\}$.

Suppose $\underline{t} > 0$ and let $S_- := \{0 < t < \underline{t} \mid u(t) > 0\}$. Since $q(t) < a$ for $t < \underline{t}$, and q is absolutely continuous, $S_- \cap [\underline{t} - \delta, \underline{t}]$ has positive measure for every $\delta > 0$. Hence, there exists a sequence $t_n \nearrow \underline{t}$ with $p_q(t_n) + t_n\tilde{M}_2(q(t_n)) \geq 0$ and $p_q(0) + m_p(t_n) \geq H^{p_U}(t_n)$ for all n. By continuity, the first two equalities in the Lemma follow for $\underline{t} > 0$. For $\bar{t} < 1$ set $S_+ := \{\bar{t} < t < 1 \mid u(t) > 0\}$. $S_+ \cap [\bar{t}, \bar{t} + \delta]$ has positive measure for every $\delta > 0$. Hence, there exists a sequence $t_n \searrow \bar{t}$ with $p_q(t_n) + t_n\tilde{M}_2(q(t_n)) \geq 0$ and $p_q(0) + m_p(t_n) \geq H^{p_U}(t_n)$ for all n. By continuity, the first two equations in the Lemma follow for $\bar{t} < 1$.

To show (B.26), note that for almost every $t \in S_-$, $p_q(t) + t\tilde{M}_2(q(t)) \geq 0$. (B.22) yields

$$p_q(\underline{t}) = p_q(t) - \int_t^{\underline{t}} M_1^{p_U}(\theta) - \tilde{M}_2(q(\theta))d\theta - \underline{t}\tilde{M}_2(q(\underline{t})) + t\tilde{M}_2(q(t)).$$

With $p_q(\underline{t}) = -\underline{t}\tilde{M}_2(q(\underline{t}))$ and $p_q(t) + t\tilde{M}_2(q(t)) \geq 0$ this implies

$$\int_t^{\underline{t}} M_1^{p_U}(\theta) - \tilde{M}_2(q(\theta))d\theta \geq 0,$$

for almost every $t \in S_-$. If this inequality is fulfilled, there must be a $t' \in [t, \underline{t}]$ with

$$M_1^{p_U}(t') - \tilde{M}_2(q(t')) \geq 0.$$

As $S_- \cap [\underline{t} - \delta, \underline{t}]$ has positive measure for every $\delta > 0$, t and hence t' can be chosen arbitrarily close to $\underline{t}$. By continuity this implies

$$M_1^{p_U}(\underline{t}) - \tilde{M}_2(q(\underline{t})) \geq 0.$$

To show (B.27), note that for almost every $t \in S_+$, $p_q(t) + t\tilde{M}_2(q(t)) \geq 0$. (B.22) yields

$$p_q(t) = p_q(\bar{t}) - \int_{\bar{t}}^{t} M_1^{p_U}(\theta) - \tilde{M}_2(q(\theta))d\theta - t\tilde{M}_2(q(t)) + \bar{t}\tilde{M}_2(q(\bar{t})).$$

With $p_q(\bar{t}) = -\bar{t}\tilde{M}_2(q(\bar{t}))$ and $p_q(t) + t\tilde{M}_2(q(t)) \geq 0$ this implies

$$\int_{\bar{t}}^{t} M_1^{p_U}(\theta) - \tilde{M}_2(q(\theta))d\theta \leq 0,$$

for almost every $t \in S_+$. As above there exists $t' \in [\bar{t}, t]$ such that the integrand is non-positive at t'. t and t' can be chosen arbitrarily close to $\bar{t}$. Therefore, by continuity

$$M_1^{p_U}(\bar{t}) - \tilde{M}_2(q(\bar{t})) \leq 0.$$

□

Lemma 9 implies that there cannot be an interval where q is constant and $q \in (0,1)$ if x_{p_U} is strictly increasing.

Lemma 10. *Suppose $u(t) = K$ for almost every $t \in (\underline{t}, \bar{t})$, $\underline{t} < \bar{t}$. Let $(\underline{t}, \bar{t})$ be chosen maximally. Then for $t = \underline{t}$ and for $t = \bar{t}$ if $\bar{t} < 1$,*

$$p_q(t) + t\tilde{M}_2(q(t)) = 0,$$

for $t = \underline{t}$ if $\underline{t} > 0$ and for $t = \bar{t}$ if $\bar{t} < 1$

$$p_q(0) + m_q(t) = H^{p_U}(t).$$

Furthermore,

$$M_1^{p_U}(\underline{t}) \leq \tilde{M}_2(q(\underline{t})), \quad \text{if } \underline{t} > 0, \tag{B.28}$$

$$\text{and} \quad M_1^{p_U}(\bar{t}) \geq \tilde{M}_2(q(\bar{t})), \quad \text{if } \bar{t} \in [0, 1]. \tag{B.29}$$

Proof. The proof is very similar to the proof of the preceding Lemma. To show the first equality for $\underline{t} = 0$, the transversality condition can be used to obtain $p_q(0) \leq 0$. For $\bar{t} = 1$, (B.29) follows from $M_1^{p_U}(1) \geq \bar{v}$ and $\tilde{M}_2(q(t)) \leq \bar{v}$. □

Setting $q(t) = x_0(t)$ for $t \geq t_1^0$ and $q(t) = 0$ otherwise, yields the optimal solution of Myerson (1981). This is not surprising because p_U would be zero if the incentive compatibility constraint for the deadline were ignored. The following Lemma, which does not depend on the maximum principle, excludes solutions that have lower winning probabilities than the undistorted solution x_0.

Lemma 11. *For $K > K^0$, let $b \geq t_1^0$ be the unique solution to $(b - t_1^0)K = x_0(b)$. If $q(t) \leq x_0(t)$ for all $t \in [t_1^0, 1]$ and $q(t) < x_0(t)$ for some $t \in [b, 1]$, then q is not optimal.*

Proof. Suppose by contradiction that q is an optimal solution with the properties stated in the Lemma. Let $b' \in [0, b]$ be the unique solution to $q(t_1^0) + (b' - t_1^0)K = x_0(b')$. Define

$$\tilde{q}(t) = \begin{cases} q(t), & \text{if } t < t_1^0, \\ q(t_1^0) + (t - t_1^0)K, & \text{if } t \in [t_1^0, b'], \\ x_0(t), & \text{if } t > b'. \end{cases}$$

Obviously, $\tilde{q} \in \mathcal{L}^K$ and $\tilde{U}(1) \geq \bar{U}$. Since x_0 is the optimal solution absent constraints, $\tilde{q}$ yields higher revenue than q. This contradicts the optimality of q. □

Lemma 12. *If $\bar{U} < \bar{v}$, then $p_U \leq \bar{p}_U := 1 + \max_{t \in [0,1]} \frac{\bar{v} - v_1(t)}{v_1'(t)} < \infty$.*

Proof. Suppose to the contrary that, $p_U > \bar{p}_U$. Then $M_1^{p_U}(t) > \tilde{M}_2(1) = \bar{v}$ for all $t \in [0, 1]$. By Lemma 8.ii, this implies $q(\underline{t}) \geq t_2^0$ if $u(t) \in (0, K)$ on a maximal interval $[\underline{t}, \bar{t}]$. By Lemma 8.i, this implies $q(t) = x_{p_U}(t)$, for all $t \in [\underline{t}, \bar{t}]$, but this contradicts $u(t) > 0$ if $M_1^{p_U}(t) > \bar{v}$. Hence we have $u(t) \in \{0, K\}$ for all $t \in [0, 1]$.

