Career Patterns and Career Concerns

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PRELIMINARY AND INCOMPLETE: NOT FOR CIRCULATION

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Abstract

This paper examines reputational incentives in the context of traditional promotion systems. As in Holmström (1999), a worker's productive abilities are revealed over time through output, and wages are based on expected output, and so on assessed ability. Specifically, work increases the probability that a skilled worker achieves an observable breakthrough. In the presence of a deadline, it is shown that career concerns not only affect the amount of work produced, but also its timing: while it would be best to have the worker put in effort early, it is optimal for him to do so midway through his probationary period if this effort is not observed, and late if it is observed. Committing to a deadline is shown to be welfare-increasing, even if it curtails learning. In the observable case, it is better to condition termination on time than on assessed ability.

I Introduction

Promotion policies in professional service firms are typically based on an "up-or-out-system" (law firms, accounting firms, consulting firms, etc.). Employees are expected to obtain promotion to partner in a certain time period; if not, they are expected to quit, when they are not dismissed forthright. While alternative theories have been put forth (e.g., tournament models), agency theory provides an appealing framework to analyze such systems (see Fama, 1980, or Fama and Jensen, 1983). This paper investigates the incentives of employees, how they evolve over time,

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how they depend on the work performance measurement, as well as on the other provisions on the labor contract. We then turn to the optimal design of such policies.

Our model borrows its key ingredients from Holmström (1999).[1] There are no explicit output-contingent contracts. The firm, or market, must pay the worker, or agent, a competitive wage, given his expected output, which in turn is based on his assessed ability. Information about ability is symmetric at the start. We depart from Holmström in the specification of the learning process. Skill and output are binary, and only a skilled agent can achieve a high output. The time at which this output arrives –the *breakthrough*– follows an exponential distribution, whose instantaneous intensity increases with the worker's effort. Once the agent succeeds, and so proves himself, he is promoted and gets a constant compensation. While in some respects more stylized than Holmström's, this specification implies that effort increases not only expected output, but also the speed of learning, unlike in the Gaussian set-up. We view it as a plausible alternative modeling for labor markets in which frequently revised effort decisions provide highly informative signals infrequently. Furthermore, we consider a somewhat richer environment, which includes an "apprenticeship period," a time period in which the worker must succeed; else, the employment stops, with the worker incurring a termination cost, a fixed penalty that can be thought of as diminished career opportunities. We analyze the equilibria of the resulting game under two sets of assumptions. First, we assume that the firm does not observe the worker's effort –only breakthroughs, if ever, are observed; second, we consider the case in which effort is observed as well, so that information remains symmetric, independently of the history of effort choices by the worker.

Effort and skill interact in a rich way. When the market places more emphasis on a worker's output (i.e., when it anticipates greater effort), the intertemporal incentives of the agent are affected. Higher wages can lead to procrastination, by which the agent chooses to postpone putting in effort to a later date, and they affect the overall amount of effort as well. In particular we establish the following results.

1 Employees work too little, too late. The social optimum involves an effort path that puts in high effort early on, to achieve a breakthrough as soon as possible; if no such breakthrough occurs, so that the agent grows sufficiently convinced that his ability is low, it is best to stop costly effort altogether. When effort is not observed by the firm, the equilibrium involves a different pattern: the employee does not work early on, then increases his effort over time,

[1] See Gilson and Mnookin (1989) for a vivid account of associate career patterns in law firms, and the relevance of Holmström's model as a possible explanation.

before reverting to low effort as the deadline nears. Finally, when effort is observed by the firm, the worker puts in no effort until the deadline is sufficiently close, at which point he starts to work hard. Despite these contrasting work patterns, the amount of work is always smaller in the equilibrium than in the social optimum, whether effort is observed or not. Further, as in Holmström, better monitoring reduces effort, i.e., total effort exerted is higher when effort choices are not observed.

More specifically, we prove that, with unobservable effort, the equilibrium is unique, and the equilibrium path can be divided into phases that correspond to the described effort pattern. Uniqueness is a little surprising in our set-up in which effort affects the speed of learning, in light of the results of Dewatripont, Jewitt and Tirole (1999a,b). In the case of observable effort, attention is restricted to Markovian equilibria. Those are not unique, but share some features described in Section IV.

We then turn to contract design, and ask whether having a deadline, corresponding to a rigid apprenticeship period, is useful. Why not keep the employee past the deadline, adjusting the wage for the diminished incentives and lower assessed ability?[2] However:

2 Deadlines are desirable, whether the agent can commit to them or not. When the worker commits to his favorite deadline, effort increases as the deadline nears (i.e., in the unobservable case, effort does not let down after having been high).

In fact, with unobserved effort, under some circumstances, it is even in the worker's best interest to have relatively high penalties for failing to achieve a breakthrough by the deadline, as it endows the worker with some commitment power. While the characterization of the optimal deadline turns out to be surprisingly intricate, it is worth noting that it is often best to set it such that, at least initially, the worker exerts no effort.

Absent a deadline, the only "fundamental," that is, the only payoff-relevant information is the assessment of the worker's ability. This determines, in particular, the socially optimal level of effort and termination time, given the cost of termination. A natural alternative to a deadline, therefore, would be to use a stopping rule based on this assessed ability, at least in the case in which this assessment is public, as is the case with observable effort. However, we show that:

3 Deadlines are always better than stopping rules based on assessed ability.

[2] See Gilson and Mnookin (1989) for a discussion of this puzzle for the case of law firms.

"Finishing lines," based on assessed ability, provide precisely the wrong incentives, as it turns out. Working hard hastens learning, and so termination, which is most useful when wages are low, that is, when assessed ability is low. With such a stopping rule, the worker puts in low effort when his assessed ability is high, and high effort when it is low, the opposite of what would be optimal.

The most related paper is Holmström, as discussed. See also Jovanovic (1979) and Murphy (1986) for related model. Our paper shares with Gibbons and Murphy (1989) the interplay of implicit incentives (career concerns) and explicit incentives (termination penalty). It shares with Prendergast and Stole (1996) the existence of a finite horizon, and thus, of complex dynamics related to seniority. See also Bar-Isaac for reputational incentives in a model in which survival depends on reputation. The binary set-up is reminiscent of Mailath and Samuelson (2005). A theory of up-or-out contracts, based on asymmetric learning and promotion incentives, is investigated in Ghosh and Waldman (2010), while Chevalier and Ellison (1999) provide evidence of the sensitivity of termination to performance.

There is a growing literature on reputation in teams, which is certainly relevant for professional service firms, in which associates routinely engage in joint projects with partners. See Bar-Isaac (2007), Jeon (1996), Landers, Rebitzer and Taylor (1996), Levin and Tadelis (2005), Morrison and Wilhelm (2004), and Tirole (1996). Extending our set-up to allow for team work is subject to ongoing research.

II The Model

We shall consider the incentives of a single agent, or worker, to exert effort, or work. Time is continuous, and the horizon finite: $t \in [0, T]$. However, the game, or project, can end before $t = T$, in case the agent's effort is successful. Specifically, we assume that there is a binary state of the world $\omega = 0, 1$ that is interpreted as the underlying skill, or ability of the agent. If the state is $\omega = 0$, the agent is bound to fail, no matter how much effort he exerts. If the state is $\omega = 1$, a success arrives at a time that is exponentially distributed, with an intensity that increases in the instantaneous level of effort exerted by the agent. The state is 1 with probability $p^0 \in [0, 1]$.

Effort is a (measurable) function from time to the interval $[0, \bar{u}]$, where $\bar{u}$ represents an upper bound to the instantaneous effort that the agent can exert. If the agent exerts effort u_t over the time interval $[t, t + dt]$, the probability of a success over that time interval is $(\lambda + u_t)dt$, where

$\lambda \geq 0$ can be viewed as luck. Formally, the instantaneous arrival rate of a breakthrough at time t is given by $\omega \cdot (\lambda + u_t)$.

While the game has not ended, the agent receives a flow wage w_t. For now, let us think of this wage as an exogenous (measurable) function of time only that accrues to the agent as long as the game has not ended, though equilibrium constraints will later be imposed on this function, as this wage will reflect the market's expectations of the agent's effort and ability, given that the value of a success is normalized to one.

In addition to this wage, the agent incurs a linear cost for effort: exerting effort level u_t over the time interval $[t, t + dt]$ entails a flow cost $\alpha \cdot u_t dt$, $\alpha > 0$. Furthermore, achieving a success is desirable on two accounts: first, a known high-ability agent can expect a flow outside wage of v, so that this outside option v is a (flow) opportunity cost incurred as long as no success has been achieved. Second, reaching the deadline (without achieving a success) entails a fixed penalty of k, representing diminished career opportunities to workers with such poor records. There is no discounting.

Thus, the worker chooses $u : [0, T] \to [0, \bar{u}]$, measurable, to maximize

$$\mathbb{E}_u \left[\int_0^{T \wedge \tau} (w_t - v - \alpha u_t) dt - \chi_{\tau \geq T} k \right],$$

where $\mathbb{E}_u$ is the expectation conditional on the worker's strategy u, and τ is the time at which a success occurs –a random variable that is exponentially distributed, with instantaneous intensity at time t equal to 0 if the state is 0, and to $\lambda + u_t$ if the state is 1, and χ_E is the indicator of event E.

Of course, at time t effort is only exerted, and the wage collected, conditional on the event that no success has been achieved. We shall omit to say so explicitly, as those histories are the only nontrivial ones. Given his past effort choices, the agent can compute his belief p_t that he is of high ability by using Bayes' rule. It is standard to show that, in this continuous-time environment, Bayes' rule is equivalent to the ordinary differential equation (O.D.E.)

$$\dot{p}_t = -p_t (1 - p_t) (\lambda + u_t),$$

with initial condition $p_0 = p^0$. By the law of iterated expectations, we can then rewrite our objective as

$$\int_0^T e^{-\int_0^t p_s(\lambda+u_s)ds}(w_t - v - \alpha u_t)dt - ke^{-\int_0^T p_t(\lambda+u_t)dt}.^{3}$$

While the belief p_t is a variable that is easy to interpret, the problem is more conveniently analyzed by using the log likelihood ratio

$$x_t := \ln \frac{1-p_t}{p_t},$$

which measures the relative likelihood that the state is 0. Thus, $x_t \in \mathbb{R}$ increases as the agent becomes more pessimistic about his ability. With some abuse, we shall refer to x as the belief as well (keeping in mind that it varies in the opposite direction of p over time). The main benefit is that the evolution of x is linear, namely,

$$\dot{x}_t = \lambda + u_t,$$

and we let $x^0 := \ln\left(1-p^0\right)/p^0$. Furthermore, the objective function simplifies considerably to

$$\int_0^T \left(1+e^{-x_t}\right)\left(w_t - \alpha u_t - v\right)dt - ke^{-x_T},^{4} \tag{1}$$

which the worker seeks to maximize, given w, over all measurable $u : [0,T] \to [0,\bar{u}]$ such that

$$\dot{x}_t = \lambda + u_t, x_0 = x^0. \tag{2}$$

First, we shall derive the social optimum. Then, we will turn to the strategic problem, in which the worker maximizes his own payoff only.

[3]To see this, note that the probability that no success has occurred by time t is given by

$$e^{-\int_0^t p_s(\lambda+u_s)ds}.$$

[4]This is the objective function up to an additive constant, as well as, more importantly, to a multiplicative constant $\left(1+e^{-x^0}\right)^{-1}$, which is ignored here, as it does not affect the optimality of a given strategy. However, for consistency, we shall re-introduce those constants when we apply the optimality principle. See Section V.

III Benchmarks

We first examine two special cases: the social planner's solution, and the single agent's optimal strategy when the wage is given exogenously.

A The Social Planner

What is the expected value of a success? Remember that we normalized the value of a success to 1. However, a success only arrives with instantaneous probability

$$p_t(\lambda + u_t) = \frac{\lambda + u_t}{1 + e^{x_t}},$$

as a success only occurs at rate $\lambda + u_t$ if the agent is of high ability. This, therefore, is the expected value of success, and the objective becomes

$$\max_u \int_0^T \left(1 + e^{-x_t}\right)\left(\frac{\lambda + u_t}{1 + e^{x_t}} - \alpha u_t - v\right) dt - k e^{-x_T}.$$

That is, the planner maximizes

$$\int_0^T [e^{-x_t}(\lambda + u_t) - (1 + e^{-x_t})(v + \alpha u_t)]dt - k e^{-x_T},$$

over all measurable $u : [0, T] \to [0, \bar{u}]$ such that $\dot{x}_t = \lambda + u_t, x_0 = x^0$. The solution to this problem follows easily from Pontryagin's maximum principle. The proof of the next lemma and of all formal results, can be found in appendix. A strategy u is extremal if it only takes extreme values: $u_t \in \{0, \bar{u}\}$, for all t.

Lemma 1 *Optimal effort is extremal. Furthermore, there are (at most) two intervals, corresponding to maximum and zero effort. Maximum effort precedes zero effort if and only if* $v/\lambda > \alpha$.

Of course, one of the intervals might be empty: it might well be that the optimum involves no effort, or effort always.

Note that neither the initial belief, nor the terminal cost (k) affect whether maximum effort is exerted first or last. Of course, they affect the total amount of effort, but given this amount, they do not affect its timing. The role of the sign $\alpha\lambda - v$ in the ordering of these intervals is easy to see: consider exerting some bit of effort now or at the next instant (thus, keeping the total amount of planned effort fixed); by waiting, a loss vdt is incurred; on the other hand, with probability λdt, the marginal cost of this effort, α, will be saved. Therefore, if

$$v/\lambda > \alpha,$$

it is better to work early than late, if at all. From now on, we shall focus on the case $v/\lambda > \alpha$. This way, effort is efficient even far from the deadline. An example of such a path is given by the left panel in Figure 1. The right panel gives the corresponding path for the value of output (i.e., $p_t(\lambda + \bar{u}_t)$).

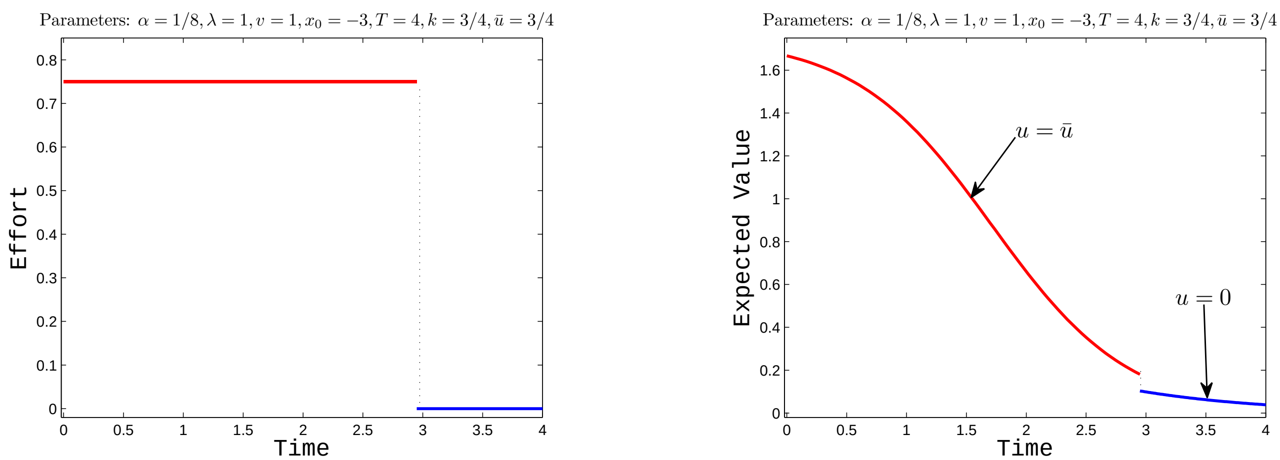


Figure 1: Effort and expected value at the social optimum

Whether effort is still exerted at the deadline depends on how pessimistic the social planner is at that point. By the transversality condition from Pontryagin's principle, full effort is exerted at the deadline if and only if

$$(1+k)\, e^{-x_T} \geq \alpha \left(1 + e^{-x_T}\right), \text{ or } p_T\,(1+k) \geq \alpha, \tag{3}$$

recalling the definition of x_T. This simply states that the expected marginal social gains from effort (one success, and avoiding the loss) should exceed the marginal cost. If the social planner becomes too pessimistic, he gives up before the end. Note that the flow loss v no longer plays a role at that time, as the terminal (lump-sum) penalty overshadows any such flow cost.

Let x^* denote the threshold terminal belief that satisfies (3) with equality, assuming such a value exists. To rule out uninteresting cases, we shall assume that x^* is finite. In fact, to narrow down the possibilities, while focusing on the most interesting ones, we shall maintain throughout the following set of assumptions on the parameters.[5]

Assumption 1 *The parameters α, k, v and λ are such that*

$$1+\alpha > v/\lambda > k > \alpha.$$

It is straightforward to solve for the switching time (or switching belief) more generally. For all terminal beliefs $x_T > x^*$, for which no effort is exerted at the deadline, the switching belief between equilibrium phases is determined by

$$(1+k-\alpha)e^{-x_T} - \alpha = \int_x^{x_T} e^{-s} \frac{\alpha\lambda - v}{\lambda} ds,$$

which gives as value of x (as a function of t)

$$x\,(t) = \ln\left((1+k-v/\lambda)e^{-\lambda(T-t)} - (\alpha - v/\lambda)\right) - \ln\alpha.$$

This represents a frontier in (t, x) space that the equilibrium path will cross from below for sufficiently long deadlines. Consistent with the fact that, in the optimum, a switch to zero effort is irreversible, when $u_t = 0$ and $\dot{x}_t = \lambda$, the path leaves this locus (i.e., it holds that $x'\,(t) < \lambda$).

The switching belief $x\,(t)$ decreases in T: the longer the deadline, the longer maximum effort will be exerted (recall that x measures pessimism). This belief decreases in α and increases in v and k: the higher the cost of failing, or the lower the cost of effort, the longer effort is exerted.

B Exogenous Wages

Before solving for an equilibrium in which wages are set by a competitive market, we derive the worker's optimal effort path given an exogenous wage path w_t. This will allow us to understand

[5] These assumptions are not necessary for all results. The inequality $k > \alpha$ allows for full effort to be exerted at some terminal belief x_T in the non-cooperative problem. The inequality $v/\lambda > k$ means the loss at the deadline is not overwhelming compared to the agent's potential talent λ and payoff v. The last inequality implies the first zero-effort phase of Section IV is not empty for sufficiently low values of x^0 (*i.e.* optimistic enough initial beliefs). If this were not the case, then full effort would obtain for all paths leading to $x_T < x^*$. Note that, if v was the flow payoff of the agent on a similar task, once his high ability were known, it would be equal to λ, as this is his expected productivity. In this case, our assumption can be thought of as requiring just $\alpha < k < 1$, where 1 is the value of a success.

under which circumstances high effort is optimal. Recall that the worker maximizes

$$\int_0^T \left(1+e^{-x_t}\right)(w_t - \alpha u_t - v)\,dt - ke^{-x_T},$$

subject to $\dot{x}_t = \lambda + u_t, x_0 = x^0$. Pontryagin's theorem states the existence of an absolutely continuous function $\gamma : [0,T] \to \mathbb{R}$, such that (a.e.):[6]

$$\begin{aligned} u_t > (<) & \quad 0 \text{ if } \gamma_t > (<) \alpha \left(1+e^{-x_t}\right), \\ \dot{\gamma}_t &= e^{-x_t}(w_t - \alpha u_t - v), \\ \gamma_T &= ke^{-x_T}. \end{aligned} \tag{4}$$

It is often more convenient to work with the "incentive function"

$$\phi_t := \gamma_t - \alpha\left(1+e^{-x_t}\right),$$

such that the agent works at full effort (i.e., sets $u = \bar{u}$) whenever ϕ_t is positive. The second term in ϕ_t is the marginal cost of effort α multiplied by the probability that the game will last until t.[7]

Let us start with a "technical" result.

Lemma 2 *A solution to* (1)—(2) *exists and is unique. Furthermore, the necessary conditions* (4) *are also sufficient.*

First note that the transversality condition in (4) implies that the agent works at the deadline if and only if

$$x_T \le \ln \frac{k-\alpha}{\alpha}, \text{ or } p_T k \ge \alpha.$$

This is similar to the social planner's decision rule at the deadline, except that the worker does not take into account the lump-sum value of success (compare with (3)).

What determines the instantaneous *amount* of work? Integrating (4) and using the transver-

[6]Note that the problem cannot be abnormal, since there is no restriction on the terminal value of the state variable. See Note 5, Ch. 2, Seierstad and Sydsaeter. It will be understood from now on that statements about derivatives only hold almost everywhere.

