Complexity and Mixed Strategy Equilibria

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Abstract

Unpredictable behavior is central to optimal play in many strategic situations because predictable patterns leave players vulnerable to exploitation. A theory of unpredictable behavior based on differential mental abilities is presented in the context of repeated two-person zero-sum games. Each player's mental ability is represented by a set of arithmetic functions. That set is assumed to be closed under computability. A strategy in the repeated game is feasible for a player only if it is simple relative to that player's mental ability. When one player is more competent than the other, no equilibrium exists without one player fully exploiting the other. If each player has an incompressible sequence (relative to the opponent) according to the Kolmogorov complexity, an equilibrium exists in which equilibrium payoffs equal to those of the stage game and all equilibrium strategies are unpredictable. A criterion called stochasticity is used to tightly characterize history-independent equilibrium strategies.

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1 Introduction

Unpredictable behavior is central to optimal play in many strategic situations, especially in social interactions with conflict of interests. There are many illustrative examples from competitive sports, such as the direction of tennis serves and penalty kicks in soccer. Other relevant examples include secrecy in military affairs, bluffing behavior in Poker, and tax auditing. A prototype example is the matching pennies game: two players simultaneously present a coin and one player wins if the sides of the coins coincide while the other wins if the sides differ. In these situations, it seems optimal for players to aim at being unpredictable to avoid detectable patterns that leave them vulnerable to exploitation. This intuition has been around since the beginning of game theory. Von Neumann and Morgenstern [22] point out that, in a matching pennies game, a player will concentrate on avoiding his or her intentions being found out even against a moderately intelligent opponent.

The standard model of unpredictable behavior is mixed strategies. Since von Neumann and Morgenstern [22], randomization has become the conventional interpretation of mixed strategies. But this interpretation has encountered serious criticisms. Aumann [2] argues "the idea that serious people would base important decisions on the flip of a coin is difficult to accept," and many authors share this opinion (see, for example, Rubinstein [28]).[1] Another theory of mixed strategies has become more influential in the context of one-shot games—the *belief* interpretation, which identifies mixed strategies with beliefs. That theory makes predictions about beliefs instead of actions (see, for example, Harsanyi [12] and Aumann and Brandenburger [3]). However, the belief interpretation is almost completely unrelated to the pattern-detection intuition for unpredictable behavior. Because patterns exist only in repeated plays, to have a theory of unpredictable behavior based on pattern detection in one-shot games seems very difficult, if not impossible.

This paper proposes a new theory of unpredictable behavior in the context of (infinitely) repeated two-person zero-sum games. In this context, unpredictable behavior is

[1]These criticisms, however, are specifically against the use of randomization devices in *one-shot* games, but not against the notion that unpredictability is useful in repeated strictly competitive games.

often discussed in terms of statistical patterns. My model, however, is not based on statistical patterns for the following reasons. Consider, for example, the repeated matching pennies game. The play that alternates between heads and tails has an obvious pattern and is not expected to be played in equilibrium. But such a sequence has the same probability as any other sequence according to the uniform distribution (which the analysis of statistical patterns for repeated matching pennies would naturally use). That sequence is excluded because of its *simple* pattern relative to other sequences, not because of its probability of occurrence. Thus, my theory does not begin with statistical patterns but with differential *mental abilities*, so that a pattern can be simple to one player while it is complicated and not detectable to the other.

Each player is endowed with a set of arithmetic functions in my model. This set represents the player's mental ability to formulate and implement his or her strategies.[2] A natural requirement for players' mental abilities is closure under simplicity: if a function is feasible for a player, that is, if that function belongs to the player's endowed set, so is any function *simpler* than that function. I employ the computability relation to define complexity classes over arithmetic functions, and function f is simpler than function g if f is computable from g. Intuitively, this means that there is a mechanical procedure that transforms g into f. Such a procedure can in principle be performed by any rational agent. Only strategies that are simple relative to a player's mental ability are feasible for that player in a repeated game. Given those feasible strategies, Nash equilibrium is then directly applicable.[3] Those feasible strategies also determine detectable patterns through equilibrium. For example, there is clearly a mechanical process to produce the sequence that alternates between heads and tails, and hence that strategy is feasible for both players. As a result, it is excluded as an equilibrium strategy.

Therefore, a more competent player can implement more complicated strategies and detect more complicated patterns than a less competent one. Indeed, a player without

[2]In my representation only arithmetic functions (whose ranges are also natural numbers) are included because any strategy can be identified with an arithmetic function under a adequate coding of actions and finite histories of previous plays.

[3]However, unlike the standard model, no mixed strategies are allowed because of my motivation.

limits on his or her mental ability can perfectly predict any strategy played by the opponent in equilibrium. Equilibrium of the repeated game, then, depends on the joint mental abilities. In fact, it is not clear whether equilibrium exists in this framework. The main contribution here is to give sufficient conditions on players' mental abilities for equilibrium existence. Moreover, I show that equilibrium strategies thus obtained have to be unpredictable, and give a characterization of optimal unpredictable behavior.

My first result is a necessary condition for the existence of an equilibrium. I show that if the two players share the same mental ability, then there is no equilibrium in any repeated zero-sum game whose stage game has no equilibrium in pure strategies. The proof is rather straightforward and I shall demonstrate the intuition in the context of repeated matching pennies. Because they share the same set of feasible functions, player 1 can simulate any player 2' strategy (via a computable operation) and has a strategy to win against it at every stage. This implies that in equilibrium, if there exists one, player 1 does win at every stage (as a result of the Minimax theorem). But a symmetric argument shows that player 2 wins at every stage in equilibrium. This leads to a contradiction. A similar logic shows that if one player is more competent than the other, then the more competent player will fully exploit the other in equilibrium, if it exists. In this case, the strategy played by the less competent player is perfectly predictable by the more competent. To obtain an equilibrium where neither player fully exploits the other, it is then necessary that each player has a feasible function that the other player cannot compute. Moreover, in such an equilibrium, no equilibrium strategy is perfectly predictable by the opponent.

But this *mutual uncomputability* condition does not seem sufficient. Not every sequence that is incomputable from the opponent's perspective is optimal. For example, in the repeated matching pennies game, the sequence that plays heads at all stage games with even indexes but uses an incomputable sequence of actions at odd indexes does not seem optimal—it is vulnerable to exploitation along an identifiable subsequence of plays. To resolve this issue I use another notion of complexity that refines the computability relation, the Kolmogorov complexity [15], which measures how much a sequence can be compressed. It refines the computability relation in that any sequence that is feasible for a player is fully compressible from that player's perspective while infeasible sequences can have different degrees of compressibility. Whether an incompressible sequence exists from a player perspective, of course, depends on that player's mental ability.

My main result is a sufficient condition for equilibrium existence using the Kolmogorov complexity, for which I need one additional assumption. An upper bound on players' mental abilities is imposed: for each player, that there is a feasible function so that any other feasible function is computable from it. It implies that the set of feasible functions is countable. This assumption guarantees that there are incompressible sequences from each player's perspective, and those sequences have no detectable patterns. I show, under this assumption, that if each player has a feasible sequence that is incompressible from the other's perspective, then the repeated game has a Nash equilibrium whose equilibrium payoffs are the same as those of the stage game.[4]

This sufficient condition is called *mutual complexity*. Under mutual complexity, equilibrium strategies are not perfectly predictable by the opponent, but not every incomputable strategy is optimal. I characterize equilibrium strategies with a concept called *stochasticity*.[5] To explain this concept, consider the repeated matching pennies game. A sequence is stochastic from a player's perspective with respect to the distribution $(\frac{1}{2}, \frac{1}{2})$ over heads and tails if, along all identifiable subsequences for that player's mental ability, the frequency of heads and tails is $(\frac{1}{2}, \frac{1}{2})$. I show that any $(\frac{1}{2}, \frac{1}{2})$-stochastic sequence is optimal. A partial converse to this result is also obtained. These results give a formal correspondence between frequencies of equilibrium strategies in a repeated game under mutual complexity and equilibrium mixed-strategies in the associated stage game.

Here I give two remarks on the existence result. First, equilibrium strategies in my framework are uncomputable and hence cannot be fully characterized constructively. This result, however, is consistent with the requirement proposed by McKelvey [20] that an equilibrium notion should be 'publication-proof,' that is, it should survive its own pub-

[4]The stage game, being a finite zero-sum game, always has a Nash equilibrium in mixed strategies and a unique equilibrium payoff under that equilibrium.

[5]This concept has appeared in von Mises [21] and is closely related to the calibration literature (see, for example, Foster and Vohra [11]).

lication. In my context, this implies that equilibrium strategies cannot be simulated by finite algorithms and hence have to be uncomputable. The aim here, therefore, is to pin down the essential properties of equilibrium unpredictable behavior based on the intuition commonly found in early game theoretic justifications of mixed strategies by limiting players' mental abilities in a precise sense. Based on Turing-computability and *descriptive complexity* that limits how compactly a strategy can be described,[6] my results show that the essential properties of equilibrium unpredictable behavior are characterized by stochasticity.

The second remark is concerned with payoff specification and the length of the repeated game. It can be shown that the discounting criterion is not consistent with equilibrium existence in this framework and the long-run average criterion is adopted for the existence result. This implies that exact equilibrium is not obtainable in finitely repeated games, but it does not exclude the existence of ε-equilibria. In fact, my main results, given their asymptotic nature, suggest that the framework can be amended to accommodate long but finitely repeated games with ε-equilibrium. This would provide a solution to the problem with discounting as well.[7]

These results also have implications for the empirical literature (including O'Neill [25], Brown and Rosenthal [5], Walker and Wooders and [30], and Palacios-Huerta [26]) that tests the equilibrium hypothesis in the context of a repeated zero-sum games. While this literature generally employs statistical tests according to the i.i.d. distribution, I show that there are equilibrium strategies that are inconsistent with any i.i.d. process in terms of statistical regularities. Thus, not all statistical regularities are relevant to the equilibrium hypothesis; those based on stochasticity are more relevant than others. Moreover, the empirical findings suggest that equilibrium unpredictable behavior appears

[6]It seems possible to go below Turing-computability and consider computationally hard problems to model unpredictable behavior. However, this approach does not resolve the issue of publication-proofness, because no matter what notion of computability we begin with, equilibrium strategies cannot be computable according to that notion. Turing-computability allows my analysis to focus on essential properties of unpredictable behavior instead of detailed computational issues, and derive results that can be modified to accommodate those issues in the future.

[7]See Section 4.3 for more details.

among professional sports players but not in experiments. My results suggest that, for unpredictable behavior to emerge in equilibrium, it requires sufficient complexities in players' mental abilities to handle the strategic interactions, and such complexities would appear only after a fair amount of experiences and competitions.

Here I turn to some related literature. The idea that players cannot comprehend the full pattern in any sequence has already appeared in Piccione and Rubinstein [27], but both their formal treatment and purpose are quite distinct from mine. In terms of formal analysis, my model is closely related to that in Ben-Porath [4]. In that model, players use finite automata to generate strategies in an infinitely repeated game. Hence, one can interpret those automata as a representation of players' mental abilities. That model analyzes the effect of strategic complexity on equilibrium behavior. However, because complexity is measured by the number of states in players' automata, only one player can have more complicated strategies than the other. Mixed strategies are necessary to obtain equilibrium existence, and hence it cannot provide a theory of unpredictable behavior.

The rest of the paper is organized as follows. Section 2 formulates repeated games with mental abilities and presents the nonexistence results. Section 3 formulates the notion of incompressible sequences using Kolmogorov complexity, gives the general existence result, and then gives a characterization of equilibrium unpredictable behavior. Section 4 discusses the results. Proofs appear in Section 5.

2 Mental abilities and repeated zero-sum games

This section formulates the model with two steps: the first step gives a representation of players' mental abilities; at the second step these abilities are used to determine the set of feasible strategies for players in a repeated zero-sum game. This section ends with a necessary condition for equilibrium existence.

2.1 Mental ability represented by arithmetic functions

In my model players use their mental abilities to formulate and implement strategies in a repeated game. The representation of mental abilities in my framework satisfies the following two properties: (i) it does not depend on the game being played; (ii) the ability is such that if a player can implement a more complex strategy, he or she can also implement any simpler strategy. Property (i) is desirable because mental ability can then be regarded as an inherent characteristic of a player. Moreover, this property allows my analysis to provide uniform conditions for equilibrium existence or nonexistence across different games. Property (ii) requires players' strategies be closed under simplicity, which is a plausible requirement for a theory of rational players.

The representation is rather straightforward: each player is endowed with a set of arithmetic functions. A function belonging to the set given to a player is said to be *feasible* for that player. This representation does not depend on the game being played (property (i)). On the other hand, because any strategy in a repeated game can be regarded as an arithmetic function with actions and histories adequately encoded into natural numbers, players' strategies in a repeated game can be determined by their abilities. Property (ii) requires a complexity structure on these functions. The *computability* relation is used for this purpose: a function is more complex than another if the second function is computable from the first. Closure under simplicity implies that if a function is feasible for a player, so is any function computable from it.

Now I formalize the representation. I include also functions over vectors of natural numbers in the set of feasible functions for a player. Because there is a constructive method to encode vectors of natural numbers with natural numbers, this inclusion is not substantial. However, it does simplify many definitions.[8] The set of all arithmetic functions is denoted by $\mathcal{F} = \bigcup_{k=1}^{\infty}\{f : \mathbb{N}^k \to \mathbb{N}\}$, where $\mathbb{N} = \{0, 1, 2, ...\}$ is the set of natural numbers. I give player i a subset $\mathcal{P}^i$ of $\mathcal{F}$, the set of feasible functions for player

[8]For example, the projection function $U_i^n(x_1, ..., x_n) = x_i$ is very simple to express and understand as a function over vectors of natural numbers, but it can look complicated if we translate it into a function over natural numbers.

i.

The computability relation provides a classification scheme for functions in $\mathcal{F}$ according to their complexity. This relation is based on two notions: *basic functions* and *basic operations.* The basic functions include (1) the zero function Z, defined by $Z(t) \equiv 0$; (2) the successor function S, defined by $S(t) = t + 1$; (3) the projection functions U_{ki}, defined by $U_{ki}(t_1, ..., t_k) = t_i$, with $k > 0,\ i = 1, .., k$. The basic operations include (1) composition: from functions $f(s_1, ..., s_l)$, $g_1(t_1, ..., t_k)$, ..., and $g_l(t_1, ..., t_k)$, obtain $h = f(g_1(t_1, ..., t_k), ..., g_l(t_1, ..., t_k))$; (2) primitive recursion: from functions f and g, obtain h defined as $h(0, t_1, ..., t_k) = f(t_1, ..., t_k)$ and $h(s, t_1, ..., t_k) = g(s-1, h(s-1, t_1, ..., t_k), t_1, ..., t_k)$ for $s > 0$; (3) minimization: from f satisfying that for any $(t_1, ..., t_k)$ there is some s such that $f(s, t_1, ..., t_k) = 0$, obtain g defined as $g(t_1, ..., t_k) = \min\{s : f(s, t_1, ..., t_k) = 0$ and $f(r, t_1, ..., t_k)$ is not zero for all $r < s\}$. The computability relation is defined as follows.