Suppose $u(t) = 0$ on a maximal interval $[\underline{t}, \bar{t}]$. By Lemma 9, this implies $\bar{t} = 1$. If $u(t) = K$ on a maximal interval $[\underline{t}, \bar{t}]$, Lemma 10 implies $\underline{t} = 0$. Therefore, there exists $a \in [0, 1]$ such that $u(t) = K$ for $t < a$ and $u(t) = 0$ for $t > a$. Suppose $a > 0$. Lemma 10 implies $p_q(0) = 0$ if $a > 0$. As $M_1^{p_U}(t) > \tilde{M}_2(q(t))$ for all t, we have $p_q(t) + m_q(t) < H^{p_U}(t)$ for all $t > 0$. Hence, $u(t) = 0$ for all $t > 0$ and $a = 0$.

If $q(t) = q$ is constant, Lemma 11 implies that $q > t_2^0$. Therefore, $p_q(0) = 0$ by the transversality condition. Using (B.22), we get $p_q(1) = -\int_0^1 M_1^{p_U}(t)dt < 0$. The transversality condition and $p_U > 0$ imply $U(1) = \bar{U}$. This yields $q = \frac{\bar{U}}{\overline{v}}$. If $q < 1$, then $\mu[0,1] = 0$, and hence, $p_q(1) = -\tilde{M}_2(q(1)) > -\int_0^1 M_1^{p_U}(t)dt$ by the transversality condition. So we must have $q = 1$ and hence $\bar{U} = \overline{v}$, which is ruled out by assumption. $\square$

Note that $|x'_{p_U}(t)| \leq \frac{M_1'(t)+p_U|v_1'(t)|}{\min_{x\in[0,1]}|M_2'(x)|}$. Defining $\overline{K} := \max_{t\in[0,1]} \frac{M_1'(t)+\bar{p}_U|v_1'(t)|}{\min_{x\in[0,1]}|M_2'(x)|}$ we have $x_{p_U} \in \mathcal{L}^{\overline{K}}$ for all $p_U \leq \bar{p}_U$.

Lemma 13. *Let $(\underline{t}, \overline{t})$ be a maximal interval such that $u(t) = K$ for all $t \in (\underline{t}, \overline{t})$ and $K > \overline{K}$. Then $q(t) < \max\{t_2^0, x_{p_U}(t)\}$ for all $t \in [\underline{t}, \overline{t})$. If $\underline{t} > 0$, then $q(\underline{t}) < t_2^0$. Furthermore $\overline{t} < 1$.*

Proof. If $q(t) \geq \max\{t_2^0, x_{p_U}(t)\}$, then $q(\overline{t}) > \max\{t_2^0, x_{p_U}(\overline{t})\}$ because $K > \overline{K}$. Hence $\tilde{M}_2(q(\overline{t})) > M_1^{p_U}(\overline{t})$, a contradiction by Lemma 10. If $\underline{t} > 0$, $q(\underline{t}) < t_2^0$ because otherwise (B.28) and $K > \overline{K}$ would imply $q(\overline{t}) \geq \max\{t_2^0, x_{p_U}(\overline{t})\}$, which is a contradiction. Finally, $\overline{t} = 1$ would imply $q(t) < x_0(t)$ for all $t \in [t_1^0, 1)$. This is also a contradiction by Lemma 11. $\square$

Lemma 14. *For $K > K^0$, $q(1) = 1$.*

Proof. Suppose $q(1) < 1$. By Lemma 11, $q(1) > t_2^0$. By the transversality condition, $p_q(1) = -\tilde{M}_2(q(1))$. Differentiating $p_q(t) + t\tilde{M}_2(q(t))$, we get $\frac{d}{dt}(p_q(t) + t\tilde{M}_2(q(t))) = p_q'(t) + \tilde{M}_2(q(t)) + t\tilde{M}_2'(q(t))q'(t) = \tilde{M}_2(q(t)) - M_1^{p_U}(t)$. As $q(1) < x_{p_U}(1)$ we have $\frac{d}{dt}(p_q(t) + t\tilde{M}_2(q(t))) < 0$, and thus $p(t) + t\tilde{M}_2(q(t)) > 0$ for t sufficiently close to one. Hence $u(t) = K$ on a maximal interval $[\underline{t}, 1]$. As $K > K^0$, $\underline{t} > 0$ and hence $q(\underline{t}) < t_2^0$ by Lemma 13. This contradicts optimality by Lemma 11. $\square$

Define $c := \min\{t \,|\, q(t) = 1\}$. By the preceding Lemma, this is well defined for $K > K^0$.

Lemma 15. *For $K > \overline{K}$,*

$$p_q(0) + m_q(c) = H^{p_U}(c),$$
$$p_q(c) + c\tilde{M}_2(q(c)) = 0,$$
$$M_1^{p_U}(c) = \tilde{M}_2(1).$$

Proof. If $c < 1$ the first two equations are implied by Lemma 9. If $c = 1$, $u(t) \notin \{0, K\}$ for a set of types with positive measure, arbitrarily close to one. ($u(t) = 0$ is ruled out by $c = 1$, $u(t) \neq K$ follows from the same argument as in the proof of Lemma 14). Hence, the first two equalities hold for t close to c and by Lemma 8 also the third equality holds for t close to c. By continuity the equalities also hold for c. If $c < 1$, $M_1^{p_U}(c) \geq \tilde{M}_2(q(c))$ by Lemma 9. For $K > \overline{K}$, $u(t) = K$ for a maximal interval $[\underline{t}, c]$ is not possible as Lemma 10 requires $M_1^{p_U}(\underline{t}) \leq \tilde{M}_2(q(\underline{t}))$. Hence $u(t) \notin \{0, K\}$ for a set of types with positive measure, arbitrarily close to c. By Lemma 8 and continuity, the third equality follows for c. $\square$

Lemma 16. *Let (U, q, u) be an optimal solution to $\mathcal{P}_C^K$ for $K > \overline{K}$.*

(i) Let $\underline{b} = \min\{q(t) \geq t_2^0\}$. *Then there exists* $\overline{b} \in [\underline{b}, c]$ *such that* $u(t) = K$ *for* $t \in [\underline{b}, \overline{b}]$, *and* $\tilde{M}_2(q(t)) = \bar{M}^{p_U}_{[\overline{b},1]}(t)$ *for* $t \in [\overline{b}, c]$. *Furthermore,* $c = \min\{t \mid \bar{M}^{p_U}_{[\overline{b},1]}(t) = \overline{v}\}$.