[7]Recall that this probability is given by $\exp(-\int_0^t p_s(\lambda+u_s)\,ds) \propto (1+e^{-x_t})$.

sality condition yields that effort is optimal whenever

$$\alpha \leq \frac{ke^{-x_T} + \int_t^T e^{-x_s}\left(v - w_s\right)ds - \alpha \int_t^T e^{-x_s} u_s ds}{1 + e^{-x_t}}. \tag{5}$$

As is clear from (5), unlike in Holmström, but as in Dewatripont, Jewitt and Tirole, future compensation does affect the incentives of the agent to put in high effort: increasing the wedge between the future rewards from success and failure ($v - w_s$) encourages high effort, *ceteris paribus*. Similarly, a higher penalty for termination or a lower cost of effort provide stronger incentives.

Let us now consider the intertemporal allocation of effort. Because our model has more than two periods, it allows us to examine not only how much, but when effort is exerted. This can be read off the derivative of ϕ:

$$\dot{\phi}_t = e^{-x_t}\left(w_t - \alpha u_t - v\right) + \alpha\left(\lambda + u_t\right)e^{-x_t}.$$

We can interpret a negative (positive) change in incentives to work as the agent's desire to anticipate (resp. procrastinate) effort. Indeed, if the agent is working today, and $\dot{\phi}_t > 0$ (procrastinating), then he clearly will work tomorrow as well. Similarly, if he intends to work tomorrow and $\dot{\phi}_t < 0$ (anticipating), then he will work today. As is clear from this equation, the horizon length and termination cost do not affect the timing of work; flow wages and outside option affect incentives to anticipate as incentives to work: higher-powered incentives lead to earlier (as well as more) effort.

We can further clarify the link between $\dot{\phi}_t$ and timing of effort by considering the incentives to procrastinate. If effort today dominates effort tomorrow, then

$$p_t \cdot \underbrace{\alpha u_{t+dt}}_{\text{cost saved}} \geq \alpha \cdot \underbrace{p_t\left(\lambda + u_t\right)}_{\text{Pr. of success at } t} - \underbrace{p_t\left(v - w_t\right)}_{\textit{net} \text{ loss due to delay}}, \tag{6}$$

which states that frontloading effort saves costs tomorrow at a rate p_t, while procrastinating saves costs (today) at a rate α with probability $p_t(\lambda + u_t)$ and reduces the probability in jumping from flow w_t to flow v at a rate p_t. For example, when $w_t \geq v$, the agent always prefers to procrastinate.[8] Again, we see that the trade-off does not depend on the belief p_t, which cancels,

[8] Rearranging the previous inequality, we obtain the condition for anticipating effort $\dot{\phi}_t \leq 0$ (up to a positive denominator $1 - p_t$, which appears because of the probability of reaching time t at all).

nor on the actual effort exerted, provided it is locally continuous. Condition (6) simplifies to

$$v - w_t \geq \alpha\lambda,$$

so that procrastination becomes less attractive over time if the wage is decreasing. If anticipating effort dominates today, procrastinating effort cannot be optimal later on. Thus, maximum effort must be concentrated on a single time interval.

Therefore, based on the procrastination incentives, we can conclude that

$$\dot{\phi}_t = \frac{p_t}{1 - p_t} \left(w_t - v + \alpha\lambda \right)$$

determines the possible effort phases. If w_t is increasing (resp. decreasing), then ϕ_t is quasiconvex (resp. quasiconcave), that is, lack of effort (resp. effort) occurs in at most one interval. More formally,

Lemma 3 *If the wage is strictly monotone, the optimal policy is extremal. If the wage is increasing (resp. decreasing), the optimal policy involves at most one interval in which effort is zero (resp., maximal).*

If w_t is increasing, and at time t the agent prefers to procrastinate ($\dot{\phi} > 0$), he will always prefer to do so in the future. Therefore, he cannot shirk, start working, and then stop: when he starts working, $\dot{\phi}_t > 0$, meaning he likes to procrastinate. Were he to stop (before the deadline), he could benefit by delaying his effort phase altogether.

Quite naturally, increasing wages can lead to work, stop, and then pick it up again when the deadline is close. If the agent does not work at the deadline, then he only has at most one (initial) effort phase.

If the wage is strictly decreasing, then $\dot{\phi}_t$ cannot be first positive, then negative, and then positive again. In fact, once the agents prefers to anticipate effort, he always prefers to anticipate it (if he's working at all). Now, when he stops working, we must have $\dot{\phi} < 0$ (so he would prefer to anticipate, if he had to work). Therefore, there cannot be a gap between effort phases. Once the agent stops working, he cannot resume work later (or he would benefit from anticipating the second phase).

Finally, if the wage is constant, the agent always prefers to procrastinate or to anticipate effort.[9] A very high wage (relative to v) means he prefers to delay effort, and leads to a zero

[9]We can think of the social planner's problem as an instance in which the "wage" (the value of success) is

followed by full work pattern (depending on how short the deadline is). A very low wage leads to a work then stop pattern.

To conclude, even when wages are monotone, the worker's incentives need not be so over time. While the equilibrium wage path of the next section fails to be monotone, the trade-off laid out in (6) remains decisive.

IV The Equilibrium with Unobservable Effort

In this section, the wage is no longer taken as exogenous. There is a competitive market, which pays the worker the expected value of output. The market does not observe the worker's past effort, but only that the worker has not succeeded so far. Therefore, the wage is given by

$$w_t = \mathbb{E}_t[p_t(\lambda + u_t)],$$

where p_t and u_t are the agent's belief and effort, respectively, at time t, given his private history of past effort (of course, it is assumed that he has had no successes so far). Given Lemma 2, the agent will not use a chattering control (i.e., a distribution over measurable functions (u_t)), but rather a single function. Therefore, we may write

$$w_t = \hat{p}_t(\lambda_t + \hat{u}_t),$$

where $\hat{p}_t$ and $\hat{u}_t$ denote the belief and anticipated effort at time t, as viewed from the market.

Equilibrium requires that expected effort coincide with actual effort. That is, for every t,

$$\hat{u}_t = u_t,$$

and therefore, also, $\hat{p}_t = p_t$ at all t. Note that, off path, the market might hold wrong beliefs. Fortunately, on path, this problem can be analyzed with the help of Pontryagin's maximum principle. This states that there exists an absolutely continuous function ϕ such that effort is maximum (resp. zero) if $\phi > 0$ (resp. < 0). This function must satisfy the differential equation

$$\dot{\phi}_t = e^{-x_t}(w_t + \alpha\lambda - v), \tag{7}$$

constant.

as well as the transversality condition

$$\phi_T = (k - \alpha)e^{-x_T} - \alpha,$$

which must hold at the deadline. Finally, in equilibrium,

$$w_t = p_t(\lambda + u_t) = \frac{\lambda + u_t}{1 + e^{x_t}}. \tag{8}$$

We now note:

Lemma 4 *The equilibrium path consists of at most four phases, for some* $0 \leq t_1 \leq t_2 \leq t_3 \leq T$*:*

1. *during* $[0, t_1]$*, no effort is exerted;*
2. *during* $(t_1, t_2]$*, effort is interior, i.e.* $u_t \in (0, \bar{u})$*;*
3. *during* $(t_2, t_3]$*, effort is maximal;*
4. *during* $(t_3, T]$*, no effort is exerted.*

Any of these intervals might be empty.[10]

Lemma 4 describes the overall structure of the equilibrium outcome. As is stated, any of the intervals might be empty, and it is easy to compute instances of each of the different possibilities. See Figure 2 for an example of effort (left panel) and corresponding wage dynamics (right panel). The parameters are the same as those used in Figure 1 above).

What is happening along the (most complicated) equilibrium path? Being very optimistic at the start, the agent sees little reason to exert effort: his wage is high anyhow, and luck might suffice. As time goes by, he becomes more pessimistic, and at some point, if he kept being paid a wage that would correspond to low effort, he would find it optimal to switch to high effort. However, the worker's actual and expected efforts are strategic substitutes: if the market expects the worker to exert high effort, the worker gets a high wage, which depresses his incentives to exert effort (as effort is likely to end the stream of wages collected, recall (6)). Therefore, if the market expected the worker to exert high effort, low effort would be optimal; if the market expected low effort, high effort would be optimal. As a result, the equilibrium involves an interior

[10]Here and elsewhere, the choices at the extremities of the intervals are irrelevant, and our specification is arbitrary in this respect.

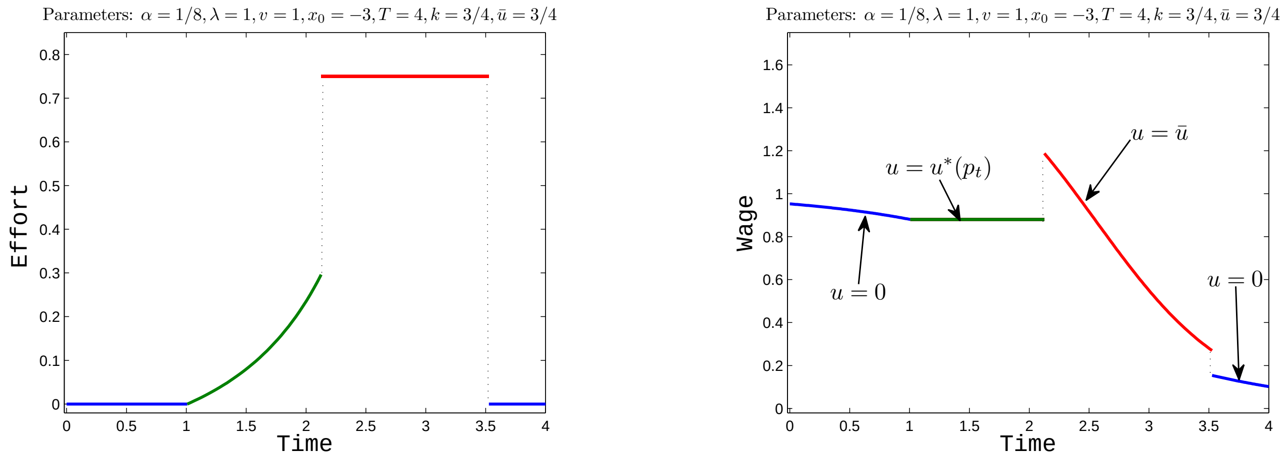


Figure 2: Effort and wages in the non-observable case

level of effort, which rises continuously in this second phase, keeping the wage constant, so that the worker remains indifferent between high and low effort. At some point, the agent becomes sufficiently pessimistic that no such tension persists, and he switches to maximum effort; and possibly, once the deadline looms even closer, he then switches back one last time to low effort, as the final penalty no longer provides sufficient incentives given the ambient pessimism.

Note that these effort dynamics imply that wages are decreasing over time, with the exception of a possible jump upward, as effort switches to $\bar{u}$. Note also that, during the phase in which effort is interior, the wage is constant.

As mentioned, not all those phases might exist; he might switch from low to maximum effort without intermediate phase; he might reach the deadline while exerting maximum effort, etc. Furthermore, there are two distinct sets of circumstances under which high effort is optimal (see Section VI, Figure 6).

A more formal account of the equilibrium structure is given in the proof of Lemma 5. To sketch how to proceed in order to solve for an equilibrium, suppose for simplicity that there is no interval with maximum effort. In the region with interior effort, the wage must be set such that $\dot{\phi} = 0$ (see (7)), i.e. $w = v - \alpha\lambda$. This pins down the level of effort, as in equilibrium the wage equals expected output (see (8)), and so we obtain a differential equation for the belief,

$$\dot{x} = (1 + e^x)(v - \alpha\lambda), \text{ or } e^{-x_t} = Ce^{(v-\alpha\lambda)t} - 1, \tag{9}$$

for some constant C. If $t_1 \in (0, t_2)$, then it is determined by the property that effort is continuous at t_1. Indeed, because $\lim_{t\uparrow t_1} \dot{\phi}_t \geq \lim_{t\downarrow t_1} \dot{\phi}_t$, it must be that $0 = \lim_{t\uparrow t_1} u_t \leq \lim_{t\downarrow t_1} u_t$. On the other hand, there is no reason for effort to be continuous at t_2, which is determined by the solution of $\phi_{t_2} = 0$. Because $u_{t_1} = 0$, it follows from (8) that

$$1 + e^{x_{t_1}} = \frac{\lambda}{v - \alpha\lambda},$$

and $x_{t_1} = x^0 + \lambda t_1$, which gives two equations for x_{t_1} and t_1. On the last interval $(t_3, T]$, effort is zero and $\phi_{t_3} = 0$, so that we can integrate (7) and use the transversality condition, to get

$$\phi_T - \phi_{t_3} = (k - \alpha)e^{-x_T} - \alpha = \int_{t_3}^{T} e^{-x_t} \left(\frac{\lambda}{1 + e^{x_t}} + \alpha\lambda - v \right) dt,$$

or

$$(v - \lambda(1 + \alpha))(1 + e^{-x_{t_3}}) = (1 + e^{-x_T})(v - \lambda(1 + k)) + \lambda \ln \frac{1 + e^{-x_T}}{1 + e^{-x_{t_3}}} + \lambda k,$$

which, given that $x_T - x_{t_3} = \lambda(T - t_3)$, provides another relationship between x_{t_3} and t_3. Furthermore, applying (9) to the endpoints of the interval $[t_1, t_2]$ gives

$$\frac{1 + e^{-x_{t_1}}}{1 + e^{-x_{t_2}}} = e^{-(v - \alpha\lambda)(t_2 - t_1)},$$

and $x_{t_3} - x_{t_2} = (\lambda + \bar{u})(t_3 - t_2)$, provides two equations for x_{t_2} and t_2. Therefore, at least in principle, we can solve for all the switching times and beliefs, and check whether all phases are well-defined given the parameters.

We then establish the following result.

Lemma 5 *The equilibrium exists and is unique.*

The proof of Lemma 5 relies on showing that the final belief x_T is a strictly increasing function of the deadline: extending the deadline unambiguously increases the amount of learning (though the total amount of work might increase or decrease).

Note that we have not specified the equilibrium strategy of the worker, because we have not derived his behavior following his own (unobservable) deviations. Yet it is not difficult to describe the worker's optimal behavior off-path. In Section VI, we shall briefly do so.

One might wonder whether the penalty k is really hurting the worker. After all, it endows him with some commitment to work. A thorough analysis is provided in Appendix C. As explained there, increasing k increases the amount of work performed; furthermore, if parameters are such that working at some point is optimal, then the optimal (i.e. payoff-maximizing) termination penalty is strictly positive.

Special cases One might also wonder how restrictive our assumptions on the parameters really are. In order to illustrate why we focus on what we believe are the more economically significant scenarios, we now discuss how our results would change in a number of special cases that do not fit our maintained assumptions.

1. $k < \alpha$. When the penalty is too low relative to the cost of effort, the agent never works at the deadline. The equilibrium can only be characterized by at most four phases, but zero effort must take place in the last one. The rest of our analysis remains unchanged.

2. $v < \alpha\lambda$. When the cost of waiting is low, the agent always has an incentive to delay effort, or to procrastinate. Formally, we have $\dot{\phi}_t > 0$ for any level of the wage. Therefore, provided he works at all, the agent works at the end, and effort is forward absorbing. In other words, for long enough deadlines (so that $x_T > x^*$) no effort is exerted, while for shorter ones the agent follows a zero-full pattern. In fact, because the agent is never indifferent with respect to the intertemporal allocation of a his work, interior levels of effort are not exerted in equilibrium.

3. $T = \infty$. When the deadline is infinite, the value of the penalty is irrelevant, because it will be paid in equilibrium with probability $1 - p^0$ irrespective of effort. Since beliefs always evolve at rate λ (or faster), learning will be complete in equilibrium. The effort pattern is similar to the case of a low penalty value. In particular, we have $\lim \phi_T = -\alpha$, and the equilibrium path involves zero-interior-(full)-zero effort. A simple condition determines whether effort is exerted at all. In particular, effort provision requires that

$$\alpha \leq \frac{(v/\lambda - 1)\, e^{-x_0} + \ln\left(1 + e^{-x_0}\right)}{1 + e^{-x_0}}.$$

   (Note that if $v = \lambda$, a sufficient condition is $\alpha \leq e^{-1}$).

4. $\lambda = 0$. In this case, talent is not productive without effort. This is the technically more challenging case, because beliefs "freeze" when the agent stops working. Nevertheless, the equilibrium construction is quite intuitive. In particular, the agent cannot stop working before the deadline, if the resulting $x_T < x^*$. In this case, the agent would strictly prefer

to work at T and would be better off anticipating his effort. Formally, note that we have

$$\dot{\phi}_t = e^{-x_t}\left(w_t - v\right) = e^{-x_t}\left(\frac{\hat{u}_t}{1+e^{\hat{x}_t}} - v\right).$$

and so, once the agent shirks, he shirks throughout. Therefore, either no effort is ever exerted, or the agent follows a work-shirk pattern. If $\bar{u} < v$, the agent always strictly prefers to anticipate effort. If not, then interior effort levels are possible, in which case the equilibrium is of the mixed-full-zero type. Intuitively, the agent does not shirk early on, because without the "luck and talent" component, the procrastination motive is completely removed. At the same time, the "luck" component is precisely the basis of the agent's reputation, i.e. the ability of achieving breakthroughs independently of his own effort.

V Observable Effort *ex post*

Let us contrast this equilibrium with the outcome when effort is observable. That is, we maintain here the assumption that the competitive wage is paid up-front, but the resulting effort is observed as soon as it is exerted. This implies that the belief of the market is always correct, $\hat{p}_t = p_t$, on and off the equilibrium path, and the payment flow is given by

$$w_t = p_t(\lambda + \hat{u}_t),$$

where p_t is the common belief, and $\hat{u}_t$ is the effort level that the market expects the agent to exert in the next instant. Up to a constant, the agent then maximizes

$$\int_0^T [e^{-x_t}(\lambda + \hat{u}_t) - (\alpha u_t + v)(1 + e^{-x_t})]dt - ke^{-x_T}.$$[11]

Note that, because the worker's actions are observed, wages are no longer a function of time only, as they are affected by a deviation by the worker. At the very least, then, we must describe wages, and behavior, as a function of time t and current belief x. In fact, we shall restrict attention to equilibria in Markov strategies

$$u : \mathbb{R} \times [0, T] \to [0, \bar{u}],$$

[11] We use here that $(1 + e^{-x_t})w_t = (1 + e^{-x_t})(\lambda + \hat{u}_t)/(1 + e^{x_t}) = e^{-x_t}(\lambda + \hat{u}_t)$.

such that u is upper semicontinuous in (x,t), and such that the value function

$$V(x,t) = \sup_u \left\{ \int_t^T \frac{1+e^{-x_s}}{1+e^{-x}} \left[\frac{\lambda + u(x_s,s)}{1+e^{x_s}} - v - \alpha u(x_s,s) \right] ds - k \frac{1+e^{-x_T}}{1+e^{-x}} \right\},$$

is piecewise differentiable.[12] We shall prove that such equilibria exist. We first argue that if the agent ever exerts low effort, he has always done so in the past.

Lemma 6 *Fix an equilibrium. If $u(x,t) = 0$ on some open set $\Omega \subset \mathbb{R} \times [0,T]$, then also $u(x',t') = 0$ if the equilibrium trajectory that starts at (x',t') intersects Ω.*

This lemma implies that every equilibrium has a simple structure: if the agent is ever willing to exert high effort, he keeps being willing to do so at any later time, at least on the equilibrium path. In any equilibrium involving extremal effort levels only, there are at most two phases: first, the worker exerts no effort, then, full effort. This is precisely the opposite of the structure for the social planner, in which high effort comes first (see lemma 1). The social planner works, until he becomes sufficiently pessimistic, if ever. The agent shirks, until it is time to panic, if ever. He can only be trusted by the market to put in high effort if he is "back to the wall," so that maximum effort will remain optimal at any later time, no matter what he does until then; if the market paid the worker for high effort, yet he was supposed to let up his effort later on, then the worker would gain by deviating to low effort, pocketing the high wage in the process; because the observable deviation to no effort would make everyone more optimistic, it would only increase his incentives to exert high effort and thus increase his wage at later times.

This, of course, relies heavily on the Markovian assumption. Nevertheless, as the next theorem states, there are multiple Markovian equilibria.