Definition 2.1. *A function $f \in \mathcal{F}$ is computable from a function $g \in \mathcal{F}$* if there is a sequence $f_1, ..., f_n$ of functions in $\mathcal{F}$ such that $f_1 = g$, $f_n = f$, and for each $i = 2, ..., n$, f_i is either a basic function or is obtained from functions in $f_1, ..., f_{i-1}$ through a basic operation.

In principle, any player can implement these basic functions and basic operations. If a function g can be transformed into another function f via these basic operations (with the aid of basic functions), then feasibility of g should imply feasibility of f for any player. The following assumption requires players' mental ability be closed under simplicity defined via computability.

Assumption A1 If $g \in \mathcal{P}^i$ and if f is computable from g, then $f \in \mathcal{P}^i$.

A natural way to compare different players' mental abilities is by set inclusion: player i is more competent than player j if $\mathcal{P}^j \subseteq \mathcal{P}^i$. Because the computability relation is transitive, (A1) implies that if any function feasible for player j is simpler (in terms of computability) than some function that is feasible for i, then i is more competent than j. Also, it gives a lower bound on players' abilities: any Turing-computable function is feasible for both players. The computability relation gives a partial order but not a complete one, and hence the two players' abilities may not be comparable.

2.2 Repeated games with mental abilities

Here I propose a model of repeated two-person zero-sum games with mental abilities. The stage game is a finite two-person zero-sum game $g = \langle X, Y, h \rangle$, where $X = \{x_1, ..., x_m\}$ is the set of player 1's actions, $Y = \{y_1, ..., y_n\}$ is the set of player 2's actions, and $h : X \times Y \to \mathbb{Q}$ is the von Neumann-Morgenstern utility function for player 1, with $\mathbb{Q}$ being the set of rational numbers. Throughout this paper, equilibrium is referred to Nash equilibrium in pure strategies unless explicitly specified otherwise. My analysis focuses on repeated games without equilibria in the stage games.[9] Player 1 is said to fully exploit player 2 in the repeated game if player 1's equilibrium payoff is $\min_{y \in Y} \max_{x \in X} h(x, y)$.

In a repeated game with mental abilities, a strategy is available to a player if and only if it is feasible as an arithmetic function for that player. Hence, the complexity of strategies feasible for a player is determined by his or her mental ability. This approach generalizes the machine game literature which model mental abilities with finite automata. In that literature, the complexities of strategies can be linearly ordered. In contrast, by assuming closure under computability, there can be incomparable pairs of mental abilities in my approach. The repeated games with mental abilities are formally defined as follows.

Definition 2.2. Let $g = \langle X, Y, h \rangle$ be a finite zero-sum game and let $\mathcal{P}^1$, $\mathcal{P}^2$ be two sets of functions satisfying (A1). The *repeated game with mental abilities* $\mathcal{P}^1$, $\mathcal{P}^2$ *based on the stage game g, denoted by* $RG(g, \mathcal{P}^1, \mathcal{P}^2)$, is a triple $\langle \mathcal{X}, \mathcal{Y}, u_h \rangle$ such that

(a) $\mathcal{X} = \{a : Y^{<\mathbb{N}} \to X : a \in \mathcal{P}^1\}$ is the set of player 1's strategies;

(b) $\mathcal{Y} = \{b : X^{<\mathbb{N}} \to Y : b \in \mathcal{P}^2\}$ is the set of player 2's strategies;[10]

[9] If the stage game has an equilibrium, there are trivial equilibria in the repeated game as well where unpredictability plays no role.

[10] Here the implicit assumption is that the sets X and Y are identified with a subset of natural numbers, and any history, as a finite sequence of actions, is identified with a code number. Hence each strategy in

(c) $u_h : X^{\mathbb{N}} \times Y^{\mathbb{N}} \to \mathbb{R}$ is player 1's payoff function defined as

$$u_h(\xi, \zeta) = \liminf_{T\to\infty} \sum_{t=0}^{T-1} \frac{h(\xi_t, \zeta_t)}{T}, \tag{1}$$

while player 2's payoff function is $-u_h$.

I adopt the long-run average criterion (1) for payoff specification in the repeated game.[11] Hu [14] gives an axiomatization of this criterion based on a frequentist interpretation of the von Neumann-Morgenstern expected utility theory. Nash equilibrium is then directly applicable, and my analysis focuses on the existence of a Nash equilibrium and its properties. An alternative criterion to the payoffs would be the discounting criterion; however, that criterion is not consistent with the existence of an equilibrium, as the following proposition shows.

Proposition 2.1. *Let $g = \langle X, Y, h\rangle$ be a two-person zero-sum game without any (pure) equilibrium. If $\mathcal{P}^1$ and $\mathcal{P}^2$ satisfy (A1), then there is no equilibrium in the game $\langle \mathcal{X}, \mathcal{Y}, v_h\rangle$, where $\mathcal{X}$ and $\mathcal{Y}$ are defined as in Definition 2.2, and v_h is defined by $v_h(\xi, \zeta) = (1 - \delta)\sum_{t=0}^{\infty} \delta^t h(\xi_t, \zeta_t)$.*

The proof of this proposition is given in Section 5. The crucial observation is that (A1) implies that each player can simulate any finite initial segment of the opponent's strategy and hence can fully exploit that part, and under the discounting criterion, that is sufficient for a player to obtain an overall payoff that is close to full exploitation. However, this does not mean that my framework cannot be used to study the discounting criterion. One possibility is to consider ε-equilibrium with more restrictive sets of feasible functions that do not satisfy (A1). See more discussions on this in Section 4.3.

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the repeated game can be identified with a function in $\mathcal{F}$. Assumption (A1) ensures that the encoding method is irrelevant. Moreover, notice that a strategy only depends on previous actions from the opponent instead of on the joint actions. This definition is without loss of generality because the two are equivalent, and this equivalence is preserved under computability. Again, (A1) is crucial for this to hold.

[11] Because the sequence $\{\sum_{t=0}^{T-1} \frac{h(\xi_t,\zeta_t)}{T}\}_{T=1}^{\infty}$ may not have a limit, it is necessary to introduce limit inferior, limit superior, or other such notions that extend long-run averages. My main results are robust to any such notion between the limit inferior and limit superior.

Now I begin the equilibrium analysis of repeated games with mental abilities. First notice that a new structure is added to Nash equilibrium in my formulation: a pair of strategies constitute a Nash equilibrium if for each player, there is no detectable pattern in the opponent's strategy that the player can employ another strategy to exploit it better than his or her equilibrium strategy. Of course, whether a pattern is detectable depends on the player's mental ability. My first result (Proposition 2.2) shows how this dependence affects equilibrium: strategies with simple patterns are not played in equilibrium, unless one player is fully exploited by the other. Here, a pattern is simple for a player if it can be generated by a function feasible for that player. This result confirms the intuition that simple strategies, like the one that alternates between heads and tails in the repeated matching pennies game, should not be observed in equilibrium. The proof of Proposition 2.2 is given in Section 5.

Proposition 2.2. *Let* $g = \langle X, Y, h \rangle$ *be a two-person zero-sum game without any equilibrium. Suppose that* $\mathcal{P}^1$ *and* $\mathcal{P}^2$ *satisfy (A1).*

(a) If $\mathcal{P}^1 = \mathcal{P}^2$*, then there is no equilibrium in* $RG(g, \mathcal{P}^1, \mathcal{P}^2)$*.*

(b) If $\mathcal{P}^2 \subseteq \mathcal{P}^1$*, then the equilibrium payoff (if an equilibrium exists) in* $RG(g, \mathcal{P}^1, \mathcal{P}^2)$ *for player 1 is* $\min_{y \in Y} \max_{x \in X} h(x, y)$.[12]

The intuition behind the proof is rather straightforward. Consider the matching pennies game $g^{MP} = \langle \{Heads, Tails\}, \{Heads, Tails\}, h \rangle$ with

$$h(Heads, Heads) = 1 = h(Tails, Tails) \text{ and } h(Heads, Tails) = 0 = h(Tails, Heads).$$

Suppose that $\mathcal{P}^1 = \mathcal{P}^2$ and suppose that, by contradiction, that there is an equilibrium in the repeated game. For any player 1's equilibrium strategy, player 2 has a feasible function that simulate that strategy and so player 2 can generate a strategy that exactly mismatch that strategy. Hence, the equilibrium payoff for 2 has to be 1. On the other hand, a symmetric argument holds for any player 2's equilibrium strategy and hence the equilibrium payoff for player 1 has to be 1 as well. This leads to a contradiction. Part (b)

[12]There is no information about the existence of an equilibrium here, because in this case the existence depends on the payoff specification.

follows a similar logic. Although this intuition does not seem to rely on the computability structure, assumption (A1) is indispensable in this proposition. Arbitrary restrictions on players' strategy sets will not, obviously, generate these results. The crucial observation is that simulating the other player's strategy is a computable operation.

As a corollary of Proposition 2.2, if an equilibrium exists and neither player fully exploits the other in equilibrium, then each player's equilibrium strategy is not feasible for the opponent and hence each player has a feasible function infeasible for the opponent. This result gives a necessary condition for the existence of an equilibrium where neither player fully exploits the other: players' mental abilities have to be incomparable, or, in symbols, $\mathcal{P}^1 - \mathcal{P}^2 \neq \emptyset$ and $\mathcal{P}^2 - \mathcal{P}^1 \neq \emptyset$. I call this condition *mutual uncomputability*. This condition, however, does not seem sufficient. Consider the repeated game based on matching pennies. Suppose that player 1 plays heads at odd periods and uses an uncomputable sequence (from player 2's perspective) at even periods. The resulting strategy is uncomputable from player 2's perspective, but player 2 can easily exploit it by playing tails at odd periods. This kind of partial predictions can be a barrier to equilibrium existence. In the next section, I introduce another notion of complexity to resolve this difficulty and obtain an existence result.

3 Complexity and unpredictable behavior

So far I have focused on the complexity of players' mental abilities in terms of computability. Through this notion a necessary condition is derived for the equilibrium existence, but that condition does not seem sufficient. In this section I strengthen this condition so that each player has a function that is not only infeasible for the other player, but is highly complicated from the other player's perspective. This condition is formalized with *Kolmogorov complexity*, which measures how incompressible a function is. I show that if each player has a function that is highly incompressible from the other player's perspective, then there exists an equilibrium. After the existence result I give a characterization of equilibrium strategies.

3.1 Kolmogorov complexity and existence

In this section I give a sufficient condition on players' mental abilities that guarantee the existence of an equilibrium in a repeated game with mental abilities. This condition on mental abilities for existence is uniform across all repeated games with finite stage games, and hence shows a close connection between complexity and unpredictability. The idea for this sufficient condition comes from the Algorithmic Randomness literature.[13] The central theme in that literature is to find a satisfactory notion of *random sequences*, and the goal is to identify crucial properties of a deterministic sequence that fit our intuition of a random sequence. Computability structure plays a crucial role in its development.

For my analysis, the essential insight from that literature is the connection between complexity and unpredictability—a sequence is unpredictable if it is complex. A more precise notion of unpredictability, called stochasticity, will be discussed later in this section; roughly speaking, a sequence is stochastic if there is no detectable statistical pattern in it in terms of long-run frequencies. The notion of complexity is the prefix-free Kolmogorov complexity (Chaitin [7]), which refines the computability relation and measures complexity of incomputable sequences according to their compressibility. Here I give a quick overview of Kolmogorov complexity.

Kolmogorov complexity [15] measures the complexity of a finite object in terms of its minimum description length for a given language. A language is a mapping from words (typically finite strings over $\{0,1\}$) to objects that are to be described. Here I consider the complexity of finite binary sequences $\sigma \in \{0,1\}^{<\mathbb{N}}$. Hence a function $L : \{0,1\}^{<\mathbb{N}} \to \{0,1\}^{<\mathbb{N}}$ is called a *language*. Its domain consists of words and its range consists of objects to be described. It turns out that *prefix-free languages* give fruitful results in terms of unpredictability (see, for example, Chaitin [7]). To formalize prefix-free languages, we need partial functions $L :\subseteq \{0,1\}^{<\mathbb{N}} \to \{0,1\}^{<\mathbb{N}}$, which may not be defined over some strings. The set of strings over which L is defined is called the *domain* of L, denoted by $dom(L)$. A languages L is prefix-free if its domain has the following property:

[13] For an overview of that literature, see Nies [23]. That literature begins with Kolmogorov [15], Martin-Löf [19], and Chaitin [7], and the ideas can be traced back to ideas in von Mises [21].

for any words $\sigma,\ \tau \in dom(L)$, σ is not an initial segment of τ. Notice that if a language L is prefix-free, then L is a partial function, i.e., $dom(L) \neq \{0,1\}^{<\mathbb{N}}$. The Kolmogorov complexity of a sequence σ is defined as

$$K_L(\sigma) = \min\{|\tau| : \tau \in \{0,1\}^{<\mathbb{N}},\ L(\tau) = \sigma\},$$

and $K_L(\sigma) = \infty$ if there is no $\tau \in dom(L)$ such that $L(\tau) = \sigma$.

To apply the Kolmogorov complexity in my framework, a language is said to be feasible for a player if, as an arithmetic function, it is computable from some feasible function for that player. But to consider prefix-free languages, the definition of computability needs to be expanded to incorporate partial function. This can be done with an extended version of the minimization operator:

(3') minimization: from function f obtain g defined as

$g(t_1, ..., t_k) = \min\{s : f(s, t_1, ..., t_k) = 0$ and $f(r, t_1, ..., t_k)$ is defined for all $r < s\}$
if for some s, $f(s, t_1, ..., t_k) = 0$;
$g(t_1, ..., t_k)$ is undefined otherwise.

Adding this new minimization operator as a basic operator, the definition of computability in Definition 2.1 is extended to partial functions as well.[14] A prefix-free language L is feasible for $\mathcal{P}^i$ if and only if f_L is computable from some $g \in \mathcal{P}^i$ under this new definition, where f_L is the representation of L with codes for binary strings.[15]

When $\mathcal{P}^i$ consists of Turing computable functions only, there is a *universal* prefix-free language among feasible functions for $\mathcal{P}^i$ that gives shortest descriptions asymptotically and hence is an optimal language to use. However, this is not the case for general $\mathcal{P}^i$'s. One more assumption is added to guarantee the existence of a universal language:

Assumption A2 There is a function $f^* \in \mathcal{P}^i$ such that any function $f \in \mathcal{P}^i$ is computable from f^*.

[14]In the extended definition of computability, partial functions are also allowed in composition and primitive recursion. In those applications, an output is defined only if all the inputs are.

[15]One encoding is as follows: for the string $\sigma \in \{0,1\}^{<\mathbb{N}}$, the code for σ is $\sum_{t=0}^{|\sigma|-1}(\sigma_t + 1)2^t$.