(ii) Let $\overline{t}_1^0 := \max\{t \mid \bar{M}_1^{p_U}(t) \leq 0\}$ *and* $\overline{t}_1^0 = 0$ *if* $\bar{M}_1^{p_U}(0) > 0$. *Then* $\underline{b} \to \overline{t}_1^0$ *and* $\overline{b} \to \overline{t}_1^0$ *as* $K \to \infty$.

(iii) For almost every $t < \underline{b}$,

$$u(t) \quad \begin{cases} = 0, & \textit{if } p_q(0) < H^{p_U}(t), \\ \in [0, K], & \textit{if } p_q(0) = H^{p_U}(t), \\ = K, & \textit{if } p_q(0) > H^{p_U}(t). \end{cases}$$

Proof. (iii) follows directly from (B.23)–(B.25) as $q(t) \leq t_2^0$ for $t < \underline{b}$ and hence $m_q(t) = 0$.

If $p_q(0) < H^{p_U}(t)$ for all $t \in [0,1]$, then $p_q(0) < 0$ and therefore $q(0) = 0$ by the transversality condition. Hence $p_q(0)+m_q(t) < H^{p_U}(t)$, and $q(t) = 0$ for all t, contradicting Lemma 11. Therefore $p_q(0) \geq \min_t H^{p_U}(t)$.

To show (i), we first show that $\tilde{M}_2(q(t)) = \bar{M}^{p_U}_{[\overline{b},c]}(t)$ for all $t \in [\overline{b}, c]$. Three cases have to be considered. To do this we need the following definitions:

$$\underline{p}_q := \begin{cases} \min\{p_q \mid \lambda\{H^{p_U}(t) \leq p_q\}K \geq t_2^0\}, & \text{if } \lambda\{H^{p_U}(t) \leq 0\}K \geq t_2^0, \\ 0, & \text{otherwise,} \end{cases}$$

$$b^{\max} := \max\{b \mid \underline{p}_q \geq H^{p_U}(b)\},$$

where λ denotes the Lebesgue measure on $[0,1]$.

Case 1: $H^{p_U}(t) > 0$ for all $t > 0$. ($\Rightarrow \underline{p}_q = 0$, $b^{\max} = 0$)

In this case, $q(0) \geq t_2^0$. Otherwise $p_q(0) + m_q(t) < H^{p_U}(t)$ for all $t > 0$. This would imply $q(1) = q(0) < 1$, a contradiction. Suppose $u(t) = K$ for a maximal interval $[\underline{t}, \overline{t}]$. By Lemma 13, $\underline{t} > 0$ would imply $q(\underline{t}) < t_2^0$. Hence $\underline{t} = 0$. Also by Lemma 13, $q(t) \leq x_{p_U}(t)$ for all $t \in [\underline{t}, \overline{t}]$ and hence $q(0) < x_{p_U}(0)$. This implies $p_q(0) + m_q(t) < H^{p_U}(t)$ for t close to zero, contradicting $u(t) = K$. Hence $u(t) < K$ for all $t \in [0,1]$. This requires $p_q(0)+m_q(t) \leq H(t)$ for all t by (B.23)–(B.25), and by Reid's Lemma, we have $\tilde{M}_2(q(t)) = M^{p_U}_{[0,c]}(t)$ for all $t \in [0, c]$. With $\underline{b} = \overline{b} = 0$, this shows $\tilde{M}_2(q(t)) = \bar{M}^{p_U}_{[\overline{b},c]}(t)$ for all $t \in [\overline{b}, c]$ in case 1.

Case 2: $H^{p_U}(t) \leq 0$ for some $t > 0$ and $M_1^{p_U}(b^{\max}) = 0$.

In this case, $q(b^{\max}) = t_2^0$. Suppose to the contrary that $q(b^{\max}) < t_2^0$. This implies $p_q(0) \leq \underline{p}_q$. Hence $p_q(0) + m_q(t) \leq \underline{p}_q < H^{p_U}(t)$ for all $t > b^{\max}$. This is a contradiction to optimality. Next, suppose that $q(b^{\max}) > t_2^0$. This implies $p_q(0) \geq \underline{p}_q$ and therefore $p_q(0) + m_q(b^{\max}) > H^{p_U}(b^{\max})$. Therefore $b^{\max}$ is contained in an interval $[\underline{t}, \overline{t}]$ where $u(t) = K$. By Lemma 13, this is a contradiction. Therefore $q(b^{\max}) = t_2^0$. By (iii) we must have $p_q(0) = \underline{p}_q$ and hence $p_q(0) + m_q(b^{\max}) = \underline{p}_q = H^{p_U}(b^{\max})$. Set $\underline{b} = \overline{b} = b^{\max}$. Lemma 13 also implies that $p_q(0)+m_q(t) \leq H^{p_U}(t)$ for all $t \in [b^{\max}, c]$. Reid's Lemma then implies that $\tilde{M}_2(q(t)) = \bar{M}^{p_U}_{[\overline{b},c]}(t)$ for all $t \in [\underline{b}, c]$ for case 2.

Case 3: $H^{p_U}(t) \leq 0$ for some $t > 0$ and $M_1^{p_U}(b^{\max}) > 0$.

In this case, $q(b^{\max}) > t_2^0$ because otherwise $q(1) = q(b^{\max}) < 1$, which is a contradiction.

This implies $\underline{b} < b^{\max}$ and $p_q(0) \geq \underline{p}_q$. Since $p_q(0) \geq \underline{p}_q$, $p_q(0) + m_q(b^{\max}) > H(b^{\max}) = \underline{p}_q$. Hence $b^{\max}$ is in the interior of a maximal interval $[\underline{t}, \overline{t}]$ such that $u(t) = K$ for all $t \in [\underline{t}, \overline{t}]$. By Lemma 13, $q(\underline{t}) < t_2^0$. This implies that $\underline{b} \in (\underline{t}, b^{\max})$. By Lemma 10, $p_q(0) + m_q(\overline{t}) = H(\overline{t})$ and by Lemma 13, $p_q(0) + m_q(t) \leq H(t)$, for $t \in [\overline{t}, c]$. Hence, we can set $\overline{b} = \overline{t}$ and have thus shown $\tilde{M}_2(q(t)) = \bar{M}^{p_U}_{[\overline{b},c]}(t)$ for all $t \in [\overline{b}, c]$ for case 3.