Theorem 1 *Given T, there exists continuous, increasing $\underline{x}, \bar{x} : [0,T] \to \mathbb{R}$, $\underline{x}_t \leq \bar{x}_t$, such that:*

i All equilibria involve maximum effort below $\underline{x}$:

$$x < \underline{x}_t \Rightarrow u(x,t) = \bar{u};$$

ii All equilibria involve no effort above $\bar{x}$:

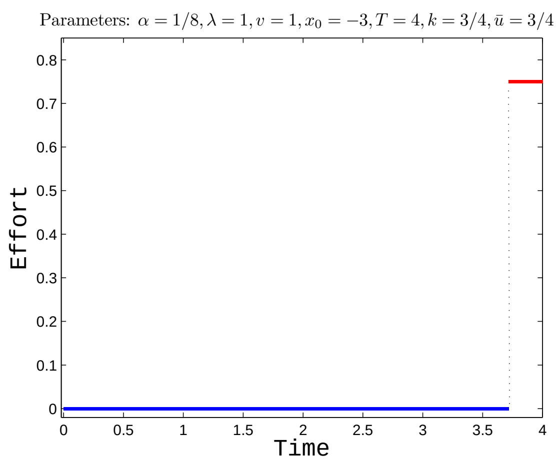
$$x > \bar{x}_t \Rightarrow u(x,t) = 0;$$

[12] That is, there exists a finite partition of $\mathbb{R} \times [0,T]$ into closed subsets S_i with non-empty interior, such that V is differentiable on the interior of S_i, and the intersection of any pair S_i, S_j is either empty or a smooth 1-dimensional manifold.

iii These bounds are tight: there exists an equilibrium $\underline{\sigma}$ (resp. $\bar{\sigma}$) in which effort is either 0 *or $\bar{u}$ depending on whether x is above or below $\underline{x}$ (resp. $\bar{x}$).*

Given Lemma 6, it should come as no surprise that, whenever finite, the functions $\underline{x}$, $\bar{x}$ (which are actually differentiable) have slopes greater than $\lambda + \bar{u}$, so that their hypographs are absorbing for the trajectories (x, t) under any strategy. The proof of the theorem provides an explicit description of these boundaries. Note that these functions take value on the extended real line: indeed, they both converge to $-\infty$ as $t \downarrow t^*$, for some common value t^* (where $T - t^*$ is independent of T): if the deadline is large enough, the equilibrium always involves no effort initially. It is worth pointing out that $\bar{x}_T = \underline{x}_T = x^*$, where $e^{x^*} = (k - \alpha)/\alpha$.

These results are illustrated in Figure 3 for the same parameters as in Figure 2 in the unobservable case.



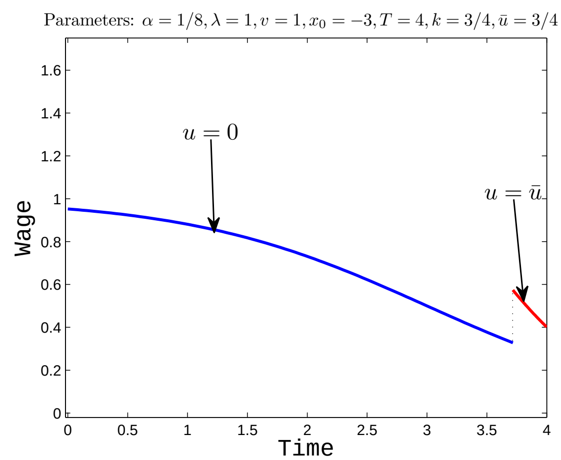


Figure 3: Effort and wages in the observable case

It is worth noting that, while $\underline{\sigma}$ and $\bar{\sigma}$ provide lower and upper bounds on the equilibrium effort exerted in an equilibrium (in the sense of (i)–(ii)), these equilibria are not the only ones. There exist other equilibria involving only extremal effort levels, whose boundary is between $\underline{x}$ and $\bar{x}$, as well as equilibria in which interior effort levels are exerted at some states. In particular, the proof builds an equilibrium in which the agent exerts an interior amount of effort at all times t for all values of x in $[\underline{x}_t, \bar{x}_t]$. This effort is decreasing in t and increasing in x. It is equal to $\bar{u}$ at $\bar{x}_t$, decreases continuously along the equilibrium trajectory from that point on, until the lower boundary is reached (which, unless a success occurs, necessarily happens before time T), at which point the effort level jumps up to $\bar{u}$.[13]

In the extremal equilibria, wages are decreasing over time, except for an upward jump at the point at which effort jumps up to $\bar{u}$. In the interior-effort equilibrium described in the proof (in which effort is interior everywhere between $\underline{x}$ and $\bar{x}$), wages decreases continuously over time.

The intuition behind the equilibrium multiplicity is straightforward. The market only believes that the worker will exert high effort if he is sufficiently optimistic and the deadline is close enough (time to go must not exceed t^*). These states are thus rewarded with a high wage, which makes them relatively desirable. If it is possible at all to reach them, putting in low effort is the best way to do so. This incentive to shirk disappears as soon as this region is reached (as it is such that no feasible strategy leaves it). This wedge in incentives, then, provides scope for a range of specification, from an equilibrium that minimizes effort (if $\underline{\sigma}$ specifies any effort at all at some state, all equilibria do specify at least as much), to an equilibrium that maximizes effort (if $\bar{\sigma}$ specifies less than full effort at some state, all other equilibria specify even less effort at that state).

The threshold $\bar{x}$ is decreasing in the cost of effort α, and increasing in the outside option v and penalty k, as one would expect. Considering the equilibrium with maximum effort, the agent works more, the more desirable success is.

VI Comparison

Along the equilibrium path, the dynamics of effort look very different when one compares the social planner, the agent when effort is unobservable, and the agent when effort is observable. Yet it turns out that effort can easily be ranked across those cases. To do so, the key is to describe effort in all three cases in terms of the state (x, t), i.e., the belief and the time.

For the observable case, it is enough to focus on the maximal full effort region, defined by the frontier $\bar{x}$, as it will turn out that even this equilibrium specifies less effort than what happens with a social planner, or an agent whose effort is not observed.

The boundaries (in (x, t)-space) that characterize the optimal strategy in the unobservable case are more complicated. An important distinction is whether, in the unobservable case, a full

[13]It is not possible to strengthen (4) further to the statement that, once maximum effort is exerted, it is exerted throughout: there is considerable leeway in specifying equilibrium strategies between $\underline{x}$ and $\bar{x}$, and nothing precludes maximum effort to be followed by interior effort. (Of course, as soon as $\underline{x}$ is crossed, effort is maximal.)

effort region occurs right before the terminal belief $x_T = x^*$. This depends on the sign of

$$\phi'(x^* \mid \bar{u}) = \frac{1}{1+e^{x^*}} - \frac{v - \alpha\lambda}{\lambda + \bar{u}} = \frac{\alpha}{k} - \frac{v-\alpha\lambda}{\lambda+\bar{u}} \lesseqgtr 0.$$

Consider the equation defining the full effort frontier in the unobservable case $x_2(t)$

$$(k-\alpha)\, e^{-x_2-(\lambda+u)(T-t)} - \alpha - \int_{x_2}^{x_2+(\lambda+u)(T-t)} e^{-x} \left(\frac{1}{1+e^x} - \frac{v-\alpha\lambda}{\lambda+\bar{u}} \right) dx. \tag{10}$$

If $\phi'(x^* \mid \bar{u}) > 0$ then the slope $x_2'(t)$ at T is less than $\lambda + \bar{u}$, meaning we would leave the full effort region in a neighborhood of (T, x^*). There is then no full effort phase at x^* and the full effort regions are disconnected. See Figure 6 and compare with Figure 4 and 5. These figures use as parameters $\bar{u} = 1/2$, $\alpha = 1/5$, $v = \lambda = 1$, $x_0 = -4$, $T = 5$ and, depending on the figure, $k \in \{.3, .4, .6\}$. If instead $\phi'(x^* \mid \bar{u}) < 0$, then there is a connected full effort region and full effort is exerted at x^* as well.

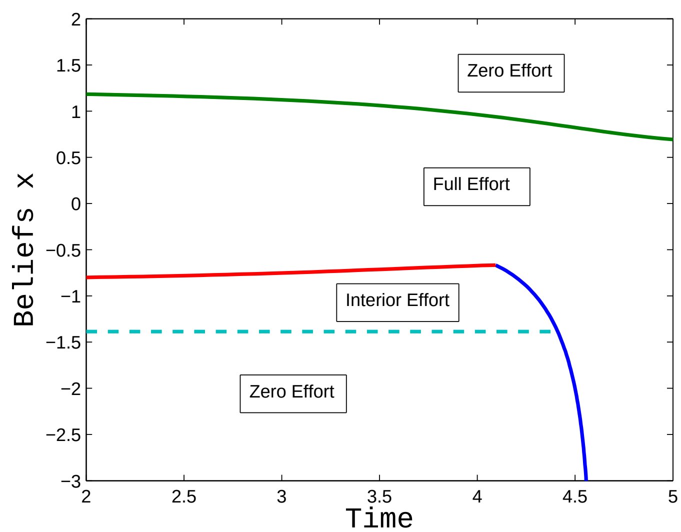


Figure 4: High k ($k = .6$)

We can now state our first comparison result.

Lemma 7

1. *The maximal effort region for the observable case is contained in the full effort region for the unobservable case.*

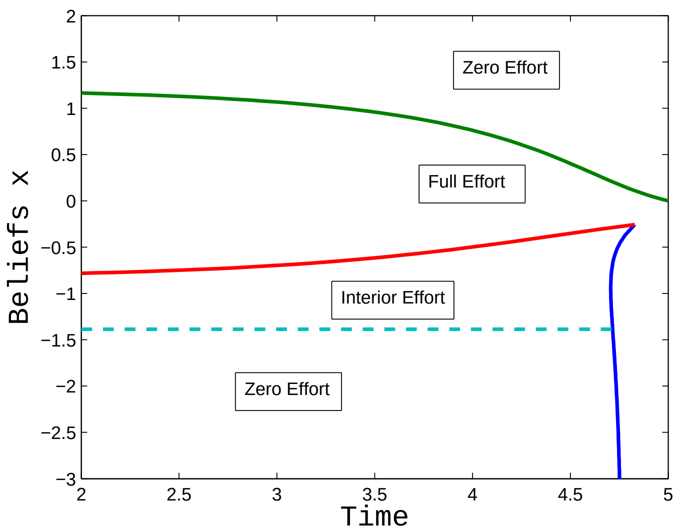


Figure 5: Medium k ($k = .4$)

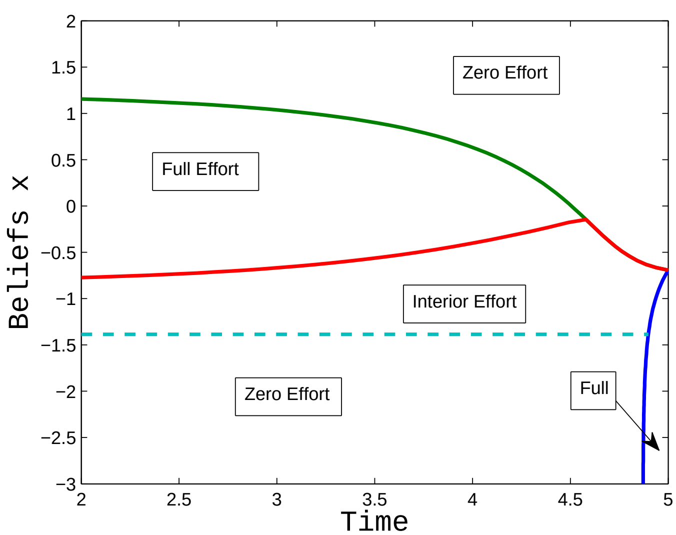


Figure 6: Low k ($k = .3$)

2. *All effort regions in the unobservable case are in turn contained in the full effort region of the social optimum.*

As shown in the pictures, Lemma 7 confirms the intuition that observability of effort reduces the incentives to work. In particular, the highest effort equilibrium in the observable case involves unambiguously lower effort levels than the (unique) equilibrium in the unobservable case. Furthermore, Lemma 7 establishes that the social planner exerts more effort both on the extensive and the intensive margin. The overall effort region in both the observable and unobservable cases are smaller than the planner, who in addition always exerts maximum effort.

We now turn to a comparison of the boundaries in the unobservable case.

Lemma 8

1. *The no effort frontier $x_3(t)$ is increasing in k and v. It is decreasing in α and λ.*

2. *The full effort frontier $x_2(t)$ is increasing in α, λ and $\bar{u}$. It is decreasing in k and v.*

This result holds regardless of whether the full effort region is connected. It confirms the intuition that (in terms of beliefs) the agent works longer when the prize and the penalty are higher, and works less when the marginal cost of effort and the luck component are more significant.

Finally, we turn to the comparative statics of the boundaries in the observable case.

Lemma 9 *The boundary of the maximal effort equilibrium $\bar{x}(t)$ is increasing in k and v and decreasing in α and λ.*

Note that the effect of the maximum effort level $\bar{u}$ is ambiguous, as $\bar{u}$ enters twice. On the one hand, time goes by faster, so the boundary tends to be closer to the deadline T. On the other hand, $\bar{x}(t)$ is also equal to the belief level at which the agent can be indifferent with respect to effort provision, when the market expects effort $\bar{u}$. Increasing the upper bound on effort relaxes this constraint and pushes the boundary out.

Finally, one might wonder whether increasing the termination penalty k can increase welfare, for some parameters, as it might help resolve the commitment problem. Unlike in the non-observable case, this turns out never to occur, at least in the maximum-effort equilibrium: increasing the penalty decreases welfare, though it unambiguously increases total effort. The proof is in Appendix C.

Similarly, increasing v, the value of succeeding, increases effort (in the maximum-effort equilibrium), though it decreases the worker's payoff.

VII Endogenous Deadlines

We now relax the assumption of an exogenous deadline, which can be seen as a partial output-contingent contract that induces the agent to exert effort close to the deadline. We now explore how long the agent will choose to work (i.e. be in a relationship) as a function of his commitment power and the observability of his effort. We confirm the earlier result that, in a specific sense, more effort is exerted without observability.

A The Commitment Case

In this section we impose the additional parameter restriction

$$k + \frac{\alpha}{k} - \frac{v + \alpha \bar{u}}{\lambda + \bar{u}} \geq 0,$$

or v not too large, which ensures the monotonicity of $x_2(t)$. In this sense, this is a natural assumption. It states that when the agent is more pessimistic at the deadline, he works at full speed for a shorter period of time (he has strict incentives to work over a smaller range of beliefs).

We summarize the qualitative properties of the optimal deadline when effort is unobservable, and relegate a more detailed treatment in an appendix.

Lemma 10 *Let T^* denote the optimal deadline with commitment and unobservable effort*

1. *T^* is finite.*

2. *If effort is ever exerted, it is maximal, and exerted at T^*.*

3. *For x_0 sufficiently low, no effort is exerted at $t = 0$.*

4. *For k sufficiently close to α no effort is ever exerted.*

We then turn to the case of observable effort.

Lemma 11 *Let $\tilde{T}$ denote the optimal deadline with commitment and unobservable effort*

1. *$\tilde{T}$ is finite.*

2. *If effort is ever exerted, it is exerted at $\tilde{T}$.*

3. *For x_0 sufficiently low, no effort is exerted at $t = 0$.*

Thus the effort patterns under the optimal deadline are similar in the two cases.

B The Non-commitment Case

In the equilibrium of the game without commitment, the agent must choose to quit at the deadline T. Given an equilibrium deadline T, we fix the off-equilibrium beliefs to specify $\hat{u}_t = \bar{u}$ if $x_t < x^*$ for all $t > T^*$, and $\hat{u}_t = 0$ otherwise. In other words, the market does not react to a failure to quit, anticipates the agent quitting immediately afterwards and expects instantaneous effort to be determined as if $x = x_T$.

Let $\bar{v} := \alpha \left(\lambda \left(1 + k\right) + \bar{u}\right) / k$ and $\underline{v} := \alpha\lambda \left(1 + k\right) / k$. Also denote by

$$\hat{x} := \ln \left(\frac{\lambda + \bar{u}}{\alpha \bar{u} + v} \left(1 + k\right) - 1 \right)$$

the belief level that makes the agent indifferent between continuing or stopping when exerting maximal effort. Similarly, let

$$x' := \ln \left(\frac{\lambda}{v} \left(1 + k\right) - 1 \right)$$

denote the belief level that makes the agent indifferent between continuing or stopping when exerting zero effort. We then have the following characterization in terms of the final beliefs x_T

Lemma 12 *The optimal deadline in the absence of commitment is given by*

$$x_{T^*} = \begin{cases} \max\{x_0, \hat{x}\} & \text{if} \quad v > \bar{v} \\ x^* & \text{if} \quad v \in [\underline{v}, \bar{v}] \\ x' & \text{if} \quad v < \underline{v} \end{cases}$$

if effort is unobservable, and by

$$x_{T^*} = \begin{cases} \max\{x_0, \hat{x}\} & \text{if} \quad v \geq \bar{v} \\ x' & \text{if} \quad v < \bar{v} \end{cases}$$

if effort is observable.

An immediate consequence of Lemma 12 is that the total amount of effort exerted in equilibrium is weakly higher in the unobservable case. Thus, the comparison result carries over to the case of endogenous termination of the relationship. In the unobservable case, the effort patterns can then be traced back to x_0. In particular, they imply a (zero-mix)-full equilibrium in the first case, a (zero)-mix equilibrium in the second, and a zero effort equilibrium in the third.

C Finishing Lines

We now compare setting a deadline with a finishing line. A deadline T is a time at which the game stops. A finishing line, instead, is a value of the belief, $\hat{x}$, at which the game stops, and the penalty k is incurred. Given some finishing line, what is the optimal strategy of the worker? As a consequence, what is the optimal finishing line, and is setting a finishing line preferable to a deadline? For brevity, we relegate the explicit description of the optimal strategy given a finishing line to the proof of the following lemma. Attention is restricted to Markovian strategies, which, given the absence of deadline, reduce to measurable functions $u(\cdot)$ of the (public) belief only. As usual, equilibrium requires that the expected effort that determines the wage coincides with optimal effort.

Lemma 13

1. *Given the finishing line $\hat{x}$, the optimal strategy involves first no effort, then interior and increasing effort, then full effort;*
2. *The optimal finishing line involves stopping immediately: $\hat{x} = x^0$. Therefore, it is always better to set a deadline than a finishing line.*

Effort increases continuously as a function of x. As the lemma states, it is zero for low enough values of x, and maximum for high enough values. These thresholds do not depend on $\hat{x}$ (of course, they only make sense if $\hat{x}$ is larger). Why is effort highest precisely when it has no chance in achieving a success? The point is, the more pessimistic the worker, the lower his wage (for a given amount of expected effort); while for high values of x, the penalty is unavoidable, increasing effort has the advantage of bringing about the end of the game faster, thereby avoiding the flow loss v. For lower values of x instead, the wage is higher, and so the incentive to terminate the game is lower. Hence, effort is highest precisely when success is not to be expected. When the market and the agent are optimistic, the agent has little incentive to work. As a result, flow payoffs are always negative (work is exerted "when it does not pay") and it is best to stop the project as soon as possible.

VIII Concluding Remarks

Rather than summarize our findings, let us point out what we view as the most promising extensions of this agenda.

First and foremost, one might wonder how results would change if the project had to be completed by a team. After all, in law or consulting firms, projects are often assigned to a team of employees that combine partners with junior associates. This raises several issues. The team must achieve several possibly conflicting objectives: incentivizing both the partner and the associate, and eliciting information about the associate's ability. How should profits be shared in the team to do so? When should the project be terminated, or the junior associate replaced? Is it indeed optimal to combine workers whose assessed ability differs, as opposed to workers about whom information is symmetric? Analyzing such questions raises a new modeling challenge, as results are likely to be sensitive to the technology that combines the agents' abilities and effort.

Relatedly, it seems important to "close" the model. For now, the deadline is taken as exogenous, or chosen by the worker. In most settings, the firm is the one that controls the length of the probationary period. Firms have a cost of hiring (or firing) workers –possibly due to the delay in filling a vacancy– but derive a surplus from the worker in excess of the competitive wage they have to pay. Studying the efficiency properties and the characteristics of the resulting labor market (composition of the working force, duration of unemployment) seems to be an interesting undertaking.