Assumption (A1) gives a lower bound on what $\mathcal{P}^i$ has to contain; assumption (A2), on the other hand, gives an upper bound. In particular, (A2) implies that $\mathcal{P}^i$ is countable. Given (A2), it can be shown that there is a universal prefix-free language $L^{\mathcal{P}^i}$ for $\mathcal{P}^i$ in the following sense: for any prefix-free language L for $\mathcal{P}^i$, there is a constant b_L for which $K_{L^{\mathcal{P}^i}}(\sigma) \leq K_L(\sigma) + b_L$ for all binary strings σ.[16]

Now we can discuss Kolmogorov complexity from player i's perspective. To apply Kolmogorov complexity to functions, I translate a function into a binary sequence as follows: first identify a function f with its graph $\{(0, f(0)), (1, f(1)), ..., (t, f(t)), ...\}$, which can be further identified with an infinite sequence ξ^f over $\{0, 1\}$.[17] It is known that functions which are feasible have low Kolmogorov complexity: if $f \in \mathcal{P}^i$, then $\lim_{T\to\infty} K_{L^{\mathcal{P}^i}}(\xi^f[T])/T = 0$. On the other hand, sequences whose initial segments exhibiting high Kolmogorov complexities are unpredictable. I borrow that insight and apply this notion in my framework.

Definition 3.1. Let $\mathcal{P}^i$ be a set of functions satisfying (A1) and (A2). A sequence $\xi \in \{0, 1\}^{\mathbb{N}}$ is an *incompressible sequence relative to* $\mathcal{P}^i$ if there is a constant b_L such that for all $T > 0$, $K_{L^{\mathcal{P}^i}}(\xi[T]) \geq T - b_L$.

Two sets $\mathcal{P}^1$, $\mathcal{P}^2$ are *mutually complex* if there are functions $f^1 \in \mathcal{P}^1$, $f^2 \in \mathcal{P}^2$ such that for both $i = 1, 2$, ξ^{f^i} (recall that ξ^{f^i} is the binary sequence corresponding to the graph of the function f^i) is an incompressible sequence relative to $\mathcal{P}^{-i}$. First I give a proposition regrading the existence of mental abilities satisfying mutual complexity. Its proof is given in Section 5.

Proposition 3.1. *There are uncountably many different pairs of sets $\mathcal{P}^1$ and $\mathcal{P}^2$ that satisfy (A1), (A2), and mutual complexity.*

Mutual complexity, together with (A1) and (A2), guarantees the existence of an equilibrium, as stated in Theorem 3.1. Its proof is given in Section 5.

[16] The argument for the case where $\mathcal{P}^i$ consists of Turing-computable functions can be found in Downey *et al.* [10]. The general case is an easy extension of the arguments there.

[17] More precisely, the infinite binary sequence ξ^f corresponding to f is defined as $\xi^f_t = 1$ if and only if $t = 2^s(2f(s) + 1) - 1$ for some s.

Theorem 3.1. *(Existence) Let g be a finite zero-sum game. Suppose that $\mathcal{P}^1$, $\mathcal{P}^2$ satisfy both (A1) and (A2) and are mutually complex. Then there exists a (pure) equilibrium in $RG(g, \mathcal{P}^1, \mathcal{P}^2)$ with the same equilibrium payoffs as those in the mixed equilibrium of g.*

The proof of Theorem 3.1 consists of two steps: first I give a lemma which states that if mutual complexity holds, then for any frequency of actions, each player can generate a sequence that is unpredictable from the other player's perspective and is consistent with that frequency. This lemma is rather technical and requires knowledge in the Algorithmic Randomness literature. Then I give a theorem which states that any sequence that is unpredictable and is consistent with a stage-game mixed equilibrium frequency is an equilibrium strategy in the repeated game.

The precise notion of unpredictability that is relevant for the analysis here is called *stochasticity.* To define this notion we need one more concept. For any given finite set X, a *selection function for* X is a total function $r : X^{<\mathbb{N}} \to \{0, 1\}$, which, as mentioned earlier, can be identified with a total function over natural numbers.[18] Given a sequence $\xi \in X^{\mathbb{N}}$, r can be used to choose a subsequence ξ^r from ξ as follows: $\xi^r_t = \xi_{g(t)}$, where $g(0) = \min\{t : r(\xi[t]) = 1\}$, and $g(t) = \min\{s : r(\xi[s]) = 1,\ s > g(t-1)\}$ for $t > 0$.[19] Obviously, such a selection function may produce not an infinite subsequence, but only a finite initial segment. I use $\Delta(X)$ to denote the set of probability distributions over X with rational probability values, i.e., $\Delta(X) = \{p \in ([0, 1] \cap \mathbb{Q})^X : \sum_{x \in X} p_x = 1\}$. Stochasticity is defined formally as follows.

Definition 3.2. Let $p \in \Delta(X)$ be a probability distribution and let $\mathcal{P}$ be a set of functions. A sequence $\xi \in X^{\mathbb{N}}$ is *p-stochastic relative to* $\mathcal{P}$ if for any selection function $r \in \mathcal{P}$ for X such that ξ^r is an infinite sequence,

$$\lim_{T \to \infty} \sum_{t=0}^{T-1} \frac{c_x(\xi^r_t)}{T} = p_x \text{ for all } x \in X,$$

where $c_x(y) = 1$ if $x = y$, and $c_x(y) = 0$ otherwise.

[18] Selection functions are closely related to the notion of *checking rules* studied in Lehrer [16] and Sandroni *et al.* [29]. The only difference is that there are no forecasts in my model.

[19] If at some t, there is no $s > g(t-1)$ such that $r(\xi[s]) = 1$, then $g(t)$ is undefined.

The following lemma states that if $\mathcal{P}^1$ and $\mathcal{P}^2$ satisfies (A1), (A2), and mutual complexity, then each player can generate a stochastic sequence consistent with any frequency relative to the other player's mental ability.

Lemma 3.1. *Let $\mathcal{P}^i$ be a set of functions satisfying (A1) and (A2). Suppose that ξ is incompressible relative to $\mathcal{P}^i$. Then for any $p \in \Delta(X)$, there exists a p-stochastic sequence relative to $\mathcal{P}^i$ that is computable from ξ.*

The proof of this lemma is in the Appendix. In fact, two more powerful theorems (Theorem 6.2 and Theorem 6.5), of which Lemma 3.1 is a direct corollary, are proved there and will be useful for later results. Those theorems, which are more technical in nature and requires more background knowledge, generalize the main results in Zvonkin and Levin [31] in two directions. First, only binary sequences are discussed there while here I accommodate sequences based on any finite set. Second, results there only apply to only stochasticity relative to Turing computability (which is the minimum set satisfying (A1) and (A2)) while here the result applies to any set satisfying (A1) and (A2).

The next step is to show that if $p \in \Delta(X)$ is an equilibrium mixed strategy in the stage game for player 1, then a p-stochastic sequence will guarantee player 1 a minimum payoff of the stage-game equilibrium payoff in the repeated game under mutual complexity. Aside from the formal statement some informal arguments in the context of the matching pennies game g^{MP} may be useful. To simplify the arguments, let's assume that players can only use history-independent strategies. To be an equilibrium strategy, a strategy $\xi \in \{Heads, Tails\}^{\mathbb{N}}$ has to satisfy the following condition: for any player 2's strategy ζ, ξ can guarantee an overall payoff of $\frac{1}{2}$. The crucial observation is that the strategy ζ is equivalent to two selection functions r^H and r^T defined as $r^H(t) = 1$ if and only if $\zeta_t = Heads$ and $r^T(t) = 1 - r^H(t)$ for all t. If ξ is $(\frac{1}{2}, \frac{1}{2})$-stochastic relative to $\mathcal{P}^2$, then both ξ^{r^H} and ξ^{r^T} have the same frequency $(\frac{1}{2}, \frac{1}{2})$. This guarantees that the payoff against ζ is at least $\frac{1}{2}$. Of course, one has to deal with history-dependent strategies as well in the repeated game, and that is taken care of in the proof. Notice that in the definition of stochasticity, the selection functions are history-dependent as well.

The following theorem (Theorem 3.2), whose proof appears in Section 5, shows that stochasticity is a sufficient property for a sequence to be optimal in repeated games under mutual complexity. Here the constructed equilibrium strategies are history-independent: a strategy (for player 1) $a : Y^{<\mathbb{N}} \to X$ is *history-independent* if there is a sequence $\xi \in X^{\mathbb{N}}$ such that for all $\sigma \in Y^{<\mathbb{N}}$, $a(\sigma) = \xi_{|\sigma|}$. I identify such a strategy a with the sequence ξ. Similar definition applies to player 2's strategies as well.

Theorem 3.2. *(Stochasticity as a sufficient requirement for optimality) Suppose that $\mathcal{P}^1$, $\mathcal{P}^2$ satisfy (A1), (A2), and mutual complexity. For any player 1's equilibrium mixed-strategy $p \in \Delta(X)$ in g, the set of p-stochastic sequences relative to $\mathcal{P}^2$ in $\mathcal{X}$, which is not empty, is a subset of player 1's equilibrium strategies in $RG(g, \mathcal{P}^1, \mathcal{P}^2)$.*

Theorem 3.1 is then a direct corollary of Lemma 3.1 and Theorem 3.2. Theorem 3.1 gives a sufficient condition on mental abilities for equilibrium existence that is uniform across all games. Moreover, under mutual complexity, it shows that there exists an equilibrium where neither player fully exploits the other (for repeated games whose stage games have no equilibria), and hence, together with Proposition 2.2, it implies that any equilibrium strategy is uncomputable from the other player's perspective. But not every uncomputable strategy is an equilibrium strategy. The rest of this section is devoted to examine what *kind* of patterns could and could not exist in an equilibrium strategy.

3.2 Unpredictable behavior under mutual complexity

Here I give a tight characterization of equilibrium history-independent strategies in repeated games under mutual complexity. In addition to the fact that history-independent strategies are easier to characterize, these strategies are of special interest in my context. These strategies capture the intuition that in a repeated zero-sum game where unpredictability is crucial for optimality, players would focus on avoiding exploitable patterns instead of detecting patterns in the opponent's behavior. The main results here suggest that the criterion stochasticity is the crucial property, and that the usual i.i.d. requirement of unpredictability is too strong for optimality.

The first theorem here gives a partial converse to Theorem 3.2. One more definition is needed for that: for any $p = (p_x)_{x \in X} \in \Delta(X)$, $\xi \in X^{\mathbb{N}}$ is a *p-sequence* if for all $x \in X$, $\lim_{T\to\infty} \frac{\sum_{t=0}^{T-1} c_x(\xi_t)}{T} = p_x$.

Theorem 3.3. *Suppose that g has no pure equilibrium and that $\mathcal{P}^1$, $\mathcal{P}^2$ satisfy (A1), (A2), and mutual complexity. Then, for any equilibrium p-sequence $\xi \in \mathcal{X}$ in $RG(g, \mathcal{P}^1, \mathcal{P}^2)$ and any selection function $r \in \mathcal{P}^2$ for X such that $\{r(\xi[t])\}_{t=0}^{\infty}$ is a $(\alpha, 1-\alpha)$-sequence with $\alpha < 1$, if ξ^r is a p'-sequence, then p' is an equilibrium mixed-strategy in g.*

Theorem 3.3, whose proof appears in Section 5, gives a stronger condition than uncomputability for sequences that have limit frequencies. It excludes all uncomputable sequences with limit frequencies that are inconsistent with any equilibrium mixed-strategy of the stage game. However, it is clearly weaker than stochasticity. I am not able to characterize equilibrium strategies without well-defined limit frequencies because they seem to rely on how the long-run average criterion is extended to sequences of average payoffs without limits. Moreover, different subsequences may have different limit frequencies because of possible multiplicity of stage-game mixed-equilibria. If player 1 has two equilibrium mixed-strategies p and p' in g, then playing a p-stochastic sequence at odd periods and playing a p'-stochastic sequences at even periods is an equilibrium strategy in the repeated game. This naturally follows from the fact that any convex combination of p and p' is also an equilibrium strategy in the stage game.[20] This issue, of course, does not arise if the stage game has a unique equilibrium, and in that case we have $p' = p$ in Theorem 3.3.

These results show that stochasticity is the relevant properties for optimality, but it is not exactly those used in the empirical literature to test the equilibrium hypothesis in the context of repeated zero-sum games. Instead, tests based on statistical regularities according to the i.i.d. distribution generated by a mixed equilibrium of the stage game are generally employed for that purpose.[21] Stochasticity can be regarded as a subclass of

[20] I do not give a formal proof of this fact. However, the proof is available upon request.

[21] A statistical regularity can be regarded as a *describable* event that has probability 1. The term 'describable' can be defined with respect to the set of feasible functions in a similar fashion as I have done for Kolmogorov complexity, and in the Appendix (Section 6.1) I give a formal definition of statistical tests in this sense.

those statistical regularities, but are all statistical regularities that are expected from an i.i.d. distribution satisfied by the equilibrium strategies?

In the Appendix (see Theorem 6.2) I show that, under mutual complexity, there exists an equilibrium strategy that satisfies all the statistical regularities with respect to the i.i.d. distribution generated by a stage-game mixed equilibrium in the repeated game under mutual complexity. Moreover, I show there that mutual complexity is also necessary for that to happen. Here I show that there are equilibrium strategies that violate a particular statistical regularity called the Law of the Iterated Logarithm (LIL). Consider a finite set X and a distribution $p \in \Delta(X)$. The i.i.d. distribution μ_p over $X^{\mathbb{N}}$ generated by p is defined as follows: for all $\sigma \in X^{<\mathbb{N}}$, $\mu_p(\{\zeta \in X^{\mathbb{N}} : \sigma = \zeta[|\sigma|]\}) = \prod_{t=0}^{|\sigma|-1} p_{\sigma_t}$. LIL states that the following condition holds for almost all ξ in $X^{\mathbb{N}}$ (with respect to μ_p):

$$\limsup_{T\to\infty} \frac{|\sum_{t=0}^{T} c_x(\xi_t) - Tp_x|}{\sqrt{2p_x(1-p_x)T\log\log T}} = 1. \tag{2}$$

This law gives the exact convergence rate of the frequency in an i.i.d. sequence. The following theorem shows that there is an equilibrium strategy in the repeated game under mutual complexity that violates this regularity. Its proof is given in Section 5.

Theorem 3.4. *Suppose that $\mathcal{P}^1$, $\mathcal{P}^2$ satisfy (A1), (A2), and mutual complexity. Let p^* be an equilibrium mixed-strategy of player 1 in g such that $0 < p^*_x < 1$ for some $x \in X$. There exists an equilibrium strategy ξ in $RG(g, \mathcal{P}^1, \mathcal{P}^2)$ that is p^*-stochastic and satisfies*

$$\lim_{T\to\infty} \frac{\sum_{t=0}^{T} c_x(\xi_t) - Tp^*_x}{\sqrt{2p^*_x(1-p^*_x)T\log\log T}} = \infty. \tag{3}$$

Theorem 3.4 shows that, although certain patterns are detectable, it may not be feasible to transform them into a strategy that exploits them. Moreover, this result shows that, in repeated zero-sum games failure of certain statistical regularities does not entail the rejection of the equilibrium hypothesis. As a result, the common practice which rejects the equilibrium hypothesis by using statistical tests may require further reconsideration.