Claim: $\tilde{M}_2(q(t)) = \bar{M}^{p_U}_{[\overline{b},1]}(t)$ for all $t \in [\overline{b}, c]$.
Note that $\bar{M}^{p_U}_{[\overline{b},1]}(c) \leq \bar{M}^{p_U}_{[\overline{b},c]}(c)$. To show the converse, note that as q is constant on $[c, 1]$, $p_q(0) + m_q(t) + \mu[0, t) \leq H^{p_U}(t)$ for a.e. $t \geq c$. This implies

$$p_q(0) + m_q(c) + (t - c)\overline{v} + \mu[c, t) \leq H^{p_U}(c) + \int_c^t M_1^{p_U}(s)ds,$$

$$\Leftrightarrow \qquad \int_c^t M_1^{p_U}(s)ds \geq \overline{v}(t - c) + \mu[c, t). \tag{B.30}$$

If $\bar{M}^{p_U}_{[\overline{b},1]}(c) < \overline{v}$, then $\int_c^t M_1^{p_U}(s)ds = H^{p_U}(t) - H^{p_U}(c) \leq H^{p_U}(t) - \bar{H}^{p_U}(c) < (t - c)\overline{v}$ for some $t > c$. This would contradict (B.30) so we must have $\bar{M}^{p_U}_{[\overline{b},1]}(c) \geq \overline{v} = \bar{M}^{p_U}_{[\overline{b},c]}(c)$. If $\bar{M}^{p_U}_{[\overline{b},c]}(c) = \bar{M}^{p_U}_{[\overline{b},1]}(c)$ we must have $\bar{M}^{p_U}_{[\overline{b},c]}(t) = \bar{M}^{p_U}_{[\overline{b},1]}(t)$ for all $t \in [\overline{b}, c]$. This proves the claim and $c = \min\{t \mid \bar{M}^{p_U}_{[\overline{b},1]}(t) = \overline{v}\}$ follows immediately. Hence we have shown (i).

It remains to show (ii): $\underline{p}_q \to \min_{t\in[0,1]} H^{p_U}(t)$ as $K \to \infty$. This implies that $b^{\max} \to \overline{t}_1^0$. Furthermore $\overline{b} \geq b^{\max} \geq \underline{b}$ and $\underline{b} - \overline{b} < \frac{1}{K}$. Hence $\underline{b} \to \overline{t}_1^0$ and $\overline{b} \to \overline{t}_1^0$ as $K \to \infty$. □

Now we can turn to the limiting solution as $K \to \infty$.

Proof of Theorem 6. The reduced form of $\bar{x}_i$ as defined in (4.3) is

$$\bar{q}_1(v_1, 2) = \begin{cases} 0, & \text{if } \bar{J}_1^{p_U}(v_1) < 0 \\ \underline{x}_1^0 F_2(v_2^0), & \text{if } \bar{J}_1^{p_U}(v_1) = 0 \\ F_2(J_2^{-1}(\bar{J}_1^{p_U}(v_1)), & \text{if } 0 < \bar{J}_1^{p_U}(v_1) \leq \overline{v}, \\ 1, & \text{otherwise,} \end{cases}$$

$$\bar{q}_2(v_2, 1) = \begin{cases} 0, & \text{if } J_2(v_2) < 0, \\ F_1((\bar{J}_1^{p_U})^{-1}(J_2(v_2))), & \text{otherwise.} \end{cases}$$

Changing variables, we have

$$\bar{q}_1(t) = \begin{cases} 0, & \text{if } t < \underline{t}_1^0, \\ \underline{x}_1^0 t_2^0, & \text{if } t \in [\underline{t}_1^0, \overline{t}_1^0], \\ M_2^{-1}(\bar{M}_1^{p_U}(t)), & \text{if } 0 < \bar{M}_1^{p_U}(t) \leq \overline{v}, \\ 1, & \text{otherwise,} \end{cases}$$

$$\bar{q}_2(t) = \begin{cases} 0, & \text{if } M_2(t) < 0, \\ (\bar{M}_1^{p_U})^{-1}(M_2(t)), & \text{otherwise,} \end{cases}$$

where $\underline{t}_1^0 = \min\{t | \bar{M}_1^{p_U}(t) \geq 0\}$.

Obviously, $\bar{q}_2(t) = \bar{q}_1^{-1}(t)$ if $t \geq t_2^0$ and $\bar{q}_2(t) = 0$ otherwise. Therefore, by Lemma 4, we only have to show optimality of $\bar{q}_1$. Let (q_1^n, q_2^n) be a sequence of optimal solutions of $\mathcal{P}_2^{K_n}$ where $\overline{K} < K_n \to \infty$ as $n \to \infty$. Denote the adjoint variables in these solutions by p_U^n and p_q^n, respectively, and let (q_1, q_2) be the a.e.-limit of the sequence. By Theorem 5, (q_1, q_2) is an optimal solution. We will show that $(\bar{q}_1, \bar{q}_2)$ yields the same expected revenue as the limit of any such sequence. Since $\bar{M}^{p_U}_{[\bar{t}_1^0,1]}(t) = \bar{M}_1^{p_U}(t)$ for all $t \in [\bar{t}_1^0, 1]$, Lemma 16 implies that $q_1(t) = \bar{q}_1(t)$ for $t > \bar{t}_1^0$ where $p_U = \lim_{n\to\infty} p_U^n$.

Next we consider the limiting solution for $t < \bar{t}_1^0$. If $\underline{t}_0^1 > 0$, then $q_1(0) = 0$ and $u(t) = 0$ for $t \leq \underline{t}_1^0$ as for $\bar{q}_1$. Now suppose that $\underline{t}_1^0 < \bar{t}_1^0$.

Claim: If $q_1(t)$ is not constant at $t \in [\underline{t}_1^0, \underline{t}_1^0]$, then $H^{p_U}(t) = \min_\theta H^{p_U}(\theta)$.
Suppose to the contrary that $H^{p_U}(t) > \min_\theta H^{p_U}(\theta)$. Then there exist $\varepsilon > 0$ and $\delta > 0$ such that $H^{p_U}(\tau) > \min_\theta H^{p_U}(\theta) + \delta$ for all $\tau \in (t-\varepsilon, t+\varepsilon)$. Since $p_q^n(0) \to \min_\theta H^{p_U}(\theta)$ for $n \to \infty$, there exists $N > 0$ and $\varepsilon' \in (0, \varepsilon)$ such that for all $n > N$, $p_q^n(0) < H^{p_U^n}(\tau)$ for all $\tau \in (t-\varepsilon', t+\varepsilon')$. This implies that q_1^n is constant on $(t-\varepsilon', t+\varepsilon')$ for $n > N$, and hence q_1 is constant on $(t-\varepsilon', t+\varepsilon')$, which is a contradiction. This proves the claim.

Now set $\underline{q}_1^0 = \left[(v_1(\bar{t}_1^0) - v_1(\underline{t}_1^0))\right]^{-1} \int_{\underline{t}_1^0}^{\bar{t}_1^0} q_1(s) v_1'(s) ds$ and let $[\underline{t}, \bar{t}]$ be the interval where $q_1(t) = \underline{q}_1^0$ (if $q_1(t) \neq \underline{q}_1^0$ for all t, set $\underline{t} = \bar{t}$ such that $q_1(t) < \underline{q}_1^0$ if $t < \underline{t}$ and $q_1(t) > \underline{q}_1^0$ if $t > \underline{t}$). With this definition, $q_1(t) < \underline{q}_1^0$ for $t < \underline{t}$ and $q_1(t) > \underline{q}_1^0$ for $t > \underline{t}$, and q_1 is not constant at $\underline{t}$ and $\bar{t}$. The claim implies that $[\underline{t}_1^0, \underline{t}]$ and $[\bar{t}, \bar{t}_1^0]$ are unions of intervals $[a,b]$ such that either $M_1^{p_U}(t) = 0$ for all $t \in [a,b]$, or q_1 is constant on $[a,b]$ and $\int_a^b M_1^{p_U}(t)dt = 0$. Hence, setting $q_1(t) = \underline{q}_1^0$ does not change the value of the objective and by definition of $\underline{q}_1^0$, $U_1(1)$ is left unchanged. Since, $\underline{q}_1^0 = \underline{x}_1^0 t_2^0$, the (q_1, q_2) yields the same expected revenue as $(\bar{q}_1, \bar{q}_2)$. Uniqueness of p_U and $\underline{x}_1^0$ are obvious.