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A Appendix

Proof of Lemma 1 The social planner maximizes

$$\int_0^T (1+e^{-x_t})(\frac{\lambda+u_t}{1+e^{x_t}} - \alpha u_t - v)dt - ke^{-x_T}, \text{ s.t. } \dot{x}_t = \lambda + u_t.$$

Let γ_t be the costate variable. The Hamiltonian for this problem is

$$H(x,u,\gamma,t) = e^{-x_t}(\lambda+u_t) - (1+e^{-x_t})(v+\alpha u_t) + \gamma_t(\lambda+u_t).$$

Define $\phi_t := \partial H/\partial u_t = (1-\alpha)e^{-x_t} - \alpha + \gamma_t$. Note that given x_t and γ_t, the value of ϕ_t does not depend on u_t. Pontryagin's principle applies, and yields

$$u_t = \bar{u} \ \ (u_t = 0) \Leftrightarrow \phi_t := \frac{\partial H}{\partial u_t} = (1-\alpha)e^{-x_t} - \alpha + \gamma_t > (<) 0,$$

as well as

$$\dot{\gamma}_t = e^{-x_t}(\lambda - v + (1-\alpha)u_t), \gamma_T = ke^{-x_T}.$$

Differentiating ϕ_t with respect to time, and using the last equation gives

$$\dot{\phi}_t = e^{-x_t}(\alpha\lambda - v), \phi_T = (1+k-\alpha)e^{-x_T} - \alpha.$$

Note that ϕ is either increasing or decreasing depending on the sign of $\alpha\lambda - v$. Therefore, the equilibrium is either maximum effort–no effort, or no effort–maximum effort. $\square$

Proof of Lemma 2 We address the three claims in turn.
Existence: Note that both the integrand of the objective and the state equation are linear in the control u. Therefore, the Filippov-Cesari existence theorem applies (see thm. 8 of Seierstad and Sydsaeter: linearity ensures that the set $N(x,U,t)$ is convex.)
Uniqueness: We can equivalently write the objective as, up to constant terms,

$$\int_0^T (1+e^{-x_t})(w_t - v - \alpha(\dot{x} - \lambda))dt - ke^{-x_T},$$

or, integrating out, letting $g_t := w_t - v + \alpha\lambda$, in terms of the likelihood ratio $l_t = p_t/(1-p_t)$,

$$\int_0^T l_t g_t dt - (k-\alpha)l_T + \alpha \ln l_T + \text{ Constant.}$$

Because the first two terms are linear in l while the last is strictly concave, it follows that there exists a unique optimal terminal odds ratio $l_T^* := l_T$. Furthermore, if there are two paths l, l', with $l_0 = l_0' = p^0/(1-p^0)$, and $l_T = l_T' = l_T^*$, then there exists a third path l^* that strictly improves on either path. Indeed, let $\alpha_t = 1$ if $l_t \geq l_t'$, and $\alpha_t = 0$ otherwise, and define $l_t^* := \alpha_t l_t + (1-\alpha_t) l_t'$. Then

$$\int_0^T (l_t^* - l_t') g_t dt = \int_0^T \alpha_t (l_t - l_t') g_t dt > 0.$$

Sufficiency: While the optimization program described above is not necessarily concave in x, observe that, given $l_t := p_t/(1-p_t)$, it is equivalent to

$$\int_0^T l_t (w_t - \alpha u_t - v)\, dt - k (l_T - 1),$$

such that $\dot{l}_t = -l_t (\lambda + \bar{u})$, so that the maximized Hamiltonian is concave in l, and sufficiency then follows from the Arrow sufficiency theorem (see Seierstad and Sydsaeter (1987), Thm. 3.17). □

Proof of Lemma 4 We prove the following:

1. If there exists $t \in (0, T)$ such that $\phi_t > 0$, then there exists $t' \in [t, T]$ such that $u_s = \bar{u}$ for $s \in [t, t']$, $u_s = 0$ for $s \in (t', T]$.

2. If there exists $t \in (0, T)$ such that $\phi_t < 0$, then either $u_s = 0$ for all $s \in [t, T]$ or $u_s = 0$ for all $s \in [0, t]$,

which easily implies our decomposition. For the first part, note that either $u_s = \bar{u}$ for all $s > t$, or there exists t'' such that both $\phi_{t''} > 0$ (so in particular $u_{t''} = \bar{u}$) and $\dot{\phi}_{t''} < 0$. Because p_t decreases over time, and $u_s \leq u_{t''}$ for all $s > t''$, it follows that $w_s < w_{t''}$, and so $\dot{\phi}_s < \dot{\phi}_{t''} < 0$. Hence ϕ can cross 0 only once for values above t, establishing the result. For the second part, note that either $u_s = 0$ for all $s \geq t$, or there exists $t'' \geq t$ such that $\phi_{t''} < 0$ (so in particular $u_{t''} = 0$) and $\dot{\phi}_{t''} > 0$. Because p_t decreases over time, and $u_s \geq u_{t''}$ for all $s < t''$, it follows that $w_s \geq w_{t''}$, and so $\dot{\phi}_s > \dot{\phi}_{t''} > 0$. For all $s < t''$, $\phi_s < 0$ and $\dot{\phi}_s > 0$. Hence, $u_s = 0$ for all $s \in [0, t]$. □

Proof of Lemma 5 We address the two claims in reverse order.

Uniqueness: assume an equilibrium exists, and note that, given a final belief x_T, the pair of differential equations for ϕ and x (along with the transversality condition) admit a unique solution, pinning down, in particular, the effort exerted by, and the wage received by the agent. Therefore, if two (or more) equilibria existed for some values (x_0, T), it would have to be the case that each of them is associated with a different terminal belief x_T. However, we shall show that, for any x_0, the time it takes to reach a terminal belief x_T is a continuous, strictly increasing function $T(x_T)$; therefore, no two different terminal beliefs can be reached in the same time T. We start with a very optimistic initial belief $x_0 < x_1$, as this allows for the richest paths (the other cases are subsets of these).

Clearly, we have $T(x_0) = 0$. As long as $x_0 < x^*$, we have a first range for x_T over which full effort is always exerted. For these terminal beliefs, we have $T(x_T) = (x_T - x_0)/(\lambda + \bar{u})$, increasing. If for all $x_T \leq x^*$ the following expression is strictly positive

$$(k - \alpha) e^{-x_T} - \alpha - \int_{x_0}^{x_T} e^{-x} \left(\frac{1}{1 + e^x} - \frac{v - \alpha\lambda}{\lambda + \bar{u}} \right) dx, \tag{11}$$

then we always have full effort, until $x_T = x^*$. If so, go to the section "Long Terminal Beliefs." Otherwise, go to the section "Short Terminal Beliefs."

Short Terminal Beliefs

For these beliefs, we have a full effort phase at the end. We assume $x_0 < x_1 < x^*$, as the other cases are subsets of those discussed here. Full effort is exerted at the end typically for short deadlines. If $x_T < x^*$ then the full effort region is given by $[x_2, x_T]$, where x_2 solves

$$(k - \alpha) e^{-x_T} - \alpha - \int_{x_2}^{x_T} e^{-x} \left(\frac{1}{1 + e^x} - \frac{v - \alpha\lambda}{\lambda + \bar{u}} \right) dx = 0.$$

Therefore, we have

$$\frac{dx_2}{dx_T} = \left(\frac{1}{1 + e^{x_2}} - \frac{v - \alpha\lambda}{\lambda + \bar{u}} \right)^{-1} \left(k - \alpha + \frac{1}{1 + e^{x_T}} - \frac{v - \alpha\lambda}{\lambda + \bar{u}} \right) e^{x_2 - x_T}.$$

The denominator is positive by construction ($\phi(x)$ hits zero going backwards).

1. Suppose $x_2 > x_1$. Then the time to get to x_T is given by

$$T(x_T) = \frac{x_T - x_2}{\lambda + \bar{u}} + \int_{x_1}^{x_2} \frac{dx}{\lambda + u(x)} + \frac{x_1 - x_0}{\lambda}.$$

Using the formula for interior effort,

$$u(x) = (v - \alpha\lambda)(1 + e^x) - \lambda,$$

we can write

$$\begin{aligned}
T'(x_T) &= \frac{1}{\lambda + \bar{u}} + \frac{dx_2}{dx_T}\frac{\bar{u} - u(x_2)}{(\lambda + \bar{u})(\lambda + u(x_2))} \\
&\propto \lambda + u(x_2) + \frac{dx_2}{dx_T}(\bar{u} - u(x_2)) \\
&= (v - \alpha\lambda)(1 + e^{x_2}) + (\bar{u} - u(x_2))\frac{dx_2}{dx_T}.
\end{aligned}$$

We want to show $T'(x_T) > 0$. Clearly, if $dx_2/dx_T > 0$, we are done. If not, then we have

$$\begin{aligned}
T'(x_T) &> (v - \alpha\lambda)(1 + e^{x_2}) + (\lambda + \bar{u} - (v - \alpha\lambda)(1 + e^{x_2}))\frac{dx_2}{dx_T}e^{-(x_2 - x_T)} \\
&= (v - \alpha\lambda)(1 + e^{x_2}) + (1 + e^{x_2})(\lambda + u)\left(k - \alpha + \frac{1}{1 + e^{x_T}} - \frac{v - \alpha\lambda}{\lambda + \bar{u}}\right) \\
&\propto k - \alpha + \frac{1}{1 + e^{x_T}} > 0.
\end{aligned}$$

2. Now suppose $x_0 < x_2 < x_1$, and so no effort is exerted on $[x_0, x_2]$. Notice that if $x_2(x_T) \leq x_0$ then $T(x_T)$ is clearly increasing, in x_T (since we have full effort throughout). If $x_2(x_T) > x_0$, the time necessary to reach the terminal belief is given by

$$T(x_T) = \frac{x_T - x_2}{\lambda + \bar{u}} + \frac{x_2 - x_0}{\lambda}.$$

Therefore,

$$\lambda(\lambda + \bar{u})T'(x_T) = \lambda + \bar{u}\frac{dx_2}{dx_T}.$$

It is immediate that if x_2 is increasing in x_T then $T'(\cdot) > 0$. If not, then we have

$$\begin{aligned}
T'(x_T) &\propto \lambda + \bar{u}\frac{dx_2}{dx_T} > \lambda + \bar{u}\frac{dx_2}{dx_T}e^{-(x_2 - x_T)} \\
&\propto \lambda\left(\frac{1}{1 + e^{x_2}} - \frac{v - \alpha\lambda}{\lambda + \bar{u}}\right) + \bar{u}\left(k - \alpha + \frac{1}{1 + e^{x_T}} - \frac{v - \alpha\lambda}{\lambda + \bar{u}}\right).
\end{aligned}$$

We also know $e^{x_2} < e^{x_1} = \lambda / (v - \alpha\lambda) - 1$, and thus

$$\begin{aligned} T'(x_T) &> \lambda \left(\frac{v - \alpha\lambda}{\lambda} - \frac{v - \alpha\lambda}{\lambda + \bar{u}} \right) + \bar{u} \left(k - \alpha + \frac{1}{1 + e^{x_T}} - \frac{v - \alpha\lambda}{\lambda + \bar{u}} \right) \\ &= \bar{u} \left(k - \alpha + \frac{1}{1 + e^{x_T}} \right) > 0. \end{aligned}$$

Longer Terminal Beliefs

For $x_T > x^*$ we can have four possible patterns: never work (in which case the time to x_T is clearly increasing), zero-mixed-zero, zero-mixed-full-zero, or zero-full-zero. We now show that $T(x_T)$ is increasing under any of these patterns. In addition the times at which the equilibrium path switches between the various effort regions are continuous functions of x_T, so it suffices to establish $T'(x_T)$ in each of these cases separately.

Zero and Mixed Effort Phases

We again consider the time necessary to reach a given terminal belief x_T. We consider beliefs $x_T > x^*$, for which the agent does not work at the end. If there is no full effort phase, the agent works at a rate

$$u(x) = (v - \alpha\lambda)(1 + e^x) - \lambda$$

until the switching belief x_3, then stops until x_T. The two thresholds are linked by the equation

$$(k - \alpha) e^{-x_T} - \alpha - \int_{x_3}^{x_T} e^{-x} \left(\frac{1}{1 + e^x} + \alpha - \frac{v}{\lambda} \right) dx = 0.$$

From the state equation, we know beliefs increase at rate $\lambda + u(x)$ in the first phase, and at rate λ afterwards. The time to x_T is therefore given by

$$T(x_T) = \int_{x_1}^{x_T} \frac{1}{\lambda + u(x)} dx = \int_{x_1}^{x_3(x_T)} \frac{1}{(v - \alpha\lambda)(1 + e^x)} dx + \frac{x_T - x_3(x_T)}{\lambda}.$$

Consider the derivative of T with respect to x_T,

$$\lambda T'(x_T) = 1 + \left(\frac{\lambda}{\lambda + u(x_3)} - 1 \right) \frac{dx_3}{dx_T},$$

where dx_3/dx_T is given by

$$\frac{dx_3}{dx_T} = \left(\frac{1}{1 + e^{x_3}} - \frac{v}{\lambda} + \alpha \right)^{-1} \left(k + \frac{1}{1 + e^{x_T}} - \frac{v}{\lambda} \right) e^{x_3 - x_T}. \tag{12}$$

Now, we know $(1+e^{x_3})^{-1}+\alpha-v/\lambda<0$ for all $x>x_1$. Therefore, if $(1+e^{x_3})^{-1}+k-v/\lambda>0$, the whole expression is positive. (Note that our assumption $\lambda k<v$ does not determine the sign of dx_3/dx_T.) Conversely, suppose that $(1+e^{x_3})^{-1}+k-v/\lambda<0$. We then check whether $T'(x_T)$ can be negative. We obtain

$$\begin{aligned}\lambda T'(x_T) &= 1-e^{x_3-x_T}\frac{u(x_3)}{\lambda+u(x_3)}\left(\frac{v}{\lambda}-\frac{1}{1+e^{x_T}}-k\right)\bigg/\left(\frac{v}{\lambda}-\frac{1}{1+e^{x_3}}-\alpha\right)\\ &> 1-\frac{u(x_3)}{\lambda+u(x_3)}\left(\frac{v}{\lambda}-\frac{1}{1+e^{x_T}}-k\right)\bigg/\left(\frac{v}{\lambda}-\frac{1}{1+e^{x_3}}-\alpha\right).\end{aligned}$$

Now plug in the expression for $u(x_3)$, notice that the x_3 drops out, and obtain

$$\lambda T'(x_T)>\lambda\frac{k-\alpha}{v-\alpha\lambda}>0.$$

Full and Mixed Effort Phases

Now suppose the path involves mixing on $[x_1,x_2]$, full effort on $[x_2,x_3]$ and zero effort on $[x_3,x_T]$. The time it takes to reach x_T is then given by

$$\lambda T(x_T)=\int_{x_1}^{x_2(x_T)}\frac{\lambda}{(v-\alpha\lambda)(1+e^{x})}dx+\frac{\lambda}{\lambda+\bar{u}}\left(x_3(x_T)-x_2(x_T)\right)+x_T-x_3(x_T).$$

Hence

$$\lambda T'(x_T)=1-\frac{\bar{u}}{\lambda+\bar{u}}\frac{dx_3}{dx_T}+\frac{dx_2}{dx_T}\left(\frac{\lambda}{\lambda+u(x_2)}-\frac{\lambda}{\lambda+\bar{u}}\right).$$

Notice that x_2 is the solution to

$$\int_{x_2}^{x_3}e^{-x}\left(\frac{1}{1+e^{x}}+\frac{\alpha\lambda-v}{\lambda+\bar{u}}\right)dx=0. \tag{13}$$

We then have

$$\frac{dx_2}{dx_T}=-\frac{e^{-x_3}\left(\frac{1}{1+e^{x_3}}+\frac{\alpha\lambda-v}{\lambda+\bar{u}}\right)}{e^{-x_2}\left(\frac{1}{1+e^{x_2}}+\frac{\alpha\lambda-v}{\lambda+\bar{u}}\right)}\frac{dx_3}{dx_T}, \tag{14}$$

and so

$$\lambda T'(x_T)=1-\frac{dx_3}{dx_T}\left(\frac{\bar{u}}{\lambda+\bar{u}}-\frac{dx_2}{dx_T}\left(\frac{\lambda}{\lambda+u(x_2)}-\frac{\lambda}{\lambda+\bar{u}}\right)\right).$$

Plug in the explicit formula for $u(x_2)$ and for dx_3/dx_T to obtain the following expression for

$\lambda T'(x_T)$:

$$\frac{\left(e^{x_T}(v-k\lambda)-(k+1)\lambda+v\right)\left(-\lambda e^{x_2}\left(\bar{u}+e^{x_3}(\alpha\lambda-v)-v+\alpha\lambda+\lambda\right)-\bar{u}\left(e^{x_3}+1\right)e^{x_3}(v-\alpha\lambda)\right)}{e^{x_T}\left(\bar{u}+\lambda\right)\left(e^{x_T}+1\right)\left(v-\alpha\lambda\right)\left(e^{x_3}(v-\alpha\lambda)+v-(\alpha+1)\lambda\right)}+1.$$

To simplify, let $V=\lambda/(v-\alpha\lambda)-1, U=(\lambda+\bar{u})/(v-\alpha\lambda)-1$, $k=\alpha(K+1)$, and $X_i=e^{x_i}$ to get

$$1-\frac{\left(K(V+1)(X_3+1)\alpha+V-X_T\right)\left(U\left(VX_2+X_2+X_3^2+X_3\right)-X_3(V(X_2+X_3+1)+X_2)\right)}{(U+1)X_T(X_T+1)(V-X_3)}.$$

The constraints are: $0<V<X_2<U<X_3<X_T$, $0<K<X_T$, and $\alpha>0$. Note that the conditions $v>\alpha\lambda$ and $\bar{u}>0$ follow from $U>V>0$. The condition $\alpha<k<\alpha(X_T+1)$ is captured by $0<K<X_T$. Finally, note that if $v>k\lambda$ (which is equivalent to $\alpha K<(1+V)^{-1}$) then this expression is positive, as it is linear in $A=K(1+V)\alpha$, and it is positive both for $A=0,1$.[14]

Only the Full Effort Phase

In this case, the incentives to exert effort hit zero when beliefs are at a level that does not allow mixing, or $x_2<x_1$. The candidate equilibrium involves zero-full-zero effort. The time required

[14]This requires a little bit of work. Consider the case $A=0$. The derivative w.r.t. U of the expression is

$$-\frac{(1+V)\left(X_3+1\right)\left(X_3+X_2\right)\left(X_T-V\right)}{(1+U)^2\left(X_3-V\right)X_T\left(1+X_t\right)}<0,$$

so the expression is minimized by choosing U as high as possible given the constraints, i.e. $U=X_3$, in which case the expression simplifies to

$$\frac{X_3V+X_T\left(1+X_T-X_3\right)}{X_T\left(1+X_T\right)}>0.$$

Consider now $A=1$. Similarly, the derivative w.r.t. U does not depend on U itself, so the expression is minimized at one of the extreme values of U; if $U=X_3$, it is equal to

$$\frac{X_3\left(1+X_3+V\right)+X_T\left(X_T-X_3+1\right)}{X_T\left(1+X_T\right)}>0;$$

if $U=X_2$, the resulting expression's derivative w.r.t. X_2 is independent of X_2, so we can again plug in one of the two extreme cases, $X_2=X_3$ or $X_2=V$; the values are then, respectively,

$$\frac{X_3\left(1+V+X_3\right)+X_T\left(X_T-X_3+1\right)}{X_T\left(1+X_T\right)}>0$$

and

$$\frac{X_T\left(1+X_T+V\right)-V\left(1+V+X_3\right)}{X_T\left(1+X_T\right)}\geq\frac{X_3\left(X_3+1\right)-V\left(V+1\right)}{X_T\left(1+X_T\right)}\geq 0.$$

is then given by

$$\begin{aligned}\lambda T\left(x_T\right) &= x_T - x_3 + \left(x_3 - x_2\right)\frac{\lambda}{\lambda+\bar{u}} + x_2 - x_0\\ &= x_T - \left(x_3 - x_2\right)\frac{\bar{u}}{\lambda+\bar{u}} - x_0,\end{aligned}$$

where x_3 and x_2 solve the same equations as before. Therefore,

$$\lambda T'\left(x_T\right) = 1 - \frac{\bar{u}}{\lambda+\bar{u}}\left(\frac{dx_3}{dx_T} - \frac{dx_2}{dx_T}\right),$$

where the last two terms are given by equations (12) and (14) respectively. Again, if $dx_3/dx_T > 0$, we are done by the same argument as before (remember that the ratio in parentheses is negative). Therefore, let $dx_3/dx_T < 0$. Want to show the term in parentheses is positive. Substituting the expressions in (12) and (14), and using the same change of variable as before, we want to show that

$$1 - \frac{X_2\left(X_2+1\right)\left(U-V\right)\left(U-X_3\right)\left(X_T(K(V+1)\alpha-1)+K(V+1)\alpha+V\right)}{(U+1)X_T\left(X_T+1\right)\left(U-X_2\right)\left(V-X_3\right)} > 0.$$

To establish this inequality, it is simpler to bound α. Setting the expression to zero, this is equivalent to requiring that

$$\alpha K < \frac{(U+1)X_T\left(U-X_2\right)\left(V-X_3\right)}{(V+1)X_2\left(X_2+1\right)\left(U-V\right)\left(U-X_3\right)} - \frac{1}{X_T+1} + \frac{1}{V+1},$$

a sufficient condition for this is that $\alpha K < (1+V)^{-1}$, which is equivalent to $v > k\lambda$.