4 Discussion

This section discusses the implications of my results for the literature, and gives some interpretations of Nash equilibrium in my framework. The discussion ends with a potential extension.

4.1 Unpredictability in game theory

The proposed theory is intended to capture the intuition that unpredictable behavior is optimal in repeated zero-sum games because it avoids detectable patterns. My approach formalizes this intuition with a model of players with limited mental abilities. The main result is a sufficient condition for the existence of an equilibrium, which requires players' mental abilities be sufficiently complex relative to one another. Under this condition, unpredictable behavior emerges in equilibrium. Unpredictable strategies are optimal because the opponent has limited mental abilities and hence cannot detect patterns to exploit them. Moreover, the characterization result suggests that stochasticity is the relevant concept of unpredictability in repeated zero-sum games. Compared to the standard theory of mixed strategies, in my approach equilibrium strategies are deterministic sequences and they are *ex post* optimal. The standard theory can only discuss the probability of a set of sequences being observed in equilibrium, while my theory predicts a definite set of sequences that would emerge in equilibrium.

Empirically, my approach gives different implications than the standard theory. First, Turing-computable strategies are ruled out as equilibrium strategies. This result captures the intuition that *simple* strategies are vulnerable to exploitations, and this approach formalizes what simple strategies means by the computability structure. The standard theory does not have similar implications. The second empirical implication is concerned with statistical regularities in the context of repeated zero-sum games. It is generally believed that only i.i.d. sequences are consistent with the equilibrium hypothesis. However, my results suggest that not all tests are relevant to the equilibrium hypothesis., but only those for stochasticity.[22] They also show that not all detectable patterns can be transformed into exploitable opportunities.

Of course, these empirical implications are subject to conditions on players' mental abilities. Unpredictable behavior emerges only if those abilities are mutually incomputable, and equilibrium exists with unpredictable behavior if those abilities are mutually complex. These conditions are at least in principle testable (see Theorem 4.1 below). Therefore, whether equilibrium unpredictable behavior will emerge becomes an empirical question contingent on both players' mental abilities. In the literature, there is some empirical support of equilibrium unpredictable behavior in zero-sum games played by professional players (Walker and Wooders [30], Hsu *et al.* [13], and Palacios-Huerta [26]), while the results are generally negative in experiments. My results suggest that this difference, at least partially, is related to different mental abilities of the players in these different situations. Experienced professional players may have developed sufficient complexity in their mental abilities to handle relevant strategic situations in the field, while players in experiments usually do not have sufficient experiences of the game being played and of players they encounter.

4.2 Interpretation of Nash equilibrium

My model adopts Nash equilibrium as the solution concept. Here equilibrium can be interpreted as a stability condition. Formally this condition says that, given the opponent's strategy, no player has a profitable deviation. What is conceptually novel in my framework, however, is about deviations that come from detectable patterns. In games such as matching pennies, only this kind of deviations is relevant. Equilibrium guarantees that there be no detectable pattern in the opponent's strategy that is not used to increase a player's payoff, and thus can be interpreted as a stability requirement in the sense that the system has no detectable pattern that is not exploited.

[22] Tests based on checking rules may be useful to develop tests for stochasticity. However, the extant literature on checking rules (see, for example, Lehrer [16] and Sandroni *et al.* [29]) focuses on testing experts and hence may not be directly applicable.

A related interpretation appeals to the maximin criterion. In a zero-sum game, a strategy is an equilibrium strategy if and only if it maximizes the security levels, defined as the minimum payoff that could result from playing that strategy. If players in a repeated game are convinced that mutual complexity holds, then Theorem 3.2 gives a recipe to construct the optimal strategies out of the incompressible sequences (relative to the opponents' abilities) they have.[23] The following theorem, which is proved in Section 5, shows that mutual complexity can be verified at least in principle:

Theorem 4.1. *Suppose that $\mathcal{P}^1$ and $\mathcal{P}^2$ satisfy (A1), (A2), and mutual complexity. There is a sequence $\xi \in \{0,1\}^{\mathbb{N}}$ in $\mathcal{P}^1$ such that, if $\zeta \in \mathcal{P}^2$ is incompressible relative to $\mathcal{T}$, then $\xi \oplus \zeta$ defined as $(\xi \oplus \zeta)_{2t} = \xi_t$ and $(\xi \oplus \zeta)_{2t+1} = \zeta_t$ for all $t \in \mathbb{N}$ is incompressible relative to $\mathcal{T}$.*[24]

Theorem 4.1 gives several potentially falsifiable implications of mutual complexity. First, both players have a feasible incompressible sequence (relative to Turing computability). Moreover, among these pairs of incompressible sequences, at least one of them can be combined into a incompressible sequence.

4.3 Extensions to finite sequences

Many results in the paper can be extended to finitely repeated games. One approach to accomplish this is to consider asymptotic properties of unpredictable behavior in long but finitely repeated games. One way to model mental abilities in finitely repeated games is to give resource restrictions on players' computational powers but maintain the basic structure of mutual complexity. The proof of Theorem 3.2 suggests that, given any resource restrictions, any pair of equilibrium strategies that are stochastic also constitute an ε-equilibrium in a long but finitely repeated game (taking their initial segments as

[23] Another interpretation is to take $\mathcal{P}^1$ and $\mathcal{P}^2$ as technological constraints on strategy implementation instead of as mental abilities, such as in Chatterjee and Sabourian [8]. Under this interpretation players can be free from mental constraints, and Maximin principle is directly applicable.

[24] Recall that $\mathcal{T}$ is the set of Turing-computable functions. It is also the minimum subset (in terms of set-inclusion) of $\mathcal{F}$ that satisfies both (A1) and (A2).

the strategies in the finite game) with ε vanishing as the length of the game approaches infinity, as long as that pair is feasible as well. Notice that if ε-equilibrium can be obtained in a finitely repeated game, then ε-equilibrium with discounting is not hard to get by manipulating the ε's.

5 Proofs

Proof of Proposition 2.1: Because the stage game g has no pure equilibrium, $v_1 =_{def} \max_{x\in X}\min_{y\in Y} h(x,y) < v_2 =_{def} \min_{y\in Y}\max_{x\in X} h(x,y)$. Suppose, by contradiction, that (a^*, b^*) is an equilibrium. Let $c = \max_{(x,y)\in X\times Y}|h(x,y)|$. For each $T \in \mathbb{N}$, I will construct a strategy $b^T \in \mathcal{Y}$ such that $v_h(a^*, b^T) \leq v_1 + \delta^{T+1}(c - v_1)$. b^T is constructed as follows: for all $\sigma \in X^{<\mathbb{N}}$, $b^T(\sigma) = y_{\zeta_{|\sigma|}}$, with ζ defined as

$$\begin{aligned}
&\zeta_0 = \min\{i : y_i \in \arg\min_{y\in Y} h(a^*(\epsilon), y)\}(\epsilon \text{ is the empty string}); \qquad (4)\\
&\text{for } t = 1, ..., T,\ \zeta_t = \min\{i : y_i \in \arg\min_{y\in Y} h(a^*(y_\zeta[t]), y)\};\\
&\text{for } t > T+1,\ \zeta_t = 1.
\end{aligned}$$

Because ζ_t is constant for all $t > T$, $\zeta \in \mathcal{P}^2$ and so $b^T \in \mathcal{Y}$. For each T,

$$v(a^*, b^T) \leq (1-\delta)\sum_{t=0}^{T}\delta^t v_1 + (1-\delta)\sum_{t=T+1}^{\infty} c = v_1 + \delta^{T+1}(c - v_1).$$

Because (a^*, b^*) is an equilibrium, $v(a^*, b^*) \leq v(a^*, b^T) \leq v_1 + \delta^{T+1}(c - v_1)$ all T. Hence,

$$v(a^*, b^*) \leq \lim_{T\to\infty} v_1 + \delta^{T+1}(c - v_1) = v_1.$$

Symmetrically, $v(a^*, b^*) \geq v_2$. It follows that $v_1 < v_2 \leq v(a^*, b^*) \leq v_1$, a contradiction. □

Proof of Proposition 2.2:

(a) This part is implied by part (b) as follows. Because g has no pure equilibrium, $v_1 = \max_{x\in X}\min_{y\in Y} h(x,y) < \min_{y\in Y}\max_{x\in X} h(x,y) = v_2$. By part (b) implies that, if equilibrium exists, the equilibrium payoff of player 1 is v_1 if $\mathcal{P}^1 \subset \mathcal{P}^2$ while it is v_2 if $\mathcal{P}^2 \subset \mathcal{P}^1$. So $\mathcal{P}^1 = \mathcal{P}^2$ implies that, if equilibrium exists, $v_1 = v_2$. This leads to a contradiction.

(b) Index the actions in X as $X = \{x_1, ..., x_m\}$. Suppose that there is an equilibrium, and hence the equilibrium payoff for player 1 is $\inf_{b \in \mathcal{Y}} \sup_{a \in \mathcal{X}} u_h(a, b)$. I will show that for any $b \in \mathcal{Y}$, there is strategy $a' \in \mathcal{X}$ such that $u_h(a', b) \geq v_2$. Given b, a' is constructed as follows: for all $\sigma \in Y^{<\mathbb{N}}$, $a'(\sigma) = x_{\zeta_{|\sigma|}}$, with ζ defined as

$$\zeta_0 = \min\{i : x_i \in \arg\max_{x \in X} h(x, b(\epsilon))\}, \text{ and}$$
$$\text{for } t > 0,\ \zeta_t = \min\{i : x_i \in \arg\max_{x \in X} h(x, b(x_\zeta[t]))\},$$

with $x_\zeta[t] = (x_{\zeta_0}, ..., x_{\zeta_{t-1}})$. $\zeta \in \mathcal{P}^2$ because $b \in \mathcal{P}^2$ and $\mathcal{P}^2$ is closed under computability. $a' \in \mathcal{X}$ because $\mathcal{P}^2 \subset \mathcal{P}^1$. By construction, for all $t \in \mathbb{N}$, $h(\zeta_t, b(x_\zeta[t])) \geq \min_{y \in Y} \max_{x \in X} h(x, y) = v_2$. Hence, $u_h(a', b) \geq \liminf_{T \to \infty} v_2 = v_2$. It follows that $\sup_{a \in \mathcal{X}} u_h(a, b) \geq v_2$. Now, let $y^* \in \arg\min_{y \in Y}(\max_{x \in X} h(x, y))$. Let $b \in \mathcal{P}^2$ be such that $b(\tau) = y^*$ for all $\tau \in X^{<\mathbb{N}}$. Then $\sup_{a \in \mathcal{X}} u_h(a, b) = v_2$ and $\min_{b \in \mathcal{Y}} \sup_{a \in \mathcal{X}} u_h(a, b) = v_2$. □

Proof of Proposition 3.1: (In this proof I use Martin-Löf randomness [19]; its definition and basic properties are presented in Section 6.1.) By Theorem 6.1, for any $\xi, \zeta \in \{0,1\}^{\mathbb{N}}$, if ξ is $\mu_{(\frac{1}{2},\frac{1}{2})}$-random relative to $C(\zeta)$ ($C(\zeta)$ denotes the set of functions computable from ζ), then ξ is incompressible relative to $C(\zeta)$. Moreover, by Theorem 12.17 in Downey *et al.* [10], if $\xi \oplus \zeta$ (where $\xi \oplus \zeta_{2t} = \xi_t$ and $\xi \oplus \zeta_{2t+1} = \zeta_t$ for all t) is $\mu_{(\frac{1}{2},\frac{1}{2})}$-random, then ξ is $\mu_{(\frac{1}{2},\frac{1}{2})}$-random relative to $C(\zeta)$ and vice versa. Hence, $C(\xi)$ and $C(\zeta)$ are mutually complex. Now, by Proposition 6.1, the set

$$A = \{\xi \oplus \zeta : \xi \oplus \zeta \text{ is } \mu_{(\frac{1}{2},\frac{1}{2})}\text{-random}\} \subset \{\xi \oplus \zeta : C(\xi) \text{ and } C(\zeta) \text{ are mutually complex}\}$$

has measure 1. Therefore, the set A is uncountable. Because for any ξ, the set of sequences ξ' such that $C(\xi) = C(\xi')$ is countable, there are uncountably many different pairs of sets satisfying (A1) and (A2) that are mutually complex. □

Now I turn to the existence results. As mentioned in Section 3, Lemma 3.1 is a direct corollary of Theorem 6.2 and Theorem 6.5 in the Appendix and hence its proof is skipped. To prove Theorem 3.2, I first give a lemma (Lemma 5.1) concerning expected values that is used repeatedly in what follows. Recall that Theorem 3.1 is a direct corollary of Lemma 3.1 and Theorem 3.2.

Lemma 5.1. *Let X be a finite set. Suppose that $\mathcal{P}$ satisfies (A1) and (A2). Let $p \in \Delta(X)$ be a distribution and let $h : X \to \mathbb{Q}$ be a function. If ξ is a p-stochastic sequence relative to $\mathcal{P}$, then, for any selection function r for X in $\mathcal{P}$ such that $\xi^r \in X^{\mathbb{N}}$,*

$$\lim_{T\to\infty} \sum_{t=0}^{T-1} \frac{h(\xi_t^r)}{T} = \sum_{x \in X} p_x h(x).$$

Proof. Let r be a selection function in $\mathcal{P}$ such that $\xi^r \in X^{\mathbb{N}}$. Then, for any $x \in X$, $\lim_{T\to\infty} \sum_{t=0}^{T-1} \frac{c_x(\xi_t^r)}{T} = p_x$. Therefore, $\lim_{T\to\infty} \sum_{t=0}^{T-1} \frac{h(\xi_t^r)}{T} = \lim_{T\to\infty} \sum_{x\in X} \sum_{t=0}^{T-1} \frac{c_x(\xi_t^r)h(x)}{T} = \sum_{x \in X} p_x h(x)$. □

Proof of Theorem 3.2: For any strategy profile $(a, b) \in \mathcal{X} \times \mathcal{Y}$, let $\theta^{a,b}$ be the sequence of actions obtained by playing this profile, i.e., $\theta_t^{a,b} = (\theta_t^{a,b,X}, \theta_t^{a,b,Y})$ with $\theta_t^{a,b,X} = a(\theta^{a,b,Y}[t])$ and $\theta_t^{a,b,Y} = b(\theta^{a,b,X}[t])$. By Lemma 3.1, there exists a p^*-stochastic sequences $\xi^* \in \mathcal{X}$ relative to $\mathcal{P}^2$ and a q^*-stochastic sequence $\zeta^* \in \mathcal{Y}$ relative to $\mathcal{P}^1$. I show that (ξ^*, ζ^*) is an equilibrium (as history-independent strategies).