For the proof of (ii) and (iii) note that π_2 can be written as

$$\pi_2(\bar{U}) = \int_0^1 \left[\bar{x}_{p_{\bar{U}}}(t) M_1(t) + \int_{\bar{x}_{p_{\bar{U}}}(t)}^1 \tilde{M}_2(q) dq \right] dt.$$

We first show that $\pi_2(\bar{U})$ is Lipschitz. For $\bar{U}' > \bar{U}$,

$$\begin{aligned}
\left|\pi_2(\bar{U}') - \pi_2(\bar{U})\right| &= \left| \int_0^1 \int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}'}}(t)} M_1(t) - \tilde{M}_2(q) dq dt \right|, \\
&\leq \int_0^1 \left| \int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}'}}(t)} \underbrace{M_1(t) - \tilde{M}_2(q)}_{|.| \leq M < \infty} dq \right| dt, \\
&\leq M \int_0^1 \bar{x}_{p_{\bar{U}'}}(t) - \bar{x}_{p_{\bar{U}}}(t) dt, \\
&\leq \frac{M}{\underline{v}_1'} (\bar{U}' - \bar{U}),
\end{aligned}$$

where $\underline{v}_1' = \min_{t\in[0,1]} v_1'(t) > 0$ by our assumptions on the type distributions. Next we show that $\pi_2'(\bar{U}) = -p_U$. for almost every $\bar{U}$. For $h > 0$,

$$\begin{aligned}
\frac{1}{h}(\pi_2(\bar{U}+h) - \pi_2(\bar{U})) &= \frac{1}{h}\int_0^1 \int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}+h}}(t)} M_1(t) - \tilde{M}_2(q)dqdt, \\
&= \frac{1}{h}\int_{\underline{t}_1^0(\bar{U}+h)}^{c(\bar{U})} \int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}+h}}(t)} M_1(t) - \tilde{M}_2(q)dqdt, \\
&= -p_{\bar{U}}\frac{1}{h}\underbrace{\int_{\underline{t}_1^0(\bar{U}+h)}^{c(\bar{U})} \int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}+h}}(t)} v_1'(t)dqdt}_{=h} + \\
&\quad + \int_{\underline{t}_1^0(\bar{U}+h)}^{c(\bar{U})} \frac{1}{h}\int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}+h}}(t)} M_1^{p_U}(t) - \tilde{M}_2(q)dqdt.
\end{aligned}$$

A similar expression can be derived for $h < 0$. $\underline{t}_1^0$ and c are are continuous in $\bar{U}$ for almost every $\bar{U}$ (for all $\bar{U}$ if $M_1^{p_U}$ is strictly increasing). Hence, by the Lebesgue differentiation theorem and dominated convergence, for almost every $\bar{U}$ (every $\bar{U}$ if $M_1^{p_U}$ is strictly increasing),

$$\begin{aligned}
\pi_2'(\bar{U}) = \lim_{h\to 0}\frac{1}{h}(\pi_2(\bar{U}+h) - \pi_2(\bar{U})) &= -p_{\bar{U}} + \int_{\underline{t}_1^0}^{c} M_1^{p_U}(t) - \tilde{M}_2(\bar{x}_{p_{\bar{U}}}(t))dt, \\
&= -p_{\bar{U}} + \int_{\underline{t}_1^0}^{c} \bar{M}_1^{p_U}(t) - \tilde{M}_2(\bar{x}_{p_{\bar{U}}}(t))dt, \\
&= -p_{\bar{U}}.
\end{aligned}$$

Since $\pi_2(\bar{U})$ is Lipschitz continuous it is absolutely continuous and $\pi_2(\bar{U}) = \pi_2(0) - \int_0^{\bar{U}} p_U(s)ds$. Therefore, as $p_U(\bar{U})$ is non-decreasing, π_2 is weakly concave. If $\{t|\bar{M}_1^{p_U}(t) = 0\}$ is a singleton $p_U(\bar{U})$ is strictly increasing an hence π_2 strictly concave. □

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Appendix C. Supplementary Material

C.1. Omitted Proofs

Proof of Lemma 1. $\hat{x}$ is derived from x as follows. Whenever a buyer is allocated a unit before his deadline is reached, this allocation is postponed to the deadline. Whenever a buyer is allocated unit after his deadline has elapsed, the unit is withheld under the new allocation rule. In all other cases, the new allocation rule is the same as the old one.

This implies that buyers who report their deadline truthfully enjoy the same expected payoff in both mechanisms:

$$\forall a \in \{1, \dots T\}, d \in \{a, \dots T\}, \forall v \in [0, \overline{v}] : \quad \sum_{\tau=a}^{d} \hat{q}_a(v, d) = \sum_{\tau=a}^{d} q_a(v, d).$$

On the other hand, for $d' \neq d$, we have

$$\sum_{\tau=a}^{d} \hat{q}_a(v, d') \leq \sum_{\tau=a}^{d} q_a(v, d').$$

Hence,

$$\hat{U}_a(v, d) = U_a(v, d) \geq U_a(v, d, v', d') \geq \hat{U}_a(v, d, v', d').$$

□

Proof of Theorem 4. Substituting V_2^{opt} into the objective function we get

$$\pi_1(U) = \max_{q_1(.,1)} V_2^{\text{opt}} + \int_0^{\overline{v}} q_1(v, 1) \left(J_1(v) - V_2^{\text{opt}} \right) f_1(v) dv, \tag{C.1}$$

subject to $q_1(v, 1) \in [0, 1]$, $\forall v \in [0, \overline{v}]$, ($\text{M}_1$), ($\text{PE}_1$) and ($\text{ICD}_\text{U}^\text{d}$). This is a control problem with state $U_1(v) = U_1(v, 1)$ and measurable control $q_1(v) = q_1(v, 1)$. The law of motion for the state is $U_1'(v) = q_1(v)$. We account for $q_1(v, 1) \in [0, 1]$ and ($\text{ICD}_\text{U}^\text{d}$) by imposing the state constraint $U_1(v) \leq U(v)$, requiring the state to start at zero, $U_1(0) = 0$, and the control to take values between zero and one, $q_1(v) \in [0, 1]$. (M_1) will be neglected for the moment.