Existence: We have established that the time necessary to reach the terminal belief is a continuous and strictly increasing function. Therefore, the terminal belief reached in equilibrium is itself given by a strictly increasing function

$$x_T\left(T\right) : \mathbb{R}_+ \to [x_0, \infty).$$

Since there exists a unique path consistent with optimality for each terminal belief, given a deadline T we can establish existence by constructing the associated equilibrium outcome, and in particular, the equilibrium wage path. Existence and uniqueness of an optimal strategy for the worker, after any (on or off-path) history, follows then from Lemma 2. □

Proof of Lemma 6 Suppose that the equilibrium effort is zero on some open set Ω. Consider the sets $\Omega_{t'} = \{(x,s) : s \in (t', T]\}$ such that the trajectory starting at (x,s) intersects Ω. Suppose that u is not identically zero on Ω_0 and let $\tau = \inf\{t' : \ u = 0 \text{ on } \Omega_{t'}\}$. That is, for all $t' < \tau$, there exists $(x,s) \in \Omega_{t'}$, $u(x,s) > 0$. Suppose first that we can take $s = t$. We can write the payoff

$$V(x,t) = \int_x^{x_\tau} \frac{1+e^{-s}}{1+e^{-x}} \left(\frac{\lambda}{1+e^s} - v \right) \frac{1}{\lambda} ds + \frac{1+e^{-x_\tau}}{1+e^{-x}} V(x_\tau, \tau),$$

or, rearranging,

$$\left(1+e^{-x}\right) V(x,t) = -\left(e^{-x_\tau} - e^{-x}\right)\left(1 - \frac{v}{\lambda}\right) - \frac{v}{\lambda}(x_\tau - x) + \left(1+e^{-x_\tau}\right) V(x_\tau, \tau),$$

where $(x_\tau, \tau) \in \Omega$ and $V(x_\tau, \tau)$ is differentiable. The Hamilton-Jacobi-Bellman equation (a function of (x,t)) can be derived from

$$\begin{aligned} V(x,t) &= \frac{\lambda + \hat{u}}{1+e^x} dt - v dt \\ &+ \max_u \left[-\alpha u dt + \left(1 - \frac{\lambda+u}{1+e^x}\right)(V(x,t) + V_x(x,t)(\lambda+u)dt + V_t(x,t)dt) \right] + o(dt), \end{aligned}$$

which gives, taking limits,

$$0 = \frac{\lambda+\hat{u}}{1+e^x} - v + \max_{u\in[0,\bar{u}]} \left[-\alpha u - \frac{\lambda+u}{1+e^x} V(x,t) + V_x(x,t)(\lambda+u) + V_t(x,t) \right].$$

Therefore, if $u(x,t) > 0$,

$$-\frac{V(x,t)}{1+e^x} - \alpha + V_x(x,t) > 0, \text{ or } \left(1+e^{-x}\right) V_x(x,t) - e^{-x} V(x,t) > \alpha\left(1+e^{-x}\right),$$

or finally,

$$\frac{\partial}{\partial x}\left[\left(1+e^{-x}\right) V(x,t)\right] - \alpha\left(1+e^{-x}\right) > 0. \tag{15}$$

Notice, however, by direct computation, that, because low effort is exerted from (x,t) to (x_τ, τ), for all points (x_s, s) on this trajectory, $s \in (t, \tau)$,

$$\frac{\partial}{\partial x}\left[\left(1+e^{-x_s}\right) V(x_s, s)\right] - \alpha\left(1+e^{-x_s}\right) = e^{-x_\tau}\left(1 - \frac{v}{\lambda} - V(x_\tau, \tau)\right) - \alpha - \left(1 + \alpha - \frac{v}{\lambda}\right) e^{-x_s} \leq 0,$$

so that, because $x_t < x_s$, and $1 + \alpha - v/\lambda > 0$,

$$\frac{\partial}{\partial x}\left[\left(1+e^{-x}\right)V\left(x,t\right)\right]-\alpha\left(1+e^{-x}\right)<0,$$

a contradiction to $u\left(x,t\right) > 0$. If instead $u\left(x,t\right) = 0$ for all $\left(x,t\right) \in \Omega_t$, then there exists $\left(x',t'\right) \to \left(x,t\right) \in \Omega_t$, $u\left(x',t'\right) > 0$. Because u is upper semi-continuous, for every $\varepsilon > 0$, there exists a neighborhood $\mathcal{N}$ of $\left(x,t\right)$ such that $u < \varepsilon$ on $\mathcal{N}$. Hence

$$\lim_{(x',t')\to(x,t)}\frac{\partial}{\partial x}\left[\left(1+e^{x'}\right)V\left(x',t'\right)\right]-\alpha\left(1+e^{x'}\right)=\frac{\partial}{\partial x}\left[\left(1+e^{-x}\right)V\left(x,t\right)\right]-\alpha\left(1+e^{-x}\right)<0,$$

a contradiction. $\square$

Proof of Theorem 1 We start with (i). That is, we show that $u\left(x,t\right) = \bar{u}$ for $x < \underline{x}_t$ in all equilibria. We first define $\underline{x}$ as the solution to the differential equation

$$(\lambda(1+\alpha)-v+\left(\lambda+\bar{u}\right)\alpha e^{\underline{x}(t)}+\bar{u}-((1+k)(\lambda+\bar{u})-(v+\alpha\bar{u}))e^{-(\lambda+\bar{u})(T-t)})\left(\frac{\underline{x}'\left(t\right)}{\lambda+\bar{u}}-1\right)=-\bar{u}, \quad (16)$$

subject to $\underline{x}\left(T\right) = x^*$. This defines a strictly increasing function of slope larger than $\lambda + \bar{u}$, for all $t \in (T - t^*, T]$, with $\lim_{t \uparrow t^*} \underline{x}\left(T-t\right) = -\infty$.[15] Given some equilibrium, and an initial value $\left(x_t, t\right)$, let $u\left(\tau; x_\tau\right)$ denote the value at time $\tau \geq t$ along the equilibrium trajectory. For all t, let

$$\tilde{x}\left(t\right):=\sup\left\{x_t:\forall\tau\geq t:u\left(\tau;x_t\right)=\bar{u}\text{ in all equilibria}\right\},$$

with $\tilde{x}\left(t\right) = -\infty$ if no such x_t exists. By definition the function $\tilde{x}$ is increasing (in fact, for all $\tau \geq t$, $\tilde{x}\left(\tau\right) \geq \tilde{x}\left(t\right) + \left(\lambda+\bar{u}\right)\left(\tau - t\right))$, and so it is a.e. differentiable (set $\tilde{x}'\left(t\right) = +\infty$ if x jumps at t). Whenever finite, let $s\left(t\right) = \tilde{x}'\left(t\right)/\left(\tilde{x}'\left(t\right)-\lambda\right) > 0$. Note that, from the transversality condition, $\tilde{x}\left(T\right) = x^*$. In an abuse of notation, we also write $\tilde{x}$ for the set function $t \to [\lim_{t'\uparrow t}\tilde{x}\left(t'\right), \lim_{t'\downarrow t}\tilde{x}\left(t'\right)]$.

We first argue that the incentives to exert high effort decrease in x (when varying the value

[15]The differential equation for $\underline{x}$ can be implicitly solved, which yields

$$\begin{aligned}\ln\frac{k-\alpha}{\alpha} &= \left(\underline{x}_t+\left(\lambda+\bar{u}\right)\left(T-t\right)\right)+\frac{\bar{u}}{\lambda\left(1+\alpha\right)+\bar{u}-v}\ln\left(k-\alpha\right)\bar{u}\left(\lambda+\bar{u}\right)\\ &-\frac{\bar{u}}{\lambda\left(1+\alpha\right)+\bar{u}-v}\ln\left(\begin{array}{c}e^{(\lambda+\bar{u})(T-t)}\left(\lambda\left(1+\alpha\right)+\bar{u}-v\right)\left(\lambda\left(1+\alpha\right)-v+\alpha\left(\lambda+\bar{u}\right)e^{\underline{x}_t}\right)\\ -\left(\lambda\left(1+\alpha\right)-v\right)\left(\lambda\left(1+\alpha\right)+\bar{u}-v+\left(k-\alpha\right)\left(\lambda+\bar{u}\right)\right)\end{array}\right).\end{aligned}$$

x of an initial condition (x,t) for a trajectory along which effort is exerted throughout). Indeed, recall that high effort is exerted iff

$$\frac{\partial}{\partial x}\left(V(x,t)\left(1+e^{-x}\right)\right)\geq\alpha\left(1+e^{-x}\right).$$

The value $V^H(x,t)$ obtained from always exerting (and being paid for) high effort is given by

$$\begin{aligned}(1+e^{-x})V^H(x,t) &= \int_t^T (1+e^{-x_s})\left[\frac{\lambda+\bar{u}}{1+e^{x_s}}-v-\alpha\bar{u}\right]ds-k(1+e^{-x_T})\\ &= \left(e^{-x}-e^{-x_T}\right)\left(1-\frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\right)-(T-t)\left(v+\alpha\bar{u}\right)-k(1+e^{-x_T}) \quad (17)\end{aligned}$$

where $x_T=x+(\lambda+\bar{u})(T-t)$. Therefore, using (15), high effort is exerted if and only if

$$k-\left(1+k-\frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\right)\left(1-e^{-(\lambda+\bar{u})(T-t)}\right)\geq\alpha\left(1+e^{x}\right).$$

Note that the left-hand side is independent of x, while the right-hand side is increasing in x. Therefore, if high effort is exerted throughout from (x,t) onward, then it is also from (x',t) for all $x'<x$.

Fix an equilibrium and a state (x_0,t_0) such that $x_0+(\lambda+\bar{u})(T-t_0)<x^*$. Note that the equilibrium trajectory must eventually intersect some state $(\tilde{x}_t,t)$. We start again from the formula for the payoff

$$\begin{aligned}\left(1+e^{-x_0}\right)V(x_0,t_0) &= \int_{t_0}^{t}\left[e^{-x_s}\left(\lambda+u(x_s,s)\right)-\left(1+e^{-x_s}\right)\left(v+\alpha u(x_s,s)\right)\right]ds\\ &\quad+\left(1+e^{-\tilde{x}_t}\right)V^H(\tilde{x}_t,t).\end{aligned}$$

Let $W(\tilde{x}_t)=V^H(\tilde{x}_t,t)$ (since $\tilde{x}$ is strictly increasing, it is well-defined). Differentiating with respect to x_0, and taking limits as $(x_0,t_0)\to(\tilde{x}_t,t)$, we obtain

$$\begin{aligned}&\lim_{(x_0,t_0)\to(\tilde{x}_t,t)}\frac{\partial\left(1+e^{-x_0}\right)V(x_0,t_0)}{\partial x_0}\\ &= \left[e^{-\tilde{x}_t}\lambda-\left(1+e^{-\tilde{x}_t}\right)v\right]\frac{s(\tilde{x}_t)-1}{\lambda}+s(\tilde{x}_t)\left[W'(\tilde{x}_t)\left(1+e^{-\tilde{x}_t}\right)-W(\tilde{x}_t)e^{-\tilde{x}_t}\right].\end{aligned}$$

If less than maximal effort can be sustained arbitrarily close to, but before the state $(\tilde{x}_t,t)$ is reached, it must be that this expression is no more than $\alpha\left(1+e^{-\tilde{x}_t}\right)$ in some equilibrium, by

(15). Rearranging, this means that

$$\left(1 - W(x) + (1+e^x)\left(W'(x) - \frac{v}{\lambda}\right)\right) s(x) + \left(\frac{v}{\lambda} - \alpha\right) e^x \leq 1 + \alpha - \frac{v}{\lambda},$$

for $x = \tilde{x}_t$. Given the explicit formula for W (see (17)), and since $s(\tilde{x}_t) = \tilde{x}'_t / (\tilde{x}'_t - \lambda)$, we can rearrange this to obtain an inequality for $\tilde{x}_t$. The derivative $\tilde{x}'_t$ is smallest, and thus the solution $\tilde{x}_t$ is highest, when this inequality binds for all t. The resulting ordinary differential equation is precisely (16).

Next, we turn to (ii). That is, we show that $u(x,t) = 0$ for $x > \bar{x}_t$ in all equilibria. We define $\bar{x}$ by

$$\bar{x}_t = \ln\left[k - \alpha + \left(\frac{v + \bar{u}\alpha}{\lambda + \bar{u}} - (1+k)\right)\left(1 - e^{-(\lambda+\bar{u})(T-t)}\right)\right] - \ln\alpha, \tag{18}$$

which is well-defined as long as $k - \alpha + \left(\frac{v+\bar{u}\alpha}{\lambda+\bar{u}} - (1+k)\right)\left(1 - e^{-(\lambda+\bar{u})(T-t)}\right) > 0$. This defines a minimum time $T - t^*$ mentioned above, which coincides with the asymptote of $\underline{x}$ (as can be seen from (16)). It is immediate to check that $\bar{x}$ is continuous and strictly increasing on $[T - t^*, T]$, with $\lim_{t \uparrow t^*} \bar{x}_{T-t} = -\infty$, $x_T = x^*$, and for all $t \in (T - t^*, T)$, $\bar{x}'_t > \lambda + \bar{u}$.

Let us define $W(x,t) = e^{-x} V(x,t)$, and re-derive the HJB equation. The payoff can be written as

$$W(x,t) = \left[(\lambda + u(x,t)) e^{-x} - \left(1 + e^{-x}\right)(v + \alpha u)\right] dt + W(x + dx, t + dt),$$

which gives

$$0 = (\lambda + u(x,t)) e^{-x} - v\left(1 + e^{-x}\right) + W_t(x,t) + \lambda W_x(x,t) + \max_{u \in [0,\bar{u}]} \left(W_x(x,t) - \alpha\left(1 + e^{-x}\right)\right) u.$$

As we already know (see (15)), effort is maximum or minimum depending on $W_x(x,t) \lesseqgtr \alpha(1 + e^{-x})$. Let us rewrite the previous equation as

$$\begin{aligned} & v - \alpha\lambda - W_t(x,t) \\ = \; & ((1+\alpha)\lambda - v + u(x,t)) e^{-x} + \lambda\left(W_x(x,t) - \alpha\left(1 + e^{-x}\right)\right) + \left(W_x(x,t) - \alpha\left(1 + e^{-x}\right)\right)^+ \bar{u}. \end{aligned}$$

Given W_x, W_t is maximized when effort $u(x,t)$ is minimized: the lower $u(x,t)$, the higher $W_t(x,t)$, and hence the lower $W(x, t - dt) = W(x,t) - W_t(x,t)\, dt$. Note also that, along any equilibrium trajectory, no effort is never strictly optimal (by (iv)). Hence, $W_x(x,t) \geq \alpha(1 + e^{-x})$, and therefore, again $u(x,t)$ (or $W(x, t - dt)$) is minimized when $W_x(x,t) = \alpha(1 + e^{-x})$: to minimize $u(x,t)$, while preserving incentives to exert effort, it is best to be indifferent whenever possible.

Hence, integrating over the equilibrium trajectory starting at (x,t),

$$\begin{aligned} & v-\alpha\lambda(T-t)-k\left(1+e^{-x_T}\right)+W(x,t) \\ = & \int_t^T u(x_s,s)\,e^{-x_s}ds+\int_t^T\left[((1+\alpha)\lambda-v)\,e^{-x_s}+(\lambda+\bar{u})\left(W_x(x_s,s)-\alpha\left(1+e^{-x_s}\right)\right)^+\right]ds. \end{aligned}$$

We shall construct an equilibrium in which $W_x(x_s,s)=\alpha\left(1+e^{-x_s}\right)$ for all $x>\underline{x}_t$. Hence, this equilibrium minimizes

$$\int_t^T u(x_s,s)\,e^{-x_s}ds,$$

along the trajectory, and since this is true from any point of the trajectory onward, it also minimizes $u(x_s,s)$, $s\in[t,T]$; the resulting $u(x,t)$ will be shown to be increasing in x, and equal to $\bar{u}$ at $\bar{x}_t$.

Let us construct this interior effort equilibrium. Integrating (15) over any domain with non-empty interior, we obtain that

$$(1+e^x)V(x,t)=e^x(\alpha x+c(t))-\alpha, \tag{19}$$

for some function $c(t)$. Because the trajectories starting at (x,t) must cross $\underline{x}$ (whose slope is larger than $\lambda+\bar{u}$), value matching must hold at the boundary, which means that

$$(1+e^{\underline{x}_t})V^H(\underline{x}_t,t)=e^{\underline{x}_t}(\alpha\underline{x}_t+c(t))-\alpha,$$

which gives $c(t)$ (for $t\geq T-t^*$). From (19), we then back out $V(x,t)$. The HJB equation then reduces to

$$v-\alpha\lambda=\frac{\lambda+u(x,t)}{1+e^x}+V_t(x,t),$$

which can now be solved for $u(x,t)$. That is, the effort is given by

$$\begin{aligned}\lambda+u(x,t) &= (1+e^x)(v-\alpha\lambda)-\frac{\partial}{\partial t}\left[(1+e^x)V(x,t)\right] \\ &= (1+e^x)(v-\alpha\lambda)-e^xc'(t).\end{aligned}$$

It follows from simple algebra (c' is detailed below) that $u(x,t)$ is increasing in x. Therefore, the upper end $\bar{x}_t$ cannot exceed the solution to

$$\lambda + \bar{u} = \left(1 + e^{\bar{x}}\right)(v - \alpha\lambda) - e^{\bar{x}} c'(t),$$

and so we can solve for

$$e^{\bar{x}} = \frac{\lambda(1+\alpha) - v + \bar{u}}{v - \alpha\lambda - c'(t)},$$

Note that, from totally differentiating the equation that defines $c(t)$,

$$\begin{aligned} c'(t) &= \underline{x}'(t) e^{-\underline{x}(t)} \left[\left(W'(\underline{x}(t)) - \alpha\right)\left(e^{\underline{x}(t)} + 1\right) - W(\underline{x}(t)) \right] \\ &= v - \alpha\lambda + e^{-\underline{x}(t)} \left(v - (1+\alpha)\lambda\right), \end{aligned}$$

where we recall that $\underline{x}$ is the lower boundary below which effort must be maximal, and $W(\underline{x}) = V^H(\underline{x}_t, t)$. This gives

$$e^{\bar{x}} = e^{\underline{x}} \frac{\lambda(1+\alpha) - v + \bar{u}}{\lambda(1+\alpha) - v}, \text{ or } e^{\underline{x}} = \frac{\lambda(1+\alpha) - v}{\lambda(1+\alpha) - v + \bar{u}} e^{\bar{x}}.$$

Because (16) is a differential equation characterizing $\underline{x}$, we may substitute for $\bar{x}$ from the last equation to obtain a differential equation characterizing $\bar{x}$, namely

$$\begin{aligned} & \frac{\bar{u}}{1 - \frac{\bar{x}'(t)}{\lambda + \bar{u}}} + \left((1+k)(\lambda + \bar{u}) - (v + \alpha\bar{u})\right) e^{-(\lambda+\bar{u})(T-t)} \\ = \;& \lambda(1+\alpha) + \bar{u} - v + \frac{\alpha(\lambda + \bar{u})(\lambda(1+\alpha) - v)}{\lambda(1+\alpha) - v + \bar{u}} e^{\bar{x}}, \end{aligned}$$

with boundary condition $\bar{x}(T) = x^*$. It is simplest to plug in the formula given by (18) and verify that it is indeed the solution of this differential equation.

Finally, we prove (iii). The same procedure applies to both, so let us consider $\bar{\sigma}$, the strategy that exerts high effort as long as $x < \bar{x}_t$, (and no effort above). We shall do so by "verification." Given our closed-form expression for $V^H(x,t)$ (see (17)), we immediately verify that it satisfies the (15) constraint for all $x \leq \bar{x}_t$ (remarkably, $\bar{x}_t$ is *precisely* the boundary at which the constraint binds; it is strictly satisfied at $\underline{x}_t$, when considering $\underline{\sigma}$). Because this function $V^H(x,t)$ is differentiable in the set $\{(x,t) : x < \bar{x}_t\}$, and satisfies the HJB equation, as well as the boundary condition $V^H(x,T) = 0$, it is a solution to the optimal control problem in this region (remember that the set $\{(x,t) : x < \bar{x}_t\}$ cannot be left under any feasible strategy, so that no further boundary condition needs to be verified). We can now consider the optimal control problem with exit region $\Omega := \{(x,t) : x = \bar{x}_t\} \cup \{(x,t) : t = T\}$ and salvage value $V^H(\bar{x}_t, t)$ or 0, depending on the exit point. Again, the strategy of exerting no effort satisfies the HJB equation, gives a differentiable value on $\mathbb{R} \times [0,T] \setminus \Omega$, and satisfies the boundary conditions. Therefore, it is a solution to the optimal control problem. $\square$

Proof of Lemma 7 (1.) Plug the expression for $\bar{x}(t)$ given by (18) into (10) and notice that (10) cannot be equal to zero unless $\bar{x}(t) = x^*$ and $t = T$, or $\bar{x}(t) \to -\infty$. Therefore, the two frontiers cannot cross before the deadline T, but they have the same vertical asymptote.

Now suppose that $\phi'(x^* \mid \bar{u}) > 0$ so that the frontier $x_2(t)$ goes through (T, x^*). Consider the slopes of $x_2(t)$ and $\bar{x}(t)$ evaluated at (T, x^*). We obtain

$$[\bar{x}'(t) - x_2'(t)]_{t=T} \propto (\bar{u} + \lambda)(k - \alpha) > 0,$$

so the unobservable frontier lies above the observable one for all t.

Next, suppose $\phi'(x^* \mid \bar{u}) < 0$, so there is no mixing at x^* and the frontier $x_2(t)$ does not go through (T, x^*). In this case, we still know the two cannot cross, and we also know a point on $x_2(t)$ is the pre-image of (T, x^*) under full effort. Since we also know the slope $\bar{x}'(t) > \lambda + \bar{u}$, we again conclude that the unobservable frontier $x_2(t)$ lies above $\bar{x}(t)$.