First I show that for all $a \in \mathcal{X}$,

$$\limsup_{T\to\infty} \sum_{t=0}^{T-1} \frac{h(\theta_t^{a,\zeta^*})}{T} \le h(p^*, q^*). \tag{5}$$

Suppose that $a \in \mathcal{X}$, and so $a \in \mathcal{P}^1$. For each $x \in X$, let $r^x : Y^{<\mathbb{N}} \to \{0, 1\}$ be the selection function such that $r^x(\sigma) = 1$ if $a(\sigma) = x$, and $r^x(\sigma) = 0$ otherwise. Define

$$L_x(T) = |\{t \in \mathbb{N} : 0 \le t \le T-1,\ r^x(\zeta^*[t]) = 1\}| \text{ and } \zeta^x = (\zeta^*)^{r^x}.$$

It is easy to see that r^x is in $\mathcal{P}^1$ because a is. Let

$$\mathcal{E}^1 = \{x \in X : \lim_{T\to\infty} L_x(T) = \infty\} \text{ and } \mathcal{E}^2 = \{x \in X : \lim_{T\to\infty} L_x(T) < \infty\}.$$

For each $x \in \mathcal{E}^2$, let $B_x = \lim_{T\to\infty} L_x(T)$ and let $C_x = \sum_{t=0}^{B_x} h(x, \zeta_t^x)$. Then, for any $x \in \mathcal{E}^1$, by Lemma 5.1, $\lim_{T\to\infty} \sum_{t=0}^{T-1} \frac{h(x,\zeta_t^x)}{T} = h(x, q^*) \le h(p^*, q^*)$. (Because (p^*, q^*) is an equilibrium of the game g, $h(x, q^*) \le h(p^*, q^*)$ for all $x \in X$.)

I claim that for any $\varepsilon > 0$, there is some T' such that $T > T'$ implies that

$$\sum_{t=0}^{T-1} \frac{h(a(\zeta^*[t]), \zeta_t^*)}{T} \le h(p^*, q^*) + \varepsilon. \tag{6}$$

Fix some $\varepsilon > 0$. Let T_1 be so large that $T > T_1$ implies that, for all $x \in \mathcal{E}^1$,

$$\sum_{t=0}^{T-1} \frac{h(x, \zeta_t^x)}{T} \leq h(p^*, q^*) + \frac{\varepsilon}{|X|}, \tag{7}$$

and, for all $x \in \mathcal{E}^2$, $\frac{C_x}{T} < \frac{\varepsilon}{|X|}$. Let T' be so large that, for all $x \in \mathcal{E}_1$, $L_x(T') > T_1$. If $T > T'$, then

$$\begin{aligned} \sum_{t=0}^{T-1} \frac{h(a(\zeta^*[t]), \zeta_t^*)}{T} &= \sum_{x \in \mathcal{E}_1} \frac{L_x(T)}{T} \sum_{t=0}^{L_x(T)-1} \frac{h(x, \zeta_t^x)}{L_x(T)} + \sum_{x \in \mathcal{E}_2} \sum_{t=0}^{L_x(T)-1} \frac{h(x, \zeta_t^x)}{T} \\ &\leq \sum_{i \in \mathcal{E}_1} \frac{L_x(T)}{T} (h(p^*, q^*) + \frac{\varepsilon}{|X|}) + \sum_{x \in \mathcal{E}_2} \frac{\varepsilon}{|X|} \leq h(p^*, q^*) + \varepsilon. \end{aligned} \tag{8}$$

Notice that L_x is weakly increasing, and $L_x(T) \leq T$ for all T. Thus, $T > T'$ implies that $L_x(T) \geq L_x(T') > T_1$, and so $T > T_1$. This proves the inequality (6), and it in turn implies (5).

Now I show that (ξ^*, ζ^*) is an equilibrium. Inequality (5) implies that for all $a \in \mathcal{X}$,

$$\liminf_{T \to \infty} \sum_{t=0}^{T-1} \frac{h(\theta_t^{a, \zeta^*})}{T} \leq h(p^*, q^*). \tag{9}$$

Symmetric arguments show that for all $b \in \mathcal{Y}$, $\limsup_{T \to \infty} \sum_{t=0}^{T-1} \frac{-h(\theta_t^{\xi^*, b})}{T} \leq -h(p^*, q^*)$, where ξ^* is identified with the strategy a^* is such that $a^*(\sigma) = \xi_{|\sigma|}$ for all $\sigma \in Y^{<\mathbb{N}}$. This implies that for all $b \in \mathcal{Y}$,

$$\liminf_{T \to \infty} \sum_{t=0}^{T-1} \frac{h(\theta_t^{\xi^*, b})}{T} \geq h(p^*, q^*). \tag{10}$$

By (9), for all $a \in \mathcal{X}$, $u_h(a, \zeta^*) \leq h(p^*, q^*)$. By (10), for all $b \in \mathcal{Y}$, $u_h(\xi^*, b) \geq h(p^*, q^*)$. Therefore, $u_h(\xi^*, \zeta^*) \leq h(p^*, q^*) \leq u_h(\xi^*, \zeta^*)$. This, together with (9) and (10), implies that (ξ^*, ζ^*) is an equilibrium.

On the other hand, if $\xi' \in \mathcal{X}$ is a p'-stochastic sequence for some equilibrium mixed-strategy p' in g, then, because g is a zero-sum game, $h(p', q^*) = h(p^*, q^*)$. All the above arguments hold for (ξ', ζ^*) as well and hence ξ' is also an equilibrium strategy. □

Proof of Theorem 3.3: I consider two cases (recall that $\{r(\xi[t])\}_{t=0}^{\infty}$ is an $(\alpha, 1-\alpha)$-sequence with $\alpha < 1$, ξ is an equilibrium p-sequence, and ξ^r is a p'-sequence):

Case 1: $\lim_{T\to\infty}\sum_{t=0}^{T-1} r(\xi[t])/T = 1$. Suppose, by contradiction, that p' is not an equilibrium mixed-strategy of g. Then for some action $y_1 \in Y$, $h(p', y_1) < v$, where v is the equilibrium payoff for player 1 in g. Then the sequence $\mathbf{y_1}$ can be identified with the strategy b defined as $b(\sigma) = y_1$ for all $\sigma \in X^{<\mathbb{N}}$. Let $1 - r$ be the selection function such that $(1 - r)(\sigma) = 1 - r(\sigma)$ for all $\sigma \in X^{<\mathbb{N}}$, and let $S(T) = \sum_{t=0}^{T-1} r(\xi[t])$ for all $T \in \mathbb{N}$.

Then, $\lim_{T\to\infty}\sum_{t=0}^{T-1} \frac{h(\xi_t, \mathbf{y_1}_t)}{T} = \lim_{T\to\infty}\sum_{t=0}^{S(T)-1} \frac{h(\xi_t^r, y_1)}{T} + \lim_{T\to\infty}\sum_{t=0}^{T-S(T)-1} \frac{h(\xi_t^{1-r}, y_1)}{T}$. Clearly, $\lim_{T\to\infty}\sum_{t=0}^{S(T)-1} \frac{h(\xi_t^r, y_1)}{T} = \lim_{T\to\infty}\sum_{t=0}^{S(T)-1} \frac{h(\xi_t^r, y_1)}{S(T)} \lim_{T\to\infty} \frac{S(T)}{T} = h(p', y_1) < v$. Let $C = \max_{x\in X, y\in Y} |h(x, y)|$. Then, $\sum_{t=0}^{T-S(T)-1} \frac{|h(\xi_t^{1-r}, y_1)|}{T} \leq \frac{(T-S(T))C}{T} \to 0$, and so $\lim_{T\to\infty}\sum_{t=0}^{T-S(T)-1} \frac{h(\xi_t^{1-r}, y_1)}{T} = 0$. Therefore, $u_h(\xi, \mathbf{y_1}) < v$ and hence ξ is not an equilibrium strategy, a contradiction.

Case 2: $\lim_{T\to\infty}\sum_{t=0}^{T-1} r(\xi[t])/T = 1 - \alpha < 1$.

First I show that the sequence ξ^{1-r} has limit relative frequency for all $x \in X$. For each $x \in X$ (recall that $p_x = \lim_{T\to\infty}\sum_{t=0}^{T-1} \frac{c_x(\xi_t)}{T}$ and $S(T) = \sum_{t=0}^{T-1} r(\xi[t])$ for all $T \in \mathbb{N}$),

$$\lim_{T\to\infty}\sum_{t=0}^{T-1} \frac{c_x(\xi_t^{1-r})}{T} = \lim_{T\to\infty}\sum_{t=0}^{T-1} \frac{c_x(\xi_t)(1 - r(\xi[t]))}{T - S(T)}$$

$$= \lim_{T\to\infty}\sum_{t=0}^{T-1} \frac{c_x(\xi_t)(1 - r(\xi[t]))}{T} \lim_{T\to\infty}\sum_{t=0}^{T-1} \frac{T}{T - S(T)}$$

$$= (\lim_{T\to\infty}\sum_{t=0}^{T-1} \frac{c_x(\xi_t)}{T} - \lim_{T\to\infty}\sum_{t=0}^{T-1} \frac{c_x(\xi_t) r(\xi[t])}{S(T)} \lim_{T\to\infty} \frac{S(T)}{T}) \lim_{T\to\infty}\sum_{t=0}^{T-1} \frac{T}{T - S(T)}$$

$$= (p_x - (1 - \alpha)p'_x)\frac{1}{\alpha}.$$

Suppose, by contradiction, that p' is not an equilibrium mixed-strategy of g. Then for some action $y_1 \in Y$, $h(p, y) < v$, where v is the equilibrium payoff for player 1 in g, and for some action $y_2 \in Y$, $h(\frac{1}{\alpha}(q - (1 - \alpha)p), y_2) \leq v$. Define $b \in \mathcal{Y}$ as follows: $b(\sigma) = y_1$ if $r(\sigma) = 1$ and $b(\sigma) = y_2$ if $r(\sigma) = 0$. Then ($S(T) = \sum_{t=0}^{T-1} r(\xi[t])$ for all $T \in \mathbb{N}$),

$$\lim_{T\to\infty}\sum_{t=0}^{T-1} \frac{h(\theta_t^{\xi,b})}{T} = \lim_{T\to\infty}\sum_{t=0}^{S(T)-1} \frac{h(\xi_t^r, y_1)}{T} + \lim_{T\to\infty}\sum_{t=0}^{T-S(T)-1} \frac{h(\xi_t^{1-r}, y_2)}{T}$$

$$= \lim_{T\to\infty}\sum_{t=0}^{S(T)-1} \frac{h(\xi_t^r, y_1)}{S(T)} \lim_{T\to\infty} \frac{S(T)}{T} + \lim_{T\to\infty}\sum_{t=0}^{T-S(T)-1} \frac{h(\xi_t^{1-r}, y_2)}{T - S(T)} \lim_{T\to\infty} \frac{T - S(T)}{T}$$

$$= (1-\alpha)h(p, y_1) + \alpha h(\frac{1}{\alpha}(q-(1-\alpha)p), y_2) < v.$$

Therefore, $u_h(\xi, b) < v$ and so ξ is not an equilibrium strategy, a contradiction. □

Proof of Theorem 3.4: (I use the Martin-Löf randomness directly in this proof; see Section 6.1 for a formal definition.) Recall that p^* is an equilibrium strategy with $0 < p^*_x < 1$. Let $x' \neq x$ be such that $p^*_{x'} > 0$. Such x' exists because $p^*_x < 1$.

For any real number s, let $\llcorner s \lrcorner$ be the largest integer less than or equal to s. Construct the sequence $\mathbf{p} = (p^0, p^1, ...)$ as follows ($\bar{t}$ is the smallest t such that $\llcorner t^{0.4} \lrcorner > \frac{1}{p^*_x}$):

(a) $p^t_y = p^*_y$ if $y \neq x$ and $y \neq x'$;

(b) $p^t_x = p^*_x$ if $t \leq \bar{t}$ and $p^t_x = p^*_x - \frac{1}{\llcorner t^{0.4} \lrcorner}$ otherwise;

(c) $p^t_{x'} = p^*_{x'}$ if $t \leq \bar{t}$ and $p^t_{x'} = p^*_{x'} + \frac{1}{\llcorner t^{0.4} \lrcorner}$ otherwise.

By construction, $p^t_x = 0$ if and only if $p^*_x = 0$, and $\lim_{t\to\infty} p^t = p^*$. Clearly, $\mathbf{p}$ is computable. By Theorem 6.2, there is a $\mu_{\mathbf{p}}$-random sequence ξ relative to $\mathcal{P}^2$ in $\mathcal{P}^1$. Now, let $X_0 = \{y \in X : p_y > 0\}$, then the sequence ξ is $\mu_{\mathbf{p}}$-random can be regarded as a sequence in $X_0^{\mathbb{N}}$. Therefore, Theorem 6.5 is applicable and so ξ is p^*-stochastic. Thus, by Theorem 3.2, ξ is an equilibrium strategy for player 1 in $RG(g, \mathcal{P}^1, \mathcal{P}^2)$.

By Theorem 6.6,

$$\limsup_{T\to\infty} \frac{|\sum_{t=0}^{T-1}(c_x(\xi_t) - p^t_x)|}{\sqrt{2(\sum_{t=0}^{T-1} p^t_x(1-p^t_x)) \log\log \sqrt{(\sum_{t=0}^{T-1} p^t_x(1-p^t_x))}}} = 1. \quad (11)$$

For any $T > \bar{t}$, $\frac{\sum_{t=0}^{T-1} c_x(\xi_t) - Tp^*_x}{\sqrt{2Tp^*_x(1-p^*_x)\log\log T}} = \frac{\sum_{t=0}^{T-1}(c_x(\xi_t) - p^t_x)}{\sqrt{2Tp^*_x(1-p^*_x)\log\log T}} + \frac{\sum_{t=\bar{t}+1}^{T-1} \frac{1}{\llcorner t^{0.4} \lrcorner}}{\sqrt{2Tp^*_x(1-p^*_x)\log\log T}}$.

I claim that

$$\lim_{T\to\infty} \frac{\sum_{t=\bar{t}+1}^{T-1} \frac{1}{\llcorner t^{0.4} \lrcorner}}{\sqrt{2Tp^*_x(1-p^*_x)\log\log T}} = \infty; \quad (12)$$

and there exists some $B > 0$ such that for all T large enough,

$$\frac{|\sum_{t=0}^{T-1}(c_x(\xi_t) - p^t_x)|}{\sqrt{2Tp^*_x(1-p^*_x)\log\log T}} < B. \quad (13)$$

The theorem follows directly from (12) and (13).

Now I prove the claim. For all t, $\llcorner t^{0.4} \lrcorner \leq t^{0.4}$ and so $\frac{1}{t^{0.4}} \leq \frac{1}{\llcorner t^{0.4} \lrcorner}$. Then,

$$\sum_{t=1}^{T-1} \frac{1}{\llcorner t^{0.4} \lrcorner} \geq \sum_{t=1}^{T-1} \frac{1}{t^{0.4}} \geq \int_{x=1}^{T-1} x^{-0.4} dx - 1 \geq (T-1)^{0.6} - 2.$$

Therefore, for T large enough,

$$\frac{\sum_{t=\bar{t}+1}^{T-1} \frac{1}{\llcorner t^{0.4} \lrcorner}}{\sqrt{2Tp_x^*(1-p_x^*) \log\log T}} \geq \frac{0.5T^{0.6}}{\sqrt{2Tp_x^*(1-p_x^*) \log\log T}} = C \frac{T^{0.1}}{\sqrt{\log\log T}} \tag{14}$$

for some constant $C > 0$. Because $\lim_{T\to\infty} \frac{T^{0.1}}{\sqrt{\log\log T}} = \infty$, (14) implies that (12).