The Hamiltonian of this problem is

$$\mathcal{H}(U_1, q_1, p, v) = pq_1 + q_1 \left(J_1(v) - V_2^{\text{opt}} \right) f_1(v)$$

where p is the adjoint variable of the state U_1. Let (U_1, q_1) be an optimal solution. By the Pontryagin maximum principle (c.f. Clarke, 1983, pp. 211-212) we have that $p(v) = p$ is constant and $p + \mu[0, \overline{v}] = 0$, where μ is a non-negative measure supported on the set $\{v \mid U_1(v) = U(v)\}$. Furthermore, for almost every v, $q_1(v)$ maximizes $\mathcal{H}(U_1(t), q_1, p + \mu[0, v), v)$. This implies that for almost every v,

$$\begin{aligned}
& q_1(v) = 1, && \text{if } p + \mu[0, v) + (J_1(v) - V_2^{\text{opt}}) f_1(v) > 0, \\
& q_1(v) \in [0, 1], && \text{if } p + \mu[0, v) + (J_1(v) - V_2^{\text{opt}}) f_1(v) = 0, \\
\text{and} \quad & q_1(v) = 0, && \text{if } p + \mu[0, v) + (J_1(v) - V_2^{\text{opt}}) f_1(v) < 0.
\end{aligned}$$

Since $p+\mu[0,v] \le 0$, $q_1(v) = 0$ if $J_1(v) < V_2^{\text{opt}}$. But if $J_1(v) \ge V_2^{\text{opt}}$, Assumption 2 implies that $(J_1(v) - V_2^{\text{opt}})f_1(v)$ is strictly increasing. Since $\mu[0,v)$ is non-decreasing, $p+\mu[0,v)+(J_1(v)-V_2^{\text{opt}})f_1(v) = 0$ implies $p+\mu[0,v')+(J_1(v')-V_2^{\text{opt}})f_1(v') > 0$ for all $v' > v$. Therefore there is a unique value r_1 such that

$$q_1(v_1) = \begin{cases} 0, & \text{if } v_1 < r_1 \\ 1, & \text{if } v_1 > r_1. \end{cases}$$

Obviously, any such solution satisfies (M_1). r_1 can be determined without resorting to optimal control theory. As the mechanism is deterministic, it is the lowest value such that $J_1(r_1) \ge V_2^{\text{opt}}$ and $U_1(\overline{v}, 1) = \overline{v} - r_1 \le U(\overline{v})$. This yields the solution stated in the Theorem.

If we set $r_1 = \max\{J_1^{-1}(V_2^{\text{opt}}), \overline{v} - \bar{U}\}$ and insert the optimal solution in the objective function we obtain

$$\pi_1(\bar{U}) = \int_{r_1}^{\overline{v}} J_1(v) f_1(v) dv + V_2^{\text{opt}} F_1(r_1).$$

$$\pi_1'(\bar{U}) = \begin{cases} (J_1(\overline{v} - \bar{U}) - V_2^{\text{opt}}) f_1(\overline{v} - \bar{U}), & \text{if } J_1(\overline{v} - \bar{U}) > V_2^{\text{opt}}, \\ 0 & \text{otherwise.} \end{cases}$$

For $\bar{U} \to \overline{v} - J_1^{-1}(V_2^{\text{opt}})$ we have $\pi_1'(\bar{U}) \to 0$ since f_1 is bounded. Hence, $\pi_1'(\bar{U})$ is continuous. Using Assumption 2, we conclude that $\pi_1'(\bar{U})$ is strictly decreasing if $J_1(\overline{v} - \bar{U}) > V_2^{\text{opt}}$ and hence π_1 is strictly concave. □

Proof of Lemma 4. (B.4) can be rewritten as

$$\forall t_2 \in [0,1]: \quad \int_{t_2}^1 q_2(\theta) d\theta \le \min_{t_1 \in [0,1]} \left[1 - t_1 t_2 - \int_{t_1}^1 q_1(\theta) d\theta \right].$$

On the right-hand side we minimize a convex function. Therefore, the first order condition is sufficient for a minimum and we have $t_2 \in [q_1(t_1^-), q_1(t_1^+)]$ for all $t_2 \in [q_1(0), q_1(1)]$, $t_1 = 0$ if $t_2 < q_1(0)$ and $t_1 = 1$ if $t_2 > q(1)$. Hence $t_1 = q_1^{-1}(t_2)$ is a minimizer for all t_2. Substituting this into (B.4) yields

$$\forall t_2 \in [0,1]: \quad \int_{t_2}^1 q_2(\theta) d\theta \le 1 - q_1^{-1}(t_2) t_2 - \int_{q_1^{-1}(t_2)}^1 q_1(\theta) d\theta. \tag{C.2}$$

q_2^* fulfills this constraint with equality for all $t_2 \in [0,1]$.

Now consider an alternative solution $\tilde{q}_2$ that differs from q_2^* on a set of positive measure. If $\tilde{q}_2(t) > 0$ for some $t < t_2^0$, than it is not a maximizer. So suppose $\tilde{q}_2(t) = 0$ for $t < t_2^0$. By (C.2) we must have $\int_t^1 \tilde{q}_2(\theta) d\theta \le \int_t^1 q_2^*(\theta) d\theta$ for all $t \in [0,1]$. Since $\tilde{q} \ne q^*$, on a set of positive measure, $\int_a^1 \tilde{q}_2(\theta) d\theta < \int_a^1 q_2^*(\theta) d\theta$ for some $a \in [t_2^0, 1]$. Let $Q(t)$ be the concave hull of

$$t \longmapsto \begin{cases} \int_t^1 \tilde{q}_2(\theta) d\theta, & \text{if } t \ne a, \\ \int_a^1 q_2^*(\theta) d\theta, & \text{if } t = a, \end{cases}$$

and define $\hat{q}_2(t) = -\frac{dQ(t)}{dt}$ for almost every t. By definition, $Q(t) = \int_t^1 \hat{q}_2(\theta)d\theta \leq \int_t^1 q_2^*(\theta)d\theta$. Hence $\hat{q}_2$ is a solution. Furthermore, there are $\underline{a}, \bar{a}$ such that

$$\hat{q}_2(t) = \begin{cases} \tilde{q}_2(t), & \text{if } t \notin [\underline{a}, \bar{a}], \\ \frac{\int_{\underline{a}}^1 \tilde{q}_2(\theta)d\theta - \int_a^1 q_2^*(\theta)d\theta}{a-\underline{a}}, & \text{if } t \in [\underline{a}, a), \\ \frac{\int_a^1 q_2^*(\theta)d\theta - \int_{\bar{a}}^1 \tilde{q}_2(\theta)d\theta}{\bar{a}-a}, & \text{if } t \in (a, \bar{a}]. \end{cases}$$

Hence $\hat{q}_2(t) < \tilde{q}_2(t)$ for $t \in (\underline{a}, a)$, $\hat{q}_2(t) > \tilde{q}_2(t)$ for $t \in (a, \bar{a})$ and $\hat{q}_2(t) = \tilde{q}_2(t)$ otherwise. Furthermore,

$$\int_{\underline{a}}^{a} \hat{q}_2(\theta) - \tilde{q}_2(\theta)d\theta = \int_a^{\bar{a}} \tilde{q}_2(\theta) - \hat{q}_2(\theta)d\theta.$$

This implies that we have constructed $\hat{q}_2$ from $\tilde{q}_2$ by shifting winning probability from types in $[\underline{a}, a]$ to types in $[a, \bar{a}]$. By Assumption 1, this increases the objective function. Hence $\tilde{q}_2$ cannot be optimal. □