Finally, consider the equation defining the no effort frontier $x_3(t)$,

$$(k - \alpha) e^{-x_3 - \lambda(T-t)} - \alpha - \int_{x_3}^{x_3 + \lambda(T-t)} e^{-x} \left(\frac{1}{1+e^x} - \frac{v - \alpha\lambda}{\lambda} \right) dx = 0. \tag{20}$$

Totally differentiating with respect to t shows that $x_3'(t) < \lambda$ (might be negative). Therefore, the no effort region does not intersect the full effort region defined by $\bar{x}(t)$ in the observable case.

(2.) To compare the effort regions in the unobservable case and the full effort region in the social optimum, consider the planner's frontier $x_P(t)$, which is given by

$$x_P(t) = \ln\left((1 + k - v/\lambda)e^{-\lambda(T-t)} - (\alpha - v/\lambda)\right) - \ln \alpha.$$

The slope of the planner's frontier is given by

$$x_P'(t) = \lambda \frac{(1 + k - v/\lambda)e^{-\lambda(T-t)}}{(1 + k - v/\lambda)e^{-\lambda(T-t)} + v/\lambda - \alpha} \in [0, \lambda].$$

In the equilibrium with unobservable effort, all effort ceases above the frontier $x_3(t)$ defined in (20) above, which has the following slope

$$x_3'(t) = \lambda \frac{\left(\left(1+e^{x_3+\lambda(T-t)}\right)^{-1} + k - v/\lambda\right) e^{-\lambda(T-t)}}{\left(\left(1+e^{x_3+\lambda(T-t)}\right)^{-1} + k - v/\lambda\right) e^{-\lambda(T-t)} + v/\lambda - \alpha - \left(1+e^{x_3}\right)^{-1}}.$$

We also know $x_3(T) = x^*$ and $x_P(T) = \ln((1+k-\alpha)/\alpha) > x^*$. Now suppose towards a contradiction that the two frontiers crossed at a point (t,x). Plugging in the expression for $x_P(t)$ in both slopes, we obtain

$$x_3'(t) = \left(1 + \frac{v/\lambda - \alpha - s(t)}{(1+k-v/\lambda+(1-s(t)))e^{-\lambda(T-t)}}\right)^{-1} > \left(1 + \frac{v/\lambda - \alpha}{(1+k-v/\lambda)e^{-\lambda(T-t)}}\right)^{-1} = x_P'(t),$$

with

$$s(t) = 1/\left(1+e^{x_P(t)}\right) \in [0,1],$$

meaning the unobservable frontier would have to cross from below, a contradiction. $\square$

Proof of Lemma 8 (1.) Fix a terminal belief $x_T = x_3 + \lambda(T-t)$ and consider the equation defining the no effort frontier, which is given in (20). The left hand side of (20) is decreasing in x_3, because fixing x_T the derivative is simply given by $\phi'(x_3 \mid u = 0)$, which is negative by construction. In addition, it is immediate to show that the left hand side of (20) is increasing in k and v and decreasing in α and λ, which establishes the result.

(2.) We analyze the cases of $x_T \leq x^*$ and $x_T > x^*$ separately.

Fix a terminal belief $x_T \leq x^*$ and consider the definition of the full effort frontier, which is obtained by setting $x_0 = x_2$ in equation (11). The left hand side of (11) is increasing in x_2, because fixing x_T the derivative is simply given by $\phi'(x_2 \mid u = \bar{u})$, which is positive by construction. In addition, it is immediate to show that the left hand side of (11) is increasing in k and v and decreasing in α, λ, and $\bar{u}$,which establishes the result.

Fix a terminal belief $x_T > x^*$ and consider the equation defining the full effort frontier, which in this case is given in (13) and depends on $x_3(x_T)$ as well. The left hand side of (13) is increasing in x_2, because fixing x_T and hence $x_3(x_T)$ the derivative is simply given by $\phi'(x_2 \mid u = \bar{u})$, which is positive by construction. In addition, it is immediate to show that the integrand in (11) is increasing in α, λ, and $\bar{u}$, and decreasing in v. Finally, the left hand side of (11) is decreasing in x_3 (the derivative is given by $\phi'(x_3 \mid u = \bar{u}) < 0$). Combining these facts with the comparative statics of x_3 from part (1.) establishes the result. $\square$

Proof of Lemma 9 The results can be obtained directly by differentiating expression (18) for the frontier $\bar{x}(t)$. $\square$

Proof of Lemma 12 In the observable case, suppose the agent quits before x^*. Then he must quit while exerting maximal effort. This can only occur at $x_T = \hat{x}$, which requires no mixing at x^*. Therefore, if $\phi'(x^* \mid \bar{u}) < 0$ and $x_0 < \hat{x}$, the agent quits at $\hat{x}$. If we have mixing at x^*, then the agent can quit at x^*. This requires $\phi'(x^* \mid \bar{u}) < 0$, so that v is not too high and interior effort is exerted at x^*, but also $V'(x^*) < 0$, where the payoff is computed assuming the market expects zero effort (for $x > x^*$) and the agent does not work going forward. The parameter conditions for this case to occur are given given by in equation (27). If on the other hand we have $V'(x^*) \geq 0$, then the agent has an incentive to continue the relationship at x^*, and will therefore quit while exerting zero effort. This occurs at $x_T = x'$, and since in this case $x_1 > x' > x^*$, the agent exerts zero effort throughout.

In the observable case, if $v < \bar{v}$, the agent will never be paid, as he would never stop while being paid, at least for values of x consistent with him working if he were to stop right after this instant. Therefore, the market never pays the worker, and the worker chooses the optimal deadline accordingly. If $v \geq \bar{v}$, then the final belief must be exactly the one that makes him indifferent ($\hat{x}$). $\square$

Proof of Lemma 13 Let $\hat{x}$ denote the stopping belief, fixed exogenously for now. The payoff to be maximized is

$$\int_x^{\hat{x}} \frac{(\lambda + u(x)) e^{-x} - (1 + e^{-x})(v + \alpha u)}{\lambda + u} dx - k\left(1 + e^{-\hat{x}}\right),$$

where $u(x)$ is the expected effort given state x and u is the control variable (equal to $u(x)$ at x in equilibrium). Transversality requires that $u = u(\hat{x})$ maximizes

$$\frac{(\lambda + u(\hat{x})) e^{-\hat{x}} - \left(1 + e^{-\hat{x}}\right)(v + \alpha u)}{\lambda + u},$$

whose derivative w.r.t. u is proportional to

$$\left(1 + e^{\hat{x}}\right)(v - \alpha\lambda) - \lambda - u(\hat{x}).$$

Hence,

$$u(\hat{x}) = \begin{cases} \bar{u} \text{ if } 1+e^{\hat{x}} > \frac{\lambda+\bar{u}}{v-\alpha\lambda}, \\ u \in (0,\bar{u}) \text{ if } u \text{ solves } 1+e^{\hat{x}} = \frac{\lambda+u}{v-\alpha\lambda}, \\ 0 \text{ if } 1+e^{\hat{x}} < \frac{\lambda}{v-\alpha\lambda}. \end{cases}$$

The intuition is straightforward: if $\hat{x}$ is high enough, there is no chance of success, but effort makes the game end sooner; there is no hope avoiding the penalty k, but at least the worker can avoid enduring the flow loss v. For a fixed value of expected effort, the flow cost of delay is relatively more important than the flow wage for high values of $\hat{x}$, because the market pays then a smaller wage, expecting success to be unlikely.

The Hamilton-Jacobi-Bellman equation is

$$(\lambda+u)e^{-x} - v\left(1+e^{-x}\right) + \lambda W'(x) + \max_{u\in[0,\bar{u}]}\left(W'(x) - \alpha\left(1+e^{-x}\right)\right)u,$$

and it follows upon integration in the relevant intervals that, more generally,

$$u(x) = \begin{cases} \bar{u} \text{ for } 1+e^{x} > \frac{\lambda+\bar{u}}{v-\alpha\lambda}, \\ (1+e^{x})(v-\alpha\lambda) - \lambda \text{ if } 1+e^{x} \in \left[\frac{\lambda}{v-\alpha\lambda}, \frac{\lambda+\bar{u}}{v-\alpha\lambda}\right], \\ 0 \text{ if } 1+e^{x} < \frac{\lambda}{v-\alpha\lambda}, \end{cases}$$

for all relevant values of x (i.e., values such that $x < \hat{x}$). Assuming $1+e^{\hat{x}} > \frac{\lambda+\bar{u}}{v-\alpha\lambda}$, the resulting value function has derivative (w.r.t. $\hat{x}$)

$$e^{-\hat{x}}\left(1 - \left(1+e^{\hat{x}}\right)\frac{\alpha\bar{u}+v}{\lambda+\bar{u}}\right),$$

for initial values of $x < \hat{x}$ such that $1+e^{x} > \frac{\lambda+\bar{u}}{v-\alpha\lambda}$. Note that

$$1 - \left(1+e^{\hat{x}}\right)\frac{\alpha\bar{u}+v}{\lambda+\bar{u}} < 1 - \frac{\alpha\bar{u}+v}{v-\alpha\lambda} = -\alpha\frac{\lambda+\bar{u}}{v-\alpha\lambda} < 0,$$

and so increasing the finishing line $\hat{x}$ decreases the value: it is best to set it equal to x. By the principle of optimality, we can rule out finishing lines $\hat{x}$ such that

$$1+e^{\hat{x}} > \frac{\lambda+\bar{u}}{v-\alpha\lambda}.$$

Similarly, assuming that both $1+e^{x}$ and $1+e^{\hat{x}}$ are in $\left[\frac{\lambda}{v-\alpha\lambda}, \frac{\lambda+\bar{u}}{v-\alpha\lambda}\right]$, the derivative of the value (w.r.t. $\hat{x}$) is

$$-\alpha\left(1+e^{-x}\right)<0;$$

hence, such finishing lines in this interval cannot be optimal either. Finally, if both $1+e^{x}$ and $1+e^{\hat{x}}$ are below $\frac{\lambda}{v-\alpha\lambda}$, this derivative equals

$$e^{-\hat{x}}\left(1-\frac{v}{\lambda}\left(1+e^{\hat{x}}\right)\right)<e^{-\hat{x}}\left(1-\frac{v}{v-\alpha\lambda}\right)=-\frac{\alpha\lambda e^{-\hat{x}}}{v-\alpha\lambda}<0,$$

so that, in that case as well, it is best to stop immediately.

To summarize, if one must choose a finishing line $\hat{x}$, it is best to set it equal to x^0, with a resulting value of $-k$. Because choosing a deadline $T=0$ is always possible, it follows immediately that deadlines always improve on finishing lines. $\square$

B Appendix: Endogenous Deadlines

A Unobservable Effort

First, we define the function $x_2(x_T)$ as the solution to

$$(k-\alpha)e^{-x_T}-\alpha-\int_{x_2(x_T)}^{x_T}e^{-x}\left(\frac{1}{1+e^x}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)dx=0. \tag{21}$$

Thus, $x_2(x_T)$ is the lowest belief at which the agent has strict incentives to work, given that his terminal belief is $x_T\le x^*$. We will also make use of the inverse function $x_2^{-1}(x)$ to indicate terminal beliefs x_T.

Throughout this section, we maintain the two following assumptions

$$\begin{aligned} x_2'(x_T) &> 0 \text{ for all } x_T\le x^*,\\ x_0 &< x^*. \end{aligned}$$

Totally differentiating (21), we obtain

$$\frac{dx_2}{dx_T}=\frac{k-\alpha+\frac{1}{1+e^{x_T}}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}}{\frac{1}{1+e^{x_2}}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}}e^{x_2-x_T}.$$

Using the definition of $1/\left(1+e^{x^*}\right)=\alpha/k$, it is then immediate to see that the first assumption is equivalent to the following:

$$k-\alpha+\frac{\alpha}{k}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\ge 0. \tag{22}$$

In particular, (22) allows for both full effort regions and mixing at $x_T=x^*$.

We now look for the optimal deadline under commitment. In our uniqueness proof, we have shown the time required to reach a final belief x_T is strictly increasing. Therefore, we now analyze the equivalent problem of finding the optimal terminal belief x_T. Let the agent's payoff be denoted by

$$V(x_T):=\int_{x_0}^{x_T}\frac{1+e^{-x}}{\lambda+u(x)}\left(\frac{\lambda+u(x)}{1+e^x}-\alpha u(x)-v\right)dx-ke^{-x_T}. \tag{23}$$

We start by establishing some results we use repeatedly.

Lemma 14 *Let the equilibrium be of the (zero)-interior-full effort type. Then $V'(x_T)<0$.*

Proof. Notice that such an equilibrium must involve $x^* \geq x_T \geq x_2(x_T) \geq x_1$. In the mixing region, effort is given by

$$\lambda + u(x) = (v - \alpha\lambda)(1 + e^x).$$

Substituting into (23), we can write the payoff as

$$\begin{aligned} V(x_T) \;:\; &= \int_{x_0}^{x_1} \frac{1+e^{-x}}{\lambda}\left(\frac{\lambda}{1+e^x} - v\right)dx - \int_{x_1}^{x_2(x_T)} \alpha\left(e^{-x}+1\right)dx \\ &+ \int_{x_2(x_T)}^{x_T} \frac{1+e^{-x}}{\lambda+\bar{u}}\left(\frac{\lambda+\bar{u}}{1+e^x} - \alpha\bar{u} - v\right)dx - ke^{-x_T}. \end{aligned}$$

Now, the derivative is given by

$$\begin{aligned} \frac{dV}{dx_T} &= (1+k)e^{-x_T} - \left(1+e^{-x_T}\right)\frac{\alpha\bar{u}+v}{\lambda+\bar{u}} - \frac{dx_2}{dx_T}\left(\frac{1+e^{-x_2}}{\lambda+\bar{u}}\left(\frac{\lambda+\bar{u}}{1+e^{x_2}} - \alpha\bar{u} - v\right) + \left(1+e^{-x_2}\right)\alpha\right) \\ &= -e^{x_2-x_T}\left(1+k-\frac{\alpha\bar{u}+v}{\lambda+\bar{u}}\right) + \frac{1+e^{x_2}}{1+e^{x_T}} - \frac{\alpha\bar{u}+v}{\lambda+\bar{u}}. \end{aligned}$$

Integrating the expression for x_2 in (21), we obtain

$$\left(1+k-\frac{\alpha\bar{u}+v}{\lambda+\bar{u}}\right)e^{x_2-x_T} - \alpha\left(1+e^{x_2}\right) - e^{x_2}\ln\frac{e^{-x_T}+1}{e^{-x_2}+1} - 1 + \frac{\alpha\bar{u}+v}{\lambda+\bar{u}} = 0.$$

Substituting the first term into the derivative, we obtain

$$\begin{aligned} \frac{dV}{dx_T} &= -\alpha\left(1+e^{x_2}\right) - e^{x_2}\ln\frac{e^{-x_T}+1}{e^{-x_2}+1} - 1 + \frac{1+e^{x_2}}{1+e^{x_T}} \\ &< -e^{x_2}\ln\frac{e^{-x_T}+1}{e^{-x_2}+1} - 1 + \frac{1+e^{x_2}}{1+e^{x_T}} \leq 0,\ \forall x_T \geq x_2. \end{aligned} \tag{24}$$

This ends the proof. ■

Therefore, we know payoffs are decreasing in x_T whenever we have an interior effort region.

Corollary 2 *Let the equilibrium be of the zero-full type. Then $V'(x_T) < 0$ for $x_2(x_T)$ sufficiently close to x_1.*

Proof. Such an equilibrium must involve $x_2(x_T) \leq x_1$. We know that the equilibrium effort is continuous at x_1, so that $u(x_1) = 0$. Since the right hand side of (24) if strictly negative for $x_2 \in [x_1, x_T]$, it follows that payoffs are decreasing in x_T whenever $x_2(x_T)$ is in a neighborhood of x_1. ■

Lemma 15 *Let the equilibrium be of the zero-full type. Then*

$$\lim_{x_0 \to -\infty} V'\left(x_2^{-1}(x_0)\right) = +\infty.$$

Proof. Then consider the asymptote of the full effort region we computed earlier,

$$k - \alpha + \left(\frac{v + \bar{u}\alpha}{\lambda + \bar{u}} - (1+k)\right)\left(1 - e^{-(\lambda+\bar{u})(T-t)}\right) = 0,$$

solve for $(\lambda + \bar{u})(T - t)$ and obtain

$$e^{x_2^{-1}(x_0)} \to e^{x_0}\left(\frac{\frac{v+\bar{u}\alpha}{\lambda+\bar{u}} - (1+k)}{\frac{v+\bar{u}\alpha}{\lambda+\bar{u}} - (1+k) + (k-\alpha)}\right).$$

Now collect consider the derivative of the zero-full effort payoffs

$$\frac{dV}{dx_T} = (1+k)\, e^{-x_T} - \left(1 + e^{-x_T}\right)\frac{\alpha\bar{u} + v}{\lambda + \bar{u}} - \frac{dx_2}{dx_T}\left(1 + e^{-x_2}\right)\frac{\bar{u}\,(v - \alpha\lambda)}{\lambda\,(\bar{u} + \lambda)},$$

plug in $x_2 = x_0$ and the expression for $x_T = x_2^{-1}(x_0)$. Then let $x_0 \to -\infty$. We obtain

$$\frac{dV}{dx_T} \to \infty \cdot \operatorname{sign}\left[(\lambda(1+\alpha) - v)\,\frac{(\bar{u} + \lambda(1+k) - v + \bar{u}(k-\alpha))}{\lambda(\lambda(1+\alpha) - v + \bar{u})}\right] = +\infty.$$

This ends the proof. ■

This means it is profitable to stretch the deadline (even if it means introducing a shirking region) for sufficiently optimistic initial beliefs. Finally, we consider the range of short deadlines over which full effort is always exerted.

Lemma 16 *Consider equilibria in which full effort is exerted throughout. Payoffs are increasing in x_T if and only if*

$$e^{x_T} \le e^{\hat{x}} = \frac{\lambda + \bar{u}}{\alpha\bar{u} + v}(1+k) - 1.$$

Furthermore,

$$\hat{x} < x^* \iff \frac{1}{1 + e^{x^*}} - \frac{v - \alpha\lambda}{\lambda + \bar{u}} = \phi'(x^* \mid \bar{u}) < 0.$$

Proof. In this case, the payoff as a function of x_T is given by

$$V(x_T) := \int_{x_0}^{x_T} \frac{1 + e^{-x}}{\lambda + \bar{u}}\left(\frac{\lambda + \bar{u}}{1 + e^{x}} - \alpha\bar{u} - v\right) dx - ke^{-x_T}.$$

Differentiating with respect to x_T yields

$$\frac{dV}{dx_T} = -\left(1+e^{-x_T}\right)\frac{\alpha\bar{u}+v}{\lambda+\bar{u}} + (1+k)\,e^{-x_T},$$

which is equal to zero at $x_T = \hat{x}$. Comparing $\hat{x}$ with x^*, we find

$$x^* - \hat{x} \propto \frac{v-\alpha\lambda}{\lambda+\bar{u}} - \frac{1}{1+e^{x^*}}.$$

This ends the proof. ■

This result allows us to conclude that whenever there is full effort at $x_T = x^*$, the optimal $\hat{x} < x^*$ and (more importantly) whenever we have mixing at x^*, payoffs under full effort are increasing for all $x_T \leq x^*$. It is therefore convenient to divide the analysis in two cases, depending on whether $\hat{x} \lesseqgtr x^*$, or in other words, on whether we have full effort regions at $x_T = x^*$.

A.1 Interior Effort (when $x_T = x^*$)

In this case ($\hat{x} > x^*$), it is useful to remember that, since there are no full effort regions at x^*, the threshold $x_2(x^*)$ is exactly equal to x^*. We highlight the role of x_0 and x_1 in what follows.

- Suppose $x_0 < x_1 < x^*$. Payoffs are clearly increasing in x_T as long as $x_2(x_T) < x_0$, as in that case only full effort is exerted. For longer deadlines, the function $V(\cdot)$ has a kink when $x_2(x_T) = x_0$. At that point, we need to consider the derivative of $V(\cdot)$ under zero-full effort. We know $V'\left(x_2^{-1}(x_1)\right) < 0$ and $\lim_{x\to-\infty} V'\left(x_2^{-1}(x)\right) = +\infty$, so (need to show quasiconcavity here, but pictures help) we conjecture there exists a unique

  $$\tilde{x} < x_1 : V'\left(x_2^{-1}(\tilde{x})\right) = 0.$$

  Clearly, if $x_0 > \tilde{x}$, then the payoff under zero-full is decreasing at $x_T = x_2^{-1}(x_0)$, and therefore we can identify

  $$x_T^* = \max\left\{x_2^{-1}(\tilde{x}), x_2^{-1}(x_0)\right\}, \tag{25}$$

  as the optimal deadline. This deadline yields either a zero-full or a full effort equilibrium. In other words, if x_0 is far away from x_1 the payoff $V(\cdot)$ might still increase at the kink.