Because of (11), to prove (13), it suffices to show that for T large enough,

$$\frac{\sqrt{2(\sum_{t=0}^{T-1} p_x^t(1-p_x^t)) \log\log \sqrt{(\sum_{t=0}^{T-1} p_x^t(1-p_x^t))}}}{\sqrt{2Tp_x^*(1-p_x^*) \log\log T}} \tag{15}$$

is bounded. Now, for T large enough,

$$\sum_{t=0}^{T-1} p_x^t(1-p_x^t)) = Tp_x^*(1-p_x^*) + (2p_x^* - 1) \sum_{t=\bar{t}+1}^{T-1} \frac{1}{\llcorner t^{0.4} \lrcorner} - \sum_{t=\bar{t}+1}^{T-1} (\frac{1}{\llcorner t^{0.4} \lrcorner})^2. \tag{16}$$

Because for t large enough, $\frac{1}{2}t^{0.4} < \llcorner t^{0.4} \lrcorner$, there is a constant $A > 0$ such that

$$\sum_{t=\bar{t}+1}^{T-1} \frac{1}{\llcorner t^{0.4} \lrcorner} < \sum_{t=\bar{t}+1}^{T-1} \frac{2}{t^{0.4}} + A < 2T^{0.6} + A.$$

Similarly, there is a constant $A' > 0$ such that

$$\sum_{t=\bar{t}+1}^{T-1} (\frac{1}{\llcorner t^{0.4} \lrcorner})^2 < \sum_{t=\bar{t}+1}^{T-1} \frac{4}{t^{0.8}} + A' < 4T^{0.2} + A'.$$

Hence, $\frac{\sum_{t=0}^{T-1} p_x^t(1-p_x^t)}{Tp_x^*(1-p_x^*)} < 1 + \frac{2|2p_x^*-1|}{T^{0.4}p_x^*(1-p_x^*)} + \frac{|2p_x^*-1|A+A'}{Tp_x^*(1-p_x^*)} + \frac{4}{T^{0.8}p_x^*(1-p_x^*)}$, and so, for T large enough, $\frac{\sum_{t=0}^{T-1} p_x^t(1-p_x^t))}{Tp_x^*(1-p_x^*)} < 2$. Equation (16) also implies that, for T large enough,

$$\sum_{t=0}^{T-1} p_x^t(1-p_x^t) \leq (2|2p_x^* - 1| + 5 + p_x^*(1-p_x^*))T = A''T,$$

and hence

$$\frac{\log\log \sqrt{(\sum_{t=0}^{T-1} p_x^t(1-p_x^t))}}{\log\log T} \leq \frac{\log(\log T + \log A'')}{\log\log T} \leq \frac{\log 2 + \log\log T}{\log\log T}.$$

So for T large enough, $\frac{\log\log\sqrt{(\sum_{t=0}^{T-1} p_x^t(1-p_x^t))}}{\log\log T} \leq 2$. Thus, the expression in (15) is bounded by 2, and this proves (13). $\square$

Proof of Theorem 4.1: (I use the Martin-Löf randomness directly in this proof; see Section 6.1 for a formal definition.) Let $\xi \in \mathcal{P}^1$ be incompressible relative to $\mathcal{P}^2$. ξ exists because of mutual complexity. Let $\zeta \in \mathcal{P}^2$ be incompressible relative to $\mathcal{T}$. This is equivalent to say that ξ is ML-random relative to ζ and ζ is ML-random in Nies [23] (see also Section 6.1 for a definition of ML-randomness and its basic properties; in Nies [23], ML-randomness refers to ML-randomness w.r.t. the uniform distribution). By the van Lambalgen's theorem (Theorem 3.4.6 in Nies [23]), $\xi \oplus \zeta$ is then incompressible relative to $\mathcal{T}$. $\square$

6 Appendix

The Appendix gives some preliminary background knowledge in Algorithmic Randomness (for a comprehensive overview of that literature, see Nies [23]) and presents some results that are extensions of extant results in that literature. Results here are technical in nature but are extensively used in the proofs in the first part.

6.1 Martin-Löf randomness

Here I discuss *Martin-Löf randomness* [19] (henceforth ML randomness) that is used in various proofs in Section 5. This concept defines random sequences in terms of statistical regularities. Its definition begins with a formulation of idealized statistical tests, which requires some further notations. Let X be a finite set. The set of infinite sequences $X^{\mathbb{N}}$ over X is endowed with the product topology. Any open set can be written as a union of basic sets, where a basic set has the form $N_\sigma = \{\zeta \in X^{\mathbb{N}} : \sigma = \zeta[|\sigma|]\}$ for some $\sigma \in X^{<\mathbb{N}}$. A formal definition of ML-randomness is given in the following.

Definition 6.1. Let X be a finite set and let $\mathcal{P}$ be a set of functions satisfying (A1) and

(A2). Suppose that μ is a computable probability measure over $X^{\mathbb{N}}$,[25] i.e., the mapping $\sigma \mapsto \mu(N_\sigma)$ is recursive. A sequence of open sets $\{V_t\}_{t=0}^{\infty}$ is *a μ-test relative to $\mathcal{P}$* if it satisfies the following conditions:

(1) There is a function $f : \mathbb{N} \to \mathbb{N} \times X^{<\mathbb{N}}$ in $\mathcal{P}$ such that for all $t \in \mathbb{N}$,

$$V_t = \bigcup\{N_\sigma : (\exists n)(f(n) = (t, \sigma))\}.$$

(2) For all $t \in \mathbb{N}$, $\mu(V_t) \leq 2^{-t}$.

A sequence $\xi \in X^{\mathbb{N}}$ is *μ-random relative to $\mathcal{P}$* if it passes all μ-tests relative to $\mathcal{P}$, i.e., for any μ-test $\{V_t\}_{t=0}^{\infty}$ relative to $\mathcal{P}$, $\xi \notin \bigcap_{t=0}^{\infty} V_t$.

The crucial part of this definition is to define a "describable" event with probability 1.[26] Martin-Löf [19] defines "describable" by saying that an event with probability 1 is describable if there is a constructive proof for that fact. A test $\{V_t\}_{t=0}^{\infty}$ is used to establish the statistical regularity corresponding to the complement of $\bigcap_{t=0}^{\infty} V_t$. Conditions (1) and (2) require that this test, as a sequence of open sets, be generated by functions in $\mathcal{P}$, and hence is constructive with respect to $\mathcal{P}$. A sequence is random if it passes all such tests. If $\mathcal{P}$ satisfies both (A1) and (A2), then for any computable probability measure μ, the set of μ-random sequences relative to $\mathcal{P}$ has probability 1 with respect to μ (see Downey *et al.* [10]), as stated in the following proposition. Its proof for a special case can be found in Martin-Löf [19] (with some minor modifications to accommodate general computable measures). See also Nies [23], p.123. The proof can be easily extended to cover the general case.

Proposition 6.1. *Suppose that X is a finite set and μ is a computable measure over $X^{\mathbb{N}}$. Let $\mathcal{P}$ satisfy (A1) and (A2). Then* $\mu(\{\xi \in X^{\mathbb{N}} : \xi$ *is μ-random relative to* $\mathcal{P}\}) = 1$.

[25] In the definition, I implicitly assume that $\mu(N_\sigma)$ is always a rational number for μ to be computable. In the literature, the computability of a measure is defined more generally, but this definition is sufficient for my purpose.

[26] The quantifier "describable" is necessary (because there exists no sequence belonging to all events with probability 1), but hard to define.

Here the most relevant measures are Bernoulli measures, measures generated by a distribution over X in an i.i.d. manner. In what follows, ML-randomness, without reference to a specific probability measure, means ML randomness with respect to an i.i.d. measure.

6.2 Generating random sequences

Here I give a theorem which shows that incompressible sequences can be used to generate ML-random sequences. I first cite a result (Theorem 6.1) which shows that a sequence is incompressible if and only if it is random w.r.t. the uniform distribution. Its proof can be found in Nies [23], p. 122.

Theorem 6.1. *Suppose that $\mathcal{P}$ satisfies (A1) and (A2). Then, $\xi \in \{0,1\}^{\mathbb{N}}$ is an incompressible sequence relative to $\mathcal{P}$ if and only if ξ is a λ-random sequence relative to $\mathcal{P}$, where $\lambda(N_\sigma) = 2^{-|\sigma|}$ for all $\sigma \in \{0,1\}^{<\mathbb{N}}$.*

The next theorem (Theorem 6.2) is the main result of this section. It shows that, if mutual complexity holds, each player can generate a $\mu_{\mathbf{p}}$-random sequence relative to the other player for any $\mathbf{p}$, where $\mu_{\mathbf{p}}$ is defined as $\mu_{\mathbf{p}}(\sigma) = \prod_{t=0}^{|\sigma|-1} p^t(\sigma_t)$, with $\mathbf{p} = (p^0, p^1, ...)$ being a sequence of probability measures over X. The proof for the case $|X| = 2$ can be found in Zvonkin and Levin [31]. I follow a similar logic, which is based on a computable version of the well-known result that one can generate a random variable with an arbitrary distribution function from the uniform distribution. In the following theorem and the subsequent pages, I assume that $\mathbf{p}$ is computable and $p^t_x > 0$ for all $x \in X$ and all $t \in \mathbb{N}$ unless stated otherwise.

To prove this theorem, some additional tools from Recursion theory is necessary. For my purpose here, it is more convenient to consider *Turing machines* with *oracle inquiries* (c.f. Odifreddi [24]). Given a finite set X, an element ξ in $X^{\mathbb{N}}$ is regarded a *Turing oracle*. Thus, if $\mathcal{P}$ satisfies (A1) and (A2), then $\mathcal{P} = \{f : \mathbb{N}^k \to \mathbb{N} |\ f \text{ is computable from } \xi\}$ for some oracle ξ.[27] Here, however, I consider also partial functions. I use $C(\xi)$ to denote the

[27] By (A2) we know that this is true for some total function f; we can then use the graph of f, which is also computable from f, to construct the oracle ξ. For more details, see Odifreddi [24].

set of partial functions that are computable from ξ, and $\varphi_e^{(k),\xi}$ denotes the partial function that is computed by the machine with Gödel number e and with oracle ξ. It is known that the predicate $\varphi_{e,s}^{(k),\xi[s]}(t_1, ..., t_k) = r$, which indicates whether the machine with index e halts (using $(t_1, .., t_k)$ as inputs and ξ as the oracle) within s steps and produce output r, is computable. If it halts within s steps, then it is denoted by $\varphi_{e,s}^{(k),\xi[s]}(t_1, ..., t_k)\downarrow$. This result is usually called the *Enumeration Theorem* in the literature (see Downey *et al.* [10]). I can then define *recursively enumerable* sets, which will be useful in what to come. Consider a sequence $\{V_t\}_{t=0}^{\infty}$ of subsets of $X^{\mathbb{N}}$. Such a sequence is *recursively enumerable in* ξ, or is of $\Sigma_1^{0,\xi}$, if there is a total function $f : \mathbb{N} \to \mathbb{N} \times X^{<\mathbb{N}}$ in $C(\xi)$ such that for all $t \in \mathbb{N}$ and for all $\zeta \in X^{\mathbb{N}}$,

$$\zeta \in V_t \Leftrightarrow (\exists n)(f(n) = (t, \sigma) \wedge \sigma = \zeta[|\sigma|]). \tag{17}$$

In this case, there is a total function $h \in \mathcal{T}$ such that[28]

$$\zeta \in V_t \Leftrightarrow (\exists s)\varphi_{h(t)}^{(1),\xi\oplus\zeta[s]}(0)\downarrow, \tag{18}$$

This result transform the relation in (17) into a parameterized form in (18).

Theorem 6.2. *Suppose that $\mathcal{P}$ satisfies (A1) and (A2). Let X be a finite set. Suppose that $\xi \in \{0,1\}^{\mathbb{N}}$ is an incompressible sequence relative to $\mathcal{P}$. Then, there is a $\mu_{\mathbf{p}}$-sequence $\zeta \in X^{\mathbb{N}}$ relative to $\mathcal{P}$ in $C(\xi)$.*

Proof. First I remark a known fact: there is a λ^X-random sequence $\xi' \in X^{\mathbb{N}}$ relative to $\mathcal{P}$ that is in $C(\xi)$, where $\lambda^X(N_\sigma) = |X|^{-|\sigma|}$ for all $\sigma \in X^{<\mathbb{N}}$ (i.e., the uniform distribution over $X^{\mathbb{N}}$). For a proof, see Calude [6], Theorem 7.18.

I will construct a partial computable functional $\Phi : X^{\mathbb{N}} \to X^{\mathbb{N}}$ such that if ξ' is a λ^X-random sequence, then $\Phi(\xi')$ is a $\mu_{\mathbf{p}}$-random sequence. Φ is constructed through a computable function $\phi : X^{<\mathbb{N}} \to X^{<\mathbb{N}}$ such that

$$\text{for any } \sigma, \tau \in X^{<\mathbb{N}},\ \sigma \subset \tau \text{ implies } \phi(\sigma) \subset \phi(\tau). \tag{19}$$

[28] This follows directly from the Parametrization Theorem for relative computability. See Downey *et al.* [10] for a more detailed discussion.

Then I define Φ as $\Phi(\xi)_t = \phi(\xi[\min\{k : |\phi(\xi[k])| \geq t\}])_t$. Notice that Φ is well defined if $\lim_{t\to\infty} \phi(\xi[t]) = \infty$ and it outputs a finite sequence otherwise (i.e., for large t's, $\Phi(\xi)_t$ is not defined). I will show that Φ satisfies the following properties:

1. Φ is well-defined over any sequence in $X^{\mathbb{N}}$ that is not computable.

2. $\lambda^X(\Phi^{-1}(A)) = \mu_{\mathbf{p}}(A)$ for any measurable A.