Proof of Theorem 8. (i) Let $(q_1^n, q_2^n)_{n\in\mathbb{N}}$ be a sequence of solutions of $\mathcal{P}_2'$ such that $R[q_1^n, q_2^n] \to \pi_2(\bar{U})$ for $n \to \infty$. By Helly's Theorem, for $i = 1, 2$ there exists a sub-sequence $\left(q_i^{n_j}\right)_{j\in\mathbb{N}}$ and a non-decreasing function $q_i : [0,1] \to [0,1]$, such that $q_i^{n_j} \to q_i$ almost everywhere. If we consider the q_i as elements of $L_2([0,1])$, the set of winning probabilities that satisfy (B.4) is weakly-compact (cf. Mierendorff (2011) and Border (1991)). Therefore, after taking sub-sequences again, $q_i^{n_j} \rightharpoonup q_i$ and q_i is feasible. As $M_i \in L_2([0,1])$ and $v_1' \in L_2([0,1])$, weak convergence of $q_i^{n_j}$ implies that q_1 fulfills (B.5)–(B.6), and $R[q_1, q_2] = \pi_2(\bar{U})$. Therefore (q_1, q_2) is an optimal solution.

(ii) Let $(q_1^n, q_2^n)_{n\in\mathbb{N}}$ be a sequence of solutions of $\mathcal{P}_2^K$ such that $R[q_1^n, q_2^n] \to \pi_2^K(\bar{U})$. After taking sub-sequences we can assume that this sequence converges to a solution satisfying (B.2)–(B.6) as in (i). Let q_1 be the limit of q_1^n. Since $q_i^n \in \mathcal{L}^K$, for all $s, t \in [0,1]$, $|q_1(t) - q_1(s)| = \lim_{n\to\infty} |q_1^n(t) - q_1^n(s)| \leq K|t-s|$. Hence $q_1 \in \mathcal{L}^K$. □

C.2. Extension of Lemma 2 to the Case of Many Objects

In order to show that Lemma 2 holds for $T = 2$ and $K \geq 2$, we need some additional notation. A state is now given by $s_t = (H_t, K_t)$ where K_t is the remaining number of available objects.

For a given state s_t, define $c_{(1)}^t \geq \ldots \geq c_{(K)}^t$ as the K highest virtual valuations among the buyers $i \in I_{\leq t}$ with deadlines $d_i = t$. If K_2 units are available in period two, we have

$$V_2(H_2, K_2) = \sum_{k=1}^{K_2} \max\left\{0, c_{(k)}^2\right\}.$$

The marginal expected revenue in period two from the k^{th} unit is given by

$$\begin{aligned}\Delta(s_1, k) &= E_{s_2}\left[V_2(s_2)|s_1, K_2 = k\right] - E_{s_2}\left[V_2(s_2)|s_1, K_2 = k-1\right] \\ &= E_{s_2}\left[\max\left\{0, c_{(k)}^2\right\}\Big|s_1\right].\end{aligned}$$

Obviously, $\Delta(s_1, k)$ is decreasing in k. Hence, the optimal number of units to be retained for period two is determined by

$$c^1_{(K-K^*_2)} > \Delta(s_1, K^*_2 + 1) \quad \text{if } K^*_2 < K,$$
$$\text{and} \qquad c^1_{(K-K^*_2+1)} \leq \Delta(s_1, K^*_2) \quad \text{if } K^*_2 > 0.$$

If we denote the identities of the buyers with virtual valuations $c^t_{(1)}, \ldots, c^t_{(K)}$ by $i^t_{(1)}, \ldots, i^t_{(K)}$, the set of winning buyers in periods $t = 1, 2$ is given by

$$W^*_1(s_1) := \left\{ i^1_{(1)}, \ldots, i^1_{(K-K^*_2(s_1))} \right\},$$
$$W^*_2(s_2) := \left\{ i^2_{(1)}, \ldots, i^2_{(K^*_2(s_1))} \right\} \cap \{ i \in I_{\leq 2} | J_{a_i}(v_i) \geq 0 \}.$$

Now we fix some buyer $i \in I_1$. His virtual valuation determines whether he is in the set of winning buyers at his deadline d_i for a given number of retained objects K_2, but it may also have an influence on the number of retained objects. The critical virtual valuation is given by

$$\zeta^1_1(H^{-i}_1) = \inf \left\{ \zeta \middle| i \in W^*_1(((H^{-i}_1, (1, J^{-1}_a(\zeta), 1)), K) \right\},$$
$$= \inf \left\{ \zeta \middle| \zeta \geq c^1_{(K-K^*_2(H^{-i}_1,(1,J^{-1}_a(\zeta),1)))} \right\},$$

for $d_i = 1$, and by

$$\zeta^2_1(H^{-i}_2) = \inf \left\{ \zeta \middle| i \in W^*_2((H^{-i}_2, (1, J^{-1}_a(\zeta), 2)), K^*_2(H^{-i}_1, (1, J^{-1}_1(\zeta), 2))) \right\},$$
$$= \inf \left\{ \zeta \middle| \zeta \geq c^2_{(K^*_2((H^{-i}_1,(1,J^{-1}_a(\zeta),1)))} \right\},$$

for $d_1 = 2$.

Proof of Lemma 2 for $T = 2$ and $K \geq 2$. Let $c^{t,-i}_{(k)}$ denote the k$^{\text{th}}$ highest virtual valuation among the buyers with deadline t in $I_{\leq t} \backslash \{i\}$. Fix any state $s_1 = (H_1, K)$ and a buyer $i \in I_1$. Suppose that in state $(H^{-i}_1, K-1)$, i.e., if buyer i is not present and one objects less is available, $\bar{K}_1$ objects are allocated in period one and $\bar{K}_2 = K - \bar{K}_1$ objects are retained for period two. We distinguish two sub-cases.

Case A—In state (H^{-i}_1, K), $\bar{K}_1$ units are allocated in the first period and $\bar{K}_2 + 1$ units are retained for the second period: If, in state $((H^{-i}_1, (1, v_i, 1)), K)$, buyer i gets a unit in the first period, then the remaining $K-1$ units are allocated as in state $(H^{-i}_1, K-1)$. This means that $\bar{K}_1$ units are allocated to buyers other than i in period one and $\bar{K}_2$ units are retained. Hence, i's virtual valuation must exceed the option value of retaining the $\bar{K}_2 + 1^{\text{st}}$ unit in order to get a unit in the first period. We have

$$\zeta^1_1(H^{-i}_1) = E_{s_2} \left[\max \left\{ 0, c^{2,-i}_{(\bar{K}_2+1)} \right\} \middle| H^{-i}_1 \right]$$

In state $((H^{-i}_1, (1, v_i, 2)), K)$, the number of units that are allocated in the first period must also be $\bar{K}_1$. It is obvious that the arrival of buyer i with $d_i = 2$ cannot increase the number of units allocated in the first period. On the other hand, suppose that in state