- Suppose $x_1 < x_0 < x^*$. Then we know $V'(x_T) > 0$ if $x_2(x_T) < x_0$ but we also know payoffs are decreasing after the kind at $x_2(x_T) = x_0$, because in that case we would have interior effort on $[x_0, x_2]$. In this case, it is immediate to conclude that

$$x_T^* = x_2^{-1}(x_0).$$

Therefore, these two subcases can be summarized by (25).

- A different case is one in which $x_1 > x^*$. By assumption we then have $x_0 < x_1$. Consider the derivative of the payoff at the zero-full equilibrium when $x_T = x^*$ (and consequently the full effort region vanishes). Rearranging, we obtain

$$V'(x^*) \propto (\alpha\lambda - (v - \alpha\lambda) k)^2 - \left(\left(\alpha - k\alpha + k^2\right)(v - \alpha\lambda) k - \alpha^2\lambda\right)\bar{u}. \tag{26}$$

It is easy to check that – under the current assumptions – all three terms in parentheses are positive, and the first one is proportional to the difference $x_1 - x^*$ (which we in fact assume positive in this case). Thus, for x_1 in a neighborhood of x^* we still have $V'(x^*) < 0$. However, for longer deadlines $x_T > x^*$ the payoff is given by

$$V(x_T) := \int_{x_0}^{x_T} \frac{1 + e^{-x}}{\lambda}\left(\frac{\lambda}{1 + e^{x}} - v\right) dx - ke^{-x_T}.$$

This is the case because no effort can be exerted if $x_T \in [x^*, x_1]$. More generally, for $x_1 > x^*$, the equation defining x_3

$$(k - \alpha) e^{-x_T} - \alpha - \int_{x_3}^{x_T} e^{-x}\left(\frac{1}{1 + e^{x}} - \frac{v - \alpha\lambda}{\lambda}\right) dx = 0,$$

does not have a solution for x_T close enough to x_1 (the first terms are $\phi(x_T) < 0$ and the integrand is positive for all $x \geq x_1$).

When the equilibrium is characterized by no effort throughout, the derivative of the payoff is given by

$$V'(x_T) = (1 + k) e^{-x_T} - \left(1 + e^{-x_T}\right)\frac{v}{\lambda}.$$

In particular, we have

$$V'(x^*) = \left(1 + k - \frac{v}{\lambda}\right)\frac{\alpha}{k - \alpha} - \frac{v}{\lambda} > 0, \tag{27}$$

so that V has an upward kink at $x_T = x^*$. The payoff under no effort has a local maximum at

$$1 + e^{x'} = \frac{1+k}{v/\lambda},$$

and it is easy to show that

$$x' < x_1,$$

so we have the right expression.

To summarize, if $V'(x^*)$ is negative from the left and positive from the right, we have a bimodal payoff function, and the optimal deadline is given by

$$x' \text{ or } \max\left\{x_2^{-1}(\tilde{x}), x_2^{-1}(x_0)\right\}.$$

Notice that if the optimal deadline is given by x', then the equilibrium involves no effort throughout.

As we increase the distance $x_1 - x^*$, for example by bringing k closer to α, the left derivative of $V(x^*)$ becomes positive. (In particular, there exists a critical $\tilde{k}$ for which $\tilde{x} = x^*$ and $V'(x^*) = 0$.) In that case, the payoff function is again single peaked and the optimal deadline is given by x'.

A.2 Full Effort (when $x_T = x^*$)

In this case, notice that $\hat{x} < x^*$ implies $x_1 < x^*$. Since x_1 and $\hat{x}$ play a more significant role, we now analyze several subcases. Many of the results follow from the previous subsection.

- Suppose $x_1 > x_2(\hat{x})$. Then:
  - $x_2(\hat{x}) \leq x_0 \Rightarrow x_T^* = \hat{x}$, because in this case we can have the (payoff-maximizing) full effort equilibrium.
  - $x_2(\hat{x}) > x_0 \Rightarrow$ for x_0 in a (left) neighborhood of $x_2(\hat{x})$, the optimal deadline cannot yield a zero-full equilibrium. Indeed, we know the payoff function has a kink when $x_2(x_T) = x_0$, while the derivative of the payoff function under full effort is continuous. Therefore, for x_0 close to $x_2(\hat{x})$, we must have $x_T^* = x_2^{-1}(x_0)$. If x_0 is very low, then we know by the earlier Lemma that eventually $V'\left(x_2^{-1}(x_0)\right) \geq 0$ even under zero-full effort. That occurs precisely when $x_0 = \tilde{x}$. To summarize, in this case we have

$$x_T^* = \max\left\{x_2^{-1}(\tilde{x}), x_2^{-1}(x_0)\right\}.$$

- Suppose $x_1 < x_2(\hat{x})$.
  - $x_2(\hat{x}) < x_0 \Rightarrow x_T^* = \hat{x}$ because, as before, we can have the (payoff-maximizing) full effort equilibrium.
  - $x_2(\hat{x}) > x_0 > x_1 \Rightarrow x_T^* = x_2^{-1}(x_0)$ because in this case we know payoffs are increasing when $x_T = x_2^{-1}(x_0) < \hat{x}$, but also decreasing if we extend x_T further. Indeed, we would introduce an interior effort region, so after the kink, the payoff function is downward sloping.
  - $x_2(\hat{x}) > x_1 > x_0 \Rightarrow x_T^* = \max\left\{x_2^{-1}(\tilde{x}), x_2^{-1}(x_0)\right\}$. This case is the most similar one to the previous subsection. Extending the deadline past $x_2^{-1}(x_0)$ introduces a no effort region. Therefore, if $V'\left(x_2^{-1}(x_0)\right) < 0$, the optimal deadline is $x_2^{-1}(x_0)$. If not, then we can extend x_T until $x_2 = \tilde{x} < x_1$.

B Observable Effort

Let us start by assuming commitment to a deadline. At what belief is a player indifferent between stopping immediately and one instant later? Denote the threshold belief $\hat{x}$. If effort is exerted at that moment, we can derive $\hat{x}$ by computing the derivative with respect to T of (17), and setting it equal to for $t = T$. We obtain

$$e^{\hat{x}} = \frac{\lambda + \bar{u}}{\alpha\bar{u} + v}(1+k) - 1,$$

which is lower than x^*, as it must, iff

$$v > \bar{v} := \frac{\alpha}{k}(\lambda(1+k) + \bar{u}).$$

On the other hand, if

$$v < \underline{v} := \frac{\alpha}{k}\lambda(1+k),$$

then we must define

$$e^{\hat{x}} = \frac{\lambda}{v}(1+k) - 1 > e^{x^*},$$

while if $v \in [\underline{v}, \bar{v}]$, we set $\hat{x} = x^* = \ln(k-\alpha)/\alpha$.

To conclude, if the choice to the agent were to stop or continue for an instant, he would stop if and only if $x > \hat{x}$. However, what this means in terms of the optimal deadline will now hinge on whether this involves work or not at that time, i.e., whether $v \lesseqgtr \underline{v}$.

Let us first assume that $v > \underline{v}$, so that he would work at the belief at which he is indifferent between stopping and continuing. If $x \geq \hat{x}$, we are done: the agent chooses to stop immediately. Suppose so $x < \hat{x}$. To minimize possibilities, we focus throughout on the equilibrium that maximizes effort, i.e. so that, given the deadline, the agent works if and only if $x < \bar{x}_t$.

Suppose first that the value of T that solves

$$x + (\lambda + \bar{u})\, T = \hat{x}$$

is such that, given a deadline of T, $x < \bar{x}_0$. In that case, it is clear that this is the optimal deadline for the worker. What if $x > \bar{x}_0$? Maintain this assumption. Note that choosing the optimal deadline given x is equivalent to choosing, given T large enough, the value of t that maximizes $V(t,x)$. The question is then, were is this maximum achieved? For (t,x) such that $x < \bar{x}_t$, $V_t(t,x) < 0$, and the agent prefers to have more time (a lower value of t). The key computation, then, is to the immediate left of $(t, \bar{x}_t)$. Recall that the equilibrium payoff in this region (in which low effort is exerted) is given by

$$\left(1 + e^{-x}\right) V(x,t) = \int_x^{x_\tau} \left(e^{-s} - \frac{v}{\lambda}\left(1 + e^{-s}\right)\right) ds + \left(1 + e^{-x_\tau}\right) V(x_\tau),$$

where $V(x_\tau)$ is the value on the boundary where effort switches from zero to maximum effort, and $x_\tau = x + \lambda(\tau - t)$ is the belief at that time. Differentiating with respect to t gives

$$\left[e^{-x_\tau} - \frac{v}{\lambda}\left(1 + e^{-x_\tau}\right) + \frac{d}{dx_\tau}\left(\left(1 + e^{-x_\tau}\right) V(x_\tau)\right)\right] \frac{dx_\tau}{dt},$$

but recall that, at the boundary, the inner derivative is identically equal to $\alpha\left(1 + e^{-x_\tau}\right)$. Note that, for all $x < x^*$,

$$e^{-x_\tau} + \left(\alpha - \frac{v}{\lambda}\right)\left(1 + e^{-x_\tau}\right) > 0,$$

while $dx_\tau/dt < 0$ (increasing t shortens the time until the boundary is hit, and thus lower the corresponding belief). Hence, the payoff strictly decreasing in t to the left of the boundary, at least for values of t sufficiently close to the boundary. This implies that the optimum deadline is then set such that the worker shirks for a while, at the beginning. This optimum deadline T, however, must be finite.

If $v < \underline{v}$, on the other hand, the worker has a choice: either he insist on working until the belief reaches $\hat{x}$ (or any value of x such that $x_T > x^*$), and then he must always put in zero effort; or he must choose a deadline that forces him to stop at a belief no larger than x^*. In the former case, which occurs if x is high enough, he sets the deadline so that

$$x + \lambda T = \hat{x};$$

in the latter case, the analysis is as in the case $v > \underline{v}$: if the value of T that solves

$$x + (\lambda + \bar{u})\, T = x^*$$

is such that $x < \bar{x}_0$, that value of T is the optimal deadline; if not, a longer deadline is chosen that maximizes $V(x, t)$ subject to $x_T < x^*$.

C Appendix: Comparative Statics

A Unobservable Effort

Throughout this section we fix a deadline T. We then focus on four kinds of equilibrium structures: interior-zero effort, interior-full effort, zero-full effort, and finally zero-interior-full-zero effort. We first examine the role of learning on payoffs, measured by the impact of the terminal beliefs x_T for a given T. We then turn to the effects of the penalty k and the cost of waiting v.

A.1 Zero-Interior-Zero Equilibria

For a given T, we have the following equations linking the thresholds x_T and x_2.

$$\begin{aligned}
(k-\alpha)e^{-x_T}-\alpha &= \int_{x_2}^{x_T} e^{-x}\left(\frac{1}{1+e^{x}}+\alpha-\frac{v}{\lambda}\right)dx \\
T\lambda &= x_T - x_2 + \int_{x_1}^{x_2}\frac{\lambda}{\lambda+u\left(x\right)}dx + x_1 - x_0.
\end{aligned}$$

From the second one we can compute

$$\frac{dx_2}{dx_T}=\frac{\lambda+u\left(x_2\right)}{u\left(x_2\right)}.$$

The payoff to the agent is given by

$$\begin{aligned}
V &= \int_{x_0}^{x_1}\left(e^{-x}-\left(1+e^{-x}\right)\frac{v}{\lambda}\right)dx+\int_{x_1}^{x_2}\left(e^{-x}-\left(1+e^{-x}\right)\frac{v+\alpha u\left(x\right)}{\lambda+u\left(x\right)}\right)dx \\
&\quad +\int_{x_2}^{x_T}\left(e^{-x}-\left(1+e^{-x}\right)\frac{v}{\lambda}\right)dx-ke^{-x_T}.
\end{aligned}$$

Therefore, we can express the effect of x_T on payoffs as

$$\frac{dV}{dx_T}=ke^{-x_T}+e^{-x_T}-\left(1+e^{-x_T}\right)\frac{v}{\lambda}-\frac{dx_2}{dx_T}\left(1+e^{-x_2}\right)\left(\frac{v+\alpha u\left(x_2\right)}{\lambda+u\left(x_2\right)}-\frac{v}{\lambda}\right)$$

Using the fact that in the interior effort region we have

$$\lambda+u\left(x\right)=\left(v-\alpha\lambda\right)\left(1+e^{x}\right),$$

we obtain

$$\frac{dV}{dx_T} = (1+k)\,e^{-x_T} - \left(1+e^{-x_T}\right)\frac{v}{\lambda} + \left(1+e^{-x_2}\right)\frac{v-\alpha\lambda}{\lambda}.$$

1. **The effect of k.** We have

$$\frac{dV}{dk} = -e^{-x_T} + \frac{dV}{dx_T}\frac{dx_T}{dk}$$

where, based on the first two equations,

$$\frac{dx_T}{dk} = \left(\left(k + \frac{1}{1+e^{x_T}} - \frac{v}{\lambda}\right)e^{-x_T} + \frac{v-\alpha\lambda}{\lambda}e^{-x_2}\right)^{-1} e^{-x_T}.$$

It is immediate (using $x_T > x_2$) to check that $dx_T/dk > 0$. Simplifying, we obtain

$$\frac{dV}{dk} = \frac{dx_T}{dk}\left(\frac{1}{e^{x_T}+1} - \alpha\right).$$

2. **The effect of v.** We have

$$\frac{dV}{dv} = \frac{dV}{dx_T}\frac{dx_T}{dv} - \int_{x_0}^{x_1}\frac{1+e^{-x}}{\lambda}dx - \int_{x_2}^{x_T}\frac{1+e^{-x}}{\lambda}dx$$

(notice that when effort is interior $\left(v+\alpha u\left(x\right)\right)/\left(\lambda+u\left(x\right)\right)$ does not depend on v). In addition, we have

$$\frac{dx_T}{dv} = \left(\left(k - \alpha + \frac{1}{1+e^{x_T}} - \frac{v-\alpha\lambda}{\lambda}\right)e^{-x_T} + e^{-x_2}\frac{v-\alpha\lambda}{\lambda}\right)^{-1}\frac{e^{-x_2}-e^{-x_T}}{\lambda},$$

which is also positive. Simplifying, we obtain

$$\begin{aligned}
\frac{dV}{dv} &= \frac{e^{-x_2}-e^{-x_T}}{\lambda} + \left(\frac{1}{1+e^{x_T}} - \alpha\right)\frac{dx_T}{dv} - \int_{x_0}^{x_1}\frac{1+e^{-x}}{\lambda}dx - \int_{x_2}^{x_T}\frac{1+e^{-x}}{\lambda}dx \\
&= \left(\frac{1}{1+e^{x_T}} - \alpha\right)\frac{dx_T}{dv} - \int_{x_0}^{x_1}\frac{1+e^{-x}}{\lambda}dx - \int_{x_2}^{x_T}\frac{1}{\lambda}dx.
\end{aligned}$$

A.2 Zero-Interior-Full Equilibria

The equations determining the switching threshold x_2 are given by

$$\begin{aligned}(k-\alpha)e^{-x_T}-\alpha &= \int_{x_2}^{x_T} e^{-x}\left(\frac{1}{1+e^{x}}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)dx \\ T\left(\lambda+\bar{u}\right) &= x_T-x_2+\int_{x_1}^{x_2}\frac{\lambda+\bar{u}}{\lambda+u\left(x\right)}dx+\frac{\lambda+\bar{u}}{\lambda}\left(x_1-x_0\right).\end{aligned}$$

Therefore, we have

$$\frac{dx_2}{dx_T}=-\frac{\lambda+u\left(x_2\right)}{\bar{u}-u\left(x_2\right)},$$

The agent's payoff is given by

$$\begin{aligned}V\left(k\right) &= \int_{x_0}^{x_1}\left(e^{-x}-\left(1+e^{-x}\right)\frac{v}{\lambda}\right)dx+\int_{x_1}^{x_2}\left(e^{-x}-\left(1+e^{-x}\right)\frac{v+\alpha u\left(x\right)}{\lambda+u\left(x\right)}\right)dx \\ &\quad+\int_{x_2}^{x_T}\left(e^{-x}-\left(1+e^{-x}\right)\frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\right)dx-ke^{-x_T}.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{dV}{dx_T} &= \left(1+k\right)e^{-x_T}-\left(1+e^{-x_T}\right)\frac{v+\alpha\bar{u}}{\lambda+\bar{u}}-\frac{dx_2}{dx_T}\left(1+e^{-x_2}\right)\left(\frac{v+\alpha u\left(x_2\right)}{\lambda+u\left(x_2\right)}-\frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\right) \\ &= \left(1+k\right)e^{-x_T}-\left(1+e^{-x_T}\right)\frac{v+\alpha\bar{u}}{\lambda+\bar{u}}+\left(1+e^{-x_2}\right)\frac{v-\alpha\lambda}{\lambda+\bar{u}}\end{aligned}$$

1. **The effect of k.** We have

   $$\frac{dV}{dk}=-e^{-x_T}+\frac{dV}{dx_T}\frac{dx_T}{dk}$$

   where, based on the first two equations,

   $$\frac{dx_T}{dk}=\left(\left(k+\frac{1}{1+e^{x_T}}-\frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\right)e^{-x_T}+\frac{v-\alpha\lambda}{\lambda+\bar{u}}e^{-x_2}\right)^{-1}e^{-x_T}.$$

   It is immediate (using $x_T>x_2$) to check that $dx_T/dk>0$. Simplifying, we obtain

   $$\frac{dV}{dk}=\frac{dx_T}{dk}\left(\frac{1}{e^{x_T}+1}-\alpha\right).$$

2. **The effect of v.** We have

$$\frac{dV}{dv}=\frac{dV}{dx_T}\frac{dx_T}{dv}-\int_{x_0}^{x_1}\frac{1+e^{-x}}{\lambda}dx-\int_{x_2}^{x_T}\frac{1+e^{-x}}{\lambda+\bar{u}}dx$$

(notice that when effort is interior $(v+\alpha u(x))/(\lambda+u(x))$ does not depend on v). In addition, we have

$$\frac{dx_T}{dv}=\left(\left(k-\alpha+\frac{1}{1+e^{x_T}}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)e^{-x_T}+\frac{v-\alpha\lambda}{\lambda+\bar{u}}e^{-x_2}\right)^{-1}\frac{e^{-x_2}-e^{-x_T}}{\lambda+\bar{u}},$$

which is also positive. Simplifying, we obtain

$$\begin{aligned}\frac{dV}{dv} &= \frac{(1+k)\,e^{-x_T}-\left(1+e^{-x_T}\right)\frac{v+\alpha\bar{u}}{\lambda+\bar{u}}+\left(1+e^{-x_2}\right)\frac{v-\alpha\lambda}{\lambda+\bar{u}}}{\left(k-\alpha+\frac{1}{1+e^{x_T}}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)e^{-x_T}+\frac{v-\alpha\lambda}{\lambda+\bar{u}}e^{-x_2}}\frac{e^{-x_2}-e^{-x_T}}{\lambda+\bar{u}} \\ &\quad -\int_{x_0}^{x_1}\frac{1+e^{-x}}{\lambda}dx-\int_{x_2}^{x_T}\frac{1+e^{-x}}{\lambda+\bar{u}}dx \\ &= \left(\frac{1}{1+e^{x_T}}-\alpha\right)\frac{dx_T}{dv}-\int_{x_0}^{x_1}\frac{1+e^{-x}}{\lambda}dx-\int_{x_2}^{x_T}\frac{1}{\lambda+\bar{u}}dx.\end{aligned}$$

A.3 Zero-Full Equilibria

For T short enough and x_0 low enough, we can have have a zero-full equilibrium with the threshold x_2 determined by

$$\begin{aligned}(k-\alpha)e^{-x_T}-\alpha &= \int_{x_2}^{x_T}e^{-x}\left(\frac{1}{1+e^{x}}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)dx \\ T\,(\lambda+\bar{u}) &= x_T-x_2+\frac{\lambda+\bar{u}}{\lambda}\,(x_2-x_0)\,.\end{aligned}$$

Therefore, we have

$$\frac{dx_2}{dx_T}=-\frac{\lambda}{\bar{u}},$$

and

$$V=\int_{x_0}^{x_2}\left(e^{-x}-\left(1+e^{-x}\right)\frac{v}{\lambda}\right)dx+\int_{x_2}^{x_T}\left(e^{-x}-\left(1+e^{-x}\right)\frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\right)dx-ke^{-x_T}.$$

Therefore

$$\begin{aligned}\frac{dV}{dx_T} &= (1+k)\,e^{-x_T} - \left(1+e^{-x_T}\right)\frac{v+\alpha\bar{u}}{\lambda+\bar{u}} - \frac{dx_2}{dx_T}\left(1+e^{-x_2}\right)\left(\frac{v}{\lambda} - \frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\right)\\ &= (1+k)\,e^{-x_T} - \left(1+e^{-x_T}\right)\frac{v+\alpha\bar{u}}{\lambda+\bar{u}} + \left(1+e^{-x_2}\right)\frac{v-\alpha\lambda}{\lambda+\bar{u}},\end{aligned}$$

exactly as in the previous case.