3. If ξ' is a λ^X-random sequence, then $\Phi(\xi')$ is a $\mu_{\mathbf{p}}$-random sequence.

(**Construction of** ϕ) ϕ is constructed through the distribution function of $\mu_{\mathbf{p}}$. Define the mapping Γ between $X^{\mathbb{N}}$ and $[0,1]$ as

$$\Gamma(\zeta) = \sum_{t=0}^{\infty} \iota(\zeta_t) \frac{1}{n^{t+1}},$$

where $X = \{x_1, ..., x_n\}$ and $\iota(x) = i-1$ if and only if $x = x_i$. Γ is onto but not one-to-one. However, the set $\{\zeta \in X^{\mathbb{N}} : \Gamma(\zeta) = \Gamma(\zeta') \text{ for some } \zeta' \neq \zeta\}$ is countable, since for any such ζ, $\Gamma(\zeta)$ is a rational number. Γ can be extended to $X^{<\mathbb{N}}$ by defining $\Gamma(\sigma) = \sum_{t=0}^{|\sigma|-1} \frac{\iota(\sigma_t)}{n^{t+1}}$. Given Γ, there distribution function of $\mu_{\mathbf{p}}$ over $[0,1]$ is defined by $g : [0,1] \to [0,1]$ as

$$g(r) = \mu_{\mathbf{p}}(\{\zeta : \Gamma(\zeta) \leq r\}).$$

Define $h = g^{-1}$; h exists because $\mu_{\mathbf{p}}$ has no atoms. Therefore, $r \leq g(s)$ if and only if $h(r) \leq s$. Hence,

$$\mu_{\mathbf{p}}(\Gamma^{-1}([0,r])) = g(r) = \lambda^X(\Gamma^{-1}([0, g(r)])) = \lambda^X(\Gamma^{-1}(h^{-1}([0,r]))).$$

Now I construct the function ϕ using the distribution function g. The idea of construction is the following: Because g is continuous, any open interval has a converse that is also open. Because each finite sequence in $X^{<\mathbb{N}}$ can be regarded as an open interval, it can be mapped into another via g. As the length of the finite sequence increases, the interval shrinks and finally the functional Φ obtains.

Let ϵ be the empty string. Define $g^0(\epsilon) = 0$ and $g^1(\epsilon) = 1$. For $\tau \in X^{<\mathbb{N}} - \{\epsilon\}$, define $(\sum \emptyset = 0)$

$$g^0(\tau) = \sum\{\mu_{\mathbf{p}}(N_\sigma) : \Gamma(\sigma) \leq \Gamma(\tau) - \frac{1}{n^{|\tau|}}, |\sigma| = |\tau|\};$$

$$g^1(\tau) = \sum\{\mu_{\mathbf{p}}(N_\sigma) : \Gamma(\sigma) \le \Gamma(\tau), |\sigma| = |\tau|\}.$$

For any $\zeta \in X^{\mathbb{N}}$, $\Gamma(\zeta) \le \Gamma(\tau)$ if and only if $\Gamma(\zeta[|\tau|]) \le \Gamma(\tau) - \frac{1}{n^{|\tau|}}$ or $\Gamma(\zeta) = \Gamma(\tau)$, and $\Gamma(\zeta) \le \Gamma(\tau) + \frac{1}{n^{|\tau|}}$ if and only if $\Gamma(\zeta[|\tau|]) \le \Gamma(\tau)$ or $\Gamma(\zeta) = \Gamma(\tau) + \frac{1}{n^{|\tau|}}$. Because $\mu_{\mathbf{p}}$ has no atoms, $g^0(\tau) = g(\Gamma(\tau))$ and $g^1(\tau) = g(\Gamma(\tau) + \frac{1}{n^{|\tau|}})$. Therefore, for each $t > 0$, the class of intervals

$$\{[g^0(\tau), g^1(\tau)] : \tau \in X^{<\mathbb{N}}, |\tau| = t\} \tag{20}$$

forms a partition of $[0, 1]$.

Construct ϕ as follows: given a string $\sigma \in X^{<\mathbb{N}}$ (recall that $|X| = n$), let

$$a_\sigma = \Gamma(\sigma) \text{ and } b_\sigma = \Gamma(\sigma) + \frac{1}{n^{|\sigma|}}.$$

Let $\phi(\sigma)$ be the longest τ with $|\tau| \le |\sigma|$ such that $[a_\sigma, b_\sigma] \subset [g^0(\tau), g^1(\tau)]$. $\phi(\sigma)$ is well-defined, because the intervals in (20) forms a partition and $[g^0(\epsilon), g^1(\epsilon)] = [0, 1]$. Clearly ϕ is recursive.

(Φ **satisfies property 1.**) First I show that ϕ satisfies (19). Suppose that $\sigma \subset \sigma'$ and $\tau = \phi(\sigma)$, $\tau' = \phi(\sigma')$. It is easy to check that $a_\sigma \le a_{\sigma'}$ and $b_{\sigma'} \le b_\sigma$. Now, if $\Gamma(\tau') \ge \Gamma(\tau) + \frac{1}{n^{|\tau|}}$, then $a_{\sigma'} \ge g^0(\tau') = g(\Gamma(\tau')) \ge g(\Gamma(\tau) + \frac{1}{n^{|\tau|}}) = g^1(\tau) \ge b_\sigma \ge b_{\sigma'}$, a contradiction to $a_\sigma < b_\sigma$. Hence, $\Gamma(\tau') < \Gamma(\tau) + \frac{1}{n^{|\tau|}}$.

By construction of ϕ, $|\tau'| \ge |\tau|$. If $\Gamma(\tau') < \Gamma(\tau)$, then $\Gamma(\tau') \le \Gamma(\tau) - \frac{1}{n^{|\tau|}}$, and hence, $b_\sigma \le b_{\sigma'} \le g^1(\tau') = g(\Gamma(\tau') + \frac{1}{n^{|\tau'|}}) \le g(\Gamma(\tau)) = g^0(\tau) \le a_\sigma$, a contradiction to $a_\sigma < b_\sigma$. Therefore, $\Gamma(\tau) \le \Gamma(\tau') < \Gamma(\tau) + \frac{1}{n^{|\tau|}}$ and so $\tau \subset \tau'$.

Then I show that, for any sequence ζ such that $h(\Gamma(\zeta)) \ne \frac{m}{n^t}$ for any $m, n, t \in \mathbb{N}$ (recall that $h = g^{-1}$), $\lim_{t\to\infty} \phi(\zeta[t]) = \infty$. Consider any such ζ. For any given K, there exists some $l \in \mathbb{N}$ such that $h(\Gamma(\zeta)) \in (\frac{l}{n^K}, \frac{l+1}{n^K})$. Let

$$\varepsilon = \min\{h(\Gamma(\zeta)) - \frac{l}{n^K}, \frac{l+1}{n^K} - h(\Gamma(\zeta))\}.$$

Because h is continuous, there is some T such that $t \ge T$ implies that

$$\min\{|h(b_{\zeta[t]}) - h(\Gamma(\zeta))|, |h(\Gamma(\zeta) - h(a_{\zeta[t]})|\} \le \frac{\varepsilon}{2} \text{ and so } [h(a_{\zeta[t]}), h(b_{\zeta[t]})] \subseteq (\frac{l}{n^K}, \frac{l+1}{n^K}).$$

Thus, if $t \geq \max\{T, K\}$, then

$$[a_{\zeta[t]}, b_{\zeta[t]}] \subset [g(\frac{l}{n^K}), g(\frac{l+1}{n^K})] = [g^0(\frac{l}{n^K}), g^1(\frac{l}{n^K})],$$

and so $|\phi(\zeta[t])| \geq K$. Clearly, any sequence ζ that satisfies $h(\Gamma(\zeta)) = \frac{m}{n^t}$ for some $m, n, t \in \mathbb{N}$ is computable, and so if $\Phi(\zeta)$ is not well-defined, ζ is computable.

(Φ **satisfies property 2.**) I first claim that if Φ is well-defined over ζ (the set of such ζ's is denoted by $D(\phi)$), then $\Gamma(\Phi(\zeta)) = h(\Gamma(\zeta))$. Let ε be given, and let K be so large that $\varepsilon < \frac{1}{n^{K-1}}$. Since $\zeta \in D(\phi)$, there exists T such that $t \geq T$ implies that $|\phi(\zeta[t])| \geq K$. Then, for all $t \geq T$, $h(\Gamma(\zeta)) \in [h(a_{\zeta[t]}), h(b_{\zeta[t]})] \subseteq [a_{\phi(\zeta[t])}, b_{\phi(\zeta[t])}]$, and so $h(\Gamma(\zeta)) - \Gamma(\phi(\zeta[t])) \leq \frac{1}{n^K} \leq \varepsilon$. Thus, $\Gamma(\Phi(\zeta)) = \lim_{t\to\infty} \Gamma(\phi(\zeta[t])) = h(\Gamma(\zeta))$. Moreover, for almost all $r \in [0,1]$ (except for countably many of them), there is a sequence $\zeta \in X^{\mathbb{N}}$ such that $\Gamma(\Phi(\zeta)) = r$, because h is strictly increasing and is continuous. Also,

$$\Gamma(\Phi(\zeta)) \geq \Gamma(\Phi(\zeta')) \Leftrightarrow \Gamma(\zeta) \geq \Gamma(\zeta'). \tag{21}$$

I show that $\lambda^X_\Phi = \mu_{\mathbf{p}}$ by demonstrating that they share the same distribution function g, where $\lambda^X_\Phi(A) = \lambda^X(\Phi^{-1}(A))$: for any ζ^*,

$$\lambda^X_\Phi(\{\zeta : \Gamma(\zeta) \leq \Gamma(\Phi(\zeta^*))\}) = \lambda^X(\{\zeta : \Gamma(\Phi(\zeta)) \leq \Gamma(\Phi(\zeta^*))\})$$

$$= \lambda^X(\{\zeta : \Gamma(\zeta) \leq \Gamma(\zeta^*)\}) = \Gamma(\zeta^*) = g(\Gamma(\Phi(\zeta^*))).$$

(Recall that, for all but a countable set of numbers $r \in [0,1]$, there is a ζ^* such that $\Gamma(\Phi(\zeta^*)) = r$. The gaps may be filled by assigning arbitrary values on Φ when it is not well-defined. The first equality comes from the definition of λ^X_Φ and the second comes from equation (21).)

(Φ **satisfies property 3.**) Recall that Φ is well-defined over any incomputable sequence. Thus, if ξ' is λ^X-random relative to $\mathcal{P}$, $\xi' \in D(\phi)$. Let $\zeta' = \Phi(\xi')$. Now I show that ζ' is $\mu_{\mathbf{p}}$-random relative to $\mathcal{P}$. Suppose not. Then there is a $\mu_{\mathbf{p}}$-test $\{V_t\}_{t=0}^\infty$ relative to $\mathcal{P}$ such that $\zeta' \in \bigcap_{t=0}^\infty V_t$. Specifically, suppose that $\mathcal{P} = C(\eta)$ for some $\eta \in \{0,1\}^\infty$ and for some total function h in $\mathcal{T}$,

$$V_t = \{\zeta : (\exists s)\varphi^{(1),\eta\oplus\zeta[s]}_{h(t)}(0)\downarrow\}.$$

Define U_t as ($\sigma \oplus \tau_{2t} = \sigma_t$ and $\sigma \oplus \tau_{2t+1} = \tau_t$ for all $t < |\sigma|$; the function does not halt if $|\phi(\zeta[s'])| < s$)

$$U_t = \{\zeta : (\exists s)(\exists s')\varphi_{h(t)}^{(1),\eta[s]\oplus\phi(\zeta[s'])[s]}(0)\downarrow\}.$$

Because ϕ is computable, $\{U_t\}_{t=0}^{\infty}$ is of $\Sigma_1^{0,\eta}$. It is easy to check that

$$\Phi^{-1}(V_t) \cap D(\phi) \subset U_t \subset \Phi^{-1}(V_t) \cup (X^{\mathbb{N}} - D(\phi)).$$

Because $\lambda^X(D(\phi)) = 1$ by property 1, for all t, $\lambda^X(U_t) = \mu_{\mathbf{p}}(V_t) \leq \frac{1}{2^t}$.

Therefore, $\{U_t\}_{t=0}^{\infty}$ is a λ^X-test relative to $\mathcal{P}$. But $\xi' \in \bigcap_{t=0}^{\infty} U_t$ since $\zeta' \in \bigcap_{t=0}^{\infty} V_t$, a contradiction. Since ϕ is computable, $\zeta' \in C(\xi') \subset C(\xi)$. □

Here I show that there is always an equilibrium strategy in the repeated game under mutual complexity that satisfies all statistical regularities with respect to the i.i.d. distribution generated by the stage-game mixed equilibrium.

Theorem 6.3. *Suppose that $\mathcal{P}^1$, $\mathcal{P}^2$ satisfy (A1), (A2), and mutual complexity. Let p^* be an equilibrium strategy of player 1 in g. There exists an equilibrium strategy $\xi \in \mathcal{X}$ for player 1 that is μ_{p^*}-random relative to $\mathcal{P}^2$ in $RG(g, \mathcal{P}^1, \mathcal{P}^2)$.*

Proof. Let p be an equilibrium strategy of player 1 in g. By Theorem 6.2, there exists a μ_p-random sequence relative to $\mathcal{P}^2$ in $\mathcal{X}$. By Theorem 6.5, such a sequence is p-stochastic and hence is an equilibrium strategy in both the repeated game. □

6.3 Martingales and stochastic sequences

Here I prove that any $\mu_{\mathbf{p}}$-random sequence is p-stochastic if $\lim_{t\to\infty} p^t = p$. This requires a new concept called martingales, defined as follows.

Definition 6.2. Let X be a finite set and let $\mathbf{p} = (p^0, p^1, ...)$ be a computable sequence of distributions over X such that $p_x^t > 0$ for all $x \in X$ and for all $t \in \mathbb{N}$. A function $M : X^{<\mathbb{N}} \to \mathbb{R}_+$ is a *martingale with respect to* $\mu_{\mathbf{p}}$ if for all $\sigma \in X^{<\mathbb{N}}$, $M(\sigma) = \sum_{x\in X} p_x^{|\sigma|} M(\sigma\langle x\rangle)$ ($\sigma\langle x\rangle$ denotes the sequence that adds x to σ in the end).

Let $\mathcal{P} \subset \mathcal{F}$ satisfy (A1) and (A2). A martingale M is $\mathcal{P}$-*effective* if there is a sequence of martingales $\{M_t\}_{t=0}^{\infty}$ that satisfies the following properties:

(a) $M_t(\sigma) \in \mathbb{Q}_+$ for all $t \in \mathbb{N}$ and for all $\sigma \in X^{<\mathbb{N}}$;

(b) the function g defined as $g(t, \sigma) =_{def} M_t(\sigma)$ belongs to $\mathcal{P}$;

(c) $\lim_{t\to\infty} M_t(\sigma) \uparrow M(\sigma)$ for all $\sigma \in X^{<\mathbb{N}}$.

In this case, the sequence $\{M_t\}_{t=0}^{\infty}$ is said to support M.

The following theorem characterizes randomness in terms of martingales. The proof for a special case of this theorem can be found in Downey *et al.* [10]. I provide a short sketch of the proof for self-containment.