$((H_1^{-i},(1,v_i,2)),K)$, only $\bar{K}_1-1$ units are allocated in the first period. Then

$$\begin{aligned} c^{1,-i}_{(\bar{K}_1)} &\leq E_{s_2}\left[\max\left\{0,c^2_{(\bar{K}_2+2)}\right\}\middle|(H_1^{-i},(1,v_i,2))\right], \\ &\leq E_{s_2}\left[\max\left\{0,c^2_{(\bar{K}_2+2)}\right\}\middle|(H_1^{-i},(1,\overline{v},2))\right], \\ &= E_{s_2}\left[\max\left\{0,c^2_{(\bar{K}_2+1)}\right\}\middle|H_1^{-i}\right], \\ &< c^{1,-i}_{(\bar{K}_1)}, \end{aligned}$$

where the last inequality follows from our assumption that in state $(H_1^{-i},K-1)$, $\bar{K}_1$ units are allocated in the first period. This is a contradiction. But if $\bar{K}_1$ objects are allocated in the first period, then

$$\zeta_1^2(H_2^{-i}) = \max\left\{0,c^{2,-i}_{(\bar{K}_2+1)}\right\}.$$

Hence, in case A, $E_{s_2}[\zeta_1^2(H_2^{-i})|H_1^{-i}] = \zeta_1^1(H_1^{-i})$ and $\left[\zeta_1^1(H_1^{-i})|H_1^{-i}\right] \succ_{\text{SSD}} \left[\zeta_1^2(H_2^{-i})|H_1^{-i}\right]$.

Case B—In state (H_1^{-i},K), $\bar{K}_1+1$ objects are allocated in the first period and $\bar{K}_2$ objects are retained for the second period: Again, if in state $((H_1^{-i},(1,v_i,1)),K)$, buyer i gets an object in the first period, then the remaining $K-1$ objects are allocated as in state $(H_1^{-i},K-1)$. Hence, in case B we have

$$\zeta_1^1(H_1^{-i}) = c^{1,-i}_{(\bar{K}_1+1)}.$$

In state $((H_1^{-i},(1,v_i,2)),K)$, it depends on v_i, how many objects are retained for the second period. Define z by

$$c^{1,-i}_{(\bar{K}_1+1)} = E_{s_2}\left[\max\left\{0,c^2_{(\bar{K}_2+1)}\right\}\middle|(H_1^{-i},(1,J_1^{-1}(z),2))\right].$$

If $J_1(v_i)\geq z$, then $\bar{K}_2+1$ objects are retained, otherwise only $\bar{K}_2$ objects are retained. Hence, we have

$$\zeta_1^2(H_2^{-i}) = \begin{cases} c^{2,-i}_{(\bar{K}_2+1)}, & \text{if } z < c^{2,-i}_{(\bar{K}_2+1)}, \\ z & \text{if } c^{2,-i}_{(\bar{K}_2+1)} \leq z < c^{2,-i}_{(\bar{K}_2)}, \\ c^{2,-i}_{(\bar{K}_2)} & \text{if } c^{2,-i}_{(\bar{K}_2)} \leq z. \end{cases}$$

Note that for $H_1 = (H_1^{-i},(1,J_1^{-1}(z),2))$ this equals $\max\left\{0,c^2_{(\bar{K}_2+1)}\right\}$. So we have

$$\begin{aligned} E\left[\zeta_1^2(H_2^{-i})\middle|H_1^{-i}\right] &= E_{s_2}\left[\max\left\{0,c^2_{(\bar{K}_2+1)}\right\}\middle|(H_1^{-i},(1,J_1^{-1}(z),2))\right], \\ &= c^{1,-i}_{(\bar{K}_1+1)}, \\ &= \zeta_1^1(H_1^{-i}), \end{aligned}$$

and hence $\left[\zeta_1^1(H_1^{-i})|H_1^{-i}\right] \succ_{\text{SSD}} \left[\zeta_1^2(H_2^{-i})|H_1^{-i}\right]$. □

C.3. Reduced Forms

The probability that a buyer who has arrived in period a_i, assesses for the event that the number of arrivals in period a_i is N_{a_i}, is given by[23]

$$\frac{N_{a_i}\,\nu_{a_i,N_{a_i}}}{\sum_{r=1}^{\infty} r\nu_{a_i,r}}.$$

The interim winning probability for period t of a buyer with type (a_i, v_i, d_i) is given by:

$$q^t_{a_i}(v_i,d_i) = \sum_{(N_1,\dots,N_t)\in\mathbb{N}_0^t}\left(\frac{N_{a_i}\nu_{a_i,N_{a_i}}}{\sum_{r=1}^{\infty} r\nu_{a_i,r}}\prod_{a\in\{1,\dots,t\}\setminus a_i}\nu_{a,N_a}\right)\Bigg[\sum_{d_1=a_1}^{T}\cdots\sum_{d_{i-1}=a_{i-1}}^{T}$$
$$\sum_{d_{i+1}=a_{i+1}}^{T}\cdots\sum_{d_{N_{\le t}}=a_{N_{\le t}}}^{T}\left(\prod_{j\in I_{\le t}\setminus i}\rho_{a_j,d_j}\right)\int\!\!\cdots\!\!\int_{v_1\dots v_{i-1}}\int\!\!\cdots\!\!\int_{v_{i+1}\dots v_{N_{\le t}}}\Big(x_i(s_1)+x_0(s_1)\Big[$$
$$\dots x_i(s_T)\Big]\Big)\prod_{j\in I_{\le t}\setminus i} f_{a_j}(v_j|d_j)dv_j\Bigg].$$

The interim expected payment of a buyer with type (a_i, v_i, d_i) is given by:

$$p_{a_i}(v_i,d_i) = \sum_{(N_1,\dots,N_t)\in\mathbb{N}_0^t}\left(\frac{N_{a_i}\nu_{a_i,N_{a_i}}}{\sum_{r=1}^{\infty} r\nu_{a_i,r}}\prod_{a\in\{1,\dots,t\}\setminus a_i}\nu_{a,N_a}\right)\Bigg[\sum_{d_1=a_1}^{T}\cdots\sum_{d_{i-1}=a_{i-1}}^{T}$$
$$\sum_{d_{i+1}=a_{i+1}}^{T}\cdots\sum_{d_{N_{\le t}}=a_{N_{\le t}}}^{T}\left(\prod_{j\in I_{\le t}\setminus i}\rho_{a_j,d_j}\right)\int\!\!\cdots\!\!\int_{v_1\dots v_{i-1}}\int\!\!\cdots\!\!\int_{v_{i+1}\dots v_{N_{\le t}}}\Bigg(\sum_{j\in I_1\cup\{0\}} x_j(s_1)y_i(s_1,k_2=j)+$$
$$+\,x_0(s_1)\Bigg[\sum_{j\in I_{\le 2}\cup\{0\}} x_j(s_2)y_i(s_2,k_3=j)+x_0(s_3)\Bigg[\dots$$
$$\sum_{j\in I_{\le T}\cup\{0\}} x_j(s_T)y_i(s_T,k_{T+1}=j)\Bigg]\Bigg]\Bigg)\prod_{j\in I_{\le t}\setminus i} f_{a_j}(v_j|d_j)dv_j\Bigg].$$

[23] See Myerson (1998) for a derivation of this expression.