1. **The effect of k.** We have
$$\frac{dV}{dk} = -e^{-x_T} + \frac{dV}{dx_T}\frac{dx_T}{dk}$$
where, based on the first two equations,
$$\frac{dx_T}{dk} = \left((k+\frac{1}{1+e^{x_T}} - \frac{v+\alpha\bar{u}}{\lambda+\bar{u}})e^{-x_T} + \frac{\lambda}{\bar{u}}e^{-x_2}\left(\frac{1}{1+e^{x_2}} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)\right)^{-1} e^{-x_T}.$$
It is immediate (using $x_T > x_2$ and $x_2 < x_1$) to check that $dx_T/dk > 0$. We then have
$$\frac{dV}{dk} = \frac{dx_T}{dk}\left(\frac{1}{1+e^{x_T}} - \alpha + e^{-x_2}\frac{\left(e^{x_2}+1\right)(v-\alpha\lambda)-\lambda}{\bar{u}\left(1+e^{x_2}\right)}\right)$$

2. **The effect of v.** We have
$$\frac{dV}{dv} = \frac{dV}{dx_T}\frac{dx_T}{dv} - \int_{x_0}^{x_2}\frac{1+e^{-x}}{\lambda}dx - \int_{x_2}^{x_T}\frac{1+e^{-x}}{\lambda+\bar{u}}dx$$
In addition, we have
$$\frac{dx_T}{dv} = \left(\left(k-\alpha+\frac{1}{1+e^{x_T}} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)e^{-x_T} + \frac{\lambda}{\bar{u}}e^{-x_2}\left(\frac{1}{1+e^{x_2}} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)\right)^{-1}\frac{e^{-x_2}-e^{-x_T}}{\lambda+\bar{u}},$$
which is also positive. Simplifying, we obtain
$$\begin{aligned}\frac{dV}{dv} &= \frac{(1+k)\,e^{-x_T} - \left(1+e^{-x_T}\right)\frac{v+\alpha\bar{u}}{\lambda+\bar{u}} + \left(1+e^{-x_2}\right)\frac{v-\alpha\lambda}{\lambda+\bar{u}}}{\left(k-\alpha+\frac{1}{1+e^{x_T}} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)e^{-x_T} + \frac{\lambda}{\bar{u}}e^{-x_2}\left(\frac{1}{1+e^{x_2}} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)}\frac{e^{-x_2}-e^{-x_T}}{\lambda+\bar{u}}\\ &\quad - \int_{x_0}^{x_1}\frac{1+e^{-x}}{\lambda}dx - \int_{x_2}^{x_T}\frac{1+e^{-x}}{\lambda+\bar{u}}dx\\ &= \left(\frac{1}{1+e^{x_T}} - \alpha + e^{-x_2}\frac{\left(e^{x_2}+1\right)(v-\alpha\lambda)-\lambda}{\bar{u}\left(1+e^{x_2}\right)}\right)\frac{dx_T}{dv} - \int_{x_0}^{x_1}\frac{1+e^{-x}}{\lambda}dx - \int_{x_2}^{x_T}\frac{1}{\lambda+\bar{u}}dx.\end{aligned}$$

A.4 Zero-Interior-Full-Zero

These equilibria can occur for $\bar{u}$ low enough, and x_0 also low. The equations determining the switching thresholds are given by

$$
\begin{aligned}
(k-\alpha)e^{-x_T}-\alpha &= \int_{x_3}^{x_T} e^{-x}\left(\frac{1}{1+e^x}-\frac{v-\alpha\lambda}{\lambda}\right)dx \\
T\lambda &= x_T - x_3 + x_1 - x_0 + \frac{x_3-x_2}{\lambda+\bar{u}}\lambda + \int_{x_1}^{x_2}\frac{\lambda}{\lambda+u(x)}dx. \\
0 &= \int_{x_2}^{x_3} e^{-x}\left(\frac{1}{1+e^x}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)dx
\end{aligned}
$$

From the second and third equations, we have

$$
\begin{aligned}
1-\frac{dx_3}{dx_T}\left(1-\frac{\lambda}{\lambda+\bar{u}}\right) &= -\frac{dx_2}{dx_T}\left(\frac{\lambda}{\lambda+u(x_2)}-\frac{\lambda}{\lambda+\bar{u}}\right) \\
\frac{dx_3}{dx_T}e^{-x_3}\left(\frac{1}{1+e^{x_3}}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\right) &= \frac{dx_2}{dx_T}e^{-x_2}\left(\frac{1}{1+e^{x_2}}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{dx_3}{dx_T} &= -\frac{e^{-x_2}\frac{v-\alpha\lambda}{\lambda}}{e^{-x_3}\left(\frac{1}{1+e^{x_3}}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)-e^{-x_2}\frac{\bar{u}}{\lambda}\frac{v-\alpha\lambda}{\bar{u}+\lambda}} \\
\frac{dx_2}{dx_T} &= -\frac{1+\frac{e^{-x_2}\frac{v-\alpha\lambda}{\lambda}}{e^{-x_3}\left(\frac{1}{1+e^{x_3}}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)-e^{-x_2}\frac{\bar{u}}{\lambda}\frac{v-\alpha\lambda}{\bar{u}+\lambda}}\left(1-\frac{\lambda}{\lambda+\bar{u}}\right)}{\left(\frac{\lambda}{(v-\alpha\lambda)(1+e^{x_2})}-\frac{\lambda}{\lambda+\bar{u}}\right)}.
\end{aligned}
$$

Now payoffs

$$
\begin{aligned}
V(k,T) &= \int_{x_0}^{x_1}\left(e^{-x}-\left(1+e^{-x}\right)\frac{v}{\lambda}\right)dx + \int_{x_1}^{x_2}\left(e^{-x}-\left(1+e^{-x}\right)\frac{v+\alpha u(x)}{\lambda+u(x)}\right)dx \\
&\quad + \int_{x_2}^{x_3}\left(e^{-x}-\left(1+e^{-x}\right)\frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\right)dx + \int_{x_3}^{x_T}\left(e^{-x}-\left(1+e^{-x}\right)\frac{v}{\lambda}\right)dx - ke^{-x_T}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{dV}{dx_T} &= (1+k)\,e^{-x_T} - \left(1+e^{-x_T}\right)\frac{v}{\lambda} - \frac{dx_3}{dx_T}\left(1+e^{-x_3}\right)\left(\frac{v+\alpha\bar{u}}{\lambda+\bar{u}} - \frac{v}{\lambda}\right) \\
&\quad - \frac{dx_2}{dx_T}\left(1+e^{-x_2}\right)\left(\frac{v+\alpha u\,(x_2)}{\lambda+u\,(x_2)} - \frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\right) \\
&= (1+k)\,e^{-x_T} - \left(1+e^{-x_T}\right)\frac{v}{\lambda} + \frac{v-\alpha\lambda}{\lambda} + \frac{e^{-x_3}\left(\frac{1}{1+e^{x_3}} - \frac{v-\alpha\lambda}{\lambda}\right)e^{-x_2}}{e^{-x_3}\left(\frac{1}{1+e^{x_3}} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right) - e^{-x_2}\frac{\bar{u}}{\lambda}\frac{v-\alpha\lambda}{\bar{u}+\lambda}}\frac{v-\alpha\lambda}{\lambda} \\
&= (1+k)\,e^{-x_T} - \left(1+e^{-x_T}\right)\frac{v}{\lambda} + \frac{v-\alpha\lambda}{\lambda}\left(1 + \frac{e^{-x_3}\left(\frac{1}{1+e^{x_3}} - \frac{v-\alpha\lambda}{\lambda}\right)e^{-x_2}}{e^{-x_3}\left(\frac{1}{1+e^{x_3}} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right) - e^{-x_2}\frac{\bar{u}}{\lambda}\frac{v-\alpha\lambda}{\bar{u}+\lambda}}\right).
\end{aligned}
$$

1. **The effect of k.** We have

$$\frac{dV}{dk} = -e^{-x_T} + \frac{dV}{dx_T}\frac{dx_T}{dk}$$

where, based on the first three equations,

$$\frac{dx_T}{dk} = \left((k-\alpha+\frac{1}{1+e^{x_T}} - \frac{v-\alpha\lambda}{\lambda})e^{-x_T} - \frac{dx_3}{dx_T}e^{-x_3}\left(\frac{1}{1+e^{x_3}} - \frac{v-\alpha\lambda}{\lambda}\right)\right)^{-1} e^{-x_T},$$

where dx_3/dx_T is given above. It is immediate (using $x_T > x_3$) to check that $dx_T/dk > 0$. Substituting, we again obtain

$$\frac{dV}{dk} = \frac{dx_T}{dk}\left(\frac{1}{1+e^{x_T}} - \alpha\right).$$

2. **The effect of v.** We have

$$\frac{dV}{dv} = \frac{dV}{dx_T}\frac{dx_T}{dv} - \int_{x_0}^{x_1}\frac{1+e^{-x}}{\lambda}dx - \int_{x_2}^{x_3}\frac{1+e^{-x}}{\lambda+\bar{u}}dx - \int_{x_3}^{x_T}\frac{1+e^{-x}}{\lambda}dx$$

In addition, we have

$$\frac{dx_T}{dv} = \frac{\frac{e^{-x_3}-e^{-x_T}}{\lambda} + \frac{e^{-x_2}-e^{-x_3}}{\bar{u}+\lambda}\frac{\left(\frac{1}{1+e^{x_3}} - \frac{v-\alpha\lambda}{\lambda}\right)e^{-x_3}}{e^{-x_3}\left(\frac{1}{1+e^{x_3}} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right) - \frac{\bar{u}}{\bar{u}+\lambda}e^{-x_2}\frac{v-\alpha\lambda}{\lambda}}}{\left(k-\alpha+\frac{1}{1+e^{x_T}} - \frac{v-\alpha\lambda}{\lambda}\right)e^{-x_T} + e^{-x_2}\frac{v-\alpha\lambda}{\lambda}\frac{e^{-x_3}\left(\frac{1}{1+e^{x_3}} - \frac{v-\alpha\lambda}{\lambda}\right)}{e^{-x_3}\left(\frac{1}{1+e^{x_3}} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right) - \frac{\bar{u}}{\bar{u}+\lambda}e^{-x_2}\frac{v-\alpha\lambda}{\lambda}}}.$$

which is positive because the second term on the denominator is decreasing in x_2 and

$x_3 > x_2$. Plugging into the payoff derivative, we obtain

$$\begin{aligned}\frac{dV}{dv} &= \frac{e^{-x_3}-e^{-x_T}}{\lambda}+\frac{e^{-x_2}-e^{-x_3}}{\bar{u}+\lambda}\frac{\left(\frac{1}{1+e^{x_3}}-\frac{v-\alpha\lambda}{\lambda}\right)e^{-x_3}}{e^{-x_3}\left(\frac{1}{1+e^{x_3}}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)-\frac{\bar{u}}{\bar{u}+\lambda}e^{-x_2}\frac{v-\alpha\lambda}{\lambda}} \\ &\quad +\frac{dx_T}{dv}\left(\frac{1}{1+e^{x_T}}-\alpha\right)-\int_{x_0}^{x_1}\frac{1+e^{-x}}{\lambda}dx-\int_{x_2}^{x_3}\frac{1+e^{-x}}{\lambda+\bar{u}}dx-\int_{x_3}^{x_T}\frac{1+e^{-x}}{\lambda}dx\end{aligned}$$

A.5 Summary

We have shown that, in all kinds of equilibria, the term $p_T \geq \alpha$ appears in the effect of k on payoffs. In other words, the potential benefits of a higher penalty depend on how likely it is to obtain a success by working harder, compared to the cost of effort (but surprisingly it does not directly depend on the size of the loss).

In conclusion, higher penalties make the agent (weakly) work harder, and the effect on payoffs depends on the productivity of this additional effort. Fix a deadline T, and define

$$\begin{aligned} x_{T_0} &: = x_0 + (\lambda + u)\, T, \text{ and} \\ x_{T_1} &: = x_1 + \left(T - \frac{x_1 - x_0}{\lambda}\right)(\lambda + \bar{u}) . \end{aligned}$$

We now summarize our findings on the role of penalties.

Lemma 17

1. *The terminal belief $x_T(k)$ is (weakly) increasing in k, and strictly increasing if the equilibrium effort is not constant.*
2. *If the equilibrium effort is constant, payoffs are strictly decreasing in k.*
3. *If the equilibrium effort is not constant, and $1 + e^{x_T(k)} > \alpha^{-1}$ then payoffs are strictly decreasing in k.*
4. *If the equilibrium involves an interior effort phase, the condition in (3.) is also necessary.*
5. *If positive effort is exerted for some t, the optimal penalty level k^* is such that*

$$\begin{aligned} &\left(1+e^{x_T(k^*)}\right)^{-1} = \alpha && \text{if} \quad \left(1+e^{x_{T_1}}\right)^{-1} < \alpha \\ &\frac{1}{1+e^{x_T(k^*)}} + \frac{\left(e^{x_2(x_T(k^*))}+1\right)(v-\alpha\lambda)-\lambda}{\bar{u}e^{x_2(x_T(k^*))}\left(1+e^{x_2(x_T(k^*))}\right)} = \alpha && \text{if} \quad \frac{1}{1+e^{x_{T_1}}} > \alpha > \frac{1}{1+e^{x_{T_0}}} + \frac{(e^{x_0}+1)(v-\alpha\lambda)-\lambda}{\bar{u}e^{x_0}(1+e^{x_0})} \\ &x_T(k^*) = x_0 + (\lambda+\bar{u})\,T && \text{if} \quad \frac{1}{1+e^{x_{T_0}}} + \frac{(e^{x_0}+1)(v-\alpha\lambda)-\lambda}{\bar{u}e^{x_0}(1+e^{x_0})} > \alpha, \end{aligned}$$

Proof of Lemma 17: (1.–4.) These statements summarize the results in the previous subsection. Notice that the only case in which $1 + e^{x_T(k)} > \alpha^{-1}$ is only sufficient for payoffs to decrease is given by "zero-full" equilibria.

(5.) We need to discuss how we move across various kinds of equilibria as k increases, for a fixed T. Start with k close to α. In this case, the agent never works at the deadline. If the deadline is too short, no effort will ever be exerted, and payoffs can only decrease in k. Similarly, if full effort is exerted throughout, the penalty has an unambiguous negative effect.

Assume that the T is large enough that some effort is exerted for all $k \in [\alpha, v/\lambda]$. Furthermore, assume that when $k = v/\lambda$ full effort is *not* exerted throughout. Under these assumptions, we start with (zero)-interior-(full)-zero equilibria. Notice that while terminal beliefs x_T increase with k, so does the critical level x^*. It is easy to check that

$$\frac{dx^*}{dk} = \frac{1}{k - \alpha} > \frac{dx_T}{dk},$$

in all the previous cases. Therefore, as k increases, we (possibly) move to (zero)-interior-full, and eventually to zero-full, if $x_0 < x_1$.

In all equilibria involving full effort at the end, the critical value of x_2 is decreasing in x_T and hence in k (see the previous analysis). Moreover, notice that when equilibria move from zero-interior-full to zero-full, the derivative of the payoff function is continuous, becomes proportional to

$$\frac{1}{1 + e^{x_T}} - \alpha + \frac{\left(e^{x_2} + 1\right)\left(v - \alpha\lambda\right) - \lambda}{\bar{u}e^{x_2}\left(1 + e^{x_2}\right)},$$

which is continuous at $x_2 = x_1$.

Finally, defining the critical values of x_T as the thresholds for a full effort and a zero-full effort equilibrium with $x_2\left(x_T\right) = x_1$ establishes the result. □

Analyzing the role of the cost of waiting is more complex, as v enters the flow payoffs and not only the thresholds and continuation values. Nevertheless, we summarize our findings in the following lemma, where we show that the cost of waiting is generally more detrimental to payoffs than the penalties. Intuitively, for higher values of the penalties, the agent works longer and reduces the likelihood of incurring the penalties. On the contrary, the cost of waiting is incurred as a flow, and even higher effort levels (hence likelihood of the breakthrough) do not compensate the agent enough.

Lemma 18 *Fix a deadline T.*

1. *The terminal belief $x_T(k)$ is (weakly) increasing in v, and strictly increasing if the equilibrium effort is not constant.*

2. *If the equilibrium does not involve an intermediate full effort phase, then*

$$\frac{dV}{dk} < 0 \Rightarrow \frac{dV}{dv} < 0.$$

Proof of Lemma 18: (1.–2.) These observations follow from the discussion in the previous section. Note that the relationship between dV/dk and dV/dv is ambiguous only when the equilibrium is of the zero-interior-full-zero type. □

B Observable Case

Attention is restricted throughout to the maximum effort equilibrium (characterized by $(\bar{x}_t)$). Note first that, if by decreasing k, we can make the agent worse off, it must be that his payoff is strictly increasing in k for some value of k, and we might so consider small changes dk. Furthermore, if such parameters existed, it must be the case that we can find them such that, prior to the decrease, $x = x^0$ is such that the agent is indifferent between working and not working, i.e. $x = \bar{x}_0$. (Without loss, we take $t = 0$). To see this, note that if x^0 is such that the agent has strict incentives to work, slightly decreasing k must make him better off (as it does not change his flow wage, but decreases his penalty in case the deadline is reached). If instead x^0 is such that he has strict incentives not to work at the beginning, then we can consider as initial conditions the point (x_t, t) at which $x_t = \bar{x}_t$: after all, the wage he will receive until then is the same whether the penalty is slightly decreased or not.

Then, decreasing the cost by dk changes the overall payoff by

$$\Delta = \left(1 - e^{-x-(\lambda+\bar{u})T}\right) - \left[\alpha\left(1+e^{-x}\right) - \frac{d}{dx}\left(\left(1+e^{-x}\right)V(x)\right) - e^{-x}\right]\bar{u}\frac{dt}{dk},$$

where $dt > 0$ is the time until the high effort region is reached, given that the cost has been decreased by $dk > 0$. By definition of the boundary (or by the envelope theorem) the first two terms in the square brackets cancel, and we can totally differentiate the formula for $\bar{x}_t$ w.r.t. t

and k to solve for dt/dk. Doing so yields

$$\begin{aligned}\Delta &= \frac{\alpha(\lambda+\bar{u})}{(1+k)(\lambda+\bar{u})-\alpha\bar{u} \quad -v \;\; - e^{(\lambda+\bar{u})T}((\alpha+1)\lambda+\bar{u}-v)} \\ &\quad + \frac{\alpha\bar{u}(\lambda+\bar{u})}{M\left((1-e^{-(\lambda+\bar{u})T})\left(\frac{\alpha\bar{u}+v}{\lambda+\bar{u}}-k-1\right)-\alpha+k\right)}+1,\end{aligned}$$

where

$$M := \lambda e^{(\lambda+\bar{u})T}((\alpha+1)\lambda+\bar{u}-v)-(2\lambda+\bar{u})((1+k)(\lambda+\bar{u})-\alpha\bar{u} \quad -v \;).$$

Letting $X = e^{-(\lambda+\bar{u})T}$, we can differentiate:

$$\begin{aligned}\frac{d\Delta}{dX} &= -\alpha(\lambda+\bar{u})\frac{(1+\alpha)\lambda-v}{(\lambda(1+\alpha-(k+1)X)+\bar{u}X(\alpha-k-1)+\bar{u}+v(X-1))^2} \\ &\quad -\alpha(\lambda+\bar{u})\frac{\lambda\bar{u}(2\lambda+\bar{u})}{(\lambda((1+\alpha)\lambda+\bar{u}-v)-X(2\lambda+\bar{u})((1+k)(\lambda+\bar{u})-\alpha\bar{u}-v))^2}.\end{aligned}$$

This is negative, so we might evaluate Δ at the highest possible value ($X=1$), to get an upper bound

$$\frac{\frac{\alpha\bar{u}}{k\lambda+(\lambda+\bar{u})(1+k-\alpha)-v}-k}{\alpha-k},$$

which is of the sign of

$$k-\frac{\alpha\bar{u}}{k\lambda+(\lambda+\bar{u})(1+k-\alpha)-v}.$$

Plainly, this is decreasing in v; set it equal to $(1+\alpha)\lambda$; the resulting expression,

$$k-\frac{\alpha\bar{u}}{(k-\alpha)(2\lambda+\bar{u})+\bar{u}},$$

is increasing in λ, so set it to zero, and we obtain

$$k-\frac{\alpha}{1+k-\alpha},$$

which is positive, as it is decreasing in α and equal to 0 when $\alpha = k$. So decreasing k always increases the payoff of the worker.