Theorem 6.4. *Suppose that $\mathcal{P}$ satisfy (A1) and (A2). Let $\mathbf{p}$ be a computable sequence such that $p_x^t > 0$ for all $x \in X$ and for all $t \in \mathbb{N}$. A sequence $\xi \in X^{\mathbb{N}}$ is $\mu_{\mathbf{p}}$-random relative to $\mathcal{P}$ if and only if for any $\mathcal{P}$-effective martingale M w.r.t. $\mu_{\mathbf{p}}$, $\limsup_{T\to\infty} M(\xi[t]) < \infty$.*

Proof. ($\Rightarrow$) Suppose that $\limsup_{T\to\infty} M(\xi[T]) = \infty$ and M is a $\mathcal{P}$-effective martingale. Let $V_t = \{\xi : (\exists s)(M(\xi[s]) > 2^t)\}$. It is easy to show that for any σ and any prefix-free set $A \subset \{\tau : \tau \subseteq \sigma\}$, $\sum_{\tau\in A} \mu_{\mathbf{p}}(N_\tau)M(\tau) \le \mu_{\mathbf{p}}(N_\sigma)M(\sigma)$. Thus, $\mu_{\mathbf{p}}(V_t) \le \frac{1}{2^t}$. It is routine to check that $\{V_t\}$ is a $\mu_{\mathbf{P}}$-test relative to $\mathcal{P}$ because M is $\mathcal{P}$ effective. Then $\xi \in \bigcap_{t=0}^{\infty} V_t$ and hence is not $\mu_{\mathbf{p}}$-random.

($\Leftarrow$) Suppose that $\xi \in \bigcap_{t=0}^{\infty} V_t$ for a $\mu_{\mathbf{P}}$-test $\{V_t\}$ relative to $\mathcal{P}$. Let $M^t(\sigma) = \mu_{\mathbf{p}}(V_t \cap N_\sigma)/\mu_{\mathbf{p}}(N_\sigma)$ and let $M = \sum_{t=0}^{\infty} M^t$. It is routine to check that M^t is a martingale. $M(\epsilon) = \sum_{t=0}^{\infty} \mu_{\mathbf{p}}(V_t) = 1$ and for any σ, $M^t(\sigma) \le M(\epsilon)/\mu_{\mathbf{p}}(N_\sigma)$ and so M is well-defined. It is easy to show that M is $\mathcal{P}$-effective because $\{V_t\}$ is a $\mu_{\mathbf{P}}$-test relative to $\mathcal{P}$. $\xi \in \bigcap_{t=0}^{\infty} V_t$ implies that $\limsup_{T\to\infty} M(\xi[T]) = \infty$. □

Here is the main result of this subsection. Theorem 6.5 shows that any $\mu_{\mathbf{p}}$-random sequence is also p-stochastic. As a corollary, any μ_p-random sequence is also p-stochastic. Moreover, by Theorem 6.2, if mutual complexity holds, then each player can generate a μ_p-random sequence for any p relative to the other player and hence can generate a p-stochastic sequence relative to the other player.

Theorem 6.5. *Suppose that $\mathcal{P}$ satisfies (A1) and (A2). Suppose that ξ is $\mu_{\mathbf{p}}$-random relative to $\mathcal{P}$ with $p_x^t > 0$ for all $t \in \mathbb{N}$ and for all $x \in X$ and $\lim_{t\to\infty} p^t = p$. Then, ξ is a p-stochastic sequence relative to $\mathcal{P}$.*

Proof. By Theorem 6.4, for any $\mathcal{P}$-effective martingale M w.r.t. $\mu_{\mathbf{p}}$, $\limsup_{T\to\infty} M(\xi[t]) < \infty$. Suppose, by contradiction, that there is a selection function r for X in $\mathcal{P}$ such that ξ^r is a total function and assume that there exists some $\varepsilon > 0$ and a sequence $\{T_k\}_{k=0}^{\infty}$ such that for all $k \in \mathbb{N}$,

$$\sum_{t=0}^{T_k - 1} \frac{c_x(\xi_t^r)}{T_k} \geq p_x + \varepsilon.$$

I construct a martingale (w.r.t. $\mu_{\mathbf{p}}$) such that $\limsup_{T\to\infty} M(\xi[t]) = \infty$.

(**Construction**) Define M as follows:

(a) $M(\epsilon) = 1$;

(b) $M(\sigma\langle x\rangle) = (1 + \kappa(1 - p_x^{|\sigma|}))M(\sigma)$ and $M(\sigma\langle y\rangle) = (1 - \kappa p_x^{|\sigma|})M(\sigma)$ for all $y \neq x$ if $r(\sigma) = 1$;

(c) $M(\sigma\langle y\rangle) = M(\sigma)$ for all $y \in X$ if $r(\sigma) = 0$.

Here, κ is a fixed rational number whose value will be determined latter. Clearly, by construction, $M \in \mathcal{P}$ because $r \in \mathcal{P}$ and $\mathbf{p}$ is computable.

(**Verify that M is a martingale**) If $r(\sigma) = 1$, then

$$\sum_{y\in X} p_y^{|\sigma|} M(\sigma\langle y\rangle) = p_x^{|\sigma|}(1 + \kappa(1 - p_x^{|\sigma|}))M(\sigma) + \sum_{y\neq x} p_y^{|\sigma|}(1 - \kappa p_x^{|\sigma|})M(\sigma)$$

$$= M(\sigma) + \kappa M(\sigma)(p_x^{|\sigma|}(1 - p_x^{|\sigma|}) - (1 - p_x^{|\sigma|})p_x^{|\sigma|}) = M(\sigma);$$

if $r(\sigma) = 0$, then $\sum_{y\in X} p_y^{|\sigma|} M(\sigma\langle y\rangle) = \sum_{y\in X} p_y^{|\sigma|} M(\sigma) = M(\sigma)$.

(M **satisfies** $\limsup_{T\to\infty} M(\xi[t]) = \infty$) For $k \geq 1$, define

$$D_k = \{t \leq k - 1 : r(\xi[t]) = 1, \xi_{t+1} = x\} \text{ and } E_k = \{t \leq k - 1 : r(\xi[t]) = 1, \xi_{t+1} \neq x\}.$$

Then, $M(\xi[k]) = \prod_{t\in D_k}(1+\kappa(1-p_x^{t+1}))\prod_{t\in E_k}(1-\kappa p_x^{t+1})$. Let $l_k = (L_r^\xi)^{-1}(T_k)$. Since ξ^r is total, l_k is well defined for all $k \in \mathbb{N}$.

Let $\delta = \min\{p_x, 1-p_x, \frac{\varepsilon}{2}\}$. Since $\lim_{t\to\infty} p^t = p$, let T be so large that $t \geq T$ implies that $|p_x^t - p_x| < \delta$. Let K be the first k such that $T_k > T$. Then, for all $k > K$,

$$M(\xi[l_k]) = \prod_{t\in D_{l_k}}(1+\kappa(1-p_x^{t+1}))\prod_{t\in E_{l_k}}(1-\kappa p_x^{t+1})$$

$$\geq \prod_{t\in D_{l_K}}(1+\kappa(1-p_x^{t+1})\prod_{t\in E_{l_K}}(1-\kappa p_x^{t+1}))(1+\kappa(1-p_x-\delta))^{|D_{l_k}-D_{l_K}|}(1-\kappa p_x-\kappa\delta)^{|E_{l_k}-E_{l_K}|}.$$

Let $A = \frac{\prod_{t\in D_{l_K}}(1+\kappa(1-p_x^{t+1})\prod_{t\in E_{l_K}}(1-\kappa p_x^{t+1}))}{(1+\kappa(1-p_x-\delta))^{L^1}(1-\kappa p_x-\kappa\delta)^{L^2}}$, where $L^1 = |D_{l_K}|$ and $L^2 = |E_{l_K}|$. Since for each k, $|D_{l_k}| \geq T_k p_x + T_k\varepsilon$,

$$M(\xi[l_k]) \geq A((1+\kappa(1-p_x-\delta))^{p_x+\varepsilon}(1-\kappa p_x-\kappa\delta)^{1-p_x-\varepsilon})^{T_k}.$$

Define $F(\kappa) = (1+\kappa(1-p_x-\delta))^{p_x+\varepsilon}(1-\kappa p_x-\kappa\delta)^{1-p_x-\varepsilon}$. We have $\ln F(0) = 1$ and $(\ln F)'(0) = (p_x+\varepsilon)(1-p_x-\delta)-(1-p_x-\varepsilon)(p_x+\delta) = \varepsilon - \delta > 0$. Thus, for κ small enough, $F(\kappa) > 1$, and so $\limsup_{T\to\infty} M(\xi[T]) = \infty$. □

6.4 Law of the Iterated Logarithm

Here I give a general Law of the Iterated Logarithm that is satisfied by any $\mu_{\mathbf{p}}$-random sequence.

Theorem 6.6. *Suppose that ξ is a $\mu_{\mathbf{p}}$-random sequence relative to $\mathcal{T}$ with $\mathbf{p} = (p^0, p^1, ..., p^t, ...)$. Then, for any $x \in X$,*

$$\limsup_{T\to\infty} \frac{|\sum_{t=0}^{T-1}(c_x(\xi_t) - p_x^t)|}{\sqrt{2(\sum_{t=0}^{T-1} p_x^t(1-p_x^t))\log\log\sqrt{(\sum_{t=0}^{T-1} p_x^t(1-p_x^t))}}} = 1, \tag{22}$$

Proof. The positive part of equation (22) is equivalent to the following two conditions:

(a) for all rational $\varepsilon > 0$,

$$(\exists S)(\forall T \geq S)\sum_{t=0}^{T-1}(c_x(\xi_t)-p_x^t) \leq \sqrt{2(1+\varepsilon)(\sum_{t=0}^{T-1} p_x^t(1-p_x^t))\log\log\sqrt{(\sum_{t=0}^{T-1} p_x^t(1-p_x^t))}}.$$

(b) for all rational $\varepsilon > 0$,

$$(\forall S)(\exists T \geq S) \sum_{t=0}^{T-1} (c_x(\xi_t) - p_x^t) \geq \sqrt{2(1-\varepsilon)(\sum_{t=0}^{T-1} p_x^t(1-p_x^t)) \log\log \sqrt{(\sum_{t=0}^{T-1} p_x^t(1-p_x^t))}}.$$

I show that (a) and (b) hold and the negative part is completely symmetric. Let

$$E_T^\varepsilon = \{\zeta : \sum_{t=0}^{T-1} (c_x(\zeta_t) - p_x^t) > \sqrt{2(1+\varepsilon)(\sum_{t=0}^{T-1} p_x^t(1-p_x^t)) \log\log \sqrt{(\sum_{t=0}^{T-1} p_x^t(1-p_x^t))}}\},$$

and

$$F_T^\varepsilon = \{\zeta : \sum_{t=0}^{T-1} (c_x(\zeta_t) - p_x^t) < \sqrt{2(1-\varepsilon)(\sum_{t=0}^{T-1} p_x^t(1-p_x^t)) \log\log \sqrt{(\sum_{t=0}^{T-1} p_x^t(1-p_x^t))}}\}.$$

Clearly, condition (a) is equivalent to $\xi \notin \bigcap_{S=0}^\infty \bigcup_{T=S}^\infty E_T^\varepsilon$ and condition (b) is equivalent to $\xi \notin \bigcup_{S=0}^\infty \bigcap_{T=S}^\infty F_T^\varepsilon$. By Theorem 7.5.1 in Chung [9],

$$\mu_{\mathbf{p}}(\bigcap_{S=0}^\infty \bigcup_{T=S}^\infty E_T^\varepsilon) = 0 \text{ and } \mu_{\mathbf{p}}(\bigcup_{S=0}^\infty \bigcap_{T=S}^\infty F_T^\varepsilon) = 0.$$

It then follows that $\mu_{\mathbf{p}}(\bigcap_{T=S}^\infty F_T^\varepsilon) = 0$ for any $S \in \mathbb{N}$. Because F_T^ε is computable (uniformly in T), $\{F_T^\varepsilon\}_{T=S}^\infty$ is a $\mu_{\mathbf{p}}$-test for any S (notice that $\mu_{\mathbf{p}}(F_T^\varepsilon)$ is also computable). Therefore, $\xi \notin \bigcup_{S=0}^\infty \bigcap_{T=S}^\infty F_T^\varepsilon$. This proves (b)

On the other hand, the set E_T^ε is computable (uniformly in T) and so the set $\bigcup_{T=S}^\infty E_T^\varepsilon$ is of Σ_1^0 (uniformly in S). For $\{\bigcup_{T=S}^\infty E_T^\varepsilon\}_{S=0}^\infty$ to be a test, we need to show that $\mu_{\mathbf{p}}(\bigcup_{T=S}^\infty E_T^\varepsilon)$ has a computable upper bound for all S. From the proof in Theorem 7.5.1 in Chung [9], we know that there exists a constant $A > 0$ and a number $\overline{k} > 0$ such that for all $k \geq \overline{k}$ (with the provision that $c^2(1+\frac{\varepsilon}{2}) < 1+\varepsilon$), c.f. p. 216),

$$\mu_{\mathbf{p}}(\bigcup_{T=T_k}^{T_{k+1}-1} E_T^\varepsilon) < \frac{A}{(k\log c)^{1+\frac{\varepsilon}{2}}},$$

where $T_k = \max\{T : \sqrt{\sum_{t=0}^T p_y^t(1-p_y^t)} \leq c^k\}$ and $c = 1 + \frac{\varepsilon}{10}$ (for ε small enough, $c^2(1+\frac{\varepsilon}{2}) < 1+\varepsilon$).

Let's define $G_0 = \bigcup_{T=0}^{T_1-1} E_T^{\varepsilon}$ and $G_k = \bigcup_{T=T_k}^{T_{k+1}-1} E_T^{\varepsilon}$ for $k > 0$. Clearly,

$$\bigcap_{S=0}^{\infty} \bigcup_{k=S}^{\infty} G_k = \bigcap_{S=0}^{\infty} \bigcup_{T=S}^{\infty} E_T^{\varepsilon}.$$

Now, because T_k is a computable function of k, $\{\bigcup_{k=S}^{\infty} G_k\}_{S=0}^{\infty}$ is also a sequence of uniformly Σ_1^0 sets. I now show that there is a computable mapping $i \mapsto S_i$ so that $\mu_{\mathbf{p}}(\bigcup_{k=S_i}^{\infty} G_k) \leq \frac{1}{2^i}$. It is easy to verify that

$$\sum_{k=S}^{\infty} \frac{A}{(k \log c)^{1+\frac{\varepsilon}{2}}} \leq \int_{x=S-1}^{\infty} \frac{A}{(x \log c)^{1+\frac{\varepsilon}{2}}} = (S-1)^{-\frac{\varepsilon}{2}} (\log c)^{-1-\frac{\varepsilon}{2}}.$$

Let $B \in \mathbb{N}$ be such that $B > (A(\log c)^{-1-\frac{\varepsilon}{2}})^{\frac{2}{\varepsilon}}$ and let $N \in \mathbb{N}$ be such that $N > \frac{2}{\varepsilon}$. Take $S_i = B2^{Ni} + 1$, and it follows that $\mu_{\mathbf{p}}(\bigcup_{k=S_i}^{\infty} G_k) \leq \frac{1}{2^i}$.

This shows that $\{\bigcup_{k=S}^{\infty} G_k\}_{S=0}^{\infty}$ is a $\mu_{\mathbf{p}}$-test, and so $\xi \notin \bigcap_{S=0}^{\infty} \bigcup_{k=S}^{\infty} G_k = \bigcap_{S=0}^{\infty} \bigcup_{T=S}^{\infty} E_T^{\varepsilon}$. This proves (a). $\square$

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